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Everything is illuminated

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We study geometrical properties of translation surfaces: the finite blocking property, bounded blocking property, and illumination properties. These are elementary properties which can be fruitfully studied using the dynamical behavior of the $\mathrm{SL}(2,\mathbb{R})$–action on the moduli space of translation surfaces. We characterize surfaces with the finite blocking property and bounded blocking property, completing work of the second-named author. Concerning the illumination problem, we also extend results of Hubert, Schmoll and Troubetzkoy, removing the hypothesis that the surface in question is a lattice surface, thus settling a conjecture of theirs. Our results crucially rely on the recent breakthrough results of Eskin and Mirzakhani and of Eskin, Mirzakhani and Mohammadi, and on related results of Wright.

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1 Introduction

A *translation surface* M is a finite union of polygons, glued along parallel edges by translations, up to a cut-and-paste equivalence. These structures arise in the study of billiards, interval exchange transformations, and various problems in group theory and geometry. See Masur and Tabachnikov [5], Zorich [18] and Yoccoz [17] for comprehensive introductions and detailed definitions. The purpose of this paper is to apply recent breakthrough results of Eskin and Mirzakhani [1] and Eskin, Mirzakhani and Mohammadi [2], on the dynamics of a group action on the moduli space of translation surfaces, to some elementary geometrical questions concerning translation surfaces. We begin with some definitions.

A pair of points $(x, y) \in M \times M$ is *finitely blocked* if there exists a finite set $B \subset M$ which does not contain x or y and intersects every straight-line trajectory connecting x and y. A set B with this property is called a *blocking set* for (x, y), and the minimal cardinality of a blocking set is called the *blocking cardinality of* (x, y) and is denoted by $\mathrm{bc}(x, y)$. A translation surface M has the *finite blocking property* if any pair $(x, y) \in M \times M$ is finitely blocked, and the *bounded blocking property* if there

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is a number n such that any pair $(x,y)\in M\times M$ is finitely blocked with blocking cardinality at most n. If x and y are finitely blocked with blocking cardinality zero, that is, if there is no straight-line path on M from x to y, then we say that x and y *do not illuminate each other*. A translation surface M is a *torus cover* if there is a surjective translation map from M to a torus (the singularities of M may project to one or several points on the torus). Equivalently (see eg Monteil [6]), the subgroup of $\mathbb{R}^2$ generated by holonomies of absolute periods on M is discrete.

Our first result settles a question of the second-named author; see [6; 7].

Theorem 1 *For a translation surface M, the following are equivalent:*

(1) *M is a torus cover.*
(2) *M has the finite blocking property.*
(3) *There is an open set $U\subset M\times M$ such that any pair of points in U is finitely blocked.*
(4) *M has the bounded blocking property.*

Hubert, Schmoll and Troubetzkoy [3] have constructed an example of a translation surface M which is not a torus cover, and in which there are infinitely many pairs of points which do not illuminate each other. In fact, there is an involution $\tau\colon M\to M$ such that for any $x\in M$, there is no straight line between x and $\tau(x)$. See Section 6.1 for similar examples. This shows that in (3) it is not enough to suppose that U is infinite.

Our second result concerns questions of illumination. The classical illumination problem was first posed in the 1950s, when it was asked whether there exists a polygonal room with a pair of points which do not illuminate each other. First examples were found by Tokarsky [11] and Boshernitzan (unpublished), and this raised the question of classification and possible cardinality of pairs of points which do not illuminate one another on translation surfaces. We refer to [3] or the Wikipedia page `http://en.wikipedia.org/wiki/Illumination_problem` for a brief history. We show:

Theorem 2 *For any translation surface M, and any point $x\in M$, the set of points y which are not illuminated by x is finite.*

Moreover, the set

$$\{(x,y) : x \text{ and } y \text{ do not illuminate each other}\}$$

is the union of a finite set with finitely many translation surfaces S embedded in $M\times M$, such that the projections $p_i|_S\colon S\to M$ are both finite-degree covers of the complement of a finite set in M.

Here $p_i \colon M \times M \to M$, $i = 1, 2$, are the natural projections onto the first and second factors, respectively.

Theorem 2 strengthens results of [3], which deal with surfaces which have a large group of translation automorphisms. Namely, Theorem 2 was proved in [3] under the additional hypothesis that M is a lattice surface, and when M is a prelattice surface, the first assertion of the theorem was shown, with "countable" in place of "finite" (for the definitions see Section 2.3). The first assertion of Theorem 2 settles [3, Conjecture 1]. In Section 5 we deduce Theorem 2 from the more general Theorem 10. In Section 6 we give examples which elaborate on related examples given in [3].

A standard "unfolding" technique (see Masur and Tabachnikov [5] and Zorich [18]) leads to the following result, which justifies the title of this paper. It settles a special case of [8, Conjecture 1].

Corollary 3 *Let P be a rational polygon. Then for any $x \in P$ there are at most finitely many points y for which there is no geodesic trajectory between x and y.*

There is a moduli space $\mathcal{H}$ parametrizing all translation surfaces sharing some topological data, and this space is equipped with an action of the group $G := \mathrm{SL}(2, \mathbb{R})$. The breakthrough work of Eskin and Mirzakhani [1] and of Eskin, Mirzakhani and Mohammadi [2] has made it possible to analyze the dynamics of this action in great detail. Our analysis depends crucially on this work, as well as on additional work of Wright [16].

We note that the crucial feature which makes our analysis possible is that the geometric properties we consider give rise to subsets of $\mathcal{H}$ which are closed and G–invariant. It has long been known that a detailed understanding of the G–action would shed light on the illumination problem, as well as on many similar "elementary" problems. For more papers applying the dynamics of the G–action to the analysis of closed and G–invariant geometrical properties of translation surfaces, see Veech [13], Vorobets [14], Monteil [6; 7], Hubert, Schmoll and Troubetzkoy [3], Smillie and Weiss [10; 9] and Lelièvre and Weiss [4].

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2 Preliminaries

We begin by briefly recalling the definitions of translation surfaces and strata, and refer to [5; 18; 17] for more details. Fix a topological orientable surface S of genus g, a finite subset $\Sigma = \{x_1, \dots, x_k\}$ of S, and nonnegative integers $\alpha_1, \dots, \alpha_k$ so that $\sum_i \alpha_i = 2g-2$. We allow some of the α_i to be zero and require $k \neq 0$. A *translation surface M of type $\vec{\alpha} = (\alpha_1, \dots, \alpha_k)$* is a surface M homeomorphic to S, with k labeled singular points $\{\xi_1, \dots, \xi_k\}$, equipped with an equivalence class of atlases of *planar charts*, ie maps from open subsets forming a cover of $M \smallsetminus \{\xi_1, \dots, \xi_k\}$ to $\mathbb{C}$, such that:

- Transition maps for the charts are translations.
- At each ξ_i the charts give rise to a cone-type singularity of angle $2\pi(\alpha_i + 1)$.

As usual, two atlases are considered equivalent if their union is also an atlas of the same type, and two translation surfaces are considered equivalent if there is a homeomorphism from one to the other which is a translation in charts and maps the distinguished finite set $\{\xi_i\}$ of one translation surface bijectively to the other in a way which respects the numbering. Note that an atlas of planar charts on $M \smallsetminus \Sigma$ naturally induces a translation structure on $(M \smallsetminus \Sigma) \times (M \smallsetminus \Sigma)$, with charts taking values in $\mathbb{C}^2$, and for which transition maps are translations. We will call this the *Cartesian product translation structure on M^2*.

The points ξ_i are called *singularities*. Note that we have allowed singularities with cone angle 2π (as happens when $\alpha_i = 0$). Such singularities are sometimes referred to as *marked points*. Note also that in contrast to the convention used by some authors, our convention is that singularities are labeled.

A homeomorphism $S \to M$ which maps each x_i to ξ_i is called a *marking*. We can use a marking and the planar charts of M to evaluate the integrals of directed paths on S beginning and ending in Σ. Such an integral is a complex number whose real and imaginary components measure, respectively, the total horizontal and vertical distance traveled when moving in M along the image of the path. Denote by $\mathcal{H}(\vec{\alpha})$ the set of translation surfaces of type $\vec{\alpha}$. It is called a *stratum* and is equipped with a natural topology defined as follows. The discussion above shows that the marking gives rise to a map

$$\mathcal{H}(\vec{\alpha}) \to H^1(S, \Sigma; \mathbb{C}).$$

It is known that the maps above constitute an atlas of charts which endow $\mathcal{H}(\vec{\alpha})$ with the structure of a linear orbifold. We will call these coordinates *period coordinates*. With respect to period coordinates, the change of a marking constitutes a change of coordinates via a unimodular integral matrix, so $\mathcal{H}(\vec{\alpha})$ is naturally endowed with a Lebesgue measure and a $\mathbb{Q}$–structure. It is known that each stratum has finitely many connected components. Our convention mentioned above, that singular points on a translation surface are labeled, implies that a stratum, with our conventions, is a finite cover of the strata considered by other authors. We will pass to a further finite cover in Section 2.1 below.

The group G acts on each stratum component $\mathcal{H}$ by postcomposition of planar charts. That is, identifying the field of complex numbers with the plane $\mathbb{R}^2$ in the usual way, each $g \in G$ is a linear map of $\mathbb{R}^2$ and we use it to replace each chart $M \supset U \xrightarrow{\varphi} \mathbb{C} \cong \mathbb{R}^2$ with the chart $g \circ \varphi \colon U \to \mathbb{R}^2$. For each stratum component $\mathcal{H}$, the subset $\mathcal{H}^{(1)}$ consisting of area-1 surfaces is a suborbifold which in period coordinates is cut out by a quadratic condition. It is preserved by the G–action, and G acts ergodically, preserving a natural smooth finite measure obtained from the Lebesgue measure by a cone construction. Given a translation surface M and a positive real number t, we denote by tM the translation surface obtained by multiplying all planar charts of M by the scalar t.

2.1 Adding marked points

We will need some notation for the operation of covering a stratum by a corresponding stratum with one or two additional marked points.

Given a stratum component $\mathcal{H}$, we denote by $\mathcal{H}'$ the corresponding stratum component of surfaces with one additional marked point, and by $\mathcal{H}''$ the corresponding stratum component of surfaces with two additional marked points. More formally, this is defined as follows. Suppose $\mathcal{H}$ is a component of $\mathcal{H}(\vec{\alpha})$, where $\vec{\alpha} := (\alpha_1, \dots, \alpha_k)$ and $\Sigma := \{x_1, \dots, x_k\}$ is a finite subset of cardinality k in the topological surface S. Let x_{k+1}, x_{k+2} denote two distinct points on $S \smallsetminus \Sigma$, set $\alpha_{k+1} = \alpha_{k+2} = 0$, and set

$$\Sigma' := \Sigma \cup \{x_{k+1}\}, \qquad \vec{\alpha}' := (\alpha_1, \dots, \alpha_{k+1}),$$
$$\Sigma'' := \Sigma' \cup \{x_{k+2}\}, \quad \vec{\alpha}'' := (\alpha_1, \dots, \alpha_{k+2}).$$

For any translation surface $M' \in \mathcal{H}'$, a simply connected neighborhood $\mathcal{U}$ of ξ_{k+1}, punctured at ξ_{k+1}, can be covered by charts from the atlas, and since $\alpha_{k+1} = 0$, one can add an additional chart to the atlas covering all of $\mathcal{U}$. The resulting translation surface M belongs to the stratum $\mathcal{H}$. If M' is marked by the pair (S, Σ') then M is naturally marked by the pair (S, Σ). Thus we get a natural map $\varphi' \colon \mathcal{H}(\vec{\alpha}') \to \mathcal{H}(\vec{\alpha})$, called the *forgetful map* since it corresponds to forgetting the location of the marked point ξ_{k+1}. Similarly, we have forgetful maps

$$\varphi'' \colon \mathcal{H}(\vec{\alpha}'') \to \mathcal{H}(\vec{\alpha}') \quad \text{and} \quad \varphi := \varphi' \circ \varphi'' \colon \mathcal{H}'' \to \mathcal{H},$$

which correspond respectively to forgetting the location of ξ_{k+2} and ξ_{k+1}, ξ_{k+2}. The three maps φ', φ'', φ are bundle maps for the bundles $\mathcal{H}(\vec{\alpha}')$, $\mathcal{H}(\vec{\alpha}'')$, $\mathcal{H}(\vec{\alpha}'')$ with bases $\mathcal{H}(\vec{\alpha})$, $\mathcal{H}(\vec{\alpha}')$, $\mathcal{H}(\vec{\alpha})$ and fibers $S \smallsetminus \Sigma$, $S \smallsetminus \Sigma'$, $(S \smallsetminus \Sigma)^2 \smallsetminus \Delta$, respectively ($\Delta$ is the diagonal). Finally, we let $\mathcal{H}', \mathcal{H}''$ be the connected components of $\mathcal{H}(\vec{\alpha}')$ and $\mathcal{H}(\vec{\alpha}'')$ covering the component $\mathcal{H}$.

We will sometimes start with surfaces $M \in \mathcal{H}$ and form surfaces in $\mathcal{H}'$ by choosing a nonsingular point $x \in M$ and specifying it as the marked point, thus obtaining a surface in $\mathcal{H}'$, which we will denote by (M, x). Similarly, starting with $M \in \mathcal{H}$ and a pair x, y of distinct nonsingular points on M, we will form (M, x, y) as a point in $\mathcal{H}''$. We caution that this may not be a well-defined operation in case M has a nontrivial translation automorphism. To explain the difficulty, suppose $h\colon M \to M$ is a nontrivial homeomorphism which is a translation in charts and h fixes points of Σ. By definition, the two translation surfaces given by the initial structure on M and the one obtained by precomposing all charts with h are considered equivalent, and thus, having chosen $M \in \mathcal{H}$ and $x \in M$, the point $h(x)$ is indistinguishable from x as a point of M and we cannot unambiguously write (M, x). To resolve this ambiguity we always pass to a finite cover of $\mathcal{H}$ in which surfaces have no nontrivial translation homeomorphisms. Such a cover is sometimes called a *stratum with a level-n structure,* and can also be obtained by quotienting the space of marked translation surfaces by a finite-index torsion-free subgroup of the mapping class group. When discussing strata we will have in mind a finite cover as above. See [18; 17] for details.

One easily checks from the definitions that the maps φ, φ', φ'' are G–equivariant, and that the fibers are linear manifolds in period coordinates. Moreover, note that the linear structure on a fiber $\varphi'^{-1}(M) \cong S \smallsetminus \Sigma$ coincides with the translation structure afforded by the translation charts on M, and similarly the linear structure on a fiber $\varphi^{-1}(M) \cong (S \smallsetminus \Sigma)^2 \smallsetminus \Delta$ coincides with the Cartesian product translation structure on M^2. In the sequel we will refer to x_{k+1} and x_{k+2} as the first and second marked points for the covers $\mathcal{H}'' \to \mathcal{H}' \to \mathcal{H}$. Note that we allow $\mathcal{H}$ to contain additional marked points.

2.2 Recent dynamical breakthroughs

We now state the results of [1; 2; 16] mentioned in the introduction. This requires some terminology. We say that a subset $\mathcal{L}_0 \subset \mathcal{H}$ is a *complex linear properly immersed manifold defined over* $\mathbb{R}$ if there is a manifold $\mathcal{N}$ and a proper immersion $f\colon \mathcal{N} \to \mathcal{H}$ such that $\mathcal{L}_0 = f(\mathcal{N})$, each $x \in \mathcal{N}$ has a neighborhood U such that the image of $f(U)$ under any of the charts $\mathcal{H} \to H^1(S, \Sigma; \mathbb{C}) \cong \mathbb{C}^N$ is an affine subspace whose linear part is a $\mathbb{C}$–linear vector space defined over $\mathbb{R}$, and the set of $y \in \mathcal{L}_0$ for which

$|f^{-1}(y)| \geq 2$ has zero measure with respect to the Lebesgue measure class on these affine subspaces. Note that the real dimension of a complex linear manifold is even. Given $\mathcal{L} \subset \mathcal{H}^{(1)}$, we denote

$$\mathbb{R}^*_+\mathcal{L} := \{tM' : t > 0, M' \in \mathcal{L}\}.$$

If ν is a measure on $\mathcal{H}$ then $\mu(A) = \nu(\{tx : x \in A,\ t \in (0,1]\})$ is a measure on $\mathcal{H}^{(1)}$ and we say that μ *is obtained by coning off* ν. We say that $\mathcal{L} \subset \mathcal{H}^{(1)}$ is an *affine invariant manifold* if it is G–invariant, is the support of an ergodic G–invariant measure μ, $\mathbb{R}^*_+\mathcal{L}$ is a complex linear properly immersed submanifold defined over $\mathbb{R}$, and μ is obtained by coning off Lebesgue measure on $\mathbb{R}^*_+\mathcal{L}$.

Theorem 4 (Eskin, Mirzakhani and Mohammadi) *For each stratum component $\mathcal{H}$ and each $M \in \mathcal{H}^{(1)}$, the orbit closure $\mathcal{L} := \overline{GM}$ is an affine invariant manifold. The collection of affine invariant manifolds of $\mathcal{H}$ obtained as orbit-closures for the G–action is countable. If $\mathcal{L}_n$, $n \geq 1$, is a sequence of distinct affine invariant manifolds of some dimension k contained in $\mathcal{H}$, then, after passing to a subsequence, the set of accumulation points*

$$\{M \in \mathcal{H} : \text{there exists } M_n \in \mathcal{L}_n \text{ such that } M_n \to M\}$$

is an affine invariant manifold $\mathcal{L}_\infty$ with $\dim \mathcal{L}_\infty > k$ and $\{M_n\} \subset \mathcal{L}_\infty$.

Note that the results of [2] work for strata with marked points, ie they allow $\alpha_i = 0$ for some i.

Suppose that the number of singularities k is at least two. Let $H_1(S)$ and $H_1(S,\Sigma)$ denote, respectively, the absolute and relative homology groups. Then we have $H_1(S) \subset H_1(S,\Sigma)$ and we can restrict each 1–cocycle in $H^1(S,\Sigma;\mathbb{C})$ to the subspace $H_1(S)$; that is, we get a natural restriction map $H^1(S,\Sigma;\mathbb{C}) \to H^1(S;\mathbb{C})$. The kernel REL of this map is a subspace of $H^1(S,\Sigma;\mathbb{C})$ of real dimension $2(k-1)$, and we have a foliation of $H^1(S,\Sigma;\mathbb{C})$ by cosets of REL. Since the restriction map $H^1(S,\Sigma;\mathbb{C}) \to H^1(S;\mathbb{C})$ is topological, the space REL is independent of a marking, that is, can be used to unequivocally define a linear foliation of $\mathcal{H}(\vec{\alpha})$ using period coordinates. This foliation of $\mathcal{H}(\vec{\alpha})$ is called the *REL foliation*. The G–action respects the REL foliation and hence we have a linear foliation of $\mathcal{H}$ by leaves tangent to $\mathfrak{g} \oplus \mathrm{REL}$, where we use $\mathfrak{g}$ to denote the tangent to the foliation by G–orbits. We denote this foliation by $G \oplus \mathrm{REL}$. Following [16], if a closed G–invariant and G–ergodic linear manifold $\mathcal{L}$ is contained in a single leaf of the foliation $G \oplus \mathrm{REL}$, we say that it is *of cylinder rank one*. A translation surface M is *completely periodic* if in any cylinder direction on M there is a complete cylinder decomposition.

Theorem 5 (Wright [16, Theorems 1.5 and 1.6]) *A linear manifold $\mathcal{L}$ as above is of cylinder rank one if and only if any surface in $\mathcal{L}$ is completely periodic.*

We will need the following lemma. Note that its assertion would be trivial if the fiber of φ were compact.

Lemma 6 *Let $M \in \mathcal{H}$ and $M'' \in \varphi^{-1}(M) \subset \mathcal{H}''$. Let $\mathcal{L} := \overline{GM}$ and $\mathcal{L}'' := \overline{GM''}$. Then $\varphi|_{\mathcal{L}''}$ is an open mapping and hence $\dim \varphi(\mathcal{L}'') = \dim \mathcal{L}$.*

Proof According to [2], there are Borel probability measures μ and μ'' on $\mathcal{H}$ and $\mathcal{H}''$, respectively, such that $\mathcal{L} = \operatorname{supp} \mu$ and $\mathcal{L}'' = \operatorname{supp} \mu''$. We first claim that $\mu = \varphi_* \mu''$. To this end, note that [2, Theorems 2.6 and 2.10] provide an averaging method converging to μ and μ''; that is, in both of these theorems, one finds probability measures ν_T on G such that for any continuous compactly supported functions f and f'' on $\mathcal{H}$ and $\mathcal{H}''$, respectively, we have

$$\int_G f(gM)\, d\nu_T(g) \to \int_{\mathcal{H}} f\, d\mu \quad \text{as } T \to \infty,$$
$$\int_G f''(gM'')\, d\nu_T(g) \to \int_{\mathcal{H}''} f''\, d\mu'' \quad \text{as } T \to \infty.$$

By a standard argument, we may assume that this is also true if f'' is continuous and has a finite limit at infinity; in particular, for $f \in C_c(\mathcal{H})$ we may take $f'' = f \circ \varphi$. Thus by equivariance we have

$$\int_{\mathcal{H}} f\, d\mu \leftarrow \int_G f(gM)\, d\nu_T(g) = \int_G f''(gM'')\, d\nu_T(g) \to \int_{\mathcal{H}''} f \circ \varphi\, d\mu'',$$

and this implies that $\mu = \varphi_* \mu''$.

When expressed in period coordinates, the restriction to charts of the map $\varphi|_{\mathcal{L}''} \colon \mathcal{L}'' \to \mathcal{L}$ is an affine map of affine manifolds. In order to show that it is open it suffices to show that its derivative is surjective at every point $x \in \mathcal{L}''$. If not, then there is a neighborhood $\mathcal{U}$ of x in $\mathcal{L}''$ such that $\varphi(\mathcal{U})$ is contained in a proper affine submanifold of $\mathcal{L}$. Such a proper affine submanifold must have zero measure for the flat measure class on $\mathcal{L}$, ie $\mu(\varphi(\mathcal{U})) = 0$. By the preceding paragraph this implies $\mu''(\mathcal{U}) = 0$, which is impossible. □

2.3 The Veech group, lattice surfaces, and periodic points

An *affine automorphism* of a translation surface M is a homeomorphism $\varphi \colon M \to M$ which is affine in charts. In this case, by connectedness, its derivative $D\varphi$ is a constant

2×2 matrix of determinant ± 1. We denote by $\mathrm{Aff}^+(M)$ the group of orientation-preserving affine automorphisms, ie those for which $D\varphi \in G$. We say that φ is a *parabolic automorphism* if $D\varphi$ is a parabolic matrix, ie is not the identity but is conjugate to an upper triangular matrix with 1 on the diagonal. The *Veech group* of M is the image under the homomorphism $D\colon \mathrm{Aff}^+(M) \to G$ of the group of orientation-preserving affine automorphisms. We say that M is a *lattice surface* if its Veech group is a lattice in G. Equivalently, by a theorem of Smillie (see [13; 9]), the orbit GM is closed. Following [3] we say that M is a *prelattice surface* if $\mathrm{Aff}^+(M)$ contains two noncommuting parabolic automorphisms. Veech [12] showed that a lattice surface is a prelattice surface, justifying the terminology. A point $x \in M$ is called *periodic* if its orbit under $\mathrm{Aff}^+(M)$ is finite.

Example In Lemma 6 we showed that $\varphi''|_{\mathcal{L}''}\colon \mathcal{L}'' \to \mathcal{L}$ is an open map. Given that $\mathcal{L}$ is connected, this leads to the question of whether $\varphi|_{\mathcal{L}''}$ is surjective. The following example of Alex Wright shows that an open affine map of orbit-closures need not be surjective. Let $M \in \mathcal{H}$ be a lattice surface which admits an involution τ (eg M could be a surface of genus 2 and τ could be the hyperelliptic involution). Let $\mathcal{L} = GM$ be the orbit of M (which in this case coincides with the orbit closure), let $x \in M$ be a nonperiodic point, and let $M' := (M, x)$ be the surface in $\mathcal{H}'$ obtained by marking the point x. It was proved in [3], and follows easily from Theorem 4, that $\mathcal{L}' := \overline{GM'}$ coincides with $\varphi'^{-1}(GM)$ (ie all surfaces in GM marked at all nonsingular points). Now let $y := \tau(x) \neq x$, let $M'' := M(x, y)$ be the surface in $\mathcal{H}''$ obtained by marking M at the two points x, y, let $\mathcal{L}'' := \overline{GM''}$, and let $\varphi''\colon \mathcal{H}'' \to \mathcal{H}'$ be the affine map which forgets the second marked point. We have

$$\mathcal{L}'' \subset \{(M_0, x_0, y_0) \in \mathcal{H}'' : M_0 \in \mathcal{L},\ \tau(x_0) = y_0 \neq x_0\},$$

since the set on the right-hand side is closed and G–invariant. This implies that $\varphi''(\mathcal{L}'') \subset \{(M_0, x_0) : M_0 \in GM,\ \tau(x_0) \neq x_0\}$, and in particular $\varphi''|_{\mathcal{L}''}$ is not surjective. However, the proof of Lemma 6 shows that $\varphi''|_{\mathcal{L}''}$ is open.

Using one additional marked point one can find similar examples that show that, in Lemma 6, one need not have $\varphi(\mathcal{L}'') = \mathcal{L}$ in general.

3 Bounded blocking defines closed sets

Let M be a translation surface with singularity set Σ, and let

$$\widehat{M^2} = \{(x, y) \in (M \smallsetminus \Sigma)^2 : x \neq y\}.$$

If Z is a topological space and $A \subset B$ are subsets of Z, when we say that A *is closed as a subset of* B, we mean that A is closed in the relative topology, ie $A = B \cap \overline{A}$.

Lemma 7 *For any fixed integer $n \geq 0$, the following hold:*

(I) *For a fixed translation surface M, the set*

$$F_n(M) := \{(x, y) \in \widehat{M^2} : \mathrm{bc}(x, y) \leq n\}$$

is closed as a subset of $\widehat{M^2}$.

(II) *For a fixed translation surface M, and a fixed nonsingular $x \in M$, the set*

$$F_n(M, x) := \{y \in M \smallsetminus (\Sigma \cup \{x\}) : \mathrm{bc}(x, y) \leq n\}$$

is closed as a subset of $M \smallsetminus (\Sigma \cup \{x\})$.

(III) *The set $\mathcal{F}_n \subset \mathcal{H}''$ consisting of all surfaces on which the first and second marked points are finitely blocked of blocking cardinality at most n is closed in $\mathcal{H}''$.*

(IV) *For a fixed stratum $\mathcal{H}$, the set of $M_0 \in \mathcal{H}$ for which any pair $(x, y) \in \widehat{M_0^2}$ satisfies $\mathrm{bc}(x, y) \leq n$ is closed in $\mathcal{H}$.*

(V) *For any stratum $\mathcal{H}$, the subset BB_n of surfaces which have the bounded blocking property, with blocking cardinality at most n, is closed in $\mathcal{H}$.*

(VI) *There is ℓ, depending only on n and the stratum containing M, such that if the set*

(1) $$E_n := \{(x, y) \in M^2 : \mathrm{bc}(x, y) \leq n\}$$

is dense in M^2, then M has the bounded blocking property with blocking cardinality at most ℓ.

Proof We will denote a surface in $\mathcal{H}''$ by (M, x, y), where x and y are respectively the first and second marked points on M. The topology on $\mathcal{H}''$ is such that, when $(M_k, x_k, y_k) \to (M, x, y)$, for any parametrized line segment $\{\sigma(t) : t \in [0, 1]\}$ on M between x and y, for any large enough k there are parametrized line segments $\{\sigma_k(t) : t \in [0, 1]\}$ such that $\sigma_k(t) \to \sigma(t)$ for all t (and uniformly) — see [5; 18; 17] for details. Here a parametrized line segment is a constant-speed straight line in each chart and does not contain singular points in its interior. We refer to this property of the topology on $\mathcal{H}''$ as the *basic fact about line segments* (for $(M_k, x_k, y_k) \to (M, x, y)$).

We begin with the proof of (III). Let (M_k, x_k, y_k) be a sequence that converges to (M, x, y) in $\mathcal{H}''$, where (x_k, y_k) belongs to $F_n(M_k)$ for all k. Let $\{b_k^{(1)}, \dots, b_k^{(n)}\} \subset M_k$ be a blocking set for (x_k, y_k). Passing to a subsequence, we may assume that $b_k^{(i)}$ converges to a point $b^{(i)} \in M$ for each i. By the above description of the topology of $\mathcal{H}''$, if $\{b^{(1)}, \dots, b^{(n)}\}$ does not contain x or y then it is a blocking set for (x, y) in M and we are done.

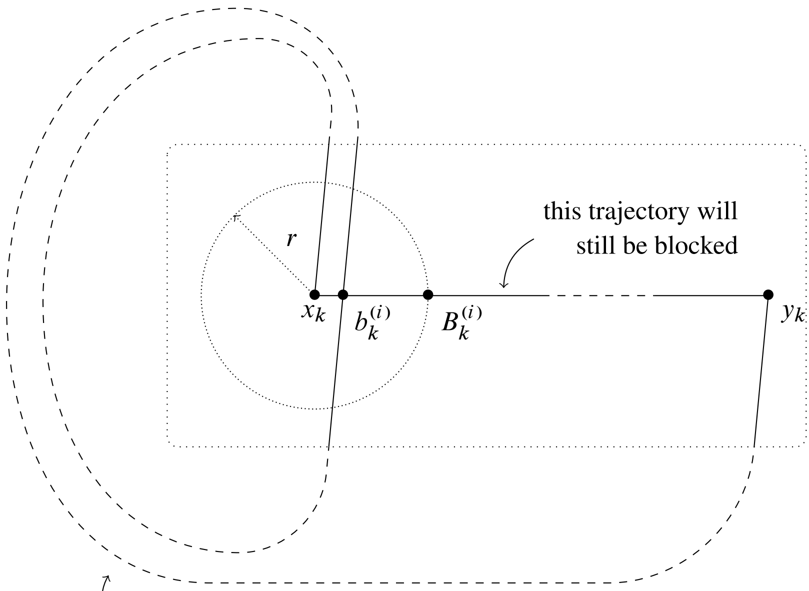

Figure 1: Proof of Lemma 7: prelimit surface (M_k, x_k, y_k).

We now discuss the case that some of the $b^{(i)}$ are equal to x or y. We modify the set $\{b^{(1)}, \dots, b^{(n)}\}$ as follows. For any i for which $b^{(i)}$ is different from both x and y, we set $B^{(i)} = b^{(i)}$. Suppose i is such that $b^{(i)} = x$. Let $r > 0$ be smaller than half the length of the shortest saddle connection on M. Since x and y are marked points, this implies that r is smaller than half the distance between x and y, and that there is no singularity in the ball $B(x,r)$ with center x and radius r.

For k large enough, $B(x_k, r)$ is an embedded flat disk in M_k that contains $b_k^{(i)}$, and there is a unique trajectory $\delta_k^{(i)}$ from x_k to $b_k^{(i)}$ that stays within this disk. Let $B_k^{(i)}$ be the point on $\delta_k^{(i)}$ at distance r from x_k. Passing again to a subsequence, we assume that $B_k^{(i)}$ converges to a point $B^{(i)}$ in M. Note that this point is distinct from x and y for each such i. We repeat this procedure for each i for which $b^{(i)}$ is equal to either x or y, passing at each stage to a further subsequence. See Figure 1.

Let us prove that $\{B^{(1)}, \dots, B^{(n)}\}$ is a blocking set for (x,y) in M. Let σ be a trajectory from x to y. We can assume without loss of generality that σ is simple, ie does not intersect itself. Let σ_k be the segment between x_k and y_k that converges pointwise to σ. If σ_k meets one of the $B_k^{(i)}$ for infinitely many k, $B^{(i)}$ belongs to σ and we are done.

Assume by contradiction that there is an index i such that, for infinitely many k, σ_k meets $b_k^{(i)}$ but not any $B_k^{(j)}$. In particular, $b_k^{(i)}$ converges to either x or y. Suppose for concreteness that it converges to x. Since $B_k^{(i)}$ does not belong to σ_k, the subsegment σ_k' of σ_k between x_k and $b_k^{(i)}$ is not equal to the segment $\delta_k^{(i)}$ defined above. In particular, the length of this subsegment is bounded below and it converges to a nontrivial subsegment σ' of σ, which is a (possibly multiple) loop from x to x. This contradicts the simplicity of σ, completing the proof of (III).

Clearly (III) $\Longrightarrow$ (I) $\Longrightarrow$ (II) and (III) $\Longrightarrow$ (IV). It remains to prove (V) and (VI). For both of these assertions, we will need to modify the argument for case (III) given above in various ways. We let x, y be points in M. We will consider separately the following cases:

Case 0 The points x, y are distinct and nonsingular.

Case 1 The points x, y are distinct and exactly one of them is a singularity.

Case 2 The points x, y are distinct singularities.

Case 3 $x = y$ is a nonsingular point.

Case 4 $x = y$ is singular.

We begin with assertion (V). We take a sequence $M_k \in \mathrm{BB}_n \subset \mathcal{H}$ converging to $M \in \mathcal{H}$, take $x, y \in M$ and need to show that $\mathrm{bc}(x, y) \le n$. In each of the five cases above, we can take $x_k, y_k \in M_k$ satisfying the same case distinction as that satisfied by M, x, y, and such that (M_k, x_k, y_k) converges to (M, x, y) in a suitable space. For example, in Case 0, when x, y are nonsingular and distinct on M we take x_k, y_k nonsingular and distinct on M_k such that, as elements of $\mathcal{H}''$, the sequence $M_k'' = (M_k, x_k, y_k)$ converges to $M'' = (M, x, y)$. This case was treated in the proof of assertion (III). In Case 1, suppose x is singular and y is nonsingular. We take x_k, y_k to be points on M_k such that x_k is singular, the label of the singularity x_k is the same as the label of the singularity x, y_k is nonsingular, and $M_k' = (M_k, y_k)$ converges to $M' = (M, y)$ in the space $\mathcal{H}'$.

The arguments given above dealt precisely with Case 0 but can be modified in a straightforward manner to deal with the other cases. That is, if $\{b_k^{(i)} : i = 1, \dots, n\}$ is a blocking set for x_k, y_k on M, we will form a modified set $\{B_k^{(i)} : i = 1, \dots, n\}$ as before and pass to subsequences to get convergence $B_k^{(i)} \to B^{(i)}$ as $k \to \infty$. We need to explain the definition of the modified sets and show that the set $\{B^{(i)} : i = 1, \dots, n\}$ will be a blocking set for x, y on M.

In Case 1, we define r and the points $B_k^{(i)}$ as before. We are working with x_k and x singular with the same label, and $(M_k, y_k) \to (M, y)$ as elements of $\mathcal{H}'$. With these definitions, the basic fact about line segments is still valid (note that it would no longer be valid if we were to take x_k nonsingular). The set $B = B(x_k, r)$ is not an embedded flat disk because x_k is a singular point, but rather it is a topological disk which is metrically a finite cover of a flat disk, branched over its center point x_k. Then B is star-shaped with respect to its center point x_k and it is still the case that there is a unique straight segment from x_k to any point in B which is contained in B. We can thus define the segment $\delta_k^{(i)}$ as in the proof of (III), and the same argument applies. Case 2 is almost identical.

In Case 3, we have $x_k = y_k$ and $x = y$, and these are nonsingular points. We have $(M_k, x_k) \to (M, x)$ as points of $\mathcal{H}'$, and with this topology the basic fact about line segments still holds. We define r and define the modified points $B_k^{(i)}, B^{(i)}$ as before. Note that since we have taken $x_k = y_k$, the segment $\delta_k^{(i)}$ is unambiguously defined. The argument given before goes through. Case 4 is similar.

To prove assertion (VI) we let E_n be as in (1) and let $x, y \in M$. We will show that $\mathrm{bc}(x, y) \le \ell$, where ℓ depends only on n and the stratum containing M; the definition of ℓ and the proof that $\mathrm{bc}(x, y) \le \ell$ will be done separately for each of the cases 0–4 above.

Case 0 follows from the arguments above used for proving statement (III) with $\ell = n$. In Case 1, suppose the point x is a singularity of cone angle $\pi\tau$ for some positive integer τ. Let r be as before and let $\mathcal{U}_1, \dots, \mathcal{U}_{\tau+1}$ be open half-disks centered at x such that $\bigcup \mathcal{U}_s = B(x, r) \smallsetminus \{x\}$. In particular, the sets $\mathcal{U}_s$ are open convex subsets of M whose closure contain x.

We now choose sequences $x_k^{(s)}$ such that $x_k^{(s)} \in \mathcal{U}_s \cap E_n$ and $x_k^{(s)} \to x$ as $k \to \infty$, and a sequence $y_k \to y$ such that $\mathrm{bc}(x_k^{(s)}, y_k) \le n$ for each s and k. Such sequences exist because E_n is dense. For each choice of $s \in \{1, \dots, \tau + 1\}$ we perform the procedure explained in the proof of (III). Namely, we take blocking sets $\{b_k^{(i,s)} : i = 1, \dots, n\}$ which block all segments between $x_k^{(s)}$ and y_k, pass to subsequences to assume that $\lim_k b_k^{(i,s)}$ exists for each i, s, and define $B^{(i,s)}$ to be this limit if it is distinct from x and y. If the limit is x we modify $b_k^{(i,s)}$ by letting $B_k^{(i,s)}$ be the unique point of distance r from x along the continuation of the unique segment $\delta_k^{(i,s)}$ which connects x and $b_k^{(i,s)}$ and which passes through $\mathcal{U}_s$. Then we take $B^{(i,s)}$ to be the limit $\lim_k B_k^{(i,s)}$ (passing to subsequences if necessary). This procedure gives us a set

$$\left\{B^{(i,s)} : i \in \{1, \dots, n\}, s \in \{1, \dots, \tau + 1\}\right\},$$

which we claim is a blocking set for x, y.

Indeed, for each segment σ from x to y, there is some s such that $\sigma(t)$ belongs to $\mathcal{U}_s$ for all $t > 0$ small enough. Then for large enough k there are segments σ_k from $x_k^{(s)}$

to y_k which approach σ pointwise. Working with these segments as in the proof of (III), we see that σ is blocked by $B^{(i,s)}$ for some i. This argument shows that if we take ℓ to be $n(\tau(\mathcal{H})+1)$, where $\pi\tau(\mathcal{H})$ is the greatest cone angle for surfaces in $\mathcal{H}$, then $\mathrm{bc}(x,y)\le\ell$. This concludes the proof in Case 1.

In Case 2 we give a similar argument where we take a union of finitely many open half-disks covering neighborhoods of both x and y, and construct sequences $x_k^{(s)}, y_k^{(t)}\in E_n$, where $s,t\in\{1,\dots,\tau(\mathcal{H})+1\}$, $x_k^{(s)}\to x$ and $y_k^{(t)}\to y$ as $k\to\infty$, and such that $x_k^{(s)}$ (respectively, $y_k^{(t)}$) belongs to the s^{th} half-disk near x (respectively, the t^{th} half disk near y). Repeating the argument of Case 1, we find that $\mathrm{bc}(x,y)\le\ell$, where $\ell=n(\tau(\mathcal{H})+1)^2$.

In Case 3 we have $x_k\to x$, $y_k\to x$ and $\mathrm{bc}(x_k,y_k)\le n$. We will take $\ell=2n$ and show that $\mathrm{bc}(x,x)\le\ell$. We construct the blocking points $B^{(i)}$ as follows. Passing to subsequences, we assume the existence of each of the limits $b_i=\lim_{k\to\infty}b_k^{(i)}$, and when $b_i\neq x$ we set $B^{(i)}=b_i$ as before. When $b_i=x$, in place of the short segments $\delta_k^{(i)}$ appearing in the proof of assertion (III), we consider two segments — one from x_k to $b_k^{(i)}$ and one from y_k to $b_k^{(i)}$. We denote these by $\delta_{k,1}^{(i)}$ and $\delta_{k,2}^{(i)}$, and construct points $B_{k,1}^{(i)}$ and $B_{k,2}^{(i)}$ by "sliding" $b_k^{(i)}$ along these segments as in the preceding argument. Taking limits, in each case in which $b_i=x$ we get two limit points, so the number of points $B^{(i)}$ is at most ℓ, and it remains to show that the set $\{B^{(i)}\}$ is a blocking set.

Let σ be a segment from x to x which does not contain any of the $B^{(i)}$, and let $r>0$ be as before. The segment σ is not contained in the ball $B=B(x,r)$. Let σ_k be a sequence of parametrized line segments from x_k to y_k converging to σ. We can assume that none of these segments contains any of the $B_{k,j}^{(i)}$, $i=1,\dots,n$, $j=1,2$. The only place in the proof of (III) in which we used that $x\neq y$ is where we needed to know that the subsegment σ' of σ constructed in the proof is a proper subsegment of σ. In the case $x=y$ there are two subsegments σ_k' (respectively σ_k'') between x_k and $b_k^{(i)}$ (respectively, between $b_k^{(i)}$ and y_k), neither of which is equal to $\delta_k^{(i)}$, since σ_k does not contain any of the $B_{k,j}^{(i)}$. In particular, each of them leaves the disk $B(x_k,r)$ and hence has length at least r. So in the limit they both converge to nontrivial (possibly multiple) loops σ',σ'' from x to itself, whose concatenation is σ. This gives the desired contradiction to the simplicity of σ.

Case 4 is proved combining the arguments used in Cases 1 and 3, resulting in a bound $\ell=2n(\tau(\mathcal{H})+1)^2$. We leave the details to the reader. □

A similar argument also shows:

Proposition 8 *Let M be a translation surface, ξ a singular point on M and $n \geq 0$ an integer. Recalling our convention that singularities on translation surfaces are labeled, we can use the notation ξ for a singular point of any other surface in $\mathcal{H}$. Let $\mathcal{F}'_n \subset \mathcal{H}'$ denote the set of surfaces on which the marked point y satisfies $\mathrm{bc}(\xi, y) \leq n$. Then $\mathcal{F}'_n$ is closed in $\mathcal{H}'$. In particular, $\{y \in M \smallsetminus \Sigma : \mathrm{bc}(\xi, y) \leq n\}$ is closed as a subset of $M \smallsetminus \Sigma$.*

Proof We repeat the proof of Lemma 7(III), replacing everywhere x with ξ and also x_k with ξ.

In this case the set $B = B(\xi, r)$ is a topological disk which is metrically a finite cover of a flat disk, branched over its center point ξ. Then B is star-shaped with respect to its center point ξ and it is still the case that there is a unique straight segment from ξ to any point in B which is contained in B. We can thus define the segment $\delta_k^{(i)}$ as in the proof of (III), and the same argument applies. □

4 Characterization of the finite blocking property

In this section we will prove Theorem 1. A translation surface is *purely periodic* if it is completely periodic and all cylinders in such a decomposition have commensurable circumferences. The following was proved in [7]:

Proposition 9 (Monteil) *If M has the finite blocking property then M is purely periodic.*

Proof of Theorem 1 The implication (1) $\Longrightarrow$ (2) is proved in [6], and it is immediate that (2) $\Longrightarrow$ (3). We first show (4) $\Longrightarrow$ (1); that is, we assume that M has the bounded blocking property and we show that it is a torus cover.

Let $\mathcal{L} := \overline{GM}$. By assumption, there is n such that $M \in \mathrm{BB}_n$. Clearly $GM \subset \mathrm{BB}_n$, and by Lemma 7(V) this means $\mathcal{L}$ is contained in BB_n. By Proposition 9 this means that every surface in $\mathcal{L}$ is completely periodic, and, by Theorem 5, $\mathcal{L}$ is of cylinder rank one.

Recall that the *field of definition* of $\mathcal{L}$ is the smallest field such that, in any coordinate chart U on $\mathcal{H}$ given by period coordinates, the connected components of $U \cap \mathcal{L}$ are cut out by linear equations with coefficients in k (see [15]). By [16, Theorem 1.9], for any completely periodic surface $M' \in \mathcal{L}$ and any cylinder decomposition on M' with circumferences $c_1, \dots, c_r$, the field of definition k of $\mathcal{L}$ satisfies

$$k \subset \mathbb{Q}\big[\{c_i/c_j : i, j = 1, \dots, r\}\big].$$

By Proposition 9, any surface in $\mathcal{L}$ is purely periodic, so $k = \mathbb{Q}$. Therefore $\mathcal{L}$ contains a surface with rational holonomies, ie a square-tiled surface M'. Since M' is square-tiled, the holonomy of absolute periods on M' is a discrete subset of $\mathbb{C}$. Motion in the $G \oplus \mathrm{REL}$ leaf only changes the holonomy of absolute periods by a linear map, and therefore for any M in $\mathcal{L}$, the holonomy of absolute periods is discrete, ie any $M \in \mathcal{L}$ is a torus cover. This proves (4) $\Longrightarrow$ (1).

Now we prove (3) $\Longrightarrow$ (4). We have an open set U_1 in $M \times M$ consisting of pairs of points on M blocked from each other by finitely many points, that is,

$$U_1 \cap \widehat{M^2} \subset \bigcup_n F_n(M).$$

Each $F_n(M)$ is closed as a subset of $\widehat{M^2}$ by Lemma 7(I), so, by Baire category, there is n such that $F_n(M)$ contains an open set U_2. Each pair of points (x, y) in U_2 defines a surface in $\mathcal{H}''$, namely $M'' = (M, x, y)$. Let $\mathcal{L}(M'') := \overline{GM''} \subset \mathcal{H}''$. By Theorem 4, $\mathbb{R}^*_+\mathcal{L}(M'')$ is a linear manifold of even dimension contained in $\mathcal{F}_n$ and the collection of such linear submanifolds is countable. By Lemma 7(III), $\mathbb{R}^*_+\mathcal{L}(M'') \subset \mathcal{F}_n$.

The fiber $\varphi^{-1}(M)$ is a linear submanifold of $\mathcal{H}''$ identified with $\widehat{M^2}$. Therefore, $\Omega(M'') := \varphi^{-1}(M) \cap \mathbb{R}^*_+\mathcal{L}(M'')$ is also a linear submanifold for any M'', and its dimension is $0, 2$ or 4. We have covered U_2, an open subset of a four-dimensional manifold, by countably many linear manifolds of dimensions at most four. By Baire category, there is M'' for which $\Omega(M'')$ is a linear manifold of dimension four. In particular $\Omega(M'')$ is open in $\varphi^{-1}(M)$, and by Lemma 7(I), it is also closed. Since $\varphi^{-1}(M)$ is connected, it coincides with $\Omega(M'')$.

We have proved that

$$\varphi^{-1}(M) = \Omega(M'') \subset \mathbb{R}^*_+\mathcal{L}(M'') \subset \mathcal{F}_n;$$

that is, any two distinct nonsingular points in M are of blocking cardinality at most n. Applying Lemma 7(VI), we see that M has the bounded blocking property. □

5 Illumination

In this section we will study some illumination problems. Recall that two points x, y on a translation surface M do not illuminate each other if and only if they are finitely blocked with blocking cardinality zero. Also recall that p_1, p_2 denote the projections onto the first and second factors of $M \times M$. The following result is the main result of this section:

Theorem 10 *Let M be a translation surface and let n be a nonnegative integer. Then:*

(i) *For any $x \in M$, the set $\{y \in M : \mathrm{bc}(x,y) \le n\}$ is either finite or contains $M \smallsetminus (\Sigma \cup \{x\})$.*

(ii) *The set $\{(x,y) \in M^2 : \mathrm{bc}(x,y) \le n\}$ either contains $\widehat{M^2}$ or is contained in a union of finitely many points and finitely many 2–dimensional translation surfaces embedded affinely in M^2. The translation surfaces in M^2 are either of the form $\{x\} \times M$ or $M \times \{y\}$, where $x, y \in M$, or are the diagonal embedding $\Delta = \{(x,x) : x \in M \smallsetminus \Sigma\}$, or they are closures of a surface S in $\widehat{M^2}$. In the latter case, for $i = 1,2$, the image of $\tau_i = p_i|_S \colon S \to M$ is the complement of finitely many points in M, τ_i is a finite-degree covering map of its image, and there is a scalar λ with $\lambda^2 \in \mathbb{Q}$, such that for every $x \in \tau_1(S)$, and every $y \in \tau_1^{-1}(x)$, the derivative map $(d_y \tau_2) \circ (d_x \tau_1^{-1})$ is equal to multiplication by λ.*

Theorem 10 implies Theorem 2 We apply Theorem 10 with $n = 0$. It is clear that the second alternative in (i) cannot hold, since for any x all nearby points illuminate x. Also, in (ii), the cases $F \times M$ and $M \times F$ do not arise, since any point illuminates some other point. □

Proof of Theorem 10 Keep the notation of Section 2.1 and Lemma 7. We will first prove (i) in case x is a regular point of M. Let $M' \in \varphi'^{-1}(M) \subset \mathcal{H}'$ denote the surface with first marked point at x. We need to show that

$$A := \{y \in M \smallsetminus (\Sigma \cup \{x\}) : \mathrm{bc}(x,y) \le n\},$$

which we may identify with $\mathcal{F}_n \cap \varphi''^{-1}(M')$, is either finite or coincides with $\varphi''^{-1}(M')$. Let us assume $A \subsetneq \varphi''^{-1}(M')$. Since $\mathcal{F}_n$ is closed and G–invariant, A is a union of at most countably many linear manifolds, which are of the form $\mathcal{L}(M_0'') := \overline{GM_0''}$ for $M_0'' \in A$. For each M_0'', the intersection $\mathcal{L}(M_0'') \cap \varphi''^{-1}(M')$ is a linear manifold of dimension 0 or 2 by Theorem 4. If the dimension were 2, A would coincide with the fiber $\varphi''^{-1}(M)$ by connectedness. Therefore A is countable, and we need to show that A is finite.

To this end we first show that the intersection of A with each individual orbit-closure $\mathcal{L}(M_0'')$ is finite. Let $\mathcal{L}' := \overline{GM'} \subset \mathcal{H}'$ and let $p \colon \mathcal{L}(M_0'') \to \mathcal{L}'$ denote the restriction of φ'' to $\mathcal{L}(M_0'')$. Since p is an affine map, the dimension of $\mathcal{L}(M_0'')$ is the sum of the dimensions of the image of p and the fiber of p, and hence, using Lemma 6, the dimension of each $\mathcal{L}(M_0'')$ is the same as the dimension of $\mathcal{L}'$. The projection p is a covering map, ie there is a connected neighborhood $\mathcal{V}$ of M' in $\mathcal{L}'$ such that the connected components of $p^{-1}(\mathcal{V})$ each map homeomorphically and affinely under p to $\mathcal{V}$. Since $\mathcal{L}(M_0'')$ is the support of the measure induced by coning off the Lebesgue measure on $\mathbb{R}_+^*\mathcal{L}$, each component of $p^{-1}(\mathcal{V})$ must have the same measure, and since this measure is finite there can only be finitely many preimages of $\mathcal{V}$. In particular, $\mathcal{L}(M_0'') \cap A$ is finite.

Now suppose if possible that A contains points from infinitely many distinct orbit-closures $\mathcal{L}(M_0'')$, all of the same dimension. By Theorem 4, A must contain accumulation points belonging to an affine invariant manifold $\mathcal{L}_\infty$ of bigger dimension, contradicting the fact that each affine invariant manifold $\mathcal{L}(M_0'')$ has the same dimension. This proves the finiteness of A.

In case $x = \xi$ is a singularity we repeat the argument, using Proposition 8 instead of Lemma 7, $\mathcal{F}_n'$ instead of $\mathcal{F}_n$, φ' instead of φ'' and $\mathcal{H}'$ instead of $\mathcal{H}''$. We leave the details to the reader.

We now prove (ii). Suppose that

$$\widehat{M^2} \not\subset A := \{(x,y) \in M^2 : \mathrm{bc}(x,y) \le n\}.$$

Applying Theorem 4 as in the proof of assertion (i), we see that $A \smallsetminus \Delta$ is the union of countably many 0–dimensional and countably many 2–dimensional linear manifolds. To show that these countable collections are in fact finite, we repeat the argument given above, using the map φ instead of the map φ''. It remains to show that all of the 2–dimensional manifolds have the stated form.

Let $S \subset \widehat{M^2}$ be a 2–dimensional linear manifold in A. By Theorem 4, S is $\mathbb{C}$–linear, ie for any $w = (w_1, w_2) \in S$ there is a neighborhood U of w in M^2 such that, in the translation charts, $U \cap S$ is the set of solutions of an equation of the form

$$az_1 + bz_2 = 0 \tag{2}$$

(up to a translation). Moreover, S is defined over $\mathbb{R}$, so we can take $a, b \in \mathbb{R}$. If $a = 0$ then any connected component of S is of the form $M \times \{y\}$ for some $y \in M$. Similarly, if $b = 0$ then S has the form $\{x\} \times M$. Now we consider the case when a, b are both nonzero.

Since the transition maps for the translation atlas are translations, a and b can actually be taken to be independent of the neighborhood, and the Cartesian product translation structure on M^2, restricted to S, endows S with a natural structure of a translation surface (see [3, Section 3] for more details), where S is locally modeled on the plane (2). Since a and b are both nonzero, each of the projections $\tau_i = p_i|_S$ has a nonsingular derivative, so is an open map. For each x in the image of τ_i, the fiber $\tau_i^{-1}(x)$ is finite by (i). Therefore each point in the image of τ_i is evenly covered, ie τ_i is a covering map of its image.

We now show that the complement of the image of each τ_i is finite. For concreteness we set $i=1$, the proof for $i=2$ being identical. For each $\xi \in \Sigma$, the set $\overline{S} \cap p_2^{-1}(\xi)$ is finite by Proposition 8 and part (i). Therefore $F = \overline{S} \cap p_2^{-1}(\Sigma)$ and $p_1(F)$ are finite. We will show that $\tau_1(S) \smallsetminus p_1(F)$ is open and closed relative to $M \smallsetminus (\Sigma \cup p_1(F))$, and this will show, by connectedness of $M \smallsetminus (\Sigma \cup p_1(F))$, that the complement of the image of τ_1 is contained in $\Sigma \cup p_1(F)$. Since τ_1 is an open map we only have to show that $\tau_1(S) \smallsetminus p_1(F)$ is closed relative to $M \smallsetminus (\Sigma \cup p_1(F))$. Let $x_k \in \tau_1(S) \smallsetminus p_1(F)$ with $x_k \to x$ and assume that $x \notin p_1(F) \cup \Sigma$. Let $y_k \in M$ such that $(x_k, y_k) \in S$. By passing to a subsequence we can assume that $y_k \to y \in M$. If $y \in \Sigma$ then we have $x \in p_1(F)$, contrary to the assumption. Since S is relatively closed in $\widehat{M^2}$ and $x \notin \Sigma$, we have $(x, y) \in S$, so $x \in \tau_1(S)$, as required.

Finally, we prove the last assertion in the description of S. Since each τ_i has constant derivative, it is a closed map as well, and by connectedness, the image of τ_i is $M \smallsetminus \Sigma$. The plane (2) can be identified with $\mathbb{C}$ in many ways and thus the translation surface structure on S is only naturally defined up to a scalar multiple. However, for any fixed choice of translation structure on S, each of the maps τ_i is the composition of a dilation and a translation covering. Let k_i be the degree of the covering map τ_i, and let λ_i be the associated dilation. The choice of the λ_i depends on a choice of the translation structure on S, but since the derivative of $\tau_2 \circ \tau_1^{-1}$ is the map $z_1 \mapsto -(a/b) z_1$, we have $\lambda := \lambda_2/\lambda_1 = -a/b$. We can compute the area of S using each of the maps τ_i, to obtain

$$\operatorname{area}(S) = \frac{k_i}{\lambda_i^2} \operatorname{area}(M).$$

Comparing these formulae for $i = 1, 2$, we see that $\lambda^2 = (a/b)^2 = k_2/k_1 \in \mathbb{Q}$. ◻

6 Examples and questions

Let T be the standard torus, obtained from the unit square $[0,1]^2$ by gluing opposite sides to each other by translations. It has been known for a long time (see [6] and the references therein) that T has the finite blocking property. We describe explicitly what is known for this example, ie we describe blocking cardinalities of pairs of points in T and blocking sets realizing them.

Denote by π the projection from $\mathbb{R}^2$ to T. For any nonzero integer n, notice that the map $\mathbb{R}^2 \to \mathbb{R}^2$, $x \mapsto nx$ descends to a map $m_n \colon T \to T$ which multiplies both components by n in $\mathbb{R}/\mathbb{Z}$, and is therefore n^2-to-1.

Lemma 11 (a) *If x and y are distinct points on T, their blocking cardinality is* $\mathrm{bc}(x,y)=4$.

(b) *It is realized by the blocking set $B(x,y)=m_2^{-1}(x+y)$, which contains the midpoint of any geodesic from x to y.*

(c) *This is the unique blocking set of size* 4.

Proof Let $\tilde{x}$, $\tilde{y}$ denote points in $\mathbb{R}^2$ which project to x, y on T. Let $u=(1,0)$, $v=(0,1)$, $w=(1,1)$. The four segments from $\tilde{y}$ to the four points

$$\tilde{x}, \quad \tilde{x}+u, \quad \tilde{x}+v, \quad \tilde{x}+w$$

(four corners of a unit square) project to segments with disjoint interiors on T, so at least 4 points are required to block the pair (x,y). On the other hand, any line segment in T from x to y is the projection of a line segment in $\mathbb{R}^2$ from $\tilde{x}$ to $\tilde{y}+au+bv$, with a and b in $\mathbb{Z}$. Such a segment has midpoint $\frac{1}{2}(\tilde{x}+\tilde{y}+au+bv)$. This midpoint in $\mathbb{R}^2$ projects to one of the points

$$\tfrac{1}{2}(x+y), \quad \tfrac{1}{2}(x+y+u), \quad \tfrac{1}{2}(x+y+v), \quad \tfrac{1}{2}(x+y+w),$$

which are the four points in T comprising $m_2^{-1}(x+y)$. This proves that the set $B(x,y)$ is a blocking set and that $\mathrm{bc}(x,y)\leq 4$. So (a) and (b) are proved.

We now prove (c). We saw that the four segments from $\tilde{y}$ to $\tilde{x}$, $\tilde{x}+u$, $\tilde{x}+v$, $\tilde{x}+w$ project to segments on T with disjoint interiors, so a blocking set for (x,y) must contain at least a point in each of them. Consider the segment from $\tilde{y}+v$ to $\tilde{x}+u$. The only intersection of its projection to T with the interiors of our four segments is its midpoint m, which is also the midpoint of the segment from y to $y+w$. So a blocking set not containing m would need to contain at least five points. Similar reasoning proves the other three points in the proposed set $B(x,y)$ have to be in a blocking set of cardinality four. □

The following two lemmas extend this description to configurations blocking a point from itself, and describe larger blocking sets on T. They are proved by similar arguments and we leave the details to the reader.

Lemma 12 (a) *If $x=y$, then the blocking cardinality is* $\mathrm{bc}(x,x)=3$.

(b) *It is realized by the blocking set $B(x,x)=m_2^{-1}(2x)\smallsetminus\{x\}$, which is the set of midpoints of all primitive geodesics from x to x. This blocking set can also be described as $B(x,x)=x+B_0$, where $B_0=B(0,0)=m_2^{-1}(0)\smallsetminus\{0\}$.*

(c) *This is the unique blocking set of size* 3.

Lemma 13 (a) *Let n and a be relatively prime integers with $1 \leq a < n$. For any pair of points (x, y) with $x \neq y$, the set $B = m_n^{-1}(ax + (n-a)y)$ is a blocking set of cardinality n^2 for the pair (x, y). It contains the point located a/n of the way along each line segment from x to y on T.*

(b) *Let $n \geq 2$ be an integer. For the pair of points (x, x) with $x = 0$, the set*

$$B_0 = m_n^{-1}(0) \smallsetminus \{0\} = \big\{(a/n, b/n) : 0 \leq a < n,\ 0 \leq b < n,\ (a, b) \neq (0, 0)\big\}$$

is a blocking set of cardinality $n^2 - 1$.
For the pair of points (x, x) with $x \neq 0$, the set $B = x + B_0$ is a blocking set of cardinality $n^2 - 1$, also equal to $m_n^{-1}(nx)$.

We will use these computations to compute blocking configurations on branched covers of T. Recall that if $M \to T$ is a branched translation cover, a singularity of M corresponds to a ramification point of the cover, and if the angle at a singularity x is $2\pi k$ then k is called the *ramification index* of x.

Lemma 14 *Suppose M is a torus cover of degree d, with arbitrary branch locus and ramification type, and let p: $M \to T$ denote the covering map.*

(a) *For a pair (x, y) of points of M such that $p(x) \neq p(y)$, if B' is a blocking set for $(p(x), p(y))$ on T, then $B = p^{-1}(B')$ is a blocking set for (x, y), of cardinality at most d times that of B', with equality when B contains no zero of M, ie no ramification point of p.*

(b) *In particular:*
- *For almost every pair (x, y) of points of M, $\mathrm{bc}(x, y) \leq 4d$.*
- *For pairs (x, y) of points of M such that the set $B(p(x), p(y))$ contains branch points of p, the bound above is decreased by the sum of the ramification indices of the ramification points above these branch points.*

(c) *For a pair of points (x, y) on M such that $p(x) = p(y)$ (whether $x = y$ or not), $p^{-1}(B(p(x), p(x)))$ is a blocking set, so that $\mathrm{bc}(x, y) \leq 3d$. As above, when $B(p(x), p(y))$ contains branch points of p, the bound is decreased by the sum of the ramification indices of the ramification points above these branch points.*

Proof Both (a) and (b) are easy, and (c) follows from the following observation. When $p(x) = p(y)$, any geodesic path γ from x to y projects to a geodesic γ' from $p(x)$ to itself, possibly nonprimitive. Considering the restriction of the geodesic γ, if γ' is not primitive, to its initial part until it first reaches a point projecting to $p(x)$, we see that (c) holds. □

6.1 Examples

Example 1 The following example shows that quite general maps τ_1, τ_2 may arise in Theorem 10.

Proposition 15 *Let a, b be positive integers with $\gcd(a,b)=1$, let $n=a+b$, and let*

$$X = \{(-ax, bx) : x \in T\} \subset T \times T.$$

Also let $p\colon M \to T$ be a translation cover with branching locus $m_n^{-1}(0)$, and nontrivial ramification at each preimage of each branch point, and let

$$Y = (p \times p)^{-1}(X) \subset M \times M.$$

Then no two points in Y illuminate each other.

Proof For $x \in \mathbb{R}^2$, the point 0 is a/n along the geodesic in $\mathbb{R}^2$ from $-ax$ to bx. Thus, by Lemma 13, the set $B = m_n^{-1}(0)$ is a common blocking set, of cardinality n^2, for all pairs of points in X. Thus the statement follows from Lemma 14. □

Example 2 The following examples show that the map $\tau_2 \circ \tau_1^{-1}$ could be a translation. Let $M = T$ be the torus, and consider

$$N = \{(x, y) \in M^2 : \mathrm{bc}(x, y) \le 3\}.$$

Then according to Lemma 12, N contains the diagonal $\{(x, x) : x \in M\}$, but, according to Lemma 11, $N \neq M^2$. Therefore the diagonal is one of the linear submanifolds appearing in Theorem 10, and we can have $\tau_2 \circ \tau_1^{-1} = \mathrm{Id}$.

Similar examples in which $\tau_2 \circ \tau_1^{-1}$ is a nontrivial translation can be obtained by taking M to be a cyclic cover of T, for example the Escher staircase (see Figure 2). This surface admits a degree-3 cover $p\colon M \to T$ and it has a nontrivial translation automorphism $D\colon M \to M$ moving one step up the ladder. Let x and y be any two points such that $D(x) = y$. Then $p(x) = p(y)$, and, according to Lemma 14(c), $\mathrm{bc}(x, y) \le 9$. It is not hard to find an explicit pair of points x, y for which $\mathrm{bc}(x, y) > 9$. This shows that if we take this surface M and $n = 9$, then we can have a subsurface N for which $D = \tau_2 \circ \tau_1^{-1}$ is a translation automorphism.

Example 3 Using the torus and Lemmas 11 and 12 we easily find sequences $x_k \to x$, $y_k \to y$ for which $\mathrm{bc}(x, y) < \lim_k \mathrm{bc}(x_k, y_k)$, ie the blocking cardinality is not continuous. The following example shows that it is not even lower semicontinuous, ie it may increase when taking limits. It also shows that in Lemma 7(I) we cannot replace $\widehat{M^2}$ with M^2, and moreover that in Theorem 10(ii) the extension of the maps τ_i to the closure of S need not be surjective.

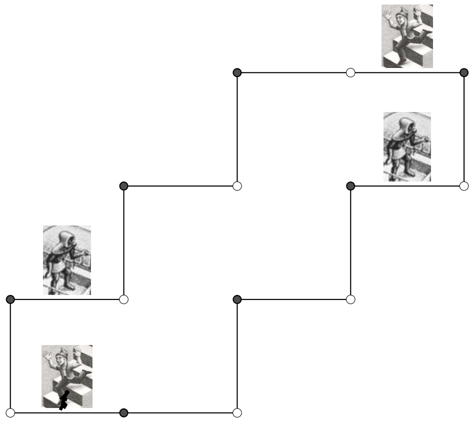

Figure 2: The Escher double staircase. Sides marked with identical stair-climbers are identified; unmarked sides are identified with the corresponding opposite sides.

Let M be a surface in $\mathcal{H}(2)$. Then M admits a hyperelliptic involution h, whose set of fixed points consists of the unique singularity ξ and five nonsingular Weierstrass points. We claim that whenever $h(x) = y$, $x \neq y$, we have $\mathrm{bc}(x, y) \leq 5$. Indeed, in this case, the action of h swaps x and $h(x)$, and acts by rotation by π. So h maps any segment σ between x and y to another segment from x to y, of the same length and in the same direction. Since x and y are distinct regular points there is only one such segment, ie h maps σ to itself, reversing the orientation on it. So its midpoint must be fixed by h, that is, the Weierstrass points form a blocking set for the pair (x, y).

On the other hand, by constructing explicit disjoint segments it is not hard to show that $\mathrm{bc}(\xi, \xi) \geq 9$. For example, we can present M as the union of six triangles (as part of an L–shaped presentation made of three parallelograms), and the edges of these triangles consist of nine disjoint segments from ξ to ξ (see Figure 3). Now, taking $x_k \to \xi$, we have $y_k = h(x_k) \to \xi$, and

$$5 \geq \lim_k \mathrm{bc}(x_k, y_k), \qquad \mathrm{bc}(\lim_k x_k, \lim_k y_k) = \mathrm{bc}(\xi, \xi) \geq 9.$$

6.2 Questions

Question 1 As in Theorem 10(ii), let $S \subset M \times M$ be a 2–dimensional linear submanifold, and let λ_1, λ_2 be the derivatives of the translation maps τ_1, τ_2. The quotient $\lambda = \lambda_1/\lambda_2$ is called the *slope* of N. In Example 2 the slope is 1, and in Example 1 the slope can be an arbitrary negative rational number. It would be interesting to know whether other slopes are possible. In particular, do the cases $\lambda = 0$, $\lambda = \infty$ actually

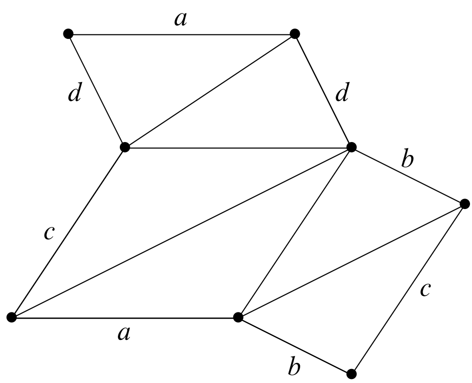

Figure 3: A surface in $\mathcal{H}(2)$ and nine disjoint saddle connections on it.

arise in connection with blocking configurations? Do positive rational slopes arise, except for $\lambda = 1$?

Question 2 As we saw in Example 1, infinitely many pairs of points on a translation surface may not illuminate each other; that is, the case of a 2–dimensional surface as in the second assertion of Theorem 2 may arise. Earlier examples of this phenomenon were obtained in [3]. However, these examples do not arise from rational billiards. So it is natural to ask whether, in connection with Corollary 3, there is a rational polygon P and infinitely many pairs of points $(x, y) \in P^2$ such that there is no geodesic trajectory between x and y.

Question 3 More generally, suppose $S \subset M \times M$ is an embedded translation surface for which the maps $\tau_i \colon S \to M$ are the composition of a dilation and a translation, and let λ be the derivative of the composition $\tau_2 \circ \tau_1^{-1}$. In the proof of Theorem 10 we showed that $\lambda^2 \in \mathbb{Q}$. Is it possible that λ is irrational?

Question 4 In connection with Example 3, does there exist a similar example in which the point ξ is nonsingular? That is, an example of a surface M with a regular point ξ and two sequences x_k, y_k converging to ξ, such that $\mathrm{bc}(\xi, \xi) > \lim_k \mathrm{bc}(x_k, y_k)$?

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