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Large caps with free pairs in dimensions five and six

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Abstract

A cap in $\mathrm{PG}(N,q)$ is said to have a free pair of points if any plane containing that pair contains at most one other point from the cap. In an earlier paper we determined the largest size of caps with free pairs for $N=3$ and 4. In this paper we use product constructions to prove similar results in dimensions 5 and 6 that are asymptotically as large as possible. If $q>2$ is even, we determine exactly the largest size of a cap in $\mathrm{PG}(5,q)$ with a free pair. In $\mathrm{PG}(5,3)$ we give constructions of a maximal size 42-cap having a free pair and of the complete 48-cap that contains it. Additionally, we give some sporadic examples in higher dimensions.

`Keywords:` cap, free pair, Galois space

`MSC 2000:` 51E22

1 Introduction

Throughout we assume that q is a prime power, $q>2$. The problem which we study is solved for $q=2$ [8, Theorem 2.2]. An n-cap $C\subset\mathrm{PG}(N,q)$ is a set of n points no three of which are collinear. An n-cap is said to be complete if it is not contained in an $(n+1)$-cap in the same space. The largest size of a cap in $\mathrm{PG}(N,q)$ is denoted $m_2(N,q)$, and the second-largest size of a complete cap, $m_2'(N,q)$.

We say that $\{x,y\}\subset C$ is a *free pair of points* if for each $z\in C\setminus\{x,y\}$ the plane xyz does not contain any other point of C.

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Caps with free pairs of points are of great interest in the design of experiments in statistics, specifically in the study of fractional factorial designs [17, 18]. Pairs of points not participating in any coplanar quadruple of points of C have certain advantages. So, it is natural to ask what the maximum cardinality is for a cap containing a free pair of points in a given projective space.

For given N and q, we use the notation $m_2^+(N,q)$ for the maximum cardinality (number of points) of a cap in $\mathrm{PG}(N,q)$ that contains at least one free pair of points. An upper bound for $m_2^+(N,q)$ is known and is included in the next section as Theorem 2.3.

One way to find large caps with free pairs is to use geometric or other arguments to construct a cap while ensuring that one pair remains free; this is the method employed in [8]. Another approach is to take known large caps (not necessarily possessing a free pair) and to delete points from the cap until a pair becomes free. We refer to Bierbrauer [3] for an excellent survey of large caps. We present new results using this latter strategy. In particular, we show that we asymptotically meet the upper bound in $\mathrm{PG}(5,q)$ and $\mathrm{PG}(6,q)$. Further, we determine the exact value for $m_2^+(5,q)$, for even q. For $\mathrm{PG}(5,3)$, we are able to give a construction of a $m_2^+(5,3)$-cap with a free pair and of the complete $m_2'(5,3)$-cap that contains it.

2 Known results

Theorem 2.1 (Theorem 10 in [5]). *Assume there is an n-cap $\mathcal{A} \subset \mathrm{PG}(k,q)$ and an m-cap $\mathcal{B} \subset \mathrm{PG}(\ell,q)$, each possessing a tangent hyperplane. Then there is an $(nm-1)$-cap in $\mathrm{PG}(k+\ell,q)$.*

Specifically, the authors prove the following. Let $(a:1)$ and $(b:1)$ be the typical representatives of the affine points of $\mathcal{A}$ and $\mathcal{B}$, respectively. Denote by $(\alpha:0)$ the representative of $\mathcal{A}$ on the tangent hyperplane (assumed to be $x_k=0$) and $(\beta:0)$ the representative of $\mathcal{B}$ on the tangent hyperplane. Then the set

$$\mathcal{C}_1 = \{(a:b:1), (a:\beta:0) \text{ and } (\alpha:b:0)\} \tag{1}$$

forms a cap in $\mathrm{PG}(k+\ell,q)$.

The other result we need is the Mukhopadhyay product construction.

Theorem 2.2 (Mukhopadhyay [15]). *Assume there is an n-cap $\mathcal{A} \subset \mathrm{AG}(k,q)$ and an m-cap $\mathcal{B} \subset \mathrm{PG}(\ell,q)$. Then there is an nm-cap in $\mathrm{PG}(k+\ell,q)$.*

The proof for this theorem is only slightly different than the first. Here, the set of points that is shown to be a cap is

$$\mathcal{C}_2 = \{(a : b)\}, \tag{2}$$

where b is a typical representative of $\mathcal{B}$ and $(a : 1)$, of $\mathcal{A}$.

Finally, we mention the following upper bound from [8].

Theorem 2.3. *For each N we have*

$$m_2^+(N, q) \leq q^{N-2} + q^{N-3} + \cdots + q + 3. \tag{3}$$

3 New results in dimensions 5 and 6

In [8] the bound in Theorem 2.3 is shown to be sharp for $N \leq 4$. In this section we show that the bound is attained asymptotically in projective dimensions 5 and 6. Secondly, we prove that for even q it is possible to precisely meet the bound in $\mathrm{PG}(5, q)$.

We obtain these results as special cases of two more general theorems which extend the product constructions of Theorems 2.1 and 2.2 for the purpose of constructing caps with free pairs.

Theorem 3.1. *Let $\mathcal{A} \subset \mathrm{PG}(k, q)$ and $\mathcal{B} \subset \mathrm{PG}(\ell, q)$ be caps, each possessing a tangent hyperplane, and assume that $|\mathcal{B}| \geq 3$. Further assume that a tangent hyperplane to the cap $\mathcal{B}$ exists at the point $P \in \mathcal{B}$, and that Q is another point of $\mathcal{B}$. Let ν denote the number of distinct planes $\pi \subset \mathrm{PG}(\ell, q)$ such that $\{P, Q\} \subset \pi$ and $|\pi \cap \mathcal{B}| \geq 3$. Then there exists a cap $\mathcal{K} \subset \mathrm{PG}(k + \ell, q)$ such that $|\mathcal{K}| = |\mathcal{A}||\mathcal{B}| - |\mathcal{B}| + \nu + 1$ and $\mathcal{K}$ contains a free pair of points.*

Proof. Let us briefly outline the proof strategy. We begin with a large cap $\mathcal{C}_1$ of the type described by equation (1) in Theorem 2.1. We then fix a pair of points $\{P_1, P_2\} \subset \mathcal{C}_1$ which we seek to make free. That is, for any plane $\rho \subset \mathrm{PG}(k + \ell, q)$ such that $\{P_1, P_2\} \subset \rho$ we eliminate all but one of the points in $\mathcal{C}_1 \cap \rho \setminus \{P_1, P_2\}$ from $\mathcal{C}_1$ in order to "liberate" the pair $\{P_1, P_2\}$.

We now present the details of the proof. We will work with concrete representatives of the cap so that we may establish that two specific cap points form a free pair. Throughout the paper the projective coordinates of $\mathrm{PG}(n, q)$ will be denoted $x_0, x_1, \ldots, x_n$. By 0^n we will denote the zero vector in $\mathbb{F}_q^n$. Without loss of generality, we take

$$(\alpha, 0) = (0^{k-1}, 1, 0)$$

to be the point of $\mathcal{A}$ on the tangent hyperplane $x_k = 0$ and

$$P = (\beta, 0) = (0^{\ell-1}, 1, 0)$$

to be the point of $\mathcal{B}$ on the tangent hyperplane $x_\ell = 0$. Since $|\mathcal{B}| \geq 3$ by assumption, we see that $\ell \geq 2$. Thus we can assume that

$$Q = (b, 1), \quad b \neq 0^\ell$$

is another point of $\mathcal{B}$, and that the vectors $(0^k, 1)$ and $(0^\ell, 1)$ represent points on $\mathcal{A}$ and $\mathcal{B}$, respectively.

Let

$$P_1 = (0^k, \beta, 0) \text{ and } P_2 = (\alpha, b, 0)$$

be the two fixed points that will form a free pair.

For the points P_1, P_2, P_3 and P_4 to be coplanar, we must have

$$\lambda_1(0^k, \beta, 0) + \lambda_2(\alpha, b, 0) + \lambda_3 P_3 + \lambda_4 P_4 = 0^{k+\ell+1}, \qquad \lambda_i \neq 0. \tag{4}$$

Since $\mathcal{C}_1$ is a cap, if any one of the λ_i is zero, they are all zero. For the remainder of the proof, we choose explicit representatives for the projective points P_i.

The structure of the points in $\mathcal{C}_1$ naturally breaks the proof into cases:

Case 1: $P_3 = (a_3, b_3, 1)$, $P_4 = (a_4, b_4, 0)$. Since P_3 is the only point with $x_{k+\ell} = 1$, $\lambda_3 = 0$.

Case 2: $P_3 = (a_3, b_3, 1)$, $P_4 = (a_4, b_4, 1)$. Then (4) implies $\lambda_2(\alpha, 0) + \lambda_3(a_3, 1) + \lambda_4(a_4, 1) = 0^{k+1}$. Since $\mathcal{A}$ is a cap, $\lambda_2 = 0$.

Case 3: $P_3 = (a_3, \beta, 0)$, $P_4 = (a_4, \beta, 0)$. Then the second coordinate section of (4) implies $\lambda_2(b) + (\lambda_1 + \lambda_3 + \lambda_4)(\beta) = 0^\ell$. So $(\lambda_1 + \lambda_3 + \lambda_4)(\beta, 0) + \lambda_2(b, 1) - \lambda_2(0^\ell, 1) = 0^{\ell+1}$, which implies that $\lambda_2 = 0$ since these three vectors are all representatives of points in $\mathcal{B}$ and since $\mathcal{B}$ is a cap.

Case 4: $P_3 = (a_3, \beta, 0)$, $P_4 = (\alpha, b_4, 0)$. Here the first coordinate section of (4) implies $\lambda_3(a_3) + (\lambda_2 + \lambda_4)(\alpha) = 0^k$. So $(\lambda_2 + \lambda_4)(\alpha, 0) + \lambda_3(a_3, 1) - \lambda_3(0^k, 1) = 0^{k+1}$, which implies that $\lambda_3 = 0$ since $\mathcal{A}$ is a cap.

Case 5: $P_3 = (\alpha, b_3, 0)$, $P_4 = (\alpha, b_4, 0)$. The first coordinate section of (4) implies $\lambda_2 + \lambda_3 + \lambda_4 = 0$. Suppose then that $\lambda_2(b, 1) + \lambda_3(b_3, 1) + \lambda_4(b_4, 1) = (u, 0)$, $u \in \mathbb{F}_q^\ell$. We have a nontrivial solution to (4) if and only if $u = -\lambda_1(\beta)$.
In other words, (4) has a nontrivial solution if and only if $(b_3, 1)$ and $(b_4, 1)$ are on a secant plane π of the cap $\mathcal{B}$ through the points $P = (\beta, 0)$ and $Q = (b, 1)$. In order to ensure that $\{P_1, P_2\}$ is a free pair, for each such plane π we must remove from $\mathcal{C}_1$ all points of the form $(\alpha, b^*, 0)$ such that $(b^*, 1) \in \pi \cap \mathcal{B} \setminus \{Q\}$, except for one. Let ν denote the number of planes in $\mathrm{PG}(\ell, q)$ that contain P, Q and at least one other point of $\mathcal{B}$. Then we must remove $|\mathcal{B}| - \nu - 2$ points from $\mathcal{C}_1$.

The cap $\mathcal{K}$ obtained this way has $|\mathcal{C}_1| - (|\mathcal{B}| - \nu - 2) = |\mathcal{A}||\mathcal{B}| - |\mathcal{B}| + \nu + 1$ points, and it contains the free pair $\{P_1, P_2\}$. □

We can use Theorem 3.1 to reprove our earlier results:

Theorem 3.2 ([8]). *For all q we have*

(i) $m_2^+(3, q) = q + 3$,

(ii) $m_2^+(4, q) = q^2 + q + 3$.

Proof. For part (i) take for $\mathcal{A}$ and $\mathcal{B}$ two points in $\mathrm{PG}(1, q)$ and an oval, respectively. Then apply Theorem 3.1 (with $\nu = 1$) and note that the upper bound of Theorem 2.3 is attained.

For part (ii) take for $\mathcal{A}$ and $\mathcal{B}$ two points in $\mathrm{PG}(1, q)$ and an ovoid, respectively. Then apply Theorem 3.1 (with $\nu = q + 1$) and note that the upper bound of Theorem 2.3 is attained. □

A further application of Theorem 3.1 is in proving that the upper bound of Theorem 2.3 is asymptotically tight in projective dimensions 5 and 6.

Theorem 3.3. *For all q we have*

(i) $m_2^+(5, q) \geq q^3 + q^2 + 2$,

(ii) $m_2^+(6, q) \geq q^4 + q^2 + q + 2$.

Proof. For part (i) take for $\mathcal{A}$ and $\mathcal{B}$ an ovoid and an oval, respectively, and apply Theorem 3.1 (with $\nu = 1$). For part (ii) take for both $\mathcal{A}$ and $\mathcal{B}$ an ovoid and apply Theorem 3.1 (with $\nu = q + 1$). □

Next, we modify Theorem 2.2 to apply to caps with free pairs. The reader will notice that the theorem below has one requirement for the base caps that Mukhopadhyay's construction does not need, namely that the projective cap $\mathcal{B}$ has a tangent hyperplane.

Theorem 3.4. *Let $\mathcal{A} \subset \mathrm{AG}(k, q)$ and $\mathcal{B} \subset \mathrm{PG}(\ell, q)$ be caps, $|\mathcal{B}| \geq 2$. Further, assume that a tangent hyperplane to the cap $\mathcal{B}$ exists at the point $P \in \mathcal{B}$, and assume that Q is another point of $\mathcal{B}$. Let ν denote the number of distinct planes $\pi \subset \mathrm{PG}(\ell, q)$ such that $\{P, Q\} \subset \pi$ and $|\pi \cap \mathcal{B}| \geq 3$. Then there exists a cap $\mathcal{K} \subset \mathrm{PG}(k + \ell, q)$ such that $|\mathcal{K}| = |\mathcal{A}||\mathcal{B}| - |\mathcal{A}| - |\mathcal{B}| + \nu + 3$ and $\mathcal{K}$ contains a free pair of points.*

Proof. We begin with a large cap $\mathcal{C}_2$ of the type described by equation (2) in Theorem 2.2. Again, our strategy will be to fix a particular pair of points of the cap to make free.

Without loss of generality, we take

$$P = \beta = (0^{\ell-1}, 1, 0)$$

to be the point of $\mathcal{B}$ on the tangent hyperplane $x_\ell = 0$, and we assume that no points of $\mathcal{A}$ lie in the hyperplane $x_k = 0$. Also, we assume that $(0^k, 1)$ and $Q = (0^\ell, 1)$ represent points on $\mathcal{A}$ and $\mathcal{B}$, respectively. Recall that the points in $\mathcal{C}_2$ are of the form $(a : b)$, where $(a : 1) \in \mathcal{A}$ and $b \in \mathcal{B}$.

Let

$$P_1 = (0^{k+\ell}, 1) \text{ and } P_2 = (0^k, \beta) = (0^{k+\ell-1}, 1, 0)$$

be the two fixed points that will form a free pair. For the points P_1, P_2, P_3 and P_4 to be coplanar, we must have

$$\lambda_1(0^{k+\ell}, 1) + \lambda_2(0^k, \beta) + \lambda_3 P_3 + \lambda_4 P_4 = 0^{k+\ell+1}, \qquad \lambda_i \neq 0. \tag{5}$$

We break this proof into three cases. In both of the first two cases, we eliminate points from $\mathcal{C}_2$. In Case 3 we show that no more points need to be removed.

Case 1: $P_3 = (0^k, b_3)$, $P_4 = (a_4, b_4)$. If $a_4 \neq 0^k$, then $\lambda_4 = 0$. So supposing $a_4 = 0^k$, we see that $\lambda_1(0^\ell, 1) + \lambda_2\beta + \lambda_3 b_3 + \lambda_4 b_4 = 0^{\ell+1}$, *i.e.*, b_3 and b_4 lie on a secant plane of $\mathcal{B}$ through the points $(0^\ell, 1)$ and β.
As before, for each such plane containing $(0^\ell, 1)$, β and other points from $\mathcal{B}$ (say b_5, b_6, b_7, $\ldots$) we must remove from $\mathcal{C}_2$ the points corresponding to b_6, b_7, $\ldots$, namely $(0^k, b_6)$, $(0^k, b_7)$, $\ldots$. Let ν denote the number of planes in $\mathrm{PG}(\ell, q)$ that contain 0^ℓ, β and at least one other point of $\mathcal{B}$. Then we must remove $|\mathcal{B}| - \nu - 2$ points.
(Notice that this argument works properly also in the case $\ell = 1$ since then we have $|\mathcal{B}| = 2$ by assumption, and the fact that no points are removed from $\mathcal{C}_2$ when $\ell = 1$ then corresponds to the natural definition of $\nu = 0$ for this special case.)

Case 2: $P_3 = (a_3, 0^\ell, 1)$, $P_4 = (a_4, b_4)$, $a_3, a_4 \neq 0^k$. The second coordinate section reveals that $(\lambda_1 + \lambda_3)(0^\ell, 1) + \lambda_2\beta + \lambda_4 b_4 = 0^{\ell+1}$. Hence, $b_4 = (0^\ell, 1)$ or $b_4 = \beta$ since $\mathcal{B}$ is a cap. If $b_4 = (0^\ell, 1)$, then $\lambda_2 = 0$. So, $b_4 = \beta$. Now, from the first coordinate section $\lambda_3 a_3 + \lambda_4 a_4 = 0^k$, which is true if and only if $\lambda_3(a_3, 1) + \lambda_4(a_4, 1) - (\lambda_3 + \lambda_4)(0^k, 1) = 0^{k+1}$. Since $\mathcal{A}$ is a cap, $a_3 = a_4$. Hence, for each of the $|\mathcal{A}| - 1$ choices for a_3, we must delete exactly one point from $\mathcal{C}_2$.

Case 3: $P_3 = (a_3, b_3)$, $P_4 = (a_4, b_4)$, $a_3, a_4 \neq 0^k$, $b_3, b_4 \neq (0^\ell, 1)$. Repeating the argument based on the first coordinate section which was used in Case 2 again yields $a_3 = a_4$, implying $\lambda_3 = -\lambda_4$. The second coordinate section now gives $\lambda_1(0^\ell, 1) + \lambda_2\beta + \lambda_3 b_3 - \lambda_3 b_4 = 0^{\ell+1}$. Clearly, $b_3 \neq b_4$. Since $\mathcal{B}$ is a cap, $b_3, b_4 \neq \beta$. By examining the last coordinate of each point, we see that $\lambda_1 + \lambda_3 + \lambda_4 = 0$, implying $\lambda_1 = 0$ since $\lambda_3 = -\lambda_4$.

The cap $\mathcal{K}$ obtained this way has $|\mathcal{C}_2| - (|\mathcal{B}| - \nu - 2) - (|\mathcal{A}| - 1) = |\mathcal{A}||\mathcal{B}| - |\mathcal{A}| - |\mathcal{B}| + \nu + 3$ points, and it contains the free pair $\{P_1, P_2\}$. □

Theorem 3.5. *For even q we have*

$$m_2^+(5, q) = q^3 + q^2 + q + 3.$$

Proof. Take for $\mathcal{A}$ and $\mathcal{B}$ a hyperoval in $\mathrm{AG}(2, q)$ and an ovoid in $\mathrm{PG}(3, q)$. Applying Theorem 3.4 (with $\nu = q + 1$) gives the desired result. □

More applications of Theorems 3.1 and 3.4 are given in Section 5.

4 More results for $\mathrm{PG}(5, 3)$

4.1 Background

Theorem 3.3 says that $m_2^+(5, q) \geq q^3 + q^2 + 2$. This is the best known lower bound for odd q except in the case of $q = 3$. In [8] we mentioned that a 42-cap in $\mathrm{PG}(5, 3)$ with a free pair was found via a computer search. Hence, by Theorem 2.3:

Theorem 4.1. *We have $m_2^+(5, 3) = 42$.*

The construction of this 42-cap was not presented in [8], and we present all details of the construction in this section.

Interestingly, our 42-cap is contained in a complete 48-cap in $\mathrm{PG}(5, 3)$. This is significant because it was recently shown [1] that 48 is the largest size of a complete cap in $\mathrm{PG}(5, 3)$ other than the projectively unique 56-cap of Hill [10] or, using the proper notation, that $m_2'(5, 3) = 48$. To the best of our knowledge only two papers in the literature describe complete 48-caps. Bierbrauer and Edel construct a family of $(q + 1)(q^2 + 3)$-caps in [5]. At that time it was not known that $m_2'(5, 3) < 49$, so the significance of this construction in the ternary case was overlooked. A group of authors independently discovered a complete 48-cap via a computer search in [12]. Although it is claimed that two distinct complete 48-caps were found in this manner [14], only one appears explicitly in the literature, and that one is projectively equivalent to the Bierbrauer-Edel 48-cap.

We note that a maximum subset of the Hill cap which contains a free pair of points has 38 points only. For the other complete 48-cap this value is 37.

In keeping with the structure in the previous section, we will first describe our complete 48-cap which contains a 42-cap with a free pair. In fact we give both a combinatorial and a geometric construction of the 48-cap.

As a final introductory comment, we remark that searching for complete 48-caps in $\mathrm{PG}(5,3)$ in a naive way, say using a pure backtrack search, is extremely unlikely to be fruitful. However, searching for 42-caps having a free pair is much easier because of the added restriction. It is still currently computationally impossible to find all such caps, but finding *some* is not unreasonable. Searching for complete caps by first searching for caps with free pairs represents a new paradigm that may be helpful in future study of caps.

4.2 The complete 48-cap: A combinatorial construction

There are similarities between the construction of our 48-cap and the description of the Hill cap given by Bierbrauer in Chapter 16 of [2]. In both cases the caps are subsets of the elliptic quadric $Q^-(5,3) = \{x \in \mathrm{PG}(5,3) : \sum_{i=1}^{6} x_i^2 = 0\}$. For completeness we include a summary of Bierbrauer's description here.

Let $\mathcal{A}$ be the sixteen points in $\mathrm{PG}(5,3)$ of Hamming weight six and an even number of entries that are 2. A second set $\mathcal{B}$ consists of all the weight three points whose support is one of the triads in B_1 or B_2 defined as

$$B_1 = \{134, 136, 145, 235, 246\}$$
$$B_2 = \{125, 126, 234, 356, 456\}.$$

The set $\mathcal{A} \cup \mathcal{B}$ has $16 + 10 \cdot 4 = 56$ points and is, in fact, the Hill cap.

We need only a slight modification of this construction to create a complete 48-cap. Specifically, let $\mathcal{C}$ contain all of the weight three points whose support is in $B_1 \cup \{156\}$. Additionally, $\mathcal{C}$ contains half of the points with support in

$$B_3 = \{123, 124, 125, 126\},$$

specifically, the ones in which the first two coordinates are equal. So if $\mathcal{S} = \mathcal{A} \cup \mathcal{C}$, then $|\mathcal{S}| = 16 + (6 \cdot 4 + 4 \cdot 2) = 48$, and $\mathcal{S}$ has 40 points in common with the Hill cap.

It is easy to verify computationally the following proposition[1]. The Magma

[1] For any reference we make to computational verification, we compiled a Magma source file which will be permanently available from `http://www.cecm.sfu.ca/~lisonek/48cap.txt` .

code available at the address in the footnote also shows a set of 6 points whose removal from $\mathcal{S}$ yields a 42-cap with a free pair.

Proposition 4.2. *The set $\mathcal{S}$ constructed above is a complete 48-cap in $\mathrm{PG}(5,3)$.*

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1
1 2 2 2 2 1 1 1 1 1 1 2 2 2 2 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 0 0 0 0 1 1 1 1 1 1 1 1
1 2 1 1 1 2 2 2 1 1 1 2 2 2 1 2 1 1 2 2 1 1 2 2 0 0 0 0 1 1 2 2 0 0 0 0 0 0 0 0 1 2 0 0 0 0 0 0
1 1 2 1 1 2 1 1 2 2 1 2 2 1 2 2 1 2 1 2 0 0 0 0 1 1 2 2 0 0 0 0 1 1 2 2 0 0 0 0 0 0 1 2 0 0 0 0
1 1 1 2 1 1 2 1 2 1 2 2 1 2 2 2 0 0 0 0 0 0 0 0 1 2 1 2 1 2 1 2 0 0 0 0 1 1 2 2 0 0 0 0 1 2 0 0
1 1 1 1 2 1 1 2 1 2 2 1 2 2 2 2 0 0 0 0 1 2 1 2 0 0 0 0 0 0 0 0 1 2 1 2 1 2 1 2 0 0 0 0 0 0 1 2

Table 1: An explicit representation of the 48-cap

4.3 The complete 48-cap: A geometric construction

4.3.1 Notes on ovals and ovoids in $\mathbf{PG(3,3)}$

We present three lemmas that lead to Theorem 4.6, which we utilize in Section 4.3.3. The first lemma is proved by elementary counting while the second and third lemmas follow from Lemma $18.4.3$ and Lemma $16.1.6$, respectively, in [13]. We use the notation $P+Q$ and $P+2Q$ to denote the two points on the line PQ other than P and Q.

Lemma 4.3. *If P and Q are points in $\mathrm{PG}(2,3)$ and ℓ (ℓ') is a line through P (Q) other than PQ, then there are precisely two ovals containing P and Q and having ℓ and ℓ' as tangent lines. Further, in one of the ovals $P+Q$ is an external point, and in the other, an internal point.*

Lemma 4.4. *Let π and π' be planes in $\mathrm{PG}(3,3)$ intersecting in the line PQ, and let O be an oval in π containing P and Q and O' be an oval in π' containing P and Q. If $P+Q$ is both an external (internal) point of O and an internal (external) point of O', then there are exactly two ovoids in $\mathrm{PG}(3,3)$ containing O and O'. Otherwise, there is exactly one such ovoid.*

Lemma 4.5. *Let P and Q be two points on an ovoid $\mathcal{O}$ in $\mathrm{PG}(3,q)$, q odd. Then each of the $q+1$ planes through PQ intersects $\mathcal{O}$ in an oval, and $P+Q$ is an external point in exactly half of these cases.*

Theorem 4.6. *Fix the points P and Q in $\mathrm{PG}(3,3)$, and let π (π') be a plane containing P (Q) but not the line $\ell = PQ$. There are exactly six ovoids in $\mathrm{PG}(3,3)$ containing the points P and Q such that π and π' are tangent planes. Further, any one of these ovoids intersects one of the other ovoids only in $\{P,Q\}$ and the other four ovoids in six points.*

Proof. Let $\mathcal{O}$ be an ovoid satisfying the required criteria. Let π_0 and π_1 be distinct planes intersecting in ℓ. Then π_0 meets $\mathcal{O}$ in an oval containing P and Q and having $\ell_p = \pi_0 \cap \pi$ and $\ell_q = \pi_0 \cap \pi'$ as tangents. By Lemma 4.3, there are two such ovals in π_0, namely O_0 with $P + Q$ external and O_0' with $P + Q$ internal. Similarly, there are ovals O_1 and O_1' in π_1.

By Lemma 4.4 there are two ovoids containing O_0 and O_1', two ovoids containing O_0' and O_1, one ovoid containing O_0 and O_1 and one ovoid containing O_0' and O_1'. This gives six ovoids total. If we denote the other two planes containing ℓ by π_2 and π_3 and let O_2, O_2', O_3 and O_3' be ovals defined as above, then Lemma 4.5 says that the six ovoids are:

$$\begin{aligned} \mathcal{O}_0 &= O_0 \cup O_1 \cup O_2' \cup O_3' & \mathcal{O}_1 &= O_0' \cup O_1' \cup O_2 \cup O_3 \\ \mathcal{O}_2 &= O_0 \cup O_1' \cup O_2' \cup O_3 & \mathcal{O}_3 &= O_0' \cup O_1 \cup O_2 \cup O_3' \\ \mathcal{O}_4 &= O_0 \cup O_1' \cup O_2 \cup O_3' & \mathcal{O}_5 &= O_0' \cup O_1 \cup O_2' \cup O_3. \end{aligned}$$

Notice that $\mathcal{O}_0 \cap \mathcal{O}_1 = \mathcal{O}_2 \cap \mathcal{O}_3 = \mathcal{O}_4 \cap \mathcal{O}_5 = \{P, Q\}$; otherwise, $|\mathcal{O}_i \cap \mathcal{O}_j| = 6$, $i \neq j$. □

Let $A \triangle B := (A \setminus B) \cup (B \setminus A)$ denote the symmetric difference of A and B. Since $|\mathcal{O}_0 \triangle \mathcal{O}_1| = |\mathcal{O}_2 \triangle \mathcal{O}_3| = |\mathcal{O}_4 \triangle \mathcal{O}_5| = 16$ and there are only $40 - (2 \cdot 13 - 4) - 2 = 16$ points to choose from in $\mathrm{PG}(3,3) \setminus (\pi \cup \pi' \cup \ell)$, it follows that each of these pairs of ovoids actually partitions the set of possible points. We will call such a pair *complementary*.

4.3.2 The Γ_4 cap in $\mathrm{PG}(4,3)$

In this section we review a cap in $\mathrm{PG}(4,3)$ which we will use as an important building block later. We note that a 3-flat is sometimes called a *solid.* Additionally, for a given cap, a k-solid or k-hyperplane is a solid or hyperplane that intersects the cap in exactly k points.

In [16] Pellegrino showed that the largest cap in $\mathrm{PG}(4,3)$ is a 20-cap. Hill [11] later classified all 20-caps in $\mathrm{PG}(4,3)$ into one of eight isomorphically distinct types, for which he introduced the names $\Gamma_1, \ldots, \Gamma_7$ and Δ. The Γ caps are constructed as follows. First take the ten points, $Q_1, \ldots, Q_{10}$, of an ovoid in a hyperplane, H_0, and any point V_0 not in H_0. Choosing any two of the three points other than V_0 on the lines V_0Q_i, $i = 1, \ldots, 10$ gives the points of the cap. For example if one always chooses the two points not in H_0, denoted $Q_i + V_0$ and $Q_i + 2V_0$, then the Γ_2 cap is constructed (which makes it the largest cap in $\mathrm{AG}(4,3)$). If exactly two points in H_0, say Q_1 and Q_2, are selected in the place of $Q_1 + V_0$ and $Q_2 + V_0$, then the Γ_4 cap is constructed. This is the cap which we use. Let $P_1, \ldots, P_{20}$ be the points of Γ_4, with P_1 and P_3 being the two points Q_1 and Q_2 from above, and let $P_2 = P_1 + 2V_0$ and $P_4 = P_3 + 2V_0$.

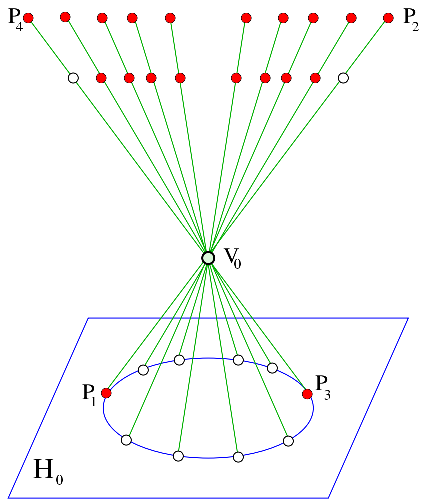

Figure 1: The Γ_4 cap in $\mathrm{PG}(4,3)$

Each of the original ovoid points in H_0 has exactly one plane in H_0 that is tangent to the ovoid at that point. The point V_0 together with each of these ten planes forms a 2-solid of Γ_4. Also, the hyperplane H_0 meets Γ_4 in exactly two points bringing the total number of such hyperplanes to eleven. The Γ_4 cap is the only 20-cap in $\mathrm{PG}(4,3)$ with this feature; all others have exactly ten 2-solids.

Let π_3 be the plane in H_0 that is tangent to the base ovoid at P_3. Then H_0 is the unique solid containing P_1 and π_3. Similarly, we name the solids formed by joining the other three points on the line P_1V_0 to π_3; specifically, H_1 when P_2 is joined to π_3, H_3 when V_0 is joined to π_3 and H_2 when the fourth point, call it V_1, is joined to π_3. Notice that H_1 contains 10 points of Γ_4 (namely, the base ovoid projected through V_0 into H_1) and H_2 contains nine points of Γ_4 (which taken together with V_1 are the points of the base ovoid projected through V_0 into H_2). H_0 and H_3 are 2-solids of Γ_4.

4.3.3 Moving into $\mathrm{PG}(5,3)$

In this section we describe how to use two Γ_4 caps along with ten other points to construct a complete 48-cap. To avoid confusion we use H to denote 4-flats (hyperplanes) and T to denote 3-flats (solids).

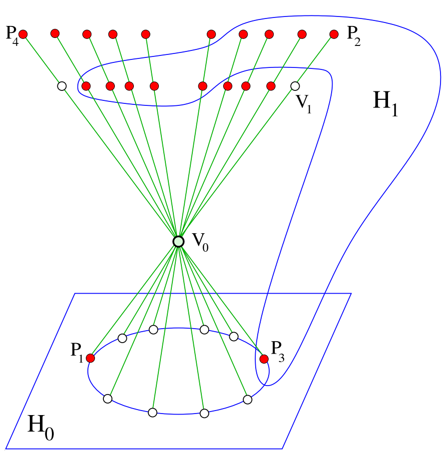

Figure 2: A 10-solid of the Γ_4 cap

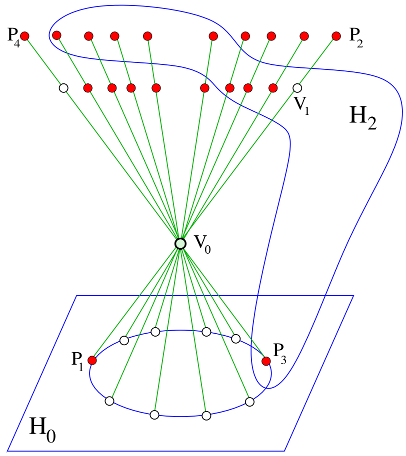

Figure 3: A 9-solid of the Γ_4 cap

To begin we take a Γ_4 cap (hereafter Γ_{4a}) in a hyperplane H_4. Let P be any point not in H_4. In Section 4.3.2 H_0, H_1, H_2 and H_3 denoted hyperplanes in $\mathrm{PG}(4,3)$, *i.e.*, solids. We extend these solids to 4-flats in $\mathrm{PG}(5,3)$, keeping the same notation, using P. Specifically, H_0 now denotes the unique 4-flat containing all the points of H_0 in Section 4.3.2 as well as the point P, and similarly for H_1, H_2 and H_3. If we use T_0 to denote the unique solid containing the plane π_3 (from Section 4.3.2) and P, then we note that $H_0, \ldots, H_3$ are the four hyperplanes containing T_0. We are also interested in another hyperplane H_5, the unique 4-flat containing P and the 2-solid in H_4 that intersects Γ_{4a} in exactly $\{P_1, P_2\}$.

We construct a second Γ_4 cap (hereafter Γ_{4b}) in H_5 as follows. The base ovoid will be in the solid $T_1 = H_1 \cap H_5$. Let P_5 be one of the nine points, including P, in $(T_1 \cap H_0) \setminus H_4$. For a fixed choice of P_5, let P_{35} and P'_{35} be points on the line P_3P_5. Recall that the ten points of Γ_{4a} in H_1 are the projection of Γ_{4a}'s base ovoid through V_0 into H_1. Call this projection $\mathcal{O}$. Then the projection of $\mathcal{O}$ through P_{35} into H_5 is the ovoid that we choose as a base ovoid for Γ_{4b}. Hence, we have $9 \times 2 = 18$ (number of choices for P_5 $\times$ number of choices for P_{35}) possible choices for Γ_{4b}'s base ovoid. It turns out that regardless of which choice is made here, the resulting 48-caps can be verified computationally to be isomorphic under $PGL(6,3)$, so we may choose any one of them at this stage and fix this choice for the rest of the construction. Notice that P_2 projects to itself since it is in T_1 and, hence, it is a member of the base ovoid. The points P_2 and P_5 will be the two points from the base ovoid included in Γ_{4b}.

We continue constructing Γ_{4b} by choosing an appropriate vertex. Recall that the point V_0 is on the line P_1P_2. The fourth point on this line we have already named V_1 and now choose as the vertex of Γ_{4b}. We must then take P_1 to be a point in Γ_{4b} since choosing V_0 would violate the cap conditions when Γ_{4a} and Γ_{4b} are viewed together. The final choice to be made is for P_6, the second point on the line P_5V_1 to be included in Γ_{4b}. We postpone this decision for now.

Modulo the choice for P_6, we have created the 38-cap depicted in Figure 4. While not drawn explicitly in Figures 4 and 5, H_4 is on the left-hand side of the figure and H_5 on the right-hand side. The only points of consequence in their intersection are the ones on the line P_1P_2, which we draw twice for clarity.

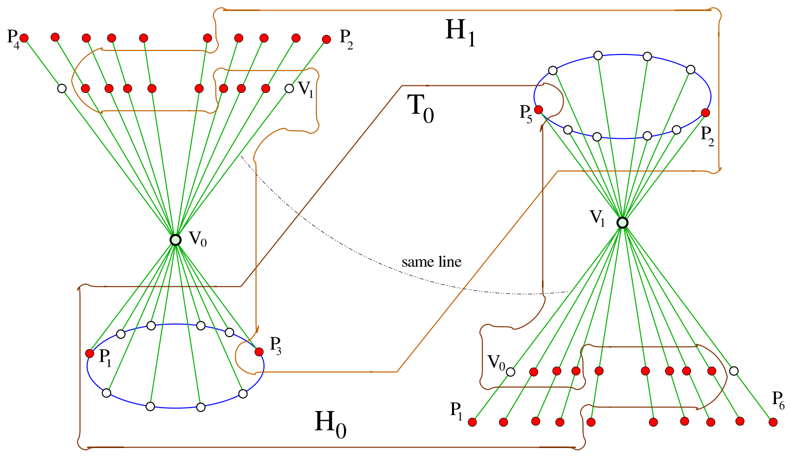

Figure 4: Γ_{4a} and Γ_{4b} form a 38-cap in $\mathrm{PG}(5,3)$

We now extend the 38-cap to a 46-cap by adding points from T_0. More specifically, since $m_2(4,3) = 20$ we cannot add any more points in either H_4 or H_5, and we limit our focus to the 16 points in $T_0 \backslash ((H_4 \cup H_5) \cup P_3P_5)$. Before adding one of these points to the current cap, we must be sure that it does not lie on a line with any pair of the 38 points. With the exception of the pairs with one of the points in Γ_{4a} but not in $(H_0 \cup H_1)$ and the other point in Γ_{4b} but not in $(H_0 \cup H_1)$, all of the pairs obviously would not give rise to collinear triples with one of the 16 candidate points. The following argument shows how to avoid the exceptional pairs and still keep eight of the 16 candidate points.

Recall that the unique hyperplane containing T_0 and V_1 is H_2 and that $(H_2 \cap H_4)$ is a 9-solid of Γ_{4a}. In fact those nine points and V_1 are the projection of Γ_{4a}'s base ovoid into H_2; call this projection $\mathcal{O}_1$. Notice that P_6, regardless of the choice, is also in H_2 since it is on the line P_5V_1.

This means that we can project $\mathcal{O}_1$ through P_6 into T_0, resulting in a new

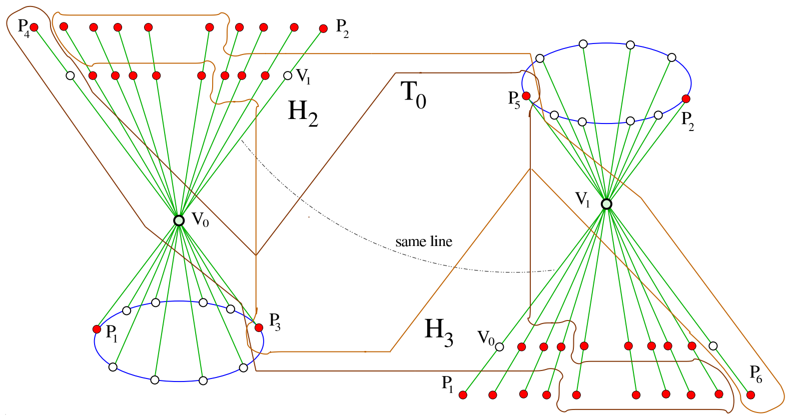

Figure 5: Γ_{4a} and Γ_{4b} with H_2 and H_3 highlighted

ovoid $\mathcal{O}_1'$. Clearly, under this projection V_1 projects to P_5, P_3 projects to itself and π_3 is a tangent plane of $\mathcal{O}_1'$ at P_3. Additionally, $\pi_5 = (T_0 \cap H_5)$ is a tangent plane to $\mathcal{O}_1'$ at P_5 since $V_1, P_6 \in H_5$ and no other points of $\mathcal{O}_1$ are in H_5. Hence, $\mathcal{O}_1'$ is one of the six ovoids (see Theorem 4.6) in T_0 having π_3 as a tangent plane at P_3 and π_5 as a tangent plane at P_5.

Similarly, it can be seen that H_3, the 4-flat containing T_0 and V_0, is a 9-solid of Γ_{4b} and that these nine points and V_0 are points of an ovoid $\mathcal{O}_2$ which is a projection of Γ_{4b}'s base ovoid through V_1 into H_3. Since $P_4 \in H_3$ we can project $\mathcal{O}_2$ through P_4 into T_0, again resulting in an ovoid, $\mathcal{O}_2'$, which is one of the six ovoids in T_0 with π_5 tangent to it at P_5 and π_3 tangent at P_3.

The immediate question is how $\mathcal{O}_2'$ compares with $\mathcal{O}_1'$. Computationally, we observe that one of the choices for P_6 causes $\mathcal{O}_1' = \mathcal{O}_2'$ and the other choice results in $\mathcal{O}_1'$ and $\mathcal{O}_2'$ being complementary, *i.e.*, intersecting only in $\{P_3, P_5\}$. We fix now the former choice for P_6 so that $\mathcal{O}_1' = \mathcal{O}_2'$, and we use $\mathcal{O}$ to denote the complementary ovoid of $\mathcal{O}_1$.

By construction all the new points in $\mathcal{O}$ may be added to the 38-cap resulting in the 46-cap depicted in Figure 6. It is of note that H_0, H_1, H_2 and H_3 are all 19-hyperplanes of this 46-cap. Hence, if any more points of $\mathrm{PG}(5,3)$ can be added to the 46-cap, there can be at most one from each of these hyperplanes. It is an easy computation to locate one point in H_2 and one point in H_3 that may be added to the cap, resulting in the final 48-cap. Incidentally, in both of these hyperplanes, the new point extends the cap of the 19 old points to a Γ_4 cap.

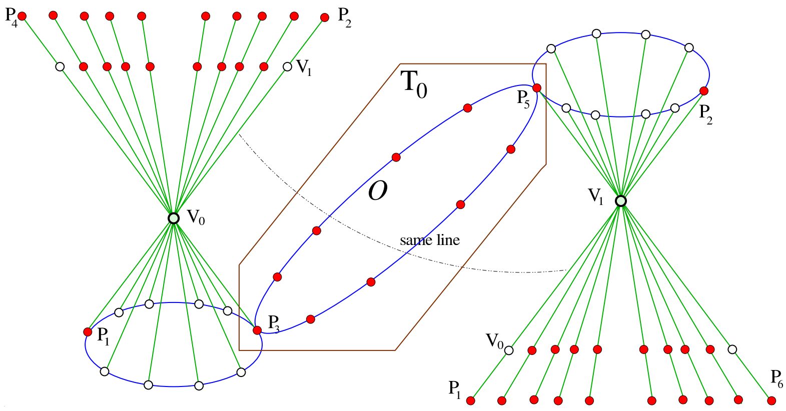

Figure 6: The 46-cap in $\mathrm{PG}(5,3)$ of Γ_{4a}, Γ_{4b} and $\mathcal{O}$

4.4 A 42-cap with a free pair

Since we were not concerned with having a free pair in our cap, we chose all eight additional points from the points of $\mathcal{O}$ in Section 4.3.3. However, if we let $\pi_1, \ldots, \pi_4$ be the four planes in T_0 through P_3P_5 and we take one of the two points in $(\mathcal{O} \cap \pi_i) \setminus \{P_3, P_5\}$, $i = 1, \ldots, 4$, then we can make $\{P_3, P_5\}$ a free pair in T_0 and in the entire 42-cap in $\mathrm{PG}(5,3)$. As mentioned in the introduction, this construction gives the same cap as was found by computer search in [8].

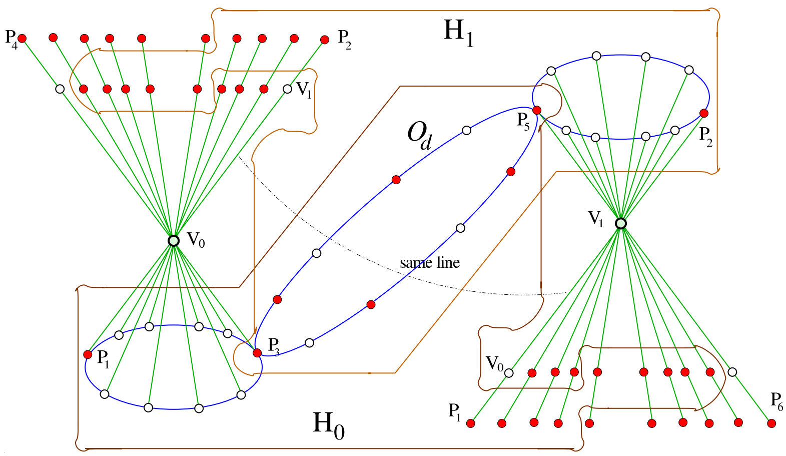

Figure 7: The entire 42-cap in $\mathrm{PG}(5,3)$ with free pair $\{P_3, P_5\}$

5 Higher dimensions

We use some of the largest known caps in higher dimensions and perform computer searches to derive large caps with free pairs. Explicit representations of many large caps can be found on Y. Edel's homepage [4].

Example 5.1. The largest known cap in $\mathrm{PG}(7,3)$ is a 248-cap discovered by Bierbrauer and Edel [6] as an extension of the Calderbank-Fishburn cap in $\mathrm{AG}(7,3)$. By removing 34 points from the former cap, we obtain a 214-cap in $\mathrm{PG}(7,3)$ with a free pair.

Example 5.2. The largest cap in $\mathrm{AG}(4,4)$ is of size 40 [7]. We use this cap together with points from an ovoid in $\mathrm{PG}(3,4)$ to create a large cap in $\mathrm{PG}(7,4)$ via the strategy in Theorem 2.2. It turns out that removing a set of 25 certain points from this cap gives a cap of size 655 with a free pair.

Example 5.3. There is a 66-cap [6] in $\mathrm{PG}(4,5)$ with two tangent hyperplanes. Taking this for $\mathcal{A}$ and the ovoid for $\mathcal{B}$, Theorem 3.1 gives a 1697-cap in $\mathrm{PG}(7,5)$.

Example 5.4. There is a 208-cap [6] in $\mathrm{AG}(4,8)$; let us call it $\mathcal{C}$. The weight distribution of the code generated by the matrix whose columns are the points of $\mathcal{C}$ was computed by Edel [4]. It turns out that this code contains codewords of weight 206 [4]. That means that there exists a 207-point subset of $\mathcal{C}$ which has a tangent hyperplane. Taking this for $\mathcal{A}$ and the ovoid for $\mathcal{B}$, Theorem 3.1 gives a 13400-cap in $\mathrm{PG}(7,8)$. This cap has only 120 points less than the largest known cap in $\mathrm{PG}(7,8)$ [4].

Example 5.5. The Hill cap together with the q^2 affine points of the ovoid in $\mathrm{PG}(3,3)$ can be used to create a 504-cap in $\mathrm{PG}(8,3)$ via the strategy in Theorem 2.2. By some clever arguments and with the help of a computer, Bierbrauer and Edel [6] extend this cap to a 532-cap in $\mathrm{PG}(8,3)$. It turns out that removing a set of 7 certain points from this cap gives a free pair. Once these points are removed, one other point may be added to achieve a 526-cap with a free pair.

In Table 2 we review the current bounds on $m_2^+(N,q)$ for some small values of N and q.

The upper bounds are obtained from Theorem 2.3. The lower bounds are from:

(i) $\mathrm{PG}(5,3)$—Section 4.4

(ii) $\mathrm{PG}(5,2^r)$—Theorem 3.5

(iii) $\mathrm{PG}(5,q)$, q odd—Theorem 3.3

N	q	$m_2^+(N,q) \geq$	$m_2^+(N,q) \leq$
5	3	42	42
5	4	87	87
5	5	152	158
6	3	95	123
6	4	278	343
6	5	657	783
7	3	214	366
7	4	655	1367
7	5	1697	3908
7	8	13400	37451
8	3	526	1095

Table 2: Bounds on $m_2^+(N,q)$.

(iv) $\mathrm{PG}(6,q)$—Theorem 3.3

(v) $\mathrm{PG}(7,3)$—Example 5.1

(vi) $\mathrm{PG}(7,4)$—Example 5.2

(vii) $\mathrm{PG}(7,5)$—Example 5.3

(viii) $\mathrm{PG}(7,8)$—Example 5.4

(ix) $\mathrm{PG}(8,3)$—Example 5.5

In (iv)–(ix) above, the upper bound for $m_2^+(N,q)$ is significantly greater than the largest known caps in $\mathrm{PG}(N,q)$. Specifically,

(iv) $123 > 112$ in $\mathrm{PG}(6,3)$ (the doubled Hill cap) and $q^4+q^3+q^2+q+3 > q^4+2q^2$ when $q > 3$ (Theorem 2.1 applied to two ovoids);

(v) $366 > 248$ in $\mathrm{PG}(7,3)$ [6];

(vi) $1367 > 756$ in $\mathrm{PG}(7,4)$ (the product of a hyperoval with the Glynn cap [9]);

(vii) $3908 > 1715$ in $\mathrm{PG}(7,5)$ [6];

(viii) $37451 > 13520$ in $\mathrm{PG}(7,8)$ [6];

(ix) $1095 > 532$ in $\mathrm{PG}(8,3)$ [6].

In other words, although we meet the upper bound exactly for $N = 3, 4$ and meet it asymptotically for $N = 5, 6$, for higher dimensions we fail to come close to the upper bound not because of problems inherent with caps having free pairs but rather because of a lack of knowledge of *any* cap of the required $\Theta(q^{N-2})$ cardinality in $\mathrm{PG}(N, q)$, where in the notation $\Theta(q^{N-2})$ we mean that the dimension N is fixed and q is arbitrary. (We say that $f(q)$ is $\Theta(g(q))$ if $f(q)/g(q)$ is bounded from above and from below by positive real constants for all q.)

6 Future work

The observation in the previous paragraph raises a provoking question. It has been an open question for quite some time whether caps of size $\Theta(q^{N-1})$ exist in $\mathrm{PG}(N, q)$ for $N \geq 4$. From our work here and in [8], we have shown that the upper bound of Theorem 2.3 is attainable asymptotically through dimension $N = 6$, thus giving reason to investigate whether it is true in general. It is a reasonable suggestion, then, that some effort be focused on finding caps, with or without free pairs, of size $\Theta(q^{N-2})$ in $\mathrm{PG}(N, q)$ for some fixed values N; perhaps such work could give insight leading to a solution of the original question.

Acknowledgment

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