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The Weyl tensor of gradient Ricci solitons

XIAODONG CAO
HUNG TRAN

This paper derives new identities for the Weyl tensor on a gradient Ricci soliton, particularly in dimension four. First, we prove a Bochner–Weitzenböck-type formula for the norm of the self-dual Weyl tensor and discuss its applications, including connections between geometry and topology. In the second part, we are concerned with the interaction of different components of Riemannian curvature and (gradient and Hessian of) the soliton potential function. The Weyl tensor arises naturally in these investigations. Applications here are rigidity results.

53C44; 53C21, 53C25

1 Introduction

The Ricci flow, which was first introduced by R Hamilton in [31], describes a one-parameter family of smooth metrics $g(t)$, $0 \leq t < T \leq \infty$, on a closed n–dimensional manifold M^n, by the equation

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(t). \tag{1-1}$$

The subject has been studied intensively, particularly in the last decade thanks to seminal contributions by G Perelman in his proof of the Poincaré conjecture (see [40; 41]). It also gained popularity after playing a key role in the proofs of the classification theorem for manifolds with 2–positive curvature operators due to C Böhm and B Wilking [6], and the Differentiable sphere theorem of S Brendle and R Schoen [9; 8].

As a weakly parabolic system, the Ricci flow can develop finite-time singularities and consequently, the study of singularity models becomes crucial. In this paper, we are concerned with gradient Ricci solitons (GRS), which are self-similar solutions of Hamilton's Ricci flow (1-1) and arise naturally in the analysis of singularities. A GRS (M, g, f, λ) is a Riemannian manifold endowed with a special structure given by a (soliton) potential function f, a constant λ, and the equation

$$\mathrm{Rc} + \mathrm{Hess}\, f = \lambda g. \tag{1-2}$$

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Depending on the sign of λ, a GRS is called shrinking (positive), steady (zero), or expanding (negative). In particular an Einstein manifold N can be considered as a special case of a GRS where f is a constant and λ becomes the Einstein constant. A less trivial example is a Gaussian soliton $(\mathbb{R}^k, g_{\mathrm{sd}}, \lambda|x|^2/2, \lambda)$, with g_{sd} being the standard metric on Euclidean space. It is interesting to note that λ can be an arbitrary real number and that the Gaussian soliton can be either shrinking, steady or expanding. Furthermore, a combination of those two above, in the notation of P Petersen and W Wylie [43], is called a *rank k rigid GRS*, namely a quotient of $N \times \mathbb{R}^k$. Other nontrivial examples of GRS are rare and mostly Kähler; see [12; 26].

In recent years, following the interest in the Ricci flow, there have been various efforts to study the geometry and classification of GRSs; for example, see Cao [13] and the citations therein. In particular, the low-dimensional cases ($n = 2, 3$) are relatively well-understood. For $n = 2$, Hamilton [33] completely classified shrinking gradient solitons with bounded curvature and showed that they must be either the round sphere, projective space, or Euclidean space with the standard metric. For $n = 3$, utilizing the Hamilton–Ivey estimate, Perelman [41] proved an analogous theorem. Other significant results include recent development of Brendle [7] showing that a non-collapsed steady GRS must be rotationally symmetric and is, therefore, isometric to the Bryant soliton.

In higher dimensions, the situation is more subtle mainly due to the non-triviality of the Weyl tensor (W) which is vacuously zero for dimension less than four. One general approach to the classification problem so far has been to impose certain restrictions on the curvature operator. An analogue of Hamilton–Perelman results was obtained by A Naber proving that a four-dimensional complete non-compact shrinking GRS with bounded nonnegative curvature operator must be a finite quotient of $\mathbb{R}^4$, $S^2 \times \mathbb{R}^2$ or $S^3 \times \mathbb{R}$ [38]. In [34], B Kotschwar classified all rotationally symmetric GRSs with given diffeomorphic types on $\mathbb{R}^n$, $S^{n-1} \times \mathbb{R}$ or S^n. Note that any rotationally symmetric Riemannian manifold has vanishing Weyl tensor.

Thus, a natural development is to impose conditions on that Weyl tensor in higher dimensions. A complete shrinking GRS with vanishing Weyl tensor must be a finite quotient of $\mathbb{R}^n$, or $S^{n-1} \times \mathbb{R}$ or S^n. That follows from the works of Ni and Wallach [39], Zhang [51], Cao, Wang and Zhang [18] and Petersen and Wylie [44]. A steady GRS is flat or rotationally symmetric (that is, a Bryant soliton) by Cao and Chen [14]. The assumption $\mathrm{W} \equiv 0$ can be weakened to $\delta \mathrm{W} \equiv 0$. In that case, a closed or non-compact shrinking GRS must be rigid; see Cao, Wang and Zhang [18], Fernández-López and García-Río [27], and Munteanu and Sesum [37]. Furthermore, in dimension four, the vanishing of the self-dual part of Weyl tensor has been studied by Chen and Wang [22]. They show that a shrinking GRS with bounded curvature must be a finite quotient of $\mathbb{R}^4$, $S^3 \times \mathbb{R}$, S^n, or $\mathbb{CP}^2$, and a steady GRS must be a Bryant soliton or flat. There are other classifications based on, for instance, Bach flatness (see Cao and Chen [15]) or assumptions on the radial sectional curvature (see Petersen and Wylie [44]).

Having highlighted the importance of the non-triviality of the Weyl tensor, this paper is devoted to studying the delicate role of that tensor within a gradient soliton structure. Our perspective here is to view a GRS as both a generalization of an Einstein manifold as well as a self-similar solution to the Ricci flow. In particular, this paper derives several new identities on the Weyl tensor of GRS in dimension four. In the first part, we prove the a Bochner–Weitzenböck-type formula for the norm of the self-dual Weyl tensor using flow equations and some ideas related to Einstein manifolds.

Theorem 1.1 *Let (M, g, f, λ) be a four-dimensional GRS. Then we have the Bochner–Weitzenböck formula*

$$\begin{aligned} \Delta_f |\mathrm{W}^+|^2 &= 2|\nabla \mathrm{W}^+|^2 + 4\lambda|\mathrm{W}^+|^2 - 36 \det \mathrm{W}^+ - \langle \mathrm{Rc} \circ \mathrm{Rc}, \mathrm{W}^+ \rangle \\ &= 2|\nabla \mathrm{W}^+|^2 + 4\lambda|\mathrm{W}^+|^2 - 36 \det \mathrm{W}^+ - \langle \mathrm{Hess}\, f \circ \mathrm{Hess}\, f, \mathrm{W}^+ \rangle. \end{aligned} \tag{1-3}$$

For the relevant notation, see Section 2. Identity (1-3) potentially has several applications and we will present a couple of them in this paper, including a gap theorem. More precisely, if the GRS is not locally conformally flat and the divergence of the Weyl tensor is relatively small, then the L_2–norm of the Weyl tensor is bounded below by a topological constant (see Theorem 4.1). The proof, in a similar manner to that of Gursky [29], uses some ideas from the solution to the Yamabe problem.

In the second part, we are mostly concerned with the interaction of different curvature components, gradient and Hessian of the potential function. In particular, an interesting connection is illustrated by the following integration by parts formula.

Theorem 1.2 *Let (M, g, f, λ) be a closed GRS. Then we have*

$$\begin{aligned} \int_M \langle \mathrm{W}, \mathrm{Rc} \circ \mathrm{Rc} \rangle &= \int_M \langle \mathrm{W}, \mathrm{Hess}\, f \circ \mathrm{Hess}\, f \rangle = \int_M \mathrm{W}(\mathrm{Hess}\, f, \mathrm{Hess}\, f) \\ &= \int_M \mathrm{W}_{ijkl} f_{ik} f_{jl} = \frac{1}{n-3} \int_M \langle \delta \mathrm{W}, (n-4)M + (n-2)P \rangle. \end{aligned} \tag{1-4}$$

In particular, in dimension four, the identity becomes

$$\int_M \langle \mathrm{W}, \mathrm{Rc} \circ \mathrm{Rc} \rangle = 4 \int_M |\delta \mathrm{W}|^2. \tag{1-5}$$

Remark 1.1 For definitions of M and P, see Section 5.

Remark 1.2 This result exposes the intriguing interaction between the Weyl tensor and the potential function f on a GRS. It will be interesting to extend those identities to a (possibly non-compact) smooth metric measure space or generalized Einstein manifold.

Remark 1.3 In dimension four, the statement also holds replacing W by $\mathrm{W}^{\pm}$; see Corollary 5.8.

The interactions of various curvature components and the soliton potential function can be applied to study the classification problem. For example, Theorem 6.1 asserts the rigidity of the Ricci curvature tensor in dimension four. More precisely, if the Ricci tensor at each point has at most two eigenvalues with multiplicity one and three, then any such closed GRS must be rigid. It is interesting to compare this result with classical classification results of the Codazzi tensor, which require both distribution of eigenvalues and information on the first derivative (see [5, Chapter 16, Section C]).

This paper is organized as follows. In Section 2, we fix our notation and collect some preliminary results. Section 3 provides a proof of Theorem 1.1 using the Ricci flow technique. Section 4 gives some immediate applications of the new Bochner–Weitzenböck type formula including the aforementioned gap theorem. In Section 5, we first discuss a general framework to study the interaction of different components of the curvature with the potential function, and then prove Theorem 1.2. In Section 6, we apply our framework to obtain various rigidity results. Finally, in the appendix, we collect a few related formulas.

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2 Notation and preliminaries

In this section, we will fix the notation and conventions that will be used throughout the paper.

$\mathrm{R}, \mathrm{W}, \mathrm{Rc}, \mathrm{S}$ and E will stand for the Riemannian curvature operator, Weyl tensor, Ricci curvature, scalar curvature, and the traceless part of the Ricci tensor respectively.

For a finite-dimensional real vector space (bundle), $\Lambda_2(V)$ denotes the space of bi-vectors or two-forms. In our case, the space of interest is normally the tangent bundle and when the context is clear, the dependence on V is omitted.

Given an orthonormal basis $\{E_i\}_{i=1}^n$ of T_pM, it is well-known that we can construct an orthonormal frame about p such that $e_i(p) = E_i$ and $\nabla e_i \mid_p = 0$. Such a frame is called normal at p. Also, e_{12} is the shorthand notation for $e_1 \wedge e_2 \in \Lambda_2$.

The modified Laplacian is defined as

$$\Delta_f = \Delta - \nabla_{\nabla f}.$$

For any $(m,0)$–tensor T, its divergence operator is defined as

$$(\delta T)_{p_2 \dots p_m} = \sum_i \nabla_i T_{ip_2 \dots p_m},$$

while its interior product by a vector field X is defined as

$$(i_X T)_{p_2 \dots p_m} = T_{Xp_2 \dots p_m}.$$

Furthermore, we will interchange the perspective of a vector and a covector freely, ie a $(2,0)$ tensor will also be seen as a $(1,1)$ tensor. Similarly, a $(4,0)$ tensor such as R or W can be interpreted as an operator on bi-vectors; that is, a map from $\Lambda_2(TM) \to \Lambda_2(TM)$. Consequently, the norm of these operators is agreed to be sum of all eigenvalues squared (this agrees with the tensor norm defined in [23] for $(2,0)$ tensors but differs by a factor of $\frac{1}{4}$ for $(4,0)$ tensors). More precisely,

$$|\mathrm{W}|^2 = \sum_{i<j;k<l} \mathrm{W}_{ijkl}^2.$$

In addition, the norm of the covariant derivative and divergence on these tensors can be defined accordingly,

$$|\nabla \mathrm{W}|^2 = \sum_i \sum_{a<b;c<d} (\nabla_i \mathrm{W}_{abcd})^2, \qquad |\delta \mathrm{W}|^2 = \sum_i \sum_{a<b} ((\delta \mathrm{W})_{iab})^2.$$

For a tensor $T\colon \Lambda_2(TM) \otimes (TM) \to \mathbb{R}$, we define

$$\langle T, \delta \mathrm{W} \rangle = \sum_{i<j;k} T_{ijk} (\delta \mathrm{W})_{kij}, \tag{2-1}$$

$$\langle T, i_X \mathrm{W} \rangle = \sum_{i<j;k} T_{ijk} (i_X \mathrm{W})_{kij}. \tag{2-2}$$

Finally, when the context is clear, we will omit the measure when integrating.

2.1 Gradient Ricci solitons

In this subsection, we recall some well-known identities for GRSs. A GRS is characterized by the Ricci soliton equation

$$\mathrm{Rc} + \nabla\nabla f = \lambda g. \tag{2-3}$$

Algebraic manipulation of (2-3) and application of the Bianchi identities lead to the following formulas (for a proof see [23]):

$$\mathrm{S} + \triangle f = n\lambda. \tag{2-4}$$

$$\tfrac{1}{2}\nabla_i \mathrm{S} = \nabla^j \mathrm{R}_{ij} = \mathrm{R}_{ij}\,\nabla^j f. \tag{2-5}$$

$$\mathrm{Rc}(\nabla f) = \tfrac{1}{2}\nabla \mathrm{S}. \tag{2-6}$$

$$\mathrm{S} + |\nabla f|^2 - 2\lambda f = \text{constant}. \tag{2-7}$$

$$\triangle \mathrm{S} + 2|\mathrm{Rc}|^2 = \langle \nabla f, \nabla \mathrm{S}\rangle + 2\lambda \mathrm{S}. \tag{2-8}$$

Remark 2.1 If $\lambda \geq 0$, then $\mathrm{S} \geq 0$ by the maximum principle and equation (2-8). Moreover, a complete GRS has positive scalar curvature unless it is isometric to the flat Euclidean space [45].

One main motivation for the study of GRSs is that they arise naturally as self-similar solutions to the Ricci flow. For a fixed GRS given by (2-3) with $g(0) = g$ and $f(0) = f$, we define $\rho(t) := 1 - 2\lambda t > 0$, and let $\phi(t)\colon M^n \to M^n$ be a one-parameter family of diffeomorphisms generated by $X(t) := (1/\rho(t))\nabla_{g(0)} f$. By pulling back, we get

$$g(t) = \rho(t)\phi(t)^* g(0), \qquad \mathrm{Rc}(t) = \phi^* \,\mathrm{Rc}(0) = \frac{\lambda}{\rho(t)} g(t) - \operatorname*{Hess}_{g(t)} f(t).$$

Then $(M, g(t))$, $0 \leq t < T$, is a solution to the Ricci flow of (1-1), where $T = 1/(2\lambda)$ (resp. $T = \infty$) if $\lambda > 0$ (resp. $\lambda \leq 0$). Other important quantities along the flow are:

- $f(t) = f(0) \circ \phi(t) = \phi(t)^* f$.
- $\mathrm{S}(t) = \mathrm{trace}(\mathrm{Rc}(t)) = \dfrac{n\lambda}{\rho(t)} - \Delta_{g(t)} f(t)$.
- $f_t = |\nabla f|^2_{g(t)}$.
- $\tau(t) = T - t = \dfrac{\rho(t)}{2\lambda}$.
- $u = (4\pi\tau)^{-n/2} e^{-f}$.
- $\Psi(g, \tau, f) = \displaystyle\int_M \big(\tau\big(|\nabla f|^2 + \mathrm{S}\big) + f - n\big) u \, d\mu = -\tau C(t) \int_M u \, d\mu$.

2.2 Four-manifolds

In this subsection, we give a brief review of the algebraic structure of curvature and geometry on an oriented four-manifold (M, g).

First we recall the Kulkarni–Nomizu product for $(2,0)$ symmetric tensors A and B,

$$(A \circ B)_{ijkl} = A_{ik} B_{jl} + A_{jl} B_{ik} - A_{il} B_{jk} - A_{jk} B_{il}.$$

Then we have the following decomposition of curvature,

$$\mathrm{R} = \mathrm{W} + \frac{\mathrm{S} g \circ g}{2n(n-1)} + \frac{\mathrm{E} \circ g}{n-2} = \mathrm{W} - \frac{\mathrm{S} g \circ g}{2(n-2)(n-1)} + \frac{\mathrm{Rc} \circ g}{n-2}. \tag{2-9}$$

In dimension four this becomes

$$\begin{aligned}
&\mathrm{R} = \mathrm{W} + \frac{\mathrm{S}}{24} g \circ g + \frac{1}{2} \mathrm{E} \circ g = \mathrm{W} + U + V, \\
&|\,\mathrm{R}\,|^2 = |\mathrm{W}|^2 + |U|^2 + |V|^2, \\
&|U|^2 = \frac{1}{2n(n-1)} \mathrm{S}^2 = \frac{1}{24} \mathrm{S}^2, \\
&|V|^2 = \frac{1}{n-2} |\mathrm{E}|^2 = \frac{1}{2} |\mathrm{E}|^2.
\end{aligned}$$

An important feature in dimension four is that the Hodge star operator decomposes the space Λ_2 of bi-vectors orthogonally according to the eigenvalues ± 1. The Riemannian curvature inherits this decomposition, and consequently has a special structure. To be more precise, let $\{e_i\}_{i=1}^4$ be an orthonormal basis of the tangent space at any arbitrary point on M. Then one pair of orthonormal bases of bi-vectors is given by

$$\begin{aligned}
&\left\{ \frac{1}{\sqrt{2}}(e_{12} + e_{34}), \frac{1}{\sqrt{2}}(e_{13} - e_{24}), \frac{1}{\sqrt{2}}(e_{14} + e_{23}) \right\} \quad \text{for } \Lambda_2^+, \\
&\left\{ \frac{1}{\sqrt{2}}(e_{12} - e_{34}), \frac{1}{\sqrt{2}}(e_{13} + e_{24}), \frac{1}{\sqrt{2}}(e_{14} - e_{23}) \right\} \quad \text{for} \Lambda_2^-.
\end{aligned} \tag{2-10}$$

Accordingly, the curvature now is

$$\mathrm{R} = \begin{pmatrix} A^+ & C \\ C^T & A^- \end{pmatrix}, \tag{2-11}$$

with C essentially the traceless part. It is easy to observe that $\mathrm{W}(\Lambda_2^\pm) \in \Lambda_2^\pm$, so we may unambiguously define $\mathrm{W}^\pm := \mathrm{W}^{|\Lambda_\pm}$. In particular,

$$\mathrm{W}^\pm(\alpha, \beta) = \mathrm{W}(\alpha^\pm, \beta^\pm), \tag{2-12}$$

with $\alpha^\pm$ and $\beta^\pm$ the projection of α, β onto $\Lambda_2^\pm$.

Furthermore, as W is traceless and satisfies the first Bianchi identity, there is a normal form due to M Berger [4] (see also [47], [19]). That is, there exists an orthonormal basis $\{e_i\}_{i=1}^4$ of T_pM, and consequently a basis $\{e_{12}, e_{13}, e_{14}, e_{34}, e_{42}, e_{23}\}$ of Λ_2, such that

$$\mathrm{W} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \tag{2-13}$$

for $A = \mathrm{diag}(a_1, a_2, a_3)$ and $B = \mathrm{diag}(b_1, b_2, b_3)$, with $a_1+a_2+a_3 = b_1+b_2+b_3 = 0$. Then, by (2-12),

$$\mathrm{W}^{\pm} = \begin{pmatrix} \dfrac{A \pm B}{2} & \dfrac{B \pm A}{2} \\ \dfrac{B \pm A}{2} & \dfrac{A \pm B}{2} \end{pmatrix}.$$

Using the basis given in (2-10), we get

$$\mathrm{W} = \begin{pmatrix} A + B & 0 \\ 0 & A - B \end{pmatrix}.$$

Hence we obtain the following well-known identities.

Lemma 2.1 *Let (M^4, g) be a four-dimensional Riemannian manifold. Then the following tensor equations hold:*

$$(\mathrm{W}^{\pm})_{ikpq}(\mathrm{W}^{\pm})_j{}^{kpq} = |\mathrm{W}^{\pm}|^2 g_{ij}, \tag{2-14}$$

$$(\mathrm{W}^{\pm})_{ikpq}(\mathrm{W}^{\pm})^{kpq}{}_j = \tfrac{1}{2}|\mathrm{W}^{\pm}|^2 g_{ij}. \tag{2-15}$$

Proof Note that these identities only depend on the decomposition of these tensors. In particular, it suffices to prove them for the Weyl tensor. Using the normal form discussed above, we calculate that

$$\mathrm{W}_{1kpq}\mathrm{W}^{kpq}{}_1 = \sum_{i=1}^{3} a_i^2 - 2(b_1b_2 + b_2b_3 + b_3b_1) = \sum_{i=1}^{3}(a_i^2 + b_i^2).$$

Similar calculations for all other pairs of indexes verify the statements. □

Remark 2.2 The first identity can also be found in [24, Section 2, Equation (31)].

In addition, in reference to the decomposition of curvature in (2-11), we have the following relations:

$$A^{\pm} = \mathrm{W}^{\pm} + \frac{\mathrm{S}}{12} I^{\pm}, \qquad |A^{\pm}|^2 = |\mathrm{W}^{\pm}|^2 + \frac{\mathrm{S}^2}{48},$$
$$|\mathrm{E}|^2 = |\mathrm{Rc}|^2 - \frac{\mathrm{S}^2}{4} = 4|C|^2 = 4\,\mathrm{tr}(CC^T).$$

If the manifold is closed, the Gauss–Bonnet–Chern formula for the Euler characteristic and Hirzebruch formulas for the signature (see [5] for more details) are given by

$$8\pi^2\chi(M) = \int_M (|\mathrm{W}|^2 - |V|^2 + |U|^2) = \int_M \Big(|\mathrm{W}|^2 - \frac{1}{2}|\mathrm{E}|^2 + \frac{S^2}{24}\Big) = \int_M (|\,\mathrm{R}\,|^2 - |\mathrm{E}|^2), \tag{2-16}$$

$$12\pi^2\tau(M) = \int_M (|\mathrm{W}^+|^2 - |\mathrm{W}^-|^2). \tag{2-17}$$

Remark 2.3 It follows immediately that if M admits an Einstein metric $\mathrm{E} = 0$, then we have the Hitchin–Thorpe inequality

$$|\tau(M)| \le \tfrac{2}{3}\chi(M).$$

The Hodge operator in dimension four is related to a decomposition on the tangent bundle. Let $\{\alpha_i\}_{i=1}^3$ be a positively oriented orthogonal basis of Λ_2^+ with $|\alpha_i| = \sqrt{2}$. When seen as an operator on a vector field X and given $\mathrm{sign}(i,j,k) = 1$, those bi-vectors satisfy the following identities (see [1]):

$$\alpha_i^2 = -\mathrm{Identity},$$
$$\alpha_i\alpha_j = \alpha_k = -\alpha_j\alpha_i,$$
$$\langle \alpha_i(X), \alpha_j(X)\rangle = \langle X, -\alpha_i\alpha_j X\rangle = \langle X, \alpha_k X\rangle = 0.$$

Here $\mathrm{sign}(i,j,k)$ is the sign of the permutation of $\{1,2,3\}$. The positive orientation is just to agree with the sign convention. An example of such a basis is given by multiplying by $\sqrt{2}$ the basis given in (2-10). Consequently, we have the following result.

Lemma 2.2 *Suppose (M, g) is a four-dimensional Riemannian manifold and X is a vector field on M. At any point p such that $X_p \neq 0$,*

$$T_pM = X_p \oplus \Lambda_2^+(X_p),$$

with $\Lambda_2^+(X) = \{\alpha(X_p), \alpha \in \Lambda_2^+\}$.

Proof Pick an orthogonal basis of Λ_2^+ as above. Then $\{\alpha_i(X_p)\}_{i=1}^3$ are three orthogonal vectors, and each is perpendicular to X_p. The statement follows. □

Remark 2.4 By symmetry, the statement also holds for Λ_2^-. When the context is clear, we normally omit the sub-index of the point.

2.3 New sectional curvature

In this subsection, we first prove some results in dimension four to illustrate that classical techniques for Einstein 4–manifolds can be adapted to study GRSs.

For a four-dimensional GRS (M, g, f, λ), define

$$H = \operatorname{Hess} f \circ g. \tag{2-18}$$

Then a straightforward calculation leads to the following decomposition.

Lemma 2.3 *With respect to the decomposition given by (2-10), we have*

$$H = \begin{pmatrix} A & B \\ B^T & A \end{pmatrix},$$

with $A = \dfrac{\Delta f}{2}\,\mathrm{Id}$ *and*

$$B = \begin{pmatrix} \frac{f_{11}+f_{22}-f_{33}-f_{44}}{2} & f_{23}-f_{14} & f_{24}+f_{13} \\ f_{23}+f_{14} & \frac{f_{11}+f_{33}-f_{22}-f_{44}}{2} & f_{34}-f_{12} \\ f_{24}-f_{13} & f_{34}+f_{12} & \frac{f_{11}+f_{44}-f_{22}-f_{33}}{2} \end{pmatrix}.$$

Remark 2.5 In particular, $\langle H, \mathrm{W}\rangle = 0$.

We further define a new "curvature" tensor $\overline{\mathrm{R}}$ by

$$\begin{aligned} \overline{\mathrm{R}} = \mathrm{R} + \frac{1}{2}H &= \mathrm{W} + \frac{\mathrm{S}}{24} g \circ g + \frac{1}{2}\Big(\mathrm{Rc} - \frac{\mathrm{S}}{4} g\Big) \circ g + \frac{1}{2}H \\ &= \mathrm{W} - \frac{\mathrm{S}}{12} g \circ g + \frac{1}{2}\lambda g \circ g \\ &= \mathrm{W} + \Big(\frac{\lambda}{2} - \frac{\mathrm{S}}{12}\Big) g \circ g. \end{aligned} \tag{2-19}$$

Thus, it follows immediately that, with respect to (2-10),

$$\overline{\mathrm{R}} = \begin{pmatrix} \bar{A}^+ & 0 \\ 0 & \bar{A}^- \end{pmatrix},$$

with $\bar{A}^\pm = \mathrm{W}^\pm + (\lambda - \mathrm{S}/6)\mathrm{Id} = \mathrm{W}^\pm + (\Delta f/4 + \mathrm{S}/12)\,\mathrm{Id}$. Furthermore, following the argument in [4], we obtain:

Proposition 2.4 *There exists a normal form for $\overline{\mathrm{R}}$. More precisely, at each point, there exists an orthonormal base $\{e_i\}_{i=1}^4$ such that as an operator on 2–forms, with respect to the corresponding base $\{e_{12}, e_{13}, e_{14}, e_{34}, e_{42}, e_{23}\}$ for Λ^2,*

$$\overline{\mathrm{R}} = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

with $A = \operatorname{diag}(a_1, a_2, a_3)$ and $B = \operatorname{diag}(b_1, b_2, b_3)$. Moreover, $a_1 = \min \overline{K}$, $a_3 = \max \overline{K}$ and $|b_i - b_j| \le |a_i - a_j|$, where $\overline{K}$ is the "sectional curvature" of $\overline{\mathrm{R}}$; that is, $\overline{K}(e_1, e_2) = \overline{\mathrm{R}}_{1212}$ for any orthonormal vectors e_1 and e_2.

Remark 2.6 Can a GRS be characterized by the existence of such a function f with $\overline{\mathrm{R}}$ constructed as above having the normal form?

Next, we investigate the assumption of having a lower bound on this new sectional curvature similar to [30]. For $\epsilon < 1/3$, suppose that

$$\overline{K} \ge \epsilon\lambda, \tag{2-20}$$

or equivalently, for any orthonormal pair e_i and e_j,

$$\overline{\mathrm{R}}_{ijij} \ge \epsilon\lambda \iff \mathrm{R}_{ijij} + \frac{f_{ii} + f_{jj}}{2} \ge \epsilon\lambda. \tag{2-21}$$

Then we have the following lemma.

Lemma 2.5 *Let (M, g, f, λ) be a GRS. Then assumption (2-20) implies the following:*

- $\mathrm{S} + 3\Delta f \ge 12\epsilon\lambda$.
- $\mathrm{S} \le 6(1-\epsilon)\lambda$.
- $\Delta f \ge 2(3\epsilon - 1)\lambda$.
- $\frac{1}{\sqrt{6}}\big(|\mathrm{W}^+| + |\mathrm{W}^-|\big) \le 2(1-\epsilon)\lambda - \frac{\mathrm{S}}{3}$.

In the last formula, equality holds if and only if $\mathrm{W}^\pm$ has the form $a^\pm \operatorname{diag}(-1,-1,2)$, with $a^\pm \ge 0$, and

$$a^+ + a^- = 2(1-\epsilon)\lambda - \frac{\mathrm{S}}{3}.$$

Proof All inequalities follow from tracing Equation (2-21) and the soliton equation $S + \Delta f = 4\lambda$ except the last one.

For the last inequality, first note that any two form ϕ can be written as a simple wedge product of 1–forms if and only if $\phi \wedge \phi = 0$. In dimension four, with respect to

(2-10), that is equivalent to $\phi = \phi^+ + \phi^-$ and $|\phi^+| = |\phi^-|$. Therefore, in light of Equation (2-13), assumption (2-20) is equivalent to

$$a^+ + a^- + 2\lambda - \frac{\mathrm{S}}{3} \geq 2\epsilon\lambda, \tag{2-22}$$

where a^+, a^- are the smallest eigenvalues of $\mathrm{W}^{\pm}$. Using the algebraic inequalities

$$-a^+ \geq \frac{1}{\sqrt{6}}|\mathrm{W}^+|, \tag{2-23}$$

$$-a^- \geq \frac{1}{\sqrt{6}}|\mathrm{W}^-|, \tag{2-24}$$

we obtain

$$2(1-\epsilon)\lambda - \frac{\mathrm{S}}{3} \geq \frac{1}{\sqrt{6}}\big(|\mathrm{W}^+| + |\mathrm{W}^-|\big).$$

Equality holds if and only if equality holds in (2-22) and (2-23) (or (2-24)). The result then follows immediately. □

Lemma 2.6 *Let (M, g, f, λ) be a closed GRS satisfying (2-20). Then*

$$\int_M \big(|\mathrm{W}^+| + |\mathrm{W}^-|\big)^2 \leq \int_M \frac{2\mathrm{S}^2}{3}\, d\mu - 8(1-\epsilon)(1+3\epsilon)\lambda^2 V(M).$$

Again, equality holds if $\mathrm{W}^{\pm}$ *has the form* $a^{\pm}\operatorname{diag}(-1,-1,2)$, *with* $a^{\pm} \geq 0$ *and*

$$a^+ + a^- = 2(1-\epsilon)\lambda - \frac{\mathrm{S}}{3}.$$

Proof Applying Lemma 2.5, we compute

$$\begin{aligned}\int_M \Big(2(1-\epsilon)\lambda - \frac{\mathrm{S}}{3}\Big)^2 &= 4(1-\epsilon)^2\lambda^2 V(M) - \frac{4(1-\epsilon)\lambda}{3}\int_M \mathrm{S} + \int_M \frac{\mathrm{S}^2}{9} \\ &= 4(1-\epsilon)^2\lambda^2 V(M) - \frac{4(1-\epsilon)\lambda}{3} 4\lambda V(M) + \int_M \frac{\mathrm{S}^2}{9} \\ &= 4(1-\epsilon)\lambda^2 V(M)\Big(-\epsilon - \frac{1}{3}\Big) + \int_M \frac{\mathrm{S}^2}{9}.\end{aligned}$$ □

Remark 2.7 If we use $\mathrm{S} \leq 6(1-\epsilon)\lambda$, then

$$\int_M (|\mathrm{W}^+| + |\mathrm{W}^-|)^2 \leq \Big(\int_M \mathrm{S}^2\, d\mu\Big)\Big(\frac{2}{3} - \frac{2(1+3\epsilon)}{9(1-\epsilon)}\Big) = \frac{4(1-3\epsilon)}{9(1-\epsilon)}\int_M \mathrm{S}^2.$$

Lemma 2.7 *Let (M, g, f, λ) be a closed GRS. Then*

$$\int_M |\mathrm{Rc}|^2 = \int_M \frac{\mathrm{S}^2}{2} - 4\lambda^2 V(M).$$

Proof Using Equation (2-8), we compute

$$\begin{aligned} 2\int_M |\mathrm{Rc}|^2 \, d\mu = \int_M (2\lambda \mathrm{S} + \langle \nabla f, \nabla \mathrm{S}\rangle) \, d\mu &= 2\lambda 4\lambda V(M) - \int_M \Delta f \, \mathrm{S} \, d\mu \\ &= 8\lambda^2 V(M) - \int_M (4\lambda - \mathrm{S})\mathrm{S} \, d\mu \\ &= -8\lambda^2 V(M) + \int_M \mathrm{S}^2 \, d\mu. \end{aligned}$$ □

The above results lead to the following estimate on the Euler characteristic.

Proposition 2.8 *Let (M, g, f, λ) be a closed non-flat GRS with unit volume, satisfying assumption (2-20). Then*

$$8\pi^2 \chi(M) < \frac{7}{12} \int_M \mathrm{S}^2 d\mu + 2\lambda^2 (12\epsilon^2 - 8\epsilon - 3).$$

Proof By the Gauss–Bonnet–Chern formula,

$$\begin{aligned} 8\pi^2 \chi(M) &= \int_M \left(|\mathrm{W}|^2 - \frac{1}{2}|\mathrm{E}|^2 + \frac{\mathrm{S}^2}{24} \right) d\mu \\ &\leq \int_M \left(|\mathrm{W}^+| + |\mathrm{W}^-| \right)^2 d\mu - \frac{1}{2} \int_M |\mathrm{Rc}|^2 \, d\mu + \int_M \frac{\mathrm{S}^2}{6} \, d\mu. \end{aligned}$$

Applying Lemmas 2.6 and 2.7 yields the inequality.

We now claim that equality can not happen. Suppose otherwise. Then $|\mathrm{W}^+||\mathrm{W}^-| = 0$, and equality also holds in Lemma 2.6. By the regularity theory for solitons [3], we can choose an orientation such that $|\mathrm{W}^-| \equiv 0$. Hence $\mathrm{W}^+ = \mathrm{diag}(-a^+, -a^+, 2a^+)$, with $a^+ = 2(1-\epsilon)\lambda - \mathrm{S}/3$. Then by [22, Theorem 1.1], we have $\mathrm{W}^+ = 0$ or $\mathrm{Rc} = 0$.

In the first case, by the classification of locally conformally flat four-dimensional closed GRSs as discussed in the introduction, (M, g) is flat; this is a contradiction.

In the second case, $\mathrm{Rc} = 0$ implies $\mathrm{S} = 0 = \lambda$, and since equality holds in Lemma 2.6, $\mathrm{W}^+ = 0$. Hence the above argument applies. □

Remark 2.8 The Euler characteristic of a closed Ricci soliton has been studied by Derdzinski [25]. If the manifold is Einstein and $\epsilon = 0$, we recover some results of Gursky and LeBrun [30].

3 A Bochner–Weitzenböck formula

In this section, we prove Theorem 1.1, a new Bochner–Weitzenböck formula for the Weyl tensor of GRSs, which generalizes the one for Einstein manifolds. Bochner–Weitzenböck formulas have proven to be a powerful tool to find connections between topology and geometry under certain curvature conditions (for example, see [28; 42; 48]).

Particularly, in dimension four, if $\delta \mathrm{W}^+ = 0$ (this contains all Einstein manifolds), we have the following well-known formula (see [5, 16.73]):

$$\Delta |\mathrm{W}^+|^2 = 2|\nabla \mathrm{W}^+|^2 + \mathrm{S}|\mathrm{W}^+|^2 - 36 \det \mathrm{W}^+. \tag{3-1}$$

This equation plays a crucial role in obtaining an L_2–gap theorem of the Weyl tensor and in studying the classification problem of Einstein manifolds (see [29; 30; 50]).

Our first technical lemma gives a formula for $\Delta_f \mathrm{W}$ in a local frame. Also note that the Einstein summation convention is used repeatedly here.

Lemma 3.1 *Let (M, g, f, λ) be a GRS and $\{e_i\}_{i=1}^n$ be a local normal frame. Then the following formula holds:*

$$\begin{aligned} \Delta_f \mathrm{W}_{ijkl} = 2\lambda \mathrm{W}_{ijkl} &- 2(C_{ijkl} - C_{ijlk} + C_{ikjl} - C_{iljk}) \\ &- \frac{2}{(n-2)^2} g^{pq} \big(\mathrm{Rc}_{ip}\,\mathrm{Rc}_{qk}\, g_{jl} - \mathrm{Rc}_{ip}\,\mathrm{Rc}_{ql}\, g_{jk} \\ &\qquad + \mathrm{Rc}_{jp}\,\mathrm{Rc}_{ql}\, g_{ik} - \mathrm{Rc}_{jp}\,\mathrm{Rc}_{qk}\, g_{il}\big) \\ &+ \frac{2\mathrm{S}}{(n-2)^2} (\mathrm{Rc}_{ik}\, g_{jl} - \mathrm{Rc}_{il}\, g_{jk} + \mathrm{Rc}_{jl}\, g_{ik} - \mathrm{Rc}_{jk}\, g_{il}) \\ &- \frac{2}{n-2} (\mathrm{R}_{ik}\,\mathrm{R}_{jl} - \mathrm{R}_{jk}\,\mathrm{R}_{il}) \\ &- \frac{2(\mathrm{S}^2 - |\mathrm{Rc}|^2)}{(n-1)(n-2)^2} (g_{ik} g_{jl} - g_{il} g_{jk}), \end{aligned} \tag{3-2}$$

where $C_{ijkl} = g^{pq} g^{rs} \mathrm{W}_{pijr} \mathrm{W}_{slkq}$.

Proof First, we recall how a GRS can be realized as a self-similar solution to the Ricci flow (1-1), as in Section 2.1.

Let $\tau(t) = 1 - 2\lambda t$ and suppose that $\phi(x,t)$ is a family of diffeomorphisms generated by $X = (1/\tau)\nabla f$. For $g(0) = g$, $g(t) = \tau(t)\phi^*(t) g$, we have that $(M, g(t))$ is a solution to the Ricci flow. Furthermore, $\mathrm{W}(t) = \tau\phi^* \mathrm{W}$. Let p be a point in M and let $\{e_i\}_{i=1}^n$ be a basis of T_pM. We obtain a local normal frame via extending e_i to a neighborhood by parallel translation along geodesics with respect to $g(0)$. First, we observe that

$$\frac{d}{dt}\mathrm{W}(t)_{ijkl}\Big|_{t=0} = \Big(\frac{d}{dt}\tau\phi^*\mathrm{W}\Big)_{ijkl}\Big|_{t=0} = -\frac{2\lambda}{\tau}\mathrm{W}_{ijkl} + (L_{\nabla f}\mathrm{W})_{ijkl}, \tag{3-3}$$

where L_X is the Lie derivative with respect to X. Furthermore, by definition,

$$\begin{aligned} L_{\nabla f}\mathrm{W}_{ijkl} = \nabla f(\mathrm{W}_{ijkl}) &- \mathrm{W}([\nabla f, e_i], e_j, e_k, e_l) - \mathrm{W}(e_i, [\nabla f, e_j], e_k, e_l) \\ &- \mathrm{W}(e_i, e_j, [\nabla f, e_k], e_l) - \mathrm{W}(e_i, e_j, e_k, [\nabla f, e_l]). \end{aligned} \tag{3-4}$$

We calculate that

$$\mathrm{W}([\nabla f, e_i], e_j, e_k, e_l) = \mathrm{W}(\nabla_{\nabla f} e_i - \nabla_{e_i}\nabla f, e_j, e_k, e_l) = -\mathrm{W}(\nabla_{e_i}\nabla f, e_j, e_k, e_l).$$

By the soliton structure, $\nabla_{e_i}\nabla. f = -\mathrm{Rc}(e_i,\cdot) + \lambda g(e_i,\cdot)$. Thus,

$$\begin{aligned} \mathrm{W}([\nabla f, e_i], e_j, e_k, e_l) &= -\mathrm{W}(\lambda e_i - \mathrm{Rc}(e_i), e_j, e_k, e_l) \\ &= -\lambda \mathrm{W}_{ijkl} + g^{pq}\,\mathrm{Rc}_{ip}\,\mathrm{W}_{qjkl}. \end{aligned} \tag{3-5}$$

Combining (3-3), (3-4), and (3-5) we obtain

$$\begin{aligned} \frac{d}{dt}\mathrm{W}(t)_{ijkl}\Big|_{t=0} &= \nabla f(\mathrm{W}_{ijkl}) + 2\lambda \mathrm{W}_{ijkl} \\ &\quad - g^{pq}(\mathrm{Rc}_{ip}\,\mathrm{W}_{qjkl} + \mathrm{Rc}_{jp}\,\mathrm{W}_{iqkl} + \mathrm{Rc}_{kp}\,\mathrm{W}_{ijql} + \mathrm{Rc}_{ip}\,\mathrm{W}_{qjkl}). \end{aligned}$$

Along the Ricci flow, the Weyl tensor is evolving according to the following equation (for example, see [21, Proposition 1.1]):

$$\begin{aligned} \frac{d}{dt}\mathrm{W}(t)_{ijkl}\Big|_{t=0} &= \Delta(\mathrm{W}_{ijkl}) + 2(C_{ijkl} - C_{ijlk} + C_{ikjl} - C_{iljk}) \\ &\quad - g^{pq}(\mathrm{Rc}_{ip}\,\mathrm{W}_{qjkl} + \mathrm{Rc}_{jp}\,\mathrm{W}_{iqkl} + \mathrm{Rc}_{kp}\,\mathrm{W}_{ijql} + \mathrm{Rc}_{ip}\,\mathrm{W}_{qjkl}) \\ &\quad + \frac{2}{(n-2)^2} g^{pq}(\mathrm{Rc}_{ip}\,\mathrm{Rc}_{qk}\,g_{jl} - \mathrm{Rc}_{ip}\,\mathrm{Rc}_{ql}\,g_{jk} \\ &\qquad + \mathrm{Rc}_{jp}\,\mathrm{Rc}_{ql}\,g_{ik} - \mathrm{Rc}_{jp}\,\mathrm{Rc}_{qk}\,g_{il}) \\ &\quad + \frac{2\mathrm{S}}{(n-2)^2}(\mathrm{Rc}_{ik}\,g_{jl} - \mathrm{Rc}_{il}\,g_{jk} + \mathrm{Rc}_{jl}\,g_{ik} - \mathrm{Rc}_{jk}\,g_{il}) \\ &\quad + \frac{2}{n-2}(\mathrm{R}_{ik}\,\mathrm{R}_{jl} - \mathrm{R}_{jk}\,\mathrm{R}_{il}) \\ &\quad + \frac{2(\mathrm{S}^2 - |\mathrm{Rc}|^2)}{(n-1)(n-2)^2}(g_{ik}g_{jl} - g_{il}g_{jk}). \end{aligned} \tag{3-6}$$

The result then follows. ☐

Furthermore, in dimension four, we are able to obtain significant simplification due to the special structure given by the Hodge operator. That leads to the proof of our first main theorem.

Proof of Theorem 1.1 We observe that

$$\langle \mathrm{W}^+, \Delta_f \mathrm{W}^+\rangle = \langle \mathrm{W}^+, \Delta \mathrm{W}^+\rangle - \langle \mathrm{W}^+, \nabla_{\nabla f} \mathrm{W}^+\rangle = \langle \mathrm{W}^+, \Delta \mathrm{W}^+\rangle - \tfrac{1}{2}\nabla_{\nabla f}|\mathrm{W}^+|^2.$$

Therefore,

$$\Delta_f |\mathrm{W}^+|^2 = \Delta |\mathrm{W}^+|^2 - \nabla_{\nabla f}|\mathrm{W}^+|^2 = 2\langle \mathrm{W}^+, \Delta_f \mathrm{W}^+\rangle + 2|\nabla \mathrm{W}^+|^2.$$

To calculate the first term of the right hand side, we use the normal form of the Weyl tensor (2-13). As usual, a local normal frame is obtained by parallel translation along geodesic lines. Then (2-10) gives a basis of eigenvectors $\{\alpha_i\}_{i=1}^3$ of W^+ with corresponding eigenvalues $\lambda_i = a_i + b_i$. Consequently,

$$\langle \mathrm{W}^+, \Delta_f \mathrm{W}^+\rangle = \sum_i \lambda_i \Delta_f \mathrm{W}^+(\alpha_i, \alpha_i). \tag{3-7}$$

In order to use Lemma 3.1, it is necessary to calculate the C_{ijkl} terms. By the normal form, we have

$$\begin{aligned} C_{1212} &= a_1^2 + b_2^2 + b_3^2, & C_{1234} &= -2a_1 b_3,\\ C_{1221} &= -2b_2 b_3, & C_{1243} &= 2a_1 b_2,\\ C_{1122} &= 2a_2 a_3, & C_{1324} &= 2a_2 b_3,\\ C_{1221} &= -2b_2 b_3, & C_{1423} &= -2a_3 b_2. \end{aligned}$$

Thus,

$$\begin{aligned} \Delta_f \mathrm{W}_{1212} &= 2\lambda a_1 - 2(a_1^2 + b_1^2 + 2a_2 a_3 + 2b_2 b_3) - \frac{1}{2}\sum_p (\mathrm{Rc}_{1p}^2 + \mathrm{Rc}_{2p}^2)\\ &\qquad + \frac{\mathrm{S}}{2}(\mathrm{Rc}_{11} + \mathrm{Rc}_{12}) - (\mathrm{Rc}_{11}\,\mathrm{R}_{22} - \mathrm{Rc}_{12}^2) - \frac{1}{6}(\mathrm{S}^2 - |\mathrm{Rc}|^2),\\ \Delta_f \mathrm{W}_{1234} &= 2\lambda b_1 - 4(a_1 b_1 + a_2 b_3 + a_3 b_2) + (\mathrm{Rc}_{13}\,\mathrm{Rc}_{24} - \mathrm{Rc}_{23}\,\mathrm{Rc}_{14}). \end{aligned}$$

Therefore,

$$\Delta_f \mathrm{W}^+(\alpha_1, \alpha_1) = 2\lambda\lambda_1 - 2\lambda_1^2 - 4\lambda_2\lambda_3 - \tfrac{1}{12}\big(|\mathrm{Rc}|^2 - \mathrm{S}^2\big) - T_1, \tag{3-8}$$

in which

$$\begin{aligned} 2T_1 &= \mathrm{Rc}_{11}\,\mathrm{Rc}_{22} + \mathrm{Rc}_{33}\,\mathrm{Rc}_{44} + 2\,\mathrm{Rc}_{13}\,\mathrm{Rc}_{24} - \mathrm{Rc}_{12}^2 - 2\,\mathrm{Rc}_{23}\,\mathrm{Rc}_{14} - \mathrm{Rc}_{34}^2\\ &= (\mathrm{Rc}\circ\mathrm{Rc})(\alpha_1, \alpha_1). \end{aligned}$$

Similar calculations hold when replacing α_1 by α_2, α_3,

$$\Delta_f \mathrm{W}^+(\alpha_2,\alpha_2) = 2\lambda\lambda_2 - 2\lambda_2^2 - 4\lambda_1\lambda_3 - \tfrac{1}{12}(|\mathrm{Rc}|^2 - \mathrm{S}^2) - \tfrac{1}{2}\,\mathrm{Rc}\circ\mathrm{Rc}(\alpha_2,\alpha_2), \tag{3-9}$$

$$\Delta_f \mathrm{W}^+(\alpha_3,\alpha_3) = 2\lambda\lambda_3 - 2\lambda_3^2 - 4\lambda_1\lambda_2 - \tfrac{1}{12}(|\mathrm{Rc}|^2 - \mathrm{S}^2) - \tfrac{1}{2}\,\mathrm{Rc}\circ\mathrm{Rc}(\alpha_3,\alpha_3). \tag{3-10}$$

Combining (3-7), (3-8), (3-9) and (3-10) yields

$$\begin{aligned}\langle \mathrm{W}^+, \Delta_f \mathrm{W}^+\rangle &= 2\lambda|\mathrm{W}^+|^2 - 18\det \mathrm{W}^+ - \sum_i T_i\lambda_i \\ &= 2\lambda|\mathrm{W}^+|^2 - 18\det \mathrm{W}^+ - \tfrac{1}{2}\langle \mathrm{Rc}\circ\mathrm{Rc}, \mathrm{W}^+\rangle.\end{aligned}$$

The first equality then follows. The second equality comes from the soliton equation, the property that W^+ is trace-free and Remark 2.5. □

4 Applications of the Bochner–Weitzenböck formula

4.1 A gap theorem for the Weyl tensor

In [29], under the assumptions $\mathrm{W}^+ \neq 0$, $\delta\mathrm{W}^+ = 0$, and the positivity of the Yamabe constant, M Gursky proved the inequality

$$\int_M |\mathrm{W}^+|^2\, d\mu \geq \tfrac{4}{3}\pi^2(2\chi(M) + 3\tau(M)), \tag{4-1}$$

relating $\|\mathrm{W}^+\|_{L_2}$ with topological invariants of a closed four-manifold. Our main result in this section is to prove an analog for GRSs. It is noted that the particular structure of a GRS allows us to relax the harmonic self-dual condition above at the cost of a worse coefficient due to the absence of an improved Kato's inequality.

Theorem 4.1 *Let (M, g, f, λ) be a closed four-dimensional shrinking GRS with*

$$\int_M \langle \mathrm{W}^+, \mathrm{Hess}\, f \circ \mathrm{Hess}\, f\rangle\, d\mu \leq \frac{2}{3}\int_M \mathrm{S}|\mathrm{W}^+|^2. \tag{4-2}$$

Then, unless $\mathrm{W}^+ \equiv 0$,

$$\int_M |\mathrm{W}^+|^2\, d\mu > \tfrac{4}{11}\pi^2\big(2\chi(M) + 3\tau(M)\big). \tag{4-3}$$

Remark 4.1 By Remark 1.3, the condition (4-2) is equivalent to

$$\int_M |\delta \mathrm{W}^+|^2 d\mu \leq \int_M \frac{\mathrm{S}}{6}|\mathrm{W}^+|^2 d\mu.$$

Thus, it is clearly weaker than the condition of being harmonic self-dual.

To prove Theorem 4.1, we follow an idea of [29] and introduce a Yamabe-type conformal invariant. First, the conformal Laplacian is given by,

$$L = -6\Delta + \mathrm{S}.$$

Furthermore, we define

$$F_{a,b} = a\mathrm{S} - b|\mathrm{W}^+|, \qquad L_{a,b} = -6a\Delta_g + F_{a,b} = aL - b\mathrm{W}^+,$$

where a and b are constants to be determined later. Under a conformal transformation as described in (A-1), for any function Φ we have:

- $\widetilde{L}(\Phi) = u^{-3}L(\Phi u)$.
- $\widetilde{L}_{a,b}\Phi = u^{-3}L_{a,b}(\Phi u)$.
- $\widetilde{F}_{a,b} = u^{-3}(-6a\Delta_g + F_{a,b})u$.
- $\displaystyle\int_M \widetilde{F}_{a,b}\, d\tilde{\mu} = \int_M u(-6a\Delta_g + F_{a,b})u\, d\mu = \int_M (F_{a,b}u^2 + 6a|\nabla u|^2)\, d\mu$

The Yamabe problem is, for a given Riemannian manifold (M, g), to find a constant scalar curvature metric in its conformal class $[g]$. That is equivalent to finding a critical point of the following functional. For any C^2 positive function u, let $\tilde{g} = u^2 g$, define

$$Y_g[u] = \frac{\langle u, Lu\rangle_{L_2}}{\|u\|^2_{L_4}} = \frac{\int_M \widetilde{\mathrm{S}}\, d\tilde{\mu}}{\sqrt{\int_M d\tilde{\mu}}}.$$

Then the conformal invariant Y is defined as

$$Y(M, [g]) = \inf\{Y_g[u] : u \text{ is a positive } C^2 \text{ function on } M\}.$$

For an expository account on the Yamabe problem, see [35].

As $F_{a,b}$ conformally transforms like the scalar curvature, in analogy with the discussion above, we can define the following conformal invariant.

Definition 4.2 Given a Riemannian manifold (M, g), define

$$\widehat{Y}_{a,b}(M, [g]) = \inf\{(Y_{a,b})_g[u] : u \text{ is a positive } C^2 \text{ function on } M\},$$

where

$$(\widehat{Y}_{a,b})_g[u] = \frac{\langle u, L_{a,b} u\rangle_{L_2}}{\|u\|_{L_4}^2} = \frac{\int_M \widetilde{F_{a,b}}\, d\tilde{\mu}}{\sqrt{\int_M d\tilde{\mu}}}.$$

In the case of interest, we shall denote

$$F = F_{1,6\sqrt{6}} = \mathrm{S} - 6\sqrt{6}|\mathrm{W}^+|, \qquad \widehat{Y}(M) = \widehat{Y}_{1,6\sqrt{6}}(M,[g]),$$

when the context is clear. First we note the following simple inequality.

Lemma 4.3 *Let (M^n, g) be a closed n–dimensional Riemannian manifold which is not locally conformally flat, and (S^n, g_{sd}) the sphere with the standard metric. Then*

$$\widehat{Y}(M,[g]) \le Y(M,[g]) < Y(S^n,[g_{sd}]) = \widehat{Y}(S^n,[g_{sd}]). \tag{4-4}$$

Proof The first inequality follows from the definition and the following observation. Given a metric g, a positive function u and $b \ge 0$, then

$$\langle u, Lu\rangle_{L_2} - \langle u, L_{1,b} u\rangle_{L_2} = \int_M b|\mathrm{W}^+|u^2\, d\mu \ge 0.$$

The second inequality is a result of T Aubin [2] and R Schoen [46]. The last inequality is an immediate consequence of the fact that the standard metric on S^n is locally conformally flat ($\mathrm{W} = 0$). □

On a complete gradient shrinking soliton, the scalar curvature is positive unless the soliton is isometric to the flat Euclidean space [45]. Therefore, if the GRS is not flat then the existence of a solution to the Yamabe problem [35] implies that $Y_g > 0$. This observation is essential because of the following result.

Proposition 4.4 *Let (M, g) be a closed four-dimensional Riemannian manifold. If $Y(M) > 0$ and $\widehat{Y}(M) \le 0$, then there is a smooth metric $\tilde{g} = u^2 g$ such that*

$$\int_M \widetilde{\mathrm{S}}^2\, d\tilde{\mu} \le 216 \int_M |\widetilde{\mathrm{W}}^+|^2\, d\tilde{\mu}. \tag{4-5}$$

Furthermore, equality holds only if $\widehat{Y}(M) = 0$ and $\widetilde{\mathrm{S}} = 6\sqrt{6}|\widetilde{\mathrm{W}}|$.

Proof The proof is almost identical to that of [29, Proposition 3.5]. Thus, we provide a brief argument here. Through a conformal transformation, the Yamabe problem can be solved via a variational approach for an appropriate eigenvalue PDE problem. In particular, the existence of a solution under the assumption $Y(M) < Y(S^n)$ depends solely on the analysis of regularity of the Laplacian operator (but not on the reaction term); see [35, Theorem 4.5].

In our case, F conformally transforms as scalar curvature and Lemma 4.3 holds, so there exists a minimizer v for $\hat{Y}_g[\cdot]$ such that under normalization $\|v\|_{L_4} = 1$, the metric $\tilde{g} = v^2 g$ satisfies the equation $\tilde{F} = \tilde{\mathrm{S}} - 6\sqrt{6}|\widetilde{\mathrm{W}}^+| = \hat{Y}(M)$. Applying $Y(M) > 0$ and $\hat{Y}(M) \le 0$, we obtain

$$\begin{aligned}\int_M \tilde{\mathrm{S}}^2\, d\tilde{\mu} &= \int_M 6\sqrt{6}|\widetilde{\mathrm{W}}^+|\tilde{\mathrm{S}}\, d\tilde{\mu} + \hat{Y}(M)\int_M \tilde{\mathrm{S}}\, d\tilde{\mu} \\ &\le \int_M 6\sqrt{6}|\widetilde{\mathrm{W}}^+|\tilde{\mathrm{S}}\, d\tilde{\mu} \\ &\le 6\sqrt{6}\left(\int_M |\widetilde{\mathrm{W}}^+|^2\, d\tilde{\mu}\right)^{1/2}\left(\int_M |\tilde{\mathrm{S}}|^2\, d\tilde{\mu}\right)^{1/2}.\end{aligned}$$

Therefore,

$$\int_M \tilde{\mathrm{S}}^2\, d\tilde{\mu} \le 216\int_M |\widetilde{\mathrm{W}}^+|^2\, d\tilde{\mu}.$$

Equality is attained if only if $\tilde{g}$ attains the infimum, $\hat{Y}(M) = 0$ and $\tilde{\mathrm{S}} = 6\sqrt{6}|\widetilde{\mathrm{W}}|$. □

Proposition 4.5 *Let (M, g, f, λ) be a closed four-dimensional shrinking GRS satisfying (4-2) and $\mathrm{W}^+ \ne 0$. Then $\hat{Y}(M) \le 0$. Moreover, equality holds only if W^+ has the form $\omega\,\mathrm{diag}(-1,-1,2)$ for some $\omega \ge 0$ at each point.*

Proof By Theorem 1.1, we have

$$\Delta_f |\mathrm{W}^+|^2 = 2|\nabla \mathrm{W}^+|^2 + 4\lambda|\mathrm{W}^+|^2 - 36\det\nolimits_{\Lambda^2_+} \mathrm{W}^+ - \langle \mathrm{Rc}\circ\mathrm{Rc}, \mathrm{W}^+\rangle.$$

Integrating both sides and applying (4-2) yields

$$\int_M \Delta_f |\mathrm{W}^+|^2 d\mu \ge \int_M \left[2|\nabla \mathrm{W}^+|^2 + \left(\frac{\mathrm{S}}{3} + \Delta f\right)|\mathrm{W}^+|^2 - 36\det\nolimits_{\Lambda^2_+} \mathrm{W}^+\right].$$

Via integration by parts, we have

$$\int_M \nabla f(|\mathrm{W}^+|^2)\, d\mu = \int_M \langle \nabla f, \nabla|\mathrm{W}^+|^2\rangle\, d\mu = -\int_M \Delta f|\mathrm{W}^+|^2\, d\mu.$$

Therefore, we arrive at

$$0 \ge \int_M \left(2|\nabla \mathrm{W}^+|^2 + \frac{\mathrm{S}}{3}|\mathrm{W}^+|^2 - 36\det\nolimits_{\Lambda^2_+} \mathrm{W}^+\right).$$

We also have the pointwise estimates

$$|\nabla \mathrm{W}^+|^2 \ge |\nabla|\mathrm{W}^+||^2, \qquad -18\det \mathrm{W}^+ \ge -\sqrt{6}|\mathrm{W}^+|^3.$$

The first one is the classical Kato's inequality while the second one is purely algebraic. Thus, for $u = |\mathrm{W}^+|$,

$$\int_M \left(\tfrac{1}{3} F u^2 + 2|\nabla u|^2\right) d\mu \le 0.$$

Hence if $|\widetilde{\mathrm{W}}^+| > 0$ everywhere then the statement follows.

If $|\widetilde{\mathrm{W}}^+| = 0$ somewhere, let M_ϵ be the set of points at which $|\widetilde{\mathrm{W}}^+| < \epsilon$. By the analyticity of a closed GRS (see [3]), $\mathrm{Vol}(M_\epsilon) \to 0$ as $\epsilon \to 0$. Let $\eta_\epsilon \colon [0,\infty) \to [0,\infty)$ be a C^2 positive function which is $\epsilon/2$ on $[0,\epsilon/2]$, identity on $[\epsilon,\infty)$ and satisfies $0 \le \eta_\epsilon' \le 10$. If $u_\epsilon = \eta_\epsilon \circ u$, then u_ϵ is C^2 and positive. In addition, we have

- $\displaystyle\int_M F u_\epsilon^2 d\mu \le \int_{M-M_\epsilon} F u^2 d\mu + C\epsilon^2\,\mathrm{Vol}(M_\epsilon),$
- $\displaystyle\int_M |\nabla u_\epsilon|^2 d\mu = \int_M |\eta_\epsilon' \nabla u|^2 d\mu \le \int_{M-M_\epsilon} |\nabla u|^2 d\mu + C\,\mathrm{Vol}(M_\epsilon),$

where C is a constant depending on the metric. Therefore, we have

$$\inf_{\epsilon>0}\left\{\int_M (F u_\epsilon^2 + 6|\nabla u_\epsilon|^2)\, d\mu\right\} \le 0.$$

Consequently, $\widehat{Y}(M) \le 0$.

Now, equality holds only if $\int_M \left(\tfrac{1}{3} F u^2 + 2|\nabla u|^2\right) d\mu = 0$ and equality holds in each point-wise estimate above. The result then follows. □

We are now ready to prove the main result of this section.

Proof of Theorem 4.1 By Proposition 4.5, we have $\widehat{Y}(g) \le 0$ and $Y(M) > 0$. Otherwise $\mathrm{S} = 0$ and the GRS is flat by [45], which is a contradiction to $\mathrm{W}^+ \ne 0$. Therefore, by Proposition 4.4 there is a conformal transformation $\tilde{g} = u^2 g$ with

$$\int_M \widetilde{\mathrm{S}}^2\, d\tilde{\mu} \le 216 \int_M |\widetilde{\mathrm{W}}^+|^2\, d\tilde{\mu}. \tag{4-6}$$

According to (2-16) and (2-17),

$$\begin{aligned} 2\pi^2(2\chi(M) + 3\tau(M)) &= \int_M |\widetilde{\mathrm{W}}^+|^2\, d\tilde{\mu} - \frac{1}{4}\int_M |\widetilde{\mathrm{E}}|^2\, d\tilde{\mu} + \frac{1}{48}\int_M \widetilde{\mathrm{S}}^2\, d\tilde{\mu} \\ &\le \int_M |\widetilde{\mathrm{W}}^+|^2\, d\tilde{\mu} + \frac{1}{48}\int_M \widetilde{\mathrm{S}}^2\, d\tilde{\mu} \\ &\le \left(1 + \frac{9}{2}\right)\int_M |\widetilde{\mathrm{W}}^+|^2\, d\tilde{\mu}. \end{aligned} \tag{4-7}$$

Here we used (4-6) in the last step. Since $\|\mathrm{W}^+\|_{L_2}$ is conformally invariant, (4-3) then follows.

Now the equality holds only if all equalities hold in (4-7), (4-6) and (4-2). The first one implies that $\widetilde{g}$ is Einstein. Therefore, by [30, Theorem 1], inequality (4-6) is strict unless $\mathrm{S} \equiv 0$. But this is a contradiction to $Y(M) > 0$. Thus the inequality is strict. □

4.2 Isotropic curvature

Another application is the following inequality, which is an improvement of [49, Proposition 2.6].

Proposition 4.6 *Let (M, g, f, λ) be a four-dimensional GRS. Then we have*

$$\Delta_f u \leq \Big(2\lambda + \frac{3}{2}u - \mathrm{S}\Big)u - \frac{1}{4}|\mathrm{Rc}|^2 \tag{4-8}$$

in the distribution sense, where $u(x)$ is the smallest eigenvalue of $S/3 - 2\mathrm{W}_{\pm}$.

Proof Let $X_{1234} = \mathrm{S}/3 - 2\mathrm{W}(e_{12}+e_{34}, e_{12}+e_{34})$ for any 4–orthonormal basis. We use the normal form discussed in (2-13) and obtain a local frame by parallel translation along geodesic lines.

We denote by $\{\alpha_i\}_{i=1}^3$ the basis of Λ_2^+ as in (2-10) with corresponding eigenvalues $\lambda_i = a_i + b_i$. Without loss of generality we can assume $a_1 + b_1 \geq a_2 + b_2 \geq a_3 + b_3$ and thus $u(x) = X_{1234}(x)$. Using Lemma 3.1, we compute

$$\begin{aligned}
\Delta_f \mathrm{W}_{1212} = 2\lambda a_1 - 2(a_1^2 + b_1^2 + 2a_2a_3 + 2b_2b_3)& \\
- \frac{1}{2}\sum_p (\mathrm{Rc}_{1p}^2 + \mathrm{Rc}_{2p}^2) + \frac{\mathrm{S}}{2}(\mathrm{Rc}_{11} + \mathrm{Rc}_{12})& \\
- (\mathrm{Rc}_{11}\,\mathrm{R}_{22} - \mathrm{Rc}_{12}^2) - \frac{1}{6}(\mathrm{S}^2 - |\mathrm{Rc}|^2)&, \\
\Delta_f \mathrm{W}_{1234} = 2\lambda b_1 - 4(a_1b_1 + a_2b_3 + a_3b_2) + (\mathrm{Rc}_{13}\,\mathrm{Rc}_{24} - \mathrm{Rc}_{23}\,\mathrm{Rc}_{14})&.
\end{aligned}$$

Let us recall that $\Delta_f \mathrm{S} = 2\lambda\mathrm{S} - 2|\mathrm{Rc}|^2$. Thus, for $2T_1 = (\mathrm{Rc}\circ\mathrm{Rc})(\alpha_1, \alpha_1)$, we have

$$\begin{aligned}
\Delta_f(X_{1234}) &= 2\lambda\frac{\mathrm{S}}{3} - \tfrac{2}{3}|\mathrm{Rc}|^2 - 4\lambda(a_1 + b_1) + 4\lambda_1^2 + 8\lambda_2\lambda_3 + \tfrac{1}{6}(|\mathrm{Rc}|^2 - \mathrm{S}^2) + T_1 \\
&= 2\lambda X_{1234} - \tfrac{1}{2}|\mathrm{Rc}|^2 + 4\lambda_1^2 + 8\lambda_2\lambda_3 - \tfrac{1}{6}\mathrm{S}^2 + T_1.
\end{aligned}$$

Next we observe that $\lambda_2+\lambda_3=-\lambda_1$ and $8\lambda_2\lambda_3\le 2\lambda_1^2$. By the Cauchy–Schwartz inequality, $T_1\le\frac{1}{4}|\mathrm{Rc}|^2$. Therefore,

$$\begin{aligned}\Delta_f(X_{1234}) &\le 2\lambda X_{1234}-\tfrac{1}{4}|\mathrm{Rc}|^2+6\left(\frac{\frac{1}{3}\mathrm{S}-X_{1234}}{2}\right)^2-\tfrac{1}{6}\mathrm{S}^2\\ &\le 2\lambda X_{1234}+\tfrac{3}{2}X_{1234}^2-\mathrm{S}X_{1234}-\tfrac{1}{4}|\mathrm{Rc}|^2\\ &= u(2\lambda+\tfrac{3}{2}u-\mathrm{S})-\tfrac{1}{4}|\mathrm{Rc}|^2.\end{aligned}$$

Since $\Delta_f u\le\Delta_f(X_{1234})$ in the barrier sense of E Calabi (see [11]), the result then follows. □

5 A framework approach

In this section, we shall propose a framework to study interactions between components of the curvature operator and the potential function on a GRS (M,g,f,λ). In particular, we represent the divergence and the interior product $i_{\nabla f}$ on each curvature component as linear combinations of four operators P,Q,M,N. The geometry of these operators, in turn, gives us information about the original objects. It should be noted that some identities here have already appeared elsewhere.

Let (M^n,g) be an n–dimensional oriented Riemannian manifold. Using the point-wise induced inner product, any anti-symmetric (2,0) tensor α (a two-form) can be seen as an operator on the tangent space by

$$\alpha(X,Y)=\langle -\alpha(X),Y\rangle=\langle X,\alpha(Y)\rangle=\langle \alpha, X\wedge Y\rangle.$$

In particular, a bi-vector acts on a vector X by

$$(U\wedge V)X=\langle V,X\rangle U-\langle U,X\rangle V.$$

For instance, in dimension four, the complete description is given by the table below.

(5-1)

	$e_{12}+e_{34}$	$e_{13}-e_{24}$	$e_{14}+e_{23}$	$e_{12}-e_{34}$	$e_{13}+e_{24}$	$e_{14}-e_{23}$
e_1	$-e_2$	$-e_3$	$-e_4$	$-e_2$	$-e_3$	$-e_4$
e_2	e_1	e_4	$-e_3$	e_1	$-e_4$	e_3
e_3	$-e_4$	e_1	e_2	e_4	e_1	$-e_2$
e_4	e_3	$-e_2$	e_1	$-e_3$	e_2	e_1

In a similar manner, any symmetric $(2,0)$ tensor b can be seen as an operator on the tangent space,

$$b(X,Y)=\langle b(X),Y\rangle=\langle X,b(Y)\rangle.$$

Consequently, when b is viewed as a 1–form valued 1–form, $d_\nabla b$ denotes the exterior derivative (a 1–form valued 2–form). That is,

$$(d_\nabla b)(X,Y,Z)=(\nabla b)(X,Y,Z)+(-1)^1(\nabla b)(Y,X,Z)=\nabla_X b(Y,Z)-\nabla_Y b(X,Z).$$

Now we can define the fundamental tensors of interest to us here, first via a local frame and then using operator language. Let $\alpha \in \Lambda_2$, X, Y, $Z \in TM$, and $\{e_i\}_{i=1}^n$ be a local normal orthonormal frame on a GRS (M^n, g, f, λ).

Definition 5.1 The tensors $P, Q, M, N\colon \Lambda_2 TM \otimes TM \to \mathbb{R}$ are defined as

$$\begin{aligned}
P_{ijk} &= \nabla_i \mathrm{Rc}_{jk} - \nabla_j \mathrm{Rc}_{ik} = \nabla_j f_{ik} - \nabla_i f_{jk} = \mathrm{R}_{jikp}\nabla^p f, \qquad (5\text{-}2)\\
P(X\wedge Y, Z) &= -\mathrm{R}(X,Y,Z,\nabla f) = (d_\nabla \mathrm{Rc})(X,Y,Z) = \delta\mathrm{R}(Z,X,Y),\\
P(\alpha, Z) &= \mathrm{R}(\alpha, \nabla f\wedge Z) = \delta\,\mathrm{R}(Z,\alpha);\\
Q_{ijk} &= g_{ki}\nabla_j \mathrm{S} - g_{kj}\nabla_i \mathrm{S} = 2(g_{ki}\mathrm{R}_{jp} - g_{kj}\mathrm{R}_{ip})\nabla^p f, \qquad (5\text{-}3)\\
Q(X\wedge Y, Z) &= 2(X,Z)\,\mathrm{Rc}(Y,\nabla f) - 2(Y,Z)\,\mathrm{Rc}(X,\nabla f),\\
Q(\alpha, Z) &= -2\,\mathrm{Rc}(\alpha(Z),\nabla f) = -2\langle \alpha Z, \mathrm{Rc}(\nabla f)\rangle;\\
M_{ijk} &= \mathrm{R}_{kj}\nabla_i f - \mathrm{R}_{ki}\nabla_j f, \qquad (5\text{-}4)\\
M(X\wedge Y, Z) &= \mathrm{Rc}(Y,Z)\nabla_X f - \mathrm{Rc}(X,Z)\nabla_Y f = -\mathrm{Rc}((X\wedge Y)\nabla f, Z),\\
M(\alpha, Z) &= -\mathrm{Rc}(\alpha(\nabla f), Z) = -\langle \alpha\nabla f, \mathrm{Rc}(Z)\rangle;\\
N_{ijk} &= g_{kj}\nabla_i f - g_{ki}\nabla_j f, \qquad (5\text{-}5)\\
N(X\wedge Y, Z) &= \langle Y,Z\rangle\nabla_X f - \langle X,Z\rangle\nabla_Y f = \langle (X\wedge Y)Z, \nabla f\rangle,\\
N(\alpha, Z) &= \langle \alpha Z, \nabla f\rangle = -\alpha(Z,\nabla f).
\end{aligned}$$

Remark 5.1 The tensors $P^\pm, Q^\pm, M^\pm, N^\pm\colon \Lambda_2^\pm TM \otimes TM \to \mathbb{R}$ are defined by restricting $\alpha \in \Lambda_2^\pm TM$. They can be seen as operators on Λ_2 by standard projection.

Remark 5.2 Before proceeding further, let us remark on the essence of these tensors. $P \equiv 0$ if and only if the curvature is harmonic; $Q \equiv 0$ if and only if the scalar curvature is constant; $N \equiv 0$ if and only if the potential function f is constant; finally, $M \equiv 0$ if and only if either $\nabla f = 0$ or Rc vanishes on the orthogonal complement of ∇f.

5.1 Decomposition lemmas

Using the framework above, we now can represent the interior product $i_{\nabla f}$ on components of the curvature tensor as follows. Again the Einstein summation convention is used here.

Lemma 5.2 *Let (M, g, f, λ) be a GRS. For P, Q, M, N as in Definition 5.1, in a local normal orthonormal frame we have*

$$\mathrm{R}_{ijkp}\nabla^p f = \mathrm{R}(e_i, e_j, e_k, \nabla f) \tag{5-6}$$
$$= -P_{ijk} = \nabla^p \mathrm{R}_{ijkp} = -\delta\,\mathrm{R}(e_k, e_i, e_j),$$

$$(g \circ g)_{ijkp}\nabla^p f = (g \circ g)(e_i, e_j, e_k, \nabla f) = -2N_{ijk}, \tag{5-7}$$

$$(\mathrm{Rc} \circ g)_{ijkp}\nabla^p f = (\mathrm{Rc} \circ g)(e_i, e_j, e_k, \nabla f) = \tfrac{1}{2}Q_{ijk} - M_{ijk}, \tag{5-8}$$

$$\mathrm{H}_{ijkp}\nabla^p f = H(e_i, e_j, e_k, \nabla f) = M_{ijk} - \tfrac{1}{2}Q_{ijk} - 2\lambda N_{ijk}, \tag{5-9}$$

$$\mathrm{W}_{ijkp}\nabla^p f = \mathrm{W}(e_i, e_j, e_k, \nabla f) \tag{5-10}$$
$$= -P_{ijk} - \frac{Q_{ijk}}{2(n-2)} + \frac{M_{ijk}}{(n-2)} - \frac{\mathrm{S}N_{ijk}}{(n-1)(n-2)}.$$

Proof The first formula is well-known (see [16]), and follows from the soliton equation and Bianchi identities. For the second, we compute

$$(g \circ g)_{ijkp}\nabla^p f = 2(g_{ik}g_{jp} - g_{ip}g_{jk})\nabla^p f$$
$$= 2g_{ik}\nabla_j f - 2g_{jk}\nabla_i f = -2N_{ijk}.$$

For the third, we use (2-6) to calculate

$$(\mathrm{Rc} \circ g)_{ijkp}\nabla^p f = (\mathrm{Rc}_{ik}\, g_{jp} + \mathrm{Rc}_{jp}\, g_{ik} - \mathrm{Rc}_{ip}\, g_{jk} - \mathrm{Rc}_{jk}\, g_{ip})\nabla^p f$$
$$= \mathrm{Rc}_{ik}\,\nabla_j f + \tfrac{1}{2}(g_{ik}\nabla_j \mathrm{S} - g_{jk}\nabla_i \mathrm{S}) - \mathrm{Rc}_{jk}\,\nabla_i f$$
$$= \tfrac{1}{2}Q_{ijk} - M_{ijk}.$$

The next formula is a consequence of the above formulas, the definition of H in (2-18) and the soliton equation (2-3). Finally, the last one comes from the decomposition of the curvature operator (2-9) and the previous formulas; it appeared, for example, in [22]. □

In addition, the divergence on these components can be written as linear combinations of P, Q, M, N.

Lemma 5.3 *Let (M, g, f, λ) be a GRS. For P, Q, M, N as in Definition 5.1, in a local normal orthonormal frame we have*

$$\nabla^p \mathrm{R}_{ijkp} = -P_{ijk}, \tag{5-11}$$

$$\nabla^p(\mathrm{S}g \circ g)_{ijkp} = 2Q_{ijk}, \tag{5-12}$$

$$\nabla^p(\mathrm{Rc} \circ g)_{ijkp} = -\nabla^p \mathrm{H}_{ijkp} = -P_{ijk} + \tfrac{1}{2}Q_{ijk}, \tag{5-13}$$

(5-14) $$\nabla^p \mathrm{W}_{ijkp} = -\frac{n-3}{n-2} P_{ijk} - \frac{n-3}{2(n-1)(n-2)} Q_{ijk} := -\frac{n-3}{n-2} C_{ijk}.$$

Proof The first formula is well-known and comes from the second Bianchi identity (see [16]). For the second, we compute

$$\begin{aligned}\nabla^p (\mathrm{S} g \circ g)_{ijkp} &= 2\nabla^p (\mathrm{S} g_{ik} g_{jp} - \mathrm{S} g_{ip} g_{jk}) \\ &= 2 g_{ik} g_{jp} \nabla^p \mathrm{S} - g_{jk} g_{ip} \nabla^p \mathrm{S} \\ &= 2 g_{ik} \nabla_j \mathrm{S} - g_{jk} \nabla_i \mathrm{S} = 2 Q_{ijk}.\end{aligned}$$

For the next one, we use (2-5) to calculate

$$\begin{aligned}\nabla^p (\mathrm{Rc} \circ g)_{ijkp} &= \nabla^p (\mathrm{Rc}_{ik}\, g_{jp} + \mathrm{Rc}_{jp}\, g_{ik} - \mathrm{Rc}_{ip}\, g_{jk} - \mathrm{Rc}_{jk}\, g_{ip}) \\ &= g_{jp} \nabla^p \mathrm{Rc}_{ik} + g_{ik} \nabla^p \mathrm{Rc}_{jp} - g_{jk} \nabla^p \mathrm{Rc}_{ip} - g_{ip} \nabla^p \mathrm{Rc}_{jk} \\ &= \nabla_j \mathrm{Rc}_{ik} + \tfrac{1}{2}(g_{ik} \nabla_j \mathrm{S} - g_{jk} \nabla_i \mathrm{S}) - \nabla_i \mathrm{Rc}_{jk} \\ &= \tfrac{1}{2} Q_{ijk} - P_{ijk}.\end{aligned}$$

Finally, the last one comes from the decomposition of curvature (2-9) and previous formulas; it also appeared, for example, in [24, Equation (9)]. □

Remark 5.3 C as defined in (5-14) is also called the Cotton tensor in the literature.

Remark 5.4 By the standard projection, and

$$(\delta \mathrm{W})^{\pm} = \delta(\mathrm{W}^{\pm}), \qquad (i_{\nabla f} \mathrm{W})^{\pm} = i_{\nabla f} \mathrm{W}^{\pm},$$

the analogous identities hold replacing W, P, Q, M, N in Lemmas 5.2 and 5.3 by $\mathrm{W}^{\pm}$, $P^{\pm}$, $Q^{\pm}$, $M^{\pm}$, $N^{\pm}$, respectively.

The following observation is an immediate consequence of Lemma 5.3.

Proposition 5.4 *Let (M^n, g, f, λ), $n > 2$, be a GRS and let H be given by (2-18). Then the tensor*

$$F = \mathrm{W} + \frac{n-3}{n-2} \mathrm{H} + \frac{n(n-3)\mathrm{S}}{4(n-1)(n-2)} g \circ g$$

is divergence free.

Remark 5.5 The result can be viewed as a generalization of the harmonicity of the Weyl tensor on an Einstein manifold.

Lastly, we introduce the following tensor D which plays a crucial role in the classification problem (see [15; 14; 22]):

(5-15)
$$\begin{aligned} D_{ijk} &= -\frac{Q_{ijk}}{2(n-1)(n-2)} + \frac{M_{ijk}}{n-2} - \frac{\mathrm{S} N_{ijk}}{(n-1)(n-2)} \\ &= C_{ijk} + \mathrm{W}_{ijkp} \nabla^p f. \end{aligned}$$

5.2 Norm calculations

Lemma 5.5 *Let (M, g, f, λ) be a GRS. Then the following identities hold:*

$$\begin{aligned} 2\langle P, Q\rangle &= -|\nabla \mathrm{S}|^2, & 2\langle N, N\rangle &= 2(n-1)|\nabla f|^2, \\ 2\langle P, N\rangle &= \langle \nabla f, \nabla \mathrm{S}\rangle, & 2\langle Q, M\rangle &= |\nabla \mathrm{S}|^2 - 2\mathrm{S}\langle \nabla f, \nabla \mathrm{S}\rangle, \\ 2\langle Q, Q\rangle &= 2(n-1)|\nabla \mathrm{S}|^2, & 2\langle Q, N\rangle &= -2(n-1)\langle \nabla f, \nabla \mathrm{S}\rangle, \\ 2\langle M, M\rangle &= 2|\mathrm{Rc}|^2|\nabla f|^2 - \tfrac{1}{2}|\nabla \mathrm{S}|^2, & 2\langle M, N\rangle &= 2S|\nabla f|^2 - \langle \nabla f, \nabla \mathrm{S}\rangle. \end{aligned}$$

Furthermore, if M is closed, then

- $\displaystyle\int_M 2\langle P, P\rangle e^{-f} = \int_M |\nabla \mathrm{Rc}|^2 e^{-f}$,
- $\displaystyle\int_M 2\langle P, M\rangle = 2\int_M (\lambda|\mathrm{Rc}|^2 - \mathrm{Rc}^3) + \int_M \langle \nabla f, \nabla |\mathrm{Rc}|^2\rangle + \frac{1}{2}\int_M |\nabla \mathrm{S}|^2.$

Proof The main technique is to compute under a normal orthonormal local frame. For example,

$$\begin{aligned} 2\langle P, Q\rangle &= P_{ijk} Q_{ijk} \\ &= (\nabla_i \mathrm{Rc}_{jk} - \nabla_j \mathrm{Rc}_{ik})(g_{ki} \nabla_j \mathrm{S} - g_{kj} \nabla_i \mathrm{S}) \\ &= 2(\nabla_i \mathrm{Rc}_{jk} - \nabla_j \mathrm{Rc}_{ik}) g_{ki} \nabla_j \mathrm{S} \\ &= 2\nabla_j \mathrm{S}(\nabla_k \mathrm{Rc}_{kj} - \nabla_j \mathrm{Rc}_{kk}) \\ &= |\nabla \mathrm{S}|^2 - 2|\nabla \mathrm{S}|^2 = -|\nabla \mathrm{S}|^2. \end{aligned}$$

Other equations follow from similar calculations.

When M is closed, we can integrate by parts. In particular, the first equation was first derived in [16]. For the second, we compute that

$$\begin{aligned} \int_M 2\langle P, M\rangle &= 2\int_M (\nabla_i \mathrm{Rc}_{jk} - \nabla_j \mathrm{Rc}_{ik}) \mathrm{Rc}_{kj} \nabla_i f \\ &= \int_M \nabla_i f \nabla_i \mathrm{Rc}^2_{jk} - 2\int_M \nabla_j \mathrm{Rc}_{ik} \mathrm{Rc}_{kj} \nabla_i f, \end{aligned}$$

$$\int_M \nabla_j \operatorname{Rc}_{ik} \operatorname{Rc}_{kj} \nabla_i f = -\int_M \operatorname{Rc}_{ik} \operatorname{Rc}_{kj} f_{ij} - \int_M \operatorname{Rc}_{ik} f_i \nabla_j \operatorname{Rc}_{kj}$$
$$= -\int_M (\lambda |\operatorname{Rc}|^2 - \operatorname{Rc}^3) - \frac{1}{4}\int_M |\nabla \mathrm{S}|^2.$$

Hence, the statement follows. □

Remark 5.6 The factor of 2 is due to our convention of calculating the norm. Some special cases of dimension four also appeared in [10, Proposition 4].

An interesting consequence of the above calculation is the following corollary, which exposes the orthogonality of Q, N versus $i_{\nabla f}\mathrm{W}$, $\delta\mathrm{W}$.

Corollary 5.6 *Let (M, g, f, λ) be a GRS.*

(a) *At each point, we have*

$$0 = \langle Q, i_{\nabla f}\mathrm{W}\rangle = \langle N, i_{\nabla f}\mathrm{W}\rangle = \langle Q, \delta\mathrm{W}\rangle = \langle N, \delta\mathrm{W}\rangle.$$

(b) *If M is closed, then*

$$\int_M 2|\delta\mathrm{W}|^2 e^{-f} = \left(\frac{n-3}{n-2}\right)^2 \int_M \left(|\nabla \operatorname{Rc}|^2 - \frac{1}{(n-1)}|\nabla \mathrm{S}|^2\right) e^{-f}. \tag{5-16}$$

Proof Part (a) follows immediately from Lemmas 5.2, 5.3, 5.5, and our convention in Equation (2-1). For example,

$$\begin{aligned}
\langle Q, i_{\nabla f}\mathrm{W}\rangle &= \sum_{i<j} Q_{ijk}(i_{\nabla f}\mathrm{W})_{kij} \\
&= \sum_{i<j} Q_{ijk}\nabla^p f \mathrm{W}_{pkij} \\
&= -\sum_{i<j} Q_{ijk}\mathrm{W}_{ijkp}\nabla^p f \\
&= \left\langle Q, P + \frac{Q}{2(n-2)} - \frac{M}{n-2} + \frac{\mathrm{S}N}{(n-1)(n-2)}\right\rangle \\
&= -\frac{|\nabla \mathrm{S}|^2}{2} + \frac{(n-1)|\nabla \mathrm{S}|^2}{2(n-2)} - \frac{|\nabla \mathrm{S}|^2}{2(n-2)} + \frac{\mathrm{S}\langle \nabla f, \nabla \mathrm{S}\rangle}{n-2} - \frac{(n-1)\mathrm{S}\langle \nabla f, \nabla \mathrm{S}\rangle}{(n-1)(n-2)} \\
&= 0.
\end{aligned}$$

Other formulas follow from similar calculations.

For part (b) we observe that

$$|\delta \mathrm{W}|^2 = \left(\frac{n-3}{n-2}\right)^2 \left\langle P + \frac{Q}{2(n-1)}, P + \frac{Q}{2(n-1)} \right\rangle = \left(\frac{n-3}{n-2}\right)^2 \left\langle P + \frac{Q}{2(n-1)}, P \right\rangle.$$

Notice that we have applied part (a) in the last step. Consequently, applying Lemma 5.5 again yields

$$\begin{aligned} 2\int_M |\delta \mathrm{W}|^2 e^{-f} &= \left(\frac{n-3}{n-2}\right)^2 \int_M 2\left\langle P + \frac{Q}{2(n-1)}, P \right\rangle e^{-f} \\ &= \left(\frac{n-3}{n-2}\right)^2 \int_M \left(|\nabla \operatorname{Rc}|^2 - \frac{|\nabla \mathrm{S}|^2}{2(n-1)} \right) e^{-f}. \end{aligned}$$

□

Remark 5.7 Part (b) recovers the well-known fact that harmonic curvature implies harmonic Weyl tensor and constant scalar curvature.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 First, we observe

$$\begin{aligned} \langle \mathrm{W}, \operatorname{Hess} f \circ \operatorname{Hess} f \rangle &= \sum_{i<j;k<l} \mathrm{W}_{ijkl} (\operatorname{Hess} f \circ \operatorname{Hess} f)_{ijkl} \\ &= \frac{1}{2} \sum_{k<l;i,j} \mathrm{W}_{ijkl} (\operatorname{Hess} f \circ \operatorname{Hess} f)_{ijkl} \\ &= \sum_{k<l;i,j} \mathrm{W}_{ijkl} (f_{ik} f_{jl} - f_{il} f_{jk}) \\ &= \sum_{i,j,k,l} \mathrm{W}_{ijkl} f_{ik} f_{jl}. \end{aligned}$$

Next, subduing the summation notation, we integrate by parts,

$$\int_M \mathrm{W}_{ijkl} f_{ik} f_{jl} = -\int_M \nabla_i \mathrm{W}_{ijkl} f_k f_{jl} - \int_M \mathrm{W}_{ijkl} f_k \nabla_i f_{jl}.$$

The first term can be written as

$$\begin{aligned} \int_M \nabla_i \mathrm{W}_{ijkl} f_k f_{jl} &= \int_M \nabla_i \mathrm{W}_{ijkl} f_k (\lambda g_{jl} - \mathrm{Rc}_{jl}) \\ &= -\int_M \nabla_i \mathrm{W}_{ijkl} f_k \mathrm{Rc}_{jl} \\ &= -\frac{1}{2} \int_M (\delta \mathrm{W})_{jkl} M_{klj} \\ &= -\int_M \langle \delta \mathrm{W}, M \rangle. \end{aligned}$$

Next, we compute the second term,

$$\begin{aligned}\int_M \mathrm{W}_{ijkl} f_k \nabla_i f_{jl} &= -\int_M \mathrm{W}_{ijlk} f_k \nabla_i (g_{jl} - \mathrm{Rc}_{jl})\\ &= \int_M \mathrm{W}_{ijlk} f_k \nabla_i \mathrm{Rc}_{jl}\\ &= \frac{1}{2}\int_M \mathrm{W}_{ijlk} f_k P_{ijl}\\ &= -\int_M \left\langle i_{\nabla f}\mathrm{W}, P + \frac{Q}{2(n-1)}\right\rangle\\ &= -\frac{n-2}{n-3}\int_M \langle \delta \mathrm{W}, i_{\nabla f}\mathrm{W}\rangle\\ &= \frac{n-2}{n-3}\int_M \left\langle \delta\mathrm{W}, -P + \frac{M}{n-2}\right\rangle.\end{aligned}$$

Note that we have used Corollary 5.6 repeatedly to manipulate Q and N. To conclude, we combine the equations above to get

$$\begin{aligned}\int_M \mathrm{W}_{ijkl} f_{ik} f_{jl} &= \int_M \langle \delta \mathrm{W}, M\rangle - \frac{n-2}{n-3}\int_M \left\langle \delta\mathrm{W}, -P + \frac{M}{n-2}\right\rangle\\ &= \frac{1}{n-3}\int_M \langle \delta\mathrm{W}, (n-2)P + (n-4)M\rangle.\end{aligned}$$

If $n = 4$, then

$$\begin{aligned}\int_M \mathrm{W}_{ijkl} f_{ik} f_{jl} &= \int_M 2\langle \delta\mathrm{W}, P\rangle\\ &= \int_M 2\left\langle \delta\mathrm{W}, P + \frac{Q}{6}\right\rangle\\ &= \int_M 2\langle \delta\mathrm{W}, 2\delta\mathrm{W}\rangle\\ &= 4\int_M |\delta\mathrm{W}|^2.\end{aligned}$$

□

Remark 5.8 The formula in dimension four is also a consequence of the divergence-free property of the Bach tensor. We omit the details here.

Moreover, in dimension four, we have similar results for $\mathrm{W}^{\pm}$.

Lemma 5.7 *Let (M^4, g, f, λ) be a GRS. Then at each point, we have*

$$0 = \langle Q^{\pm}, i_{\nabla f}\mathrm{W}^{\pm}\rangle = \langle Q^{\pm}, \delta\mathrm{W}^{\pm}\rangle = \langle N^{\pm}, i_{\nabla f}\mathrm{W}^{\pm}\rangle = \langle N^{\pm}, \delta\mathrm{W}^{\pm}\rangle. \tag{5-17}$$

Proof It suffices to show the statement is true for the self-dual part.

Let $\{e_i\}_{i=1}^4$ be a normal orthonormal local frame, and let $\{\alpha_i\}_{i=1}^4$ be an orthonormal basis for Λ_2^+. Then

$$\begin{aligned}\langle Q^+, i_{\nabla f}\mathrm{W}^+\rangle &= \sum_i \sum_j Q(\alpha_i, e_j)\mathrm{W}(\nabla f \wedge e_j, \alpha_i) \\ &= -2\langle \alpha_i(e_j), \mathrm{Rc}(\nabla f)\rangle \mathrm{W}(\nabla f \wedge e_j, \alpha_i).\end{aligned}$$

Furthermore, we can choose a special basis, namely the normal form as in (2-13). Then the α_i diagonalize W^+ with eigenvalues λ_i. Consequently,

$$\mathrm{W}(\nabla f \wedge e_j, \alpha_i) = \lambda_i \alpha_i(\nabla f \wedge e_j) = \lambda_i \langle \nabla f, \alpha_i(e_j)\rangle.$$

It follows that

$$\begin{aligned}\langle Q^+, i_{\nabla f}\mathrm{W}^+\rangle &= -2\lambda_i \langle \alpha_i(e_j), \mathrm{Rc}(\nabla f)\rangle\langle \alpha_i(e_j), \nabla f\rangle \\ &= -2\eta_k \langle e_k, \mathrm{Rc}(\nabla f)\rangle \langle e_k, \nabla f\rangle,\end{aligned}$$

where

$$\eta_k = \sum_{i,j\,:\,\alpha_i(e_j) = \pm e_k} \lambda_i.$$

Now by (5-1), it is easy to see that each $\eta_k = 0$ because W^+ is traceless.

Next, we state the following fact.

Claim $$\langle P^+, Q^+\rangle = -\tfrac{1}{4}|\nabla \mathrm{S}|^2.$$

To prove this claim, we choose $\{\alpha_i\}$ as in (2-10) and observe that

$$\begin{aligned}P(\alpha_1, e_j)Q(\alpha_1, e_j) &= \tfrac{1}{2}P(e_{12}+e_{34}, e_j)Q(e_{12}+e_{34}, e_j) \\ &= -(P_{12j}+P_{34j})\langle (e_{12}+e_{34})e_j, \mathrm{Rc}(\nabla f)\rangle \\ &= -(\nabla_1 \mathrm{Rc}_{2j} - \nabla_2 \mathrm{Rc}_{1j} + \nabla_3 \mathrm{Rc}_{4j} - \nabla_4 \mathrm{Rc}_{3j}) \\ &\qquad \times \langle (e_{12}+e_{34})e_j, \mathrm{Rc}(\nabla f)\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}P(\alpha_2, e_j)Q(\alpha_2, e_j) &= -(\nabla_1 \mathrm{Rc}_{3j} - \nabla_3 \mathrm{Rc}_{1j} - \nabla_2 \mathrm{Rc}_{4j} + \nabla_4 \mathrm{Rc}_{2j}) \\ &\qquad \times \langle (e_{13}-e_{24})e_j, \mathrm{Rc}(\nabla f)\rangle, \\ P(\alpha_3, e_j)Q(\alpha_3, e_j) &= -(\nabla_1 \mathrm{Rc}_{4j} - \nabla_4 \mathrm{Rc}_{1j} + \nabla_2 \mathrm{Rc}_{3j} - \nabla_3 \mathrm{Rc}_{2j}) \\ &\qquad \times \langle (e_{14}+e_{23})e_j, \mathrm{Rc}(\nabla f)\rangle.\end{aligned}$$

Thus,

$$\langle P^+, Q^+\rangle = \sum_{i,j} P(\alpha_i, e_j)Q(\alpha_i, e_j) = -\sum_k \zeta_k \langle e_k, \mathrm{Rc}(\nabla f)\rangle,$$

for

$$\zeta_k = \sum_{i,j\,:\,\alpha_i(e_j)=e_k} \sqrt{2}P(\alpha_i, e_j) \;-\; \sum_{i,j\,:\,\alpha_i(e_j)=-e_k} \sqrt{2}P(\alpha_i, e_j).$$

Using (5-1), we can compute

$$\begin{aligned}
\zeta_1 &= \sqrt{2}(P(\alpha_1, e_2) + P(\alpha_2, e_3) + P(\alpha_3, e_4))\\
&= \nabla_1 \mathrm{Rc}_{22} - \nabla_2 \mathrm{Rc}_{12} + \nabla_3 \mathrm{Rc}_{42} - \nabla_4 \mathrm{Rc}_{32}\\
&\qquad\qquad + \nabla_1 \mathrm{Rc}_{33} - \nabla_3 \mathrm{Rc}_{13} - \nabla_2 \mathrm{Rc}_{43} + \nabla_4 \mathrm{Rc}_{23}\\
&\qquad\qquad\qquad\qquad + \nabla_1 \mathrm{Rc}_{44} - \nabla_4 \mathrm{Rc}_{14} + \nabla_2 \mathrm{Rc}_{34} - \nabla_3 \mathrm{Rc}_{24}\\
&= \nabla_1(\mathrm{S} - \mathrm{Rc}_{11}) - (\tfrac{1}{2}\nabla_1 \mathrm{S} - \nabla_1 \mathrm{Rc}_{11})\\
&= \tfrac{1}{2}\nabla_1 \mathrm{S}.
\end{aligned}$$

Similarly we have $\zeta_k = \frac{1}{2}\nabla_k \mathrm{S}$. We also have $\mathrm{Rc}(\nabla f) = \frac{1}{2}\nabla \mathrm{S}$. This proves our claim.

In addition, it is easy to see that

$$\langle Q^+, Q^+\rangle = \tfrac{3}{2}|\nabla \mathrm{S}|^2.$$

Since $\delta \mathrm{W}^+ = \frac{1}{2}P^+ + \frac{1}{12}Q^+$, it follows that

$$\langle Q^+, \delta \mathrm{W}^+\rangle = 0.$$

The statements involving N follow from analogous calculations as

$$N(\alpha_i, e_j) = \langle \alpha_i(e_j), \nabla f\rangle. \qquad \square$$

By manipulation as in the proof of Theorem 1.2, using Remark 5.4 (replacing Lemmas 5.2 and 5.3) and Lemma 5.7 (replacing Corollary 5.6), we immediately obtain the following result.

Corollary 5.8 *Let (M, g, f, λ) be a four-dimensional closed GRS. Then we have the identity*

$$\int_M \langle \mathrm{W}^+, \mathrm{Rc}\circ\mathrm{Rc}\rangle = 4\int_M |\delta \mathrm{W}^+|^2. \tag{5-18}$$

6 Rigidity results

In this section we present conditions that imply the rigidity of a GRS, using the analysis on the framework discussed in the previous section.

First, Proposition 6.10 provides a geometric way to understand the tensor D defined in Equation (5-15). In particular, it says that $D \equiv 0$ is equivalent to a special condition; namely, the normalization of ∇f (if not trivial) is an eigenvector of the Ricci tensor, and all other eigenvectors have the same eigenvalue. Such a structure will imply rigidity as the geometry of the level surface (of f) is well-described.

On the other hand, Theorem 1.2 reveals an interesting connection between the Ricci tensor and the Weyl tensor in dimension four. That allows us to obtain rigidity results using only the structure of the Ricci curvature for a GRS.

Theorem 6.1 *Let (M^4, g, f, λ) be a closed four-dimensional GRS. Assume that at each point the Ricci curvature has one eigenvalue of multiplicity one and another of multiplicity three. Then the GRS is rigid, hence Einstein.*

We also find conditions that imply the vanishing of the tensor D.

Theorem 6.2 *Let (M^n, g, f, τ), $n > 3$, be a GRS. Assume that one of these conditions holds:*

(1) $i_{\nabla f}\,\mathrm{Rc} \circ g \equiv 0$.

(2) $i_{\nabla f}\mathrm{W} \equiv 0$ *and* $\delta\mathrm{W}(\cdot, \cdot, \nabla f) = 0$.

Then at points where $\nabla f \neq 0$, we have $D = 0$.

Remark 6.1 $D \equiv 0$ can be derived from other conditions such as the vanishing of the Bach tensor (see [15, Lemma 4.1]).

Remark 6.2 For GRSs, condition (2) is a slight improvement of [20] which characterizes generalized quasi-Einstein manifolds under the assumption $\delta\mathrm{W} = i_{\nabla f}\mathrm{W} = 0$.

In dimension four, the result can be improved significantly.

Theorem 6.3 *Let (M, g, f, λ) be a four-dimensional GRS. At points where $\nabla f \neq 0$, then $\mathrm{W}^+(\nabla f, \cdot, \cdot, \cdot) = 0$ implies $\mathrm{W}^+ = 0$.*

As discussed in the last section, there are some similarities between taking the divergence and interior product $i_{\nabla f}$ of the Weyl tensor; for example, see Corollary 5.6. The following theorem is inspired by condition (1) of Theorem 6.2.

Theorem 6.4 *Let (M^n, g, f, τ), $n > 3$, be a GRS. Then $\delta(\mathrm{Rc} \circ g) \equiv 0$ if and only if the Weyl tensor is harmonic and the scalar curvature is constant.*

As an immediate consequence of the results above (plus known classifications discussed in the introduction) we obtain rigidity results.

Corollary 6.5 *Let (M^n, g, f, λ), $n \geq 4$, be a complete shrinking GRS.*

(i) *If $i_{\nabla f} \mathrm{Rc} \circ g \equiv 0$, then (M^n, g, f, λ) is Einstein.*
(ii) *If $i_{\nabla f}\mathrm{W} = 0$ and $\delta \mathrm{W}(\cdot, \cdot, \nabla f) = 0$, then (M^n, g, f, λ) is rigid of rank $k = 0, 1, n$.*
(iii) *If $\delta(\mathrm{Rc} \circ g) = 0$, then (M^n, g, f, λ) is rigid of rank $0 \leq k \leq n$.*

In particular, when the dimension is four, we have the following result.

Corollary 6.6 *Let (M, g, f, λ) be a four-dimensional complete GRS. If*

$$\mathrm{W}^+(\nabla f, \cdot, \cdot, \cdot) = 0,$$

then the GRS is either Einstein or has $\mathrm{W}^+ = 0$. Furthermore, in the second case, it is isometric to a Bryant soliton or Ricci flat manifold if $\lambda = 0$, or it is a finite quotient of $\mathbb{R}^4$, $S^3 \times \mathbb{R}$, S^4 or $\mathbb{CP}^2$ if $\lambda > 0$.

The general strategy for proving the aforementioned statements is to use the framework to study the structure of the Ricci tensor.

6.1 Eigenvectors of the Ricci curvature

Here we study various interconnections between the eigenvectors of the Ricci curvature, the Weyl tensor, and the potential function. We begin with a lemma.

Lemma 6.7 *Let (M, g) be a Riemannian manifold. Assume that at each point the Ricci curvature has one eigenvalue of multiplicity one and another of multiplicity $n-1$. Then we have*

$$\langle \mathrm{W}, \mathrm{Rc} \circ \mathrm{Rc} \rangle = 0.$$

Proof Without loss of generality, we can choose a basis $\{e_i\}_{i=1}^n$ of T_pM consisting of eigenvectors of Rc, namely $\mathrm{Rc}_{11} = \eta$ and $\mathrm{Rc}_{ii} = \zeta$ for $i = 2, \dots, n$. Then

$$\langle \mathrm{W}, \mathrm{Rc} \circ \mathrm{Rc} \rangle = \sum_{i<j;k<l} \mathrm{W}_{ijkl} \,\mathrm{Rc}_{ik}\, \mathrm{Rc}_{jl} \tag{6-1}$$

$$= \sum_{i<j} \mathrm{W}_{ijij}\, \mathrm{Rc}_{ii}\, \mathrm{Rc}_{jj} = \eta\zeta \sum_j \mathrm{W}_{1j1j} + \zeta^2 \sum_{1<i<j} \mathrm{W}_{ijij}. \tag{6-2}$$

We observe that

$$\sum_{j>1} \mathrm{W}_{ijij} = -\mathrm{W}_{1i1i}, \tag{6-3}$$

$$2 \sum_{1<i<j} \mathrm{W}_{ijij} = \sum_{i>1} \sum_{j>1} \mathrm{W}_{ijij} = -\sum_i \mathrm{W}_{1i1i} = 0. \tag{6-4}$$

The result then follows. □

Next, a consequence of our previous framework (on P, Q, M, and N) is the following characterization of the condition $\mathrm{Rc}(\nabla f) = \mu \nabla f$.

Lemma 6.8 *Let (M, g, f, λ) be a GRS. Then the following are equivalent:*

(1) $\mathrm{Rc}(\nabla f) = \mu \nabla f$
(2) $Q(\cdot, \cdot, \nabla f) = 0$
(3) $M(\cdot, \cdot, \nabla f) = 0$
(4) $\delta \mathrm{W}(\nabla f, \cdot, \cdot) = 0$
(5) $\delta \mathrm{H}(\nabla f, \cdot, \cdot) = 0$

Proof We'll show that $(1) \leftrightarrow (2)$, $(1) \leftrightarrow (3)$, $(2) \leftrightarrow (4)$, and $(2) \leftrightarrow (5)$.

$(2) \to (1)$: Let $\alpha \in \Lambda_2$, we have $0 = Q(\alpha, \nabla f) = -2\langle \alpha(\nabla f), \mathrm{Rc}(\nabla f)\rangle$. Since α can be arbitrary, $\alpha(\nabla f)$ can realize any vector in the complement of ∇f in TM. Therefore, $\mathrm{Rc}(\nabla f) = \mu \nabla f$.

$(1) \to (2)$: $Q(\alpha, \nabla f) = -2\langle \alpha(\nabla f), \mathrm{Rc}(\nabla f)\rangle = -2\langle \alpha(\nabla f), \mu \nabla f\rangle = 0$ because $\alpha(\nabla f) \perp \nabla f$.

That $(1) \leftrightarrow (3)$ follows from an identical argument.

That $(2) \leftrightarrow (4)$ follows from

$$\delta \mathrm{W}(X, Y, Z) = \frac{n-3}{n-2} P(Y, Z, X) + \frac{n-3}{2(n-1)(n-2)} Q(Y, Z, X),$$
$$P(Y, Z, \nabla f) = -\mathrm{R}(Y, Z, \nabla f, \nabla f) = 0.$$

That (2) $\leftrightarrow$ (5) follows from

$$\begin{aligned}\delta \mathrm{H}(X,Y,Z) &= -P(Y,Z,X) + \tfrac{1}{2}Q(Y,Z,X),\\ P(Y,Z,\nabla f) &= -\mathrm{R}(Y,Z,\nabla f,\nabla f) = 0.\end{aligned}$$ □

Furthermore, the rigidity of these operators Q, M, N is captured by the following result.

Proposition 6.9 *Let (M^n, g, f, τ), $n > 3$, be a GRS and let $T = aQ + bM + cN$ for some real numbers a, b, c.*

(i) *Assume that $T \equiv 0$. If $a \neq 0$ then $\mathrm{Rc}(\nabla f) = \mu \nabla f$. Moreover, if $\nabla f \neq 0$ and $b \neq 0$, then all other eigenvectors must have the same eigenvalue.*

(ii) *In dimension four, if $T_{|\Lambda_2^+ \otimes TM} \equiv 0$ then $T \equiv 0$.*

Proof Let $\{e_i\}_{i=1}^n$ be an orthonormal basis which consists of eigenvector of Rc with corresponding eigenvalues λ_i. Then we have

$$\begin{aligned}T(\alpha, e_i) &= aQ(\alpha, e_i) + bM(\alpha, e_i) + cN(\alpha, e_i)\\ &= -2a\langle \alpha(e_i), \mathrm{Rc}(\nabla f)\rangle - b\langle \alpha(\nabla f), \mathrm{Rc}(e_i)\rangle + c\langle \alpha(e_i), \nabla f\rangle\\ &= -2a\langle \alpha(e_i), \mathrm{Rc}(\nabla f)\rangle + b\langle \nabla f, \alpha(\lambda_i e_i)\rangle + c\langle \alpha(e_i), \nabla f\rangle\\ &= \langle \alpha(e_i), -2a\,\mathrm{Rc}(\nabla f) + b\lambda_i \nabla f + c\nabla f\rangle.\end{aligned} \tag{6-5}$$

(i) Without loss of generality, we can assume $\nabla f \neq 0$. Since $T(\alpha, e_i) = 0$ for arbitrary α and e_i,

$$T(\alpha, \nabla f) = 0 = \langle \alpha(\nabla f), \mathrm{Rc}(\nabla f)\rangle = Q(\alpha, \nabla f).$$

By Lemma 6.8, $e_1 = \nabla f / |\nabla f|$ is an eigenvector of Rc. Plugging into (6-5) yields

$$T(\alpha, e_i) = (-2a\lambda_1 + b\lambda_i + c)\langle \alpha(e_i), \nabla f\rangle.$$

Therefore, $-2a\lambda_1 + b\lambda_i + c = 0$. Hence, as $b \neq 0$, all other eigenvectors have the same eigenvalue.

(ii) In dimension four, fix a unit vector e_i and note that $T(\alpha, e_i) = 0$ for any $\alpha \in \Lambda_2^+$. By Lemma 2.2 and Remark 2.4, $T(\beta, e_i) = 0$ for all $\beta \in \Lambda_2^-$. As e_i is arbitrary the result then follows. □

Recall that the tensor D is a special linear combination of M, N, Q. Therefore, we obtain the following geometric characterization.

Proposition 6.10 *Let (M^n, g), $n > 3$, be a Riemannian manifold and D be defined as in (5-15). Then the following are equivalent:*

(1) $D \equiv 0$.

(2) *The Weyl tensor under the conformal change $\tilde{g} = e^{-2f/(n-2)} g$ is harmonic.*

(3) *Either $\nabla f = 0$ and the Cotton tensor $C_{ijk} = 0$, or ∇f is an eigenvector of* Rc *and all other eigenvectors have the same eigenvalue.*

Proof We shall show $(1) \leftrightarrow (2)$, $(1) \to (3)$ and $(3) \to (1)$.

$(1) \leftrightarrow (2)$: By Equation (5-15) and (5-14), we have

$$D_{ijk} = C_{ijk} + \mathrm{W}_{ijkp} \nabla^p f = \frac{n-2}{n-3} (\delta \mathrm{W})_{kij} - \mathrm{W}(\nabla f, e_k, e_i, e_j).$$

Thus, $D \equiv 0$ is equivalent to

$$\delta \mathrm{W}(X, Y, Z) - \frac{n-3}{n-2} \mathrm{W}(\nabla f, X, Y, Z) = 0.$$

Under the conformal transformation $\tilde{g} = u^2 g$ (see the appendix), $\widetilde{\mathrm{W}} = u^2 \mathrm{W}$, and

$$\delta \widetilde{\mathrm{W}}(X, Y, Z) = \delta \mathrm{W}(X, Y, Z) + (n-3) \mathrm{W}\Big(\frac{\nabla u}{u}, X, Y, Z\Big).$$

The result then follows from the last two equations.

The statement $(1) \to (3)$ follows from [15, Proposition 3.2 and Lemma 4.2].

$(3) \to (1)$: For all a, b, c, let $T = aQ + bM + cN$. For any $\alpha \in \Lambda_2$ and e_i a unit tangent vector, by (6-5) we have

$$T(\alpha, e_i) = \langle \alpha(e_i), -2a \operatorname{Rc}(\nabla f) + b \lambda_i \nabla f + c \nabla f \rangle.$$

For the tensor D,

$$a = \frac{-1}{2(n-1)(n-2)}, \qquad b = \frac{1}{n-2}, \qquad c = \frac{-\mathrm{S}}{(n-1)(n-2)}.$$

If $\nabla f = 0$ then $T \equiv 0$, hence $D \equiv 0$. If $\nabla f \neq 0$, then there exist eigenvectors $e_1 = \nabla f / |\nabla f|$ and $\{e_i\}_{i=2}^n$ of Rc, with eigenvalues ζ and η respectively. Then

$$T(\alpha, e_i) = \langle \alpha(e_i), (-2a\zeta + b\eta + c) \nabla f \rangle.$$

Since $\zeta + (n-1)\eta = \mathrm{S}$, with values of a, b, c given above, it follows that $-2a\zeta + b\eta + c = 0$. Thus, $D \equiv 0$. □

Remark 6.3 Note that our formulas differ from [24, Section 2, Equation (19)] by a sign convention.

Remark 6.4 Under that conformal change of the metric, the Ricci tensor is given by

$$\begin{aligned}\widetilde{\mathrm{Rc}} &= \mathrm{Rc} + \mathrm{Hess}\, f + \frac{1}{n-2}\, df \otimes df + \frac{1}{n-2}(\Delta f - |\nabla f|^2)g \\ &= \frac{1}{n-2}\, df \otimes df + \frac{1}{n-2}(\Delta f - |\nabla f|^2 + (n-2)\lambda)g.\end{aligned}$$

Therefore, at each point, $\widetilde{\mathrm{Rc}}$ has at most two eigenvalues. Furthermore, since $\tilde{g}$ has harmonic Weyl tensor, its Schouten tensor

$$\widetilde{\mathrm{Sc}} = \frac{1}{n-2}\Big(\widetilde{\mathrm{Rc}} - \frac{1}{2(n-1)}\tilde{\mathrm{S}}\tilde{g}\Big)$$

is a Codazzi tensor with at most two eigenvalues. Using the splitting results for Riemannian manifolds admitting such a tensor gives another proof of results in [15]. This method is inspired by [20].

Now we investigate several conditions which will imply that $\mathrm{Rc}(\nabla f) = \mu\nabla f$.

Proposition 6.11 *Let (M^n, g, f, τ), $n > 3$, be a GRS. Assume that one of these conditions holds:*

(1) $i_{\nabla f}\mathrm{W} \equiv 0$.
(2) $\delta\mathrm{W}^+ = 0$ *if* $n = 4$.

Then $\mathrm{Rc}(\nabla f) = \mu\nabla f$.

Proof The idea is to find a connection of each condition with Lemma 6.8.

Assuming (1): We claim that $\delta\mathrm{W}(\nabla f, \cdot, \cdot) = 0$.

Choosing a normal local frame $\{e_i\}_{i=1}^n$, we have

$$\begin{aligned}\delta\mathrm{W}(\nabla f, e_k, e_l) &= \sum_i (\nabla_i \mathrm{W})(e_i, \nabla f, e_k, e_l) \\ &= \sum_i \nabla_i \mathrm{W}(e_i, \nabla f, e_k, e_l) - \sum_i \mathrm{W}(e_i, \nabla_i\nabla f, e_k, e_l) \\ &= 0 - \mathrm{W}(\mathrm{Hess}\, f, e_k, e_l).\end{aligned}$$

Since $\mathrm{Hess}\, f$ is symmetric and W is anti-symmetric, $\delta\mathrm{W}(\nabla f, \cdot, \cdot) = 0$. The result then follows.

Assuming (2): First recall

$$\delta\mathrm{W}(X, Y, Z) = \tfrac{1}{2}C(Y, Z, X) = \tfrac{1}{2}P(Y \wedge Z, X) + \tfrac{1}{12}Q(Y \wedge Z, X).$$

For all $\alpha \in \Lambda^2_+$, since

$$\delta \mathrm{W}^-(X,\alpha) = \nabla_i \mathrm{W}^-(e_i \wedge X, \alpha) = 0,$$

we have

$$\delta(\mathrm{W})(X,\alpha) = \delta(\mathrm{W}^+)(X,\alpha) = \tfrac{1}{2} P(\alpha, X) + \tfrac{1}{12} Q(\alpha, X).$$

Since $0 = \mathrm{R}(Y, Z, \nabla f, \nabla f) = -P(Y \wedge Z, \nabla f)$ and $\delta \mathrm{W}^+ = 0$, we get $Q(\alpha, \nabla f) = 0$. The desired statement follows from Lemmas 2.2 and 6.8. □

6.2 Proofs of rigidity theorems

Proof of Theorem 6.1 By Lemma 6.7, we have

$$\int_M \mathrm{W}(\mathrm{Rc} \circ \mathrm{Rc}) = 0.$$

Theorem 1.2, therefore, implies that $\delta \mathrm{W} \equiv 0$. Then by the rigidity result for harmonic Weyl tensor discussed in the introduction, the result follows. □

Proof of Theorem 6.2 Assuming (1): We observe that

$$\mathrm{Rc} \circ g(X, Y, Z, \nabla f) = \tfrac{1}{2} Q(X, Y, Z) - M(X, Y, Z).$$

Therefore, the result follows from Proposition 6.9 and Proposition 6.10.

Assuming (2): By Proposition 6.11, $e_1 = \nabla f / |\nabla f|$ is a unit eigenvector. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of Rc with eigenvalues λ_i. By (5-10) and $\mathrm{W}(\nabla f, \cdot, \cdot, \cdot) = 0$,

$$P = -\frac{Q}{2(n-2)} + \frac{M}{(n-2)} - \frac{\mathrm{S} N}{(n-1)(n-2)}.$$

Therefore,

$$\begin{aligned} P(i,j,k) &= \frac{|\nabla f|}{n-2}\Big[\lambda_1(\delta_{jk}\delta_{1i} - \delta_{ik}\delta_{j1}) - \lambda_k(\delta_{j1}\delta_{ik} - \delta_{i1}\delta_{jk}) \\ &\qquad\qquad - \frac{S}{n-1}(\delta_{jk}\delta_{1i} - \delta_{ik}\delta_{j1})\Big] \\ &= \frac{|\nabla f|}{n-2}(\delta_{jk}\delta_{1i} - \delta_{ik}\delta_{j1})\Big(\lambda_1 + \lambda_k - \frac{S}{n-1}\Big). \end{aligned} \tag{6-6}$$

Using the assumption $\delta \mathrm{W}(\cdot, \cdot, \nabla f) = 0$, we obtain that

$$\Big(P + \frac{1}{2(n-1)} Q\Big)(\nabla f, \cdot, \cdot) = 0.$$

Combining with (6-6) yields

$$P(1,k,k) = -\frac{1}{2(n-1)}Q(1,k,k) = \frac{\lambda_1|\nabla f|}{(n-1)} = \frac{|\nabla f|}{n-2}\Big(\lambda_1+\lambda_k-\frac{S}{n-1}\Big).$$

Thus $\lambda_2=\lambda_3=\lambda_4=(S-\lambda_1)/(n-1)$. Proposition 6.10 then concludes the argument. □

The proof of Theorem 6.4 follows from a similar argument.

Proof of Theorem 6.4 By Equation (5-13), $\delta(\mathrm{Rc}\circ g)=0$ implies $P-\frac{1}{2}Q=0$. Thus, by Lemma 5.5,

$$2|P|^2 = 2\Big\langle P,\frac{Q}{2}\Big\rangle = -\frac{|\nabla \mathrm{S}|^2}{2}.$$

Hence $P=0=\nabla \mathrm{S}$. It then follows from Corollary 5.6 that $\delta \mathrm{W}=\delta \mathrm{S}=0$. The converse is obvious. □

Proof of Theorem 6.3 Using a normal local frame, we can rewrite the assumption as

$$\sum_i f_i \mathrm{W}^+_{ijkl}=0.$$

We pick an arbitrary index a and multiply both sides with W_{ajkl} to arrive at

$$\sum_i f_i \mathrm{W}^+_{ijkl}\mathrm{W}^+_{ajkl}=0.$$

Applying identity (2-14) yields

$$\begin{aligned}0&=\sum_{jkl}\sum_i f_i \mathrm{W}^+_{ijkl}\mathrm{W}^+_{ajkl}\\&=\sum_i f_i\sum_{jkl}\mathrm{W}^+_{ijkl}\mathrm{W}^+_{ajkl}\\&=\sum_i f_i|\mathrm{W}^+|^2 g_{ia}=f_a|\mathrm{W}^+|^2.\end{aligned}$$

Since the index a is arbitrary, we have $\nabla f=0$ or $|\mathrm{W}^+|=0$. □

Proof of Corollary 6.5 By Theorems 6.2 and 6.4, each condition implies $D\equiv 0$. Then [15, Lemma 4.2] further implies that $\delta \mathrm{W}=0$. It follows, from classification results for harmonic Weyl tensor as discussed in the introduction, that the manifold must be rigid. We now look at each case closely and observe that not all ranks can arise.

(i) In this case, Proposition 6.9 reveals that $\lambda_0 - \lambda_i = 0$, where $\mathrm{Rc}(\nabla f) = \lambda_0 \nabla_f$, and λ_i is any other eigenvalue of Rc. Therefore, the manifold structure must be Einstein.

(ii) In this case, $D \equiv 0$ implies Rc has at most two eigenvalues with one of multiplicity 1 and another of multiplicity $n-1$. So k can only be $0, 1$ or n.

(iii) In this case, there is no obvious obstruction, so all ranks can arise. □

Proof of Corollary 6.6 The statement follows immediately from Theorem 6.3, [22, Theorems 1.1, 1.2], and the analyticity of a GRS with bounded curvature [3]. □

Appendix

In this appendix, we collect a few formulas that are related to this paper; they follow from direct computation.

A.1 Conformal change calculation

In this subsection, we state the change of covariant derivative of the Weyl tensor and Bochner–Weitzenböck type formula, with respect to the conformal transformation of a metric.

We first fix our notation. Let (M^n, g) be a smooth Riemannian manifold and $u = e^f$ be a smooth positive function on M. A conformal change is defined by

$$\tilde{g} = e^{2f} = u^2 g. \tag{A-1}$$

Then, for any tensor $\mathfrak{D}$ with respect to g, the corresponding tensor for $\tilde{g}$ is denoted by $\widetilde{\mathfrak{D}}$.

We can calculate the transformation of the covariant derivative. For fixed X, Y, Z,

$$\begin{aligned}
2e^{2f}(\widetilde{\nabla}_X Y, Z)_g &= 2(\widetilde{\nabla}_X Y, Z)_{\tilde{g}} \\
&= X(Y, Z)_{\tilde{g}} + Y(Z, X)_{\tilde{g}} - Z(X, Y)_{\tilde{g}} \\
&\qquad\qquad - (Y, [X, Z])_{\tilde{g}} - (Z[Y, X])_{\tilde{g}} + (X[Z, Y])_{\tilde{g}} \\
&= 2X(f)e^{2f}(Y, Z)_g + 2Y(f)e^{2f}(Z, X)_g \\
&\qquad\qquad - 2Z(f)e^{2f}(X, Y)_g + 2e^{2f}(\nabla_X Y, Z)_g.
\end{aligned}$$

Thus,

$$\widetilde{\nabla}_X Y = \nabla_X Y + X(f)Y + Y(f)X - (X, Y)_g \nabla f. \tag{A-2}$$

Consequently, with the convention of $a \doteqdot \nabla^2 f - df \otimes df + \frac{1}{2}|\nabla f|^2 g$, we have:

- $\widetilde{\mathrm{R}} = e^{2f}\,\mathrm{R} - e^{2f}\, a \circ g$.
- $\widetilde{\mathrm{R}}_{ijk}^{l} = \mathrm{R}_{ijk}^{l} - a_i^l g_{jk} - a_{jk}\delta_i^l + a_{ik}\delta_j^l + a_j^l g_{ik}$.
- $d\widetilde{\mu} = e^{nf}\, d\mu$.
- $\widetilde{\Delta} h = e^{-2f}(\Delta h + (n-2)\nabla^k f \nabla_k h)$.
- $\widetilde{\mathrm{W}} = e^{2f}\mathrm{W}$.
- $\widetilde{\mathrm{Rc}} = \mathrm{Rc} - (n-2)a - \left(\Delta f + \frac{n-2}{2}|\nabla f|^2\right) g$.
- $\widetilde{\mathrm{S}} = e^{-2f}(\mathrm{S} - 2(n-1)\Delta f - (n-2)(n-1)|\nabla f|^2)$
  $= e^{-2f}\left(\mathrm{S} - \dfrac{4(n-1)}{n-2} e^{-\frac{1}{2}(n-2)f}\Delta\left(e^{\frac{1}{2}(n-2)f}\right)\right)$ when $n > 2$.

Now restricting our attention to dimension four, we arrive at:

- $\widetilde{\mathrm{S}} = u^3(-6\Delta_g + \mathrm{S})u$.
- $\widetilde{\mathrm{W}}_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = u^{-4}\widetilde{\mathrm{W}}_{abcd} = u^{-2}\mathrm{W}_{abcd}$.
- $\widetilde{\Delta} = u^{-2}\left(\Delta - 2\dfrac{\nabla u}{u}\nabla\right)$.
- $\det \widetilde{\mathrm{W}_+} = u^{-6} \det \mathrm{W}_+$.

Lemma A.1 *The divergence of the Weyl tensor under the above conformal change is given by*

$$\widetilde{\delta}\widetilde{\mathrm{W}}(X,Y,Z) = \delta \mathrm{W}(X,Y,Z) + (n-3)\mathrm{W}\left(\frac{\nabla u}{u}, X, Y, Z\right).$$

Next, we calculate the conformal change of the norm of the covariant derivative of the Weyl tensor.

Lemma A.2 *Let (M,g) be a four-dimensional Riemannian manifold, and $\widetilde{g} = u^2 g$, for some positive smooth function u. Then we have*

$$|\widetilde{\nabla}\widetilde{\mathrm{W}}|^2 = u^{-6}|\nabla \mathrm{W}|^2 + 18u^{-8}|\nabla u|^2|\mathrm{W}|^2 - 10u^{-7}\nabla u \nabla |\mathrm{W}|^2 + 16\langle \delta \mathrm{W}, i_{\nabla u}\mathrm{W}\rangle. \tag{A-3}$$

Proof We observe that

$$|\widetilde{\nabla}\widetilde{\mathrm{W}}|^2 = u^{-10}\big((\widetilde{\nabla}_{e_i}\widetilde{\mathrm{W}})_{abcd}\big)^2.$$

Then

$$
\begin{aligned}
(\widetilde{\nabla}_{e_i}\widetilde{\mathrm{W}})_{abcd} &= \nabla_i(u^2\mathrm{W}_{abcd}) - u^2\big[\mathrm{W}(\widetilde{\nabla}_{e_i}a,b,c,d) + \mathrm{W}(a,\widetilde{\nabla}_{e_i}b,c,d) \\
&\qquad\qquad + \mathrm{W}(a,b,\widetilde{\nabla}_{e_i}c,d) + \mathrm{W}(a,b,c,\widetilde{\nabla}_{e_i}d)\big] \\
&= u^2\nabla_i\mathrm{W}_{abcd} - 2uu_i\mathrm{W}_{abcd} + u\delta_{ia}\mathrm{W}_{\nabla ubcd} - u\mathrm{W}_{ibcd}u_a \\
&\qquad + u\delta_{ib}\mathrm{W}_{a\nabla ucd} - u\mathrm{W}_{aicd}u_b + u\delta_{ic}\mathrm{W}_{ab\nabla ud} \\
&\qquad - u\mathrm{W}_{abid}u_c + u\delta_{id}\mathrm{W}_{abc\nabla u} - u\mathrm{W}_{abci}u_d.
\end{aligned}
$$

Now summing over all the indices, using Lemma 2.1, we have:

- $(\nabla_i\mathrm{W}_{abcd})^2 = |\nabla\mathrm{W}|^2$.
- $(u_i\mathrm{W}_{abcd})^2 = |\nabla u|^2|\mathrm{W}|^2$.
- $(\delta_{ia}\mathrm{W}_{\nabla ubcd})^2 = 4(\mathrm{W}_{\nabla ubcd})^2 = 4|\nabla u|^2|\mathrm{W}|^2$.
- $(\mathrm{W}_{ibcd}u_a)^2 = |\nabla u|^2|\mathrm{W}|^2$.
- $2\nabla_i\mathrm{W}_{abcd}u_i\mathrm{W}_{abcd} = \langle\nabla|\mathrm{W}|^2,\nabla u\rangle$.
- $\nabla_i\mathrm{W}_{abcd}\delta_{ia}\mathrm{W}_{\nabla ubcd} = \langle\delta\mathrm{W}, i_{\nabla u}\mathrm{W}\rangle$.
- $\nabla_i\mathrm{W}_{abcd}\mathrm{W}_{ibcd}u_a = \langle\nabla|\mathrm{W}|^2,\nabla u\rangle - \langle\delta\mathrm{W}, i_{\nabla u}\mathrm{W}\rangle$.
- $u_i\mathrm{W}_{abcd}\delta_{ia}\mathrm{W}_{\nabla ubcd} = |\nabla u|^2|\mathrm{W}|^2$.
- $u_i\mathrm{W}_{abcd}\mathrm{W}_{ibcd}u_a = |\nabla u|^2|\mathrm{W}|^2$.
- $\delta_{ia}\mathrm{W}_{\nabla ubcd}\mathrm{W}_{ibcd}u_a = |\nabla u|^2|\mathrm{W}|^2$.
- $\delta_{ia}\mathrm{W}_{\nabla ubcd}\delta_{ib}\mathrm{W}_{a\nabla ucd} = -|\nabla u|^2|\mathrm{W}|^2$.
- $\delta_{ia}\mathrm{W}_{\nabla ubcd}\mathrm{W}_{aicd}u_b = 0$.
- $\mathrm{W}_{\nabla ubcd}\delta_{ac}\mathrm{W}_{ab\nabla ud} = \mathrm{W}_{\nabla ubid}\mathrm{W}_{bid\nabla u} = \frac{1}{2}|\nabla u|^2|\mathrm{W}|^2$.
- $\mathrm{W}_{ibcd}u_a\mathrm{W}_{aicd}u_b = -|\nabla u|^2|\mathrm{W}|^2$.
- $\mathrm{W}_{ibcd}u_a\mathrm{W}_{abid}u_c = \mathrm{W}_{ib\nabla ud}\mathrm{W}_{\nabla ubid} = \frac{1}{2}|\nabla u|^2|\mathrm{W}|^2$.

The result then follows immediately. □

We now can calculate the conformal change of the Bochner–Weitzenböck formula.

Corollary A.3 *Let (M,g) be a four-dimensional Riemannian manifold, and let $\widetilde{g} = u^2g$, for some positive smooth function u. If*

$$h = \Delta|\mathrm{W}^+|^2 - 2|\nabla\mathrm{W}^+|^2 - \mathrm{S}|\mathrm{W}^+|^2 + 36\det\mathrm{W}^+,$$

then

$$
\begin{aligned}
u^6\widetilde{h} = h - 20u^{-2}|\nabla u|^2|\mathrm{W}^+|^2 + 2u^{-1}|\mathrm{W}^+|^2\Delta u + 10u^{-1}\nabla u\nabla|\mathrm{W}^+|^2 \\
- 32u^{-1}\langle\delta\mathrm{W}^+, i_{\nabla u}\mathrm{W}^+\rangle.
\end{aligned}
$$

Proof Without risk of confusion we denote $\mathrm{W} \doteqdot \mathrm{W}^+$ for simplicity, and calculate

$$\begin{aligned}
\widetilde{\Delta}|\widetilde{\mathrm{W}}|^2 &= \widetilde{\Delta}(u^{-4}|\mathrm{W}|^2) \\
&= u^{-2}(\Delta(u^{-4}|\mathrm{W}|^2) - 2\frac{\nabla u}{u}\nabla(u^{-4}|\mathrm{W}|^2) \\
&= u^{-2}\Big(u^{-4}\Delta|\mathrm{W}|^2 + |\mathrm{W}|^2\Delta u^{-4} + 2\nabla u^{-4}\nabla|\mathrm{W}|^2 \\
&\qquad\qquad - 2|\mathrm{W}|^2\frac{\nabla u}{u}\nabla u^{-4} - 2u^{-4}\frac{\nabla u}{u}\nabla|\mathrm{W}|^2\Big) \\
&= u^{-6}\Delta|\mathrm{W}|^2 + 20u^{-8}|\mathrm{W}|^2|\nabla u|^2 - 4u^{-7}|\mathrm{W}|^2\Delta u \\
&\qquad\qquad - 10u^{-7}\nabla u\nabla|\mathrm{W}|^2 + 8u^{-8}|\nabla u|^2|\mathrm{W}|^2 \\
&= u^{-6}\Delta|\mathrm{W}|^2 + 28u^{-8}|\mathrm{W}|^2|\nabla u|^2 - 4u^{-7}|\mathrm{W}|^2\Delta u - 10u^{-7}\nabla u\nabla|\mathrm{W}|^2, \\
\widetilde{\mathrm{S}}|\widetilde{\mathrm{W}}|^2 &= u^{-6}\mathrm{S}|\mathrm{W}|^2 - 6u^{-7}|\mathrm{W}|^2\Delta u.
\end{aligned}$$

The result then follows by combining the above equations with Lemma A.2. □

A.2 Along the Ricci flow

Inspired by the simplification in Bochner–Weitzenböck formula in Theorem 1.1, we carry out a similar calculation on a Ricci flow. As a consequence, we obtain several interesting evolution equations involving the self-dual part and other components of the curvature operator. First, we state some useful lemmas.

Lemma A.4 *Let $(M^4, g(t))$, $0 \le t < T \le \infty$, be a solution to the Ricci flow of (1-1), and the curvature operator be decomposed as in (2-11). Then*

$$\frac{\partial}{\partial t}\mathrm{W}^+ = \Delta\mathrm{W}^+ + 2(\mathrm{W}^+)^2 + 4(\mathrm{W}^+)^\sharp + 2(CC^T - \tfrac{1}{3}|C|^2 I^+). \tag{A-4}$$

Remark A.5 Our convention agrees with [36] but differs from [32].

Lemma A.5 *For a four-dimensional Riemannian manifold (M, g), if the curvature is represented as in (2-11), then*

$$\langle \mathrm{W}^+, CC^T\rangle = \tfrac{1}{4}\langle \mathrm{W}^+, \mathrm{Rc}\circ\mathrm{Rc}\rangle. \tag{A-5}$$

Using the results above, we arrive at the following statement.

Theorem A.6 *Let $(M^4, g(t)$, $0 \le t < T \le \infty$, be a closed solution to the Ricci flow of (1-1). Then we have the evolution equation*

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)|\mathrm{W}^+|^2 = -2|\nabla\mathrm{W}^+|^2 + 36\det{}_{\Lambda^2_+}\mathrm{W}^+ + \langle \mathrm{Rc}\circ\mathrm{Rc}, \mathrm{W}^+\rangle. \tag{A-6}$$

Remark A.6 The Weyl tensor is considered the traceless part of the curvature operator (modding out the Ricci and scalar components). Thus, it is interesting to compare the above calculation with the evolution equation for the traceless part of the Ricci curvature $h = |\mathrm{E}|^2$ (see [17]),

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)h^2 = -2|\nabla \operatorname{Rc}|^2 + \frac{|\nabla \mathrm{S}|^2}{2} + \frac{2}{3}\mathrm{S}h - 4\mathrm{E}^3 + 4\mathrm{W}(\mathrm{E},\mathrm{E})$$
$$= -2\nabla h \nabla(\ln \mathrm{S}) - \frac{2}{\mathrm{S}^2}|\mathrm{S}\nabla \operatorname{Rc} - \operatorname{Rc}\nabla \mathrm{S}|^2 + 2h^2\Big(2|\nabla(\ln \mathrm{S})|^2 + \frac{\mathrm{S}}{3}\Big) - 4\mathrm{E}^3 + 4\mathrm{W}(\mathrm{E},\mathrm{E}).$$

A consequence of Theorem A.6 is the following statement.

Corollary A.7 *Let $(M, g(t))$, $0 \le t < T \le \infty$, be a closed solution to the Ricci flow of (1-1). Then*

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\Big(\frac{|\mathrm{W}^+|^2}{\mathrm{S}^2}\Big) = -\frac{2}{\mathrm{S}^4}|\mathrm{S}\nabla \mathrm{W}^+ - \mathrm{W}^+\nabla \mathrm{S}|^2 + \Big\langle \nabla\Big(\frac{|\mathrm{W}^+|^2}{\mathrm{S}^2}\Big), \nabla \ln \mathrm{S}^2\Big\rangle + 36\frac{\det_{\Lambda^2_+}\mathrm{W}^+}{\mathrm{S}^2} + \frac{\langle \operatorname{Rc}\circ\operatorname{Rc}, \mathrm{W}^+\rangle}{\mathrm{S}^2} - 4\frac{|\mathrm{W}^+|^2|\operatorname{Rc}|^2}{\mathrm{S}^3}. \tag{A-7}$$

Remark A.7 On a GRS, the equation becomes

$$-\Delta_f\Big(\frac{|\mathrm{W}^+|^2}{\mathrm{S}^2}\Big) = -\frac{2}{\mathrm{S}^4}|\mathrm{S}\nabla \mathrm{W}^+ - \mathrm{W}^+\nabla \mathrm{S}|^2 + \Big\langle \nabla\Big(\frac{|\mathrm{W}^+|^2}{\mathrm{S}^2}\Big), \nabla \ln \mathrm{S}^2\Big\rangle + 36\frac{\det_{\Lambda^2_+}\mathrm{W}^+}{\mathrm{S}^2} + \frac{\langle \operatorname{Rc}\circ\operatorname{Rc}, \mathrm{W}^+\rangle}{\mathrm{S}^2} - 4\frac{|\mathrm{W}^+|^2|\operatorname{Rc}|^2}{\mathrm{S}^3}. \tag{A-8}$$

An immediate application of the computation above and the maximum principle is the result below.

Proposition A.8 *Let $(M, g(t))$, $0 \le t < T \le \infty$, be a closed solution to the Ricci flow of (1-1). If $\det_{\Lambda^2_+}\mathrm{W}^+$ is nonpositive along the Ricci flow, then there exists a constant $C = C(g(0))$ such that $|\mathrm{W}^+|/\mathrm{S} < C$ is preserved along the flow.*

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Department of Mathematics, Cornell University
507 Malott Hall, Ithaca, NY 14853, USA

Department of Mathematics, University of California, Irvine
440G Rowland Hall, UCI, Irvine, CA 92697, USA

`cao@math.cornell.edu`, `hungtt1@uci.edu`

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