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# The degree of the Alexander polynomial is an upper bound for the topological slice genus

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We use the famous knot-theoretic consequence of Freedman's disc theorem — knots with trivial Alexander polynomial bound a locally flat disc in the 4–ball — to prove the following generalization: the degree of the Alexander polynomial of a knot is an upper bound for twice its topological slice genus. We provide examples of knots where this determines the topological slice genus.

57M25, 57M27

## 1 Introduction

For a knot $K$ — a smooth and oriented embedding of the unit circle $S^1$ into the unit 3–sphere $S^3$ — the *topological slice genus* $g_4^{\mathrm{top}}(K)$ is the minimal genus of locally flat oriented surfaces $S$ in the closed unit 4–ball $B^4$ with oriented boundary $K \subset \partial B^4 = S^3$. A celebrated theorem of Freedman [7, Theorem 1.13] asserts that every knot $K$ with trivial Alexander polynomial is *topologically slice*, ie $g_4^{\mathrm{top}}(K)$ equals 0; see also Freedman and Quinn [8, Section 11.7B] and Garoufalidis and Teichner [9, Appendix]. Here, the *Alexander polynomial* $\Delta_K$, first introduced by Alexander [1], is the Laurent polynomial with integer coefficients in the indeterminate $t$ defined by

$$\det\left(M\sqrt{t} - M^T\frac{1}{\sqrt{t}}\right),$$

where $M$ is any Seifert matrix for $K$ and $M^T$ is its transpose. The *degree* $\deg \Delta_K$ of the Alexander polynomial $\Delta_K$ is the difference of the largest and the smallest exponent among the exponents of the monomials of $\Delta_K$; ie $\deg \Delta_K$ is the breadth of $\Delta_K$.

**Theorem 1** *For every knot, the degree of its Alexander polynomial is greater than or equal to twice its topological slice genus.*

An appealing way of summarizing Theorem 1 and the classical genus bound of the Alexander polynomial is the following. For all knots $K$, we have

$$2g_4^{\mathrm{top}}(K) \le \deg \Delta_K \le 2g(K),$$

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where $g(K)$ denotes the genus of $K$, ie the minimal genus of Seifert surfaces for $K$. Theorem 1 determines $g_4^{\text{top}}$ for many knots including some of small crossing number; examples are provided in Section 2.

The following purely 3-dimensional proposition reduces Theorem 1 to the genus-zero case. The proof of this proposition, detailed in Section 3, is completely elementary whereas the genus-zero case uses the entire Freedman machine of infinite constructions in topological 4–manifold theory.[1]

**Proposition 2** *Let $K$ be a knot. Every Seifert surface $S$ of $K$ contains a separating simple closed curve $L$ with the following properties:*

- *The Alexander polynomial of $L$ (as a knot in $S^3$) is trivial.*
- *The connected component $C$ of $S \setminus L$ that does not contain $K$ is a Seifert surface for $L$ with*

$$2\,\text{genus}(C) = 2\,\text{genus}(S) - \deg \Delta_K.$$

The reduction of Theorem 1 to Proposition 2 and Freedman's result is rather direct. The same idea was used by Rudolph to provide examples of torus knots whose topological slice genus is smaller than their genus [12, Theorem 2].

**Proof of Theorem 1** For a given knot $K$, let $L \subset S$ be a simple closed curve in some Seifert surface $S$ of $K$ with the properties described in Proposition 2. By removing the connected component $C$ of $S \setminus L$ that does not contain $K$, one obtains a surface $\overline{S \setminus C} \subset S^3$ with boundary $K \dot{\cup} L$. By Freedman's work [7, Theorem 1.13], $L$ bounds a topological locally flat disc $D$ in $B^4$. Gluing $\overline{S \setminus C}$ and $D$ along $L$ yields a locally flat surface $S^{\text{top}}$ of genus $\frac{1}{2}\deg \Delta_K$ in $B^4$ with boundary $K \subset S^3 = \partial B^4$. To be explicit, $S^{\text{top}}$ can be given as follows: shrink $D$ by a factor of 2, yielding a disc in the 4–ball $B^4_{1/2}$ of radius $\frac{1}{2}$ with boundary $L$ viewed as a knot in the 3–sphere of radius $\frac{1}{2}$. Then embed $S \setminus C$ in $B^4 \setminus B^4_{1/2} = S^3 \times \left(\frac{1}{2}, 1\right]$ via the map

$$S \setminus C \to S^3 \times \left(\tfrac{1}{2}, 1\right], \quad x \mapsto \left(x, \tfrac{1}{2} + \frac{\operatorname{dist}(L, x)}{2\operatorname{dist}(K, x) + 2\operatorname{dist}(L, x)}\right),$$

and set $S^{\text{top}} \subset B^4$ to be the union of the shrunken $D$ and the embedded $S \setminus C$. □

We conclude the introduction by describing previous work relating $\deg \Delta_K$ and $g_4^{\text{top}}$. Borodzik and Friedl proved that $\deg \Delta_K + 1$ is an upper bound for the algebraic unknotting number $u_a$, which follows from combining their results [3, Lemma 2.3] and [2, Theorem 1.1]. Since $g_4^{\text{top}} \leq u_a$, this yields that $g_4^{\text{top}} \leq \deg \Delta_K + 1$.

[1] The editorial board of Geometry & Topology encourages the reader to refer to Theorem 1 as the "Freedman–Feller Theorem".

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## 2 Applications

Combining Theorem 1 with classical bounds for the topological slice genus, eg Kauffman and Taylor's signature bound [10, Theorem 3.13], yields simple criteria to determine the topological slice genus. Indeed, let $\sigma(K)$ denote the signature of a knot $K$ as introduced by Trotter [15], ie the signature $\sigma(M+M^T)$ of the symmetrization $M+M^T$ of any Seifert matrix $M$ for the knot $K$.

**Corollary 3** *For every knot $K$, we have*

$$|\sigma(K)| \leq 2g_4^{\mathrm{top}}(K) \leq \deg \Delta_K.$$

*In particular, if $|\sigma(K)| = \deg \Delta_K$, then*

$$g_4^{\mathrm{top}}(K) = \left|\tfrac{1}{2}\sigma(K)\right| = \tfrac{1}{2}\deg \Delta_K. \qquad \square$$

**Example 4** We describe an infinite family of knots for which $g_4^{\mathrm{top}}$ is arbitrarily large, while being arbitrarily smaller than the smooth slice genus $g_4$.

For any positive integer $g$, any integer $2g \times 2g$ matrix $M$ for which $M - M^T$ has determinant 1 describes the Seifert form on a Seifert surface $S$ bounded by some knot $K$; in fact, $S$ can be chosen to be a quasipositive Seifert surface, as proven by Rudolph [11; 13, Theorem 1.2]. If one chooses $M$ to satisfy

$$|\sigma(M+M^T)| = \deg\left(\det\left(M\sqrt{t} - M^T\frac{1}{\sqrt{t}}\right)\right) < 2g,$$

then one has examples of knots $K$ for which

$$\deg \Delta_K = 2g_4^{\mathrm{top}}(K) < 2g = 2g_4(K),$$

by Corollary 3 and the fact that quasipositive surfaces realize the smooth slice genus; see Rudolph's slice-Bennequin inequality [14]. Of course, the above examples include

knots for which $g_4^{\mathrm{top}}$ is determined by Freedman's result; eg if $K$ has trivial Alexander polynomial, or if $K$ is a connected sum of knots with trivial Alexander polynomial and knots for which $g = \left|\frac{1}{2}\sigma\right|$. However, for most knots $K$ as above, we do not know of a method that determines the topological slice genus and that does not use Theorem 1.

Next, we apply Corollary 3 to knots with small crossing number:

**Example 5** We determine the topological slice genus of the following knots, which can be represented by diagrams with 12 crossings. With designations as in *KnotInfo* [4], we have:

- $\sigma = -\deg \Delta_K = -4$ for the two knots $12n830$ and $12n750$.
- $\sigma = \deg \Delta_K = 2$ for the two knots $12n519$ and $12n411$.
- $\sigma = -\deg \Delta_K = -2$ for the two knots $12n321$ and $12n293$.

Previously, the topological slice genus appears to have been unknown for all these knots; compare [4]. We remark that, for $12n830$ and $12n750$, the smooth slice genus is known to be 3, and for $12n321$ and $12n293$ it is known to be 2; in particular, it is strictly larger than the topological slice genus; while for $12n519$ and $12n411$ the smooth slice genus appears to be unknown (it is either 1 or 2). In Section 4, we discuss the knot $12n750$ explicitly.

## 3 Proof of Proposition 2

We provide a sketch of our proof of Proposition 2. For a given knot $K$, we fix (in all of Section 3) a Seifert surface $S$ and denote its genus by $g$. We find a basis for $H_1(S,\mathbb{Z})$ for which the corresponding Seifert matrix $M$ is of the following form: $M - M^T$ is the standard symplectic form on $\mathbb{Z}^{2g}$ and the bottom right corner of $M$ is a square matrix $N$ of size $2g - \deg \Delta_K$ which represents the Seifert form of a knot with trivial Alexander polynomial. Then we represent this basis by simple closed curves such that for all pairs of curves the geometric intersection number equals the algebraic intersection number, and choose a curve $L$ that separates the curves that represent the last $2g - \deg \Delta_K$ elements of this basis. Thus, $N$ is a Seifert matrix for $L$ and, therefore, $L$ has trivial Alexander polynomial. We note that, if $\Delta_K = 1$, then $L$ is parallel to $K$ and this proof essentially reduces to the proof of [9, Lemma 4.2]; see Remark 7.

In order to provide a detailed proof of Proposition 2, we recall some facts about Seifert matrices and bilinear forms. By choosing a basis for $H_1(S,\mathbb{Z})$, ie by identifying $H_1(S,\mathbb{Z})$ with $\mathbb{Z}^{2g}$, the Seifert form becomes a bilinear form on $\mathbb{Z}^{2g}$, which

is canonically identified with a $2g \times 2g$ matrix $M$ —a *Seifert matrix*. The skew-symmetrization $M - M^T$ of $M$ represents the *intersection form* $I$ on $H_1(S, \mathbb{Z})$ (with respect to the same basis) and, therefore, has determinant 1. A change of basis amounts to changing $M$ to $A^T M A$ for some $\mathbb{Z}$–invertible $2g \times 2g$ matrix $A$ (which amounts to performing a finite number of elementary column operations and their corresponding elementary row operations on $M$). Fix a skew-symmetric bilinear form $F$ on a finitely generated free abelian group $V$, eg $I$ on $H_1(S, \mathbb{Z})$. A basis for $V$ is called *symplectic* (with respect to $F$), if the corresponding matrix representing $F$, eg $M - M^T$, is the standard symplectic form

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

A necessary and sufficient condition for the existence of a symplectic basis for $V$ is that $F$ is invertible; ie has determinant 1 when identified with a matrix.

**Proof of Proposition 2** First, we make an observation from linear algebra:

**Lemma 6** *There exists a basis $B$ for $H_1(S, \mathbb{Z})$ and a nonnegative integer $d$ such that the Seifert form on $H_1(S, \mathbb{Z})$ with respect to $B$ is given by a $2g \times 2g$ matrix of the form*

$$\left[\begin{array}{ccc|ccc|c|cc} & & & & & 0 & & & 0 \\ & M_{2d} & & & v_{g-d} & \vdots & & & \\ & & & & & 0 & & & \\ \hline & v_{g-d}^T & & 0 & 0 & & & v_1 & \vdots \\ 0 & \cdots & 0 & 1 & 0 & & & & \\ & & & & & & \ddots & & \\ & & & & & & & & 0 \\ \hline & & & & v_1^T & & & 0 & 0 \\ 0 & 0 & & & \cdots & & 0 & 1 & 0 \end{array}\right], \tag{1}$$

*where $M_{2d}$ is a $2d \times 2d$ matrix with nonzero determinant and the $v_i$ are column vectors with $2g - 2i$ entries. Furthermore, the degree of $\Delta_K$ equals $2d$.*

**Proof** Let $M_{2g}$ be a Seifert matrix representing the Seifert form on $H_1(S, \mathbb{Z})$ with respect to some basis. We consider the case $\det(M_{2g}) = 0$ as otherwise the statement is trivial. Thus, by a change of basis, we can arrange that the last column of $M_{2g}$ consists of zeros only (this is done by choosing a primitive vector in the kernel of $M_{2g}$ and extending it to a basis). From $\det(M_{2g} - M_{2g}^T) = 1$, we deduce that the greatest

common divisor of the entries of the last row of $M_{2g} - M_{2g}^T$, which is equal to the last row of $M_{2g}$, is 1. Therefore, we can change basis again (by performing elementary column operations on $M_{2g}$ simulating the Euclidean algorithm that yields the greatest common divisor of the last row) such that the corresponding Seifert matrix takes the form

$$\left[\begin{array}{ccc|cc} & & & & 0 \\ & M_{2g-2} & & w_1 & \vdots \\ & & & & 0 \\ \hline & v_1^T & & x & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{array}\right],$$

where $M_{2g-2}$ is a $(2g-2)\times(2g-2)$ matrix, $x$ is an integer, and $v_1$ and $w_1$ are column vectors with $2g-2$ entries. By changing basis once more, we can arrange that $x = 0$ and $v_1 = w_1$. The statement of Lemma 6 follows by induction on $g$, and $2d = \deg \Delta_K$ is immediate from the fact that calculating $\Delta_K$ using a Seifert matrix of the form (1) yields

$$\Delta_K = \det\left(M_{2d}\sqrt{t} - M_{2d}^T \frac{1}{\sqrt{t}}\right),$$

which has degree $2d$ since $\det(M_{2d})$ is nonzero. □

Next, we establish that there is also a symplectic (with respect to $I$) basis for $H_1(S,\mathbb{Z})$ for which the corresponding Seifert matrix is of the form (1); in fact, this follows from a version of Witt's theorem. Let $B$ be a basis as provided by Lemma 6, and let $M_{2g}$ be the corresponding Seifert matrix. We write $H_1(S,\mathbb{Z}) = V_1 \oplus V_2$, where $V_1$ denotes the subgroup spanned by the first $2d$ elements of $B$ and $V_2$ denotes the subgroup spanned by the other elements of $B$. Denote the lower right square of size $2g-2d$ of $M_{2g}$ by $N_{2g-2d}$. By (1), we have that

$$M_{2g} - M_{2g}^T = \begin{bmatrix} M_{2d} - M_{2d}^T & 0 \\ 0 & N_{2g-2d} - N_{2g-2d}^T \end{bmatrix},$$

where $N_{2g-2d} - N_{2g-2d}^T$ equals the standard symplectic form on $\mathbb{Z}^{2g-2d}$. Since $M_{2d} - M_{2d}^T$ is invertible (which follows from $M_{2g} - M_{2g}^T$ being invertible), there is a symplectic (with respect to the restriction of $I$) basis $B_{V_1}$ for $V_1$. Let $B_{\text{sympl}} = (a_1, b_1, \dots, a_g, b_g)$ denote the basis for $H_1(S,\mathbb{Z})$ obtained by replacing the first $2d$ elements of $B$ by $B_{V_1}$. By construction, $B_{\text{sympl}}$ is symplectic. The corresponding Seifert matrix $M_{\text{sympl}}$ is of the form (1) since $M_{\text{sympl}}$ is obtained from $M_{2g}$ by column (row) operations that involve only the first $2d$ columns (rows); in particular, $N_{2g-2d}$ remains unchanged.

Since $B_{\text{sympl}}$ is a symplectic basis for $I$, it can be realized geometrically; ie for all $1 \le i \le g$, there exist simple closed curves $\alpha_i$ and $\beta_i$ in $S$ representing the classes $a_i$ and $b_i$, respectively, such that $\alpha_i$ intersects $\beta_i$ once transversally and no other intersections occur; see eg Farb and Margalit [5, Theorem 6.4]. Let $L$ be any simple closed curve in $S$ separating the curves

$$K, \alpha_1, \beta_1, \dots, \alpha_d, \beta_d \quad \text{from} \quad \alpha_{d+1}, \beta_{d+1}, \dots, \alpha_g, \beta_g,$$

and denote the component of $S \setminus L$ containing $\alpha_{d+1}, \beta_{d+1}, \dots, \alpha_g, \beta_g$ by $C$. The existence of such a curve $L$ is evident since

$$S \setminus \{K \cup \alpha_1 \cup \beta_1 \cup \cdots \cup \alpha_g \cup \beta_g\}$$

is a $(g+1)$–punctured sphere. The surface $C$ has genus $g-d$ and is a Seifert surface for $L$. Furthermore, the Seifert matrix corresponding to the basis

$$([\alpha_{d+1}], [\beta_{d+1}], \dots, [\alpha_g], [\beta_g])$$

for $H_1(C, \mathbb{Z})$ is $N_{2g-2d}$. Therefore, we have

$$\Delta_L = \det\left(N_{2g-2d}\sqrt{t} - N_{2g-2d}^T \frac{1}{\sqrt{t}}\right) \overset{(1)}{=} 1. \qquad \square$$

Lukas Lewark pointed out Lemma 6. Originally, following the arguments in [6, Lemma 2] and [9, Lemma 4.2], we used changes of basis and $S$–equivalences to obtain a Seifert matrix of the form (1), which only yields the following weaker version of Proposition 2: every Seifert surface can be stabilized to a Seifert surface that contains a knot with the properties described in Proposition 2. We note that this version still suffices to establish Theorem 1.

The author greatly profited from the nice presentation of Freedman's result by Garoufalidis and Teichner [9, Appendix], where smooth $S^3$– and $B^4$–arguments are clearly separated from the application of Freedman's machinery. In fact, before discovering Proposition 2, which allows one to reduce Theorem 1 to a single application of the fact that knots with trivial Alexander polynomial are topologically slice, our proof of Theorem 1 closely followed the argument in [9, Appendix]. The following remark is related to their presentation:

**Remark 7** In the case when $\Delta_K = 1$, the proof of Proposition 2 reduces to the following slight improvement of a lemma [9, Lemma 4.2, first part] by Garoufalidis and Teichner. If a knot has trivial Alexander polynomial, then every Seifert surface has a trivial Alexander basis in the language of [9, Definition 4.1]. This follows by considering the basis $(b_1, \dots, b_g, a_1, \dots, a_g)$ instead of $B_{\text{sympl}}$.

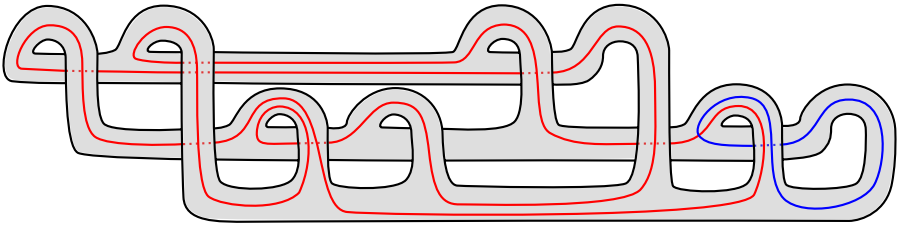

Figure 1: A quasipositive Seifert surface $S$ (gray) for the knot $12n750$ (black) containing two simple closed curves $\alpha$ (red, left) and $\beta$ (blue, right) such that a neighborhood of $\alpha \cup \beta$ bounds a knot $L$ with trivial Alexander polynomial.

## 4 Explicit example: the knot $12n750$ and its genera

For the knot $K = 12n750$, which is the closure of the 3–braid

$$aaaba^{-1}baaaba^{-1}b, \tag{2}$$ [2]

we exhibit the curve $L$ from Proposition 2 explicitly. Let $S$ be the Seifert surface of $K$ depicted in Figure 1. The Seifert surface $S$ has genus 3 and it realizes the genus and the smooth slice genus of $K$ since it is quasipositive [14, slice-Bennequin inequality]; in fact, $S$ is the quasipositive surface canonically associated with the strongly quasipositive braid word given in (2); compare [11; 13]. Let $\alpha$ and $\beta$ be the two once-intersecting simple closed curves depicted in Figure 1. A neighborhood of their union is a one-holed torus $T$ with a boundary curve $L = \partial T$ that has trivial Alexander polynomial. The latter follows since the Seifert form on $T$ with respect to the basis given by the homology classes of $\alpha$ and $\beta$ is $\left[\begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix}\right]$. Now, (as in the proof of Theorem 1) one can modify $S$ by replacing $T$ by a locally flat disc in $B^4$ to find a locally flat surface $S^{\mathrm{top}}$ of genus 2 in $B^4$ with boundary $K$. Calculating the signature of $K$ (it is $-4$) shows that $S^{\mathrm{top}}$ realizes $g_4^{\mathrm{top}}(K) = 2$.

## References

[1] **J W Alexander**, *Topological invariants of knots and links*, Trans. Amer. Math. Soc. 30 (1928) 275–306 MR1501429

[2] **M Borodzik**, **S Friedl**, *On the algebraic unknotting number*, Trans. London Math. Soc. 1 (2014) 57–84 MR3296484

[2] Here, $a$ and $b$ denote the standard generators of the 3-strand braid group corresponding to a positive crossing of the first two and the last two strands, respectively.

[3] **M Borodzik**, **S Friedl**, *The unknotting number and classical invariants, I*, Algebr. Geom. Topol. 15 (2015) 85–135 MR3325733

[4] **J C Cha**, **C Livingston**, *KnotInfo: Table of knot invariants* (2015) Available at http://www.indiana.edu/~knotinfo

[5] **B Farb**, **D Margalit**, *A primer on mapping class groups*, Princeton Mathematical Series 49, Princeton Univ. Press (2012) MR2850125

[6] **M H Freedman**, *A surgery sequence in dimension four; the relations with knot concordance*, Invent. Math. 68 (1982) 195–226 MR666159

[7] **M H Freedman**, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982) 357–453 MR679066

[8] **M H Freedman**, **F Quinn**, *Topology of* 4*–manifolds*, Princeton Mathematical Series 39, Princeton Univ. Press (1990) MR1201584

[9] **S Garoufalidis**, **P Teichner**, *On knots with trivial Alexander polynomial*, J. Differential Geom. 67 (2004) 167–193 MR2153483

[10] **L H Kauffman**, **L R Taylor**, *Signature of links*, Trans. Amer. Math. Soc. 216 (1976) 351–365 MR0388373

[11] **L Rudolph**, *Constructions of quasipositive knots and links, I*, from: "Knots, braids and singularities", (C Weber, editor), Monogr. Enseign. Math. 31, Enseignement Math., Geneva (1983) 233–245 MR728589

[12] **L Rudolph**, *Some topologically locally-flat surfaces in the complex projective plane*, Comment. Math. Helv. 59 (1984) 592–599 MR780078

[13] **L Rudolph**, *Quasipositivity and new knot invariants*, Rev. Mat. Univ. Complut. Madrid 2 (1989) 85–109 MR1012106

[14] **L Rudolph**, *Quasipositivity as an obstruction to sliceness*, Bull. Amer. Math. Soc. 29 (1993) 51–59 MR1193540

[15] **H F Trotter**, *Homology of group systems with applications to knot theory*, Ann. of Math. 76 (1962) 464–498 MR0143201

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