

Kontsevich's Swiss cheese conjecture

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We prove a conjecture of Kontsevich, which states that if A is an $E_d - 1$ algebra then the Hochschild cochain object of A is the universal E_d algebra acting on A . The notion of an E_d algebra acting on an E_{d-1} algebra was defined by Kontsevich using the Swiss cheese operad of Voronov. The degree 0 and 1 pieces of the Swiss cheese operad can be used to build a cofibrant model for A as an $E_{d-1} - A$ -module. The theorem amounts to the fact that the Swiss cheese operad is generated up to homotopy by its degree 0 and 1 pieces.

[13D03](#), [18D50](#); [18G55](#)

1 Introduction

Gerstenhaber [10] showed that the Hochschild cohomology $\mathrm{HH}^*(A)$ of an associative algebra A is a graded Lie algebra and a graded commutative algebra, and the two structures are compatible. Any graded vector space with this algebraic structure is now called a Gerstenhaber algebra. Cohen [7] showed that the homology of the little disks operad, $H_*(E_2)$, is the Gerstenhaber operad. Sinha [20] has also shown this. Deligne later asked if the action of $H_*(E_2)$ on $\mathrm{HH}^*(A)$ descends from a natural action at the level of chains. In other words, is there a natural algebra structure on $\mathrm{CH}^*(A)$ of $\mathrm{Chains}(E_2)$ which recovers the structure discovered by Gerstenhaber after passing to (co)homology?

Already, this question is evidently in the realm of homotopy theory. So let us replace the associative algebras by E_1 algebras. This makes it clear that the question is fundamentally one about the relationship between the operads E_1 and E_2 . Indeed, we can generalize to consider the relationship between E_d and E_{d-1} algebras. For any E_{d-1} algebra in a sufficiently rich homotopical category \mathcal{C} we can make sense of its Hochschild cochains as an object of \mathcal{C} . The Hochschild cochain object of A is denoted by $\mathrm{Hoch}(A)$ and is an object of the same category to which A belongs. This terminology is based on the case where \mathcal{C} is the category of differential graded vector spaces and A is an associative algebra. In that case, $\mathrm{Hoch}(A)$ is the usual Hochschild cochain complex of A .

The original Deligne conjecture, where A is an E_1 algebra in the category of chain complexes, has been solved by Tamarkin [22], Kontsevich and Soibelman [17], Voronov [26], McClure and Smith [19], Berger and Fresse [2] and Kaufmann and Schwell [15]. A cyclic version is also due to Kaufmann [14]. Both cyclic and non-cyclic versions can be found in Batanin and Berger [1]. Vallette [24] generalized the theorem to include certain other Koszul operads. The generalized version, where A is an E_d algebra in a general category like \mathcal{C} , has been proven in the ∞ -operad setting by Lurie [18]. We show here that $\text{Hoch}(A)$ is not just an E_d algebra, but comes equipped with a universal property. It is the universal E_d algebra acting on the E_{d-1} algebra A . A chain-level realization of the Swiss cheese action is constructed in Dolgushev, Tamarkin and Tsygan [9].

The notion of an E_d algebra acting on an E_{d-1} algebra was introduced by Kontsevich [16]. This notion uses the Swiss cheese operad SC_d of Voronov [25]. This is a 2-colored operad which interpolates between E_d and E_{d-1} . A Swiss cheese algebra is a pair (B, A) where B is an E_d algebra, A is an E_{d-1} algebra and there is some extra structure compatible with these, as seen in Definition 2.1.9. We refer to this extra structure as an *action* of B on A .

The case $d = 1$ is enlightening. For simplicity, let us work in the category of vector spaces. A (non-unital) E_0 algebra A in vector spaces is just a vector space with no extra data. The Hochschild object in this case is $\text{hom}(A, A)$. If B is an associative algebra, it is in particular an E_1 algebra. An SC_1 structure on the pair (B, A) then amounts to the choice of a B -module structure on A .

In this case, the Swiss cheese conjecture merely states that $\text{hom}(A, A)$ is an associative algebra and giving an SC_1 structure on (B, A) is equivalent to giving a map of associative algebras $B \rightarrow \text{Hoch}(A) = \text{hom}(A, A)$. We prove the analog of this when B is an E_d algebra and A is an E_{d-1} algebra, $d \geq 1$.

Joseph Hirsh brought to the attention of the author the following helpful characterization of the results of this paper. Given a bifibrant E_{d-1} algebra A the functor from E_d algebras to spaces

$$B \mapsto \{\text{the space of Swiss cheese actions of } B \text{ on } A\}$$

is represented by the Hochschild cochain object of A .

1.1 Outline of the paper

In Section 2 we define the E_d and SC_d operads and give both an imprecise and a precise statement of the theorem we will prove. We also outline the idea of the proof.

In [Section 3](#) we define the Hochschild cochain object for E_d algebras and show that we can use the Swiss cheese operad to construct a model for the Hochschild cochain object. We use this model in [Section 4](#) to prove a “universal cheese” theorem which applies to an arbitrary operad acting on Hochschild cochains. In [Section 5](#) we show that E_d does indeed act, up to homotopy, on Hochschild cochains, and the universal cheese theorem specializes to the main theorem of this paper. Finally, in [Section 6](#) we prove the main theorem that allows the homotopy E_d action: the Swiss cheese operad is freely generated up to homotopy by its degree 0 and 1 pieces.

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2 The Swiss cheese operad

We will define K -colored operads in general and the Swiss cheese operad in particular. We also describe algebras over the Swiss cheese operad and state the main theorem of this paper.

2.1 The colored operad Swiss cheese

Fix a set K . A K -colored set is a pair (I, col) where I is a set and $\text{col}: I \rightarrow K$ is a map of sets, called the *coloring*. We will often denote such a colored set simply by I , leaving the coloring implicit. Let $\text{aut } I$ be the group of bijections on the set I which preserve its coloring. Since we have left the coloring implicit, we use the notation $I^\#$ to refer to the underlying uncolored set.

Let (\mathcal{S}, \otimes) be a symmetric monoidal category. We can speak of categories enriched over \mathcal{S} . In particular, suppose \mathcal{O} is a category enriched over \mathcal{S} and suppose the objects of \mathcal{O} are finite K -colored sets. We let $\mathcal{O}(I; J)$ denote the \mathcal{S} object of morphisms in \mathcal{O} from I to J . If we further suppose that disjoint union of finite K -colored sets extends to an \mathcal{S} -enriched symmetric monoidal structure on \mathcal{O} , then each $\mathcal{O}(I; J)$ is a right $\text{aut}(I)$ - and left $\text{aut}(J)$ -module in a natural way. In addition, if \mathcal{S} contains finite coproducts, the symmetric monoidal structure on morphisms in \mathcal{O} is specified by equivariant maps

$$(1) \quad \coprod_{f: I^\# \rightarrow J^\#} \bigotimes_{j \in J} \mathcal{O}(f^{-1}\{j\}; \{j\}) \rightarrow \mathcal{O}(I; J).$$

The following definition follows Boardman and Vogt [6].

2.1.1 Definition Let (\mathcal{S}, \otimes) be a symmetric monoidal category containing all finite coproducts where \otimes distributes over finite coproducts. Let K be a set. The data of a K -colored operad \mathcal{O} in the symmetric monoidal category \mathcal{S} is a symmetric monoidal category, denoted by \mathcal{O} , which is enriched over \mathcal{S} and whose objects are K -colored finite sets $I \rightarrow K$. This data must satisfy two conditions. First, on objects, the symmetric monoidal structure of \mathcal{O} is the disjoint union of sets over K . Second, the map in (1) must be an isomorphism for every I and J .

2.1.2 Remark We will use the unmodified noun *operad* to mean K -colored operad when the coloring set K is clear from context. The reader should note that this differs from an equally plausible convention, where *operad* is always used to denote $\{*\}$ -colored operads.

2.1.3 Notation Let \underline{n} denote the finite set $\{1, \dots, n\}$. Typically K will be $K = \{f, h\}$, where f stands for *full-disk* and h stands for *half-disk*. In this case we use (n, m) to denote the K -colored set where $\{1, \dots, n\}$ has color f and $\{n+1, \dots, m\}$ has color h . If $K \simeq \{*\}$, then a K -colored operad will simply be called a 1-colored operad. Any 1-colored operad \mathcal{E} gives, for each $m, m' \in \mathbb{Z}_{\geq 0}$, objects $\mathcal{E}(\underline{m}'; \underline{m}) \in \mathcal{S}$. We denote $\mathcal{E}(\underline{m}'; \underline{m})$ simply by $\mathcal{E}(m'; m)$ and $\mathcal{E}(m; 1)$ simply by $\mathcal{E}(m)$.

Any $\{f, h\}$ -colored operad \mathcal{O} gives, for each $n, m, n', m' \in \mathbb{Z}_{\geq 0}$, objects

$$\mathcal{O}(n', m'; n, m) := \mathcal{O}((n', m'); (n, m)) \in \mathcal{S}.$$

We denote $\mathcal{O}(n, m; 0, 1)$ by $\mathcal{O}^h(n, m)$ and $\mathcal{O}(n, m; 1, 0)$ by $\mathcal{O}^f(n, m)$.

2.1.4 Definition Let (Top, \times) denote the symmetric monoidal category of compactly generated Hausdorff topological spaces with the cartesian product.

2.1.5 Example Let K be the one-point set $\{f\}$ and let $(\mathcal{S}, \otimes) = (\text{Top}, \times)$. The operad E_d is an $\{f\}$ -colored operad in the category Top . Let \bar{D}^d be the closed unit disk inside \mathbb{R}^d . Call a map $f: \bar{D}^d \rightarrow \bar{D}^d$ a *little full-disk* (or *little d disk* or simply *little disk*) if f is of the form $f(x) = rx + c$ for some $0 < r \leq 1$ and $c \in \mathbb{R}^d$. Given a finite set I , the underlying set of $E_d(I) := E_d^f(I)$ is the set of embeddings

$$f: \coprod_{i \in I} \bar{D}^d \rightarrow \bar{D}^d,$$

where each restriction $f_i: \bar{D}^d \rightarrow \bar{D}^d$ is a little full-disk. Using [Notation 2.1.3](#), any isomorphism $I \rightarrow \underline{n}$ allows us to identify $E_d(I)$ with a subset of \mathbb{R}^{n+dn} . This gives

each $E_d(I)$ a topology. The operadic structure is given by composing little d disks as maps $\bar{D}^d \rightarrow \bar{D}^d$. The identity of E_d is the little full-disk id: $\bar{D}^d \rightarrow \bar{D}^d$. This is the unital version of E_d , so $E_d(0) = *$ and $E_d(1)$ consists of more than just the identity.

2.1.6 Example Let $K = \{f, h\}$. The K -colored operad SC_d is called the (d -dimensional) Swiss cheese operad and is the principal subject of this paper. Like [Example 2.1.5](#), it is an operad in (Top, \times) .

By [Definition 2.1.1](#) and formula (1), to define SC_d as a K -colored operad we only need to define the spaces $SC_d^f(I)$ and $SC_d^h(I)$ for every K -colored set $(I, \text{col}: I \rightarrow K)$. First, we define the “full-disk output” part of SC_d ,

$$SC_d^f(I) = \begin{cases} E_d(I) & \text{if } \text{col}^{-1}(f) = I, \\ \emptyset & \text{else.} \end{cases}$$

To define the “half-disk output” part of SC_d , that is $SC_d^h(I)$, we first need the notion of *little half-disks*. Let \bar{D}_+^d be the closed d -dimensional half-disk,

$$\bar{D}_+^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid |x| \leq 1 \text{ and } x_d \geq 0\}.$$

A *little half-disk* is defined to be a map $f: \bar{D}_+^d \rightarrow \bar{D}_+^d$ of the form $f(x) = rx + c$ for some $0 < r \leq 1$ and $c \in \mathbb{R}^{d-1} \times \{0\}$. As a set, we define $SC_d^h(I)$ to consist of embeddings

$$f: \coprod_{i \in I} \bar{D}^d \rightarrow \bar{D}_+^d,$$

where each restriction $f_i: \bar{D}^d \rightarrow \bar{D}_+^d$ is a little full-disk ([Example 2.1.5](#)) if $\text{col}(i) = f$ or a little half-disk if $\text{col}(i) = h$. It is clear that if $|\text{col}^{-1}(f)| = n$ and $|\text{col}^{-1}(h)| = m$, then $SC_d^h(I)$ can be naturally embedded inside \mathbb{R}^N , where $N = (d + 1)n + dm$. We give $SC_d^h(I)$ the subspace topology inherited from such an embedding.

Following [Notation 2.1.3](#), a point in $SC_d^h(n, m)$ is given by n labeled full-disks and m labeled half-disks in the unit half-disk, where none of the disks intersect and the half-disks all lie on the bottom. We allow the degenerate configuration when $(n, m) = (0, 1)$, which is the unit half-disk contained in itself. Note that we have $SC_d^h(0, 0) = *$ and $SC_d^h(1, 0)$ contains more than one point. Thus we are using the unital Swiss cheese operad. This differs from [\[16; 25\]](#).

Composition in SC_d is given by substituting full-disks and half-disks into each other. More precisely, we have maps

$$(2) \quad E_d(n) \times E_d(k_1) \times \dots \times E_d(k_n) \rightarrow E_d(k_1 + \dots + k_n)$$

and

$$(3) \quad SC_d^h(n, m) \times E_d(k_1) \times \cdots \times E_d(k_n) \times SC_d^h(k_{n+1}, \ell_1) \times \cdots \times SC_d^h(k_{n+m}, \ell_m) \\ \longrightarrow SC_d^h(k_1 + \cdots + k_{n+m}, \ell_1 + \cdots + \ell_m).$$

Notice that we can identify $SC_d^h(0, m)$ with $E_{d-1}(m)$ so that the restriction of SC_d to the spaces $SC_d^h(0, \bullet)$ is isomorphic to the operad E_{d-1} . We say that E_{d-1} is the h color of SC_d and E_d is the f color of SC_d . We think of SC_d as interpolating between E_d and E_{d-1} .

2.1.7 Definition Suppose \mathcal{O} is a K -colored operad in \mathcal{S} and \mathcal{C} is a symmetric monoidal category enriched over \mathcal{S} . An *algebra* over \mathcal{O} in the category \mathcal{C} is a strong symmetric monoidal functor $\mathcal{O} \rightarrow \mathcal{C}$. A morphism of \mathcal{O} algebras is a monoidal natural transformation. The category of \mathcal{O} algebras in \mathcal{C} will be denoted by $\text{Alg}_{\mathcal{O}}(\mathcal{C})$.

2.1.8 Example If \mathcal{C} is a symmetric monoidal category enriched over Top , we can consider algebras over SC_d in \mathcal{C} . Such an algebra gives the data of a pair (B, A) of objects in \mathcal{C} together with maps of topological spaces

$$E_d(n) \rightarrow \text{map}_{\mathcal{C}}(B^{\otimes n}, B)$$

and

$$SC_d^h(n, m) \rightarrow \text{map}_{\mathcal{C}}(B^{\otimes n} \otimes A^{\otimes m}, A),$$

where $\text{map}_{\mathcal{C}}(C, C')$ is the topological space of maps between two objects C and C' in \mathcal{C} . These data must satisfy conditions guaranteeing they assemble into a strong symmetric monoidal functor $SC_d \rightarrow \mathcal{C}$.

The object B corresponds to the object $\{f\}$ of SC_d and the object A corresponds to $\{h\}$. Together these form a K -colored operad $\text{End}(B, A)$ in Top , where, using [Notation 2.1.3](#),

$$(4) \quad \text{End}^f(B, A)(n, m) = \text{map}(B^{\otimes n} \otimes A^{\otimes m}, B),$$

$$(5) \quad \text{End}^h(B, A)(n, m) = \text{map}(B^{\otimes n} \otimes A^{\otimes m}, A).$$

The definition of the operad morphisms on $\text{End}(B, A)$ are left to the reader. The collection of SC_d algebra structures on a fixed pair (B, A) is the collection of strong symmetric monoidal functors

$$SC_d \rightarrow \text{End}(B, A),$$

which restrict to the identity on the set of objects. Simply put, a Swiss cheese algebra (B, A) is an E_d algebra B , an E_{d-1} algebra A and some chosen mixing of these structures. We refer to this mixing as an action of B on A .

2.1.9 Definition (Kontsevich [16]) Let B be an E_d algebra and A an E_{d-1} algebra. A *Swiss cheese action of B on A* is the structure of a Swiss cheese algebra on the pair (B, A) extending the given E_d and E_{d-1} structures. We may also simply call this “an action of B on A ”.

2.2 Statement of main theorem

Now we can informally state the conjecture proven in this paper.

Informal statement of Swiss cheese conjecture *The Hochschild cochain object $\text{Hoch}(A)$ of an E_{d-1} algebra A is the universal E_d algebra acting on A . In other words, for any E_{d-1} algebra A , there is an E_d algebra structure on $\text{Hoch}(A)$ such that, for any E_d algebra B , giving a map of E_d algebras $B \rightarrow \text{Hoch}(A)$ is equivalent to giving the structure of an SC_d algebra on the pair (B, A) extending the given E_d and E_{d-1} structures.*

The basic structure of the proof is outlined in this diagram:

$$\begin{array}{ccccc}
 \text{Op}_K & \longrightarrow & \text{Op}(\text{Top}^\Sigma) & \longrightarrow & \text{Op}(\text{Top}^{\Sigma \leq 1}) \\
 (6) \quad SC_d & \longleftarrow & SC_d^h & \longrightarrow & SC_d^{h1} \\
 SC_d^{h\infty} \times E & \xleftarrow{\text{transfer of structure}} & SC_d^{h\infty} & \longleftarrow & SC_d^{h1}
 \end{array}$$

The categories of operads shown in the diagram are defined precisely in Section 2.3. The category Top^Σ consists of symmetric sequences of topological spaces. Justification will be given later, but for now think of $\text{Op}(\text{Top}^\Sigma)$ as consisting of K -colored operads whose f -colored output is trivial. Further, we will introduce a category $\text{Op}(\text{Top}^{\Sigma \leq 1})$, consisting of K -colored operads whose f -colored inputs total 0 or 1. The forgetful functors are presented below as straight arrows; there is a left adjoint shown as a bent arrow.

Boardman and Vogt's W construction [6] is an explicit cofibrant replacement functor, which we apply to SC_d , to get an equivalent cofibrant operad $SC_d := WSC_d$. The W construction does not strictly commute with the forgetful functors in (6). In particular, $W(SC_d^h)$ is not isomorphic to $(WSC_d)^h$, but they are homotopy equivalent. We will let SC_d^h denote $W(SC_d^h)$ and $SC_d^{h1} = W(SC_d^{h1})$.

Proposition 3.2.7 shows that we can use SC_d^{h1} to construct a model for the Hochschild cochain object. This allows us to prove a weak version of the Swiss cheese theorem in Proposition 4.1.2, taking place in the context of $\text{Op}(\text{Top}^{\Sigma \leq 1})$. Next, we will take the free extension of $SC_d^{h1} \in \text{Op}(\text{Top}^{\Sigma \leq 1})$ to an operad in $\text{Op}(\text{Top}^\Sigma)$, to get $SC_d^{h\infty}$. This

immediately gives a version of the Swiss cheese theorem in the context of $\text{Op}(\text{Top}^\Sigma)$; see [Corollary 4.1.4](#). Then we use the fact that $SC_d^{\text{h}\infty}$ is freely generated by its degree 0 and 1 pieces to prove a version of the Swiss cheese theorem in the context of Op_K . None of these three versions of the Swiss cheese theorem make any use of E_d . One can think of this last “universal cheese” theorem ([Proposition 4.1.9](#)) as a construction of the universal K -colored operad built from E_{d-1} and controlling E_{d-1} -linear actions on A .

To bring E_d back into the story, we use a technical result, [Theorem 5.1.1](#), which shows that the canonical map $SC_d^{\text{h}\infty} \rightarrow SC_d^{\text{h}}$ is an equivalence. Observe that one can view SC_d as SC_d^{h} equipped with the extra structure of a right action of E_d . Now use a transfer of structure argument to construct an operad E which is equivalent to E_d and which acts on the right on $SC_d^{\text{h}\infty}$. This allows us to define $SC_d^{\text{h}\infty} \rtimes E$, which we show is equivalent to SC_d in [Lemma 5.1.6](#). The universal property of $SC_d^{\text{h}\infty} \rtimes E$ with respect to the Hochschild cochain object is stated in [Theorem 2.2.1](#) and follows from [Proposition 4.1.9](#).

It should be noted that the theorem rests on the equivalence between $SC_d^{\text{h}\infty}$ and SC_d^{h} . At first glance, these spaces seem very different. The former has its d -dimensional disks very spread out. The latter allows its d -dimensional disks to be arbitrarily close together. Each element of $SC_d^{\text{h}\infty}$ can be broken into a tree where each piece has at most one d -dimensional disk. Thus, the only interesting topology on $SC_d^{\text{h}\infty}$, aside from the topology of E_{d-1} , is given by the combinatorics of these trees. The surprising fact is that these combinatorics are rich enough to account for the topology of configuration spaces of d -dimensional disks. The best summary explanation for this equivalence was revealed to me in a conversation with Ralph Kaufmann, where he pointed out that $SC_d^{\text{h}\infty}$ resembles a kind of cellular structure relative to E_{d-1} on SC_d^{h} .

2.2.1 Theorem (Precise version of Kontsevich’s Swiss cheese conjecture) *Let \mathcal{C} be a symmetric monoidal model category tensored over Top and satisfying the conditions in [Notation 3.1.9](#). Let $A \in \text{Alg}_{E_{d-1}}(\mathcal{C})$ be cofibrant and fibrant using the projective model structure ([Definition 3.1.8](#)). There is a model of SC_d , called $SC_d^{\text{h}\infty} \rtimes E$, where $E \simeq E_d$. There is also a model for the Hochschild cochain object of A , called $\text{Hoch}(A)$, such that $\text{Hoch}(A)$ is the universal E algebra acting on A through $SC_d^{\text{h}\infty} \rtimes E$. That is, $\text{Hoch}(A)$ is an E algebra and this structure, together with the E_{d-1} algebra structure on A , can be extended to an $SC_d^{\text{h}\infty} \rtimes E$ algebra structure on $(\text{Hoch}(A), A)$ in such a way that there is an equivalence of relative categories*

$$\text{Alg}_{(SC_d^{\text{h}\infty} \rtimes E)}^A(\mathcal{C}) \cong \text{Alg}_E(\mathcal{C}) / \text{Hoch}(A).$$

The category on the left consists of E algebras B together with an action of B on A . The category on the right consists of E algebras B together with an E algebra map $B \rightarrow \text{Hoch}(A)$.

Lemma 5.1.6 shows that $SC_d^{\text{h}\infty} \rtimes E$ is equivalent to SC_d . Lemma 5.1.5 shows that E is equivalent to E_d . Proposition 3.2.7 sets up our choice of model of $\text{Hoch}(A)$. Proposition 4.1.9, together with the construction of the operad $SC_d^{\text{h}\infty} \rtimes E$ in Section 5, shows the desired equivalence of relative categories.

2.3 Defining SC_d^{h} and $SC_d^{\text{h}1}$

Recall that \mathcal{C} is a symmetric monoidal category enriched over \mathcal{S} , our basic category in which our operads live. We will assume that both \mathcal{C} and \mathcal{S} have all coproducts and that tensor products distribute over finite coproducts. In the case of operads from Op_K we have $(\mathcal{S}, \otimes) = (\text{Top}, \times)$, the symmetric monoidal category of compactly generated topological spaces with cartesian product. In the case of operads such as SC_d^{h} and $SC_d^{\text{h}1}$ from $\text{Op}(\text{Top}^\Sigma)$, we have $(\mathcal{S}, \otimes) = (\text{Top}^\Sigma, \otimes)$ from Definition 2.3.1. Finally, for $SC_d^{\text{h}1}$ and $SC_d^{\text{h}1}$ we use $\mathcal{S} = \text{Top}^{\Sigma \leq 1}$ as in Definition 2.3.2.

2.3.1 Definition Let Σ denote the category of finite sets with morphisms given by bijections and let $(\mathcal{D}, \otimes_{\mathcal{D}})$ be any symmetric monoidal category. The category of functors $\Sigma^{\text{op}} \rightarrow \mathcal{D}$, denoted by \mathcal{D}^Σ , is usually called the category of *symmetric sequences* in \mathcal{D} . Such a functor is a sequence $(X(n))_{n \geq 0}$ of objects in \mathcal{D} together with a right $\text{aut } \underline{n}$ action on $X(n)$ for each $n \geq 0$. We endow \mathcal{D}^Σ with the usual symmetric monoidal structure given by left Kan extension of $\Sigma \times \Sigma \rightarrow \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ along disjoint unions of sets $\Sigma \times \Sigma \rightarrow \Sigma$. Specifically, if $X, Y \in \mathcal{D}^\Sigma$, then $X \otimes Y \in \mathcal{D}^\Sigma$ satisfies

$$(X \otimes Y)(\underline{n}) = \coprod_{n=n_1+n_2} \text{Ind}_{S_{n_1} \times S_{n_2}}^{S_n} (X(\underline{n}_1) \otimes_{\mathcal{D}} Y(\underline{n}_2)),$$

where $S_n = \text{aut } \underline{n}$. See Harper [12, Definition 3.3] for more details.

2.3.2 Definition Let $\Sigma_{\leq 1}$ denote the full subcategory of Σ consisting of finite sets of size 0 or 1 together with bijections as morphisms. Let $\mathcal{D}^{\Sigma \leq 1}$ denote the symmetric monoidal category of functors $\Sigma_{\leq 1} \rightarrow \mathcal{D}$, with monoidal structure inherited from \mathcal{D}^Σ . Call these the *degree 0–1 symmetric sequences in \mathcal{D}* . Concretely, $\mathcal{D}^{\Sigma \leq 1}$ is just the category $\mathcal{D} \times \mathcal{D}$ endowed with the symmetric monoidal structure

$$(7) \quad (C_0, C_1) \otimes (D_0, D_1) = (C_0 \otimes_{\mathcal{D}} D_0, (C_0 \otimes_{\mathcal{D}} D_1) \amalg (C_1 \otimes_{\mathcal{D}} D_0)).$$

The braiding isomorphism $(C_0, C_1) \otimes (D_0, D_1) \rightarrow (D_0, D_1) \otimes (C_0, C_1)$ is induced from the braiding isomorphism on \mathcal{D} .

2.3.3 Definition Let Op_K denote the category of $K = \{f, h\}$ -colored operads in Top . There is a forgetful functor $\text{Op}_K \rightarrow \text{Op}(\text{Top}^\Sigma)$. This functor takes $\mathcal{O} \in \text{Op}_K$ to the operad \mathcal{O}^h whose arity- m component is the symmetric sequence $n \mapsto \mathcal{O}^h(n, m)$ (see [Notation 2.1.3](#)). We forget the spaces $\mathcal{O}^f(n, m)$ and think of elements of $\mathcal{O}^h(n, m)$ as degree- n , arity- m elements of \mathcal{O}^h .

The functor $\text{Op}(\text{Top}^\Sigma) \rightarrow \text{Op}(\text{Top}^{\Sigma \leq 1})$ is induced by the symmetric monoidal forgetful functor $\text{Top}^\Sigma \rightarrow \text{Top}^{\Sigma \leq 1}$. Write the image of \mathcal{O} in $\text{Op}(\text{Top}^{\Sigma \leq 1})$ as \mathcal{O}^{h1} . Furthermore, the functor $\text{Op}(\text{Top}^{\Sigma \leq 1}) \rightarrow \text{Op}(\text{Top})$ is induced by the symmetric monoidal functor $\text{Top}^{\Sigma \leq 1} \rightarrow \text{Top}$, $(X, Y) \mapsto X$. We denote this functor by $\mathcal{O} \mapsto \mathcal{O}^0$.

Observe that if $\mathcal{O} \in \text{Op}(\text{Top}^{\Sigma \leq 1})$ then \mathcal{O}^0 has commuting left and right actions on \mathcal{O} . By this we mean we have maps

$$\mathcal{O}(0, m'') \times \mathcal{O}(n, m'; 0, m'') \times \mathcal{O}(0, m; 0, m') \rightarrow \mathcal{O}(n, m),$$

satisfying straightforward conditions which we leave to the reader.

2.3.4 Example We outline the structure of SC_d^h as an operad in $\text{Op}(\text{Top}^\Sigma)$. Think of $\text{SC}_d^h(\bullet, m)$ as the symmetric sequence $n \mapsto \text{SC}_d^h(n, m)$. In [\(8\)](#), \otimes is the tensor product of symmetric sequences. The operad composition law is

$$(8) \quad \text{SC}_d^h(\bullet, m) \otimes \text{SC}_d^h(\bullet, \ell_1) \otimes \cdots \otimes \text{SC}_d^h(\bullet, \ell_m) \rightarrow \text{SC}_d^h(\bullet, \ell),$$

where $\ell = \sum_i \ell_i$. The degree- n component of the right-hand symmetric sequence is $\text{SC}_d^h(n, \ell)$. The degree- n component of the left-hand symmetric sequence is

$$\coprod_{n_0 + \cdots + n_m = n} \text{Ind } \text{SC}_d^h(n_0, m) \times \text{SC}_d^h(n_1, \ell_1) \times \cdots \times \text{SC}_d^h(n_m, \ell_m),$$

where Ind is the induction functor giving the correct symmetric group action. The point is that if we delete all appearances of E_d from [\(3\)](#) then it provides exactly the data of [\(8\)](#).

2.3.5 Example We outline the structure of SC_d^{h1} as a 1-colored operad in $\text{Top}^{\Sigma \leq 1}$. For each m' and m , define an object $\text{SC}_d^{h1}(m'; m) \in \text{Top}^{\Sigma \leq 1}$,

$$(9) \quad \begin{aligned} \text{SC}_d^{h1}(m'; m) &= (\text{SC}_d^h(0, m'; 0, m), \text{SC}_d^h(1, m'; 0, m)) \\ &\cong (E_{d-1}(m'; m), \text{SC}_d^h(1, m'; 0, m)). \end{aligned}$$

Using the symmetric monoidal structure from [Definition 2.3.2](#) and the identification in [\(9\)](#), we can write the operad structure maps on SC_d^{h1} as a triple of morphisms: the map of degree-0 pieces

$$E_{d-1}(m''; m) \times E_{d-1}(m'; m'') \rightarrow E_{d-1}(m'; m),$$

and the maps of degree-1 pieces

$$E_{d-1}(m''; m) \times SC_d^h(1, m'; 0, m'') \rightarrow SC_d^h(1, m'; 0, m)$$

and

$$SC_d^h(1, m''; 0, m) \times E_{d-1}(m'; m'') \rightarrow SC_d^h(1, m'; 0, m).$$

3 Hochschild cohomology from Swiss cheese

For the remainder of the paper we replace E_{d-1} , E_d and SC_d by cofibrant models given by the Boardman–Vogt W construction [6]. A proof that this gives a cofibrant replacement for certain operads can be found in Berger and Moerdijk [4]. Recall that we will denote these cofibrant replacements by E_{d-1} , E_d and SC_d . We also want to restrict our attention to Swiss cheese algebras in categories where we can do homotopy theory. In the proper context the Hochschild cochain object of an E_{d-1} algebra A has a natural model constructed from A and the degree 0–1 parts of SC_d^h . This is the content of Proposition 3.2.7.

3.1 Homotopy-theoretic context

3.1.1 Definition (Hovey [13, Definition 4.2.6]) A symmetric monoidal model category \mathcal{S} is a closed symmetric monoidal category whose monoidal structure $\otimes: \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$ is a Quillen bifunctor and where the cofibrant replacement $Q1 \rightarrow 1$ of the monoidal unit induces weak equivalences $Q1 \otimes X \rightarrow X$ for every cofibrant X .

3.1.2 Example The category (Top, \times) of compactly generated spaces with the cartesian product and Serre model structure is a symmetric monoidal model category.

3.1.3 Definition Let \mathcal{S} be a symmetric monoidal model category. A symmetric monoidal model category tensored over \mathcal{S} is a closed symmetric monoidal model category \mathcal{C} , together with a symmetric monoidal Quillen functor $\mathcal{S} \rightarrow \mathcal{C}$. For more details see [13, Definition 4.2.20].

In particular, \mathcal{C} comes equipped with functors

$$\begin{aligned} \otimes: \mathcal{C} \times \mathcal{C} &\rightarrow \mathcal{C}, & \otimes: \mathcal{S} \times \mathcal{C} &\rightarrow \mathcal{C}, \\ \underline{\text{hom}}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \times \mathcal{C} &\rightarrow \mathcal{C}, & \text{map}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \times \mathcal{C} &\rightarrow \mathcal{S}. \end{aligned}$$

The mapping spaces $\text{map}_{\mathcal{C}}(A, B)$ give \mathcal{C} the structure of a category enriched over \mathcal{S} , so we can speak of E_{d-1} , E_d and SC_d algebras in \mathcal{C} .

For any object A of \mathcal{C} , the functor $- \otimes A$ has right adjoints $\underline{\text{hom}}_{\mathcal{C}}(A, -): \mathcal{C} \rightarrow \mathcal{C}$ and $\text{map}_{\mathcal{C}}(A, -): \mathcal{C} \rightarrow \mathcal{S}$. This data satisfies Quillen's SM7 axiom [13, Section 4.2].

3.1.4 Example The category $\mathcal{C}^{\Sigma_{\leq 1}}$ from [Definition 2.3.2](#) is tensored over the symmetric monoidal category $\text{Top}^{\Sigma_{\leq 1}}$ with

$$\text{Top}^{\Sigma_{\leq 1}} \otimes \mathcal{C}^{\Sigma_{\leq 1}} \rightarrow \mathcal{C}^{\Sigma_{\leq 1}}$$

given by the analogue of (7),

$$(X_0, X_1) \otimes (C_0, C_1) = (X_0 \otimes C_0, X_0 \otimes C_1 \amalg X_1 \otimes C_0),$$

where $X_i \in \text{Top}$ and $C_i \in \mathcal{C}$.

3.1.5 Example The symmetric monoidal functor $\text{Top} \rightarrow \text{Top}^{\Sigma_{\leq 1}}$ sending X to (X, \emptyset) makes both $\text{Top}^{\Sigma_{\leq 1}}$ and $\mathcal{C}^{\Sigma_{\leq 1}}$ into symmetric monoidal model categories tensored over Top . Note that for a topological 1-colored operad \mathcal{O} , we can consider algebras over \mathcal{O} in \mathcal{C} as well as algebras over \mathcal{O} in $\mathcal{C}^{\Sigma_{\leq 1}}$.

The category of degree 0–1 symmetric sequences is naturally home to \mathcal{O} –algebras A and \mathcal{O} – A –modules M .

3.1.6 Definition Suppose \mathcal{O} is a 1-colored operad in Top and let (A, M) be an object of $\mathcal{C}^{\Sigma_{\leq 1}}$. The structure of an \mathcal{O} algebra on the degree 0–1 symmetric sequence (A, M) is the structure of an \mathcal{O} algebra on A together with the data of maps

$$\mathcal{O}(m) \otimes M \otimes A^{\otimes m-1} \rightarrow M,$$

satisfying certain conditions (see diagram (16)). We call this data the structure of an \mathcal{O} – A –module on M . Given a fixed \mathcal{O} algebra $A \in \mathcal{C}$, the category $\text{Mod}_{\mathcal{O}}^A(\mathcal{C})$ of \mathcal{O} – A –modules has objects $M \in \mathcal{C}$, together with the structure of an \mathcal{O} algebra on $(A, M) \in \mathcal{C}^{\Sigma_{\leq 1}}$ extending the given \mathcal{O} algebra structure on A .

The author was shown the following quick way to see that the category of \mathcal{O} – A –modules is naturally enriched over Top . First, one can show that this category is equivalent to the category of algebras (classical associative algebras, not E_1) over the universal enveloping monoid of A . Then this implies that the free A –module monad is a strong and enriched monad, from which it follows that $\text{Mod}_{\mathcal{O}}^A(\mathcal{C})$ is enriched.

3.1.7 Example Let \mathcal{O} be a 1-colored topological operad and let $A \in \mathcal{C}$ be an \mathcal{O} algebra; then the degree 0–1 symmetric sequence (A, A) is naturally an \mathcal{O} algebra. That is, A is naturally an \mathcal{O} – A –module.

3.1.8 Definition Let \mathcal{O} be an operad in $\text{Top}^{\Sigma_{\leq 1}}$ and let \mathcal{P} be the image of \mathcal{O} under the functor $\text{Op}(\text{Top}^{\Sigma_{\leq 1}}) \rightarrow \text{Op}(\text{Top})$ (for example, if \mathcal{O} is SC_d^{hd} then \mathcal{P} is E_{d-1}). Since $\mathcal{C}^{\Sigma_{\leq 1}}$ is enriched over $\text{Top}^{\Sigma_{\leq 1}}$ by [Example 3.1.4](#), we can consider \mathcal{O} algebras

in $\mathcal{C}^{\Sigma_{\leq 1}}$. In addition, by [Example 3.1.5](#) we can consider \mathcal{P} algebras in $\mathcal{C}^{\Sigma_{\leq 1}}$. There are adjunctions

$$\mathcal{C} \rightleftarrows \text{Alg}_{\mathcal{P}}(\mathcal{C}), \quad \mathcal{C} \rightleftarrows \text{Mod}_{\mathcal{P}}^A(\mathcal{C}) \quad \text{and} \quad \mathcal{C} \times \mathcal{C} \rightleftarrows \text{Alg}_{\mathcal{P}}(\mathcal{C}^{\Sigma_{\leq 1}}) \rightleftarrows \text{Alg}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\leq 1}}).$$

We describe the right adjoints only. On the top left the \mathcal{P} algebra A is sent to the underlying object A of \mathcal{C} . The top right functor sends the \mathcal{P} - A -module M to the underlying object M of \mathcal{C} . In the pair of composable adjunctions, a \mathcal{P} algebra (A, M) can be considered as a \mathcal{P} algebra by forgetting the structure maps in [\(13\)](#). For the final adjunction, any \mathcal{P} algebra in $\mathcal{C}^{\Sigma_{\leq 1}}$ has an underlying pair of \mathcal{C} -objects (A, M) .

3.1.9 Notation Throughout the remainder of this paper \mathcal{C} will be a cofibrantly generated symmetric monoidal model category tensored over Top such that the adjunctions in [Definition 3.1.8](#) are Quillen adjunctions.

3.1.10 Remark Spitzweck [[21](#), Theorem 6] shows that $\text{Mod}_{\mathcal{O}}^A(\mathcal{C}) \rightleftarrows \mathcal{C}$ is a Quillen adjunction if A is cofibrant in \mathcal{C} . In addition, Berger and Moerdijk [[3](#), Proposition 4.1] show that, if \mathcal{D} is a symmetric monoidal model category which is cofibrantly generated and has a cofibrant unit and a symmetric monoidal fibrant replacement functor, then the category $\text{Alg}_{\mathcal{O}}(\mathcal{D})$ has the projective model structure induced from the forgetful functor to \mathcal{D} . The operad \mathcal{O} in this theorem is an operad in \mathcal{D} . However, their result is more general, as seen in [[3](#), Remark 4.6.4]. Their argument extends without change to show that $\text{Alg}_{\mathcal{O}}(\mathcal{D})$ has the desired model structure in the case that \mathcal{O} is an operad in Top and \mathcal{D} is tensored over Top as in [Definition 3.1.1](#), and the generating trivial cofibrations of \mathcal{D} have cofibrant domains. The condition that the monoidal unit of \mathcal{D} is cofibrant is not necessary in this situation. The cofibrance of the monoidal unit in Top is enough. Taking $\mathcal{O} = \text{E}_{d-1}$ or $\mathcal{O} = \text{SC}_d^{\text{h1}}$ and $\mathcal{D} = \mathcal{C}$ or $\mathcal{D} = \mathcal{C}^{\Sigma_{\leq 1}}$, we conclude that the assumptions in [Notation 3.1.9](#), and in the main theorem of this paper, hold when \mathcal{C} is the category of compactly generated Hausdorff spaces (Top), or spectra, or chain complexes.

We use the model structure on the category of \mathcal{P} - A -modules to define the Hochschild cochain object.

3.1.11 Definition Given a 1-colored topological operad \mathcal{P} and a \mathcal{P} algebra $A \in \mathcal{C}$, let the *Hochschild cochain object* of A be

$$\text{Hoch}(A) = \underline{\text{hom}}_{\mathcal{P}}^A(A^c, A^f),$$

where $\underline{\text{hom}}_{\mathcal{P}}^A$ is given by the equalizer in [Definition 3.1.6](#). The \mathcal{P} - A -modules A^c and A^f are cofibrant and fibrant replacements for A , respectively. Note that $\text{Hoch}(A)$ is an object of \mathcal{C} .

3.2 Swiss cheese in degrees zero and one

Let \mathcal{O} be a (1-colored) operad in $\text{Top}^{\Sigma \leq 1}$. A (1-colored) \mathcal{O} algebra (see [Definition 3.1.8](#) and [Example 2.3.5](#)) is a pair (A, M) of objects of \mathcal{C} together with maps in $\mathcal{C}^{\Sigma \leq 1}$, for every m ,

$$(10) \quad (\mathcal{O}(0, m), \mathcal{O}(1, m)) \otimes (A, M)^{\otimes m} \rightarrow (A, M).$$

Alternatively, we can view the morphism in (10) as three separate maps in \mathcal{C} :

$$(11) \quad \mathcal{O}(0, m) \otimes A^{\otimes m} \rightarrow A,$$

$$(12) \quad \mathcal{O}(0, m) \otimes M \otimes A^{\otimes m-1} \rightarrow M,$$

$$(13) \quad \mathcal{O}(1, m) \otimes A^{\otimes m} \rightarrow M.$$

The condition that the maps in (10) define an \mathcal{O} structure on the pair (A, M) is the condition that the diagram (14) commutes in the $\text{Top}^{\Sigma \leq 1}$ -enriched category $\mathcal{C}^{\Sigma \leq 1}$:

$$(14) \quad \begin{array}{ccc} \mathcal{O}(m') \otimes \mathcal{O}(m; m') \otimes (A, M)^{\otimes m} & \rightarrow & \mathcal{O}(m) \otimes (A, M)^{\otimes m} \\ & \downarrow & \downarrow \\ \mathcal{O}(m') \otimes (A, M)^{\otimes m'} & \longrightarrow & (A, M) \end{array}$$

In terms of equations (11), (12) and (13), the $\mathcal{C}^{\Sigma \leq 1}$ diagram (14) splits into four diagrams in \mathcal{C} . Each diagram is determined by the degrees of the three tensor factors in the upper left-hand corner of (14). In the first the degrees are 0, 0, 0; in the second the degrees are 0, 0, 1; in the third, 1, 0, 0; and in the fourth, 0, 1, 0.

The following diagram commutes for all m' and m if and only if A is an \mathcal{O}^0 algebra (see [Definition 2.3.3](#)):

$$(15) \quad \begin{array}{ccc} \mathcal{O}(0, m') \otimes \mathcal{O}(0, m; 0, m') \otimes A^{\otimes m} & \longrightarrow & \mathcal{O}(0, m) \otimes A^{\otimes m} \\ & \downarrow & \downarrow \\ \mathcal{O}(0, m') \otimes A^{\otimes m'} & \longrightarrow & A \end{array}$$

Diagram (16) commutes if and only if M is an \mathcal{O}^0 - A -module:

$$(16) \quad \begin{array}{ccc} \mathcal{O}(0, m') \otimes \mathcal{O}(0, m; 0, m') \otimes M \otimes A^{\otimes m-1} & \rightarrow & \mathcal{O}(0, m) \otimes M \otimes A^{\otimes m'-1} \\ & \downarrow & \downarrow \\ \mathcal{O}(0, m) \otimes M \otimes A^{\otimes m-1} & \longrightarrow & M \end{array}$$

Diagram (17) shows a compatibility condition between the degree 0 and degree 1 structures:

$$(17) \quad \begin{array}{ccc} \mathcal{O}(1, m') \otimes \mathcal{O}(0, m; 0, m') \otimes A^{\otimes m} & \longrightarrow & \mathcal{O}(1, m) \otimes A^{\otimes m} \\ \downarrow & & \downarrow \\ \mathcal{O}(1, m') \otimes A^{\otimes m'} & \longrightarrow & M \end{array}$$

Diagram (18) presents another compatibility condition between the degree 0 and degree 1 structures:

$$(18) \quad \begin{array}{ccc} \mathcal{O}(0, m') \otimes \mathcal{O}(1, m; 0, m') \otimes A^{\otimes m} & \longrightarrow & \mathcal{O}(1, m) \otimes A^{\otimes m} \\ \downarrow & & \downarrow \\ \mathcal{O}(0, m') \otimes M \otimes A^{\otimes m'-1} & \longrightarrow & M \end{array}$$

3.2.1 Example In this example we construct the universal extension of an \mathcal{O}^0 algebra A to an \mathcal{O} algebra. We denote this universal pair by (A, A^o) . The composite forgetful functor

$$\text{Alg}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\leq 1}}) \rightarrow \text{Alg}_{\mathcal{O}^0}(\mathcal{C}^{\Sigma_{\leq 1}}) \rightarrow \text{Alg}_{\mathcal{O}^0}(\mathcal{C}),$$

has a left adjoint, which sends the \mathcal{O}^0 algebra A to the pair (A, A^o) , where A^o is a quotient of

$$(19) \quad \bar{A}^o = \coprod_{m \geq 0} \mathcal{O}(1, m) \otimes_{S_m} A^{\otimes m}.$$

We can think of \bar{A}^o heuristically as $\mathcal{O}(1, \bullet) \otimes A^{\otimes \bullet}$. Now, both A and $\mathcal{O}(1, \bullet)$ carry an action of \mathcal{O}^0 , so we can form the quotient $A^o := \mathcal{O}(1, \bullet) \otimes_{\mathcal{O}^0} A^{\otimes \bullet}$. More precisely, A^o is defined as the coequalizer

$$(20) \quad \coprod_{m, m'} \mathcal{O}(1, m) \otimes \mathcal{O}(0, m'; 0, m) \otimes A^{\otimes m'} \rightrightarrows \bar{A}^o \xrightarrow{\text{coeq}} A^o,$$

where one of the arrows is given by the operadic composition on Swiss cheese and the other by the \mathcal{O}^0 structure on A .

Verifying that (A, A^o) is a 1-colored \mathcal{O} algebra is a matter of using the commuting left and right actions of \mathcal{O}^0 on \mathcal{O} . By this we mean the morphism below uses both left and right actions, and can be obtained by performing the left action first then the right, or vice versa,

$$\mathcal{O}(0, m'') \times \mathcal{O}(1, m'; 0, m'') \times \mathcal{O}(0, m; 0, m') \rightarrow \mathcal{O}(1, m).$$

The left action defines a map

$$(21) \quad \mathcal{O}(0, m) \otimes \bar{A}^o \otimes A^{\otimes m-1} \rightarrow \bar{A}^o,$$

using the \circ_1 operad composition. Since the left and right \mathcal{O}^0 actions on \mathcal{O} commute, the arrow in (21) descends to give the data of an \mathcal{O}^0 - A -module structure on A^o , ie (12) for $M = A^o$. Of course, the maps from (13) with $M = A^o$ are simply given by $\bar{A}^o \rightarrow A^o$.

Now let us observe that the four diagrams (15)–(18) commute for (A, A^o) . The first diagram, (15), is trivial since A is an \mathcal{O}^0 algebra. The second diagram, (16), commutes since the left action of \mathcal{O}^0 on \mathcal{O} is indeed an action. That is, it is compatible with composition in \mathcal{O}^0 . The third diagram, (17), clearly commutes. Indeed, this diagram is the reason the coequalizer $\bar{A}^o \rightarrow A^o$ in (20) was defined in the first place. Finally, the fourth diagram, (18), certainly commutes if $M = \bar{A}^o$. In this case, note that the bottom map in diagram (18) corresponds to (21). Thus, if we pass from \bar{A}^o to A^o this last diagram still commutes since, by definition, the \mathcal{O}^0 - A -module structure on A^o is defined using the quotient map $\bar{A}^o \rightarrow A^o$ together with (21).

3.2.2 Lemma *Fix an \mathcal{O}^0 algebra A and consider A^o as an \mathcal{O}^0 - A module. There is an equivalence of relative categories*

$$\text{Mod}_{\mathcal{O}}^A(\mathcal{C}) \cong \text{Mod}_{\mathcal{O}^0}^A(\mathcal{C})_{A^o/},$$

where $\text{Mod}_{\mathcal{O}}^A(\mathcal{C})$ is the fiber over A of the forgetful functor $\text{Alg}_{\mathcal{O}}(\mathcal{C}^{\Sigma \leq 1}) \rightarrow \text{Alg}_{\mathcal{O}^0}(\mathcal{C})$.

Proof Let (A, M) be an $\text{SC}_d^{\text{h}1}$ algebra extending the existing \mathcal{O}^0 algebra structure on A . Then $M \in \text{Mod}_{\mathcal{O}^0}^A(\mathcal{C})$ and the structure maps in (13), when combined for all m , give a map $\bar{A}^o \rightarrow M$. This descends to an \mathcal{O}^0 - A -module map $A^o \rightarrow M$.

On the other hand, if M is an \mathcal{O} - A -module, then M is already equipped with the data of (12). If $A^o \rightarrow M$ is a morphism of \mathcal{O} - A -modules, then M is equipped with the data of (13). The diagram (14) commutes because of the relation defining A^o in (20) and because $A^o \rightarrow M$ is a morphism of \mathcal{O}^0 - A -modules. □

3.2.3 Corollary *Take $\mathcal{O} = \text{SC}_d^{\text{h}1}$ and denote A^o by A^{sc} . Let A be a cofibrant E_{d-1} algebra. Then A^{sc} is a cofibrant E_{d-1} - A -module.*

Proof The forgetful functor $\text{Alg}_{\text{SC}_d^{\text{h}1}}(\mathcal{C}) \rightarrow \text{Alg}_{\text{E}_{d-1}}(\mathcal{C})$ preserves fibrations (Definition 3.1.8). Thus the left adjoint of this forgetful functor, applied to the cofibrant object A , gives a cofibrant $\text{SC}_d^{\text{h}1}$ algebra (A, A^{sc}) . Thus A^{sc} is cofibrant in $\text{Mod}_{\text{SC}_d^{\text{h}1}}^A(\mathcal{C})$. Lemma 3.2.2 shows that the forgetful functor $\text{Mod}_{\text{SC}_d^{\text{h}1}}^A(\mathcal{C}) \rightarrow \text{Mod}_{\text{E}_{d-1}}^A(\mathcal{C})$ preserves pushouts. The model structures here are cofibrantly generated, so A^{sc} is also cofibrant as an object of $\text{Mod}_{\text{E}_{d-1}}^A(\mathcal{C})$. □

3.2.4 Definition Let $p(m): SC_d^h(1, m) \rightarrow SC_d^h(0, m) \simeq E_{d-1}(m)$ be the projection which forgets the single full-disk. Let p be the morphism $(p(m))_{m \geq 0}$ of operads in $\text{Top}^{\Sigma \leq 1}$. For each m consider the degree 0–1 symmetric sequence $(E_{d-1})^{\leq 1}(m) := (E_{d-1}(m), E_{d-1}(m))$. The structure of E_{d-1} as an operad in Top can be used to make $(E_{d-1})^{\leq 1}$ an operad in $\text{Top}^{\Sigma \leq 1}$. This makes $(\text{id}, p): SC_d^h \rightarrow (E_{d-1})^{\leq 1}$ into a morphism of operads.

If A is an E_{d-1} algebra we can define a morphism in \mathcal{C} ,

$$\bar{A}^{\text{sc}} = \coprod_{m \geq 0} SC_d^h(1, m) \otimes_{S_m} A^{\otimes m} \rightarrow \coprod_{m \geq 0} E_{d-1}(m) \otimes_{S_m} A^{\otimes m} \rightarrow A,$$

where the first arrow uses p and the second arrow uses the E_{d-1} algebra structure on A . This map factors to give a morphism of E_{d-1} - A -modules $p_A: A^{\text{sc}} \rightarrow A$.

3.2.5 Remark By [21, Section 5] we can conclude that $\text{Op}(\text{Top}^{\Sigma \leq 1})$ is tensored over Top . If $\mathcal{O} = (\mathcal{O}^0, \mathcal{O}^1)$ is an operad in degree 0–1 symmetric sequences of topological spaces and $K \in \text{Top}$, then $K \otimes \mathcal{O}$ is defined to be the coequalizer

$$F(F(K \otimes \mathcal{O})) \rightrightarrows F(K \otimes \mathcal{O}) \xrightarrow{\text{coeq}} K \otimes \mathcal{O},$$

where $K \otimes \mathcal{O}$ is the symmetric sequence of degree 0–1 symmetric sequences whose arity- m , degree- i component is $K \times \mathcal{O}^i(m)$, and F is the free operad functor.

If \mathcal{O} is a cofibrant operad in $\text{Top}^{\Sigma \leq 1}$, then $[0, 1] \otimes \mathcal{O}$ is a cylinder object and a homotopy $h: [0, 1] \otimes \mathcal{O} \rightarrow \mathcal{P}$ gives the data of maps $h^i(m): [0, 1] \times \mathcal{O}^i(m) \rightarrow \mathcal{P}^i(m)$, which assemble into $h^i(m'; m): [0, 1] \times \mathcal{O}^i(m'; m) \rightarrow \mathcal{P}(m'; m)$ for $i = 0, 1$ and $m', m \geq 0$. The homotopy h is compatible with operad composition in the sense that, if $\alpha \in \mathcal{O}^i(m)$ and $\beta \in \mathcal{O}^j(m'; m)$, $i + j \leq 1$ and $t \in [0, 1]$, then

$$h^i(m)(t, \alpha) \circ h^j(m'; m)(t, \beta) = h^{i+j}(m')(t, \alpha \circ \beta).$$

3.2.6 Lemma For any E_{d-1} algebra $A \in \mathcal{C}$, the map $A^{\text{sc}} \rightarrow A$ is a weak equivalence of E_{d-1} - A -modules.

Proof Abusing notation, we write $p := (\text{id}, p)$ for the morphism of operads from Definition 3.2.4. One can show that p is a weak equivalence of fibrant and cofibrant operads in degree 0–1 symmetric sequences of topological spaces. Therefore, there is a map of operads $\iota: (E_{d-1})^{\leq 1} \rightarrow SC_d^h$ and there are homotopies $h: \text{id}_{SC_d^h} \simeq \iota p$ and $g: \text{id}_{(E_{d-1})^{\leq 1}} \simeq p \iota$. Since $[0, 1] \otimes -$ distributes over coequalizers, we can use h to

define a homotopy $h_A: [0, 1] \otimes A^{\text{sc}} \rightarrow A^{\text{sc}}$:

$$\begin{array}{ccc} \coprod_m [0, 1] \otimes \text{SC}_d^h(1, m) \otimes A^{\otimes m} & \longrightarrow & \otimes A^{\text{sc}} \\ \downarrow \sqcup h^1(m) \otimes \text{id}_A^{\otimes m} & & \downarrow h_A \\ \coprod_m \text{SC}_d^h(1, m) \otimes A^{\otimes m} & \longrightarrow & A^{\text{sc}}, \end{array}$$

where $h^1(m)$ is defined from h as in Remark 3.2.5. When $t = 1$, $h_A(1, -)$ factors as

$$\begin{array}{ccc} \coprod_m \text{SC}_d^h(1, m) \otimes A^{\otimes m} & \longrightarrow & A^{\text{sc}} \\ \downarrow & \swarrow \text{---} s \text{---} & \downarrow p_A \\ \coprod_m E_{d-1}(m) \otimes A^{\otimes m} & \longrightarrow & A \\ \downarrow & & \downarrow \iota_A \\ \coprod_m \text{SC}_d^h(1, m) \otimes A^{\otimes m} & \longrightarrow & A^{\text{sc}}, \end{array}$$

where p_A is the map from Definition 3.2.4 and ι_A is the evident composite in the diagram using s . The map s is a section of the middle horizontal arrow, defined using the identity of the operad E_{d-1} ,

$$A \simeq \{1_{E_{d-1}}\} \otimes A \rightarrow E_{d-1}(1) \otimes A \rightarrow \coprod_m E_{d-1}(m) \otimes A^{\otimes m}.$$

The map ι_A is a map of E_{d-1} - A -modules and h_A is a homotopy $\text{id}_{A^{\text{sc}}} \simeq \iota_A p_A$. Similarly, g defines a homotopy $g_A: \text{id}_A \rightarrow p_A \iota_A$. □

The precise sense in which Hochschild cohomology can be obtained from the degree 0–1 pieces of the Swiss cheese operad is contained in the following proposition:

3.2.7 Proposition *Let A be a fibrant and cofibrant E_{d-1} algebra. Then the Hochschild cochain object of A can be computed as*

$$\text{Hoch}(A) \simeq \underline{\text{hom}}_{E_{d-1}}^A(A^{\text{sc}}, A).$$

Proof We are using the projective model structure from Definition 3.1.8, so A is fibrant as an object of \mathcal{C} and thus as an E_{d-1} - A -module. By Corollary 3.2.3 and Lemma 3.2.6, A^{sc} is a cofibrant replacement for A as an E_{d-1} - A -module. By Definition 3.1.11, this proves the proposition. □

4 The universal cheese theorem

We now shift our attention to K -colored algebras over SC_d^{h1} . Take the K -colored endomorphism $\text{End}(B, A)$ defined in (4) and (5). Apply the functor in Definition 2.3.3 to this operad to get $\text{End}^{h1}(B, A) \in \text{Op}(\text{Top}^{\Sigma \leq 1})$. With this endomorphism operad we have a separate notion of algebra over SC_d^{h1} . We call this notion of algebra a K -colored algebra over SC_d^{h1} . As a point of contrast, the former notion, detailed in Section 3.2, might be called a 1-colored algebra. Recall that a 1-colored SC_d^{h1} algebra on (X, Y) is equivalent to an E_{d-1} structure on X , an E_{d-1} - X structure on Y and an E_{d-1} - X -module map $X^{sc} \rightarrow Y$. We will see in Proposition 4.1.2 that a K -colored algebra on (B, A) is like a family of 1-colored algebra structures on (A, A) . More precisely, it is an E_{d-1} algebra structure on A , together with any map $B \rightarrow \underline{\text{hom}}_{E_{d-1}}^A(A^{sc}, A)$. In other words, $\text{Hoch}(A)$ is the universal object of \mathcal{C} acting on the E_{d-1} algebra A through SC_d^{h1} . This result is generalized twice, first in Corollary 4.1.4, then in Proposition 4.1.9. We refer to Proposition 4.1.9 as the universal cheese theorem, since it replaces E_d in the Swiss cheese theorem with an arbitrary operad.

4.1.1 Definition Let A be an E_{d-1} algebra. Let $\mathcal{C}/_{\text{Hoch}(A)}$ denote the over category of $\text{Hoch}(A) \in \mathcal{C}$. More precisely, the objects are \mathcal{C} -morphisms $B \rightarrow \text{Hoch}(A)$ and the morphisms are \mathcal{C} -morphisms $B \rightarrow B'$ commuting with the maps to $\text{Hoch}(A)$. In addition, let $\text{Alg}_{SC_d^{h1}}^A(\mathcal{C})$ denote the category of SC_d^{h1} algebras of the form (B, A) , where the induced E_{d-1} structure on A is the one given. Morphisms are maps of SC_d^{h1} algebras which are the identity on A .

Given a pair of objects (B, A) of \mathcal{C} , we let $\text{End}^{h1}(B, A)$ denote the operad obtained by applying the forgetful functor $\text{Op}_K \rightarrow \text{Op}(\text{Top}^{\Sigma \leq 1})$ to $\text{End}(B, A)$ from (4) and (5).

Let $\text{Alg}_{SC_d^{h1}}^A(\mathcal{C})$ denote the category of objects $B \in \mathcal{C}$ together with a morphism $SC_d^{h1} \rightarrow \text{End}^{h1}(B, A)$ in $\text{Op}(\text{Top}^{\Sigma \leq 1})$, extending the E_{d-1} structure on A . A morphism $B \rightarrow B'$ in \mathcal{C} induces a map of operads $\text{End}^{h1}(B', A) \rightarrow \text{End}^{h1}(B, A)$. Such a morphism gives a map in $\text{Alg}_{SC_d^{h1}}^A(\mathcal{C})$ if this induced map respects maps from SC_d^{h1} .

4.1.2 Proposition *There is an equivalence of relative categories*

$$\text{Alg}_{SC_d^{h1}}^A(\mathcal{C}) \cong \mathcal{C}/_{\text{Hoch}(A)}.$$

Proof The data of an algebra on the left-hand side is an object $B \in \mathcal{C}$ together with maps

$$(22) \quad SC_d^h(1, m) \otimes B \otimes A^{\otimes m} \rightarrow A$$

for each $m \geq 0$. The conditions on (22) are that the following diagrams commute:

$$(23) \quad \begin{array}{ccc} E_{d-1}(m') \otimes SC_d^h(1, m; 0, m') \otimes B \otimes A^{\otimes m} & \rightarrow & E_{d-1}(m') \otimes A^{\otimes m'} \\ \downarrow & & \downarrow \\ SC_d^h(1, m) \otimes B \otimes A^{\otimes m} & \longrightarrow & A \end{array}$$

$$(24) \quad \begin{array}{ccc} SC_d^h(1, m') \otimes E_{d-1}(m; m') \otimes B \otimes A^{\otimes m} & \rightarrow & SC_d^h(1, m) \otimes B \otimes A^{\otimes m} \\ \downarrow & & \downarrow \\ SC_d^h(1, m') \otimes B \otimes A^{\otimes m'} & \longrightarrow & A \end{array}$$

Equivalently, we can use the hom-tensor adjunction and assemble the maps in (22) to a single map $B \rightarrow \underline{\text{hom}}_C(\bar{A}^{\text{sc}}, A)$ (see (19)). The commutativity of diagram (24) is equivalent to this map lifting to

$$(25) \quad B \rightarrow \underline{\text{hom}}_C(A^{\text{sc}}, A).$$

Note that, dual to (20), $\underline{\text{hom}}_C(A^{\text{sc}}, A)$ is given by the equalizer

$$\underline{\text{hom}}_C(A^{\text{sc}}, A) \xrightarrow{\text{eq}} \underline{\text{hom}}_C(\bar{A}^{\text{sc}}, A) \rightrightarrows \underline{\text{hom}}_C\left(\coprod_{m, m'} SC_d^h(1, m') \otimes E_{d-1}(m; m') \otimes A^{\otimes m}, A\right).$$

With this observation, we can now rewrite diagram (23) as:

$$(26) \quad \begin{array}{ccc} B \otimes F_{E_{d-1}}^A(A^{\text{sc}}) & \longrightarrow & F_{E_{d-1}}^A(A) \\ \downarrow & & \downarrow \\ B \otimes A^{\text{sc}} & \longrightarrow & A \end{array}$$

Recall that $F_{E_{d-1}}^A : \mathcal{C} \rightarrow \mathcal{C}$ is the free E_{d-1} - A module functor from Definition 3.1.6. Clearly, diagram (23) commutes if and only if diagram (26) commutes and the map (25) factors through $\text{Hoch}(A) = \underline{\text{hom}}_{E_{d-1}}^A(A^{\text{sc}}, A)$. We conclude that the data of an SC_d^h algebra structure on (B, A) is the data of an E_{d-1} algebra structure on A together with a \mathcal{C} -morphism $B \rightarrow \text{Hoch}(A)$.

It is clear that a map $(B, A) \rightarrow (B', A)$ which is identity on A gives an SC_d^h algebra morphism if and only if the map $B \rightarrow B'$ commutes with the corresponding morphisms to $\text{Hoch}(A)$. □

4.1.3 Definition Define $SC_d^{\text{h}\infty} \in \text{Op}(\text{Top}^\Sigma)$ as the image of the left adjoint of $\text{Op}(\text{Top}^\Sigma) \rightarrow \text{Op}(\text{Top}^{\Sigma \leq 1})$ applied to SC_d^h . As in Definition 4.1.1, define $\text{Alg}_{SC_d^{\text{h}\infty}}^A(\mathcal{C})$ as the category of objects $B \in \mathcal{C}$ together with a morphism $SC_d^{\text{h}\infty} \rightarrow \text{End}^h(B, A)$ in $\text{Op}(\text{Top}^\Sigma)$ extending the E_{d-1} structure on A .

4.1.4 Corollary *There is an equivalence of relative categories*

$$\text{Alg}_{\text{SC}_d^{\text{h}\infty}}^A(\mathcal{C}) \cong \mathcal{C} / \text{Hoch}(A).$$

Proof The adjunction isomorphism puts operad maps $\text{SC}_d^{\text{h}1} \rightarrow \text{End}^{\text{h}1}(B, A)$ in one-to-one correspondence with operad maps $\text{SC}_d^{\text{h}\infty} \rightarrow \text{End}^{\text{h}}(B, A)$. This gives an isomorphism between $\text{Alg}_{\text{SC}_d^{\text{h}\infty}}^A(\mathcal{C})$ and $\text{Alg}_{\text{SC}_d^{\text{h}1}}^A(\mathcal{C})$. Now apply [Proposition 4.1.2](#). \square

4.1.5 Definition For $n \geq 0$, let $\text{SC}_d^{\text{h}\infty}(n, \bullet)$ denote the operad in $\text{Op}(\text{Top}^{\Sigma_{\leq 1}})$ whose arity- m component is the degree 0–1 symmetric sequence $(E_{d-1}(m), \text{SC}_d^{\text{h}\infty}(n, m))$. One may think of $\text{SC}_d^{\text{h}\infty}(n, \bullet)$ as a bimodule over E_{d-1} . Let $\text{End}(\text{SC}_d^{\text{h}\infty})$ be the symmetric sequence whose n^{th} space is the E_{d-1} bimodule maps

$$\text{End}(\text{SC}_d^{\text{h}\infty})(n) := \text{map}_{E_{d-1}}(\text{SC}_d^{\text{h}\infty}(1, \bullet), \text{SC}_d^{\text{h}\infty}(n, \bullet)).$$

4.1.6 Lemma *Operadic composition in $\text{SC}_d^{\text{h}\infty}$ induces the structure of an operad on the symmetric sequence $\text{End}(\text{SC}_d^{\text{h}\infty})$.*

Proof Define an operad $\mathcal{E} \in \text{Op}(\text{Top}^{\Sigma})$ by setting

$$\mathcal{E}(n, m) = \text{map}_{\text{Top}^{\Sigma}}(\text{End}(\text{SC}_d^{\text{h}\infty})(\bullet; n), \text{SC}_d^{\text{h}\infty}(\bullet, m)).$$

Let $n_0 + \dots + n_m = n$ and $\ell_1 + \dots + \ell_m = \ell$. The monoidal structure \otimes on symmetric sequences gives a map from $\mathcal{E}(n_0, m) \times \prod_{i=1}^m \mathcal{E}(n_i, \ell_i)$ to

$$(27) \quad \text{map}_{\text{Top}^{\Sigma}} \left(\bigotimes_{i=0}^m \text{End}(\text{SC}_d^{\text{h}\infty})(\bullet; n_i), \text{SC}_d^{\text{h}\infty}(\bullet, m) \otimes \left(\bigotimes_{i=1}^m \text{SC}_d^{\text{h}\infty}(\bullet, \ell_i) \right) \right).$$

Now push forward from (27) via the operad structure on $\text{SC}_d^{\text{h}\infty}$,

$$\text{SC}_d^{\text{h}\infty}(\bullet, m) \otimes \left(\bigotimes_{i=1}^m \text{SC}_d^{\text{h}\infty}(\bullet, \ell_i) \right) \rightarrow \text{SC}_d^{\text{h}\infty}(\bullet, \ell),$$

and pull back from (27) by

$$(28) \quad \text{End}(\text{SC}_d^{\text{h}\infty})(\bullet; n) \rightarrow \bigotimes_{i=0}^m \text{End}(\text{SC}_d^{\text{h}\infty})(\bullet; n_i).$$

This defines

$$\mathcal{E}(n_0, m) \times \prod_{i=1}^m \mathcal{E}(n_i, \ell_i) \rightarrow \mathcal{E}(n, \ell).$$

The morphism in (28) comes from the sequence of maps

$$\begin{aligned} \text{End}(\text{SC}_d^{\text{h}\infty})(k; n) &= \coprod_{f: \underline{k} \rightarrow \underline{n}} \prod_{i=1}^n \text{End}(\text{SC}_d^{\text{h}\infty})(f^{-1}(i)) \\ &= \coprod_{f: \underline{k} \rightarrow \underline{n}} \prod_{j \in \underline{0} \sqcup \underline{m}} \prod_{i \in g^{-1}(j)} \text{End}(\text{SC}_d^{\text{h}\infty})(f^{-1}(i)) \\ &\rightarrow \coprod_{\tilde{f}: \underline{k} \rightarrow \underline{0} \sqcup \underline{m}} \prod_{j \in \underline{0} \sqcup \underline{m}} \text{End}(\text{SC}_d^{\text{h}\infty})(\tilde{f}^{-1}(j); n_j). \end{aligned}$$

The first equality holds by definition; the second is a regrouping. The decomposition $\sum_{i=0}^m n_i = n$ defines a map $g: \underline{n} \rightarrow \underline{0} \sqcup \underline{m}$, where $|g^{-1}(i)| = n_i$. The third map sends the component corresponding to $f: \underline{k} \rightarrow \underline{n}$ to the component corresponding to $f g: \underline{k} \rightarrow \underline{0} \sqcup \underline{m}$. For each n and m there is a map

$$(29) \quad \text{SC}_d^{\text{h}\infty}(n, m) \rightarrow \mathcal{E}(n, m).$$

When $n = 0, 1$, the map (29) is canonical. Restricting to degrees 0 and 1 gives a map of $\text{Op}(\text{Top}^{\Sigma_{\leq 1}})$ operads $\text{SC}_d^{\text{h}1} \rightarrow \mathcal{E}^1$, where \mathcal{E}^1 is the degree 0–1 part of \mathcal{E} . Since $\text{SC}_d^{\text{h}\infty}$ is freely generated by its degree 0 and 1 pieces, we get (29) for all n , assembling into a map of operads in $\text{Op}(\text{Top}^{\Sigma})$. This guarantees that (30) can be used to define an operadic composition law on $\text{End}(\text{SC}_d^{\text{h}\infty})$,

$$(30) \quad \text{SC}_d^{\text{h}\infty}(n, m) \times \text{End}(\text{SC}_d^{\text{h}\infty})(k_1) \times \cdots \times \text{End}(\text{SC}_d^{\text{h}\infty})(k_n) \rightarrow \text{SC}_d^{\text{h}\infty}(k, m). \quad \square$$

4.1.7 Definition Let \mathcal{O} be a 1-colored topological operad and let $\rho: \mathcal{O} \rightarrow \text{End}(\text{SC}_d^{\text{h}\infty})$ be a map of operads. Define the K -colored operad $\text{SC}_d^{\text{h}\infty} \rtimes_{\rho} \mathcal{O}$ by setting

$$(\text{SC}_d^{\text{h}\infty} \rtimes_{\rho} \mathcal{O})^{\text{h}}(n, m) = \text{SC}_d^{\text{h}\infty}(n, m), \quad (\text{SC}_d^{\text{h}\infty} \rtimes_{\rho} \mathcal{O})^{\text{f}}(n, m) = \begin{cases} \mathcal{O}(n) & m = 0, \\ \emptyset & m > 0. \end{cases}$$

Composition in $\text{SC}_d^{\text{h}\infty} \rtimes_{\rho} \mathcal{O}$ uses composition in \mathcal{O} , composition in $\text{SC}_d^{\text{h}\infty}$ and the action of \mathcal{O} on $\text{SC}_d^{\text{h}\infty}$ defined by ρ ,

$$\text{SC}_d^{\text{h}\infty}(n, m) \times \mathcal{O}(k; n) \rightarrow \text{SC}_d^{\text{h}\infty}(n, m) \times \text{End}(\text{SC}_d^{\text{h}\infty})(k; n) \rightarrow \text{SC}_d^{\text{h}\infty}(k, m),$$

where the right arrow above is the one in (30).

4.1.8 Lemma Let \mathcal{O} be a topological operad and let $\rho: \mathcal{O} \rightarrow \text{End}(\text{SC}_d^{\text{h}\infty})$ be a map of operads; then the $\text{SC}_d^{\text{h}\infty}$ structure on (H, A) naturally extends to a $\text{SC}_d^{\text{h}\infty} \rtimes_{\rho} \mathcal{O}$ structure on (H, A) . In particular, $H = \text{Hoch}(A)$ inherits an \mathcal{O} algebra structure.

Proof We only need to show there is a map of operads $\text{End}(\text{SC}_d^{\text{h}\infty}) \rightarrow \text{End}(H)$ compatible with the action of $\text{End}(\text{SC}_d^{\text{h}\infty})$ on $\text{SC}_d^{\text{h}\infty}$ and the action of $\text{SC}_d^{\text{h}\infty}$ on H . Indeed, the map

$$(31) \quad \text{End}(\text{SC}_d^{\text{h}\infty})(n) \otimes H^{\otimes n} \rightarrow H$$

is adjoint to the maps, for all $m \geq 0$,

$$\text{SC}_d^{\text{h}\infty}(1, m) \otimes \text{End}(\text{SC}_d^{\text{h}\infty})(n) \otimes H^{\otimes n} \otimes A^{\otimes m} \rightarrow \text{SC}_d^{\text{h}\infty}(n, m) \otimes H^{\otimes n} \otimes A^{\otimes m} \rightarrow A,$$

where the first arrow is (30) and the second arrow is the $\text{SC}_d^{\text{h}\infty}$ structure on (H, A) . To check that (31) is compatible with composition in $\text{End}(\text{SC}_d^{\text{h}\infty})$ observe that there are two morphisms of operads in $\text{Op}(\text{Top}^\Sigma)$,

$$\text{SC}_d^{\text{h}\infty} \rightrightarrows \text{End}^h(\text{End}(\text{SC}_d^{\text{h}\infty})(H), A),$$

where $\text{End}(\text{SC}_d^{\text{h}\infty})(H)$ is the free $\text{End}(\text{SC}_d^{\text{h}\infty})$ algebra generated by H . One of the arrows uses the action of $\text{End}(\text{SC}_d^{\text{h}\infty})$ on $\text{SC}_d^{\text{h}\infty}$, while the other uses the map $\text{End}(\text{SC}_d^{\text{h}\infty})(H) \rightarrow H$ defined by (31). To check that these arrows agree, we only need to check that they agree out of $\text{SC}_d^{\text{h}\infty}(n, m)$ when $n = 0, 1$. This is because $\text{SC}_d^{\text{h}\infty}$ is freely generated in degrees 0 and 1. When $n = 0$, the maps are obviously the same. When $n = 1$, the maps are the same by definition of the $\text{SC}_d^{\text{h}1}$ structure on (H, A) . \square

4.1.9 Proposition (The universal cheese theorem) *Let \mathcal{O} be a topological operad and let $\rho: \mathcal{O} \rightarrow \text{End}(\text{SC}_d^{\text{h}\infty})$ be a map of operads. Then using the induced $\text{SC}_d^{\text{h}\infty} \rtimes_{\rho} \mathcal{O}$ structure on $(\text{Hoch}(A), A)$ from Lemma 4.1.8 gives an equivalence of relative categories*

$$\text{Alg}_{(\text{SC}_d^{\text{h}\infty} \rtimes_{\rho} \mathcal{O})}^A(\mathcal{C}) \cong \text{Alg}_{\mathcal{O}}(\mathcal{C}) / \text{Hoch}(A).$$

Proof Given any \mathcal{C} -morphism $B \rightarrow H$ we can form the following diagram (for brevity, we have deleted appearances of \otimes):

$$(32) \quad \begin{array}{ccc} \text{SC}_d^{\text{h}\infty}(1, m)\mathcal{O}(n)B^n A^m & \longrightarrow & \text{SC}_d^{\text{h}\infty}(n, m)B^n A^m \\ \swarrow \text{SC}_d^{\text{h}\infty}(1, m)\mathcal{O}(n)H^n A^m & \downarrow & \swarrow \text{SC}_d^{\text{h}\infty}(n, m)H^n A^m \\ \downarrow & \downarrow & \downarrow \\ \text{SC}_d^{\text{h}\infty}(1, m)BA^m & \longrightarrow & A \\ \swarrow \text{SC}_d^{\text{h}\infty}(1, m)HA^m & \downarrow & \swarrow = \\ \downarrow & \downarrow & \downarrow \\ \text{SC}_d^{\text{h}\infty}(1, m)HA^m & \longrightarrow & A \end{array}$$

Let (B, A) be a $\text{SC}_d^{\text{h}\infty} \rtimes_{\rho} \mathcal{O}$ algebra extending the given E_{d-1} structure on A , then by Corollary 4.1.4 we get a \mathcal{C} morphism $B \rightarrow H = \text{Hoch}(A)$ making the right face of the cube (32) commute. The front face commutes by Lemma 4.1.8. The back face

commutes by assumption. The bottom face commutes by definition, and the top face commutes trivially.

This implies that, after composition with the maps whose codomain is A , the left face of the cube commutes. By adjointness, the two maps $\mathcal{O}(B) \rightrightarrows H$ agree, implying that $B \rightarrow H$ is indeed an \mathcal{O} algebra morphism.

On the other hand, given an \mathcal{O} algebra B together with an \mathcal{O} algebra map $B \rightarrow H$, we get an $SC_d^{h\infty}$ structure on (B, A) from the underlying \mathcal{C} -morphism. We only need to check that the \mathcal{O} structure on B and the $SC_d^{h\infty}$ structure on (B, A) are compatible via ρ . Indeed, since $SC_d^{h\infty}$ is freely generated in degrees 0 and 1, it is enough to check that the back face of the cube commutes. But this holds because all other faces commute. Most importantly, the left face commutes because $B \rightarrow H$ is an \mathcal{O} algebra map.

It is easy to see that each of these constructions are natural in B and are inverse to one another. □

5 The homotopy E_d structure on $\text{Hoch}(A)$

In light of [Proposition 4.1.9](#), to prove the Swiss cheese theorem, [Theorem 2.2.1](#), we need to construct $E \simeq E_d$ and an operad morphism $E \rightarrow \text{End}(SC_d^{h\infty})$ in such a way that the corresponding K -colored operad $SC_d^{h\infty} \rtimes E$ is equivalent to SC_d . While $SC_d^{h\infty}$ has no obvious action of E_d , it is equivalent to something that does have an E_d action. The following theorem is proven in [Section 6](#).

5.1.1 Theorem *The natural map $SC_d^{h\infty} \rightarrow SC_d^h$ is an acyclic cofibration of operads in $\text{Op}(\text{Top}^\Sigma)$.*

In this section, we define the precise sense in which [Theorem 5.1.1](#) gives us our E_d action on $SC_d^{h\infty}$ up to homotopy. First, we have a lift p in the following diagram:

$$\begin{array}{ccc} SC_d^{h\infty} & \xrightarrow{\text{id}} & SC_d^{h\infty} \\ \iota \downarrow & \dashrightarrow & \\ SC_d^h & & \end{array}$$

p

We know that SC_d^h is cofibrant since it is obtained as the W construction applied to a Σ -cofibrant, well-pointed operad SC_d^h , so it fits into the context covered by Berger and Moerdijk [\[4; 5\]](#). In Spitzweck [\[21\]](#) we see that the corner axiom (or Quillen’s SM7) for monoidal model categories tensored over topological spaces applies to categories of operads in topological spaces. Thus we have an acyclic fibration

$$\text{map}(SC_d^h, SC_d^h) \xrightarrow{\iota^*} \text{map}(SC_d^{h\infty}, SC_d^h)$$

given by pre-composing with ι . Since both ιp and id live over ι , they must be homotopic. Let $h: [0, \infty] \otimes \text{SC}_d^h \rightarrow \text{SC}_d^h$ be a homotopy with $h_0 = \text{id}$ and $h_\infty = \iota p$.

We will use this h to define a homotopy right E_d -module structure on $\text{SC}_d^{h\infty}$. For this we will use a homotopy equivalent version of E_d which sits inside the W construction. For simplicity we denote it by E . First we define the category LE_d . This category is not monoidal, but will be used to build E . The letter L stands for *level trees*. The objects of the topological category LE_d are finite sets and the morphism space $LE_d(n, n')$ is defined to be a quotient of

$$\coprod_{\substack{k \geq 0 \\ n_1, \dots, n_k}} E_d(n_1, n') \times E_d(n_2, n_1) \times \dots \times E_d(n, n_k) \times [0, \infty]^k.$$

A point of the space above is given by a sequence $\alpha_i \in E_d(n_i, n_{i-1})$ for $1 \leq i \leq k + 1$ and $t_i \in [0, \infty]$ for $1 \leq i \leq k$. For convenience of notation, we set $n_0 = n'$, $n_{k+1} = n$, $t_0 = \infty$ and $t_{k+1} = \infty$. We impose the following relations:

5.1.2 Relations If $t_i = 0$, then we can delete t_i and replace $(\dots, \alpha_i, \alpha_{i+1}, \dots)$ by the composition $(\dots, \alpha_i \circ \alpha_{i+1}, \dots)$. If $n_i = n_{i-1}$ and α_i is the identity, and $t_{i-1} = \infty = t_i$, then we can delete α_i from the sequence and delete t_i from the sequence.

5.1.3 Remark In the W construction, we could always delete the appearance of an identity and sum the lengths of the surrounding edges. We do not allow that here since we do not have $h_{s+t} = h_s \circ h_t$.

Composition in the category LE_d is given by concatenating sequences, setting the new coordinate in the factor $[0, \infty]$ between the two sequences to be ∞ .

We can use the action of E_d on SC_d^h as well as the maps h_t , p and ι to define

$$(33) \quad LE_d(n, n') \rightarrow \text{map}(\text{SC}_d^{h\infty}(n', m), \text{SC}_d^{h\infty}(n, m)).$$

To do this, represent $\alpha \in LE_d(n, n')$ with a sequence $n' = n_0, n_1, \dots, n_k, n_{k+1} = n$ together with $\alpha_i \in E_d(n_i, n_{i-1})$ for $1 \leq i \leq k + 1$ and $t_i \in [0, \infty]$ for $1 \leq i \leq k$. This gives a chain of maps

$$(34) \quad \text{SC}_d^{h\infty}(n', m) \xrightarrow{\iota} \text{SC}_d^h(n_0, m) \xrightarrow{\alpha_1} \text{SC}_d^h(n_1, m) \xrightarrow{h_{t_1}} \text{SC}_d^h(n_1, m) \xrightarrow{\alpha_2} \dots \\ \dots \xrightarrow{h_{t_k}} \text{SC}_d^h(n_k, m) \xrightarrow{\alpha_{k+1}} \text{SC}_d^h(n_{k+1}, m) \xrightarrow{p} \text{SC}_d^{h\infty}(n, m).$$

The maps $\text{SC}_d^h(n_i, m) \xrightarrow{\alpha_i} \text{SC}_d^h(n_{i+1}, m)$ are defined by the action of E_d on SC_d^h ,

$$\text{SC}_d^h(n_i, m) \times E_d(n_{i+1}, n_i) \rightarrow \text{SC}_d^h(n_{i+1}, m).$$

Let us check that [Relations 5.1.2](#) in LE_d are satisfied and that composition in E_d corresponds to composition of maps of $SC_d^{h\infty}$.

Suppose $t_i = 0$ for some i . Then $h_0 = \text{id}$, so our chain of arrows contains

$$SC_d^h(n_{i-1}, m) \xrightarrow{\alpha_i} SC_d^h(n_i, m) \xrightarrow{\alpha_{i+1}} SC_d^h(n_{i+1}, m).$$

The composition of these two is equal to the map given by $\alpha_i \alpha_{i+1} \in E_d(\underline{n}_{i+1}, \underline{n}_{i-1})$. This is because $SC_d^h(-, m)$ is a right E_d -module.

If $n_i = n_{i-1}$, α_i is the identity and $t_{i-1} = t_i = \infty$, then the composition $h_{t_{i-1}} \circ \alpha_i \circ h_{t_i}$ is equal to $h_{t_{i-1}} = h_\infty$, so we are justified in deleting α_i and t_i from the sequence.

Now suppose we have some $t_i = \infty$, so that $\alpha \in LE_d(n, n')$ decomposes as $\beta_1 \beta_2$ for some $\beta_1 \in E(n_i, n')$ and $\beta_2 \in E(n, n_i)$. The chain of compositions defining the action of α from $SC_d^{h\infty}(1, m)$ to $SC_d^{h\infty}(n, m)$ contains the segment:

$$\dots SC_d^h(n_i, m) \xrightarrow{h_\infty} SC_d^h(n_i, m) \rightarrow \dots$$

The composite of the actions of β_1 and β_2 is computed by joining the chains for β_1 and for β_2 . This joined chain agrees with the chain for α except for the segment above, which is replaced with the segment:

$$\dots \rightarrow SC_d^h(n_i, m) \xrightarrow{p} SC_d^{h\infty}(n_i, m) \xrightarrow{l} SC_d^h(n_i, m) \rightarrow \dots$$

Since $h_\infty = \iota p$, these chains of maps have the same composition.

The maps [\(33\)](#) define a functor

$$(35) \quad LE_d \rightarrow \text{End}(SC_d^{h\infty}).$$

There is no obvious operad structure on LE_d , so we take the smallest operad containing LE_d . More precisely, [\(35\)](#) is a morphism of topological categories whose objects are finite sets. There is a forgetful functor from operads to the category of such topological categories. The operad E is defined to be the result of applying the left adjoint of this forgetful functor to the category LE_d .

5.1.4 Definition Let $F(LE_d)$ be the free 1-colored operad generated by the symmetric sequence $n \mapsto LE_d(n, 1)$. For each $n, n' \geq 0$ let $E(n, n')$ be the topological space given by the coequalizer

$$\coprod_{n''} LE_d(n'', n') \times LE_d(n, n'') \rightrightarrows F(LE_d)(n, n') \xrightarrow{\text{eq}} E(n, n'),$$

where the two maps are given by composition in either $F(LE_d)$ or LE_d and the inclusion of LE_d into $F(LE_d)$.

5.1.5 Lemma *The category E is an operad and is equivalent to E_d .*

Proof Given a tree with its internal edges labeled by lengths $[0, \infty]$, call it a *level tree* if edges equidistant from the root vertex have the same length. Every morphism in $LE_d(n, 1)$ can be represented by a level tree with vertices labeled by E_d . We can represent a point of $F(LE_d)$ with a tree whose vertices are labeled by level trees in LE_d . The relation defining $F(LE_d) \rightarrow E$ allows us to break up a level tree with at least one level of length ∞ into several level trees all of whose levels have finite length. We conclude that E consists of trees labeled by E_d on the vertices and $[0, \infty]$ on the internal edges, satisfying the condition that every maximal finite subtree is level.

There is an operad morphism $E \rightarrow E_d$ which collapses all edge lengths to 0. On the level of symmetric sequences, there is a homotopy inverse $E_d \rightarrow E$. The homotopy $g_t: E \rightarrow E$ first collapses lengths of the edges furthest from the root to zero. This preserves the condition that every maximal finite subtree is level. Continuing in this way, we collapse all edge lengths to zero. \square

The adjoint to (35) is an operad morphism $E \rightarrow \text{End}(SC_d^{h\infty})$, which by Definition 4.1.7 we can use to define the K -colored operad $SC_d^{h\infty} \rtimes E$.

5.1.6 Lemma *The $\{f, h\}$ -colored operad $SC_d^{h\infty} \rtimes E$ is weakly equivalent to the Swiss cheese operad.*

Proof First, note that SC_d is equivalent to the semi-direct product of SC_d^h and E_d where the action of E_d factors through the map $E_d \rightarrow E_d$ which sends all lengths of internal edges to zero. This is because the map $SC_d^h \rightarrow SC_d^h$ which collapses trees is a weak equivalence and respects the action of E_d .

The action of E on $SC_d^{h\infty}$ can be extended to an action on all of SC_d^h . The sequence $\alpha_1, t_1, \dots, t_k, \alpha_{k+1}$ acts via the composition

$$(36) \quad SC_d^h(n_0, m) \xrightarrow{\alpha_1} SC_d^h(n_1, m) \xrightarrow{h_{t_1}} SC_d^h(n_1, m) \rightarrow \dots \rightarrow SC_d^h(n_{l-1}, m) \\ \xrightarrow{h_{t_l}} SC_d^h(n_{l-1}, m) \xrightarrow{\alpha_l} SC_d^h(n_l, m) \xrightarrow{h_\infty} SC_d^h(n_l, m).$$

Define, for each $s \in [0, \infty]$, a homotopy $h^{[0,s]}: [0, \infty] \otimes SC_d^h \rightarrow SC_d^h$ by setting $h_t^{[0,s]} = h_{\min(s,t)}$. We have $h_0^{[0,s]} = \text{id}$ and $h_\infty^{[0,s]} = h_s$; therefore, we can define an action of E on $[0, \infty]_s \otimes SC_d^h$ by replacing h_{t_i} in (36) with $h_{t_i}^{[0,s]}$. Then, when $s = 0$, each $h_{t_i}^{[0,0]}$ is the identity, so the action factors through the map $E \rightarrow E_d$ collapsing all edges to 0. When $s = \infty$ we have $h_{t_i}^{[0,\infty]} = h_{t_i}$, so the action of E on SC_d^h is (36). Thus we have a diagram of equivalences

$$SC_d \leftarrow SC_d^h \rtimes_{s=0} E \rightarrow ([0, 1] \otimes SC_d^h) \rtimes E \leftarrow SC_d^h \rtimes_{s=\infty} E \rightarrow SC_d^{h\infty} \rtimes E,$$

where the map on the left collapses all edge lengths to 0. □

6 The equivalence $SC_d^{h\infty} \rightarrow SC_d^h$

This section is dedicated to proving [Theorem 5.1.1](#).

The proof uses a recasting [Definition 6.1.4](#) of [Definition 2.1.1](#) which is equivalent when considering operads in Top ; see Getzler [\[11\]](#). First we set the context for this new definition, then we prove that $SC_d^{h\infty} \rightarrow SC_d^h$ is a cofibration. Finally, we show that it is a weak equivalence.

6.1 The category of forests

The following definition is an amalgamation of those found in [\[17; 8; 11\]](#).

6.1.1 Definition Fix a set K . A K -colored young forest is an uncolored map of finite K -colored sets $x: I_x \rightarrow J_x$. A K -colored forest $f: x \rightarrow y$ is a color-preserving isomorphism $f: I_y \sqcup J_x \rightarrow J_y \sqcup I_x$ satisfying certain conditions. Before mentioning the conditions, it helps to think of I_y as the set of incoming edges of f , J_x as the set of internal vertices of f , J_y as the set of outgoing vertices of f and I_x as the set of internal edges of f .

To form the conditions f must satisfy to qualify as a K -colored forest, we use f, x and y to build an endomorphism (y, f, id, x) of $I_y \sqcup J_x \sqcup J_y \sqcup I_x$ which is y on I_y , f on J_x , the identity on J_y and x on I_x . We require that, for every $p \in I_y \sqcup J_x \sqcup J_y \sqcup I_x$, there is a $k \geq 0$ such that $(y, f, \text{id}, x)^k(p) \in J_y$. Moreover, if $p \in I_y$, we require $(y, f, \text{id}, x)^k(p)$ to be equal to $y(p)$. An example is shown in [Figure 1](#). We will abuse notation and write $(f \circ x)^k(p)$ when $p \in I_x$, $(y, f, \text{id}, x)^k(p) \in J_y$ and $(y, f, \text{id}, x)^{k-1}(p) \notin J_y$.

6.1.2 Definition Given a K -colored forest $f: x \rightarrow y$, we call $V(f) := J_x$ the set of internal vertices of f . We call $\text{in}(f) := I_y$ the set of input vertices of f and $\text{rt}(f) := J_y$ the set of root vertices of f . In addition, $\text{Edge}(f) = J_x \sqcup I_y \cong J_y \sqcup I_x$ is called the set of extended edges of f and $E(f) := J_x \times_f I_x$ is the set of internal edges of f .

6.1.3 Definition If $g: x \rightarrow y$ and $f: y \rightarrow z$ are forests, we can define a composite forest $fg: x \rightarrow z$. We use concatenation to denote this composition and \circ to denote composition of maps of finite sets. The forest fg , as a map $I_z \sqcup J_x \rightarrow J_z \sqcup I_x$, is defined by the following rule. If $i \in I_z$, then there is a $k \geq 0$ and an $\varepsilon \in \{0, 1\}$

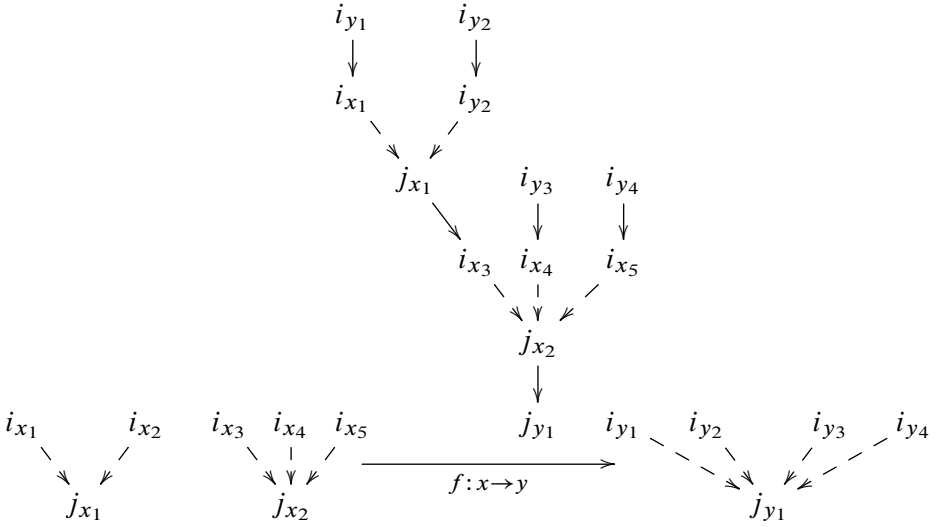


Figure 1: Here, f is a forest from x to y , where $x: I_x \rightarrow J_x$ and $y: I_y \rightarrow J_y$ are maps of finite K -colored sets for $I_x = \{i_{x_1}, i_{x_2}, i_{x_3}, i_{x_4}, i_{x_5}\}$ and $J_x = \{j_{x_1}, j_{x_2}\}$. Similarly, $I_y = \{i_{y_1}, i_{y_2}, i_{y_3}, i_{y_4}\}$ and $J_y = \{j_{y_1}, j_{y_2}\}$. The map x is denoted by dotted edges. The map y is denoted by dashed edges, and the map f is denoted, where applicable, by solid edges. The picture demonstrates the second condition for f to qualify as a forest. Namely, $f(x(f(x(f(i_{y_1})))))) = f(x(f(x(f(i_{y_2})))))) = f(x(f(i_{y_3}))) = f(x(f(i_{y_4}))) = y(i_{y_1}) = y(i_{y_2}) = y(i_{y_3}) = y(i_{y_4}) = j_{y_1}$.

such that $g^\varepsilon(f \circ g)^k(i) \in J_z \sqcup I_x$. Similarly, if $i \in J_x$ there is a k and ε such that $f^\varepsilon(g \circ f)^k(i) \in J_z \sqcup I_x$. In [23], we show that forest composition is associative. This, together with disjoint union, makes young forests the objects and forests the morphisms of a symmetric monoidal category denoted For .

6.1.4 Definition A K -colored operad \mathcal{O} is a strong symmetric monoidal functor $(\text{For}_K, \sqcup) \rightarrow (\text{Top}, \times)$. The category of operads is the category $\text{Fun}^\otimes(\text{For}_K, \text{Top})$ of symmetric monoidal functors and natural transformations. We denote this category by Op_K just as in Definition 2.3.3.

6.1.5 Remark The category of K -colored operads Op_K as defined in Definition 2.3.3 is naturally isomorphic to the functor category $\text{Fun}^\otimes(\text{For}_K, \text{Top})$. Indeed, given \mathcal{O} from Definition 2.1.1, as in Definition 2.3.3 we define $\mathcal{O}(x)$ for a young forest x to be $\bigotimes_{j \in J_x} \mathcal{O}(x^{-1}(j); j)$.

6.1.6 Remark When the set of colors K is understood, we often drop it from the notation. Forests and young forests are always K -colored for some set K . The category For_K will be abbreviated For and Op_K will be denoted by Op .

6.1.7 Definition Let (C, \otimes) be a symmetric monoidal category. Call an object $c \in C$ *indecomposable* if it cannot be written as a tensor product $c \cong c_1 \otimes c_2$ for any $c_1, c_2 \in C$. Let $\text{Fun}^\otimes(C, \text{Top})$ denote the category of strong symmetric monoidal functors $(C, \otimes) \rightarrow (\text{Top}, \times)$. We call a morphism $\psi: \mathcal{O} \rightarrow \mathcal{P}$ in $\text{Fun}^\otimes(C, \text{Top})$ a *fibration* (respectively *weak equivalence*) if $\psi(c): \mathcal{O}(c) \rightarrow \mathcal{P}(c)$ is a fibration (respectively weak equivalence) for every indecomposable $c \in C$. Define the class of *cofibrations* in the usual manner (see Hovey [13]).

For every symmetric monoidal category C we consider in this paper, we will use [Definition 6.1.7](#) to define cofibrations, fibrations and weak equivalences regardless of whether or not these form a model structure.

6.1.8 Remark A young forest x is indecomposable ([Definition 6.1.7](#)) if $J_x \simeq *$. In this case we say x is a young *tree*. We say a forest $f: y \rightarrow x$ is a *tree* if x is a young tree.

6.1.9 Definition For a category C , let C^\times denote the category with the same objects as those of C , but only invertible morphisms. Let C_{indec} denote the full subcategory (not monoidal) of C consisting indecomposable objects. Let $C_{\text{indec}}^\times = (C_{\text{indec}})^\times$. The functor category $\text{Top}^{C_{\text{indec}}^\times}$ is called the category of C -*symmetric sequences*, denoted by $C\text{-sSeq}$.

6.1.10 Remark A forest $f: x \rightarrow y$ is invertible if and only if $f(J_x) = J_y$ and $f(I_y) = I_x$. Thus an invertible forest gives a pair of isomorphisms of K -colored finite sets $J_f: J_x \rightarrow J_y$ and $I_f: I_y \rightarrow I_x$ which are compatible with the maps $x: I_x \rightarrow J_x$ and $y: I_y \rightarrow J_y$. We conclude that the category $\text{For}_{\text{indec}}^\times$ is isomorphic to the *opposite* of the category of K -colored finite sets and bijections. In the case $K \simeq \{*\}$, we get $\text{For}_{\text{indec}}^\times \cong \Sigma$, where Σ is as in [Definition 2.3.1](#). Moreover, in this case the category of For -symmetric sequences is isomorphic to the category Top^Σ of symmetric sequences.

6.1.11 Notation If $C \rightarrow \text{For}$ is any symmetric monoidal functor, we will denote the category $\text{Fun}^\otimes(C, \text{Top})$ of strong symmetric monoidal functors by $C\text{-Op}$ unless we say otherwise (for example, we do not use this notation in [Definition 6.3.4](#)). We call the objects of $C\text{-Op}$ C -*operads*. In all cases we consider, the functor $C \rightarrow \text{For}$ will be apparent from the category C , so we leave the functor out of the notation. If $C \rightarrow D \rightarrow \text{For}$ is a pair of symmetric monoidal functors, we denote the forgetful functor $D\text{-Op} \rightarrow C\text{-Op}$ by U_C^D and denote its left adjoint by F_D^C .

6.2 The W construction

We show how the W construction of [6] can be realized as a coend construction using For . In nice situations the W construction gives a cofibrant replacement for an operad, as shown in [4]. We use $[0, \infty]$ as our edge labels as in [16].

Given any young forest z there is a contravariant functor $W: \text{For}/_z \rightarrow \text{Top}$ from the over category of z to topological spaces. For any object $g: y \rightarrow z$ of this over category, set $W(g) = \text{map}(E(g), [0, \infty])$. If $f: x \rightarrow y$ is a forest, the map $W(g) \rightarrow W(gf)$ is denoted by $W_\Sigma(f)$. This map uses the sum operation on $[0, \infty]$. This is an extension of $+$ on $[0, \infty)$ such that $t + \infty = \infty = \infty + t$ for all values of t . Concretely, g and gf define maps of sets

$$E(g) \xrightarrow{\tilde{g}} J_z \sqcup I_x \xleftarrow{\tilde{gf}} E(gf)$$

We can turn a function $t \in W(g)$ to a function $W_\Sigma(t) \in W(gf)$ by pushing forward along \tilde{g} then pulling back along \tilde{gf} .

Pushing forward means summing over fibers, which is well defined since all the sets we are considering are finite.

If $h: z \rightarrow w$ is a forest there is a morphism $W_\infty(h): W(g) \rightarrow W(hg)$ which uses the maps $E(g) \hookrightarrow I_y \hookleftarrow E(hg)$. In this case we do not push forward and pull back functions. Rather, we extend a function $t: E(g) \rightarrow [0, \infty]$ to a function on $E(hg)$ by setting $t(\epsilon) = \infty$ if $\epsilon \notin E(g)$. This defines a natural transformation $W_\infty(h): W \rightarrow Wh_*$, where $h_*: \text{For}/_z^{\text{op}} \rightarrow \text{For}/_w^{\text{op}}$ is induced by $h: z \rightarrow w$. In other words, this diagram commutes for all composable triples h, g and f :

$$(37) \quad \begin{array}{ccc} W(g) & \xrightarrow{W_\Sigma(f)} & W(gf) \\ W_\infty(h) \downarrow & & \downarrow W_\infty(h) \\ W(hg) & \xrightarrow{W_\Sigma(f)} & W(hgf) \end{array}$$

Consider an operad \mathcal{O} as a collection of functors $\mathcal{O}_z: \text{For}/_z \rightarrow \text{Top}$ by setting $\mathcal{O}_z(g) := \mathcal{O}(y)$ for $g: y \rightarrow z$. For a young forest z the topological space $W\mathcal{O}(z)$ is the coend

$$(38) \quad W\mathcal{O}(z) = W \otimes_{\text{For}/_z} \mathcal{O}_z = \left(\coprod_{g: y \rightarrow z} W(g) \times \mathcal{O}(y) \right) / \sim,$$

where $(W_\Sigma(f)t, \alpha) \sim (t, \mathcal{O}(f)\alpha)$ for every $f: x \rightarrow y, t \in W(g)$ and $\alpha \in \mathcal{O}(x)$. Now a forest $h: z \rightarrow w$ with the natural transformations above gives us a map

$$(39) \quad W\mathcal{O}(z) = W \otimes_{\text{For}/_z} \mathcal{O}_z \xrightarrow{W_\infty(h) \otimes \text{id}} Wh_* \otimes_{\text{For}/_z} \mathcal{O}_z \rightarrow W \otimes_{\text{For}/_w} \mathcal{O}_w = W\mathcal{O}(w).$$

This defines $W\mathcal{O}$ as a functor $\text{For} \rightarrow \text{Top}$. This functor is symmetric monoidal, so $W\mathcal{O}$ is a K -colored operad.

6.2.1 Remark The space $W\mathcal{O}(z)$ consists of labeled forests $f: y \rightarrow z$. A labeled forest is one whose internal edges each have a length in $[0, \infty]$ and whose vertices $j \in J_x$ have a corresponding label in $\mathcal{O}(x^{-1}(j); j)$.

In the sequel, we will define several variants on the category For . Each of these variants admits a functor to For and we want a corresponding W construction for each.

6.2.2 Definition Suppose \mathcal{O} is an operad $\text{For} \rightarrow \text{Top}$ and C is any symmetric monoidal category equipped with a symmetric monoidal functor $G: C \rightarrow \text{For}$. From this data we can construct an operad $W_C\mathcal{O} \in C\text{-Op}$ using formulas analogous to (38) and (39). Specifically, let W_C be the composite $W \circ G_c: (C/c)^{\text{op}} \rightarrow (\text{For}/G(c))^{\text{op}} \rightarrow \text{Top}$, where $c \in C$. We define

$$(40) \quad W_C\mathcal{O}(c) = W_C \otimes_{C/c} (G^*\mathcal{O})_c = \left(\coprod_{f: b \rightarrow c} W(G(f)) \times \mathcal{O}(G(y)) \right) / \sim.$$

We define $W_C\mathcal{O}(c) \rightarrow W_C\mathcal{O}(d)$ for a C -morphism $g: c \rightarrow d$ just as in (39).

6.2.3 Example Define the full subcategory $D \hookrightarrow \text{For}$ to be given by those young forests x where J_x has only color h . Restricting the Swiss cheese operad to this full subcategory gives the operad SC_d^h as $W_D SC_d$. To get SC_d^{h1} , we use $C \hookrightarrow \text{For}$, the full subcategory of young forests x where J_x has color h and, for each $j \in J_x$, there is at most one element of $x^{-1}(j)$ of color f .

It should be clear that $D\text{-Op} \simeq \text{Op}(\text{Top}^\Sigma)$: the forgetful functor $\text{Op}_K \rightarrow \text{Op}(\text{Top}^\Sigma)$ from Definition 2.3.3 is given by pulling back along the symmetric monoidal functor $D \rightarrow \text{For}_K$. Finally, note that $SC_d^{h\infty}$ as defined in Definition 4.1.3 is $F_D^C SC_d^{h1}$, where F_D^C is as defined in Notation 6.1.11

6.3 Weighted forests

We need to define the category of weighted forests to prove the following half of Theorem 5.1.1:

6.3.1 Theorem *The natural map $SC_d^{h\infty} \rightarrow SC_d^h$ is a cofibration in $\text{Op}(\text{Top}^\Sigma)$.*

The proof follows the work of Berger and Moerdijk [4] closely. From an operad \mathcal{O} they construct an increasing chain of symmetric sequences

$$(41) \quad W_0\mathcal{O} \rightarrow W_0^+\mathcal{O} \rightarrow W_1\mathcal{O} \rightarrow W_1^+\mathcal{O} \rightarrow W_2\mathcal{O} \rightarrow \dots .$$

The symmetric sequence $W_0\mathcal{O}$ is just the underlying symmetric sequence of \mathcal{O} . If \mathcal{O} is cofibrant as a symmetric sequence, \mathcal{O} is said to be Σ -cofibrant. If the operadic unit maps of \mathcal{O} are cofibrations, \mathcal{O} is said to be *well-pointed*. In the case \mathcal{O} is well-pointed and Σ -cofibrant, Berger and Moerdijk show that $W\mathcal{O} := \text{colim}_k W_k\mathcal{O}$ is a cofibrant replacement of \mathcal{O} as an operad. In the course of the proof, they show that $W_k\mathcal{O}$ is a *k-operad*, which is a partial operad in a certain sense. This partial operad structure will be encoded here in the category Op_k from Definition 6.3.4. Each $W_k^+\mathcal{O}$ is an operad in Op_{k+1} and, in our context, is given by the left adjoint to a forgetful functor $\text{Op}_{k+1} \rightarrow \text{Op}_k$ applied to $W_k\mathcal{O}$. Concretely, the points of $W_k\mathcal{O}$ are given by trees with at most k internal edges whose vertices are labeled by \mathcal{O} and whose internal edges are labeled by $[0, \infty]$.

6.3.2 Definition Let $f: x \rightarrow y$ be a forest. Let $(f | x)$ denote the endomorphism of $I_y \sqcup J_y \sqcup I_x \sqcup J_x$ which is f on $I_y \sqcup J_x$, x on I_x and the identity on J_y . Since f is a forest, there is a $k \geq 0$ such that, for every i , we have $(f | x)^k(i) \in J_y$. Let $[f | x]$ denote $(f | x)^\infty$.

6.3.3 Definition For $I \subset I_y \sqcup I_x \sqcup J_x$ and $j \in J_y$, let $I(j)$ denote $I \cap [f | x]^{-1}(j)$, the set of elements of I living over j . A *weighted young forest* is a pair (x, ω_x) where x is a young forest and $\omega_x: J_x \rightarrow \mathbb{Z}_{\geq 0}$ is any function, called the *weight of x* . A *weighted forest* $f: (x, \omega_x) \rightarrow (y, \omega_y)$ is a forest $f: x \rightarrow y$ such that, for all $j \in J_y$,

$$(42) \quad \omega_y(j) \geq \#E(f) + \sum_{i \in J_x(j)} \omega_x(i).$$

If $g: (y, \omega_y) \rightarrow (z, \omega_z)$ is a weighted forest. then one can show that $gf: x \rightarrow z$ defines a weighted forest $gf: (x, \omega_x) \rightarrow (z, \omega_z)$.

6.3.4 Definition Disjoint union of forests extends to disjoint union of weighted forests. Let For_ω denote the symmetric monoidal category of weighted forests. For each $k \geq 0$, let For_k denote the full subcategory of For_ω generated by objects of the form (x, ω_x) such that $\omega_x(j) \leq k$ for every $j \in J_x$. Let Op_k denote the category of (*K-colored*) *weight-k operads*, which are strong symmetric monoidal functors $\text{For}_k \rightarrow \text{Top}$. Note that there is a symmetric monoidal functor $\text{For}_\omega \rightarrow \text{For}$ which forgets the weights.

6.3.5 Remark If a forest $f: x \rightarrow y$ has no internal edges then f is a disjoint union of isomorphisms and maps of the form $[\emptyset \rightarrow \emptyset] \rightarrow [\{\kappa\} \rightarrow \{\kappa\}]$, where $\kappa \in K$. The restriction of an operad \mathcal{O} to For_0 remembers only

- the spaces $\mathcal{O}(I; \kappa)$ for each K -colored set I and each color κ , together with the right $\text{aut } I$ action on $\mathcal{O}(I; \kappa)$, and
- the operadic unit maps $* \rightarrow \mathcal{O}(\kappa; \kappa)$ for each color $\kappa \in K$.

Thus \mathcal{O} is cofibrant (Definition 6.1.7) as an object of $\text{Fun}^\otimes(\text{For}_0, \text{Top})$ if and only if it is well-pointed and Σ -cofibrant as in [4, Section 3].

Proof of Theorem 6.3.1 In Example 6.2.3 we constructed categories C and D such that $SC_d^{h1} = W_C SC_d$, $SC_d^h = W_D SC_d$ and the map $SC_d^{h\infty} \rightarrow SC_d^h$ is

$$F_D^C W_C SC_d \rightarrow W_D SC_d.$$

Lemma 6.3.12 applies since C is a full subcategory of D , which is a full subcategory of For , and $SC_d: \text{For} \rightarrow \text{Top}$ is cofibrant as a functor $\text{For}_0 \rightarrow \text{Top}$ (see Remark 6.3.5). \square

6.3.6 Definition Let C be any full subcategory of For . Define C_ω to be the full subcategory of For_ω given by young weighted forests (x, ω_x) such that $x \in C$. The functor $\text{For}_\omega \rightarrow \text{For}$ induces a functor $C_\omega \rightarrow C$.

If $C \hookrightarrow \text{For}$ is a full subcategory, there is a map of C_ω operads $W_{C_\omega} \rightarrow U_{C_\omega}^C W_C \mathcal{O}$. Concretely, for each weighted young forest (x, ω_x) we have a map

$$(43) \quad \coprod_{f:(y,\omega_y)\rightarrow(x,\omega_x)} W(f) \times \mathcal{O}(y) \rightarrow \coprod_{f:y\rightarrow x} W(f) \times \mathcal{O}(y),$$

defined in the obvious way. This descends to give a morphism

$$(44) \quad W_{C_\omega} \mathcal{O} \rightarrow U_{C_\omega}^C W_C \mathcal{O}$$

6.3.7 Lemma For any full subcategory $C \hookrightarrow \text{For}$, the left adjoint of (44),

$$F_C^{C_\omega} W_{C_\omega} \mathcal{O} \rightarrow W_C \mathcal{O},$$

is an isomorphism.

Proof The left adjoint $F_C^{C_\omega} W_{C_\omega} \mathcal{O}$ can be computed at a young tree x as

$$\text{colim}_{k \rightarrow \infty} (W_{C_\omega} \mathcal{O})(x, k).$$

For each k , a point of $(W_{C_\omega} \mathcal{O})(x, k)$ is given by a labeled tree (Remark 6.2.1) $f: y \rightarrow x$ with at most k internal edges. Taking the colimit as k goes to ∞ , we get all labeled trees over x , which is $W_C \mathcal{O}(x)$. \square

6.3.8 Lemma *Let C be any symmetric monoidal full subcategory of For_ω . Suppose $\psi: \mathcal{O} \rightarrow \mathcal{P}$ is a morphism of operads in $C\text{-Op}$ which is a cofibration of C -symmetric sequences. Further suppose that, for every young forest $x \in C$ such that there is a tree $f: x \rightarrow y$ in C with at least one internal edge, we have that $\psi(x): \mathcal{O}(x) \rightarrow \mathcal{P}(x)$ is an isomorphism. These conditions imply that ψ is a cofibration of C -operads.*

Proof Suppose $\mathcal{Q} \rightarrow \mathcal{Q}'$ is an acyclic fibration of C -operads. By Definition 6.1.7 this means that, for every young tree $x \in C$, $\mathcal{Q}(x) \rightarrow \mathcal{Q}'(x)$ is an acyclic fibration of topological spaces. Suppose we have a commutative diagram of C -operads:

$$(45) \quad \begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{Q} \\ \downarrow & & \downarrow \\ \mathcal{P} & \longrightarrow & \mathcal{Q}' \end{array}$$

By assumption, there is a lift $\rho: \mathcal{P} \rightarrow \mathcal{Q}$ of C -symmetric sequences. We claim that ρ is automatically a morphism of C -operads. To prove this claim, it is enough to show that the square on the right in the following diagram commutes for every tree $f: x \rightarrow y$ in C :

$$(46) \quad \begin{array}{ccccc} \mathcal{O}(x) & \xrightarrow{\psi(x)} & \mathcal{P}(x) & \xrightarrow{\rho(x)} & \mathcal{Q}(x) \\ \mathcal{O}(f) \downarrow & & \mathcal{P}(f) \downarrow & & \downarrow \mathcal{Q}(f) \\ \mathcal{O}(y) & \xrightarrow{\psi(y)} & \mathcal{P}(y) & \xrightarrow{\rho(y)} & \mathcal{Q}(y) \end{array}$$

Suppose f is an isomorphism. Then we know the square commutes because ρ is a map of C -symmetric sequences. Suppose f is not an isomorphism and has no internal edges. Since f is a tree it must be of the form $[\emptyset \rightarrow \emptyset] \rightarrow [\{c\} \rightarrow \{c\}]$ for some $c \in K$. In this case we have $\mathcal{O}(x) = \mathcal{P}(x) = \mathcal{Q}(x) = *$; in particular, $\psi(x)$ is an isomorphism. Thus our assumption shows that if f is not an isomorphism then $\psi(x)$ is an isomorphism. We can deduce that the square on the right commutes in this case from the fact that the square on the left and the outer square commute. \square

Given a full subcategory $D \hookrightarrow \text{For}$ and a full subcategory $C \hookrightarrow D$, we can interpolate between C_ω and D_ω with a sequence of subcategories of For_ω . For each $k \geq 0$, let D_k be the full subcategory of For_ω given by disjoint unions of young weighted trees (x, ω_x) where either $x \in C$, or $x \in D$ and $\omega_x \leq k$. Note that $D_{-1} = C_\omega$. We have left the inclusion $C \hookrightarrow D$ implicit in the notation D_k . We have a commutative diagram of symmetric monoidal functors:

$$(47) \quad \begin{array}{ccccccc} & & C & \longrightarrow & & & D \\ & & \uparrow & & & & \uparrow \\ C_\omega = D_{-1} & \longrightarrow & D_0 & \longrightarrow & \cdots & \longrightarrow & D_\omega \end{array}$$

The bottom row consists of successively larger full subcategories of For_ω and the top row consists of full subcategories of For .

From the symmetric monoidal functors in (47) we get the categories and forgetful functors in:

$$(48) \quad \begin{array}{ccccccc} C\text{-Op} & \longleftarrow & & & & & D\text{-Op} \\ & & \downarrow & & & & \downarrow \\ C_\omega\text{-Op} = D_{-1}\text{-Op} & \longleftarrow & D_0\text{-Op} & \longleftarrow & \cdots & \longleftarrow & D_\omega\text{-Op} \end{array}$$

6.3.9 Remark For $k, \ell \in \{-1, 0, 1, \dots\} \sqcup \{\omega\}$ with $k \geq \ell$, the forgetful functor $D_k\text{-Op} \rightarrow D_\ell\text{-Op}$ clearly preserves fibrations (Definition 6.1.7); thus, the left adjoint $F_{D_k}^{D_\ell}$ preserves cofibrations.

6.3.10 Lemma For each $k \geq 0$ there is a natural map in $D_\omega\text{-Op}$,

$$F_{D_\omega}^{D_k} W_{D_k} \mathcal{O} \rightarrow W_{D_\omega} \mathcal{O}.$$

The colimit of these maps as $k \rightarrow \infty$ is an isomorphism.

Proof For each young tree x and weight $\omega_x = \ell \in \mathbb{Z}_{\geq 0}$, the space $(F_{D_\omega}^{D_k} W_{D_k} \mathcal{O})(x, \ell)$ can be described as the subspace of $(W_{D_\omega} \mathcal{O})(x, \ell)$ given by labeled trees $f: y \rightarrow x$ with at most ℓ internal edges such that, after cutting apart f at the edges labeled ∞ , each remaining subtree of f has at most k internal edges. Taking the colimit as $k \rightarrow \infty$, we get all labeled trees $f: x \rightarrow y$ with at most ℓ internal edges. That is, we get all of $(W_{D_\omega} \mathcal{O})(x, \ell)$. \square

6.3.11 Lemma Suppose $\mathcal{O} \in \text{Op}$ is cofibrant as an object of $\text{Fun}^\otimes(\text{For}_0, \text{Top})$ (see Remark 6.3.5). Then, for each $k \geq 0$, the natural map $\iota_k: F_{D_k}^{D_{k-1}} W_{D_{k-1}} \mathcal{O} \rightarrow W_{D_k} \mathcal{O}$ is a cofibration in $D_k\text{-Op}$.

Proof Observe that, if $f: x \rightarrow y$ is a tree in D_k with at least one internal edge, then $x \in D_{k-1}$, so $\iota_k(x)$ is an isomorphism. By Lemma 6.3.8 we only need to show that ι_k is a cofibration of D_k -symmetric sequences.

In the case $k = 0$, if x is a tree in $C = D_{-1}$ then $\iota_0(x, 0)$ is an isomorphism. If x is in $D - C$ then $\iota(x, 0)$ is the map $\emptyset \rightarrow \mathcal{O}(x)$. The assumption that \mathcal{O} is cofibrant as a functor $\text{Fun}^\otimes(\text{For}_0, \text{Top})$ implies in particular that each $\mathcal{O}(x)$ is $\text{aut}(x)$ -cofibrant. That is, \mathcal{O} is cofibrant as a For -symmetric sequence, so ι_0 is a cofibration of D_0 -symmetric sequences. We now consider the case $k > 0$. We follow [4, Lemma 5.4]. For a young tree $z \in \text{For}_k$ and $g: y \rightarrow z$ a tree in For_k , let $(W \times \mathcal{O})_k^+(g)$ be $W(g) \times \mathcal{O}(y)$ if g has at most $k - 1$ internal edges. Otherwise, let $(W \times \mathcal{O})_k^+(g) \subset W(g) \times \mathcal{O}(y)$ be the set of (t, α)

such that $t(\epsilon) = 0$ or $t(\epsilon) = \infty$ for some $\epsilon \in E(g)$ or $\alpha(j) = \text{id}$ for some $j \in V(g)$. Using the techniques of [4, Section 2] one can also show that if \mathcal{O} is cofibrant as an object of $\text{Fun}^{\otimes}(\text{For}_0, \text{Top})$ (see Remark 6.3.5) then $(W \times \mathcal{O})_k^+(g) \rightarrow W(g) \times \mathcal{O}(\text{dom } g)$ is an $\text{aut}(g)$ -cofibration, where $\text{aut } g$ is the automorphism group of g as an object of the category $\text{For}_{k/z}$. Define the map $(W \times \mathcal{O})_k^+(g) \rightarrow F_{D_k}^{D_{k-1}} W_{D_{k-1}} \mathcal{O}(z)$ by collapsing any edge labeled 0 and deleting any vertex labeled with the identity.

In the diagram below, $\pi_0 \text{For}_{k/z}$ is the set of isomorphism classes of forests $g: y \rightarrow z$ in For_k and the domain, y , of a forest $g: y \rightarrow z$ is denoted by $\text{dom } g$:

$$(49) \quad \begin{array}{ccc} \coprod_{[g] \in \pi_0 \text{For}_{k/z}} ((W \times \mathcal{O})_k^+(g))_{\text{aut } g} & \longrightarrow & (F_{D_k}^{D_{k-1}} W_{D_{k-1}} \mathcal{O})(z) \\ \downarrow & & \downarrow \\ \coprod_{[g] \in \pi_0 \text{For}_{k/z}} (W(g) \times \mathcal{O}(\text{dom } g))_{\text{aut } g} & \longrightarrow & (W_{D_k} \mathcal{O})(z) \end{array}$$

The square in diagram (49) is a pushout. By remarks above, we know the map on the left in (49) is an $\text{aut}(z)$ -cofibration. We conclude $F_{D_k}^{D_{k-1}} W_{D_{k-1}} \mathcal{O} \rightarrow W_{D_k} \mathcal{O}$ is a cofibration of D_k -symmetric sequences. \square

6.3.12 Lemma *If C is a full subcategory of D and D is a full subcategory of For , and \mathcal{O} is an operad $\text{For} \rightarrow \text{Top}$ which is cofibrant as a functor $\text{For}_0 \rightarrow \text{Top}$, then the natural map $F_D^C W_C \mathcal{O} \rightarrow W_D \mathcal{O}$ is a cofibration of d -operads.*

Proof By Lemma 6.3.7 we have

$$W_C \mathcal{O} \cong F_C^{C^\omega} W_{C^\omega} \mathcal{O} = F_{C^\omega}^C W_{D_{-1}} \mathcal{O}.$$

The commutative diagram (48) shows that $F_D^C F_C^{C^\omega} = F_D^{D^\omega} F_{D^\omega}^{D_{-1}}$. This gives the first equality in

$$F_D^C F_C^{C^\omega} W_{C^\omega} \mathcal{O} = F_D^{D^\omega} F_{D^\omega}^{D_{-1}} W_{D_{-1}} \mathcal{O} \rightarrow F_D^{D^\omega} \text{colim}_k F_{D^\omega}^{D_k} W_{D_k} \mathcal{O} = F_D^{D^\omega} W_{D^\omega} \mathcal{O} \cong W_D \mathcal{O}.$$

The second equality comes from Lemma 6.3.10 and the isomorphism from Lemma 6.3.7. The arrow is a cofibration: Indeed, by repeated application of Lemma 6.3.11 together with the fact that each $F_{D_k}^{D_{k-1}}$ preserves cofibrations (Remark 6.3.9) we see that

$$F_{D_k}^{D_{k-1}} \dots F_{D_0}^{D_{-1}} W_{D_{-1}} \mathcal{O} \rightarrow W_{D_k} \mathcal{O}$$

is a cofibration in $D_k\text{-Op}$. Again, by Remark 6.3.9 $F_{D^\omega}^{D_k}$ and $F_{D^\omega}^{D^\omega}$ preserve cofibrations. Thus the arrow is a cofibration in $D\text{-Op}$. \square

6.4 Weak equivalence proof

This section contains a proof of:

6.4.1 Theorem *The natural map $SC_d^{h\infty} \rightarrow SC_d^h$ is a weak equivalence of operads in $Op(\text{Top}^\Sigma)$.*

The idea of the proof is to consider the maps $p_1: SC_d^{h\infty}(n, m) \rightarrow SC_d^{h\infty}(n - 1, m)$ and $p: SC_d^h(n, m) \rightarrow SC_d^h(n - 1, m)$ given by forgetting the n^{th} disk. By induction, we can suppose $SC_d^{h\infty}(n - 1, m) \rightarrow SC_d^h(n - 1, m)$ is a weak equivalence. We continue the induction by showing that $p_1^{-1}(\alpha) \rightarrow p^{-1}(\alpha)$ is a weak equivalence for every $\alpha \in SC_d^{h\infty}(n - 1, m)$.

To make the computation of $p_1^{-1}(\alpha)$ and $p^{-1}(\alpha)$ accessible, we will collapse the n^{th} disk of $\alpha \in SC_d^h(n, m)$ to a point. Our goal in the next section is to make this precise.

6.4.1 Defining $SC_{d,\bullet}(k, \ell | n, m)$ When we collapse the n^{th} disk of $\alpha \in SC_d^h(n, m)$ to its center, we think of the result $\hat{\alpha}$ as living in a 4-colored operad, which we denote by $SC_{d,\bullet}$. We add the colors f_\bullet and h_\bullet . Let $K_\bullet = \{f_\bullet, h_\bullet, f, h\}$ be the set of colors for this new operad. The color f_\bullet stands for collapsed full-disks. It is convenient to also allow a collapsed half disk, which we color with h_\bullet . Let $(k, \ell | n, m)$ denote the K_\bullet -colored finite set with k, ℓ, n and m elements of color f_\bullet, h_\bullet, f and h , respectively. Let D_\bullet denote the full subcategory of For_{K_\bullet} with objects isomorphic to disjoint unions of the young trees

$$(50) \quad \begin{aligned} (0, 0 | n, m) &\rightarrow \{h\}, & (1, 0 | n, m) &\rightarrow \{h\}, \\ (0, 1 | n, m) &\rightarrow \{h\}, & (1, 0 | 0, 0) &\rightarrow \{h_\bullet\}. \end{aligned}$$

To define $SC_{d,\bullet}: D_\bullet \rightarrow \text{Top}$ we need the notion of the geometric realization of β in $SC_d^h(n, m)$.

6.4.2 Definition Given $\beta \in SC_d^h(n, m)$, let $|\beta|$ be its *geometric realization*. This is the subset of \mathbb{R}^d given by deleting the open disks and half-disks of β from the closed unit half-disk. More precisely, if \bar{D}_+^d is the closed unit half-disk in \mathbb{R}^d , $\{(D_f^d)_j\}_{j=1}^m$ are the open disks of β , and $\{(D_h^d)_i\}_{i=1}^n$ are the open half-disks of β considered as open disks in \mathbb{R}^d whose center lies in \mathbb{R}^{d-1} , then

$$|\beta| = \bar{D}_+^d - \left(\left(\bigcup_{i=1}^n (D_h^d)_i \right) \cup \left(\bigcup_{j=1}^m (D_f^d)_j \right) \right).$$

Let

$$\partial_h |\beta| := \partial \left(\bar{D}_+^d - \left(\bigcup_i (D_h^d)_i \right) \right)$$

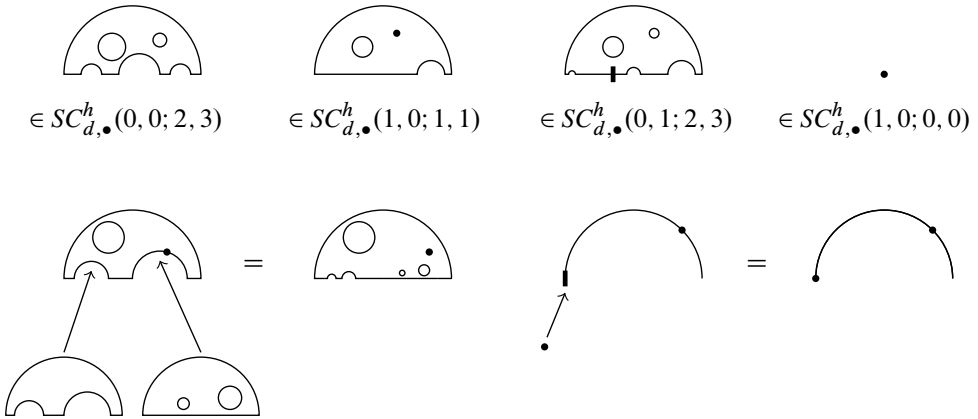


Figure 2: The collapsed disks are denoted by dots and the collapsed half-disks by tick marks. Collapsed disks are color \bullet , input edges and collapsed half-disks are color h_\bullet , input edges. To keep the collapsed disks and half-disks from coinciding, we only allow one or the other in any composition. Composition in $SC_{d,\bullet}$ takes place only in the half-disks and collapsed half disks. The only composition we can do in a collapsed half-disk is given by plugging in a collapsed disk. The result is a collapsed disk replacing the collapsed half-disk.

be the h -colored boundary of $|\beta|$. Let $\partial_{\text{rt}}(|\beta|)$ be the upper hemisphere $S_+^{d-1} \subset \partial \bar{D}_+^d$ and let $\partial_i |\beta|$ be the upper hemisphere of $\partial(D_h^d)_i$ for $1 \leq i \leq n$.

Now we can set

$$\begin{aligned}
 SC_{d,\bullet}^h(0, 0 \mid n, m) &= SC_d^h(n, m), \\
 SC_{d,\bullet}^h(1, 0 \mid n, m) &= \{(\alpha, q) \mid \alpha \in SC_d^h(n, m), q \in |\alpha|\}, \\
 SC_{d,\bullet}^h(0, 1 \mid n, m) &= \{(\alpha, q) \mid \alpha \in SC_d^h(n, m), q \in |\alpha| \cap \mathbb{R}^{d-1}\}, \\
 SC_{d,\bullet}^h(k, \ell \mid n, m) &= *.
 \end{aligned}$$

We think of the point $q \in |\alpha|$ as a collapsed disk and the point $q \in |\alpha| \cap \mathbb{R}^{d-1}$ as a collapsed half-disk. Composition in $SC_{d,\bullet}$ takes place in the half-disks and collapsed half-disks only. The un-collapsed disks play no part in composition. However, the collapsed half-disks and collapsed disks only play a part in composition when we plug a collapsed disk into a collapsed half-disk. The result is a collapsed disk which happens to live on the boundary of the geometric realization. See Figure 2.

6.4.3 Definition Let C_\bullet denote the full subcategory of D_\bullet with objects isomorphic to disjoint unions of the young forests from (50) with $n \leq 1$. Recall from Example 6.2.3

that we used D to denote the full subcategory of For_K whose young trees are isomorphic to the trees in (50) with $k = \ell = 0$. Likewise, C is the full subcategory of D given by trees isomorphic to disjoint unions of forests from (50) with $k = \ell = 0$ and $n \leq 1$. We write $x \leq 1$ if $x \in C$ or $x \in C_\bullet$. We have that $\text{SC}_d^{\text{h}1}$ is the restriction of SC_d to C and SC_d^{h} is the restriction of SC_d to D . Let $\text{SC}_{d,\bullet}^{\text{h}1}$ denote the restriction of $\text{SC}_{d,\bullet}$ to C_\bullet .

Let

$$\text{SC}_{d,\bullet}^{\text{h}}$$
 and $\text{SC}_{d,\bullet}^{\text{h}1}$

denote the W construction applied to the 4-colored operads $\text{SC}_{d,\bullet}$ and $\text{SC}_{d,\bullet}^{\text{h}1}$ via Definition 6.2.2. Let F denote Kan extension along $C_\bullet \rightarrow D_\bullet$, that is, $F = F_{D_\bullet}^{C_\bullet}$ using Notation 6.1.11.

6.4.4 Definition Define $p: D_\bullet \rightarrow D$ to send the K_\bullet -colored young forest $x: I_x \rightarrow J_x$ to the K -colored forest px with

$$I_{px} = (I_x)_{f,h} \quad \text{and} \quad J_{px} = (J_x)_{f,h},$$

where for $K' \subset K_\bullet$ we set $I_{K'} = \text{col}_I^{-1}(K')$, $\text{col}_I: I \rightarrow K_\bullet$. In (50) we see that $x(I_{px}) \subset J_{px}$, so that we can define px as the restriction of x to I_{px} . Observe that $p(1, 0 | n, m) = (n, m)$. If $f: y \rightarrow x$ is a forest, then $pf: py \rightarrow px$ is defined using f . Since f preserves the colorings, pf is indeed a forest from py to px . If f is a morphism in C_\bullet then pf is a morphism in C .

If $\beta \in \text{SC}_{d,\bullet}(z)$ for a K_\bullet -colored young forest z , then we get $p\beta \in \text{SC}_d(pz)$. To define $p\beta$ write $\beta = (\beta_j)_{j \in J_z}$, where $\beta_j \in \text{SC}_{d,\bullet}(z^{-1}(j); j)$. Each β_j is of the form (γ_j, q_j) with $q_j \in |\gamma_j|$ or of the form $\gamma_j \in \text{SC}_d(z^{-1}(j); j)$. Set $p\beta = (\gamma_j)_{j \in J_{pz}}$.

If $t \in W(f)$ and $f \in D_\bullet$, then $E(pf) \subset E(f)$ and $pt \in W(pf)$ is defined to be the pullback of $t: E(f) \rightarrow [0, \infty]$.

If $(f: y \rightarrow x, t \in W(f), \alpha \in \text{SC}_{d,\bullet}(x))$ represents a point in $\text{SC}_{d,\bullet}^{\text{h}}(x)$ then $(pf, pt, p\alpha)$ represents a point in $\text{SC}_d^{\text{h}}(x)$. This defines the map $p: \text{SC}_{d,\bullet}^{\text{h}1}(1, 0 | n, m) \rightarrow \text{SC}_d^{\text{h}}(n, m)$. The restriction of p to $F(\text{SC}_{d,\bullet}^{\text{h}1})(1, 0 | n, m)$ factors through the inclusion

$$\text{SC}_d^{\text{h}\infty}(n, m) \rightarrow \text{SC}_d^{\text{h}}(n, m).$$

Let p_1 be the induced map $F(\text{SC}_{d,\bullet}^{\text{h}1})(1, 0 | n, m) \rightarrow \text{SC}_d^{\text{h}\infty}(n, m)$.

Consider the commutative diagram of topological spaces, where the horizontal arrows do not assemble to operad maps:

$$\begin{array}{ccccc} \text{SC}_d^{\text{h}\infty}(n+1, m) & \xrightarrow{\sim} & F(\text{SC}_{d,\bullet}^{\text{h}1})(1, 0 | n, m) & \xrightarrow{p_1} & \text{SC}_d^{\text{h}\infty}(n, m) \\ \downarrow & & \downarrow & & \downarrow \iota \\ \text{SC}_d^{\text{h}}(n+1, m) & \xrightarrow{\sim} & \text{SC}_{d,\bullet}^{\text{h}}(1, 0 | n, m) & \xrightarrow{p} & \text{SC}_d^{\text{h}}(n, m). \end{array}$$

The maps p_1 and p delete the collapsed disk and, if necessary, a leftover collapsed half-disk. By induction on n we assume the right vertical arrow is an equivalence. We will show that for each $\alpha \in SC_d^{\text{h}\infty}(n, m)$ the inclusion $p_1^{-1}(\alpha) \rightarrow p^{-1}(\alpha)$ is an equivalence. Then by the long exact sequence of homotopy groups we conclude that the middle vertical arrow is an equivalence. The top left and bottom right maps collapse the n^{th} full-disk. One can show that these are equivalences. We conclude that the left vertical arrow is also an equivalence. This will prove [Theorem 6.4.1](#).

6.4.2 Computing $p^{-1}(\alpha)$ and $p_1^{-1}(\alpha)$ We have shown that the proof rests on [Proposition 6.4.5](#) below. This section is dedicated to the proof of this proposition.

6.4.5 Proposition Fix $\alpha \in SC_d^{\text{h}\infty}(n, m)$. The inclusion of the fiber $p_1^{-1}(\alpha)$ into the fiber $p^{-1}(\alpha)$ is a weak equivalence.

Combining the W construction (38) and the left adjoint $C\text{-Op} \rightarrow D\text{-Op}$ (that is, $\text{Op}(\text{Top}^{\Sigma \leq 1}) \rightarrow \text{Op}(\text{Top}^{\Sigma})$) we get

$$SC_d^{\text{h}\infty}(n, m) = \left(\coprod_{\substack{g: z \rightarrow y \\ f: y \rightarrow (n, m)}} W(g) \times SC_d(z) \right) / \sim,$$

where $y \leq 1$ ([Definition 6.4.3](#)). If $\alpha \in SC_d^{\text{h}\infty}(n, m)$ is represented by $(f, g, t, \tilde{\alpha})$ where $f: y \rightarrow (n, m)$, $g: z_{\alpha} \rightarrow y$, $t \in W(g)$ and $\tilde{\alpha} \in SC_d(z)$, then $\iota\alpha \in SC_d^{\text{h}\infty}(n, m)$ is represented by $(fg, W_{\infty}(f)t, \tilde{\alpha})$. Let $T_{\alpha} = fg: z_{\alpha} \rightarrow (n, m)$ and $t_{\alpha} = W_{\infty}(f)t$. Without loss of generality, we may assume $t_{\alpha}(i) > 0$ for every $i \in E(T_{\alpha})$ and that $\tilde{\alpha}(j) \neq \text{id}_{SC_d}$ for any $j \in J_{z_{\alpha}}$.

6.4.6 Definition Let $\text{Trees}(1, 0 | n, m)$ denote the over category $(D_{\bullet})_{/(1, 0 | n, m)}$. Similarly, let $\text{Trees}(n, m) = D_{/(n, m)}$. Let $p: \text{Trees}(1, 0 | n, m) \rightarrow \text{Trees}(n, m)$ denote the functor induced by p from [Definition 6.4.4](#).

Note that $T_{\alpha} \in \text{Trees}(n, m)$. Let $(S, \nu) \in \text{Trees}(1, 0 | n, m)_{/T_{\alpha}}$, where $S: x \rightarrow (1, 0 | n, m)$ is any K_{\bullet} -colored tree and $\nu: z_{\alpha} \rightarrow px$ is a forest such that $(pS)\nu = T_{\alpha}$. Define functors $W_{\alpha}: \text{Trees}(1, 0 | n, m)_{/T_{\alpha}}^{\text{op}} \rightarrow \text{Top}$ and $SC_{\alpha}: \text{Trees}(1, 0 | n, m)_{/T_{\alpha}} \rightarrow \text{Top}$ via the pullbacks:

$$(51) \quad \begin{array}{ccc} SC_{\alpha}(S) & \longrightarrow & SC_{d, \bullet}(x) & & W_{\alpha}(S) & \rightarrow & W(S) & \xrightarrow{p} & W(pS) \\ \downarrow & & \downarrow p & & \downarrow & & \downarrow W_{\Sigma}(\nu) & & \downarrow \\ * & \xrightarrow{\tilde{\alpha}} & SC_d(z) & \xrightarrow{SC_d(\nu)} & SC_d(px) & & * & \xrightarrow{t_{\alpha}} & W(T_{\alpha}) \end{array}$$

We want to replace $\text{Trees}(1, 0 | n, m)$ by a much smaller category. First we need the wedge operation on forests.

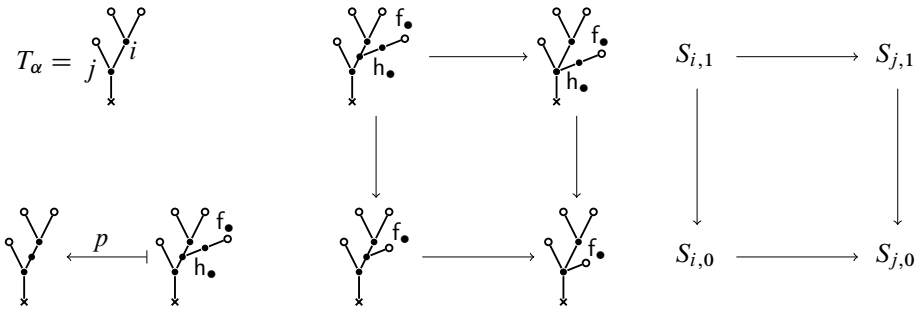


Figure 3: The edge i and vertex j of T_α give a commutative square in Trees_α . The input vertices are circles. The output vertex ends in an \times . The internal vertices are filled dots. The input and internal vertices of Γ_0 and Γ_1 are labeled with their colors. In addition, the image of $S_{i,1}$ under the functor p is shown. This makes it clear that the map $T_\alpha \rightarrow pS_{i,1}$ is given by inserting a single vertex.

6.4.7 Definition Let $f: x \rightarrow y$ be a K_f -colored forest and let $g: z \rightarrow w$ be a K_g -colored forest for some finite sets K_f and K_g . Let $\tau: J_w \rightarrow J_x$ be any map. Define $x \vee_\tau z$ to be the young $K_f \sqcup K_g$ -colored forest $(x, \tau, z): I_x \sqcup J_w \sqcup I_z \rightarrow J_x \sqcup J_z$ and define $y \vee_\tau w$ to be the young forest $(y, [f | x]\tau w): I_y \sqcup I_w \rightarrow J_y$ (see Definition 6.3.2). Finally, set $f \vee_\tau g: x \vee_\tau z \rightarrow y \vee_\tau w$ to be the forest

$$(f, g, f, \tau): I_y \sqcup I_w \sqcup J_x \sqcup J_z \rightarrow I_x \sqcup J_w \sqcup I_z \sqcup J_y.$$

6.4.8 Definition Let Γ_0 be the tree with no internal vertices and a single input vertex of color f_\bullet . Let Γ_1 be the tree with a single internal vertex of color h_\bullet and a single input vertex of color f_\bullet .

For any edge $i \in \text{Edge}(T_\alpha)_h$ define $\nu(i): T_\alpha \rightarrow T_\alpha(i)$ to be the morphism in $\text{Trees}(n, m)$ which inserts a unary vertex along i . Call this new vertex i_ν . Let $S_{i,k} = T_\alpha(i) \vee_{i_\nu} \Gamma_k$. For any internal vertex $j \in J_{z_\alpha}$ let $S_{j,k} = T_\alpha \vee_j \Gamma_k$. Note that $pS_{i,k} = T_\alpha(i)$ and $pS_{j,k} = T_\alpha$.

Let Trees_α be the full subcategory of $\text{Trees}(1, 0 | n, m) / T_\alpha$ given by the objects $S_{i,k} = (S_{i,k}, \nu(i))$ and $S_{j,k} = (S_{j,k}, \text{id}_{T_\alpha})$, where $i \in (I_{z_\alpha})_h \sqcup \{\text{rt}\}$, $j \in J_{z_\alpha}$ and $k \in \{0, 1\}$.

6.4.9 Remark The advantage of T_α is that it is easy to understand and computes the space $p^{-1}\alpha$ (Lemma 6.4.10). There is a unique morphism $S_{\ell,1}$ to $S_{\ell,0}$ for every ℓ and unique morphisms $S_{i,k} \rightarrow S_{T_\alpha^{-1}(i),k}$ and $S_{i,k} \rightarrow S_{z_\alpha(i),k}$. See Figure 3 for an illustration.

6.4.10 Lemma *The fiber $p^{-1}(\alpha)$ is given by the coend $W_\alpha \otimes_{\text{Trees}_\alpha} \text{SC}_\alpha$.*

Proof Let $\gamma = [S, s, \tilde{\gamma}] \in \text{SC}_{d,\bullet}^h(1, 0 | n, m)$, where $S: x \rightarrow (1, 0 | n, m)$ is a forest in D_\bullet , $s \in W(S)$ and $\tilde{\gamma} \in \text{SC}_{d,\bullet}(x)$. Let us assume that $\tilde{\gamma}(j) \neq \text{id}$ for all $j \in J_x$ and $s(i) > 0$ for all $i \in E(S)$. Observe that $p\gamma \in \text{SC}_d^h(n, m)$ is given by $[pS, ps, p\tilde{\gamma}]$. If $p\gamma = \alpha$ there must be some $v: T_\alpha \rightarrow pS$ in $\text{Trees}(n, m)$ such that $\text{SC}_d(v)\tilde{\alpha} = p\tilde{\gamma}$ and $W(v)ps = t_\alpha$. The condition $t_\alpha(i) > 0$ for all $i \in E(T_\alpha)$ implies that $t_\alpha \neq W_\Sigma(v)(t')$ for any t' and any v which collapses any edges. Moreover the condition $\tilde{\gamma}(j) \neq \text{id}$ for all j implies that $p\tilde{\gamma}(j) \neq \text{id}$ for all $j \in J_x$ such that $x^{-1}(j)_{f_\bullet, h_\bullet}$ is not empty. We conclude that either $v = \text{id}$ or v is the insertion of the unique unary (in pS , not in S) vertex j such that $x^{-1}(j)_{f_\bullet, h_\bullet}$ is not empty. In the former case we must have $S = S_{j,k}$ for some vertex $j \in J_x$ and some $k \in \{0, 1\}$. In the latter case we have $S = S_{i,k}$ for some edge i of T_α and some k . This defines the map $p^{-1}(\alpha) \rightarrow W_\alpha \otimes_{\text{Trees}_\alpha} \text{SC}_\alpha$. The map in the other direction is clear and the verification that they are inverses is left to the reader. \square

In diagram (52) we have h -colored edges i_1 and i_2 of T_α with $z_\alpha(i_1) = j = T_\alpha^{-1}(i_2)$. Thus we get the commutative diagram on the left. The image of this diagram under SC_α is shown on the right:

$$(52) \quad \begin{array}{ccccc} S_{i_1,1} \rightarrow S_{j,1} \leftarrow S_{i_2,1} & |\text{id}_h| \cap \mathbb{R}^{d-1} \rightarrow |\tilde{\alpha}(j)| \cap \mathbb{R}^{d-1} \leftarrow |\text{id}_h| \cap \mathbb{R}^{d-1} \\ \downarrow \quad \downarrow \quad \downarrow & \downarrow & \downarrow & \downarrow \\ S_{i_1,0} \rightarrow S_{j,0} \leftarrow S_{i_2,0} & |\text{id}_h| \longrightarrow |\tilde{\alpha}(j)| \longleftarrow |\text{id}_h| \end{array}$$

The geometric realization of the identity id_h is just S_+^{d-1} , the top half of the $(d-1)$ -sphere. The input of $\tilde{\alpha}(j)$ corresponding to i_1 is a half-disk and the map $|\text{id}_h| \rightarrow |\tilde{\alpha}(j)|$ corresponding to $S_{i_1,0} \rightarrow S_{j,0}$ is just $\partial_{i_1}|\tilde{\alpha}(j)| \rightarrow |\tilde{\alpha}(j)|$ (see Definition 6.4.2). On the other hand, the image of $S_{i_2,0} \rightarrow S_{j,0}$ is the inclusion of the output boundary $\partial_{\text{rt}}|\tilde{\alpha}(j)| \rightarrow |\tilde{\alpha}(j)|$.

6.4.11 Definition Let $\epsilon_\bullet \in E(S_{\ell,1})$ be the unique internal edge of color h_\bullet . If $i \in \text{Edge}(T_\alpha)$ (Definition 6.1.2), let i_v denote the vertex inserted by $v: T_\alpha \rightarrow pS_{i,k}$. Let i_{in} and i_{out} denote the incoming and outgoing edges of i_v , respectively, considered as internal edges of $S_{i,k}$. For any object $S_{\ell,k}$ of Trees_α , let $E_\alpha(S_{\ell,k}) = \{\epsilon_\bullet\}^k \sqcup (\{i_{\text{in}}, i_{\text{out}}\} \cap E(S_{\ell,k}))$. This defines a functor $E_\alpha: \text{Trees}_\alpha^{\text{op}} \rightarrow \text{Set}$.

The image under W_α of the square in diagram (52) is:

$$(53) \quad \begin{array}{ccccc} [0, \infty]^2 & \xleftarrow{(\text{id}, 0)} & [0, \infty] & \xrightarrow{(\text{id}, \infty)} & [0, \infty]^2 \\ (0, \text{id}) \uparrow & & \uparrow 0 & & \uparrow (0, \text{id}) \\ [0, \infty] & \xleftarrow{0} & * & \xrightarrow{\infty} & [0, \infty] \end{array}$$

More precisely,

$$(54) \quad W_\alpha(S) = \{s: E_\alpha(S) \rightarrow [0, \infty] \mid s(i_{\text{in}}) + s(i_{\text{out}}) = t_\alpha(i)\},$$

and $W_\alpha(S) \rightarrow W_\alpha(S')$ for a map $S' \rightarrow S$ in Trees_α is given by push forward of functions along the map of finite sets $E_\alpha(S) \hookrightarrow E_\alpha(S')$. There is no condition on $s(\epsilon_\bullet)$, the length of the edge of color h_\bullet . The isomorphism $W_\alpha(S_{i,1}) \rightarrow [0, \infty]^2$ sends s to $(s(\epsilon_\bullet), r(s(i_{\text{out}}), s(i_{\text{in}})))$, where

$$r(s_o, s_i) = \frac{1 - e^{-s_o}}{1 - e^{-s_i}},$$

which lands in $[0, \infty]$ because $s_o + s_i = t_\alpha > 0$. Note that $s_o = 0$ if and only if $r(s_o, s_i) = 0$ and $s_o = t_\alpha$ if and only if $r(s_o, s_i) = \infty$. Since the morphism $S_{i_1,1} \rightarrow S_{j,1}$ from diagram (52) collapses the edge $(i_1)_{\text{out}}$, we get $W_\alpha(S_{j,1}) \cong \{(r_\bullet, r) \in W_\alpha(S_{i_1,1}) \mid r = 0\}$. In the same diagram, the morphism $S_{i_2,1} \rightarrow S_{j,1}$ collapses the edge $(i_2)_{\text{in}}$, so we have $W_\alpha(S_{j,1}) \cong \{(r_\bullet, r) \in W_\alpha(S_{i_2,1}) \mid r = \infty\}$. The unique morphism $S_{i,1} \rightarrow S_{i,0}$ collapses the edge i_\bullet , so that $W_\alpha(S_{i,0}) \cong \{(r_\bullet, r) \in W_\alpha(S_{i,1}) \mid r_\bullet = 0\}$. The rest can be deduced from these cases.

6.4.12 Lemma *For any functor $F: \text{Trees}_\alpha \rightarrow \text{Top}$, the coend $W_\alpha \otimes_{\text{Trees}_\alpha} F$ is the homotopy colimit of F over Trees_α .*

Proof It is clear from diagrams (53) and (52) that $W_\alpha(S)$ is the geometric realization of the nerve of the under category of S for each object $S \in \text{Trees}_\alpha$. In addition, the maps $W_\alpha(S) \rightarrow W_\alpha(S')$ for $S' \rightarrow S$ agree with the maps obtained from the nerves of under categories. \square

6.4.13 Lemma *We can explicitly compute $p^{-1}(\iota\alpha)$ as*

$$p^{-1}(\iota\alpha) \simeq |SC_d(T_\alpha)\tilde{\alpha}| \simeq (S^{d-1})^{\vee n},$$

where $SC_d(T_\alpha)\tilde{\alpha}$ is the composition of all vertex labels from $\iota\alpha$.

Proof Let $\text{Trees}_{\alpha,0}$ denote the full subcategory of Trees_α consisting of objects $S_{j,0}$ and $S_{i,0}$ for internal vertices j and internal edges i . This category is homotopy terminal so, by Lemma 6.4.12 and Lemma 6.4.10, we have $p^{-1}(\iota\alpha) = \text{hocolim}_{\text{Trees}_{\alpha,0}} SC_\alpha$. This is the same as the homotopy colimit of the coequalizer diagram

$$\coprod_{i \in E(T_\alpha)} |\text{id}_h| \rightrightarrows \coprod_{j \in V(T_\alpha)} |\tilde{\alpha}(j)|,$$

where one arrow is given by including into output parts of the boundaries of the $|\tilde{\alpha}(j)|$ and the other arrow is given by including into input boundaries. These maps are

cofibrations with disjoint images. Each space in the coequalizer diagram is cofibrant. Thus the coequalizer diagram is already cofibrant as a functor $(\cdot \rightrightarrows \cdot) \rightarrow \text{Top}$. Thus we can compute the normal colimit. It is clear that this is the same as composing the $\tilde{\alpha}(j)$ via T_α then taking the realization of the result. In addition, $|\beta|$ is equivalent to a wedge of n spheres of dimension $d - 1$ if $\beta \in SC_d^h(n, m)$. \square

6.4.14 Definition Let $\text{Trees}_{\alpha,1}$ denote the full subcategory of Trees_α , where we discard the objects $S_{j,0}$ and $S_{i,0}$ for $j \in J_{z_\alpha}$ and $i \in E(T_\alpha)$. Define a functor $W_{\alpha,1}: \text{Trees}_{\alpha,1}^{\text{op}} \rightarrow \text{Top}$ by setting

$$W_{\alpha,1}(S) = \left\{ s: E_\alpha(S) \rightarrow [0, \infty] \mid \sum_{i \in E_\alpha(S)} s(i) = \infty \right\}.$$

6.4.15 Lemma Suppose $t_\alpha < \infty$ and $n = 1$; then $p_1^{-1}(\alpha)$ is given by the coend

$$W_\alpha \otimes_{\text{Trees}_{\alpha,1}} SC_\alpha,$$

where SC_α is the functor in [Definition 6.4.6](#) restricted to $\text{Trees}_{\alpha,1}$ and $W_{\alpha,1}$ is defined in [Definition 6.4.14](#).

Proof Let $\gamma \in F(SC_{d,\bullet}^h)(1, m)$ be such that $p_1(\gamma) = \alpha$. Pick a representative $(f, g, s, \tilde{\gamma})$, where $f: y \rightarrow (1, 0 \mid 1, m)$, $y \leq 1$, $g: z \rightarrow y$, $s \in W(g)$ and $\tilde{\gamma} \in SC_{d,\bullet}(z)$. Consider $\iota\gamma \in SC_{d,\bullet}^h(1, m)$, which is represented by $(fg, W_\infty(f)s, \tilde{\gamma})$. Recall that the condition $y \leq 1$ means that each connected component of the young forest y has at most one input whose color lives in $\{f, f_\bullet\}$. This implies that f has at least one internal edge $i \in E(f)$. Thus $W_\infty(f)s(i) = \infty$ when i is viewed as an internal edge in fg .

We know $p_1\iota\gamma = \iota\alpha$, so $\iota\gamma$ is represented by some triple $(S, s', \tilde{\gamma})$ with $S \in \text{Trees}_\alpha$, $s' \in W_\alpha(S)$ and $\tilde{\gamma} \in SC_\alpha(S)$. The relations in $SC_{d,\bullet}^h$ preserve edges of length ∞ , so we must have $s'(i) = \infty$ for some $i \in E(S)$. We are assuming $t_\alpha(i) < \infty$ for all $i \in E(T_\alpha)$, so the infinite edge in S must be in $E_\alpha(S)$. This implies $s' \in W_{\alpha,1}(S)$. Moreover, we cannot have such an infinite edge if $S = S_{j,0}$ for some vertex j or $S = S_{i,0}$ for some internal edge i . Thus $S \in \text{Trees}_{\alpha,1}$.

This defines the map from $p^{-1}(\alpha)$ to the coend. We leave the rest to the reader. \square

6.4.16 Lemma For any functor $F: \text{Trees}_{\alpha,1} \rightarrow \text{Top}$, the coend $W_{\alpha,1} \otimes_{\text{Trees}_{\alpha,1}} F$ is the homotopy colimit of F over $\text{Trees}_{\alpha,1}$.

Proof The argument here is similar to the proof of [Lemma 6.4.12](#). \square

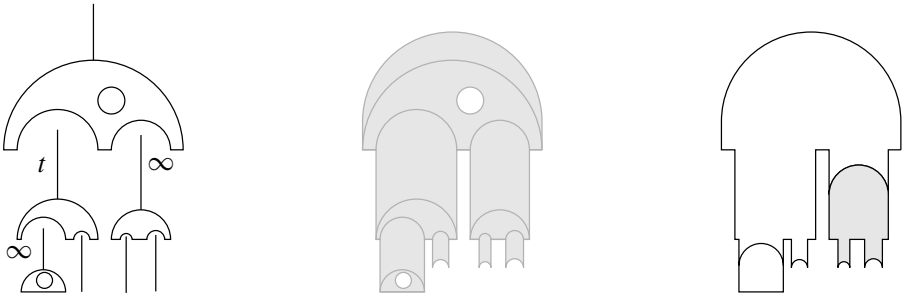


Figure 4: On the left is $\alpha \in SC_d^{\text{h}\infty}(2, 3)$. In the middle is $p^{-1}(\alpha)$ and on the right is $p_1^{-1}(\alpha)$. Both $p^{-1}(\alpha)$ and $p_1^{-1}(\alpha)$ have the homotopy type of a wedge of spheres, one for each disk in α .

6.4.17 Corollary *If $t_\alpha < 0$ and $n = 1$, then the fiber $p_1^{-1}(\alpha)$ is equivalent to $\partial_{\text{h}}|SC_d(T_\alpha)\tilde{\alpha}| \simeq S^{d-1}$.*

Proof By the same argument as in Lemma 6.4.13, $\text{hocolim}_{\text{Trees}_{\alpha,1}} SC_\alpha$ is equivalent to $\text{colim}_{\text{Trees}_{\alpha,1}} SC_\alpha$. This is easily computed as the h-colored boundary of the composite of $\tilde{\alpha}$. □

Proof of Proposition 6.4.5 Recall α is represented by $f: y \rightarrow (n, m)$, $y \leq 1$, $g: z_\alpha \rightarrow y$, $t \in W(g)$ and $\tilde{\alpha} \in SC_d(z_\alpha)$. By applying relations in $SC_d^{\text{h}\infty}$ we may assume $0 < t < \infty$. We may think of (g, t) as representing an element of $SC_d^{\text{h}1}(y)$, which we can write as $(\alpha(j))_{j \in J_y}$. If $\alpha(j) \in SC_d^{\text{h}}(n_j, m_j)$ then $n_j \leq 1$. Clearly $p_1^{-1}(\alpha(j)) \simeq p^{-1}(\alpha(j)) \simeq *$ when $n_j = 0$. Since $t_{\alpha(j)} < \infty$ we can use Corollary 6.4.17 to conclude $p_1^{-1}(\alpha(j)) \simeq \partial_{\text{h}}|(SC_d(g)(\alpha))(j)|$. The fiber $p_1^{-1}(\alpha)$ is equal to the colimit of the diagram

$$\coprod_{i \in E(f)} |1_{\text{h}}| \rightrightarrows \coprod_{j \in V(f)} p_1^{-1}(\alpha(j)),$$

where one arrow is given by $|1_{\text{h}}| \simeq \partial_i|\alpha(y(i))| \rightarrow \partial_{\text{h}}|(SC_d(g)(\alpha))(y(i))|$ and the other by $|1_{\text{h}}| \simeq \partial_\pi|\alpha(f(i))| \rightarrow \partial_{\text{h}}|(SC_d(g)(\alpha))(y(i))|$. This colimit is clearly $(S^{d-1})^{\vee n} \simeq p^{-1}(\iota\alpha)$. □

See Figure 4 for an illustration.

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