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Rigidity of convex divisible domains in flag manifolds

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In contrast to the many examples of convex divisible domains in real projective space, we prove that up to projective isomorphism there is only one convex divisible domain in the Grassmannian of p–planes in $\mathbb{R}^{2p}$ when $p > 1$. Moreover, this convex divisible domain is a model of the symmetric space associated to the simple Lie group $\mathrm{SO}(p, p)$.

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1 Introduction

The Lie group $\mathrm{PGL}_{d+1}(\mathbb{R})$ acts naturally on real projective space $\mathbb{P}(\mathbb{R}^{d+1})$ and for an open set $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ we define the *automorphism group* of Ω as

$$\mathrm{Aut}(\Omega) = \{\varphi \in \mathrm{PGL}_{d+1}(\mathbb{R}) : \varphi\Omega = \Omega\}.$$

An open set Ω is then called a *convex divisible domain* if it is a bounded convex open set in some affine chart of $\mathbb{P}(\mathbb{R}^{d+1})$ and there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ which acts properly, freely and cocompactly on Ω. The fundamental example of a convex divisible domain comes from the Klein–Beltrami model of real hyperbolic d–space $\mathbb{H}^d_{\mathbb{R}}$:

Example 1.1 Let $\mathcal{B} \subset \mathbb{P}(\mathbb{R}^{d+1})$ be the unit ball in some affine chart. The group $\mathrm{PSO}(d,1) \subseteq \mathrm{PGL}_d(\mathbb{R})$ acts transitively (and by projective transformations) on $\mathcal{B}$, and the stabilizer of a point is $\mathrm{PSO}(d)$. This gives a natural identification $\mathcal{B} \cong \mathrm{PSO}(d,1)/\mathrm{PSO}(d)$.

Further, there is a natural metric on $\mathcal{B}$, called the Hilbert metric, such that $\mathrm{PSO}(d,1) = \mathrm{Isom}(\mathcal{B})^0$. Equipped with this metric, $\mathcal{B}$ is isometric to hyperbolic d–space. Any torsion-free cocompact lattice Γ in $\mathrm{PSO}(d,1)$ will act properly discontinuously, freely and cocompactly on $\mathcal{B}$.

There are many more examples of convex divisible domains, for instance:

(1) The symmetric spaces associated to $\mathrm{SL}_d(\mathbb{R})$, $\mathrm{SL}_d(\mathbb{C})$, $\mathrm{SL}_d(\mathbb{H})$ and $E_{6(-26)}$ can all be realized as convex divisible domains. For instance, consider the convex set

$$\mathcal{P} = \{[X] \in \mathbb{P}(S_{d,d}) : X \text{ is positive definite}\},$$

where $S_{d,d}$ is the vector space of real symmetric $d \times d$ matrices. Then the group $\mathrm{SL}_d(\mathbb{R})$ acts transitively on $\mathcal{P}$ by $g \cdot [X] = [gXg^t]$ and the stabilizer of a point is $\mathrm{SO}(d)$. Hence, if $\Gamma \leq \mathrm{PSL}_d(\mathbb{R})$ is a cocompact torsion-free lattice then Γ acts properly, freely and cocompactly on $\mathcal{P}$.

(2) Let $\mathcal{B} \subseteq \mathbb{P}(\mathbb{R}^{d+1})$ be the Klein–Beltrami model of $\mathbb{H}^d_{\mathbb{R}}$. Results of Johnson and Millson [31] and Koszul [38] imply that the domain $\mathcal{B}$ can be deformed to a divisible convex domain Ω where $\mathrm{Aut}(\Omega)$ is discrete (see Benoist [5, Section 1.3] for $d > 2$ and Goldman [26] for $d = 2$).

(3) There are many examples in low dimensions (see for instance Vinberg [46] and Vinberg and Kac [47]).

(4) For every $d \geq 4$, Kapovich [32] has constructed divisible convex domains $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ such that $\mathrm{Aut}(\Omega)$ is discrete, Gromov hyperbolic, and not quasi-isometric to any symmetric space.

(5) Benoist [9] and Ballas, Danciger and Lee [2] have constructed divisible convex domains $\Omega \subset \mathbb{P}(\mathbb{R}^4)$ such that $\mathrm{Aut}(\Omega)$ is discrete, not Gromov hyperbolic, and not quasi-isometric to any symmetric space.

(6) For $d = 4, 5, 6$, Choi, Lee and Marquis [17] have constructed divisible convex domains $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ such that $\mathrm{Aut}(\Omega)$ is discrete, not Gromov hyperbolic, and not quasi-isometric to any symmetric space.

More background can be found in the survey papers by Benoist [10], Choi, Lee and Marquis [18], Marquis [39] and Quint [43].

There is a more general setting in which convex divisible domains can be studied, namely in flag manifolds: Suppose G is a connected semisimple Lie group with trivial center and compact factors. If $P \leq G$ is a parabolic subgroup then G acts by diffeomorphisms on the compact manifold G/P, which is called a *flag manifold*. Given an open set $\Omega \subset G/P$ we define the *automorphism group of* Ω to be

$$\mathrm{Aut}(\Omega) = \{g \in G : g\Omega = \Omega\}.$$

The manifold G/P admits natural affine charts given by translates of a Bruhat big cell, and a domain Ω is *convex* if it is convex in some affine chart. There are many examples of convex divisible domains in flag manifolds coming from symmetric spaces: The Harish-Chandra embedding shows that every noncompact Hermitian symmetric space X embeds as a domain Ω_X into a flag manifold G/P (and this flag manifold can be identified with the compact dual of X) such that $\mathrm{Aut}(\Omega_X) = \mathrm{Isom}_0(X)$; see eg Helgason [30, 8.7.14]. More generally, Nagano [40, Theorem 6.1] has characterized all the noncompact symmetric spaces X whose compact dual X^* can be identified with a flag manifold G/P and X embeds as a domain Ω_X into G/P such that $\mathrm{Aut}(\Omega_X) = \mathrm{Isom}_0(X)$. In all these examples the images are bounded convex domains in some affine chart of G/P [40, Theorem 6.2].

There also exist examples of symmetric spaces which embed into a flag manifold which cannot be identified with their compact dual. In particular, we have already seen above that the symmetric spaces associated to $\mathrm{SL}_d(\mathbb{R})$, $\mathrm{SL}_d(\mathbb{C})$, $\mathrm{SL}_d(\mathbb{H})$ and $E_{6(-26)}$ can all be realized as convex divisible domains in real projective spaces.

Given theses examples it is natural to ask:

Question 1.2 If G/P is a flag manifold, are there nonsymmetric convex divisible domains in G/P?

Outside of the case when G/P can be identified with real projective space or the complex projective plane, we suspect that the answer is no. In particular, outside of those two cases the action of G on G/P usually preserves some special structure. For instance, if $G = \mathrm{PSL}_{p+q}(\mathbb{R})$ and P is the stabilizer of a p–plane then G/P can be identified with $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ the Grassmannians of p–planes in $\mathbb{R}^{p+q}$. In this case the action of G on G/P preserves an "algebraic distance" given by $d(V,W) = \dim(V \cap W)$. Despite this source of rigidity, the above question seems difficult to answer in full generality.

In this paper we specialize to the particular case of real Grassmannians. As above let $G = \mathrm{PSL}_{p+q}(\mathbb{R})$ and P is the stabilizer of a p–plane then G/P can be identified with $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ the Grassmannians of p–planes in $\mathbb{R}^{p+q}$. The set of $q \times p$ real matrices $M_{q,p}(\mathbb{R})$ can be naturally identified with an affine chart of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ via

$$X \leftrightarrow \mathrm{Im}\begin{pmatrix} \mathrm{Id}_p \\ X \end{pmatrix}.$$

Now let $\mathcal{B}_{q,p}$ be the unit ball (with respect to the Euclidean operator norm) in $M_{q,p}(\mathbb{R})$. As in the real projective setting, $\mathcal{B}_{q,p}$ is a symmetric domain; in fact, $\mathcal{B}_{q,p}$ can be identified with the symmetric space $\mathrm{PSO}(p,q)/\mathrm{PS}(\mathrm{O}(p)\times\mathrm{O}(q))$. Further, under the above identification we have $\mathrm{Aut}(\mathcal{B}_{q,p}) \cong \mathrm{PSO}(p,q)$.

In contrast to the many examples of convex divisible domains in real projective space, we prove that every convex divisible domain in $\mathrm{Gr}_p(\mathbb{R}^{2p})$ is symmetric and even more precisely that, up to projective isomorphism, $\mathcal{B}_{p,p}$ is the only convex divisible domain in $\mathrm{Gr}_p(\mathbb{R}^{2p})$. The following is our main result:

Theorem 1.3 *Suppose $p > 1$ and $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$ is a bounded convex open subset of some affine chart, and there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ such that Γ acts cocompactly on Ω. Then Ω is projectively isomorphic to $\mathcal{B}_{p,p}$.*

Remark 1.4 There is much more flexibility for domains which are not bounded in an affine chart:

(1) If Ω is an entire affine chart, there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ which acts freely, properly and cocompactly on Ω (see Section 3.5 below).

(2) If $P \subseteq Q$ are parabolic subgroups in G, then there is a natural projection $\pi\colon G/P \to G/Q$. Then, for any divisible domain $\Omega \subset G/Q$, the preimage $\pi^{-1}(\Omega)$ is a divisible domain in G/P. This shows that for many flag manifolds, classifying divisible domains is at least as difficult as classifying divisible domains in real projective spaces.

(3) There are recent constructions by Guichard and Wienhard [28; 29], Guéritaud, Guichard, Kassel and Wienhard [27] and Kapovich, Leeb and Porti [33] of open domains Ω in certain flag manifolds where there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ that acts properly, freely and cocompactly on Ω. These constructions come from the theory of Anosov representations, and give many examples of nonsymmetric divisible domains Ω. However, these constructions often produce domains whose complement has positive codimension and hence are not bounded in any affine chart (see for instance [29, Proposition 8.2]).

Remark 1.5 It is well known that convex domains in real projective space are very similar to nonpositively curved Riemannian manifolds (see for instance Benoist [8; 9], Crampon [20] or Cooper, Long and Tillmann [19]). In particular, the flexibility of domains in real projective space and the rigidity of domains in $\mathrm{Gr}_p(\mathbb{R}^{2p})$ when $p > 1$ can be compared to the well-known dichotomy for the rigidity of a nonpositively curved metric based on its *Euclidean rank*. Nonpositively curved metrics of rank one are very flexible (eg negatively curved metrics), but in higher rank there is an amazing amount of rigidity. Namely, the higher-rank rigidity theorem of Ballmann [3] and Burns and Spatzier [15; 16] states that any nonpositively curved, irreducible, closed Riemannian manifold whose Euclidean rank is at least two is isometric to a locally symmetric space. In this sense convex divisible domains in $\mathrm{Gr}_p(\mathbb{R}^{2p})$ behave like irreducible nonpositively curved manifolds of higher Euclidean rank.

Remark 1.6 In Theorem 1.3 we only assume that there is a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ acting cocompactly on Ω. However, this implies that there exists a discrete group $\Gamma_0 \leq \mathrm{Aut}(\Omega)$ that acts freely, properly discontinuously and cocompactly on Ω. Namely, we construct an invariant metric for the action of $\mathrm{Aut}(\Omega)$ (see Step 1 below and Proposition 4.8), and hence $\mathrm{Aut}(\Omega)$ acts properly on Ω. Thus, if $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group and Γ acts cocompactly on Ω then Γ is finitely generated (by the Švarc–Milnor lemma; see Bridson and Haefliger [13, Chapter I.8, Proposition 8.19]). Then Selberg's lemma (see Alperin [1]) implies that Γ has a finite-index, torsion-free subgroup $\Gamma_0 \leq \Gamma$. Then Γ_0 acts freely, properly discontinuously and cocompactly on Ω.

1.1 Outline of the proof of Theorem 1.3

The proof of Theorem 1.3 uses a variety of techniques from real projective geometry, several complex variables, Riemannian geometry, Lie theory and algebraic topology. Here is an outline of the three main steps:

Step 1 (constructing an invariant metric) A convex domain Ω in an affine chart of $\mathbb{P}(\mathbb{R}^{d+1})$ that is proper (that is, does not contain any affine real lines) has a complete metric called the Hilbert metric. One of the main steps in the proof is the construction of a metric K_Ω that generalizes this classical construction.

We say a convex domain Ω in an affine chart of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ is $\mathcal{R}$–proper if it does not contain any "rank-one affine real lines" (see Definition 4.3 below).

Theorem 1.7 (Theorem 4.6 and Theorem 5.1 below) *Suppose $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is an affine chart and $\Omega \subset \mathbb{M}$ is an $\mathcal{R}$–proper convex open subset of $\mathbb{M}$. Then there exists a complete length metric K_Ω with the following properties:*

(1) **Invariance** *The group $\mathrm{Aut}(\Omega)$ acts by isometries on (Ω, K_Ω).*

(2) **Equivariance** *If $\Phi \in \mathrm{PGL}_{p+q}(\mathbb{R})$, then*
$$K_\Omega(x,y) = K_{\Phi\Omega}(\Phi x, \Phi y).$$

(3) **Continuity in the local Hausdorff topology** *If $\Omega_n \subset \mathbb{M}$ is a sequence of $\mathcal{R}$–proper convex open sets converging in the local Hausdorff topology to an $\mathcal{R}$–proper convex open set $\Omega \subset \mathbb{M}$, then K_{Ω_n} converges to K_Ω uniformly on compact subsets of Ω.*

(4) *If $p = 1$, then K_Ω coincides with the classical Hilbert metric.*

The above theorem allow us to establish an analogue of the powerful "rescaling" method from several complex variables (see the survey articles by Frankel [23] and Kim and Krantz [36]). See Remark 1.13 below for further details on this analogy (or lack thereof). We prove:

Theorem 1.8 (Theorem 5.2 below) *Suppose $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is an affine chart, $\Omega \subset \mathbb{M}$ is an $\mathcal{R}$–proper convex open subset of $\mathbb{M}$, and $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω. If $A_n \in \mathrm{Aff}(\mathbb{M}) \cap \mathrm{PGL}_{p+q}(\mathbb{R})$ and $A_n\Omega$ is a sequence of $\mathcal{R}$–proper convex sets converging in the local Hausdorff topology to an $\mathcal{R}$–proper convex open set $\widehat{\Omega}$, then there exists some $\Phi \in \mathrm{PGL}_{p+q}(\mathbb{R})$ such that $\Phi(\Omega) = \widehat{\Omega}$.*

Remark 1.9 An affine chart $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ can be identified with the vector space $M_{q,p}(\mathbb{R})$ of $q \times p$ real matrices in a way that is unique up to an affine automorphism of $M_{q,p}(\mathbb{R})$ (see Section 3.3 for details). In particular, the group $\mathrm{Aff}(\mathbb{M})$ of affine transformations of $\mathbb{M}$ is well defined (see Definition 3.3).

To explain how the properties of the metric K_Ω imply Theorem 1.8, let us sketch the proof:

Proof sketch Suppose that $A_n\Omega \to \widehat{\Omega}$. Fix a point $x_0 \in \Omega$. Since $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω, we can pass to a subsequence and find $\varphi_n \in \mathrm{Aut}(\Omega)$ such that $A_n\varphi_n x_0 \to \hat{x}_0 \in \widehat{\Omega}$. Now consider the maps $f_n := A_n\varphi_n$. By parts (1) and (2) of Theorem 1.7, each f_n induces an isometry $(\Omega, K_\Omega) \to (\Omega_n, K_{\Omega_n})$. Then by part (3) of Theorem 1.7, one can pass to a subsequence such that $f_n \to f$ and f will be an isometry $(\Omega, K_\Omega) \to (\widehat{\Omega}, K_{\widehat{\Omega}})$. A simple argument then shows that f is actually the restriction of a element in $\mathrm{PGL}_{p+q}(\mathbb{R})$. □

Theorem 1.8 should also be compared to a theorem of Benzécri from real projective geometry. Let $\mathbb{X}_d$ be the space of proper convex open sets in $\mathbb{P}(\mathbb{R}^d)$ with the Hausdorff topology. Then $\mathbb{X}_d$ is closed in the Hausdorff topology and $\mathrm{PGL}_d(\mathbb{R})$ acts on $\mathbb{X}_d$. With this notation Benzécri proved:

Theorem 1.10 [11] *Suppose Ω is a proper convex open set in $\mathbb{P}(\mathbb{R}^d)$. If $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω, then $\mathrm{PGL}_d(\mathbb{R}) \cdot \Omega$ is a closed subset of $\mathbb{X}_d$.*

It is important to note that unlike in the real projective setting, when $p, q > 1$, convexity is not invariant under the action of $\mathrm{PGL}_{p+q}(\mathbb{R})$ on $\mathrm{Gr}_p(\mathbb{R}^{p+q})$: if Ω is a convex subset of some affine chart $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ and $\phi(\Omega) \subset \mathbb{M}$ for some $\phi \in \mathrm{PGL}_p(\mathbb{R}^{p+q})$, then $\phi(\Omega)$ may not be a convex subset of $\mathbb{M}$. Thus, to preserve convexity we are forced to consider the orbit of Ω under the group $\mathrm{Aff}(\mathbb{M}) \cap \mathrm{PGL}_{p+q}(\mathbb{R})$. We compute this group and its action in Observation 3.4.

Step 2 (the automorphism group is nondiscrete) In the second step of the proof we use the rescaling theorem (Theorem 1.8) from Step 1 to show that $\mathrm{Aut}(\Omega)$ is nondiscrete when $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$ is a convex divisible domain.

We can identify $M_{p,p}(\mathbb{R})$ with the affine chart

$$\left\{ \begin{bmatrix} \mathrm{Id}_p \\ X \end{bmatrix} : X \in M_{p,p}(\mathbb{R}) \right\}$$

of $\mathrm{Gr}_p(\mathbb{R}^{2p})$. Recall that the unit ball $\mathcal{B}_{p,p} \subset M_{p,p}(\mathbb{R})$ (with respect to the Euclidean operator norm) can be identified with the symmetric domain

$$\mathrm{PSO}(p,p)/\mathrm{PS}(\mathrm{O}(p)\times \mathrm{O}(p)).$$

Note that $\mathcal{B}_{p,p}$ is a convex set and the extreme points of $\mathcal{B}_{p,p}$ are exactly the orthogonal matrices. Given an orthogonal matrix $A \in \partial \mathcal{B}_{p,p}$, define the projective transformation

$$F(X) := \begin{bmatrix} -\mathrm{Id}_p & A^{-1} \\ \mathrm{Id}_p & A^{-1} \end{bmatrix} \cdot X = (A^{-1}X + \mathrm{Id}_p)(A^{-1}X - \mathrm{Id}_p)^{-1}.$$

Then we see that

$$F(\mathcal{B}_{p,p}) = \{X \in M_{p,p}(\mathbb{R}) : X^t + X > 0\}$$

and $F(A) = 0$. Now $F(\mathcal{B}_{p,p})$ is a cone and in particular $\mathrm{Aut}(F(\mathcal{B}_{p,p}))$ contains a one-parameter group of homotheties. Translating this back to $\mathcal{B}_{p,p}$ shows that $A \in \partial \mathcal{B}_{p,p}$ is the attracting fixed point of a one-parameter group of automorphisms of $\mathcal{B}_{p,p}$.

Using the rescaling theorem (Theorem 1.8) from Step 1 we will recover these one-parameter groups for a general divisible domain. The key result is the following:

Theorem 1.11 (Theorem 7.4 below) *Suppose $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$ is an affine chart, $\Omega \subset \mathbb{M}$ is an $\mathcal{R}$–proper convex subset of $\mathbb{M}$, and $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω. If $e \in \partial\Omega$ is an extreme point, then the tangent cone of Ω at e is $\mathcal{R}$–proper.*

Now the tangent cone of Ω at e is precisely the limit of the rescaled domains

$$n(\Omega - e) + e$$

in the local Hausdorff topology. In particular, combining Theorems 1.8 and 1.11 implies the following:

Corollary 1.12 (Corollary 7.11 below) *Suppose $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$ is an affine chart, $\Omega \subset \mathbb{M}$ is an $\mathcal{R}$–proper convex subset of $\mathbb{M}$, and $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω. Then $\mathrm{Aut}(\Omega)$ is nondiscrete.*

Remark 1.13 In the several complex variable setting, rescaling can also be used to find one-parameter groups of automorphisms (see Frankel [22, Section 6] or Kim [35]). However, in this setting one obtains these automorphisms by rescaling at a point in the boundary with either C^1 or C^2 regularity. This procedure actually finds automorphisms because a complex line has two real dimensions (see the proof of [23, Lemma 6.8]).

In contrast we find a one-parameter group of automorphisms by rescaling at a point where the tangent cone is $\mathcal{R}$–proper and hence very far from being C^1. Finally, we should observe that the rescaling method cannot be used to find one-parameter groups of automorphisms in the real projective setting.

Remark 1.14 If $p \neq q$, an explicit computation for $\mathcal{B}_{p,q}$ shows that Theorem 1.11 fails in this setting. This is one of the main problems that prevent us from extending our methods to the general case.

Step 3 (showing the automorphism group is simple and acts transitively) In the final part of the proof we show that $\mathrm{Aut}_0(\Omega)$, the connected component of the identity of $\mathrm{Aut}(\Omega)$, is a simple Lie group which acts transitively on Ω.

Our approach for this step is based on work of Farb and Weinberger [21], who prove a number of remarkable rigidity results for compact aspherical Riemannian manifolds whose universal covers have nondiscrete isometry groups. In particular, we combine their approach with the representation theory of Lie groups to establish: Whenever Ω is a bounded and convex domain in an affine chart and $\Gamma \subseteq \mathrm{Aut}(\Omega)$ is discrete such that $\Gamma \backslash \Omega$ is compact, at least one of the following holds (see Theorem 8.2):

(1) A finite-index subgroup of Γ has nontrivial centralizer in $\mathrm{PGL}_{2p}(\mathbb{R})$,

(2) There exists a nontrivial abelian normal unipotent group $U \leq \mathrm{Aut}(\Omega)$ such that $\Gamma \cap U$ is a cocompact lattice in U,

(3) $p = 2$ and there exists a finite-index subgroup G' of $\mathrm{Aut}(\Omega)$ such that $G' = \mathrm{Aut}_0(\Omega) \times \Lambda$ for some discrete group Λ. Further, up to conjugation,
$$\mathrm{Aut}_0(\Omega) = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \mathrm{SL}_2(\mathbb{R}) \right\}$$
and
$$\Lambda \leq \left\{ \begin{bmatrix} a\,\mathrm{Id}_2 & b\,\mathrm{Id}_2 \\ c\,\mathrm{Id}_2 & d\,\mathrm{Id}_2 \end{bmatrix} : ad - bc = 1 \right\}.$$

(4) $p = 2$, $\mathrm{Aut}_0(\Omega) \leq \mathrm{Aut}(\Omega)$ has finite index and acts transitively on Ω, and, up to conjugation,
$$\mathrm{Aut}_0(\Omega) = \left\{ \begin{bmatrix} aA & bA \\ cA & dA \end{bmatrix} : A \in \mathrm{SL}_2(\mathbb{R}), ad - bc = 1 \right\}.$$

(5) $\mathrm{Aut}_0(\Omega)$ is a simple Lie group with trivial center that acts transitively on Ω.

In Sections 9, 10 and 11, we use the dynamics of the action of $\mathrm{PGL}_{2p}(\mathbb{R})$ on $\mathrm{Gr}_p(\mathbb{R}^{2p})$ to show that the first four cases are impossible. Finally, in Section 12 we use the classification of simple Lie groups and the representation theory of simple Lie groups to complete the proof of Theorem 1.3.

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2 Preliminaries

2.1 Notation

Given some object o we will let $[o]$ be the projective equivalence class of o, for instance: if $v \in \mathbb{R}^{d+1} \setminus \{0\}$ let $[v]$ denote the image of v in $\mathbb{P}(\mathbb{R}^{d+1})$; if $\phi \in \mathrm{GL}_{d+1}(\mathbb{R})$ let $[\phi]$ denote the image of ϕ in $\mathrm{PGL}_{d+1}(\mathbb{R})$; if $T \in \mathrm{Hom}(\mathbb{R}^{d_1+1}, \mathbb{R}^{d_2+1}) \setminus \{0\}$ let $[T]$ denote the image of T in $\mathbb{P}(\mathrm{Hom}(\mathbb{R}^{d_1+1}, \mathbb{R}^{d_2+1}))$.

2.2 The Hilbert metric

The Hilbert metric is classically only defined for convex domains in real projective space, but Kobayashi [37] gave a construction that works for any open connected domain in real projective space. In this subsection we recall Kobayashi's construction.

Given four points $a, x, y, b \in \mathbb{P}(\mathbb{R}^d)$ that are collinear, that is contained in a projective line, one can define the cross-ratio by

$$[a; x; y; b] = \log \frac{|x-b|\,|y-a|}{|x-a|\,|y-b|}.$$

The cross-ratio is $\mathrm{PGL}_d(\mathbb{R})$–invariant in the sense that

$$[a; x; y; b] = [\varphi a; \varphi x; \varphi y; \varphi b]$$

for any $\varphi \in \mathrm{PGL}_d(\mathbb{R})$.

Next consider the interval

$$I := \{[1:t] \in \mathbb{P}(\mathbb{R}^2) : |t| < 1\}$$

and the function $H_I\colon I \times I \to \mathbb{R}_{\geq 0}$ given by

$$H_I(s,t) = |\log[-1; s; t; 1]|.$$

Then H_I is a complete $\mathrm{Aut}(I)$–invariant length metric on I.

Now suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an open connected set. Let

$$\mathrm{Proj}(I, \Omega) \subset \mathbb{P}(\mathrm{End}(\mathbb{R}^2, \mathbb{R}^d))$$

be the set of projective maps T such that $I \cap \ker T = \varnothing$ and $T(I) \subset \Omega$. Then define a function $\rho_\Omega\colon \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$ as follows:

$$\rho_\Omega(x,y) := \inf\{H_I(s,t) : \text{there exists } f \in \mathrm{Proj}(I,\Omega) \text{ with } f(s) = x \text{ and } f(t) = y\}.$$

Finally, using ρ_Ω, one defines the *pseudometric* K_Ω as

$$K_\Omega(x,y) = \inf\left\{ \sum_{i=0}^{N-1} \rho_\Omega(x_i, x_{i+1}) : N > 0,\ x_0, \dots, x_N \in \Omega,\ x_0 = x,\ x_N = y \right\}.$$

Note that if $x, y \in \Omega$ are such that the projective line through x and y has unbounded intersection with Ω, then $K_\Omega(x,y) = 0$. Kobayashi proved the following:

Theorem 2.1 [37] *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is an open connected set. Then:*

(1) *K_Ω is an $\mathrm{Aut}(\Omega)$–invariant pseudometric on Ω, ie K_Ω is finite, symmetric and satisfies the triangle inequality.*

(2) *If Ω is bounded in an affine chart, then K_Ω is a metric.*

(3) *If Ω is convex and bounded in some affine chart, then K_Ω coincides with the Hilbert metric.*

(4) *K_Ω is a complete metric if and only if Ω is convex and bounded in some affine chart.*

3 The Grassmannians

In this expository section we recall the two standard models of the Grassmannians, define affine charts and describe the projective lines contained in the Grassmannians.

3.1 The matrix model

We can identify $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ with the quotient

$$\{X \in M_{p+q,p}(\mathbb{R}) : \operatorname{rank} X = p\}/\operatorname{GL}_p(\mathbb{R}),$$

where $\operatorname{GL}_p(\mathbb{R})$ acts on $M_{p+q,p}(\mathbb{R})$ by multiplication on the right and the identification with $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ is given by $X \mapsto \operatorname{Im}(X)$. Note that in this model the action of $\operatorname{PGL}_{p+q}(\mathbb{R})$ on $\mathrm{Gr}_p(\mathbb{R})$ is given by the action by multiplication on the left on $M_{p+q,p}(\mathbb{R})$.

3.2 The projective model

We have a natural embedding $\mathrm{Gr}_p(\mathbb{R}^{p+q}) \to \mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ defined by

$$\operatorname{Span}(v_1, \dots, v_p) \to [v_1 \wedge \cdots \wedge v_p].$$

Remark 3.1 The image of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ is a closed smooth algebraic subvariety of dimension pq in $\mathbb{P}(\Lambda^p \mathbb{R}^{p+q})$, which has dimension $\binom{p+q}{p} - 1$. Nevertheless, if $\mathcal{O} \subseteq \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is open, then the cone over the image of $\mathcal{O}$ in $\mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ spans $\bigwedge^p \mathbb{R}^{p+q}$.

Remark 3.2 The following characterization of the image will also be useful: for $x \in \bigwedge^p \mathbb{R}^{p+q}$, we have that $[x]$ belongs to $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ if and only if the linear map $T_x \colon \mathbb{R}^{p+q} \to \bigwedge^{p+1} \mathbb{R}^{p+q}$ given by $T_x(v) = v \wedge x$ has rank q.

It is also straightforward to describe the action of $\operatorname{PGL}_{p+q}(\mathbb{R})$ on $\mathrm{Gr}_p(\mathbb{R}^{p+q})$: any element $g \in \operatorname{PGL}_{p+q}(\mathbb{R})$ induces a natural projective linear map $\bigwedge^p g$ of $\mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ defined by

$$\textstyle\bigwedge^p g[v_1 \wedge \cdots \wedge v_p] := [g v_1 \wedge \cdots \wedge g v_p].$$

The image of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ in $\mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ is invariant under the action of $\operatorname{PGL}_{p+q}(\mathbb{R})$.

3.3 Affine charts

Suppose W_0 is a q–dimensional subspace of $\mathbb{R}^{p+q}$. Then consider the set

$$\mathbb{M} := \{U \in \mathrm{Gr}_p(\mathbb{R}^{p+q}) : U \cap W_0 = (0)\}.$$

Note that $\mathbb{M}$ is an open dense subset of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$. We call $\mathbb{M}$ an *affine chart*.

If we fix a subspace $U_0 \in \mathbb{M}$, we can identify $\mathbb{M}$ with the set $\mathrm{Hom}(U_0, W_0)$ via

$$\mathrm{Hom}(U_0, W_0) \to \mathbb{M}, \quad T \mapsto \mathrm{Graph}(T) := \{(\mathrm{Id} + T)u : u \in U_0\}.$$

Fixing bases of U_0 and W_0 gives an identification of $\mathbb{M}$ with the space of $q \times p$ real matrices. Notice that a different choice of bases or of U_0 only changes this identification by a map of the form

$$X \mapsto AXB + C, \tag{3-1}$$

where $A \in \mathrm{GL}_q(\mathbb{R})$, $B \in \mathrm{GL}_p(\mathbb{R})$ and C is a $q \times p$ matrix. This observation leads to the next definition:

Definition 3.3 For an affine chart $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ let $\mathrm{Aff}(\mathbb{M})$ be the transformations of $\mathbb{M}$ that are affine maps with respect to some (and hence any) identification of $\mathbb{M}$ with the space of $q \times p$ real matrices.

We end this subsection with some basic facts about affine charts.

Observation 3.4 For an affine chart $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$, the group

$$\mathrm{Aff}(\mathbb{M}) \cap \mathrm{PGL}_{p+q}(\mathbb{R})$$

coincides with the stabilizer of $\mathbb{M}$ in $\mathrm{PGL}_{p+q}(\mathbb{R})$.

Proof It is straightforward to see that

$$\mathrm{Aff}(\mathbb{M}) \cap \mathrm{PGL}_{p+q}(\mathbb{R}) = \left\{ \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} : A \in \mathrm{GL}_p(\mathbb{R}),\ C \in M_{q,p}(\mathbb{R}),\ D \in \mathrm{GL}_q(\mathbb{R}) \right\}$$

and that $\mathrm{Aff}(\mathbb{M}) \cap \mathrm{PGL}_{p+q}(\mathbb{R})$ stabilizes $\mathbb{M}$. So suppose that $g(\mathbb{M}) = \mathbb{M}$ and

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

for some $A \in \mathrm{GL}_p(\mathbb{R})$, $B \in M_{p,q}(\mathbb{R})$, $C \in M_{q,p}(\mathbb{R})$ and $D \in \mathrm{GL}_q(\mathbb{R})$. Then $\det(A + BX) \neq 0$ for every $X \in M_{q,p}(\mathbb{R})$, which is only possible if $B = 0$. Thus, $g \in \mathrm{Aff}(\mathbb{M}) \cap \mathrm{PGL}_{p+q}(\mathbb{R})$. □

If $\mathbb{M}$ is an affine chart then there exists $g \in \mathrm{PGL}_{p+q}(\mathbb{R})$ such that

$$g\,\mathbb{M} = \left\{ \begin{bmatrix} \mathrm{Id}_p \\ X \end{bmatrix} : X \in M_{q,p}(\mathbb{R}) \right\}$$

in the matrix model. Moreover, if $e_1, \dots, e_{p+q}$ is the standard basis of $\mathbb{R}^{p+q}$ then

$$g\,\mathbb{M} = \{[(e_1 + v_1) \wedge \cdots \wedge (e_p + v_p)] : v_1, \dots, v_p \in \operatorname{Span}\{e_{p+1}, \dots, e_{p+q}\}\} \tag{3-2}$$

in the projective model.

3.4 Projective lines in the two models

The description of an affine chart of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ as a subset of $\mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$, given by equation (3-2), shows that a generic line in $\mathbb{M}$ is not contained in a projective line in $\mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$. However, there is a natural set of lines in $\mathbb{M}$ which are. In this subsection we characterize these lines.

Lemma 3.5 *If ℓ is a projective line in $\mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ contained in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$, then there exist $v_1, \dots, v_p, w \in \mathbb{R}^{p+q}$ such that*

$$\ell = \{[v_1 \wedge \cdots \wedge v_{p-1} \wedge (v_p + tw)] : t \in \mathbb{R}\} \cup \{[v_1 \wedge \cdots \wedge v_{p-1} \wedge w]\}.$$

Proof Recall that for $x \in \bigwedge^p \mathbb{R}^{p+q}$, we have that $[x]$ belongs to $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ if and only if the linear map $T_x\colon \mathbb{R}^{p+q} \to \bigwedge^{p+1}\mathbb{R}^{p+q}$ given by $T_x(v) = v \wedge x$ has rank q.

Now since ℓ is a projective line there exist $w_1, \dots, w_p, v_1, \dots, v_p \in \mathbb{R}^{p+q}$ such that

$$\ell = \{[(v_1 \wedge \cdots \wedge v_p) + t(w_1 \wedge \cdots \wedge w_p)] : t \in \mathbb{R}\} \cup \{[w_1 \wedge \cdots \wedge w_p]\}.$$

Let

$$V = \operatorname{Span}\{v_1, \dots, v_p\} \cap \operatorname{Span}\{w_1, \dots, w_p\}$$

and $r = \dim V$. We claim that $r = p - 1$.

We can assume that $v_i = w_i$ for $1 \le i \le r$ and thus $v_1, \dots, v_p, w_{r+1}, \dots, w_p$ are all linearly independent. So, if

$$x_t = (v_1 \wedge \cdots \wedge v_p) + t(w_1 \wedge \cdots \wedge w_p)$$

and $v \wedge x_t = 0$ then either $v \in V$ or

$$v \wedge v_1 \wedge \cdots \wedge v_p = -t(v \wedge w_1 \wedge \cdots \wedge w_p) \neq 0.$$

This last case is only possible when $r = p-1$ and $v = v_p - t w_p$. Since $\dim\ker T_{x_t} = p$ and $\dim V = r \le p-1$, this implies that $r = p-1$. Then

$$[(v_1 \wedge \cdots \wedge v_p) + t(w_1 \wedge \cdots \wedge w_p)] = [v_1 \wedge \cdots \wedge v_{p-1} \wedge (v_p + t w_p)]$$

for all $t \in \mathbb{R}$, which implies the lemma. □

Corollary 3.6 *Suppose $x, y \in \mathrm{Gr}_p(\mathbb{R}^{p+q})$. Then the following are equivalent:*

(1) *There exists a projective line ℓ in $\mathbb{P}\left(\bigwedge^p \mathbb{R}^{p+q}\right)$ contained in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ such that $x, y \in \ell$.*

(2) $\dim(x \cap y) \geq p-1$.

Lemma 3.7 *Suppose $\mathbb{M}$ is an affine chart in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ and we identify $\mathbb{M}$ with the set of $q \times p$ matrices. Then:*

(1) *If ℓ is a projective line in $\mathbb{P}\left(\bigwedge^p \mathbb{R}^{p+q}\right)$ contained in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ and $\ell \cap \mathbb{M} \neq \varnothing$ then*
$$\ell \cap \mathbb{M} = \{X + tS : t \in \mathbb{R}\}$$
for some $X, S \in \mathbb{M}$ with $\operatorname{rank}(S) = 1$.

(2) *Conversely, if $X, S \in \mathbb{M}$ and $\operatorname{rank}(S) = 1$ then the closure of*
$$\{X + tS : t \in \mathbb{R}\}$$
in $\mathbb{P}\left(\bigwedge^p \mathbb{R}^{p+q}\right)$ is a projective line contained in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$.

Proof First suppose that ℓ is a projective line contained in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ and $\ell \cap \mathbb{M} \neq \varnothing$. There exists some $W_0 \in \mathrm{Gr}_q(\mathbb{R}^{p+q})$ such that $\mathbb{M} = \{U \in \mathrm{Gr}_p(V) : U \cap W_0 = (0)\}$. By Lemma 3.5 we can assume
$$\ell = \{[v_1 \wedge \cdots \wedge v_{p-1} \wedge (v_p + tw)] : t \in \mathbb{R}\} \cup \{[v_1 \wedge \cdots \wedge v_{p-1} \wedge w]\}.$$
for some $w, v_1, \ldots, v_p \in \mathbb{R}^{p+q}$. By modifying these vectors we can assume that $[v_1 \wedge \cdots \wedge v_p] \in \mathbb{M}$ and $w \in W_0$ (in particular $[w \wedge v_2 \wedge \cdots \wedge v_p] \notin \mathbb{M}$). Let $U_0 = \operatorname{Span}\{v_1, \ldots, v_p\}$ and identify $\mathbb{M}$ with $\operatorname{Hom}(U_0, W_0)$. Under this identification, $[v_1 \wedge \cdots \wedge v_{p-1} \wedge (v_p + tw)]$ corresponds to the homomorphism tS, where S is the linear map
$$S\left(\sum_{i=1}^{p} \alpha_i v_i\right) = \alpha_1 w.$$
Then $\ell \cap \mathbb{M} = \{tS : t \in \mathbb{R}\}$. Then the first part of the lemma follows from the change of coordinates formula (3-1).

Next suppose that $X, S \in \mathbb{M}$ and $\operatorname{rank}(S) = 1$. There exists a basis $v_1, \ldots, v_p \in \mathbb{R}^p$ such that $v_1, \ldots, v_{p-1} \in \ker S$ and $S v_p \neq 0$. Then $X + tS$ corresponds to the subspace
$$\operatorname{Span}\{v_1 + X(v_1), \ldots, v_{p-1} + X(v_{p-1}), v_p + X(v_p) + tS(v_p)\}$$

and hence in the projective model the line

$$[(v_1 + X(v_1)) \wedge \cdots \wedge (v_{p-1} + X(v_{p-1})) \wedge (v_p + X(v_p) + tS(v_p))].$$

So the closure of $\{X + tS : t \in \mathbb{R}\}$ in $\mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ is a projective line. $\square$

Since the lines in $\mathbb{M}$ that arise from projective lines in $\mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ will play an important role, it is convenient to make the following definition:

Definition 3.8 A *rank-one line* is a line ℓ in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ of the form of Lemma 3.7, ie such that the image of ℓ in $\mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ is a line.

3.5 A trivial example

In this subsection we observe that an entire affine chart is an example of a convex divisible domain. Using the matrix model of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ let

$$\Omega = \left\{ \begin{bmatrix} \mathrm{Id}_p \\ X \end{bmatrix} : X \in M_{q,p}(\mathbb{R}) \right\}.$$

Then

$$\Gamma = \left\{ \begin{bmatrix} \mathrm{Id}_p & 0 \\ Y & \mathrm{Id}_q \end{bmatrix} : Y \in M_{q,p}(\mathbb{Z}) \right\} \leq \mathrm{Aut}(\Omega)$$

is a discrete group which acts freely, properly discontinuously and cocompactly on Ω. Notice that the quotient $\Gamma\backslash\Omega$ can be identified with the torus of dimension pq.

Part I An invariant metric

4 The metric

The purpose of this section is to extend Kobayashi's definition of the Hilbert metric to domains in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$.

Suppose that $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is open and connected. Recall from Section 2.2 that $I \subset \mathbb{P}(\mathbb{R}^2)$ is the open interval

$$I := \{[1:t] \in \mathbb{P}(\mathbb{R}^2) : |t| < 1\}$$

and H_I is the Hilbert metric on I. Using the projective model of the Grassmannians, view Ω as a subset of $\mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ and let

$$\mathrm{Proj}(I, \Omega) \subset \mathbb{P}\big(\mathrm{End}\big(\mathbb{R}^2, \textstyle\bigwedge^p \mathbb{R}^{p+q}\big)\big)$$

be the set of projective maps such that $I \cap \ker T = \varnothing$ and $T(I) \subset \Omega$. Then define a function $\rho_\Omega\colon \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$ as follows:

$$\rho_\Omega(x,y) := \inf\{H_I(s,t) : \text{there exists } f \in \mathrm{Proj}(I,\Omega) \text{ with } f(s) = x \text{ and } f(t) = y\}.$$

We then define

$$K_\Omega^{(n)}(x,y) := \inf\left\{\sum_{i=0}^{n-1} \rho_\Omega(x_i,x_{i+1}) : x = x_0, x_1, \dots, x_{n-1}, x_n = y \in \Omega\right\}.$$

In particular $K_\Omega^{(n)}(x,y)$ is finite precisely when there is a path in Ω from x to y consisting of at most n segments of projective lines. Further, we evidently have $K_\Omega^{(n+1)} \le K_\Omega^{(n)}$ for any n, so we set

$$K_\Omega(x,y) := \lim_{n\to\infty} K_\Omega^{(n)}(x,y).$$

Note that at the moment it is not clear that K_Ω is finite, but we will prove this in Proposition 4.2(4).

Remark 4.1 For $x, y \in \Omega$ it is possible to explicitly compute $\rho_\Omega(x,y)$:

(1) If $\dim(x \cap y) < p-1$ then $\rho_\Omega(x,y) = \infty$.

(2) If $\dim(x \cap y) \ge p-1$, let ℓ be the projective line in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ containing x and y. If x and y are in different connected components of $\ell \cap \Omega$, then $\rho_\Omega(x,y) = \infty$. Finally, if x and y are contained in the same connected component $\mathcal{O}$ of $\ell \cap \Omega$, then

$$\rho_\Omega(x,y) = |\log[a;x;y;b]|,$$

where a and b are the endpoints of $\mathcal{O}$.

Proposition 4.2 *If $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is an open connected set then:*

(1) *If $\varphi \in \mathrm{PGL}_{p+q}(\mathbb{R})$ then $K_\Omega(x,y) = K_{\varphi\Omega}(\varphi x, \varphi y)$ for all $x, y \in \Omega$.*

(2) *$K_\Omega(x,y) \le K_\Omega(x,z) + K_\Omega(z,y)$ for any $x, y, z \in \Omega$.*

(3) *If $\Omega_1 \subset \Omega_2$ then $K_{\Omega_2}(x,y) \le K_{\Omega_1}(x,y)$ for all $x, y \in \Omega_1$.*

(4) *For any compact set $K \subset \Omega$ there exists $N > 0$ such that $K_\Omega^{(N)}(x,y) < \infty$ for every $x, y \in K$,*

(5) *K_Ω is continuous.*

Proof Parts (1)–(3) follow from the definition of K_Ω and the invariance of the cross-ratio.

To establish part (4) it is enough to show the following: for any $x \in \Omega$ there exist an open neighborhood U of x and a number $n = n(x)$ such that $K_\Omega^{(n)}(z, y) < \infty$ for any $z, y \in U$. Suppose that $x = [v_1 \wedge v_2 \wedge \cdots \wedge v_p]$. Then there exists $\epsilon > 0$ such that

$$U := \{[w_1 \wedge w_2 \wedge \cdots \wedge w_p] : \|v_i - w_i\| < \epsilon \text{ for } 1 \le i \le p\} \subset \Omega.$$

But then clearly $K_\Omega^{(p-1)}(z, y) < \infty$ for any $z, y \in U$.

To establish part (5), first observe that

$$|K_\Omega(x_0, y_0) - K_\Omega(x, y)| \le K_\Omega(x_0, x) + K_\Omega(y, y_0),$$

so it is enough to show that the map $x \mapsto K_\Omega(x_0, x)$ is continuous at x_0. But if $x_0 = [v_1 \wedge v_2 \wedge \cdots \wedge v_p]$ then there exists $\epsilon > 0$ such that

$$U := \{[w_1 \wedge w_2 \wedge \cdots \wedge w_p] : \|v_i - w_i\| < \epsilon \text{ for } 1 \le i \le p\} \subset \Omega.$$

But then for $[w_1 \wedge \cdots \wedge w_p] \in U$ we have

$$K_\Omega(x_0, [w_1 \wedge w_2 \wedge \cdots \wedge w_p]) \le K_U(x_0, [w_1 \wedge w_2 \wedge \cdots \wedge w_p]) \le \sum_{i=2}^{p} \log \frac{\epsilon + \|v_i - w_i\|}{\epsilon - \|v_i - w_i\|}$$

and so

$$\lim_{x \to x_0} K_\Omega(x_0, x) = 0. \qquad \square$$

The above proposition shows that K_Ω is an $\mathrm{Aut}(\Omega)$–invariant pseudometric. We will next show that K_Ω is a complete metric for certain convex subsets.

Definition 4.3 (1) Let $\mathcal{L}$ be the space of rank-one lines in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$, that is, the space of projective lines in $\mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ which are contained in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$.

(2) An open connected set $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is called $\mathcal{R}$*–proper* if

$$|\ell \setminus \ell \cap \Omega| > 1$$

for all $\ell \in \mathcal{L}$.

Remark 4.4 The definition of $\mathcal{R}$–properness should be compared to properness of a convex domain $U \subseteq \mathbb{P}(\mathbb{R}^{p+1})$, which can be characterized by the property that $|\ell \setminus \ell \cap U| > 1$ for every projective line ℓ. Since in projective geometry every line has rank one, $\mathcal{R}$–properness is thus a generalization of properness in projective space.

Example 4.5 If $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is an affine chart and Ω is a bounded subset of $\mathbb{M}$, then Ω is an $\mathcal{R}$–proper subset of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ (see Lemma 3.7 above).

Theorem 4.6 *Suppose $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is an affine chart and $\Omega \subset \mathbb{M}$ is an open convex set. Then the following are equivalent:*

(1) *Ω is $\mathcal{R}$–proper.*

(2) *K_Ω is a complete length metric on Ω.*

(3) *K_Ω is a metric on Ω.*

Remark 4.7 The above theorem should be compared to two well-known results in real projective geometry and several complex variables:

(1) For an open convex set $\Omega \subset \mathbb{R}^{d+1}$ the Hilbert metric is complete if and only if Ω does not contain any real affine lines.

(2) For an open convex set $\Omega \subset \mathbb{C}^{d+1}$ the Kobayashi metric is complete if and only if Ω does not contain any complex affine lines (see Barth [4]).

Proof Clearly (2) implies (3). Moreover, if there exists a projective line $\ell \in \mathcal{L}$ such that

$$|\ell \setminus \ell \cap \Omega| \leq 1$$

then $\rho_\Omega(x, y) = 0$ for all $x, y \in \ell \cap \Omega$. Thus, if Ω is not $\mathcal{R}$–proper then K_Ω is not a metric. Thus, (3) implies (1). The proof that (1) implies (2) can be found in Appendix A. □

The existence of an invariant metric implies that the action of $\mathrm{Aut}(\Omega)$ on Ω is proper:

Proposition 4.8 *Suppose $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is an affine chart and $\Omega \subset \mathbb{M}$ is an open convex set. If Ω is $\mathcal{R}$–proper, then*

(1) *$\mathrm{Aut}(\Omega)$ is a closed subgroup of $\mathrm{PGL}_{p+q}(\mathbb{R})$,*

(2) *$\mathrm{Aut}(\Omega)$ is a closed subgroup of $\mathrm{Isom}(\Omega, K_\Omega)$, and*

(3) *$\mathrm{Aut}(\Omega)$ acts properly on Ω.*

Proof We first observe that $\mathrm{Aut}(\Omega)$ is closed in $\mathrm{PGL}_{p+q}(\mathbb{R})$. Suppose that $\varphi_n \in \mathrm{Aut}(\Omega)$ and $\varphi_n \to \varphi$ in $\mathrm{PGL}_{p+q}(\mathbb{R})$. Then $\varphi(\Omega) \subset \overline{\Omega}$. Since Ω is convex in an affine chart, $\operatorname{int}(\overline{\Omega}) = \Omega$. Then, since φ induces a homeomorphism $\operatorname{Gr}_p(\mathbb{R}^{p+q}) \to \operatorname{Gr}_p(\mathbb{R}^{p+q})$, we must have

$$\varphi(\Omega) \subset \operatorname{int}(\overline{\Omega}) = \Omega.$$

But the same argument implies that $\varphi^{-1}(\Omega) \subset \Omega$. So $\varphi(\Omega) = \Omega$ and $\varphi \in \operatorname{Aut}(\Omega)$.

We next show that the action of $\operatorname{Aut}(\Omega)$ on Ω is proper. Suppose that $\varphi_n \in \operatorname{Aut}(\Omega)$ is a sequence of automorphisms such that

$$\varphi_n x_0 \in \{y \in \Omega : K_\Omega(x_0, y) \leq R\}$$

for some $x_0 \in \Omega$ and $R \geq 0$. We need to show that a subsequence of φ_n converges in $\operatorname{PGL}_{p+q}(\mathbb{R})$.

Since $\operatorname{Aut}(\Omega)$ acts by isometries on the metric space (Ω, K_Ω), by the Arzelà–Ascoli theorem there exist an isometry $f\colon (\Omega, K_\Omega) \to (\Omega, K_\Omega)$ and a subsequence $n_k \to \infty$ such that

$$f(x) = \lim_{k\to\infty} \varphi_{n_k}(x)$$

for all $x \in \Omega$. Since f is an isometry, it is injective.

Now let $T_k \in \operatorname{GL}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ be representatives of $\bigwedge^p \varphi_{n_k} \in \operatorname{PGL}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$. We normalize T_k such that $\|T_k\| = 1$, where $\|\cdot\|$ denotes the operator norm. By passing to another subsequence we can suppose that $T_k \to T \in \operatorname{End}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ with $\|T\| = 1$. Now for $x \in \Omega \setminus \ker T$ we have

$$T(x) = \lim_{k\to\infty} \varphi_{n_k}(x) = f(x)$$

and so T is injective on $\Omega \setminus \ker T$. This implies that $T \in \operatorname{GL}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$; see Remark 3.1. Hence, $\varphi_{n_k} \to \varphi$ in $\operatorname{PGL}_{p+q}(\mathbb{R})$ for some φ with $\bigwedge^p \varphi = [T]$. So $\operatorname{Aut}(\Omega)$ acts properly.

Notice that the above argument to prove that $T = f$ also implies that $\operatorname{Aut}(\Omega)$ is a closed subgroup of $\operatorname{Isom}(\Omega, K_\Omega)$. $\square$

5 Limits in the local Hausdorff topology and rescaling

Given a set $A \subset \mathbb{R}^d$, let $\mathcal{N}_\epsilon(A)$ denote the *ϵ–neighborhood of A* with respect to the Euclidean distance. The *Hausdorff distance* between two bounded sets A and B is given by

$$d_H(A, B) = \inf\{\epsilon > 0 : A \subset \mathcal{N}_\epsilon(B) \text{ and } B \subset \mathcal{N}_\epsilon(A)\}.$$

Equivalently,

$$d_H(A, B) = \max\Big\{ \sup_{a\in A} \inf_{b\in B} \|a-b\|, \sup_{b\in B} \inf_{a\in A} \|a-b\|\Big\}.$$

The Hausdorff distance is a complete metric on the space of compact sets in $\mathbb{R}^d$.

The space of closed sets in $\mathbb{R}^d$ can be given a topology from the local Hausdorff seminorms. For $R > 0$ and a set $A \subset \mathbb{R}^d$ let $A^{(R)} := A \cap B_R(0)$. Then define the *local Hausdorff seminorms* by

$$d_H^{(R)}(A, B) := d_H(A^{(R)}, B^{(R)}).$$

Finally, we say that a sequence of open convex sets A_n converges in the local Hausdorff topology to an open convex set A if there exists some $R_0 \geq 0$ such that $d_H^{(R)}(\bar{A}_n, \bar{A}) \to 0$ for all $R \geq R_0$.

Theorem 5.1 *Let $\mathbb{M}$ be an affine chart of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ and suppose $\Omega_n \subset \mathbb{M}$ is a sequence of $\mathcal{R}$–proper convex open sets converging to an $\mathcal{R}$–proper convex open set $\Omega \subset \mathbb{M}$ in the local Hausdorff topology. Then*

$$K_\Omega(x, y) = \lim_{n\to\infty} K_{\Omega_n}(x, y)$$

for all $x, y \in \Omega$ uniformly on compact sets of $\Omega \times \Omega$.

We provide the proof of Theorem 5.1 in Appendix B.

Theorem 5.2 *Let $\mathbb{M}$ be an affine chart of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ and suppose $\Omega \subset \mathbb{M}$ is an $\mathcal{R}$–proper open convex subset. Assume in addition that there exist a subgroup $H \leq \mathrm{Aut}(\Omega)$ and a compact set $K \subset \Omega$ such that $H \cdot K = \Omega$.*

If there exists a sequence $A_n \in \mathrm{Aff}(\mathbb{M}) \cap \mathrm{PGL}_{p+q}(\mathbb{R})$ such that $A_n\Omega$ converges in the local Hausdorff topology to an $\mathcal{R}$–proper open convex set $\widehat{\Omega} \subset \mathbb{M}$, then there exist $n_k \to \infty$ and $h_k \in H$ such that

$$\phi = \lim_{k\to\infty} A_{n_k} h_k$$

exists in $\mathrm{PGL}_{p+q}(\mathbb{R})$ and $\widehat{\Omega} = \phi(\Omega)$.

Proof Fix $y_0 \in \widehat{\Omega}$. Then we have $y_0 \in A_n\Omega$ for n sufficiently large. Pick $h_n \in H$ and $k_n \in K$ such that $y_0 = A_n\varphi_n k_n$. Let $T_n := A_n\varphi_n \in \mathrm{PGL}_{p+q}(\mathbb{R})$. Then

$$\Omega_n := T_n(\Omega) = A_n(\Omega)$$

is an $\mathcal{R}$–proper open convex subset and T_n is an isometry $(\Omega, K_\Omega) \to (\Omega_n, K_{\Omega_n})$. By Theorem 5.1,

$$K_{\Omega_n} \to K_{\widehat{\Omega}}$$

uniformly on compact sets on $\widehat{\Omega}$, so we can pass to a subsequence such that T_n converges uniformly on compact sets to an isometry $T\colon (\Omega, K_\Omega) \to (\widehat{\Omega}, K_{\widehat{\Omega}})$. Since T is an isometry, it is injective. On the other hand, since the metrics converge and closed metric balls are compact, we also see that T is onto.

Now we can pick a representative $\Phi_n \in \mathrm{GL}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ of $\bigwedge^p T_n \in \mathrm{PGL}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ such that $\|\Phi_n\| = 1$. By passing to a subsequence we can assume that $\Phi_n \to \Phi$ in $\mathrm{End}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$. The set $\bigwedge^p \mathrm{End}(\mathbb{R}^{p+q}) \subset \mathrm{End}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ is closed and so $\Phi = \bigwedge^p \phi$ for some $\phi \in \mathrm{End}(\mathbb{R}^{p+q})$. Moreover, $\Phi(x) = T(x)$ for any $x \notin \ker \Phi$. Since $\mathrm{Gr}_p(\mathbb{R}^{p+q}) \setminus \ker \Phi$ is an open dense set and Ω is open, this implies that Φ is injective on $\mathrm{Gr}_p(\mathbb{R}^{p+q}) \setminus \ker \Phi$. It follows that $\Phi \in \mathrm{GL}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ and hence $\phi \in \mathrm{GL}_{p+q}(\mathbb{R})$. Finally, we have that $\phi = T$ on Ω, so that $\widehat{\Omega} = \phi(\Omega)$. □

6 The geometry near the boundary

For the classical Hilbert metric on a convex divisible domain in real projective space, there are many connections between the shape of the boundary and the behavior of the metric (see eg [8; 7; 34]). In a similar spirit, we will prove some basic results connecting the geometry of K_Ω with the geometry of $\partial\Omega$.

As before, let $\mathcal{L}$ be the set of projective lines $\ell \subset \mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ which are contained in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$.

Definition 6.1 Suppose $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is an open connected set.

(1) Two points $x, y \in \partial\Omega$ are *adjacent*, denoted by $x \sim y$, if either $x = y$ or there exists a projective line $\ell \in \mathcal{L}$ such that x and y are contained in a connected component of the interior of $\ell \cap \partial\Omega$ in ℓ.

(2) The *$\mathcal{R}$–face of* $x \in \partial\Omega$, denoted by $\mathcal{R}\,F(x)$, is the set of points $y \in \partial\Omega$ where there exists a sequence $x = y_0, y_1, \dots, y_k = y$ with $y_i \sim y_{i+1}$.

(3) A point $x \in \partial\Omega$ is called an *$\mathcal{R}$–extreme point* if $\mathcal{R}\,F(x) = \{x\}$.

(4) Let $\mathrm{Ext}_{\mathcal{R}}(\Omega) \subset \partial\Omega$ denote the set of $\mathcal{R}$–extreme points of Ω.

As the next two results show, this relation on the boundary is connected with the asymptotic geometry of the intrinsic metric.

Proposition 6.2 *Suppose $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is an affine chart and $\Omega \subset \mathbb{M}$ is an $\mathcal{R}$–proper open convex set. If $x_n, y_n \in \Omega$ are sequences such that $x_n \to x \in \partial\Omega$ and $y_n \to y \in \partial\Omega$, and there exists $N \geq 0$ such that*

$$\liminf_{n\to\infty} K_\Omega^{(N)}(x_n, y_n) < \infty,$$

then $\mathcal{R}\,F(x) = \mathcal{R}\,F(y)$.

Proof For each n, choose a sequence $x_n = x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(N)} = y_n$ with

$$\liminf_{n\to\infty} \sum_{0\leq i\leq N-1} \rho_\Omega(x_n^{(i)}, x_n^{(i+1)}) < \infty.$$

By passing to subsequences, we can assume that $x_n^{(i)} \to x^{(i)}$ for each $1 \leq i \leq N-1$. By inducting on N, it therefore suffices to consider the case $N = 1$ and $y = x^{(1)}$, so that

$$\lim_{n\to\infty} K_\Omega^{(1)}(x_n, y_n) = \lim_{n\to\infty} \rho_\Omega(x_n, y_n) < \infty$$

and $x \neq y$. For each n let ℓ_n be the projective line in $\mathbb{P}\left(\bigwedge^p \mathbb{R}^{p+q}\right)$ containing x_n and y_n. Also let $\{a_n, b_n\} = \ell_n \cap \partial\Omega$ with labeling such that the ordering of the points along ℓ_n is given by a_n, x_n, y_n, b_n. Then

$$\rho_\Omega(x_n, y_n) = \log \frac{|x_n - b_n||y_n - a_n|}{|x_n - a_n||y_n - b_n|}.$$

By passing to a subsequence we can suppose that $a_n \to a$ and $b_n \to b$. Then, by the hypothesis we must have that $a \neq x$ and $b \neq y$. So $x \sim y$. □

Corollary 6.3 *Suppose $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is an affine chart, $\Omega \subset \mathbb{M}$ is an $\mathcal{R}$–proper open convex set, and $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω. If $x_n, y_n \in \Omega$ are sequences such that $x_n \to x \in \partial\Omega$, $y_n \to y \in \partial\Omega$ and*

$$\liminf_{n\to\infty} K_\Omega(x_n, y_n) < \infty,$$

then $\mathcal{R}\,F(x) = \mathcal{R}\,F(y)$.

Proof By passing to a subsequence we can suppose that

$$M = \sup_{n\in\mathbb{N}} K_\Omega(x_n, y_n) < \infty.$$

For $R \geq 0$ and $x \in \Omega$, let $B_R(x)$ denote the ball of radius R and center x with respect to the metric K_Ω. Since $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω there exists $R \geq 0$ such that

$$\mathrm{Aut}(\Omega) \cdot B_R(x_0) = \Omega.$$

Let $B := B_{R+M}(x_0)$ be the ball with center x_0 and radius $R+M$. By compactness of B and Proposition 4.2, we know there exists $N > 0$ such that

$$\sup_{x,y \in B} K_\Omega^{(N)}(x,y) < \infty$$

for all $x, y \in B$. But this implies that

$$\sup_{n \in \mathbb{N}} K_\Omega^{(N)}(x_n, y_n) < \infty$$

because for any $n \in \mathbb{N}$ there exists some $\varphi \in \mathrm{Aut}(\Omega)$ such that $\varphi x_n, \varphi y_n \in B$. □

Part II The automorphism group is nondiscrete

7 Extreme points and symmetry

7.1 The geometry of extreme points

In this subsection we provide a number of characterizations of $\mathcal{R}$–extreme points for domains $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$ where $\mathrm{Aut}(\Omega)$ acts cocompactly. But first a few definitions.

Suppose Ω is a convex set in a vector space and $x \in \partial\Omega$, then the *tangent cone of* Ω *at* x is the set

$$\mathcal{TC}_x\Omega := x + \bigcup_{t>0} t(\Omega - x).$$

Notice that the sets $x + t(\Omega - x)$ converge to $\mathcal{TC}_x\Omega$ in the local Hausdorff topology as $t \to \infty$.

We will also define natural hypersurfaces in $\mathrm{Gr}_p(\mathbb{R}^{p+q})$.

Definition 7.1 Given $\xi \in \mathrm{Gr}_q(\mathbb{R}^{p+q})$ define the hypersurface

$$Z_\xi := \{x \in \mathrm{Gr}_p(\mathbb{R}^{p+q}) : x \cap \xi \neq (0)\}.$$

Remark 7.2 If $p = 1$, then $Z_\xi \subset \mathbb{P}(\mathbb{R}^{q+1}) = \mathrm{Gr}_1(\mathbb{R}^{q+1})$ is the image of ξ in $\mathbb{P}(\mathbb{R}^{q+1})$. In particular, if a set $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is convex and bounded in an affine chart then for any $x \in \partial\Omega$ there exists $\xi \in \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ such that $x \in Z_\xi$ and $Z_\xi \cap \Omega = \varnothing$.

In [48], the second author proved that symmetry also implies the existence of such "supporting hypersurfaces":

Theorem 7.3 [48, Theorem 1.12] *If $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is a bounded connected open subset of some affine chart and $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω, then for all $x \in \partial\Omega$ there exists $\xi \in \mathrm{Gr}_q(\mathbb{R}^{p+q})$ such that $x \in Z_\xi$ and $Z_\xi \cap \Omega = \varnothing$.*

Henceforth we will only consider the case $p = q$. With this notation we will prove the following:

Theorem 7.4 *Suppose $p > 1$ and $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$ is an affine chart, Ω is a bounded open convex subset of $\mathbb{M}$, and $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω. If $e \in \partial\Omega$, then the following are equivalent:*

(1) *$e \in \partial\Omega$ is an $\mathcal{R}$–extreme point.*

(2) *$Z_e \cap \Omega = \varnothing$.*

(3) *$\mathcal{TC}_e\,\Omega$ is an $\mathcal{R}$–proper cone.*

(4) *There exist $\varphi_n \in \mathrm{Aut}(\Omega)$ and representatives $\widehat{\varphi}_n \in \mathrm{GL}\big(\bigwedge^p \mathbb{R}^{2p}\big)$ such that $\widehat{\varphi}_n \to S$ in $\mathrm{End}\big(\bigwedge^p \mathbb{R}^{2p}\big)$ and $\mathrm{Im}(S) = e$.*

Remark 7.5 (1) The implication (1)$\Longrightarrow$(3) fails for the symmetric domains $\mathcal{B}_{p,q} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ when $p \neq q$; see Remark 1.14.

(2) The implication (4)$\Longrightarrow$(1) fails for convex divisible domains in real projective space. In particular, by a result of Benoist [9], if $\Omega \subset \mathbb{P}(\mathbb{R}^4)$ is a convex divisible domain and $x \in \partial\Omega$, then there exist $\varphi_n \in \mathrm{Aut}(\Omega)$ and representatives $\widehat{\varphi}_n \in \mathrm{GL}_4(\mathbb{R})$ such that $\widehat{\varphi}_n \to S$ in $\mathrm{End}(\mathbb{R}^4)$ and $\mathrm{Im}(S) = x$. However, there are examples of convex divisible domains in $\mathbb{P}(\mathbb{R}^4)$ whose boundary contains nonextreme points (see [9; 2; 17]).

Proof We first show that (1)$\Longrightarrow$(4). Suppose that $e \in \partial\Omega$ is an $\mathcal{R}$–extreme point. Pick a sequence $x_n \in \Omega$ such that $x_n \to e$. Since $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω, we can find $R \geq 0$ and $\varphi_n \in \mathrm{Aut}(\Omega)$ such that

$$K_\Omega(x_n, \varphi_n x_0) \leq R$$

for all $n \geq 0$. Now for any $x \in \Omega$ we have

$$K_\Omega(\varphi_n x, x_n) \leq K_\Omega(\varphi_n x, \varphi_n x_0) + R = K_\Omega(x, x_0) + R$$

and so by Corollary 6.3 we see that $\varphi_n x \to e$. Pick representatives $\widehat{\varphi}_n \in \mathrm{GL}\big(\bigwedge^p \mathbb{R}^{2p}\big)$ of $\bigwedge^p \varphi_n$ such that $\|\widehat{\varphi}_n\| = 1$. By passing to a subsequence we can suppose that $\widehat{\varphi}_n \to S$

in $\mathrm{End}\left(\bigwedge^p \mathbb{R}^{2p}\right)$. Now if $x \in \mathcal{O} := \mathrm{Gr}_p(\mathbb{R}^{2p}) \setminus \ker S$ then $S(x) = \lim_{n\to\infty} \varphi_n x$. Since $\mathcal{O}$ is open and dense, we see that $\Omega \cap \mathcal{O}$ is dense in Ω. In particular, $\Omega \cap \mathcal{O}$ contains a basis of $\bigwedge^p \mathbb{R}^{2p}$. However, for every $x \in \Omega \cap \mathcal{O}$ we have $S(x) = e$. So $\mathrm{Im}(S) = e$. So (1)$\Longrightarrow$(4).

We next show that (4)$\Longrightarrow$(2). So suppose there exist $\varphi_n \in \mathrm{Aut}(\Omega)$ and representatives $\widehat{\varphi}_n \in \mathrm{GL}\left(\bigwedge^p \mathbb{R}^{2p}\right)$ such that $\widehat{\varphi}_n \to S$ in $\mathrm{End}\left(\bigwedge^p \mathbb{R}^{2p}\right)$ and $\mathrm{Im}(S) = e$. Notice that if $x \in \mathcal{O} := \mathrm{Gr}_p(\mathbb{R}^{2p}) \setminus \ker S$ then $S(x) = \lim_{n\to\infty} \varphi_n(x)$. Now, similar to the case of properly convex sets in projective space, we can consider the dual of Ω,

$$\Omega^* := \{\xi \in \mathrm{Gr}_p(\mathbb{R}^{2p}) : Z_\xi \cap \Omega = \varnothing\}.$$

Note that, unlike the case of domains in projective space, Ω and Ω^* are both subsets of $\mathrm{Gr}_p(\mathbb{R}^{2p})$. Since Ω is open, Ω^* is compact. Moreover, since Ω is bounded in an affine chart, Ω^* has nonempty interior: $\mathbb{M} = \mathrm{Gr}_p(\mathbb{R}^{2p}) \setminus Z_\xi$ for some ξ and since Ω is bounded in $\mathbb{M}$ we see that Ω^* contains an open neighborhood of ξ. In particular, $\Omega^* \cap \mathcal{O}$ is nonempty. But then for $\eta \in \Omega^* \cap \mathcal{O}$ we have $e = S(\eta) = \lim_{n\to\infty} \varphi_n(\eta)$. Since Ω^* is $\mathrm{Aut}(\Omega)$–invariant, we then see that $e \in \Omega^*$. So (4)$\Longrightarrow$(2).

We next show that (2)$\Longrightarrow$(3). So suppose that $e \in \partial\Omega$ and $Z_e \cap \Omega = \varnothing$. We can assume that

$$\Omega \subset \mathbb{M} := \left\{ \begin{bmatrix} \mathrm{Id}_p \\ X \end{bmatrix} : X \in M_{p,p}(\mathbb{R}) \right\}$$

and $e = 0$ in $\mathbb{M}$. Then since $Z_e \cap \Omega = \varnothing$ we see that

$$\Omega \subset \left\{ \begin{bmatrix} \mathrm{Id}_p \\ X \end{bmatrix} : \det(X) \neq 0 \right\}.$$

Since Ω is connected, by making an affine transformation, we may assume that

$$\Omega \subset \left\{ \begin{bmatrix} \mathrm{Id}_p \\ X \end{bmatrix} : \det(X) > 0 \right\}.$$

Then, since $\mathcal{TC}_0\Omega$ is open, we see that

$$\mathcal{TC}_0\Omega \subset \left\{ \begin{bmatrix} \mathrm{Id}_p \\ X \end{bmatrix} : \det(X) > 0 \right\}.$$

Now suppose for a contradiction that $\mathcal{TC}_0\Omega$ is not $\mathcal{R}$–proper. Then by Lemma 3.7 and convexity there exists a rank-one endomorphism S such that

$$\left\{ \begin{bmatrix} \mathrm{Id}_p \\ T + tS \end{bmatrix} : t \in \mathbb{R} \right\} \subset \mathcal{TC}_0\Omega$$

whenever $\begin{bmatrix} \mathrm{Id}_p & T \end{bmatrix}^t \in \mathcal{TC}_0\Omega$. So

$$\det(T+tS)>0$$

for any $\begin{bmatrix} \mathrm{Id}_p & T \end{bmatrix}^t \in \mathcal{TC}_0\Omega$ and $t\in\mathbb{R}$. Now

$$\det(T+tS)=\det(T)\det(\mathrm{Id}_p+tT^{-1}S)=\det(T)(1+t\operatorname{tr}(T^{-1}S))$$

since $T^{-1}S$ has rank one. But since $\mathcal{TC}_0\Omega$ is open there exists some $[\mathrm{Id}_p \ \ T_0]^t\in\mathcal{TC}_0\Omega$ such that $\operatorname{tr} T_0^{-1}S$ is nonzero. But then

$$\det(T_0+tS)=0$$

when $t=-(\operatorname{tr} T_0^{-1}S)^{-1}$. So we have a contradiction, and so (2)$\Longrightarrow$(3).

Finally, we show that (3)$\Longrightarrow$(1) by contraposition. If $e\in\partial\Omega$ is not an $\mathcal{R}$–extreme point then $\overline{\mathcal{TC}_e\,\Omega}$ contains an entire rank-one line. Since $\mathcal{TC}_e\,\Omega$ is convex and open this implies that $\mathcal{TC}_e\,\Omega$ contains an entire rank-one line and so $\mathcal{TC}_e\,\Omega$ is not $\mathcal{R}$–proper. □

Corollary 7.6 *Suppose $p>1$ and $\mathbb{M}\subset\mathrm{Gr}_p(\mathbb{R}^{2p})$ is an affine chart, Ω is a bounded open convex subset of $\mathbb{M}$, and $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω. Then $\mathrm{Ext}_{\mathcal{R}}(\Omega)\subset\partial\Omega$ is closed.*

Remark 7.7 This corollary fails for convex divisible domains in real projective space by the same argument as Remark 7.5(2).

Proof By the above proposition, the set of extreme points coincides with

$$\{e\in\partial\Omega : Z_e\cap\Omega=\varnothing\},$$

which is obviously closed. □

7.2 Constructing extreme points

Proposition 7.8 *Suppose $\mathbb{M}\subset\mathrm{Gr}_p(\mathbb{R}^{p+q})$ is an affine chart and $\Omega\subset\mathbb{M}$ is an open bounded convex set. Then $\mathrm{Ext}_{\mathcal{R}}(\Omega)$ spans $\bigwedge^p\mathbb{R}^{p+q}$.*

Proof Identify $\mathbb{M}$ with $M_{q,p}(\mathbb{R})$. For $x\in\partial\Omega$ let

$$V_x=x+\mathrm{Span}\{v\in M_{q,p}(\mathbb{R}) : v+x \text{ is adjacent to } x\}\subset M_{q,p}(\mathbb{R}).$$

See Definition 6.1 for the notion of adjacency. Notice that $x \in \partial\Omega$ is an $\mathcal{R}$–extreme point if and only if $\dim V_x = 0$.

Now, since rank-one lines in $M_{q,p}(\mathbb{R})$ are mapped to projective lines in $\mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$, we have the following: if v is a rank-one matrix, $t<0<s$, and $a,b,c \in \mathbb{P}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$ are the images of $x+tv, x, x+sv \in M_{q,p}(\mathbb{R})$, respectively, then the line b is contained in the span of the lines a and c. Thus, it is enough to show: for any $x \in \partial\Omega$ with $\dim V_x > 0$ there exist a rank-one matrix $v \in M_{q,p}(\mathbb{R})$ and $t<0<s$ such that $x+tv, x+sv \in \partial\Omega$ and

$$\dim V_{x+tv}, V_{x+sv} < \dim V_x. \tag{7-1}$$

Let $F_x = \partial\Omega \cap V_x$. This is a convex set which has nonempty interior in V_x.

We claim that $V_y \subset V_x$ for $y \in F_x$. Suppose that $v+y \in V_y$, that is, $v \in M_{q,p}(\mathbb{R})$ and $v+y$ is adjacent to y. Then there exists $\epsilon>0$ such that $tv+y \in \partial\Omega$ for $t \in (-\epsilon, 1+\epsilon)$. Moreover, since $y \in \partial\Omega \cap V_x$ there exists $\delta > 0$ such that $\lambda x + (1-\lambda) y \in \partial\Omega$ for $\lambda \in [0, 1+\delta]$. Then, by convexity, there exists $\epsilon_1 > 0$ such that $x+tv \in \partial\Omega$ for $t \in (-\epsilon_1, \epsilon_1)$. Thus, $x+v \in V_x$. Since $y \in V_x$, we then see that

$$x+v+(y-x) = x+y \in V_x.$$

Since $v+y \in V_y$ was arbitrary, we then see that $V_y \subset V_x$.

Notice that the above claim implies that if $y \in \partial F_x$, then $\dim V_y < \dim V_x$.

So, for $x \in \partial\Omega$ and $\dim V_x > 0$, pick a rank-one matrix v such that $x+\mathbb{R}v \subset V_x$. Then, if

$$\{x+sv, x+tv\} = \partial F_x \cap (x+\mathbb{R}v),$$

we have

$$\dim V_{x+tv}, V_{x+sv} < \dim V_x,$$

which establishes equation (7-1) and thereby completes the proof. □

Suppose $\varphi \in \mathrm{PGL}_d(\mathbb{R})$. Let $\overline{\varphi} \in \mathrm{GL}_d(\mathbb{R})$ be a representative of φ with $\det(\overline{\varphi}) = \pm 1$. Next let

$$\lambda_1(\varphi) \geq \lambda_2(\varphi) \geq \cdots \geq \lambda_d(\varphi)$$

be the absolute values of the eigenvalues (counted with multiplicity) of $\overline{\varphi}$ (notice that this does not depend on the choice of $\overline{\varphi}$). Let $m^+(\varphi)$ be the size of the largest Jordan block of $\overline{\varphi}$ whose corresponding eigenvalue has absolute value $\lambda_1(\varphi)$. Next let

$E_{\mathbb{C}}^{+}(\varphi) \subset \mathbb{C}^d$ be the span of the eigenvectors of $\overline{\varphi}$ whose eigenvalue have absolute value $\lambda_1(\varphi)$ and are part of a Jordan block with size $m^+(\varphi)$. Then let $E^+(\varphi) = E_{\mathbb{C}}^{+}(\varphi) \cap \mathbb{R}^d$. Since φ is a real matrix, the nonreal eigenvalues come in conjugate pairs and so we always have

$$E_{\mathbb{C}}^{+}(\varphi) = E^+(\varphi) + iE^+(\varphi).$$

Also define $E^-(\varphi) = E^+(\varphi^{-1})$.

Given $y \in \mathbb{P}(\mathbb{R}^d)$ let $L(\varphi, y) \subset \mathbb{P}(\mathbb{R}^d)$ denote the limit points of the sequence $\{\varphi^n y\}_{n \in \mathbb{N}}$. With this notation we have the following observation:

Proposition 7.9 *Suppose $\varphi \in \mathrm{PGL}_d(\mathbb{R})$ and $\{\varphi^n\}_{n \in \mathbb{N}} \subset \mathrm{PGL}_d(\mathbb{R})$ is unbounded, then there exists a proper projective subspace $P \subsetneq \mathbb{P}(\mathbb{R}^d)$ such that $L(\varphi, y) \subset [E^+(\varphi)]$ for all $y \in \mathbb{P}(\mathbb{R}^d) \setminus P$.*

Proof We can write $\overline{\varphi} = gJg^{-1}$, where $g \in \mathrm{GL}_d(\mathbb{C})$ and J is a Jordan matrix. We can further assume that

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix},$$

where J_1 consists of the blocks of J whose eigenvalues have absolute value $\lambda_1(\varphi)$ and have size $m^+(\varphi)$. Then let

$$V = \mathbb{R}^d \cap \left(g \ker \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

and $P = [V] \subset \mathbb{P}(\mathbb{R}^d)$. A straightforward calculation then shows that $L(\varphi, y) \subset [E^+(\varphi)]$ for all $y \in \mathbb{P}(\mathbb{R}^d) \setminus P$. □

Corollary 7.10 *Suppose Ω is an open connected set of $\mathrm{Gr}_p(\mathbb{R}^{2p})$, there exists an affine chart which contains Ω as a bounded convex set, and $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω. If $\varphi \in \mathrm{Aut}(\Omega)$ and $\{\varphi^n\}_{n \in \mathbb{N}} \subset \mathrm{PGL}_{2p}(\mathbb{R})$ is unbounded, then $E^+\big(\bigwedge^p \varphi\big) \cap \partial\Omega$ is nonempty and contains an $\mathcal{R}$–extreme point.*

Proof Let $P \subset \mathbb{P}\big(\bigwedge^p \mathbb{R}^{2p}\big)$ be as in the above proposition for $\bigwedge^p \varphi$. Since the set of $\mathcal{R}$–extreme points of $\partial\Omega$ spans $\bigwedge^p \mathbb{R}^{2p}$, there exists an $\mathcal{R}$–extreme point $e \in \partial\Omega$ such that $e \notin P$. Then any limit point of $\varphi^n e$ belongs to $E^+\big(\bigwedge^p \varphi\big)$ and is also an $\mathcal{R}$–extreme point by Corollary 7.6. □

7.3 Finding symmetry

Our goal is now to use Theorems 5.2 and 7.4 to show that for suitable domains Ω, the group $\mathrm{Aut}(\Omega)$ is not discrete.

Corollary 7.11 *Suppose $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$ is an $\mathcal{R}$–proper open convex set in the affine chart*

$$\mathbb{M} = \left\{ \begin{bmatrix} \mathrm{Id}_p \\ X \end{bmatrix} : X \in M_{p,p}(\mathbb{R}) \right\}$$

and $H \leq \mathrm{Aut}(\Omega)$ acts cocompactly on Ω. If

$$e = \begin{bmatrix} \mathrm{Id}_p \\ X_0 \end{bmatrix} \in \partial\Omega$$

is an $\mathcal{R}$–extreme point, then there exist $h_n \in H$ and $t_n \to \infty$ such that

$$\varphi = \lim_{n\to\infty} \begin{bmatrix} \mathrm{Id}_p & 0 \\ (1-e^{t_n})X_0 & e^{t_n}\,\mathrm{Id}_p \end{bmatrix} h_n$$

exists in $\mathrm{PGL}_{2p}(\mathbb{R})$ and $\varphi(\Omega) = \mathcal{TC}_e\Omega$. In particular, Ω is invariant under the one-parameter group

$$\varphi^{-1} \left\{ \begin{bmatrix} \mathrm{Id}_p & 0 \\ (1-e^{t})X_0 & e^{t}\,\mathrm{Id}_p \end{bmatrix} : t \in \mathbb{R} \right\} \varphi.$$

Proof Let

$$A_t = \begin{bmatrix} \mathrm{Id}_p & 0 \\ (1-e^{t})X_0 & e^{t}\,\mathrm{Id}_p \end{bmatrix};$$

then

$$A_t \cdot \begin{bmatrix} \mathrm{Id}_p \\ X \end{bmatrix} = \begin{bmatrix} \mathrm{Id}_p \\ e^t(X-X_0)+X_0 \end{bmatrix}.$$

So $A_t \in \mathrm{Aff}(\mathbb{M}) \cap \mathrm{PGL}_{2p}(\mathbb{R})$ and $A_t\Omega$ converges in the local Hausdorff topology to $\mathcal{TC}_e\,\Omega$ as $t \to \infty$. So the corollary follows from Theorems 5.2 and 7.4. □

Part III The automorphism group is simple

8 Initial reduction

For the rest of this section suppose $p > 1$ and $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$ is an affine chart, $\Omega \subset \mathbb{M}$ is a bounded convex open subset of $\mathbb{M}$, and there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ such that Γ acts cocompactly on Ω. Set $G := \mathrm{Aut}(\Omega)$ and let G^0 be the connected component of the identity of G.

Warning 8.1 Note that unlike in the introduction, henceforth G does not a priori denote a connected semisimple Lie group.

By Corollary 7.11, we know that $G^0 \neq 1$. The goal of this section is to use the fact that $G^0 \neq 1$ to obtain that either G^0 is simple and acts transitively on Ω, or we are in one of four very constrained situations (cases (1)–(4) in Theorem 8.2 below). In Sections 9, 10 and 11, we will prove that cases (1)–(4) cannot occur.

Theorem 8.2 *With the notation above, at least one of the following holds:*

(1) *A finite-index subgroup of Γ has nontrivial centralizer in $\mathrm{PGL}_{2p}(\mathbb{R})$.*

(2) *There exists a nontrivial abelian normal unipotent group $U \leq G$ such that $\Gamma \cap U$ is a cocompact lattice in U.*

(3) *$p=2$ and there exists a finite-index subgroup G' of G such that $G' = G^0 \times \Lambda$ for some discrete group Λ. Further, up to conjugation,*

$$G^0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \mathrm{SL}_2(\mathbb{R}) \right\}$$

and

$$\Lambda \leq \left\{ \begin{bmatrix} a\,\mathrm{Id}_2 & b\,\mathrm{Id}_2 \\ c\,\mathrm{Id}_2 & d\,\mathrm{Id}_2 \end{bmatrix} : ad-bc=1 \right\}.$$

(4) *$p=2$, $G^0 \leq G$ has finite index and acts transitively on Ω, and, up to conjugation,*

$$G^0 = \left\{ \begin{bmatrix} aA & bA \\ cA & dA \end{bmatrix} : A \in \mathrm{SL}_2(\mathbb{R}),\ ad-bc=1 \right\}.$$

(5) *G^0 is a simple Lie group with trivial center that acts transitively on Ω.*

Since the statement of Theorem 8.2 may seem unmotivated at first, let us sketch the argument. First suppose that G^0 is not semisimple. Let $G^{\mathrm{sol}} \leq G^0$ be the solvable radical of G^0 (that is, the maximal connected, closed, normal, solvable subgroup of G^0) and let N be the nilpotent radical of G^{sol} (that is, the maximal connected, normal, closed, nilpotent subgroup of G^{sol}).

Note that N contains the unipotent radical $R_{\mathrm{u}}(G^0)$ of G^0 (ie all unipotent elements of G^{sol}), and hence is an extension

$$1 \to R_{\mathrm{u}}(G^0) \to N \to N/R_{\mathrm{u}}(G^0) \to 1.$$

The group $N/R_{\mathrm{u}}(G^0)$ is the subgroup of $G^{\mathrm{sol}}/R_{\mathrm{u}}(G^0)$ whose action on the Lie algebra $\mathfrak{r}_{\mathrm{u}}(\mathfrak{g})$ is unipotent. Let Z be the center of N. We distinguish two cases, depending on whether Z is contained in $R_{\mathrm{u}}(G)$:

(1) If Z only consists of unipotent elements, we will show that Γ intersects some normal unipotent subgroup in a lattice. This corresponds to case (2) in Theorem 8.2.

(2) Otherwise, we show that a finite-index subgroup of Γ centralizes some semisimple torus in the Zariski closure of Z. This corresponds to case (1) in Theorem 8.2.

Suppose now that G^0 is semisimple. We want to show G^0 actually has to be simple and acts transitively on Ω. We do this by using the virtual cohomological dimension $\mathrm{vcd}(\Gamma)$ of Γ (see below for more information). We know that $\mathrm{vcd}(\Gamma) = \dim(\Omega) = p^2$. Then we relate $\mathrm{vcd}(\Gamma)$ to the structure of G^0 to show that G has to have finitely many components, and G^0 is simple. This latter argument only fails if $p = 2$, in which case we obtain very specific information on the structure of G^0 and its action on Ω (cases (3) and (4) in the above Theorem 8.2).

We start with the following lemma.

Lemma 8.3 *Γ is a cocompact lattice in G and $\Gamma_0 := \Gamma \cap G^0$ is a cocompact lattice in G^0.*

Proof Since Γ acts cocompactly on Ω and G acts properly on Ω (see Proposition 4.8), we see that $\Gamma \leq G$ is a cocompact lattice. Since $G^0 \leq G$ is a connected component, the set $\Gamma \cdot G^0$ is closed in G. So $\Gamma_0 \backslash G^0$ is closed in $\Gamma \backslash G$. Then, since $\Gamma \backslash G$ is compact, so is $\Gamma_0 \backslash G^0$. □

The rest of this section will be devoted to the proof of Theorem 8.2. We will assume that cases (1), (2), (3) and (4) do not hold and show that case (5) occurs.

Lemma 8.4 *$\Gamma_0 \cap Z$ is a cocompact lattice in Z.*

Proof Let $G^{\mathrm{ss}} \leq G$ be a semisimple subgroup such that $G^0 = G^{\mathrm{ss}} G^{\mathrm{sol}}$ is a Levi–Maltsev decomposition of G^0. Then let $\sigma\colon G^{\mathrm{ss}} \to \mathrm{Aut}(G^{\mathrm{sol}})$ be the action of G^{ss} by conjugation on G^{sol}. If $\ker\sigma$ has no compact factors in its identity component, then $\Gamma_0 \cap N$ is a cocompact lattice in N (see [25, Theorem 1.3(i)]). In this case, $\Gamma_0 \cap Z \leq Z$ is a cocompact lattice by [44, Proposition 2.17].

Therefore, it suffices to show $\ker\sigma$ contains no compact factors. Since $\ker\sigma \leq G^{\mathrm{ss}}$ is a normal subgroup, we see that $\ker\sigma$ is semisimple. So there is a unique maximal connected, compact, normal subgroup K_0 in $\ker\sigma$. Assume for a contradiction that $\dim K_0 > 0$. Then K_0 is also a connected normal subgroup of G^{ss} and hence of G^0, which is impossible by an argument of Farb and Weinberger [21, Claim II]. Let us sketch this proof for completeness.

Let K be a maximal compact factor of G^0. Since $\dim K_0 > 0$, we see that $\dim K > 0$. Consider the natural quotient map $\Omega \to \Omega/K$. Since Γ permutes the maximal compact factors of G^0, we see that a finite-index subgroup of Γ normalizes K. Then it is not hard to see that there is a continuous quasi-isometric inverse $\Omega/K \to \Omega$ to this quotient map. Consider the maps induced by the composition

$$\Omega \to \Omega/K \to \Omega$$

on locally finite simplicial homology. On the one hand, since this composition is a bounded distance from the identity map, the induced map on locally finite simplicial homology is the identity map. On the other hand, since Ω is the universal cover of a closed aspherical manifold, there is a fundamental class in top degree. But since $\dim K > 0$, the image of this fundamental class in $H_*(\Omega/K)$ vanishes. This is a contradiction. For full details, see the proof of Claim II in [21]. □

Lemma 8.5 *G^0 is semisimple.*

Proof As above let N be the nilpotent radical of G^{sol} and Z the center of N. If $N = 1$, then G^0 is semisimple. So suppose for a contradiction that $N \neq 1$. Then $Z \neq 1$. Next let C be the Zariski closure of Z in $\mathrm{PSL}_{2p}(\mathbb{R})$ and let C^0 be the connected component of the identity in C. Since G normalizes Z, it also normalizes C and C^0.

Since Z, is abelian so is C^0. Then, since C^0 is an abelian real algebraic group, we can write

$$C^0 = C_{\mathrm{ss}} C_{\mathrm{u}},$$

where C_{ss} is the subset of semisimple elements in C^0 and C_{u} is the subset of unipotent elements of C^0 (see eg [12, Theorem 4.7]). By [12, Corollary 4.4] both C_{ss} and C_{u} are actually groups. Since G normalizes C^0 it also normalizes C_{ss} and C_{u}.

If $C_{\mathrm{ss}} = 1$, then each element of C^0 is unipotent and thus each element of Z is unipotent. Thus, we are in case (2), which is a contradiction. Therefore we have

$C_{\mathrm{ss}} \neq 1$. But the normalizer of any semisimple torus T in $\mathrm{PGL}_{2p}(\mathbb{R})$ contains the centralizer of T with finite index [12, Corollary 8.10.2], so we know that a finite-index subgroup of G centralizes C_{ss}. Hence, we are in case (1), which contradicts our initial assumption. Thus, G^0 is semisimple. □

Lemma 8.6 *G^0 has trivial center.*

Proof Let Z be the center of G^0. First, we observe that Z is finite. Indeed, the center of any connected semisimple linear group is finite (see eg [41, page 146]). We already know that G^0 is connected and semisimple, and G^0 is linear because it is a subgroup of the linear group $\mathrm{PGL}_{2p}(\mathbb{R})$.

Next we show Z is trivial. Since G normalizes G^0, G also normalizes Z. Since Z is finite, a finite-index subgroup of G centralizes Z. Thus, if $Z \neq 1$ we are in case (1), which has been excluded by assumption. □

Next we use an argument of Farb and Weinberger to deduce:

Lemma 8.7 [21, Proposition 3.1] *G has a finite-index subgroup G' such that $G' \cong G^0 \times \Lambda$ for some discrete group Λ and Γ has a finite-index subgroup Γ' such that $\Gamma' \cong \Gamma_0 \times \Lambda$. Moreover, by possibly passing to a finite-index subgroup of G' we may assume that Λ is either trivial or infinite.*

Remark 8.8 The above lemma follows from the "triviality of the extension" part of the proof of Proposition 3.1 in [21]. This part of their proof only involves the groups and not the Riemannian metric in the statement of Proposition 3.1. In particular, this part of the argument adapts to our situation verbatim.

Now let

$$\mathrm{SL}^{\pm}_{2p}(\mathbb{R}) = \{g \in \mathrm{GL}_{2p}(\mathbb{R}) : \det g = \pm 1\}.$$

Then let $\widehat{G}$ be the inverse image of G under the map $\pi\colon \mathrm{SL}^{\pm}_{2p}(\mathbb{R}) \to \mathrm{PGL}_{2p}(\mathbb{R})$ and let $\widehat{G}^0$ be the connected component of the identity of $\widehat{G}$.

Decompose the representation $\widehat{G}^0 \curvearrowright \mathbb{R}^{2p}$ as a direct sum of irreducible representations of the semisimple group $\widehat{G}^0$:

$$\mathbb{R}^{2p} \cong \bigoplus_{\rho} V_{\rho}^{n_\rho}. \tag{8-1}$$

Here the direct sum is over nonisomorphic irreducible representations ρ of $\widehat{G}^0$ and $n_\rho \geq 0$ is the multiplicity of ρ. Now since $\widehat{G}$ normalizes $\widehat{G}^0$ we see that $\widehat{G}$ preserves each $V_\rho^{n_\rho}$.

First let us consider the situation that multiple irreducible representations contribute, say $\rho_1, \dots, \rho_k$, where $k > 1$. Consider the 1–parameter group $\{b_t : t \in \mathbb{R}\}$, where b_t acts by e^t on the $V_{\rho_1}^{n_{\rho_1}}$ factor and by the identity on all other factors. Then b_t is not a scalar matrix, and centralizes G, so we are in case (1).

Therefore, there is only one irreducible representation and $\mathbb{R}^{2p} \cong V_\rho^n$ for some irreducible representation ρ and some n.

Lemma 8.9 $n = 1$.

Proof Suppose for a contradiction that $n > 1$. We first claim that $p = 2$. Let us now consider the *virtual cohomological dimension* $\operatorname{vcd}(\Gamma)$ of Γ. Recall that the *cohomological dimension* $\operatorname{cd}(\Gamma)$ of Γ is the supremum of all numbers m such that $H^m(\Gamma, M) \neq 0$ for some Γ–module M (see for instance [14, Chapter VIII] for more information). We will only need the following properties of $\operatorname{cd}(\Gamma)$:

(1) $\operatorname{cd}(\Gamma) > 0$ if $\Gamma \neq 1$.

(2) If Γ acts freely and properly discontinuously on a contractible CW–complex X, then $\operatorname{cd}(\Gamma) \leq \dim(X)$, with equality if and only if X/Γ is compact.

(3) If $\Delta \subseteq \Gamma$, then $\operatorname{cd}(\Delta) \leq \operatorname{cd}(\Gamma)$.

(4) If $\Gamma = \Gamma_0 \times \Gamma_1$, then $\operatorname{cd}(\Gamma) \leq \operatorname{cd}(\Gamma_0) + \operatorname{cd}(\Gamma_1)$.

The virtual cohomological dimension of Γ is then the infimum of $\operatorname{cd}(\Delta)$ as Δ ranges over finite-index subgroups of Γ.

Now write $\dim V_\rho = d$. Since Γ_0 can be identified with a discrete subgroup of $\mathrm{PGL}(V_\rho)$, we have, by property (2) above,

$$\operatorname{vcd}(\Gamma_0) \leq \dim \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d) = \tfrac{1}{2}d(d+1) - 1. \tag{8-2}$$

Further, since Λ commutes with G^0 and ρ is an irreducible representation of $\widehat{G}^0$, we can identify Λ with a discrete subgroup of $\mathrm{PGL}_n(\mathbb{R})$. Therefore,

$$\operatorname{vcd}(\Lambda) \leq \dim \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n) = \tfrac{1}{2}n(n+1) - 1. \tag{8-3}$$

On the other hand, $\mathrm{vcd}(\Gamma) = \dim \Omega = p^2$ by property (2) above. Combining this with property (4) and equations (8-2) and (8-3), we have

$$\begin{aligned} 2p^2 = 2\mathrm{vcd}(\Gamma) &\leq 2(\mathrm{vcd}(\Gamma_0) + \mathrm{vcd}(\Lambda)) \\ &\leq d(d+1) - 2 + n(n+1) - 2 \\ &= d^2 + d + n^2 + n - 4. \end{aligned}$$

Using that $2p = dn$ (from the dimension count in $\mathbb{R}^{2p} \cong V_\rho^n$), we find that

$$2p^2 \leq \frac{4p^2}{n^2} + \frac{2p}{n} + n^2 + n - 4.$$

The right-hand side is a convex function of n, so that on the interval $[2, p]$, it is maximal at one of the endpoints. At either endpoint the inequality reduces to

$$p^2 - p - 2 \leq 0,$$

which is only possible if $p = 2$.

Then $(n, d) \in \{(2,2), (1,4), (4,1)\}$. We assumed that $n > 1$ and, since the representation $\widehat{G}^0 \hookrightarrow \mathrm{SL}(V_\rho)$ is injective, we must have $d > 1$. So $n = d = 2$.

Thus, $\widehat{G}^0$ is a semisimple Lie group which has a faithful irreducible representation into $\mathrm{SL}_2(\mathbb{R})$. Thus, $\widehat{G}^0$ has to be isomorphic to $\mathrm{SL}_2(\mathbb{R})$ and $\rho = \mathrm{Id}$. With respect to the decomposition $\mathbb{R}^4 = V \oplus V$ we have

$$\widehat{G}^0 = \{(\varphi, \varphi) \in \mathrm{SL}(V) \times \mathrm{SL}(V)\}$$

and hence we are in case (3), which is a contradiction. □

Since $n = 1$, we have that $\widehat{G}^0 \curvearrowright \mathbb{R}^{2p}$ is an irreducible representation. Note that Λ centralizes G^0 in $\mathrm{PGL}_{2p}(\mathbb{R})$, and hence any element of $\mathrm{GL}_{2p}(\mathbb{R})$ lying over Λ has to be scalar by Schur's lemma. It follows that Λ is trivial, so that $G' = G^0$ and thus G^0 has finite index in G. Then Γ_0 has finite index in Γ and hence acts cocompactly on Ω. Thus, $\mathrm{vcd}(\Gamma_0) = \dim(\Omega) = p^2$.

Lemma 8.10 *G^0 acts transitively on Ω.*

Proof Let $x \in \Omega$ be any point and let K_x denote its stabilizer in G^0. Then K_x is a compact subgroup of G^0 by Proposition 4.8 and the G^0–orbit X of x is diffeomorphic to G^0/K_x. Now let K be a maximal compact subgroup of G^0 containing K_x. Then $\Gamma_0 \backslash G^0 / K$ is a closed aspherical manifold with fundamental group Γ_0, so by property (2) of cohomological dimension we have $\mathrm{vcd}(\Gamma_0)=\dim(G^0/K)$. On the other hand, since $K_x \leq K$ and $G^0/K_x \cong X \subset \Omega$,

$$\mathrm{vcd}(\Gamma_0)=\dim(G^0/K)\leq \dim(G^0/K_x)=\dim(X)\leq \dim(\Omega)=\mathrm{vcd}(\Gamma_0).$$

We conclude that $\dim(X)=\dim(\Omega)$, so that X is a codimension 0 closed submanifold of Ω. Connectedness of Ω then implies that $X=\Omega$, as desired. □

Remark 8.11 The above proof shows that the stabilizer of any point $x \in \Omega$ has finite index in a maximal compact subgroup of $\mathrm{Aut}(\Omega)$.

Lemma 8.12 *G^0 is simple.*

Proof Since G^0 has trivial center, either G^0 is simple or $G^0 \cong G_1 \times G_2$ for some semisimple nontrivial Lie groups G_1 and G_2.

So suppose that $G^0 \cong G_1 \times G_2$. Let $\widehat{G}_i$ be the inverse image of $G_i \times \{\mathrm{Id}\}$ under the map $\mathrm{SL}_{2p}(\mathbb{R}) \to \mathrm{PSL}_{2p}(\mathbb{R})$. Next decompose the representation $\widehat{G}_1 \curvearrowright \mathbb{R}^{2p}$ as a direct sum of irreducible representations of the semisimple group $\widehat{G}_1$:

$$\mathbb{R}^{2p} \cong \bigoplus_{\tau} V_\tau^{n_\tau}.$$

Here the direct sum is over nonisomorphic irreducible representations τ of $\widehat{G}_1$, and $n_\tau \geq 0$ is the multiplicity of τ. Using the fact that $\widehat{G}_2$ centralizes $\widehat{G}_1$ and arguing as in Lemma 8.9, we see that $p=2$ and $\mathbb{R}^4=V_\tau^2$ for some irreducible representation τ of $\widehat{G}_1$. So $\dim V_\tau=2$ and thus $\widehat{G}_1$ is isomorphic to $\mathrm{SL}_2(\mathbb{R})$. Applying the same argument to $\widehat{G}_2$ shows that $\widehat{G}_2$ is also isomorphic to $\mathrm{SL}_2(\mathbb{R})$. Up to conjugation, we have

$$\widehat{G}_1=\left\{\begin{pmatrix} A & 0 \\ 0 & A\end{pmatrix} : A \in \mathrm{SL}_2(\mathbb{R})\right\}.$$

An easy computation shows that the centralizer of $\widehat{G}_1$ is exactly

$$\left\{\begin{pmatrix} a\,\mathrm{Id}_2 & b\,\mathrm{Id}_2 \\ c\,\mathrm{Id}_2 & d\,\mathrm{Id}_2\end{pmatrix} : ad-bc=1\right\} \cong \mathrm{SL}_2(\mathbb{R}).$$

Since $\widehat{G}_2$ centralizes $\widehat{G}_1$ and is isomorphic to $\mathrm{SL}_2(\mathbb{R})$, we must have that

$$\widehat{G}_2=\left\{\begin{pmatrix} a\,\mathrm{Id}_2 & b\,\mathrm{Id}_2 \\ c\,\mathrm{Id}_2 & d\,\mathrm{Id}_2\end{pmatrix} : ad-bc=1\right\}.$$

Hence, we are in case (4), which is a contradiction. □

9 The centralizer

In this section we prove that case (1) in Theorem 8.2 is impossible. For a subgroup $H \leq \mathrm{PGL}_{p+q}(\mathbb{R})$, let

(1) $\hat{H} = \{h \in \mathrm{GL}_{p+q}(\mathbb{R}) : [h] \in H,\ \det h = \pm 1\}$,

(2) $C_H = \{c \in \mathrm{End}(\mathbb{R}^{p+q}) : ch = hc \text{ for all } h \in \hat{H}\}$, and

(3) C_H^0 be the connected component of Id_{p+q} in $C_H \cap \mathrm{GL}_{p+q}(\mathbb{R})$.

Remark 9.1 C_H is the centralizer of H in $\mathrm{End}(\mathbb{R}^{p+q})$, and hence is a subalgebra of $\mathrm{End}(\mathbb{R}^{p+q})$, whereas C_H^0 is a subgroup of $\mathrm{GL}_{p+q}(\mathbb{R})$.

With this notation we will prove the following:

Theorem 9.2 *Suppose $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$ is an open set which is convex and bounded in some affine chart. If $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group that acts cocompactly on Ω, then $C_\Gamma^0 = \mathbb{R}_{>0}\,\mathrm{Id}_{2p}$.*

9.1 The centralizer in the general case

We begin by proving the following (which holds for any Grassmannian):

Theorem 9.3 *Suppose $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is an open $\mathcal{R}$–proper set that is convex in some affine chart. If $H \leq \mathrm{Aut}(\Omega)$ acts cocompactly on Ω, then $C_H^0 \leq \mathrm{Aut}(\Omega)$ and there is a decomposition $\mathbb{R}^{p+q} = \bigoplus_{i=1}^m V_i$ such that*

$$C_H = \bigoplus_{i=1}^m \mathbb{R}\,\mathrm{Id}_{V_i}\,.$$

Remark 9.4 In the special case where $p = 1$, the above theorem is due to Vey [45, Theorem 5]. In both proofs the main step is to show that the elements of C_H^0 are real diagonalizable, however the methods for accomplishing this are very different.

For the rest of this subsection assume that $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ and $H \leq \mathrm{Aut}(\Omega)$ satisfy the hypothesis of Theorem 9.3.

Lemma 9.5 *With the notation above, $C_H^0 \leq \mathrm{Aut}(\Omega)$.*

Proof Vey [45, page 645] proved this lemma in the case when $p = 1$ and his proof works verbatim here: Fix a compact set $K \subset \Omega$ such that $HK = \Omega$. Then there exists a neighborhood $\mathcal{O}$ of Id_{p+q} in C_H^0 such that $\mathcal{O}$ generates C_H^0 and $uK \subset \Omega$ for all $u \in \mathcal{O}$. Without loss of generality, we can assume that $\mathcal{O}$ is symmetric, ie for any $u \in \mathcal{O}$, we have $u^{-1} \in \mathcal{O}$. Then, for $u \in \mathcal{O}$, we have

$$u\Omega = uHK = HuK \subset H\Omega = \Omega.$$

Since $\mathcal{O}$ is symmetric we also see that $u^{-1}\Omega \subset \Omega$. Thus, u restricts to a diffeomorphism $\Omega \to \Omega$ and $u \in \mathrm{Aut}(\Omega)$. Since $\mathcal{O}$ generates C_H^0, we then see that $C_H^0 \leq \mathrm{Aut}(\Omega)$. □

Lemma 9.6 *With the notation above, if $c \in C_H^0$ then*

$$\sup_{x\in\Omega} K_\Omega(cx, x) < \infty.$$

Proof Fix some $x_0 \in \Omega$. Since H acts cocompactly on Ω, there exists $R > 0$ such that

$$\bigcup_{h\in H} B_R(hx_0) = \Omega.$$

If $x \in \Omega$, pick $h \in H$ such that $K_\Omega(x, hx_0) \leq R$. Then

$$\begin{aligned} K_\Omega(cx, x) &\leq K_\Omega(cx, chx_0) + K_\Omega(chx_0, hx_0) + K_\Omega(hx_0, x) \\ &\leq K_\Omega(x, hx_0) + K_\Omega(cx_0, x_0) + R \\ &\leq 2R + K_\Omega(cx_0, x_0). \end{aligned}$$

□

Lemma 9.7 *With the notation above, if $c \in C_H^0$ then c fixes every $\mathcal{R}$–extreme point of Ω.*

Proof For an $\mathcal{R}$–extreme point $x \in \partial\Omega$, choose points $p_n \in \Omega$ with $p_n \to x$. By Lemma 9.6, we have

$$\limsup_{n\to\infty} d_\Omega(cp_n, p_n) < \infty.$$

Then, by Corollary 6.3, we have $cp_n \to x$. Since c acts continuously on $\mathrm{Gr}_p(\mathbb{R}^{2p})$ and $p_n \to x$, we must have that $cx = x$. □

We will need the following elementary facts:

Lemma 9.8 *Let $p, q > 0$. The homomorphism $\bigwedge^p \colon \mathrm{GL}_{p+q}(\mathbb{R}) \to \mathrm{GL}\big(\bigwedge^p \mathbb{R}^{p+q}\big)$*

(i) *maps unipotents to unipotents and semisimple elements to semisimple elements, and*

(ii) *has kernel given by $\{\mathrm{Id}_{p+q}\}$ if p is odd and $\{\pm \mathrm{Id}_{p+q}\}$ if p is even.*

Proof Assertion (i) is obvious from the definition of $\bigwedge^p$. To see (ii), consider some $g \in \mathrm{GL}_{p+q}(\mathbb{R})$ with $\bigwedge^p g = 1$. Let $\lambda_1, \dots, \lambda_{p+q}$ be the eigenvalues of g (listed with multiplicity). Then the eigenvalues of $\bigwedge^p g$ are exactly given by the product of p eigenvalues of g, ie $\lambda_{i_1} \cdots \lambda_{i_p}$ for any choice of $1 \le i_1 < \cdots < i_p \le p+q$. We claim that $\lambda_1 = \lambda_2 = \cdots = \lambda_{p+q}$. To see this fix $1 \le i, j \le p+q$ distinct and then fix some $i_1, \dots, i_{p-1}$ such that i, j, i_1, $\dots, i_{p-1}$ are all distinct. Since $\bigwedge^p g = 1$, we have

$$\lambda_i \lambda_{i_1} \cdots \lambda_{i_{p-1}} = 1 = \lambda_j \lambda_{i_1} \cdots \lambda_{i_{p-1}},$$

so that $\lambda_i = \lambda_j$. Since i and j were arbitrary, we then have $\lambda_1 = \lambda_2 = \cdots = \lambda_{p+q}$. So

$$\lambda_1^p = \lambda_1 \cdots \lambda_p = 1.$$

In addition, λ_1 is real, so it follows that $\lambda_1 \in \{-1, 1\}$. We conclude that $g = \pm \mathrm{Id}_{p+q}$. □

Lemma 9.9 *With the notation above, every $c \in C_H^0$ is semisimple and C_H^0 is abelian.*

Proof Fix a basis $v_1, \dots, v_D$ of $\bigwedge^p \mathbb{R}^{p+q}$ such that each $[v_i]$ is an $\mathcal{R}$–extreme point of Ω (this is possible by Proposition 7.8). Then for any $c \in C_H^0$, each v_i is an eigenvector of $\bigwedge^p c$ and so $\bigwedge^p c$ is diagonalizable with respect to the basis $v_1, \dots, v_D$ of $\bigwedge^p \mathbb{R}^{p+q}$. Hence, $\bigwedge^p C_H^0$ is an abelian group.

Now, since $\bigwedge^p C_H^0$ is an abelian group, we see that $\bigwedge^p [C_H^0, C_H^0] = 1$. Then, since $\ker \bigwedge^p \subset \{\pm \mathrm{Id}_{p+q}\}$, we see that $[C_H^0, C_H^0] \subset \{\pm \mathrm{Id}_{p+q}\}$. But since C_H^0 is connected, $[C_H^0, C_H^0]$ is connected and hence must be trivial. We conclude that C_H^0 is abelian.

Next, we claim that any $c \in C_H^0$ is semisimple. If $c = su$ is the Jordan decomposition of c then $\bigwedge^p c = \left(\bigwedge^p s\right)\left(\bigwedge^p u\right)$ and by uniqueness this is the Jordan decomposition of $\bigwedge^p c$. It follows that $\bigwedge^p u = 1$, and hence $u = 1$. We conclude that $c = s$ is semisimple. □

Lemma 9.10 *With the notation above, every $c \in C_H^0$ has all real eigenvalues.*

Let us comment briefly on the strategy of the proof of Lemma 9.10 before carrying out the algebraic manipulations. Notice that the proof of Lemma 9.9 implies that if $c \in C_H^0$, then $\bigwedge^p c$ has all real eigenvalues. Therefore the product of any p distinct eigenvalues of c (counted with multiplicity) is real. Unfortunately this does not directly imply that the eigenvalues of c are real; for example, if $g \in \mathrm{GL}_4(\mathbb{R})$ has eigenvalues $\pm i$, each with multiplicity 2, then $\bigwedge^2 g$ has eigenvalues ± 1. The strategy in the proof of Lemma 9.10 is to argue by contradiction, ie assume there exists some element $c \in C_H^{\mathbf{0}}$ which has a nonreal eigenvalue and then use c to construct some other $c' \in C_H^{\mathbf{0}}$, where $\bigwedge^p c'$ has a nonreal eigenvalue.

Proof For $n \in \mathbb{N}$, $\lambda > 0$ and $\theta \in [0, 2\pi)$, let $E_n(\lambda, \theta)$ be the $2n \times 2n$ block diagonal matrix whose blocks are

$$\begin{pmatrix} \lambda \cos\theta & -\lambda \sin\theta \\ \lambda \sin\theta & \lambda\cos\theta \end{pmatrix}.$$

Now suppose for a contradiction that there exists some $c \in C_H^{\mathbf{0}}$ with a nonreal eigenvalue. Then there exist $g \in \mathrm{SL}_{p+q}(\mathbb{R})$; $n_1, \dots, n_k \in \mathbb{N}$; $\lambda_1, \dots, \lambda_r > 0$; $\theta_1, \dots, \theta_r \in [0, 2\pi)$; and $\mu_{r+1}, \dots, \mu_k \in \mathbb{R}$ such that

$$c = g \begin{pmatrix} E_{n_1}(\lambda_1, \theta_1) & & & & & \\ & \ddots & & & & \\ & & E_{n_r}(\lambda_r; \theta_r) & & & \\ & & & \mu_{r+1}\,\mathrm{Id}_{n_{r+1}} & & \\ & & & & \ddots & \\ & & & & & \mu_k\,\mathrm{Id}_{n_k} \end{pmatrix} g^{-1}.$$

We can further assume that the pairs (λ_i, θ_i) are all distinct and the μ_i are all distinct. Then we have

$$\hat{H} \le \left\{ g \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix} g^{-1} : A_i \in \mathrm{GL}_{n_i}(\mathbb{R}) \right\},$$

which implies that

$$\left\{ g \begin{pmatrix} E_{n_1}(\lambda, \theta) & \\ & \mathrm{Id}_{n_2+\dots+n_k} \end{pmatrix} g^{-1} : \lambda, \theta \in \mathbb{R} \right\} \le C_H^{\mathbf{0}}.$$

Then it is easy to construct some $c' \in C_H^{\mathbf{0}}$ such that $\bigwedge^p c'$ has a nonreal eigenvalue. So we have a contradiction. □

Lemma 9.11 *With the notation above, there is a decomposition* $\mathbb{R}^{p+q} = \bigoplus_{i=1}^m V_i$ *such that*

$$C_H = \bigoplus_{i=1}^m \mathbb{R}\,\mathrm{Id}_{V_i}.$$

Proof Since C_H^0 is abelian and every element in C_H^0 is semisimple with all real eigenvalues, there exist some $g \in \mathrm{SL}_{p+q}(\mathbb{R})$ and $n_1, \dots, n_k \in \mathbb{N}$ such that

$$C_H^0 \leq \left\{ g \begin{pmatrix} \mu_1 \operatorname{Id}_{n_1} & & \\ & \ddots & \\ & & \mu_k \operatorname{Id}_{n_k} \end{pmatrix} g^{-1} : \mu_1, \dots, \mu_k > 0 \right\}.$$

We may further assume that for every $1 \leq i < j \leq k$ there exists $c \in C_H^0$ such that

$$c = g \begin{pmatrix} \mu_1 \operatorname{Id}_{n_1} & & \\ & \ddots & \\ & & \mu_k \operatorname{Id}_{n_k} \end{pmatrix} g^{-1}$$

and $\mu_i \neq \mu_j$. Then we have

$$\hat{H} \leq \left\{ g \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix} g^{-1} : A_i \in \mathrm{GL}_{n_i}(\mathbb{R}) \right\}$$

and hence

$$C_H^0 = \left\{ g \begin{pmatrix} \mu_1 \operatorname{Id}_{n_1} & & \\ & \ddots & \\ & & \mu_k \operatorname{Id}_{n_k} \end{pmatrix} g^{-1} : \mu_1, \dots, \mu_k > 0 \right\}.$$

Now if $X \in C_H$, then there exists some $t \in \mathbb{R}$ such that $\operatorname{Id}_{p+q} + tX \in C_H^0$. Hence,

$$C_H = \left\{ g \begin{pmatrix} \mu_1 \operatorname{Id}_{n_1} & & \\ & \ddots & \\ & & \mu_k \operatorname{Id}_{n_k} \end{pmatrix} g^{-1} : \mu_1, \dots, \mu_k \in \mathbb{R} \right\},$$

which implies the lemma. □

9.2 The centralizer in $\mathrm{Gr}_p(\mathbb{R}^{2p})$

We now specialize to the case in which $p = q$ and prove Theorem 9.2. We begin by showing that we can assume that Ω is a cone in some affine chart.

Proposition 9.12 *Suppose $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is an open set which is convex and bounded in some affine chart. If $H \leq \mathrm{Aut}(\Omega)$ acts cocompactly on Ω and $C_H^0 \neq \mathbb{R}_{>0} \operatorname{Id}_{2p}$, then there exists $\varphi \in \mathrm{PGL}_{2p}(\mathbb{R})$ such that*

$$\varphi\Omega \subset \mathbb{M} = \left\{ \begin{bmatrix} \operatorname{Id}_p \\ X \end{bmatrix} : X \in M_{p,p}(\mathbb{R}) \right\}$$

and $\varphi\Omega$ is a convex cone in $\mathbb{M}$ based at 0. Moreover, we can select φ such that either

$$C^0_{\varphi H\varphi^{-1}} = \left\{ \begin{pmatrix} e^t \operatorname{Id}_p & 0 \\ 0 & e^s \operatorname{Id}_p \end{pmatrix} : s,t \in \mathbb{R} \right\}.$$

or $C^0_{\varphi H\varphi^{-1}}$ contains the subgroup

$$\left\{ \begin{pmatrix} e^t \operatorname{Id}_{p+\ell} & 0 \\ 0 & e^s \operatorname{Id}_{p-\ell} \end{pmatrix} : s,t \in \mathbb{R} \right\}$$

for some $0 < \ell < p$.

Remark 9.13 By Corollary 7.11, there exists $\varphi \in \mathrm{GL}_{2p}(\mathbb{R})$ such that $\varphi\Omega \subset \mathbb{M}$ and $\varphi\Omega$ is a convex cone in $\mathbb{M}$ based at 0. The key part of the proposition is that we can pick φ such that the centralizer $C_{\varphi H\varphi^{-1}}$ has a subgroup of a particularly nice form.

Proof We can assume that Ω is a convex bounded subset of $\mathbb{M}$. Throughout the argument we will replace Ω by translates of the form

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \Omega.$$

This transformation preserves the affine chart $\mathbb{M}$ and acts on $\mathbb{M}$ by affine transformations.

By Theorem 9.3, there exist $g_0 \in \mathrm{GL}_{2p}(\mathbb{R})$ and $0 \le \ell < p$ such that

$$T := \left\{ g_0 \begin{pmatrix} e^t \operatorname{Id}_{p+\ell} & 0 \\ 0 & e^s \operatorname{Id}_{p-\ell} \end{pmatrix} g_0^{-1} : s,t \in \mathbb{R} \right\} \le C^0_H.$$

Notice that we can choose $\ell > 0$ except when

$$C^0_H = \left\{ g_0 \begin{pmatrix} e^t \operatorname{Id}_p & 0 \\ 0 & e^s \operatorname{Id}_p \end{pmatrix} g_0^{-1} : s,t \in \mathbb{R} \right\}.$$

So, in the case when $\ell = 0$ we can also assume that $C^0_H = T$.

Now let $W := g_0 \operatorname{Span}\{e_1, \dots, e_{p+\ell}\}$. Notice that $hW = W$ for all $h \in H$. We claim that there exists an $\mathcal{R}$–extreme point e of Ω in $\mathrm{Gr}_p(W)$. Consider some

$$c = g_0 \begin{pmatrix} e^t \operatorname{Id}_{p+\ell} & 0 \\ 0 & e^s \operatorname{Id}_{p-\ell} \end{pmatrix} g_0^{-1} \in T$$

with $e^t > e^s$. Then $E^+\left(\bigwedge^p c\right) \cap \mathrm{Gr}_p(\mathbb{R}^{2p}) \subset \mathrm{Gr}_p(W)$ and by Corollary 7.10 there is an $\mathcal{R}$–extreme point e of Ω in $E^+\left(\bigwedge^p c\right) \cap \partial\Omega \subset \mathrm{Gr}_p(W)$.

Now by replacing Ω with an affine translate we can assume that

$$e = \begin{bmatrix} \mathrm{Id}_p \\ 0 \end{bmatrix},$$

which implies that $\mathrm{Span}\{e_1, \dots, e_p\} \subset W$. By construction, if $a \in T$ then $a|_W = e^t \, \mathrm{Id}_W$ for some $t \in \mathbb{R}$. So any $a \in T$ can be written as

$$a = \begin{pmatrix} e^t \, \mathrm{Id}_p & B \\ 0 & C \end{pmatrix}$$

for some $t \in \mathbb{R}$ and $B, C \in \mathrm{GL}_p(\mathbb{R})$.

Since e is an extreme point, by Corollary 7.11 there exist $t_n \to \infty$ and $h_n \in H$ such that

$$\varphi = \lim_{n\to\infty} \begin{bmatrix} \mathrm{Id}_p & 0 \\ 0 & e^{t_n} \, \mathrm{Id}_p \end{bmatrix} h_n$$

in $\mathrm{PGL}_{2p}(\mathbb{R})$ and $\varphi(\Omega) = \mathcal{T}\mathcal{C}_0 \, \Omega$. Let $\widehat{\varphi} \in \mathrm{GL}_{2p}(\mathbb{R})$ be a representative of φ and, for each $n \in \mathbb{N}$, choose a representative $\widehat{h}_n \in \mathrm{GL}_{2p}(\mathbb{R})$ of h_n such that

$$\widehat{\varphi} = \lim_{n\to\infty} \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & e^{t_n} \, \mathrm{Id}_p \end{pmatrix} \widehat{h}_n$$

in $\mathrm{GL}_{2p}(\mathbb{R})$.

Then if

$$a = \begin{pmatrix} e^t \, \mathrm{Id}_p & B \\ 0 & C \end{pmatrix} \in T,$$

we have

$$\begin{aligned}
\widehat{\varphi} a \widehat{\varphi}^{-1} &= \lim_{n\to\infty} \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & e^{t_n} \, \mathrm{Id}_p \end{pmatrix} \widehat{h}_n \begin{pmatrix} e^t \, \mathrm{Id}_p & B \\ 0 & C \end{pmatrix} \widehat{h}_n^{-1} \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & e^{-t_n} \, \mathrm{Id}_p \end{pmatrix} \\
&= \lim_{n\to\infty} \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & e^{t_n} \, \mathrm{Id}_p \end{pmatrix} \begin{pmatrix} e^t \, \mathrm{Id}_p & B \\ 0 & C \end{pmatrix} \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & e^{-t_n} \, \mathrm{Id}_p \end{pmatrix} \\
&= \begin{pmatrix} e^t \, \mathrm{Id}_p & 0 \\ 0 & C \end{pmatrix}.
\end{aligned}$$

In the second equality we used that $a \in C_H^0$.

Then, since T is abelian, we can find some $g_0 \in \mathrm{GL}_p(\mathbb{R})$ such that if

$$g = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & g_0 \end{pmatrix},$$

then

$$g\widehat{\varphi}T(g\widehat{\varphi})^{-1} = \left\{ \begin{pmatrix} e^t \operatorname{Id}_{p+\ell} & 0 \\ 0 & e^s \operatorname{Id}_{p-\ell} \end{pmatrix} : s, t \in \mathbb{R} \right\}.$$

So, replacing φ by $g\varphi$ (and hence replacing $\widehat{\varphi}$ by $g\widehat{\varphi}$), we can assume

$$\widehat{\varphi}T\widehat{\varphi}^{-1} = \left\{ \begin{pmatrix} e^t \operatorname{Id}_{p+\ell} & 0 \\ 0 & e^s \operatorname{Id}_{p-\ell} \end{pmatrix} : s, t \in \mathbb{R} \right\}.$$

Since $\widehat{\varphi}T\widehat{\varphi}^{-1} \leq C^0_{\varphi H \varphi^{-1}}$, this completes the proof. ☐

Proof of Theorem 9.2 By Proposition 9.12, we can assume that

$$\Omega \subset \mathbb{M} = \left\{ \begin{bmatrix} \operatorname{Id}_p \\ X \end{bmatrix} : X \in M_{p,p}(\mathbb{R}) \right\}$$

is a convex cone in $\mathbb{M}$ based at 0, and that C^0_Γ contains the subgroup

$$\left\{ \begin{pmatrix} e^t \operatorname{Id}_{p+\ell} & 0 \\ 0 & e^s \operatorname{Id}_{p-\ell} \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

for some $0 \leq \ell < p$. Then

$$\Gamma \leq \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A \in \mathrm{GL}_{p+\ell}(\mathbb{R}), B \in \mathrm{GL}_{p-\ell}(\mathbb{R}) \right\}.$$

Throughout the argument we will write a matrix $X \in M_{p,p}(\mathbb{R})$ as

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

where $X_1 \in M_{\ell,p}(\mathbb{R})$ and $X_2 \in M_{p-\ell,p}(\mathbb{R})$. Let

$$\Omega_2 = \left\{ \begin{bmatrix} \operatorname{Id}_p \\ 0 \\ X_2 \end{bmatrix} : \text{there exists } X_1 \text{ such that } \begin{bmatrix} \operatorname{Id}_p \\ X_1 \\ X_2 \end{bmatrix} \in \Omega \right\}.$$

Lemma 9.14 *Ω_2 is a proper convex cone in $\mathbb{M}$, ie Ω_2 does not contain any affine lines.*

Proof Since Ω_2 is open and convex, it is easy to see that

$$\{x + tv : t \in \mathbb{R}\} \subset \Omega_2 \text{ for some } x \in \Omega_2 \iff \{x + tv : t \in \mathbb{R}\} \subset \Omega_2 \text{ for all } x \in \Omega_2.$$

Hence, Ω_2 contains an affine line if and only if there exists some nonzero $v \in \mathbb{M}$ such that

$$\begin{bmatrix} \mathrm{Id}_p & 0 & 0 \\ 0 & \mathrm{Id}_\ell & 0 \\ v & 0 & \mathrm{Id}_{p-\ell} \end{bmatrix} \in \mathrm{Aut}(\Omega).$$

Thus, to complete the proof, it suffices to show that

$$\{\mathrm{Id}_{2p}\} = \left\{ \begin{bmatrix} \mathrm{Id}_{p+\ell} & 0 \\ Y & \mathrm{Id}_{p-\ell} \end{bmatrix} : Y \in M_{p-\ell,p+\ell}(\mathbb{R}) \right\} \cap \mathrm{Aut}(\Omega).$$

So suppose that

$$g := \begin{bmatrix} \mathrm{Id}_{p+\ell} & 0 \\ Y & \mathrm{Id}_{p-\ell} \end{bmatrix} \in \mathrm{Aut}(\Omega)$$

for some $Y \in M_{p-\ell,p+\ell}(\mathbb{R})$. Since Γ is a cocompact lattice in $\mathrm{Aut}(\Omega)$, there exist

$$\gamma_n := \begin{bmatrix} A_n & 0 \\ 0 & B_n \end{bmatrix} \in \Gamma$$

such that $\{\gamma_n g^n\}_n$ is bounded in $\mathrm{PGL}_{2p}(\mathbb{R})$. By picking representatives of γ_n and g^n in $\mathrm{GL}_{2p}(\mathbb{R})$ correctly, we can assume that

$$\begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{p+\ell} & 0 \\ nY & \mathrm{Id}_{p-\ell} \end{pmatrix} = \begin{pmatrix} A_n & 0 \\ nB_nY & B_n \end{pmatrix}$$

is a bounded sequence in $\mathrm{GL}_{2p}(\mathbb{R})$. This implies $\{B_n\}_n$ and $\{nB_nY\}_n$ are bounded sequences in $\mathrm{GL}_{p-\ell}(\mathbb{R})$ and $M_{p-\ell,p+\ell}(\mathbb{R})$, respectively. Therefore we must have $Y = 0$, as desired. $\square$

Since Proposition 9.12 yields different conclusions depending on whether $\ell = 0$ or $\ell > 0$, we will consider these two situations separately below.

Case 1 First suppose that $\ell = 0$. Then $\Omega = \Omega_2$ is a proper convex cone and by Proposition 9.12 we may assume that

$$C_\Gamma^0 = \left\{ \begin{pmatrix} e^t \,\mathrm{Id}_p & 0 \\ 0 & e^s \,\mathrm{Id}_p \end{pmatrix} : s,t \in \mathbb{R} \right\}.$$

Then

$$\Gamma \leq \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in \mathrm{GL}_p(\mathbb{R}) \right\}.$$

So Γ acts by linear transformations on Ω. We will now use the theory of linear automorphisms of a proper convex cone to establish a contradiction.

Define a homomorphism

$$\Phi\colon \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathrm{PGL}_{2p}(\mathbb{R}) : A, B \in \mathrm{GL}_p(\mathbb{R}) \right\} \to \mathrm{GL}(\mathbb{M})$$

by

$$\Phi\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right)(X) = BXA^{-1}.$$

Notice that Φ is injective and well-defined.

Then $\Lambda := \Phi(\Gamma)$ acts cocompactly on $\Omega \subset \mathbb{M}$. Let $\overline{\Gamma}^Z$ be the Zariski closure of Γ in $\mathrm{PGL}_{2p}(\mathbb{R})$ and $\overline{\Lambda}^Z$ the Zariski closure of Λ in $\mathrm{GL}(\mathbb{M})$. Then

$$\Phi(\overline{\Gamma}^Z) = \overline{\Lambda}^Z.$$

By possibly passing to a finite-index subgroup we can assume that $\overline{\Gamma}^Z$ is connected in the Zariski topology.

Recall that a convex cone $\mathcal{C} \subset V$ in a real finite-dimensional vector space V is called *reducible* if there exist a pair of proper subspaces $V_i \subset V$ and convex cones $\mathcal{C}_i \subset V_i$ for $i = 1, 2$ such that $V = V_1 \oplus V_2$ and $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$. A convex cone $\mathcal{C} \subset V$ is called *irreducible* if it is not reducible.

Let $C_\Lambda \le \mathrm{GL}(\mathbb{M})$ denote the centralizer of Λ in $\mathrm{GL}(\mathbb{M})$. By a result of Vey [45, Theorem 5] either Ω is an irreducible cone and $C_\Lambda = \mathbb{R}^* \mathrm{Id}_{\mathbb{M}}$ or $\dim C_\Lambda > 1$. By [6, Theorem 1.1], we see that $C_\Lambda \le \overline{\Lambda}^Z$. Now, if $[C^0_\Gamma]$ is the image of C^0_Γ in $\mathrm{PGL}_{2p}(\mathbb{R})$, we see that

$$\Phi^{-1}(C_\Lambda) \subset [C^0_\Gamma].$$

Since $\dim[C^0_\Gamma] = 1$, so we see that $\dim C_\Lambda = 1$. Thus, Ω is an irreducible cone. Then by [45, Theorem 3] (see also [6]) there exists a simple group $H \le \mathrm{GL}(\mathbb{M})$ such that

$$\overline{\Lambda}^Z = (\mathbb{R}^* \,\mathrm{Id}) H.$$

So $\overline{\Gamma}^Z \cong \mathbb{R}^* \times H$.

Now consider the projections

$$\pi_1, \pi_2\colon \overline{\Gamma}^Z \to \mathrm{PGL}_p(\mathbb{R})$$

given by

$$\pi_1\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) = A \quad \text{and} \quad \pi_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) = B.$$

Since H is simple, we see that $\ker \pi_i = \overline{\Gamma}^Z$ or

$$\ker \pi_i = \left\{ \begin{bmatrix} e^t \operatorname{Id}_p & 0 \\ 0 & e^s \operatorname{Id}_p \end{bmatrix} \in \mathrm{PGL}_{2p}(\mathbb{R}) : s,t \in \mathbb{R} \right\}.$$

Since

$$\ker(\pi_1 \times \pi_2) = \left\{ \begin{bmatrix} e^t \operatorname{Id}_p & 0 \\ 0 & e^s \operatorname{Id}_p \end{bmatrix} \in \mathrm{PGL}_{2p}(\mathbb{R}) : s,t \in \mathbb{R} \right\}$$

we must have that $\ker \pi_i \neq \overline{\Gamma}^Z$ for some $i \in \{1,2\}$. Then we see that

$$\pi_i \circ \Phi^{-1} \colon H \to \mathrm{PGL}_p(\mathbb{R})$$

is an injection and thus we obtain an injective homomorphism

$$\overline{\Gamma}^Z \hookrightarrow \mathbb{R} \times \mathrm{PGL}_p(\mathbb{R}).$$

But then

$$p^2 = \dim(\Omega) = \operatorname{vcd}(\Gamma) \leq 1 + \dim(\mathrm{SL}_p(\mathbb{R})/\operatorname{SO}(p)) = \tfrac{1}{2}p(p+1) = \tfrac{1}{2}p^2 + \tfrac{1}{2}p < p^2,$$

which is a contradiction.

Case 2 Suppose that C_Γ^0 contains the subgroup

$$\left\{ \begin{pmatrix} e^t \operatorname{Id}_{p+\ell} & 0 \\ 0 & e^s \operatorname{Id}_{p-\ell} \end{pmatrix} : s,t \in \mathbb{R} \right\}$$

for some $0 < \ell < p$.

Let

$$\Omega_1 = \left\{ \begin{bmatrix} \operatorname{Id}_p \\ X_1 \\ 0 \end{bmatrix} : \text{there exists } X_2 \text{ such that } \begin{bmatrix} \operatorname{Id}_p \\ X_1 \\ X_2 \end{bmatrix} \in \Omega \right\}.$$

Lemma 9.15 $$\Omega = \Omega_1 + \Omega_2.$$

Proof By construction,

$$\overline{\Omega} \subset \overline{\Omega}_1 + \overline{\Omega}_2.$$

Now

$$\begin{pmatrix} \operatorname{Id}_{p+\ell} & 0 \\ 0 & e^s \operatorname{Id}_{p-\ell} \end{pmatrix} \cdot \begin{bmatrix} \operatorname{Id}_p \\ X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \operatorname{Id}_p \\ X_1 \\ e^s X_2 \end{bmatrix}.$$

So, by sending $s \to -\infty$, we see that

$$\overline{\Omega} \supset \overline{\Omega}_1.$$

On the other hand,

$$\begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & e^{-s}\,\mathrm{Id}_p \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{p+\ell} & 0 \\ 0 & e^{s}\,\mathrm{Id}_{p-\ell} \end{pmatrix} \cdot \begin{bmatrix} \mathrm{Id}_p \\ X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \mathrm{Id}_p \\ e^{-s} X_1 \\ X_2 \end{bmatrix}.$$

So, sending $s \to \infty$, we see that

$$\overline{\Omega} \supset \overline{\Omega}_2.$$

Then if $X_1 \in \overline{\Omega}_1$ and $X_2 \in \overline{\Omega}_2$, we have

$$X_1 + X_2 = \tfrac{1}{2}(2X_1) + \tfrac{1}{2}(2X_2) \in \overline{\Omega}.$$

Thus, $\overline{\Omega} = \overline{\Omega}_1 + \overline{\Omega}_2$, which by convexity implies that

$$\Omega = \Omega_1 + \Omega_2. \qquad \square$$

Now if $\gamma \in \Gamma$ then we can write

$$\gamma = \begin{bmatrix} A_1 & A_2 & 0 \\ A_3 & A_4 & 0 \\ 0 & 0 & B \end{bmatrix}$$

for some $A_1 \in M_{p,p}(\mathbb{R})$, $A_2 \in M_{p,\ell}(\mathbb{R})$, $A_3 \in M_{\ell,p}(\mathbb{R})$, $A_4 \in M_{\ell,\ell}(\mathbb{R})$ and $B \in \mathrm{GL}_{p-\ell}(\mathbb{R})$. With this decomposition,

$$\begin{bmatrix} A_1 & A_2 & 0 \\ A_3 & A_4 & 0 \\ 0 & 0 & B \end{bmatrix} \cdot \begin{bmatrix} \mathrm{Id}_p \\ X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \mathrm{Id}_p \\ (A_3 + A_4 X_1)(A_1 + A_2 X_1)^{-1} \\ B X_2 (A_1 + A_2 X_1)^{-1} \end{bmatrix}.$$

Now by identifying $M_{p-\ell,p}(\mathbb{R})$ with $\mathbb{R}^{(p-\ell)p}$ we can view Ω_2 as a convex subset of $\mathbb{P}(\mathbb{R}^{(p-\ell)p+1})$. Let e be an extreme point of Ω_2 in $\mathbb{P}(\mathbb{R}^{(p-\ell)p+1}) \setminus \mathbb{R}^{(p-\ell)p+1}$. Fix a sequence of points $y_n \in \Omega_2$ which converges to e in $\mathbb{P}(\mathbb{R}^{(p-\ell)p+1})$

Next fix some $x_0 \in \Omega_1$ and consider the sequence

$$z_n = \begin{bmatrix} \mathrm{Id}_p \\ x_0 \\ y_n \end{bmatrix} \in \Omega,$$

where we view $x_0 \in M_{\ell,p}(\mathbb{R})$ and $y_n \in M_{p-\ell,p}(\mathbb{R})$.

Since Γ acts cocompactly on Ω, there exist $\gamma_n \in \Gamma$ and a compact subset K of Ω such that

$$\gamma_n^{-1} z_n \in K.$$

Suppose

$$\gamma_n = \left[\begin{pmatrix} A_1^{(n)} & A_2^{(n)} & 0 \\ A_3^{(n)} & A_4^{(n)} & 0 \\ 0 & 0 & B^{(n)} \end{pmatrix} \right].$$

Now let

$$\mathrm{GL}(\Omega_2) = \{T \in \mathrm{GL}(M_{p-\ell,p}(\mathbb{R})) : T(\Omega_2) = \Omega_2\}.$$

Since $\Omega_2 \subset M_{p-\ell,p}(\mathbb{R})$ is a proper convex cone, the Hilbert metric H_{Ω_2} is a complete $\mathrm{GL}(\Omega_2)$–invariant metric on Ω_2. Moreover, since $\Omega = \Omega_1 + \Omega_2$, we see that the linear map

$$T_n(X) = B^{(n)} X (A_1^{(n)} + A_2^{(n)} x_0)^{-1}$$

is in $\mathrm{GL}(\Omega_2)$ for all $n \geq 0$, where we again view $x_0 \in M_{p-\ell,p}(\mathbb{R})$. So there exists $R \geq 0$ such that

$$H_{\Omega_2}(y_n, B^{(n)} y_0 (A_1^{(n)} + A_2^{(n)} x_0)^{-1}) \leq R$$

for all $n \geq 0$. Since y_n converges to an extreme point of Ω_2, we see that $[T_n] \in \mathbb{P}(\mathrm{End}(M_{p-\ell,p}(\mathbb{R})))$ converges to some $T_\infty \in \mathbb{P}(\mathrm{End}(M_{p-\ell,p}(\mathbb{R})))$ and rank $T_\infty = 1$ (see either Vey [45, Lemma 4] or Theorem 7.4 above).

Now if $\sigma_1^{(n)} \geq \cdots \geq \sigma_{p-\ell}^{(n)}$ are the singular values of $B^{(n)}$ and $\mu_1^{(n)} \geq \cdots \geq \mu_p^{(n)}$ are the singular values of $(A_1^{(n)} + A_2^{(n)} x_0)^{-1}$ then T_n has singular values

$$\{\sigma_i^{(n)} \mu_j^{(n)} : 1 \leq i \leq p-\ell,\ 1 \leq j \leq p\}.$$

Then since $[T_n] \to T_\infty$ and rank $T_\infty = 1$ we must have

$$\lim_{n \to \infty} \frac{\sigma_1^{(n)} \mu_1^{(n)}}{\sigma_i^{(n)} \mu_j^{(n)}} = \infty$$

for all $1 \leq i \leq p-\ell$ and $1 \leq j \leq p$ with $(i,j) \neq (1,1)$.

In particular,

$$\lim_{n \to \infty} \frac{\mu_1^{(n)}}{\mu_2^{(n)}} = \infty.$$

So we will finish the proof by establishing the following:

Lemma 9.16
$$\limsup_{n \to \infty} \frac{\mu_1^{(n)}}{\mu_2^{(n)}} < \infty.$$

Proof Now view Ω_1 as an open subset of $\mathrm{Gr}_p(V)$, where $V=\mathrm{Span}\{e_1,\dots,e_{p+\ell}\}$. By construction, Ω_1 is an $\mathcal{R}$–proper convex open subset of some affine chart of $\mathrm{Gr}_p(V)$. Thus, K_{Ω_1} is a proper metric and there exists $R_1\geq 0$ such that

$$K_{\Omega_1}(x_0,(A_3^{(n)}+A_4^{(n)}x_0)(A_1^{(n)}+A_2^{(n)}x_0)^{-1})\leq R_1.$$

So the set

$$\left\{\begin{bmatrix} A_1^{(n)} & A_2^{(n)} \\ A_3^{(n)} & A_4^{(n)}\end{bmatrix} : n\in\mathbb{N}\right\}\subset \mathrm{PGL}(V)$$

is relatively compact in $\mathrm{PGL}(V)$. So we can pass to a subsequence and pick representatives such that

$$\begin{pmatrix} A_1^{(n)} & A_2^{(n)} \\ A_3^{(n)} & A_4^{(n)}\end{pmatrix}\to\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4\end{pmatrix}$$

in $\mathrm{GL}(V)$. Now we claim that $A_1+A_2x_0$ is an invertible matrix. Suppose this is not the case. Then for each n we can find a unit eigenvector $v_n\in\mathbb{C}^p$ such that

$$(A_1^{(n)}+A_2^{(n)}x_0)v_n\to 0$$

Since $(A_3^{(n)}+A_4^{(n)}x_0)(A_1^{(n)}+A_2^{(n)}x_0)^{-1}$ stays within a compact subset of Ω_2, we must have that $(A_3^{(n)}+A_4^{(n)}x_0)v_n\to 0$. Then we can pass to a subsequence such that $v_n\to v$. We have

$$0=\lim_{n\to\infty}\begin{pmatrix} A_1^{(n)} & A_2^{(n)} \\ A_3^{(n)} & A_4^{(n)}\end{pmatrix}\begin{pmatrix} v_n \\ x_0v_n\end{pmatrix}=\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4\end{pmatrix}\begin{pmatrix} v \\ q_0v\end{pmatrix},$$

which contradicts the fact that

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4\end{pmatrix}\in\mathrm{GL}_{p+\ell}(\mathbb{R}).$$

So $A_1+A_2q_0$ is an invertible matrix. But this implies that there exists $C>0$ such that

$$\{\mu_i^{(n)}:1\leq i\leq p\}\subset[1/C,C],$$

which implies the lemma. □

10 Unipotent subgroups

In this section we show that case (2) of Theorem 8.2 is impossible.

Theorem 10.1 *Suppose $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$ is an open set which is bounded and convex in some affine chart. If $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group which acts cocompactly on Ω, then there does not exist a nontrivial abelian normal unipotent group $U \leq \mathrm{Aut}(\Omega)$ such that $\Gamma \cap U$ is a cocompact lattice in U.*

For the rest of the section suppose $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$ and $\Gamma \leq \mathrm{Aut}(\Omega)$ satisfy the hypothesis of Theorem 10.1. Assume for a contradiction that there exists a nontrivial abelian normal unipotent group $U \leq \mathrm{Aut}(\Omega)$ such that $\Gamma \cap U$ is a cocompact lattice in U.

Since Γ is finitely generated, by passing to a finite-index subgroup we can assume that Γ is torsion-free. Then, since Γ acts properly on Ω, we see that Γ acts freely on Ω. Then, using the fact that $\Gamma \backslash \Omega$ is compact, we see that

$$\inf_{\gamma \in \Gamma, x \in \Omega} K_\Omega(\gamma x, x) > 0. \tag{10-1}$$

The basic idea of the following argument is that if $u \in U \cap \Gamma$, then the translation distance

$$\inf_{x \in \Omega} K_\Omega(ux, x)$$

should be zero, which then implies that $U \cap \Gamma = 1$. This approach is motivated by Lemma 2.8 in [9] and Proposition 2.13 in [19].

The group $\bigwedge^p U \leq \mathrm{PGL}\left(\bigwedge^p \mathbb{R}^{2p}\right)$ is also unipotent, so the set

$$E_1 = \{v \in \mathbb{P}\left(\textstyle\bigwedge^p \mathbb{R}^{2p}\right) : \left(\textstyle\bigwedge^p u\right)v = v \text{ for all } u \in U\}$$

is nonempty. Note that $\bigwedge^p U$ can be conjugated so that it is upper triangular. Since $U \cap \Gamma$ is a lattice in U, we can choose $u_0 \in U \cap \Gamma$ such that its Jordan decomposition is generic among elements of U, that is to say

$$E_1 = \{v \in \mathbb{P}\left(\textstyle\bigwedge^p \mathbb{R}^{2p}\right) : \left(\textstyle\bigwedge^p u_0\right)v = v\}.$$

Then, with the notation of Proposition 7.9,

$$E^+\left(\textstyle\bigwedge^p u_0\right) \subset E_1$$

and by Corollary 7.10 there exists an $\mathcal{R}$–extreme point $e \in E^+\left(\bigwedge^p u_0\right) \cap \partial\Omega$.

Now suppose that Ω is a bounded convex open set in the affine chart

$$\mathbb{M} = \left\{ \begin{bmatrix} \mathrm{Id}_p \\ X \end{bmatrix} : X \in M_{p,p}(\mathbb{R}) \right\}.$$

Without loss of generality we can assume $e=0$ in this affine chart. Then by Corollary 7.11, there exist $\gamma_n \in \Gamma$ and $t_n \to \infty$ such that

$$\varphi = \lim_{n\to\infty} \begin{bmatrix} \mathrm{Id}_p & 0 \\ 0 & e^{t_n}\,\mathrm{Id}_p \end{bmatrix} \gamma_n$$

exists in $\mathrm{PGL}_{2p}(\mathbb{R})$ and $\varphi\Omega \subset \mathbb{M}$ is an $\mathcal{R}$–proper convex open cone based at 0. In particular, $\mathrm{Aut}(\varphi\Omega)$ contains the one-parameter subgroup

$$a_t := \begin{bmatrix} \mathrm{Id}_p & 0 \\ 0 & e^{t}\,\mathrm{Id}_p \end{bmatrix}.$$

Now if

$$\varphi_n := \begin{bmatrix} \mathrm{Id}_p & 0 \\ 0 & e^{t_n}\,\mathrm{Id}_p \end{bmatrix} \gamma_n$$

then

$$\varphi_n^{-1}(e) = \gamma_n^{-1}(e) \in \gamma_n^{-1}E_1 \cap \gamma_n^{-1}E^+\big(\textstyle\bigwedge^p u_0\big) = E_1 \cap E^+\big(\textstyle\bigwedge^p \gamma_n^{-1}u_0\gamma_n\big),$$

so

$$\varphi_n^{-1}(e) \in E_1 \cap \bigg(\bigcup_{u\in U} E^+\big(\textstyle\bigwedge^p u\big)\bigg),$$

so sending $n\to\infty$ we see that

$$\varphi^{-1}(e) \in E_1 \cap \overline{\bigcup_{u\in U} E^+\big(\textstyle\bigwedge^p u\big)}.$$

And thus

$$e \in \varphi(E_1) \cap \overline{\bigcup_{u\in\varphi U\varphi^{-1}} E^+\big(\textstyle\bigwedge^p u\big)}.$$

In particular, since $e = \mathrm{Span}\{e_1,\dots,e_p\} \subset \varphi(E_1)$, we have

$$\varphi U \varphi^{-1} \le \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : A, B, C \in M_{p,p}(\mathbb{R}) \right\}.$$

Lemma 10.2 *If*

$$\begin{bmatrix} \mathrm{Id}_p & X \\ 0 & \mathrm{Id}_p \end{bmatrix} \in \varphi U \varphi^{-1}$$

then $X=0$.

Proof Suppose for a contradiction that there exists

$$u = \begin{bmatrix} \mathrm{Id}_p & X \\ 0 & \mathrm{Id}_p \end{bmatrix} \in \varphi U \varphi^{-1}$$

with $X \neq 0$. We claim there exist $n_k \to \infty$ and $\gamma_k \in \varphi(\Gamma \cap U)\varphi^{-1}$ such that $\gamma_k^{-1}u^{n_k} \to \mathrm{Id}_{2p}$. Indeed, consider the group $\Lambda := \langle \varphi(\Gamma \cap U)\varphi^{-1}, u\rangle$. If Λ is discrete, some power of u belongs to $\varphi(\Gamma \cap U)\varphi^{-1}$, in which case the claim obviously holds. If Λ is not discrete, we can find $\gamma_k^{-1}u^{n_k} = \lambda_k \in \Lambda$ such that $\lambda_k \to \mathrm{Id}_{2p}$. Further, it is clear that $n_k \to \infty$, for otherwise λ_k lie in a union of finitely many translates of $\varphi(\Gamma \cap U)\varphi^{-1}$, which is a discrete set. This proves the claim.

So let $\gamma_k \in \varphi(\Gamma \cap U)\varphi^{-1}$ and $n_k \to \infty$ such that $\gamma_k^{-1}u^{n_k} \to \mathrm{Id}_{2p}$. By picking representatives correctly we can assume that

$$\gamma_k = \begin{bmatrix} A_k & B_k \\ 0 & C_k \end{bmatrix}$$

and

$$\begin{pmatrix} A_k^{-1} & -A_k^{-1}B_k \\ 0 & C_k^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{Id}_p & n_k X \\ 0 & \mathrm{Id}_p \end{pmatrix} = \begin{pmatrix} A_k^{-1} & n_k A_k^{-1} X - A_k^{-1} B_k \\ 0 & C_k^{-1} \end{pmatrix} \to \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & \mathrm{Id}_p \end{pmatrix}$$

in $\mathrm{GL}_{2p}(\mathbb{R})$. So $A_k \to \mathrm{Id}_p$ and $C_k \to \mathrm{Id}_p$. But then there exist $t_k \to \infty$ such that $a_{t_k}\gamma_k a_{-t_k} \to \mathrm{Id}_{2p}$. But then, for any $p \in \varphi\Omega$,

$$\lim_{k\to\infty} K_{\varphi\Omega}(\gamma_k a_{-t_k} p, a_{-t_k} p) = \lim_{k\to\infty} K_{\varphi\Omega}(a_{t_k}\gamma_k a_{-t_k} p, p) = 0,$$

which contradicts equation (10-1). □

Lemma 10.3 $$\varphi U \varphi^{-1} \leq \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in \mathrm{GL}_p(\mathbb{R}) \right\}.$$

Proof Suppose for a contradiction that there exists

$$u = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \varphi U \varphi^{-1}$$

with $C \neq 0$.

Then

$$u' = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \lim_{t\to\infty} a_t u a_{-t} \in \varphi U \varphi^{-1}$$

and so

$$\begin{bmatrix} \mathrm{Id}_p & A^{-1}C \\ 0 & \mathrm{Id}_p \end{bmatrix} = (u')^{-1} u \in \varphi U \varphi^{-1},$$

which we just showed is impossible. □

Lemma 10.4 *If $u \in \varphi U \varphi^{-1}$ is nontrivial then*

$$E^+\big(\textstyle\bigwedge^p u\big) \cap \mathrm{Gr}_p(\mathbb{R}^{2p}) \subset \mathrm{Gr}_p(\mathbb{R}^{2p}) \setminus \mathbb{M}.$$

Proof Suppose $u = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. Then both A and B are unipotent and

$$u^m \cdot \begin{bmatrix} \mathrm{Id}_p \\ X \end{bmatrix} = \begin{bmatrix} \mathrm{Id}_p \\ B^m X A^{-m} \end{bmatrix}.$$

Since both B and A are unipotent, for a generic $X \in M_{p,p}(\mathbb{R})$ we have

$$\lim_{m \to \infty} \|B^m X A^{-m}\| = \infty,$$

which implies that $E^+\big(\bigwedge^p u\big) \cap \mathrm{Gr}_p(\mathbb{R}^{2p}) \subset \mathrm{Gr}_p(\mathbb{R}^{2p}) \setminus \mathbb{M}$. □

Now we have a contradiction because

$$e \in \mathrm{Gr}_p(\mathbb{R}^{2p}) \cap \overline{\bigcup_{u \in \varphi U \varphi^{-1}} E^+\big(\textstyle\bigwedge^p u\big)} \subset \mathrm{Gr}_p(\mathbb{R}^{2p}) \setminus \mathbb{M}$$

and $e \in \mathbb{M}$.

11 When $p = 2$

In this section we show that cases (3) and (4) of Theorem 8.2 are impossible.

Theorem 11.1 *Suppose $\Omega \subset \mathrm{Gr}_2(\mathbb{R}^4)$ is a bounded convex open subset of some affine chart of $\mathrm{Gr}_2(\mathbb{R}^4)$ and there exists a discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ such that $\Gamma \backslash \Omega$ is compact. Then the connected component of the identity in $\mathrm{Aut}(\Omega)$ is a simple Lie group with trivial center that acts transitively on Ω.*

For the rest of the section let $\Omega \subset \mathrm{Gr}_2(\mathbb{R}^4)$ and $\Gamma \leq \mathrm{Aut}(\Omega)$ be as in the hypothesis of Theorem 11.1. As in Section 8, let $G := \mathrm{Aut}(\Omega)$ and let G^0 be the connected component of the identity of G.

Define the subgroups

$$G_1 := \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \mathrm{SL}_2(\mathbb{R}) \right\}$$

and

$$G_2 := \left\{ \begin{bmatrix} a\,\mathrm{Id}_2 & b\,\mathrm{Id}_2 \\ c\,\mathrm{Id}_2 & d\,\mathrm{Id}_2 \end{bmatrix} : ad - bc = 1 \right\}.$$

By Theorem 8.2 we may assume that either

(1) G^0 is a simple Lie group with trivial center that acts transitively on Ω, or

(2) there exists a cocompact lattice $\Lambda \leq G_2$ such that $G_1 \times \Lambda$ has finite index in $\mathrm{Aut}(\Omega)$, or

(3) $G_1 \times G_2$ has finite index in $\mathrm{Aut}(\Omega)$ and acts transitively on Ω.

Lemma 11.2 *With the notation above, case* (2) *cannot occur.*

Proof Suppose not. Then there exists a cocompact lattice $\Lambda \leq G_2$ such that $G_1 \times \Lambda$ has finite index in $\mathrm{Aut}(\Omega)$. By possibly changing Γ, we may also assume that $\Gamma = \Gamma_1 \times \Lambda$ for some cocompact lattice $\Gamma_1 \leq G_1$.

For a subgroup $H \leq \mathrm{Aut}(\Omega)$ let $\mathcal{L}(H)$ denote the set of points $x \in \partial\Omega$ where there exist some $y \in \Omega$ and sequence $h_n \in H$ such that $h_n y \to x$. Recall that $\mathrm{Ext}_{\mathcal{R}}(\Omega) \subset \partial\Omega$ is the set of $\mathcal{R}$–extreme points of Ω. Then define

$$\mathrm{Ext}_{\mathcal{R}}(H) := \mathcal{L}(H) \cap \mathrm{Ext}_{\mathcal{R}}(\Omega).$$

Let $e_1, \dots e_4$ be the standard basis of $\mathbb{R}^4$. Then a direct computation (using part (4) of Theorem 7.4) shows that

$$\mathrm{Ext}_{\mathcal{R}}(G_1) = \{[(\alpha e_1 + \beta e_2) \wedge (\alpha e_3 + \beta e_4)] : \alpha, \beta \in \mathbb{R}\}$$

and

$$\mathrm{Ext}_{\mathcal{R}}(\Lambda) \subset \{[(\alpha e_1 + \beta e_3) \wedge (\alpha e_2 + \beta e_4)] : \alpha, \beta \in \mathbb{R}\}.$$

This description implies that $\overline{\mathrm{Ext}_{\mathcal{R}}(G_1)}$ and $\overline{\mathrm{Ext}_{\mathcal{R}}(\Lambda)}$ are disjoint and Γ–invariant sets. Moreover, since $\Lambda \leq G_2$ is a cocompact lattice, there exists some $\lambda \in \Lambda$ such that $\bigwedge^2 \lambda$ has a unique eigenvalue of maximum absolute value (see [42]). Then part (4) of Theorem 7.4 implies that $\mathrm{Ext}_{\mathcal{R}}(\Lambda) \neq \varnothing$. So suppose that $e \in \mathrm{Ext}_{\mathcal{R}}(\Lambda)$.

Now up to a projective isomorphism we can assume that Ω is a convex subset of the affine chart

$$\mathbb{M} = \left\{ \begin{bmatrix} \mathrm{Id}_2 \\ X \end{bmatrix} : X \in M_{2,2}(\mathbb{R}) \right\}.$$

and $e = \begin{bmatrix} \mathrm{Id}_2 & 0 \end{bmatrix}^t \in \partial\Omega$. Then by Corollary 7.11 there exist $\gamma_n \in \Gamma$ and $t_n \to \infty$ such that

$$\varphi = \lim_{n\to\infty} \begin{bmatrix} \mathrm{Id}_2 & 0 \\ 0 & e^{t_n}\,\mathrm{Id}_2 \end{bmatrix} \gamma_n$$

exists in $\mathrm{PGL}_4(\mathbb{R})$ and $\varphi(\Omega) = \mathcal{T}\mathcal{C}_e\Omega$. In particular, Ω is invariant under the one-parameter group

$$\varphi^{-1}\left\{\begin{bmatrix} \mathrm{Id}_p & 0 \\ 0 & e^t\,\mathrm{Id}_p \end{bmatrix} : t \in \mathbb{R}\right\}\varphi.$$

This implies that $\varphi^{-1}(e) \in \mathrm{Ext}_{\mathcal{R}}(G_1)$. But

$$\gamma_n^{-1}\begin{bmatrix} \mathrm{Id}_2 & 0 \\ 0 & e^{-t_n}\,\mathrm{Id}_2 \end{bmatrix} e = \gamma_n^{-1} e \subset \mathrm{Ext}_{\mathcal{R}}(\Lambda)$$

and thus

$$\varphi^{-1}(e) \in \mathrm{Ext}_{\mathcal{R}}(G_1) \cap \overline{\mathrm{Ext}_{\mathcal{R}}(\Lambda)}.$$

This is a contradiction. □

We rule out case (3) above by proving the following:

Lemma 11.3 *With the notation above, case* (3) *cannot occur.*

Proof Suppose not; then $G_1 \times G_2$ has finite index in $\mathrm{Aut}(\Omega)$. By possibly changing Γ, we may assume that $\Gamma = \Gamma_1 \times \Gamma_2$ for some cocompact lattices $\Gamma_1 \le G_1$ and $\Gamma_2 \le G_2$. Define the subgroups

$$K_1 = \left\{\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \mathrm{SO}(2)\right\}$$

and

$$K_2 = \left\{\begin{bmatrix} a\,\mathrm{Id}_2 & b\,\mathrm{Id}_2 \\ c\,\mathrm{Id}_2 & d\,\mathrm{Id}_2 \end{bmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SO}(2)\right\}.$$

Then $K_1 \times K_2 \le G_1 \times G_2$ is a maximal compact connected subgroup. Moreover, the action of $K_1 \times K_2$ on $\mathrm{Gr}_2(\mathbb{R}^4)$ has no fixed points.

Next let $K_x \le \mathrm{Aut}(\Omega)$ be the connected component of the stabilizer of some $x \in \Omega$. Since $\mathrm{Aut}(\Omega)$ acts properly on Ω (see Proposition 4.8), K_x is a compact subgroup. Moreover, since $G^0 = G_1 \times G_2$, we see that $K_x \le G_1 \times G_2$. Thus, since maximal compact subgroups are conjugate in semisimple Lie groups, there exists some $g \in G_1 \times G_2$ such that

$$gK_xg^{-1} \le K_1 \times K_2.$$

But $\dim(K_1 \times K_2) = 2$. Moreover,

$$6 - \dim(K_x) = \dim(G_1 \times G_2/K_x) \le \dim(\Omega) = 4,$$

so $\dim K_x \ge 2$. Thus, $gK_xg^{-1} = K_1 \times K_2$. This contradicts the fact that $K_1 \times K_2$ has no fixed points in $\mathrm{Gr}_2(\mathbb{R}^4)$. □

12 Finishing the proof of Theorem 1.3

Theorems 8.2, 9.2, 10.1 and 11.1 reduce the proof of Theorem 1.3 to the following:

Theorem 12.1 *Suppose $p > 1$ and $\Omega \subset \mathrm{Gr}_p(\mathbb{R}^{2p})$ is a bounded convex open subset of some affine chart of $\mathrm{Gr}_p(\mathbb{R}^{2p})$. If the connected component of the identity of $\mathrm{Aut}(\Omega)$ is a simple Lie group with trivial center which acts transitively on Ω, then Ω is projectively isomorphic to $\mathcal{B}_{p,p}$.*

For the rest of the section suppose that Ω satisfies the hypothesis of Theorem 12.1. As in Section 8, let $G := \mathrm{Aut}(\Omega)$ and let G^0 be the connected component of the identity of G. Also let $e_1, \dots, e_{2p} \in \mathbb{R}^{2p}$ be the standard basis.

Throughout the argument we will replace Ω by translates $g\Omega$ for some $g \in \mathrm{PGL}_{2p}(\mathbb{R})$. This will have the effect of replacing G by gGg^{-1}.

Fix some $x_0 \in \Omega$ and let $K \leq G^0$ be the identity component of the stabilizer of x_0. By Remark 8.11, K is a finite-index subgroup of some maximal compact subgroup of G^0. Moreover, since K is compact, by translating Ω we may assume that $K \leq \mathrm{PSO}(2p)$. Then, since $\mathrm{PSO}(2p)$ acts transitively on $\mathrm{Gr}_p(\mathbb{R}^{2p})$, we can translate Ω and assume that $x_0 = [e_1 \wedge \dots \wedge e_p]$. Then, using the fact that K is connected,

$$K \leq \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in \mathrm{SO}(p) \right\}.$$

In particular, $\dim(K) \leq p(p-1)$.

Now let $\mathrm{rank}_{\mathbb{R}}(G^0)$ be the real rank of G^0.

Lemma 12.2 *With the notation above, $\mathrm{rank}_{\mathbb{R}}(G^0) \geq p$.*

Proof Using the Cartan decomposition, we see there exists a connected abelian group $A \leq G^0$ such that $\dim(A) = \mathrm{rank}_{\mathbb{R}}(G^0)$ and $KAK = G^0$. In particular, in the matrix model of $\mathrm{Gr}_p(\mathbb{R}^{2p})$,

$$\Omega = KAK \cdot \begin{bmatrix} \mathrm{Id}_p \\ 0 \end{bmatrix} = KA \cdot \begin{bmatrix} \mathrm{Id}_p \\ 0 \end{bmatrix}.$$

Thus, we must have

$$\dim(K) + \dim(A) \geq \dim(\Omega) = p^2. \tag{12-1}$$

Since $\dim(K) \leq p(p-1)$ we then have

$$\mathrm{rank}_{\mathbb{R}}(G^0) = \dim(A) \geq p. \qquad \square$$

Lemma 12.3 *With the notation above, G^0 is isomorphic to* $\mathrm{PSO}(p,p)$.

Proof Now

$$\dim(G^0/K) = \dim(\Omega) = p^2$$

and

$$\mathrm{rank}_{\mathbb{R}}(G^0) \geq p.$$

In particular,

$$\mathrm{rank}_{\mathbb{R}}(G^0) \geq \sqrt{\dim(G^0/K)}.$$

The only two simple Lie groups of noncompact type and with trivial center with this property are $\mathrm{PSL}_{d+1}(\mathbb{R})$ for $d \geq 3$ and $\mathrm{PSO}(d,d)$ for $d \geq 2$ (see the classification of simple Lie groups in [30, Chapter X]).

If G^0 is isomorphic to $\mathrm{PSL}_{d+1}(\mathbb{R})$ then K is isomorphic to $\mathrm{PSO}(d+1)$. In particular, K is a simple Lie group and

$$\dim K = \tfrac{1}{2}d(d+1).$$

Next consider the natural projections

$$\pi_1, \pi_2\colon K \to \mathrm{PSO}(p)$$

given by

$$\pi_1\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = A \quad \text{and} \quad \pi_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = B.$$

Now since K is simple either $(\pi_1 \times \pi_2)\colon K \to \mathrm{PSO}(p) \times \mathrm{PSO}(p)$ is trivial or injective. But

$$\ker(\pi_1 \times \pi_2) \leq \left\{ \mathrm{Id}_{2p}, \begin{bmatrix} \mathrm{Id}_p & \\ & -\mathrm{Id}_p \end{bmatrix}, \begin{bmatrix} -\mathrm{Id}_p & \\ & \mathrm{Id}_p \end{bmatrix} \right\},$$

so $\pi_1 \times \pi_2$ is injective. Thus, at least one π_i has nontrivial image. Then by the simplicity of K we see that $K \cong \pi_i(K) \leq \mathrm{PSO}(p)$. So

$$\dim K \leq \tfrac{1}{2}p(p-1)$$

and so

$$d(d+1) \leq p(p-1).$$

Thus, $d \leq p+1$. But then we have a contradiction, because by equation (12-1), we have

$$p^2 \leq \mathrm{rank}(G^0) + \dim(K) \leq d + \tfrac{1}{2}p(p-1) \leq p + 1 + \tfrac{1}{2}p(p-1) = \tfrac{1}{2}p^2 + p + 1,$$

which is only true when $p=2$. Then $d=p+1=3$, but

$$\dim \mathrm{PSL}_4(\mathbb{R})/\mathrm{PSO}(4)=9\neq 4=\dim\Omega,$$

so this case is impossible.

Thus, we must have that G^0 is isomorphic to $\mathrm{PSO}(p,p)$. □

Now the inclusion $G^0\leq \mathrm{PGL}_{2p}(\mathbb{R})$ induces a representation

$$\phi\colon \mathrm{PSO}(p,p)\to \mathrm{PGL}_{2p}(\mathbb{R}).$$

Notice that replacing Ω by $g\Omega$ for some $g\in\mathrm{PGL}_{2p}(\mathbb{R})$ has the effect of replacing ϕ by $\mathrm{Ad}(g)\circ\phi$.

At this point there are a number of ways to deduce that this representation is conjugate to the standard inclusion, but we will use the representation theory of $\mathrm{SO}(2p,\mathbb{C})$ because it appears explicitly in standard references (for instance [24]).

Now K has finite index in a maximal compact subgroup of $G^0\cong\mathrm{PSO}(p,p)$ and

$$K\leq\left\{\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}: A,B\in\mathrm{SO}(p)\right\},$$

so we see that

$$K=\left\{\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}: A,B\in\mathrm{SO}(p)\right\}.$$

Then since maximal compact subgroups are conjugate in G^0 we may translate Ω to assume that

$$\phi\big(P(\mathrm{SO}(p)\times\mathrm{SO}(p))\big)=P(\mathrm{SO}(p)\times\mathrm{SO}(p)).$$

Now if $K_1=P(\mathrm{SO}(p)\times\{\mathrm{Id}_p\})$ and $K_2=P(\mathrm{SO}(p)\times\{\mathrm{Id}_p\})$ then, using the simplicity of K_1 and K_2 and the fact that $\phi(K_1),\phi(K_2)$ commute, we see that

$$\{\phi(K_1),\phi(K_2)\}=\{K_1,K_2\}.$$

So by translating Ω we may assume that $\phi(K_1)=K_1$ and $\phi(K_2)=K_2$. Now each K_i is isomorphic to $\mathrm{SO}(p)$.

It is well known that any automorphism of $\mathrm{SO}(p)$ is given by conjugation by some element of $\mathrm{O}(p)$ (eg because such an automorphism is determined by the automorphism of the Dynkin diagram). Therefore, by translating Ω, we can assume that $\phi(k)=k$ for all $k\in K_1\cup K_2$.

Now let $d(\phi)\colon \mathfrak{so}(p,p) \to \mathfrak{sl}_{2p}(\mathbb{R})$ be the corresponding Lie algebra representation. We can complexify to obtain a representation $d(\phi)\colon \mathfrak{so}(2p,\mathbb{C}) \to \mathfrak{sl}_{2p}(\mathbb{C})$. But then by the classification of irreducible representations of $\mathrm{SO}(2p,\mathbb{C})$ (see for instance [24, Chapter 19]) we see that there exists $g \in \mathrm{SL}_{2p}(\mathbb{C})$ such that

$$\mathrm{Ad}(g)d(\phi) = \iota,$$

where $\iota\colon \mathfrak{so}(2p,\mathbb{C}) \hookrightarrow \mathfrak{sl}_{2p}(\mathbb{C})$ is the standard inclusion representation. Since

$$g^{-1}\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} g = d(\phi)\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$$

for all $X_1, X_2 \in \mathfrak{so}(p)$, it is easy to see that

$$g = \begin{pmatrix} \alpha\,\mathrm{Id}_p & 0 \\ 0 & \alpha^{-1}\,\mathrm{Id}_p \end{pmatrix}$$

for some $\alpha \in \mathbb{C}^*$. Now

$$g\begin{pmatrix} A & B \\ C & D \end{pmatrix} g^{-1} = \begin{pmatrix} A & \alpha^2 B \\ \alpha^{-2} C & D \end{pmatrix}$$

and $gd(\phi)(\mathfrak{so}(p,p))g^{-1} = \mathfrak{so}(p,p)$. So $\alpha^2 \in \mathbb{R}$. So either $\alpha \in \mathbb{R}^\times$ or $\alpha = \lambda i$ for some $\lambda \in \mathbb{R}^\times$. In the latter case, we also have

$$\mathrm{Ad}(-ig)d(\phi) = \mathrm{Ad}(g)d(\phi) = \iota.$$

So by possibly replacing g by $-ig$ we can assume that $g \in \mathrm{SL}_{2p}(\mathbb{R})$.

Then, if we replace Ω by $g\Omega$, then $\phi\colon \mathrm{PSO}(p,p) \hookrightarrow \mathrm{PGL}_{2p}(\mathbb{R})$ is the standard inclusion representation and so $G^0 = \mathrm{PSO}(p,p)$.

Finally,

$$\Omega = G^0 \cdot x_0 = \mathrm{PSO}(p,p) \cdot \begin{bmatrix} \mathrm{Id}_p \\ 0 \end{bmatrix} = \mathcal{B}_{p,p}$$

and so Theorem 12.1 is proven.

Appendix A Proof of Theorem 4.6

In this section we prove that (1) implies (2) in Theorem 4.6:

Theorem A.1 *Suppose $\mathbb{M} \subset \mathrm{Gr}_p(\mathbb{R}^{p+q})$ is an affine chart and $\Omega \subset \mathbb{M}$ is an open convex set. If Ω is $\mathcal{R}$–proper, then K_Ω is a complete length metric on Ω.*

We will use some basic properties of the Hilbert metric $H_{\mathcal{C}}$ on a convex set $\mathcal{C} \subset \mathbb{R}^d$. In particular we will use:

(1) **Equivariance** If $A \in \mathrm{Aff}(\mathbb{R}^d)$ then $H_{A\mathcal{C}}(Ax, Ay) = H_{\mathcal{C}}(x, y)$.

(2) **Properness** If $x \in \partial \mathcal{C}$ and $x_n \in \mathcal{C}$ is a sequence with $x_n \to x$, then
$$H_{\mathcal{C}}(x_0, x_n) \to \infty.$$

(3) **Completeness** If $\mathcal{C}$ contains no affine lines then $H_{\mathcal{C}}$ is a complete metric.

(4) If $\mathcal{C} = \mathbb{R}^d \times \mathcal{C}'$, then
$$H_{\mathcal{C}}((x_1, y_1), (x_2, y_2)) = H_{\mathcal{C}'}(y_1, y_2).$$

All these properties follow immediately from the cross-ratio definition of the Hilbert metric.

Proof Identify $\mathbb{M}$ with the set of $q \times p$ matrices and let $\mathbb{M}_1 \subset \mathbb{M}$ be the subset of rank-one matrices. Define a function $\delta_\Omega \colon \Omega \times \mathbb{M}_1 \to \mathbb{R}_{\geq 0}$ by
$$\delta_\Omega(x; v) = \inf\{\|y - x\| : y \in \partial\Omega \cap (x + \mathbb{R}v)\}.$$

Since Ω is $\mathcal{R}$–proper, we must have that $\delta_\Omega(x; v) < \infty$ for all $x \in \Omega$ and $v \in \mathbb{M}_1$. Moreover, since Ω is convex, δ_Ω is a continuous function.

We will first show that K_Ω is a metric; using Proposition 4.2 we only need show that $K_\Omega(x, y) > 0$ for $x, y \in \Omega$ distinct. Now we can find $\epsilon > 0$ such that the closed Euclidean ball
$$B_\epsilon(x) = \{z \in \mathbb{M} : \|x - z\| \leq \epsilon\}$$
is contained in Ω but $y \notin B_\epsilon(x)$. Since δ_Ω is continuous, there exists $M > 0$ such that
$$\delta_\Omega(z; v) \leq M$$
for all $z \in B_\epsilon(x)$ and $v \in \mathbb{M}_1$.

We claim that if $[z_1, z_2] \subset B_\epsilon(x)$, then $\rho_\Omega(z_1, z_2) \geq \|z_1 - z_2\|/(\epsilon + M)$. If $z_2 - z_1 \notin \mathbb{M}_1$, then $\rho_\Omega(z_1, z_2) = \infty$. So we may assume that $z_2 - z_1 \in \mathbb{M}_1$. Then let $(a, b) = \overline{z_1 z_2} \cap \Omega$, labeled so that a, z_1, z_2, b is the ordering along the line. By relabeling we may assume that $\|a - z_1\| = \delta_\Omega(z_1, z_1 - z_2) \leq M$. Then
$$\begin{aligned}\rho_\Omega(z_1, z_2) &= \left|\log \frac{\|z_1 - a\| \|z_2 - b\|}{\|z_1 - b\| \|z_2 - a\|}\right| \geq \log \frac{\|z_2 - a\|}{\|z_1 - a\|} \\ &= \int_{\|z_1 - a\|}^{\|z_2 - a\|} \frac{dt}{t} \geq \frac{1}{\|z_2 - a\|}(\|z_2 - a\| - \|z_1 - a\|).\end{aligned}$$

Since z_1, z_2 and a are all collinear and $\|z_1 - z_2\| \leq \epsilon$, we then have

$$\rho_\Omega(z_1, z_2) \geq \frac{1}{M+\epsilon}\|z_1 - z_2\|.$$

Now we wish to show that $K_\Omega(x, y) > 0$. We claim that

$$\rho_\Omega(x, a_1) + \sum_{i=1}^{n-1} \rho_\Omega(a_i, a_{i+1}) + \rho_\Omega(a_n, y) \geq \frac{\epsilon}{M+\epsilon}$$

for any $a_1, \ldots, a_n \in \Omega$. This will imply that $d_\Omega(x, y) > 0$. Now, by definition, if $a, b \in \mathbb{M}$ and $c \in [a, b]$, then

$$\rho_\Omega(a, b) + \rho_\Omega(b, c) = \rho_\Omega(a, c).$$

So without loss of generality there exists $1 \leq l < n$ such that $a_1, \ldots, a_l \in B_\epsilon(x)$ and $a_{l+1} \in \partial B_\epsilon(x)$. Then, by the above calculation,

$$\rho_\Omega(x, a_1) + \sum_{i=1}^{l} \rho_\Omega(a_i, a_{i+1}) \geq \frac{1}{M+\epsilon}\left(\|x - a_1\| + \sum_{i=1}^{l} \|a_i - a_{i+1}\|\right) \geq \frac{\epsilon}{M+\epsilon}.$$

This shows that K_Ω is a metric.

We will next show that K_Ω is a length metric. This follows from the fact that if $x, y \in \Omega$ and $x - y \in \mathbb{M}_1$, then

$$\rho_\Omega(x, y) = \rho_\Omega(x, z) + \rho_\Omega(z, y)$$

for any $z \in [x, y]$. Thus, when $x - y \in \mathbb{M}_1$, there is a curve of length at most $\rho_\Omega(x, y)$ joining x to y. Then, by definition, for any $x, y \in \Omega$ there exists a sequence of curves σ_n joining x to y whose length converges to $K_\Omega(x, y)$.

Next we show that K_Ω is proper, that is, for any $x_0 \in \Omega$ and $R \geq 0$ the closed metric ball $B = \{x \in \Omega : K_\Omega(x, x_0) \leq R\}$ is compact. Let $x_n \in B$ be a sequence. We will show that a subsequence of x_n converges in B. By passing to a subsequence we can suppose that $x_n \to x \in \mathbb{M}$ or $x_n \to \infty$ (that is, x_n leaves every compact subset of $\mathbb{M}$).

First suppose that $x_n \to x \in \mathbb{M}$. If $x \in \Omega$, then $x \in B$ by part (5) of Proposition 4.2. Otherwise $x \in \partial\Omega$. Let H_Ω be the Hilbert metric on Ω; then $H_\Omega \leq K_\Omega$ by Kobayashi's construction of the Hilbert metric (described in Section 2.2). So

$$K_\Omega(x_0, x_n) \geq H_\Omega(x_0, x_n) \to \infty,$$

which is a contradiction.

Finally, suppose that the sequence x_n leaves every compact subset of Ω. If Ω contains no affine lines, then H_Ω is a proper metric and so

$$K_\Omega(x_0, x_n) \geq H_\Omega(x_0, x_n) \to \infty.$$

If Ω is not proper, then we can identify $\mathbb{M}$ with $\mathbb{R}^D$, where $D = pq$ and find an affine map $\Phi \in \mathrm{Aff}(\mathbb{R}^D)$ such that $\Phi\Omega = \mathbb{R}^d \times \Omega'$, where Ω' is a proper convex set and $d \leq D$. Notice that $H_\Omega(z_1, z_2) = H_{\Phi\Omega}(\Phi z_1, \Phi z_2)$ for all $z_1, z_2 \in \Omega$, but the metrics $K_{\Phi\Omega}$ and K_Ω have no clear relationship because Φ will in general not preserve the rank-one lines. Since Ω is $\mathcal{R}$–proper we must have that $d < D$. Let $\pi \colon \mathbb{R}^D \to \mathbb{R}^{D-d}$ be the projection onto the second factor. Next let $\sigma_n \colon [0,1] \to \Omega$ be a curve joining x_0 to x_n with K_Ω–length less than $R + \epsilon$.

We claim that the set $\overline{\{\pi(\Phi\sigma_n(t)) : n \in \mathbb{N}, t \in [0,1]\}}$ is a compact subset of Ω'. This follows from the fact that

$$\begin{aligned} R + \epsilon \geq K_\Omega(x_0, \sigma_n(t)) \geq H_\Omega(x_0, \sigma_n(t)) &= H_{\Phi\Omega}(\Phi x_0, \Phi\sigma_n(t)) \\ &= H_{\Omega'}\big(\pi(\Phi x_0), \pi(\Phi\sigma_n(t))\big) \end{aligned}$$

and the fact that $H_{\Omega'}$ is a proper metric on Ω'. So if $x_n = \Phi^{-1}(y_n, z_n)$, we must have $y_n \to \infty$. But then notice that

$$\delta_\Omega(x + a; v) = \delta_\Omega(x; v)$$

for all $a \in \Phi^{-1}(\mathbb{R}^d \times \{0\})$ and $v \in \mathbb{M}_1$. And so there exists $M \geq 0$ such that

$$\delta_\Omega(\sigma_n(t); v) \leq M$$

for all $n \in \mathbb{N}$, $t \in [0,1]$ and $v \in \mathbb{M}_1$. But then, arguing as before, we see that

$$\mathrm{length}(\sigma_n) \geq \frac{1}{M} \|x_0 - x_n\|.$$

Since x_n leaves every compact subset of Ω and $\mathrm{length}(\sigma_n) < R + \epsilon$, we have a contradiction.

Finally, we observe that K_Ω is a complete metric on Ω. If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (Ω, K_Ω), then we can pass to a subsequence such that

$$\sum_{n=1}^{\infty} \|x_n - x_{n+1}\| = R < \infty.$$

But then $x_n \in \{x \in \Omega : K_\Omega(x_1, x) \leq R\}$, which is a compact subset of Ω. □

Appendix B Proof of Theorem 5.1

In this section we prove Theorem 5.1:

Theorem B.1 *Let $\mathbb{M}$ be an affine chart of $\mathrm{Gr}_p(\mathbb{R}^{p+q})$ and suppose $\Omega_n \subset \mathbb{M}$ is a sequence of $\mathcal{R}$–proper convex open sets converging to an $\mathcal{R}$–proper convex open set $\Omega \subset \mathbb{M}$ in the local Hausdorff topology. Then*

$$K_\Omega(x,y) = \lim_{n\to\infty} K_{\Omega_n}(x,y)$$

for all $x, y \in \Omega$ uniformly on compact sets of $\Omega \times \Omega$.

It will be helpful to introduce an infinitesimal version of ρ_Ω. As in the proof of Theorem A.1, identify $\mathbb{M}$ with the vector space of $q \times p$ matrices and let $\mathbb{M}_1 \subset \mathbb{M}$ be the space of rank-one matrices. Next, for a $\mathcal{R}$–proper convex open set $\Omega \subset \mathbb{M}$, define a function $k_\Omega \colon \Omega \times \mathbb{M}_1 \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by

$$k_\Omega(x;v) = \frac{1}{t^+} + \frac{1}{t^-},$$

where $t^+, t^- \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfy $x + t^+ v, x + t^-(-v) \in \partial\Omega$ and we define $1/\infty = 0$. Notice that, by definition, $k_\Omega(x;\lambda v) = |\lambda| k_\Omega(x;v)$ for any $\lambda \in \mathbb{R}$.

Now if $x, x+tv \in \Omega$, $v \in \mathbb{M}_1$ and $t > 0$, then it is easy to show that

$$\rho_\Omega(x, x+tv) = \int_0^t k_\Omega(x+sv; v)\, ds. \tag{B-1}$$

The following lemma is a simple consequence of this formulation of ρ_Ω:

Lemma B.2 *With the notation in Theorem B.1, for any compact subset $K \subset \Omega$ and $\epsilon > 0$ there exists $N > 0$ such that*

$$(1-\epsilon)\rho_{\Omega_n}(x,y) \leq \rho_\Omega(x,y) \leq (1+\epsilon)\rho_{\Omega_n}(x,y)$$

for all $x, y \in K$ and $n \geq N$.

Proof By possibly increasing K, we can assume that K is convex. We first claim that there exists $N > 0$ such that

$$(1-\epsilon)k_{\Omega_n}(x;v) \leq k_\Omega(x;v) \leq (1+\epsilon)k_{\Omega_n}(x;v)$$

for all $n > N$, $x \in K$ and $v \in \mathbb{M}_1$. Suppose not; then there exist $n_k \to \infty$, $x_{n_k} \in K$ and $v_{n_k} \in \mathbb{M}_1$ such that

$$\frac{k_\Omega(x_{n_k}; v_{n_k})}{k_{\Omega_n}(x_{n_k}; v_{n_k})} \notin [1-\epsilon, 1+\epsilon].$$

We can assume that $\|v_{n_k}\| = 1$ (where $\|\cdot\|$ is the operator norm). Then we can pass to a subsequence and assume that $x_{n_k} \to x \in K$ and $v_{n_k} \to \mathbb{M}_1$. But, using the fact that Ω_n converges to Ω in the local Hausdorff topology, we have

$$\lim_{k\to\infty} \frac{k_\Omega(x_{n_k}; v_{n_k})}{k_{\Omega_n}(x_{n_k}; v_{n_k})} = \frac{k_\Omega(x; v)}{k_\Omega(x; v)} = 1.$$

So we have a contradiction.

Then the lemma follows from equation (B-1). □

Proof of Theorem B.1 Now suppose that $K \subset \Omega$ is compact. Then we can pick $R > 0$ and $x_0 \in \Omega$ such that $K \subset \{x \in \Omega : d_\Omega(x, x_0) \le R\}$. Let

$$K' = \{x \in \Omega : K_\Omega(x, x_0) \le (1+\epsilon)^2(R+1) + R + \epsilon\}.$$

Next pick $N > 0$ such that

$$(1-\epsilon)\rho_{\Omega_n}(x, y) \le \rho_\Omega(x, y) \le (1+\epsilon)\rho_{\Omega_n}(x, y)$$

for all $x, y \in K'$ and $n \ge N$. Now we claim that

$$K_{\Omega_n}(x, y) \le (1+\epsilon) K_\Omega(x, y)$$

for $x, y \in K$ and $n \ge N$. For $x, y \in K$ and $\delta \in (0, 1)$, pick $x = a_0, a_1, \dots, a_m = y$ such that

$$\rho_\Omega(x, a_1) + \rho_\Omega(a_1, a_2) + \cdots + \rho_\Omega(a_{m-1}, y) \le K_\Omega(x, y) + \delta.$$

Then $a_0, \dots, a_m \in K'$ and so

$$\rho_{\Omega_n}(x, a_1) + \rho_{\Omega_n}(a_1, a_2) + \cdots + \rho_{\Omega_n}(a_{m-1}, y) \le (1+\epsilon)(K_\Omega(x, y) + \delta)$$

for $n \ge N$. Since $\delta > 0$ was arbitrary we see that

$$K_{\Omega_n}(x, y) \le (1+\epsilon) K_\Omega(x, y)$$

for $x, y \in K$ and $n \ge N$.

Now suppose $n \geq N$, $x, y \in K$, $\delta \in (0,1)$ and $x = a_0, a_1, \dots, a_m = y \in \Omega_n$ are such that

$$\rho_{\Omega_n}(x, a_1) + \rho_{\Omega_n}(a_1, a_2) + \dots + \rho_{\Omega_n}(a_{m-1}, y) \leq K_{\Omega_n}(x, y) + \delta.$$

If $a_0, a_1, \dots, a_m \in K'$ then we immediately see that

$$K_\Omega(x, y) \leq \rho_\Omega(x, a_1) + \rho_\Omega(a_1, a_2) + \dots + \rho_\Omega(a_{m-1}, y) \leq (1+\epsilon)(K_{\Omega_n}(x, y) + \delta).$$

Otherwise we can assume that there is some a_l such that $a_l \in \partial K'$. Then $K_\Omega(a_l, x_0) = (1+\epsilon)^2(R+1) + R + \epsilon$ and so

$$\begin{aligned}(1+\epsilon)^2(R+1) + \epsilon &\leq K_\Omega(x_0, a_l) - K_\Omega(x_0, x) \leq K_\Omega(x, a_l)\\ &\leq \rho_\Omega(x, a_1) + \rho_\Omega(a_1, a_2) + \dots + \rho_\Omega(a_{l-1}, a_l)\\ &\leq (1+\epsilon)(K_{\Omega_n}(x, y) + \delta) \leq (1+\epsilon)((1+\epsilon)K_\Omega(x, y) + 1)\\ &\leq (1+\epsilon)^2(R+1),\end{aligned}$$

which is a contradiction. Thus, $a_0, a_1, \dots, a_m \in K'$ and

$$K_\Omega(x, y) \leq (1+\epsilon)(K_{\Omega_n}(x, y) + \delta).$$

Since $\delta \in (0,1)$ was arbitrary we see that

$$K_\Omega(x, y) \leq (1+\epsilon)K_{\Omega_n}(x, y). \qquad \square$$

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