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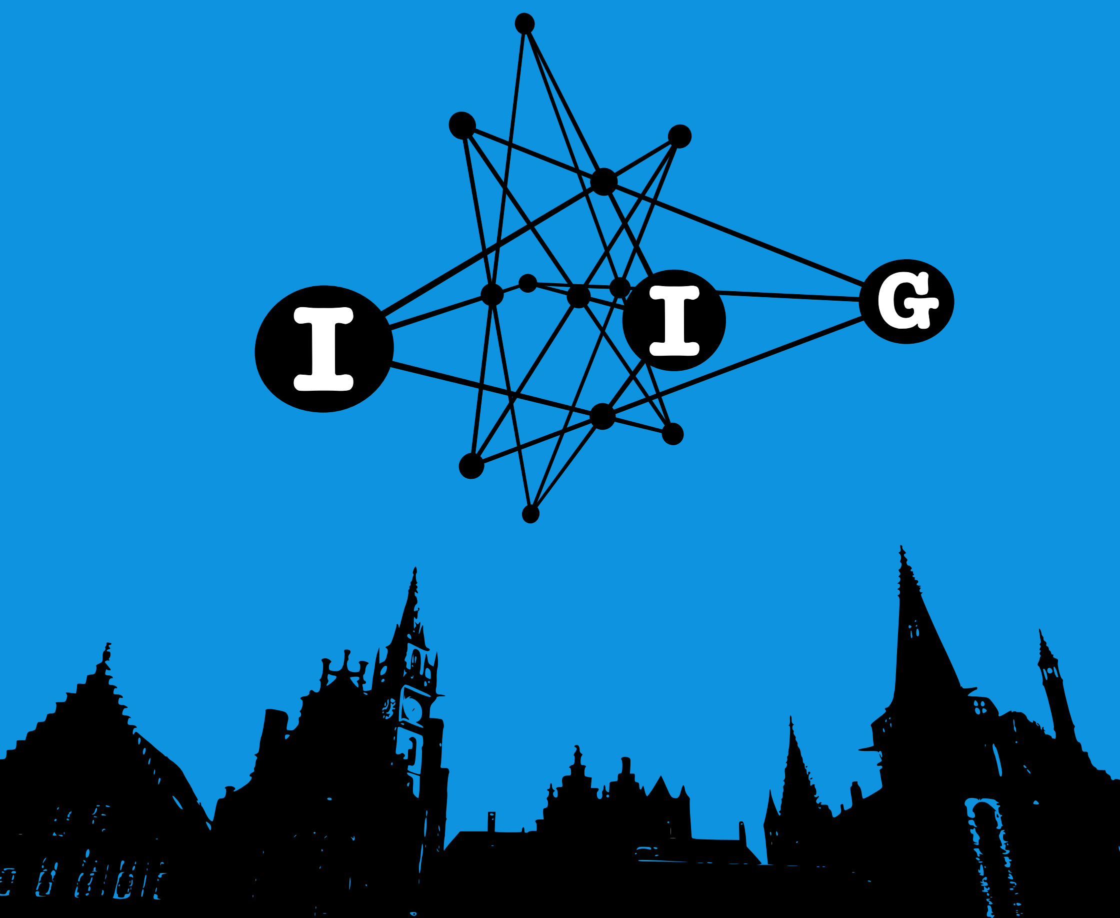

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Twisted hyperbolic flocks

Norman L. Johnson

We give a generalization of the theory of flocks of hyperbolic quadrics in $\mathrm{PG}(3, q)$ to what is called an α-twisted hyperbolic flock in an arbitrary 3-dimensional projective space over a field K. We obtain an equivalence between a set of translation planes with spreads in $\mathrm{PG}(3, K)$ that admit affine homology groups such that the axis and coaxis and one orbit is a twisted regulus. Examples and generalizations are also given.

1. Introduction

In this article, various ideas involving derivation nets and connections with other geometric objects are considered. Previously, the author established a strong connection between derivable nets and 4-dimensional vector spaces over skewfields K (see [Johnson 2000]). There is a basic embedding theory, whereby any derivable net may be embedded in a 3-dimensional projective space $\mathrm{PG}(3, K)$ over a skewfield K. There are strong connections with this idea and that of finite dual nets admitting the axiom of Pasch (see [Thas and De Clerck 1977/78]). The full group of a derivable net may be realized as $\mathrm{P\Gamma L}(4, K)_N$, where N is a fixed line of $\mathrm{PG}(3, K)$. Using this group, there is a retraction method available that realizes the original derivable net within an associated 4-dimensional vector space V_4/K (as a left vector space), where the derivable net now has a classical form:

$$x = 0,\ \ y = dx;\ \forall d \in K;\ \text{components are right spaces;}$$
$$P(a, b) = \{(ca, cb, da, db);\ \forall c, d \in K\};\ \text{Baer subspaces are left spaces.}$$

The derived net is

$$x = 0,\ \ y = xd;\ \forall d \in K;\ \text{components are left spaces;}$$
$$P(a, b) = \{(ac, bc, ad, bd);\ \forall c, d \in K\};\ \text{Baer subspaces are right spaces.}$$

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This form is a regulus in affine form when K is a field. This classical form is called a “pseudo-regulus”, when K is a noncommutative skewfield (division ring). In the finite case, De Clerck and Johnson [1992], putting together ideas of both nets and dual nets of Thas and De Clerck [1977/78], were able to show that “every finite derivable net is a regulus net”.

The reader might note that in [Donati and Durante 2020; Durante 2019], the term pseudo-regulus is used for a finite regulus that is represented in a twisted variation, called a “twisted regulus” in this article.

Actually, the results covered, more generally, what in the affine case are called “subplane covered nets”, where the Baer subplanes are replaced by “little” arbitrary subplanes. In this setting, in the finite case, and necessarily finite dimensional case, a finite subplane covered net may be embedded in $\mathrm{PG}(n+1, q)$, where the group of the subplane covered net now becomes $\Gamma\mathrm{L}(2n, q)_N$, where N is now a co-dimension 2-subspace of $\mathrm{PG}(n+1, q)$. The retraction back to the affine form now shows that the subplane covered net is within a vector space $V_{2n}/\,\mathrm{GF}(q)$, and is also a “regulus net”, where the term “regulus” is appropriately generalized.

In [Johnson 2000] we generalized all of these ideas to the arbitrary subplane covered net/derivable net and the theory is extended to show that every subplane covered net is a pseudo-regulus net in a vector space over K a skewfield (either left or right may be specified) and the dimension need not be finite for general subplane covered nets. While this theory arose from finite geometry, finite nets and derivation, it has now grown beyond finite geometry into infinite incidence geometry, noncommutative algebra, topological geometry, and many other related areas. It is in this spirit that we offer this analysis.

As every derivable net may be embedded in a 4-dimensional vector space, V_4/K, for K a skewfield, fix the 4-space and ask the question: Are there other derivable nets that (although they are pseudo-regulus nets somewhere) cannot be considered equivalent to the original derivable nets that we just embedded and retracted? This was the question the author decided to consider when writing [Johnson 2021a; Johnson 2021b]). The idea was to measure the difference of these additional derivable nets by using the original embedded/retracted pseudo-regulus net that was used to construct the 4-dimensional vector space, by asking how the two sets of Baer components (subplanes incident with the origin) those of the original pseudo-regulus net and those of the imposter derivable net are related. It turned out that the main classification result of [Johnson 2021a] established that the two derivable nets could share exactly 0, 1, 2 or at least three Baer subspaces. These imposter derivable nets were then called type i, where $i = 0, 1, 2, 3$. There are many varieties, but using the group of the original pseudo-regulus net, the type 1, 2, 3 nets are conjugates within V_4/K, of three derivable net types, which we will call the “generic types”. In this setting, we use the term “reducible” as our manner of classification shows that there are other skewfields L, related to the derivable nets (recalling that all of them are actually pseudo-regulus nets), where all of these are isomorphic to K, in these three cases. Furthermore, the associated matrices (see below) support the reducible terminology, particularly in the generic cases.

The associated skewfield L is not isomorphic to K if and only if the type becomes type 0, and there is a connection then to classical linear algebra and irreducible groups acting on vector spaces that may be used to then classify the type 0 forms. So, here, we are interested in the reducible types $1, 2, 3;$ we call these "derivative regulus nets", "twisted regulus nets" and "classical regulus nets", when K is a field and the "twisted derivative pseudo-regulus nets" , "twisted pseudo-regulus nets" and "pseudo-regulus nets" relative to K, when K is a noncommutative skewfield (or when is otherwise considered a general derivable net), respectively.

So, in this article, we consider what can be said regarding translation planes with spreads in $PG(3, K)$, for K a skewfield (this needs to be defined more carefully, if K is noncommutative) with particular attention to derivative nets and twisted nets, as well as with pseudo-regulus nets. When K is a skewfield, the analysis leads us into deep noncommutative ring theory, where Brauer groups, and cohomology are the principle usable theories. Therefore, we now specialize the question to considering only the situation when K is a field, and the associated derivable nets are coordinatized by fields isomorphic to K, but do not make any other assumptions.

Our derivable nets now have the following generic forms:

For vectors (x_1, x_2, x_3, x_4), let $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

$$x = 0, \;\; y = x \begin{bmatrix} u & A(u) \\ 0 & u \end{bmatrix}; \forall u \in K, \tag{1}$$

where A is an additive derivative function ("derivative regulus net") on K:

$$A(uv) = A(v)u + vA(u), \forall u, v \in K,$$

and A is nontrivial (not trivially the zero function). This is the our generic type 1 derivable net, also called a derivative net, and notice that the condition on A guarantees that the matrix set is a field isomorphic to K.

$$x = 0, \;\; y = x \begin{bmatrix} u^{\alpha} & 0 \\ 0 & u \end{bmatrix}; \forall u \in K, \tag{2}$$

and α is a nontrivial automorphism of K.

Again, the matrix set is isomorphic to K; this is our generic "twisted regulus net".

$$x = 0, \;\; y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}; \forall u \in K; \tag{3}$$

the generic ("classical") regulus net.

We note that for finite nets, the type 1 nets do not occur.

A hyperbolic quadric H when viewed in an affine form is a classical/generic regulus net within a 4-dimensional vector space V_4 over a finite field $\mathrm{GF}(q)$. A flock of H is a covering of H by a set of $q+1$ mutually disjoint planes in $\mathrm{PG}(3, q)$. Associated with a hyperbolic flock is a translation plane in the associated V_4, that admits an affine homology group of order $q-1$, one of whose orbits union the axis and coaxis becomes a regulus net and then all orbits union the axis and coaxis are regulus nets, the union of which define a translation plane; the translation plane of the hyperbolic flock. These two geometries, the hyperbolic flock and the translation plane are equivalent. There are exactly the following classes; the flocks are the linear flock, where the associated planes of $\mathrm{PG}(3, q)$ share a line, and the Thas flocks, with a few exceptions. The Thas flocks correspond to the regular nearfield planes and the exceptional flocks correspond to the irregular nearfield planes and are due to a number of mathematicians from different points of view [Bader 1988; Baker and Ebert 1987; Cherowitzo et al. 2017; Johnson 1989a]). There are three irregular nearfields of orders $11^2, 23^2, 59^2$ admitting the regulus-inducing subgroups, and Bader, Bonisoli and Johnson independently for all three orders, and by Baker and Ebert for orders $11^2, 23^2$ determined the flocks/translation planes by using essentially different methods. The main point here is that the associated translation planes are all Bol planes. The subject of the existence of finite Bol planes has been of considerable interest, and finally a complete classification was given in [Thas 1990; Bader and Lunardon 1989]. There are various possible formulations for this classification, depending on whether it is phrased in the associated translation plane or in the hyperbolic flock. Sometimes, names are used, sometimes the name of the algebra coordinatizing the structures are used to describe the structures. For uniformity, here we shall use the name of the coordinate structures for the translation plane version, and the names of the mathematicians finding the flocks in the flock version.

Theorem 1 [Thas 1990; Bader and Lunardon 1989]. – ***Classification of finite hyperbolic flocks/translation planes admitting regulus-inducing homology groups in* $\mathrm{PG}(3, q)$.**

Plane version: *The translation planes are exactly the nearfield planes*; *based upon the regular nearfields of order q^2 and the three irregular nearfields of orders* $11^2, 23^2$ *and* 59^2.

Flock version: *The hyperbolic flocks are exactly the Thas flocks and the flocks of Bader*, *Baker-Ebert*, *Bonisoli*, *Johnson.*

In the infinite planes not all nearfield planes need be Bol, and there is a much richer set of examples. The interested reader is referred to [Johnson et al. 2007] for additional information and for the pertinent references.

There are also some interesting infinite classes of partial spreads related to hyperbolic flocks. In particular, there are partial spreads that contain the irregular nearfields spreads of degree the order of the planes (see [Bader et al. 2002]).

It is also an interesting question on how large a partial hyperbolic flock could be. For example, could there exist a deficiency one partial hyperbolic flock; missing exactly one plane? There are geometric reasons for asking such a question. It was shwon in [Johnson 1990; 1989b] that any embeddable partial deficiency one hyperbolic flock arises from a translation planes with spread in $\mathrm{PG}(3, q)$ that admits a Baer group of order $q - 1$ are and, indeed, the two incidence structures are equivalent. The partial flock may be uniquely extended to a hyperbolic flock if and only if the net of degree $q + 1$ containing the Baer axis and coaxis is a regulus net.

Such partial geometries that are not extendable are extremely rare and occur in just a few known translation planes. These are as follows: orders 2^4 and 3^4. These translation planes are derivable by a twisted regulus net and are also transitive on the partial spread defining the partial hyperbolic flock. The most general result is the theorem in [Johnson and Cordero 2009] that classifies such translation planes of order p^4, where is p is a prime. The result is that there are exactly two such partial extendable hyperbolic flocks of deficiency one of order p^4 which are transitive and derivable, the Johnson partial flock of degree 4 in PG(2, 4) and the Johnson-Pomareda partial flock of degree 9 in PG(3, 9).

In [Royle 1998], there are four partial hyperbolic flocks of deficiency one, two in PG(3, 5) and two in PG(3, 7).

Concerning Baer groups over infinite fields or over skewfields, these have been developed somewhat in [Johnson 2000; 2010].

In the present article, the ideas of hyperbolic flocks and associated translations are extended to the twisted hyperbolic case over arbitrary fields. It would be also a natural continuation of these ideas to extend the theory of Baer groups in translation planes over arbitrary fields that connects to partial hyperbolic flocks of deficiency one.

In the finite case and considering the associated affine homology group of order $q - 1$, it might be stressed that it is required that one orbit of this group together with the axis and coaxis is a regulus (net).

In a classical sense, there is exactly one other type of derivable net that can sit in $\mathrm{PG}(3, q)$, the "twisted regulus net", where the twisting comes about using a nontrivial automorphism of $\mathrm{GF}(q)$. Could there be something like a "twisted" hyperbolic quadric form in the projective case?

For flocks of quadratic cones in the finite case, there are associated translation planes with spreads in $\mathrm{PG}(3, q)$, that admit an affine elation group, one orbit of which, together with the axis of the group is a regulus net. There are infinite versions of these conical flock planes, as well.

Here are the problems we consider in this article: Let D be any reducible derivable net in PG(3, K), for K a field. When a derivable net is reducible and of type 1 or 2, there are either 1 or 2 associated Baer subspaces (incident with the zero vector) that are shared by a classical regulus net. By "generic" we mean that if there is 1 net then the Baer subplane is $P(0, 1)$ and if there are two, these two Baer subplanes are $P(0, 1)$ and $P(1, 0)$.

(1) Are there translation planes with spreads in PG(3, K), that admit an affine homology group such that together with the axis and coaxis and one orbit form a net isomorphic to D? If so, are there associated "flocks of D" by planes within PG(3, K)?

(2) Are there translation planes with spreads in PG(3, K), that admit an affine elation group such that together with the axis and one orbit form a net isomorphic to D? If so, are there associated "flocks of D-cones" by planes within PG(3, K)?

When K is GF(q), the Desarguesian planes satisfy both of the conditions of (1) and (2), simultaneously; the associated flocks of the regulus and flocks of quadratic cones are always "linear", in that the covering planes share a line of PG(3, q).

(3) Suppose a translation plane with spread in PG(3, K) admits both an affine homology group and an elation group corresponding to the same type of derivable net. Are the translation planes known? If so, are there associated flocks and then, if so, are the flocks linear?

1.1. ***The general generic form for translation planes of D-cones.*** The generic form for translation planes of D-cones is

$$x = 0,\ \ y = x\left(\begin{bmatrix} F(t) & G(t) \\ t & 0 \end{bmatrix} + \begin{bmatrix} u^{\alpha} & A(t) \\ 0 & u \end{bmatrix}\right);\ \forall t, u \in K,$$

where F and G are functions on K, for K a skewfield. We only consider the field case, in this article.

(1) If $\alpha = 1$, then A is a derivative function (type 1, A nonidentically zero); derivative net.

(2) If $\alpha \neq 1$ then this is a type 2 or twisted regulus net (the derivative function occurs only when $\alpha = 1$, in the field case); this has been developed in [Cherowitzo and Johnson 2011]. The form for the flocks of α-cones is

$$\gamma_t : x_1 t + x_2 G(t)^{\alpha} - x_3 F(t) + x_4 = 0,$$

where (x_1, x_2, x_3, x_4) represent points of PG(3, K), using homogeneous coordinates.

(3) If $\alpha = 1$ and A identically zero, this is the conical flock type, which is known to exist in both finite and infinite versions.

1.2. ***The general form for translation planes of D-flocks.*** The general form for translation planes of D-flocks is

$$x=0,\ y=x\begin{bmatrix} v^{\alpha} & A(t) \\ 0 & v \end{bmatrix},\ y=x\left(\begin{bmatrix} F(t) & G(t) \\ 1 & t \end{bmatrix}\begin{bmatrix} u^{\alpha} & A(t) \\ 0 & u \end{bmatrix}\right);$$
$$\forall t, v, u \neq 0 \in K,$$

where F and G are functions on K, and where

(1) if $\alpha = 1$, then A is a derivative function (type 1, A nonidentically zero); derivative net;

(2) if $\alpha \neq 1$, then this is a type 2 or twisted regulus net.

We are able to extend the theory to include the type 2 or twisted regulus nets and to show equivalence between flocks of α-reguli by planes of PG(3, K), and translation planes using a twisted regulus net instead of the regulus hyperbolic quadric type plane. We also use the term twisted hyperbolic flock in this setting.

We obtain an equivalence between twisted hyperbolic flocks in PG(3, K), that admit an automorphism α and translation planes with spreads in PG(3, K), that admit an affine homology group, one of whose orbits together with the axis and coaxis is a twisted regulus net. Both finite and infinite examples are given. For the derivative type possibilities, we will construct what we shall call the "classical derivative type spread", which is a translation plane that conceivably could correspond to a flock of a D-conic and a flock of a D-derivable net, but the connections with derivable type spreads have not yet been established.

2. The main results

We begin with our definition of an α-twisted hyperbolic quadric in PG(3, K), for K an arbitrary field, and α is an automorphism of K, possibly trivial. We use (or may use) homogeneous coordinates for the definition.

Definition 2. An α-quadric Q^{α} is a nondegenerate variety of the form

$$\begin{aligned} Q^{\alpha}(x_1, x_2, x_3, x_4) = {} & Ax_1^{\alpha+1} + Bx_2^{\alpha+1} + Cx_3^{\alpha+1} + Dx_4^{\alpha+1} \\ & + Ex_1^{\alpha}x_2 + Fx_1^{\alpha}x_3 + Gx_1^{\alpha}x_4 \\ & + Hx_2^{\alpha}x_1 + Zx_2^{\alpha}x_3 + Jx_2^{\alpha}x_4 \\ & + Kx_3^{\alpha}x_1 + Lx_3^{\alpha}x_2 + Mx_3^{\alpha}x_4 \\ & + Nx_4^{\alpha}x_1 + Rx_4^{\alpha}x_2 + Sx_4^{\alpha}x_3. \end{aligned}$$

An α-conic would then be any plane intersection of Q^{α}.By the plane in question to intersect H^{α} in a nondegenerate α-conic, it is meant that the plane does not contain a line of the α-regulus (or α-twisted regulus).

We have defined generic reducible nets as those containing Baer subplanes that force the representation of the derivable to be diagonal or partially diagonal. We are concerned with translation planes that contain derivable nets that contain either 1 or 2 Baer subplanes of an associated regulus derivable net, where the derivable net is not diagonal or partially diagonal. By results from [Johnson 2021a], there are mappings in the collineation group of the associated regulus derivable net that map the type 1 or type 2 derivable net into a generic net. When a translation plane with spread in $\mathrm{PG}(3, F)$, the mappings are in $\mathrm{GL}(2, F)_{left}$, where the associated vector space is a left 4-dimensional F-space.

Theorem 3. *A translation plane π with spread in* $\mathrm{PG}(3, F)$, *for F a skewfield such that the components are left subspaces may be represented in the form $x = 0$, $y = xM$, where $M \in \lambda$ is a subset of* $\mathrm{GL}(2, F)_{left}$.

(1) *Assume that there is a left derivable net D within λ containing $x = 0,\ y = x$ and $y = x$ and assume that D is reducible with respect to the standard right pseudo-regulus net R. Then there is an element g of* $\mathrm{GL}(2, F)_{left}$ *such that g leaves R invariant maps D to the associated generic pseudo-regulus net.*

(2) *Furthermore, πg is isomorphic to π and is in* $\mathrm{PG}(3, F)$.

Why this theorem is important is that since and although a α-twisted derivable net is of type 2, it may not be in generic form. The above theorem states that any translation plane corresponding to a type 1 or type 2 derivable net and admits an affine homology group one of orbit of which and the axis and coaxis is a type 1 or 2 derivable net is isomorphic to one that admits a generic derivable net of type 1 or of type 2.

In this section, we show the equivalence between certain spreads in $\mathrm{PG}(3, K)$, for K a field and flocks of planes within $\mathrm{PG}(3, K)$ that form a cover that we call an α-twisted hyperbolic quadric.

Theorem 4. *Let Σ be a translation plane with spread in* $\mathrm{PG}(3, K)$, *for K an arbitrary field. Let α denote an automorphism of K, possibly trivial. Assume that Σ admits an affine homology group one orbit of which, together with the axis and coaxis, is a twisted regulus net. Then all orbits union $x = 0,\ y = 0$ are twisted regulus nets and the spread may be coordinatized in the following form: Let V_4 be the associated 4-dimensional vector space over K. Letting x and y denote 2-vectors, then the spread is*

$$x = 0,\ y = 0,\ y = x\begin{bmatrix} u^{\alpha} & 0 \\ 0 & u \end{bmatrix},\ \textit{and}\ \ y = x\begin{bmatrix} f(t) & g(t) \\ 1 & t \end{bmatrix}\begin{bmatrix} v^{\alpha} & 0 \\ 0 & v \end{bmatrix};$$
$$\forall u, t, v, uv \neq 0,\ \textit{of}\ K,$$

and functions f, g on K. Furthermore, f is bijective.

Proof. To obtain a spread, the differences of the matrices involved in the components must be nonsingular, this implies that f is injective. To see that f is bijective, consider the vector $(1, -a, 0, 1)$ for $a \in K$. This point is on

$$y = x \begin{bmatrix} f(t) & g(t) \\ 1 & t \end{bmatrix} \begin{bmatrix} v^\alpha & 0 \\ 0 & v \end{bmatrix},$$

for some $v \neq 0$, so that $(f(t) - a)v^\alpha = 0$, and $(g(t) - at)v = 1$, so that $f(t) = a$, for some t, proving that f is bijective. Since the twisted regulus net may be assumed to be generic and then is well known to have the given form, we have the proof. □

Theorem 5. *Let Σ denote the spread of the previous theorem. For* PG(3, K), *and points written in homogeneous coordinates, the "α-twisted hyperbolic quadric" has the form*

$$\{(x_1, x_2, x_3, x_4);\ \textit{such that } x_1 x_4^\alpha = x_2^\alpha x_3\}.$$

Then there is a flock of the α-twisted hyperbolic quadric with the flock of planes as follows of PG(3, K), as follows:

$$\pi_t : -x_1 g(t)^\alpha + x_2 f(t) - x_3 t^\alpha + x_4 = 0,\ \textit{and}\ \ \rho : x_2 = x_3,$$

where the intersection with each plane is a nondegenerate α-conic.

Proof. We may use the affine form $x = 0,\ y = x\left[\begin{smallmatrix} u^\alpha & 0 \\ 0 & u \end{smallmatrix}\right]; \forall u \in K$, of the α-regulus net. Consider the set of vectors on $y = x\left[\begin{smallmatrix} u^\alpha & 0 \\ 0 & u \end{smallmatrix}\right]$, $(x_1, x_2, x_1 u^\alpha, x_2 u)$, to note that if these vectors are also represented by (x_1, x_2, x_3, x_4), then $x_1 x_4^\alpha = x_1(x_2 u)^\alpha$ and $x_2^\alpha x_3 = x_2^\alpha(x_1 u^\alpha)$. On $x = 0$, $(0, 0, x_3, x_4)$, this relationship is still valid. This proves (1). □

Lemma 6. *The α-twisted quadric H^α is*

$$\{(x_3 x_1^\alpha, x_4^\alpha x_1, x_3 x_2^\alpha, x_4^\alpha x_2);\ \textit{where } (x_1, x_2, x_3, x_4) \textit{ are the points of } \mathrm{PG}(3, K)\}.$$

Proof. We note that $(x_3 x_1^\alpha)(x_4^\alpha x_2)^\alpha = (x_4^\alpha x_1)^\alpha (x_3 x_2^\alpha)$. It remains to show the set is onto. For this, we use the affine form $x = 0,\ y = x\left[\begin{smallmatrix} u^\alpha & 0 \\ 0 & u \end{smallmatrix}\right]; \forall u \in K$. Take $x_1 = 0$ to obtain $x = 0$, noting that this is a surjective mapping, for $x_2 \neq 0$. So, now assume that $x_1 \neq 0$. Represent $(z_1, z_2) = (x_3 x_1^\alpha, x_4^\alpha x_1)$, which clearly represents all 2-vectors. Then $(x_3 x_2^\alpha, x_4^\alpha x_2) = (z_1(x_1^{-1} x_2)^\alpha, z_2(x_1^{-1} x_2))$. Therefore, for $u = x_1^{-1} x_2$, for all $x_1, x_2 \in K$, the set represents the affine form $y = x\left[\begin{smallmatrix} u^\alpha & 0 \\ 0 & u \end{smallmatrix}\right]$; and then the homogeneous coordinates represents the projective form of the α-twisted regulus (twisted hyperbolic quadric). Note that $x_2 = 0$ is $y = 0$ so that the full affine form is obtained. The reader could check that when $\alpha = 1$, this is also an alternative method of describing the hyperbolic quadric. This completes the proof of the lemma. □

Proof. Continuing with the proof of the theorem, since Σ is a translation plane, given any vector (x_1, x_2, x_3, x_4), we know that there is a unique spread component containing this vector. Since, we are interesting in connecting the spread components other than the affine form of the α-regulus, we assume that

$$x_1 \text{ or } x_2 \text{ is nonzero and } (x_3, x_4) \neq (x_1 v^\alpha, x_2 v), \text{ for any } v \in K. \qquad (*)$$

What we are trying to do is find planes that cover the α-hyperbolic quadric, so the use of affine and projective α-regulus terms might be confusing for the reader. In the translation plane, there is the classical twisted regulus in affine form. This is not the α-hyperbolic, we will be covering in the projective 3-space. What to look for is what sort of elements $(x_3x_1^\alpha, x_4^\alpha x_1, x_3x_2^\alpha, x_4^\alpha x_2)$ are covered by the planes that arise from the components. These will not quite be sufficient for a complete cover, this is where the plane ρ comes in. So, for each vector (x_1, x_2, x_3, x_4) that satisfies $(*)$, there is a unique pair (t, v) and corresponding component such

$$(x_3, x_4) = (x_1, x_2)\begin{bmatrix} f(t) & g(t) \\ 1 & t \end{bmatrix}\begin{bmatrix} v^\alpha & 0 \\ 0 & v \end{bmatrix}.$$

Hence, we obtain $x_3 = (x_1 f(t) + x_2)v^\alpha$, and $x_4 = (x_1 g(t) + x_2 t)v$. As a result,

$$x_4^\alpha = (x_1^\alpha g(t)^\alpha + x_2^\alpha t^\alpha)v^\alpha.$$

Therefore, we have

$$x_4^\alpha((x_1 f(t) + x_2)v^\alpha) = x_3((x_1^\alpha g(t)^\alpha + x_2^\alpha t^\alpha)v^\alpha).$$

Recalling that, here $v \neq 0$, we then have

$$x_4^\alpha(x_1 f(t) + x_2) = x_3(x_1^\alpha g(t)^\alpha + x_2^\alpha t^\alpha).$$

Recall the points $(x_3x_1^\alpha, x_4^\alpha x_1, x_3x_2^\alpha, x_4^\alpha x_2)$, and note that we line up the above equation, using this order, as follows:

$$\pi_t \cap H^\alpha : x_3x_1^\alpha(-g(t)^\alpha) + x_4^\alpha x_1 f(t) - x_3x_2^\alpha t^\alpha + x_4^\alpha x_2 = 0. \qquad (**)$$

So, for all vectors (x_1, x_2, x_3, x_4) satisfying $(*)$, such that

$$(x_3, x_4) = (x_1, x_2)\begin{bmatrix} f(t) & g(t) \\ 1 & t \end{bmatrix}\begin{bmatrix} v^\alpha & 0 \\ 0 & v \end{bmatrix},$$

we obtain a plane of $PG(3, K)$, $\pi_t : x_1(-g(t)^\alpha) + x_2 f(t) - x_3 t^\alpha + x_4 = 0$, such that the intersection of π_t with H^α is $(**)$.

Lemma 7.

$$\pi_t \cap H^\alpha : x_3x_1^\alpha(-g(t)^\alpha) + x_4^\alpha x_1 f(t) - x_3x_2^\alpha t^\alpha + x_4^\alpha x_2 = 0,$$

is a nondegenerate α-conic.

Proof. So, the intersection of each π_t with H^α has the form

$$x_3x_1^\alpha(-g(t)^\alpha)+x_4^\alpha x_1 f(t)-x_3x_2^\alpha t^\alpha+x_4^\alpha x_2=0,$$

so that $-g(t)^\alpha = F$, $f(t)=N$, $-t^\alpha = I$, and $1=Z$, in the above proposition and the definition of the α-hyperbolic quadric. Assume that the plane π_t contains a line of the α-regulus. In affine form, the lines are $x=0$ and $y=x\left[\begin{smallmatrix} u^\alpha & 0 \\ 0 & u \end{smallmatrix}\right]$, for $u\in K$. Since $x=0$ is $(0,0,x_3,x_4)$, this has been excluded by $(*)$. Similarly the line $y=x\left[\begin{smallmatrix} u^\alpha & 0 \\ 0 & u \end{smallmatrix}\right]$ cannot be contained in π_t, again by $(*)$. Hence, the intersection is a nondegenerate α-conic. □

Lemma 8. *The remaining plane is* ρ; $x_2=x_3$.

Proof. So, we have used $(*)$ to find the planes π_t, from the translation plane. Then all elements of $\{(x_3x_1^\alpha, x_4^\alpha x_1, x_3x_2^\alpha, x_4^\alpha x_2)$; where (x_1,x_2,x_3,x_4) are the points of $\mathrm{PG}(3,K)\}$, such that (x_1,x_2,x_3,x_4) satisfies $(*)$. The "points" that do not satisfy $(*)$ are $x_1=x_2=0$ and $(x_3,x_4)=(x_1v^\alpha, x_2v)$, for $v\in K$. First note that $x_1=x_2$ produces simply then the zero vector. For $(x_3,x_4)=(x_1v^\alpha,x_2v)$, we have

$$\begin{aligned}(x_3x_1^\alpha, x_4^\alpha x_1, x_3x_2^\alpha, x_4^\alpha x_2) &= (x_1^{\alpha+1}v^a, x_1x_2^a v^a, x_1x_2^\alpha v^\alpha, x_2^{\alpha+1}v^a)\\ &= (x_1^{\alpha+1}, x_1x_2^\alpha, x_1x_2^\alpha, x_2^{\alpha+1}),\end{aligned}$$

in homogeneous coordinates. Hence, this is $\rho\cap H^\alpha$, and so ρ has the form $x_2=x_3$, and the intersection is $x_2^{\alpha+1}-x_1x_4^\alpha=0$, a nondegenerate α-conic, as it contains one point from each of the lines of the α-twisted conic; $(1,0,0,0)$, $(0,0,0,1)$ and one point from each of the nonzero components $y=x\left[\begin{smallmatrix} u^\alpha & 0 \\ 0 & u \end{smallmatrix}\right]$, for $u=(x_1^{-1}x_2)$. So, we see that we have a set of planes mutually disjoint on the α-hyperbolic quadric and intersect the α-hyperbolic quadric in α-conics. It remains to show that we have a cover. So assume that some point

$$(x_3x_1^\alpha, x_4^\alpha x_1, x_3x_2^\alpha, x_4^\alpha x_2)$$

is not covered. We know that the points (x_1,x_2,x_3,x_4) that satisfies $(*)$ must be within this set. This only leaves the points on $x=0$, since all of the rest have been considered and checked by looking at ρ. But $(0,0,0,1)$ is in ρ. So we are looking for $(0,0,1,0)$. Consider the question of whether $(0,0,1,0)$ is on π_t, which would say that there exists a unique t_o such that $f(t_o)=0$. Since, f is bijective, there is such a unique plane π_{t_0}. Note that when $\alpha=1$, the planes are $\pi_t: -x_1g(t)+x_2f(t)-x_3t+x_4=0$, and $\rho: x_2=x_3$. when using the form $x_1x_4=x_2x_3$. For flocks of hyperbolic quadrics, the planes are

$$\pi_t^*: x_1-x_2t+x_3f(t)-x_4g(t)=0, \rho: x_2=x_3,$$

when using form for the hyperbolic quadric of $x_1x_4 = x_2x_3$ and homogeneous coordinates (x_1, x_2, x_3, x_4), so we are using (x_4, x_3, x_2, x_1) in the form that works also with $\alpha \neq 1$, but which is clearly equivalent in the $\alpha = 1$ case. □

This completes the proof of the theorem. □

Theorem 9. *Conversely, a flock F of the α-twisted hyperbolic quadric by planes of the given form constructs a translation plane Σ as above.*

Proof. Assume that there is a flock

$$F : \pi_t : -x_1g(t)^\alpha + x_2 f(t) - x_3t^\alpha + x_4 = 0, \ \textit{and} \ \rho : x_2 = x_3.$$

We shall use the formulation of the α-hyperbolic quadric H^α, using homogeneous coordinates, as following $\{(x_3x_1^\alpha, x_4^\alpha x_1, x_3x_2^\alpha, x_4^\alpha x_2);$ where (x_1, x_2, x_3, x_4) are the points of PG(3, K)$\}$. Furthermore, we claim that the following mappings

$$h_{u,v} : (x_1, x_2, x_3, x_4) \to (x_1u^\alpha, x_2u, x_3v^\alpha, x_4v); \forall u, v \in K^*,$$

preserve H^α. To see this, note that $(x_1u^\alpha)(x_4v)^\alpha = (x_2u)^\alpha(x_3v^\alpha)$. Also, note this is the mapping describing the Baer components, which must preserve the α-hyperbolic quadric (see above for the affine form for the α-regulus net and the Baer components $P(a, b)$). Then

$$-x_3x_1^\alpha g(t)^\alpha + x_4^\alpha x_1 f(t) - x_3x_2^\alpha t^\alpha + x_4^\alpha x_2 = 0,$$

is the set of “points” of PG(3, K), then clearly, by working the proof backwards, shows that this is equivalent to having a vector (x_1, x_2, x_3, x_4) being an element of

$$y = x \begin{bmatrix} f(t) & g(t) \\ 1 & t \end{bmatrix} \begin{bmatrix} v_t^\alpha & 0 \\ 0 & v_t \end{bmatrix},$$

for some element $v_t \in K^*$. So, the α-conic of intersection of π_t, then becomes the set of 1-dimensional subspaces of this particular component. This then shows that the set of planes π_t, for all $t \in K$, will directly reconstruct a set of elements

$$\left\{ y = x \begin{bmatrix} f(t) & g(t) \\ 1 & t \end{bmatrix} \begin{bmatrix} v_t^\alpha & 0 \\ 0 & v_t \end{bmatrix}; \forall t \in K \right\},$$

where $v_t \in K^*$, depends on t. Note that $(x_3x_1^\alpha, x_4^\alpha x_1, x_3x_2^\alpha, x_4^\alpha x_2)$. Consider again

$$h_{1,u}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3u^\alpha, x_4u),$$

and the effect on the corresponding elements on H^α, then becomes

$$(x_3x_1^\alpha u^\alpha, x_4^\alpha x_1u^\alpha, x_3x_2^\alpha u^\alpha, x_4^\alpha x_2u^\alpha),$$

which is, of course, the same point of PG(3, K), but also shows that we may expand the point to belong to

$$\begin{bmatrix} f(t) & g(t) \\ 1 & t \end{bmatrix} \begin{bmatrix} v_t^{\alpha} & 0 \\ 0 & v_t \end{bmatrix} \begin{bmatrix} u^{\alpha} & 0 \\ 0 & u \end{bmatrix}.$$

In this way, from the set of planes π_t; $\forall t \in K$, and the intersections with H^{α}, we have reconstructed all components of the form of the target translation plane, with the exception of the set of elements $x = 0$, $y = x\left[\begin{smallmatrix} u^{\alpha} & 0 \\ 0 & u \end{smallmatrix}\right]$.

We now turn to the plane ρ. We first note that the element of elements of H^{α}, that correspond to the elements vectors (x_1, x_2, x_3, x_4) satisfying $(*)$, produced the elements H^{α}, $(x_3 x_1^{\alpha}, x_4^{\alpha} x_1, x_3 x_2^{\alpha}, x_4^{\alpha} x_2)$, with the same $(*)$ condition on the elements (x_1, x_2, x_3, x_4). Hence, we are missing the possible points of H^{α} such that either $x_1 = x_2 = 0$ or $(x_3, x_4) = (x_1 u^{\alpha}, x_2 u)$, for $u \in K^*$. When $x_1 = x_2 = 0$, we see there are no corresponding elements of H^{α}. We shall come back for this technicality, in a moment. Thus, assume $(x_3, x_4) = (x_1 u^{\alpha}, x_2 u)$, for $u \in K^*$. Then

$$(x_3 x_1^{\alpha}, x_4^{\alpha} x_1, x_3 x_2^{\alpha}, x_4^{\alpha} x_2) = (x_1^{\alpha+1} u^{\alpha}, x_2^{\alpha} x_1 u^{\alpha}, x_2^{\alpha} x_1 u^{\alpha}, x_2^{\alpha+1} u^{\alpha}),$$

by which is the intersection of ρ with H^{α}. Therefore, we see that we have recovered the components $y = x\left[\begin{smallmatrix} u^{\alpha} & 0 \\ 0 & u \end{smallmatrix}\right]$. While it may seem that we are missing the component $x = 0$, we note that a spread is the covering of V_4/K by mutually disjoint 2-dimensional K-subspaces. The partial spread is missing exactly the set of points $(0, 0, x_3, x_4)$. So, by the formal addition of $x = 0$, we have proved that we may recover the translation plane from the α-flock. The reader should not confuse the existence of the α-regulus in affine form with the cover of the projective version of the α-regulus. That is, since $x = 0$ in the affine form was not recovered, does not mean that the projective α-regulus was not covered. To make a point a little stronger, we consider where the $x = 0$ version of the projective version was covered. So, consider again, $(x_3 x_1^{\alpha}, x_4^{\alpha} x_1, x_3 x_2^{\alpha}, x_4^{\alpha} x_2)$. If $x_1 = 0$ then we obtain $(0, 0, x_3 x_2^{\alpha}, x_4^{\alpha} x_2)$. Thus, these points correspond to the vectors $(0, x_2, x_3, x_4)$, were covered by the π_t planes/or corresponding spread components. Similarly, $x_2 = 0$, produces $(x_3 x_1^{\alpha}, x_4^{\alpha} x_1, 0, 0)$, and then corresponds to the vector $(x_1, 0, x_3, x_4)$, which again is covered by the π_t. So, it is counter-intuitive but the projective $x = 0$ and $y = 0$, not the affine components $x = 0$, $y = 0$, are covered. So, the missing $x = 0$, the component, is recovered by realizing the covering required for a translation plane is uniquely extended by the partial spread consisting of the other components that correspond directly to the α-flock. Note that every element (x_1, x_2, x_3, x_4) is covered back in the vector space as this point produces $(x_1^{\alpha} x_3, x_4^{\alpha} x_1, x_2^{\alpha} x_3, x_4^{\alpha} x_2)$, a point on the α-hyperbolic quadric. If (x_1, x_2, x_3, x_4) is not covered in the putative spread, then the orbit of this element in the α-regulus-inducing group is not covered. But this orbit is the complete set of points corresponding to $(x_1^\alpha x_3, x_4^\alpha x_1, x_2^\alpha x_3, x_4^\alpha x_2)$ which must be covered by the α-flock. Hence, we have recovered a spread from an α-flock. This completes the proof of the theorem. □

3. Examples

In the finite case, consider the Knuth (type II (ii))/Hughes–Kleinfled semifield plane, defined as follows:

$$x = 0,\ \ y = x \begin{bmatrix} u^\alpha + as^\alpha & bs \\ s^\alpha & u \end{bmatrix}; \forall w, s \in \mathrm{GF}(q),$$

where b is a constant given by

$$u^{\alpha+1} + as^\alpha u - bt^{\alpha+1} \neq 0.$$

Switching notation, to try to match the notation in the theorem in the previous section, note that

$$\begin{bmatrix} a + t^\alpha & b \\ 1 & t \end{bmatrix} \begin{bmatrix} w^\alpha & 0 \\ 0 & w \end{bmatrix} = \begin{bmatrix} aw^\alpha + (tw = u)^\alpha & bw \\ w^\alpha & tw = u \end{bmatrix},$$

which, since $\{w^\alpha, tw\}$ are independent, as $w \neq 0$ and t vary over K, we see we have the following example of an translation plane of the form

$$x = 0,\ \ y = 0,\ \ y = x \begin{bmatrix} u^\alpha & 0 \\ 0 & u \end{bmatrix},\ \ y = x \begin{bmatrix} f(t) = a + t^\alpha & g(t) = b \\ 1 & t \end{bmatrix} \begin{bmatrix} v^\alpha & 0 \\ 0 & v \end{bmatrix};$$

$\forall u, t, v, uv \neq 0$, and functions f, g on K, where now $f(t) = t^\alpha + a$, $g(t) = bt$.

Consider the corresponding α-flock:

$$\pi_t : -x_1 b^\alpha t^\alpha + x_2(t^\alpha + a) - x_3 t^\alpha + x_4 = 0,\ \text{ and }\ \rho : x_2 = x_3.$$

We have

$$\cap\{\pi_t; t \in K\} \cap \rho = \{(x_1, x_2, x_2, -x_2 a); x_1, x_2 \in K\}.$$

Hence, the set of Knuth/Hughes–Kleinfeld semifield planes of this form defines linear α-flocks, in that the planes share a line.

There are obvious generalizations to what might be called the "generalized Knuth/Hughes–Kleinfeld semifield planes" in $\mathrm{PG}(3, K)$. These also give rise to linear α-flocks.

The recent article [Johnson 2021a] classifies derivable nets that lie in $\mathrm{PG}(3, K)$, where K is skewfield. In that article,there is a class of division ring planes $(a, b)_F$, the quaternion division ring planes that actually define derivable nets in $\mathrm{PG}(3, K)$, where K is a Galois field extension of dimension 4 over K. Recalling that the quaternion division rings are 4-dimensional over their centers F, the following was noted:

Theorem 10. *All quaternion division rings $(a,b)_F$ may be represented in the following form over $F(\sqrt{a})$, a quadratic field extension of F, where F is not of characteristic 2:*

$$x=0,\;\; y=x\begin{bmatrix} w^{\alpha} & bs \\ s^{\alpha} & w \end{bmatrix};\, \forall w,s\in F(\sqrt{a}).$$

Here b is a constant satisfying $w^{\alpha+1}-bt^{\alpha+1}\neq 0$, and α is the unique involutory automorphism of $F(\sqrt{a})$ mapping $\sqrt{a}$ to $-\sqrt{a}$.

Hence, every quaternion division ring plane of characteristic odd or 0, constructs a linear α-flock of H^{α}, where the line is now $\{(x_1,x_2,x_2,0);\, x_1,x_2\in L\}$.

4. Spreads that give both flocks of *D*-cones and flocks of *D*-derivable nets.

In this section, we consider spreads in PG(3, K), for K a field, that simultaneously are equivalent to flocks of D-cones and flocks of D-derivable nets. Many of the results are valid for arbitrary skewfields, but we consider here only when K is a field.

Let Σ be a translation plane with spread in PG(3, K), for K a field that admits both an elation group whose axis L has an orbit Δ such that $L\cup\Delta$ is an D-affine derivable net and a homology group whose coaxis L and axis M and some orbit is an D-affine derivable net. We first point out that we may always deal with generic forms of type 1 and type 2 nets by a previous theorem.

We first define a net of the derivable net flock types:

$$x=0,\;\; y=x\begin{bmatrix} u & A(u) \\ 0 & u \end{bmatrix},\;\; y=x\begin{bmatrix} tv & A(u+tv)+bv \\ v & tv \end{bmatrix},$$

for all $u,t\neq 0, v\in K$. Letting $w=u+tv$, the putative spread has the form

$$x=0,\;\; y=x\begin{bmatrix} w & A(w)+bv \\ v & w \end{bmatrix},$$

for all $w,v\in K$. This is a semifield spread if and only if $w^2-v(A(w)+bv)\neq 0$.

Definition 11. A spread of the above type shall be called a “classical derivative net spread” .

Theorem 12. *Under the above hypothesis, then Σ is one of the following translation planes*:

(0) *Σ is a Desarguesian spread (type 3-classical regulus derivable net).*

(1) *K is finite and Σ is a Knuth/Hughes–Kleinfield semifield plane (type 2).*

(2) *K is an infinite field and Σ is a generalized Knuth/Hughes–Kleinfield semifield plane* (*type* 2).

(3) *K is an infinite field of characteristic not* 2, *and Σ is a quaternion division ring plane* (*type* 2).

(4) *In cases* (0), (1), (2), *and* (3), *there is a corresponding α-flock of a twisted hyperbolic quadric and of a flock of an α-conic.*

(5) *K is infinite and Σ is a classical derivative net* (*type* 1) *plane.*

Proof. Let L be coordinatized by $x = 0$. If one of the orbits is an α-regulus choose $y = 0$ to belong to that orbit. Then the form of the elation group E is given by

$$(x, y) \to \left(x, x \begin{bmatrix} u^{\alpha} & 0 \\ 0 & u \end{bmatrix} + y \right),$$

where α is an automorphism of K, possibly trivial. This implies that all of the orbits of E are α-regulus nets that share $x = 0$. This implies that we have a translation plane corresponding to a flock of an α-cone in $\mathrm{PG}(3, K)$. Hence, the form for the translation plane is

$$x = 0, \; y = \begin{bmatrix} u^{\alpha} + f(t) & g(t) \\ t & u \end{bmatrix}; \; \forall t, u \in K.$$

Since there is also an affine homology group H with coaxis $x = 0$ and axis $y = 0$, we choose the orbit defining H to contain $y = x$. Hence, we have the homology group

$$(x, y) \to \left(x, y \begin{bmatrix} u^{\alpha} & 0 \\ 0 & u \end{bmatrix} \right); \; \forall u \in K^{*}.$$

It then follows that, when $t \neq 0$ then

$$\begin{bmatrix} u^{\alpha} + f(t) & g(t) \\ t & u \end{bmatrix} = \begin{bmatrix} t^{\alpha} + f(1) & g(1) \\ 1 & t \end{bmatrix} \begin{bmatrix} v^{\alpha} & 0 \\ 0 & v \end{bmatrix}; \forall t, v \in K, \text{ for } v \neq 0.$$

Hence, $f(1)v^{\alpha} = f(v^{\alpha})$, and $g(1)v = g(v^{\alpha})$. Therefore, the spread has the form

$$x = 0, \; y = \begin{bmatrix} u^{\alpha} + av^{\alpha} & bv \\ v^{\alpha} & u \end{bmatrix}; \; \forall u, v \in K, \text{ where } f(1) = a \text{ and } g(1) = b.$$

When K is $\mathrm{GF}(q)$, this is the definition of a Knuth/Hughes–Kleinfeld semifield plane.

To complete the proof of this result, we need to consider the possible derivable derivative nets and their associated spreads.

Now assume that we have both of the types of cones and hyperbolic quadratic generalizations that we have been discussing in the case where we have a derivative type derivable net: First, we consider the flock of the D-cone, the spread components would be

$$x=0,\ y=x\begin{bmatrix} r+F(s) & A(r)+G(s) \\ s & r \end{bmatrix};\ \forall u,s\in K,$$

for K a field. In general, to have a flock of a D-net, we would have the form

$$x=0,\ y=x\begin{bmatrix} u & A(u) \\ 0 & u \end{bmatrix},\ y=x\begin{bmatrix} f(t) & g(t) \\ 1 & t \end{bmatrix}\begin{bmatrix} v & A(v) \\ 0 & v \end{bmatrix};\ u,t,v\neq 0\in K,$$

the latter of which is

$$\begin{bmatrix} f(t)v & f(t)A(v)+g(t)v \\ v & tv \end{bmatrix}.$$

Let $tv=r$, for $v\neq 0$, and $v=s$.

Therefore, $tv+F(v)=f(t)v$; letting $t=1$, we have $f(1)v=v+F(v)$. So, $F(v)=v(f(1)-1)$, for all nonzero $v\in K$. Since $F(0)=0$, this is also valid for all $v\in K$. Then $tv+v(f(1)-1)=f(t)v$, and $f(t)=t+(f(1)-1)$. Let $f(1)-1=a$. Also,

$$f(t)A(v)+g(t)v=A(r)+G(s)=A(tv)+G(v).$$

Therefore, we have

$$(t+a)A(v)+g(t)v=A(tv)+G(v)=tA(v)+vA(t)+G(v),$$

or equivalently

$$aA(v)+g(t)v=vA(t)+G(v).$$

Letting $t=1$, we have

$$G(v)=aA(v)+g(1)v \text{ as } A(1)=0.$$

Replacing this back in the previous equation,

$$aA(v)+g(t)v=vA(t)+G(v)=vA(t)+aA(v)+g(1)v,$$

we have $g(t)v=vA(t)+g(1)v$. Thus $g(t)=A(t)+g(1)$. Letting $g(1)=b$, so $G(t)=aA(t)+bt$, we obtain

$$x=0,\ y=x\begin{bmatrix} u & A(u) \\ 0 & u \end{bmatrix},\ y=x\begin{bmatrix} t+a & A(t)+b \\ 1 & t \end{bmatrix}\begin{bmatrix} v & A(v) \\ 0 & v \end{bmatrix}$$

is

$$x=0,\ y=x\begin{bmatrix} tv+av & A(tv)+aA(t)+bv \\ v & tv \end{bmatrix}.$$

This implies that

$$A(tv)+aA(t)+bv=(t+a)A(v)+(A(t)+b)v,$$

or equivalently

$$A(tv)+aA(t)=(t+a)A(v)+A(t)v.$$

Letting $v=1$, we see that $A(t)+aA(t)=A(t)$, so that $a=0$, since A is not trivially zero. But then $A(tv)=tA(v)+A(t)v$, just the requirement for a type 1 derivative net. Hence, we have the spread

$$x=0,\ y=x\begin{bmatrix} u & A(u) \\ 0 & u \end{bmatrix},\ y=x\begin{bmatrix} t & A(t)+b \\ 1 & t \end{bmatrix}\begin{bmatrix} v & A(v) \\ 0 & v \end{bmatrix},$$

and, since $tA(v)+A(t)v=A(tv)$,

$$x=0,\ y=x\begin{bmatrix} u & A(u) \\ 0 & u \end{bmatrix},\ y=x\begin{bmatrix} tv & A(tv)+bv \\ v & tv \end{bmatrix},$$

for all $u, t\neq 0, v\in K$. Letting $w=u+tv$, the putative spread has the form

$$x=0,\ y=x\begin{bmatrix} w & A(w)+bv \\ v & w \end{bmatrix}, \forall w, v\in K.$$

Thus, we have a spread if and only $w^2-v(A(w)+bv)\neq 0$, for all $w, v\in K$. This completes the proof of the theorem. □

In [Johnson 2021a], the question was raised that amounted to when a spread of PG(3, K) could be a noncommutative skewfield (division ring), when the characteristic is not 2. We provide the explicit theorem.

Theorem 13 [Johnson 2021a]. ***The classification of matrix noncommutative skewfields.*** *Consider a set S of nonsingular matrices*

$$\begin{bmatrix} f(t,u) & g(t,u) \\ t & u \end{bmatrix}$$

in GL(2, L) *for all u, t in a "field" L of characteristic $\neq 2$, and certain functions $f, g: L\times L\to L$. Assume S is a spread of a 2-dimensional vector space over L, and assume further that S is a noncommutative skewfield.*

Then L is a quadratic extension of a field F by a root θ to an irreducible equation $x^2+\alpha x-\beta$, over F, $\theta^2=\sqrt{\alpha^2+4\beta}$ and noting that $\alpha^2+4\beta=c$ is a nonsquare in F. We may have the open form construction of an isomorphic/different copy, such as $\left[\begin{smallmatrix} u & ta \\ t & u \end{smallmatrix}\right]$; $\forall t, u\in F$, and where σ is the nontrivial element, the involution, in the Galois group of L over F maps $\sqrt{a}$ to $-\sqrt{a}$, which is nontrivial, since the characteristic is not 2. Then, an isomorphic noncommutative skewfield isomorphic to

$$\begin{bmatrix} f(t,u) & g(t,u) \\ t & u \end{bmatrix} \text{ is } \begin{bmatrix} u^\sigma & bt^\sigma \\ t & u \end{bmatrix}; \forall u, t\in L,$$

and b in F and not of the form $\{x^2-ay^2$, for any x, y in $F\}$.

Furthermore, the noncommutative skewfield S is 4*-dimensional, as a vector space, over the center $Z(S) = F$.*

Therefore, it was proved that this was the case exactly when the translation plane has the form

$$x = 0,\ y = \begin{bmatrix} u^\alpha & bv \\ t^\alpha & u \end{bmatrix} \forall t, u \in K,\ \text{ and } \alpha^2 = 1, \text{ and } \alpha \text{ not } 1 \text{ and where } K = F(\sqrt{a}).$$

It was also shown in [Johnson 2021a] that this set of translation planes are the quaternion division ring planes given by the division ring $(a, b)_F$. (The reader is directed to [Johnson 2021a] for the notation and further background).

Using these ideas, but now assuming that F is a noncommutative skewfields. The ideas may be generalized to obtain what are called "generalized quaternion division rings", where there are yet to be concrete examples.

So, it might be possible to generalize the ideas of flocks of α-conics and flocks of D-nets to the arbitrary skewfield case. In fact, this has been done in [Johnson 2000], for flocks of cones over skewfields, and the interested reader is directed there for additional information.

In any case, these would also be examples of translation planes with spreads in PG(3, K), where now K is a skewfield, that would satisfy the hypothesis of the theorem. In this setting, whereas the quaternion division rings would have F as the center in $(a, b)_F$, generalized quaternion division rings would have their center as $Z(F)$, the center of F, over which the division ring could be either finite or infinite dimensional.

We note that each of these planes correspond to two flocks:

- First, of an α-conic of this form: Planes correspond to

  $$x = 0,\ y = \begin{bmatrix} u^\alpha + at^\alpha & bt \\ t^\alpha & u \end{bmatrix}; \forall t, u \in K,$$

  $\gamma_{t^\alpha} : x_1 t^\alpha + x_2 b^\alpha t^\alpha - x_3 a t^\alpha + x_4$, and $\bigcap \gamma_{t^\alpha} = \{(-x_2 b^\alpha + x_3 a, x_2, x_3, 0)\}$.
- And an α-hyperbolic flock of the form $\pi_t : -x_1(bt)^\alpha + x_2 a t^\alpha - x_3 t^\alpha + x_4$, $\rho : x_2 = x_3$; $\bigcap \pi_t \cap \rho = \{(x_1, x_2, x_1 b^\alpha - x_2 a, 0)\}$.

Related problems

In this section, we offer a few related problems, many of which simply ask if various theories for finite incidence structures may be realized more generally over arbitrary fields or skewfields.

In [Cherowitzo and Johnson 2011], it was shown that given there are transitive parallelisms of flocks of α-conics, and of flocks of hyperbolic quadrics in any PG(3, K), for any field admitting an automorphism in the first case (also see [Cherowitzo et al. 2017]).

Problem 14. The open question for parallelisms, at this point, is whether there are always transitive parallelisms of every flock of a twisted hyperbolic quadric?

Problem 15. Determine if there can exist nonlinear α-flocks of hyperbolic quadrics over fields K.

Problem 16. Determine if there can exist flocks of derivative type cones.

Problem 17. Determine if there can exist flocks of derivative derivable nets (the hyperbolic extension).

Problem 18. Complete the study of Baer groups over arbitrary fields and extend the ideas of partial flocks of a hyperbolic quadric of deficiency one to the arbitrary field case.

Problem 19. Flocks over α-conics have been considered over arbitrary fields in [Cherowitzo et al. 2017], where Baer groups were considered in the finite case. There are also connections to the algebraic lifting theory by which any quasifibration in PG(3, K), for K a field that admits a quadratic extension $K(\theta)$ may be "lifted" to a quasifibration in PG(3, $K(\theta)$). In the finite case, lifted spreads may be characterized in terms of an elation group E and Baer group B, such that B normalizes but does not centralize E. Extend this theory to the infinite case.

Problem 20. Continuing with ideas of lifting, in this setting, there could be chains of lifted quasifibrations. This theory is considered in [Biliotti et al. 2001] for the infinite case and might be developed more completely. Of interest are the bilinear α-flokki (α-flocks) that may be obtained in the finite case by listing the Andre' spreads, using a field representation of a Desarguesian plane. Can these bilinear α-flokki be extended to the arbitrary field case?

Problem 21. In the finite case, there are connections with lifting spreads and subgeometry partitions. Extend these ideas to the infinite case. The ideas of "double Baer groups" could then possibly be extended to the arbitrary field case (see [Johnson 2010] for definitions of double Baer groups) as well as the theory of spread retraction.

Appendix

After this article was completed, the author found other examples of irreducible derivable nets arising from quaternion division rings in characteristic 2. Furthermore, a new analysis of twisted hyperbolic quadrics was found so as to develop Baer groups and deficiency one partial hyperbolic flocks. These works are mentioned in the references. Here are the pertinent notes and references.

Corollary 22 [Johnson 2021d]. *Let $K(\tau) = F(\theta, \tau)$ be a Galois extension of F of dimension* 4.

(1) *If we represent the classical $F(\theta, \tau)$-regulus in* $\mathrm{PG}(3, F(\theta, \tau))$ *as*

$$x = 0,\ y = 0,\ y = x \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}; w \in F(\theta, \tau)$$

then

$$x = 0,\ y = x \begin{bmatrix} u^{\sigma} & bt^{\sigma} \\ t & u \end{bmatrix}; \forall t, u \in F(\theta)$$

$$\sigma \neq 1, \sigma^2 = 1.$$

$$\theta^2 = \theta + a$$

$$\theta^{\sigma} = \theta + 1,$$

$$b \notin \left\{ w^{\sigma+1}; w \in F(\theta)^{*} \right\},\ b \in F.$$

This is an irreducible (*type* 0) *derivable net within* $\mathrm{PG}(3, K)$. *All quaternion division rings in characteristic* 2 *may be represented as derivable nets as above.*

Remarks. (1) Taking just the quaternion division ring plane (seen above as a derivable net), we have an additional translation plane with spread in $\mathrm{PG}(2, F(\theta))$, of characteristic 2 that admits both the elation σ-twisted regulus inducing group and the σ-hyperbolic regulus-inducing group, which then provide flocks of σ-conics and σ-twisted hyperbolic flocks.

(2) In Theorem 12(3), regarding the possible examples of spreads producing both flocks of D-cones and D-hyperbolic flocks, where D is a reducible derivable net, will not now have the restriction on characteristic and has then another set of examples.

(3) Problem 18 above, considered for Baer groups over α-twisted reguli has recently been solved in [Johnson 2021e].

(4) The analysis of the results of [Royle 1998] on the four sporadic deficiency one partial hyperbolic flocks of orders 5^2 and 7^2 is also discussed in [Johnson 2021e].

(5) Problem 20 is studied in [Johnson 2021c], where it is shown that any noncommutative skewfield F admitting nonsquares in F and in $Z(F)$, the center of F, may be lifted to new classes of semifield planes. What this means is that there are translation planes in $\mathrm{PG}(3, L)$, for L a noncommutative skewfield, that have elation twisted pseudo-regulus-inducing groups acting on translation planes. Whether there are associated flocks in $\mathrm{PG}(3, L)$ is a completely open question.

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