Desingularizing homology manifolds

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We prove that if X^n, $n \geq 6$, is a compact ANR homology n–manifold, we can blow up the singularities of X to obtain an ANR homology n–manifold with the disjoint disks property. More precisely, we show that there is an ANR homology n–manifold Y with the disjoint disks property and a cell-like map $f\colon Y \to X$.

1 Introduction

Homology n–manifolds are defined as (finite-dimensional, locally contractible) spaces whose local homology $H_*(X, X - \{x\})$ is isomorphic to $H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ for every $x \in X$. Equivalently, they can be described as spaces that satisfy a local form of Poincaré duality: every open set satisfies noncompact Poincaré duality between locally finite homology and cohomology. Originally introduced as a natural setting for the study of Poincaré duality, such spaces are important basic objects in geometric topology and arise in many investigations of structural properties of manifolds.

Until the 1990s, the only known non-manifold examples of homology manifolds were produced by singularizing manifolds: one would take disjoint, nearly contractible (technically, cell-like) subsets of a manifold and collapse them to points to obtain interesting singular spaces. Homology manifolds obtained in this manner are referred to as *resolvable*. A basic question then arises: Can every homology manifold be resolved by a manifold? In other words, given a homology n–manifold X, is there a topological n–manifold M and a surjective map $f\colon M \to X$ that is cell-like? A map f is cell-like if the inverse image of each point is contractible in every open neighborhood of itself. Quinn [19] showed that resolutions, if they exist (for $n > 4$), are unique, and that existence is detected by an obstruction in $H^0(X; Z)$ which is natural under restriction to open subsets. This implies that if a connected X has even a single manifold point, then it is resolvable.

A celebrated theorem of Edwards [9] asserts that a resolvable homology n–manifold, $n > 5$, is a topological n–manifold if and only if it has a modicum of general position: continuous maps $f, g\colon D^2 \to X$ of 2–disks into X can be approximated by maps with disjoint images. This is known as the *disjoint disks property*, DDP. In other words, for resolvable homology manifolds, the DDP provides a geometric characterization of manifolds. In Bryant [3] it was shown that the DDP leads to general position properties in all dimensions.

In earlier work [4] we showed that nonresolvable homology manifolds exist and can be classified, up to s–cobordism, by a variant of surgery theory. We also conjectured that DDP homology manifolds share many geometric features with topological manifolds, notably, homogeneity. The first main goal of this paper is to show that all homology manifolds of dimension ≥ 6 can be resolved by DDP homology manifolds. (In [4] we showed that they are all simple homotopy equivalent to DDP homology manifolds.) Even though the homogeneity conjecture remains open, one should view the "desingularization" in this paper as taming singularities by a resolution.

Alongside the DDP, which reflects a basic geometric characteristic of manifolds, another property that distinguishes manifolds among resolvable homology manifolds is the UV^1–approximation property. If one takes a degree–k map $S^n \to S^n$, one naively expects a typical point inverse image to have at least k components. The truth (see Bestvina [1]) is rather the opposite. If $n > 4$, then f is homotopic to a map with simply connected point inverses. This is predicted by the high connectivity of the homotopy fiber of the map. The point is that, rather than the usual strategy of approximating maps by smooth maps, it is sometimes more useful to consider approximations by maps that behave like space-filling "curves", which are closer models of the underlying abstract homotopy theory. The original construction of nonresolvable homology manifolds used this approach very strongly, and it is one of the themes of this paper.

Theorem A   *Every homology n–manifold X, $n \geq 6$, is the cell-like image of a DDP homology n–manifold that has the UV^1–approximation property.*

(We shall not discuss here the sense in which the UV^1 map approximates a given map: smooth maps cannot be C^0–close to UV^1 maps in the case of degree > 1 self maps of the sphere.)

An intriguing question presents itself of whether DDP is equivalent to the UV^1–approximation property in general. These are different types of conditions, but they both seem to pick out the nonsingular examples. In any case, the explicit focus on this property was critical to the constructions of resolutions of this paper, even if one was only interested in the DDP.

Technically, this paper also proves a "squeezing" theorem in controlled geometric topology, which is new even in the case of manifolds.

Theorem B *Suppose that $f: Y \to Z$ is a UV^1 map between compact, locally contractible, finite-dimensional spaces so that the inverse image of each open subset of Z is a (usually noncompact) n–dimensional Poincaré space, $n \geq 6$. Then there is a total surgery obstruction $\sigma \in H_{n-1}(Z, Y; \mathbb{L})$ that vanishes if and only if there exist a locally contractible, finite-dimensional DDP homology n–manifold X and a map $g: X \to Y$ that is an ϵ–homotopy equivalence over Z, for every $\epsilon > 0$. Moreover, there is an $\epsilon_0 > 0$ depending on Z so that σ vanishes if there is an ϵ_0–equivalence $g_{\epsilon_0}: X \to Y$ for some compact locally contractible, finite-dimensional homology n–manifold X.*

Note that if $f: Y \to Z$ is the identity map, then this theorem reduces to our resolution theorem.

No doubt, the restriction to UV^1 maps can be substantially weakened. It is a very interesting question to wonder about how much. If the map $Y \to Z$ is a fibration, then the equivalence of the epsilon and the controlled statement (in the manifold case) is equivalent to the theorem of Chapman–Hughes [5; 14] that maps between manifolds that have an ϵ–homotopy lifting property are near approximate fibrations. We expect that π_1 uniformity should suffice for the conclusion of our theorem, but we will not explore this aspect here.

Many of the ideas of this paper were already present in some form in our earlier paper, including the extensive use of controlled Poincaré duality, controlled surgery and the UV^1 approximation property of manifolds. However, the constructions here are both more efficient (we avoid Gromov–Hausdorff convergence of embeddings in Euclidean space and take a more conceptual approach to convergence problems) and more intricate (note the complexity of the statement of Theorem 8.1). The immediate technical problem to be confronted is that the sequence of approximate resolutions that were built in our previous paper in principle have different domains, and there is no immediate connection between them. A similar problem occurs in trying to produce controlled homotopy equivalences even for manifolds: the approximations are only close from the perspective of the control space, not from the perspective of the range. Essentially, we use some form of "squeezing", the α–approximation theorem or the thin h–cobordism theorem to gain the extra control needed.

2 Controlled maps

Our first goal is to describe a class of metric spaces that we shall refer to as excellent spaces. These are ANR homology manifolds obtained by a variation of the construction presented in [4]. They are inverse limits of ever finer Poincaré duality spaces, have the disjoint disks property and other useful properties. We begin by reviewing some basic facts about controlled maps.

Definition 2.1

  (i) A compact metric space X is UV^k if whenever X is embedded in a compact ANR, for each neighborhood U of X there is a smaller neighborhood V of X such that if $\alpha\colon S^\ell \to V$, $0 \le \ell \le k$, there is an extension $\bar{\alpha}\colon D^{\ell+1} \to U$. Lacher [16] shows that this property can be checked on any given embedding of X into an ANR.

  (ii) Let X be a metric space. A proper map $f\colon X \to Y$ is UV^k if f is surjective and $f^{-1}(y)$ is UV^k, for every $y \in Y$.

Remarks

  (i) A proper map is UV^0 if and only if it is surjective and all of its point-inverses are connected. A proper *PL* map is UV^k if and only if it is surjective and its point-inverses are k–connected.

  (ii) Compositions and uniform limits of UV^k maps are UV^k.

We will need to use a characterization of UV^k maps in terms of their lifting properties.

Definition 2.2 Let X be an ANR and $p\colon Y \to B$ a map from a space Y to a metric space B. Given $\epsilon > 0$, a proper map $f\colon X \to Y$ is said to be $UV^k(\epsilon)$ *over* B if whenever (P, Q) is a polyhedral pair with $\dim(P) \le k+1$, $\alpha_0\colon Q \to X$ is a map and $\alpha\colon P \to Y$ is a map with $f \circ \alpha_0 = \alpha|Q$, there is a map $\bar{\alpha}\colon P \to X$ extending α_0 so that $f \circ \bar{\alpha}$ is homotopic to α by a homotopy whose tracks have diameter $< \epsilon$ in B.

This is the same as Quinn's notion [18, Definition 5.1] of a relatively $(\epsilon, k+1)$– connected map over B.

Lemma 2.3 *Let X and Y be ANRs. A map $f\colon X \to Y$ is UV^k if and only if f is $UV^k(\epsilon)$ over $1_Y\colon Y \to Y$, for every $\epsilon > 0$. The map f is cell-like iff f is UV^k for all $k \le \dim(X)$.*

Proof See Lacher [16]. □

Definition 2.4 Let X and Y be topological spaces and let $p\colon Y \to B$ a map to a metric space B. A map $f\colon X \to Y$ is an ϵ–*homotopy equivalence over* B if there exist a map $g\colon Y \to X$ and homotopies $h_t\colon X \to X$, $k_t\colon Y \to Y$ so that $h_0 = 1_X$, $h_1 = g \circ f$, $k_0 = 1_Y$, $k_1 = f \circ g$, $\operatorname{diam}\{p \circ k_t(y)|0 \le t \le 1\} < \epsilon$ for every $y \in Y$, and $\operatorname{diam}\{p \circ f \circ h_t(x)|0 \le t \le 1\} < \epsilon$ for every $x \in X$. The map g is called an ϵ–*inverse* for f.

By a 1–Lipschitz map between metric spaces we mean a Lipschitz map with Lipschitz constant 1 ie a map that does not increase distance.

Lemma 2.5

  (i) *If* $\alpha\colon X \to Y$ *in an* ϵ–*homotopy equivalence over* Y *with* ϵ–*inverse* β, *and* β' *is* δ–*homotopic to* β *over* Y, *then* β' *is an* $(\epsilon + \delta)$–*inverse for* α.

  (ii) *If* $\alpha_i\colon X_i \to X_{i-1}$ *is a 1–Lipschitz* ϵ_{i-1}–*equivalence, for* $1 \le i \le n$, *then* $\alpha_1 \circ \cdots \circ \alpha_n\colon X_n \to X_0$ *is a* $(3(\epsilon_{n-1} + \cdots + \epsilon_1) + \epsilon_0)$–*equivalence.*

The proof of the lemma is a simple exercise.

3 Inverse systems

Let $X_1 \xleftarrow{\alpha_2} X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} \cdots$ be an inverse sequence of compact metric spaces and continuous maps, and let $X = \varprojlim X_i$ be the inverse limit. We denote the projection $X \to X_i$ by p_i, and for $i > j$, we write $\alpha_{i,j}$ for the composition $\alpha_{j+1} \circ \cdots \circ \alpha_i\colon X_i \to X_j$. For every i, $\alpha_{i,i} = \operatorname{id}_{X_i}$. We denote the metric on X_i by ρ_i and assume that $\operatorname{diam}(X_i) \le 1$ for every i.

We equip $\coprod X_i$, and hence the X_i themselves, with a new *system metric* d as follows

$$d(x, y) = \begin{cases} \sum_{j=1}^{i} \frac{1}{2^j} \rho_j\left(\alpha_{i,j}(x), \alpha_{i,j}(y)\right) & \text{if } x, y \in X_i \\ d(\alpha_{i,j}(x), y) + \left(\frac{1}{2^j} - \frac{1}{2^i}\right) & \text{if } x \in X_i,\ y \in X_j \text{ and } i > j. \end{cases}$$

This metric on $\coprod X_i$ allows us to compare sizes in different X_is. In [4], we accomplished this by embedding all X_is under consideration in a euclidean space. Here, we achieve the same goal by introducing the metric d that depends on the inverse system.

Let X_i^+ be the union of X_i and a disjoint basepoint $*_i$ whose distance from each point in X_i is 1. For every i, the map

$$x \to \big(\alpha_{i,1}(x), \alpha_{i,2}(x), \ldots, \alpha_{i,i-1}(x), x, *_{i+1}, *_{i+2}, \ldots\big),$$

embeds X_i into $\prod X_i^+$ and the images of the X_i are disjoint. These maps induce an isometric embedding of $(\coprod X_i, d)$ into $\prod X_i^+$. The closure of $\coprod X_i$ in $\prod X_i^+$ is the union of the X_is with the inverse limit $X = \varprojlim X_i$. Hence, the completion of $\coprod X_i$ with respect to this metric is $Z_X = (\coprod X_i) \cup X$. The induced metric on the inverse limit X is given by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \rho_i(p_i(x), p_i(y)).$$

Notice that the projections $p_i \colon X \to X_i$ are 1–Lipschitz in the system metric.

If the bonding maps α_i are ϵ_{i-1}–equivalences and $\sum \epsilon_i < \infty$, let $\beta_{i-1} \colon X_{i-1} \to X_i$ denote an ϵ_{i-1}–inverse to α_i. For $i \leq j$, define maps $\gamma_{i,j} \colon X_i \to X_j$ inductively by setting $\gamma_{i,i} = 1_{X_i}$ and $\gamma_{i,j+1} = \beta_j \circ \gamma_{i,j}$. Then,

$$\begin{aligned}
d(\gamma_{i,j}, \gamma_{i,j+1}) &= \frac{1}{2^{j+1}} + d(\gamma_{i,j}, \alpha_{j+1} \circ \beta_j \circ \gamma_{i,j}) \\
&= \frac{1}{2^{j+1}} + d(1_{X_j}, \alpha_{j+1} \circ \beta_j) \\
&\leq \frac{1}{2^{j+1}} + \epsilon_j.
\end{aligned}$$

Thus, for each i, the sequence $\{\gamma_{i,j}\}_{j=i}^{\infty}$ is Cauchy and converges to a map $\gamma_i \colon X_i \to X$. Here, we are abusing notation: X_i and X are viewed as subspaces of Z_X. Combining these maps for $i \geq 1$, we obtain a retraction $\gamma \colon Z_X = (\coprod X_i) \cup X \to X$.

We form a space T_X by inserting the mapping cylinder of $\alpha_i \colon X_i \to X_{i-1}$ between X_i and X_{i-1} in the space Z_X. Using the homotopy from $\beta_i \circ \alpha_{i+1}$ to the identity, we can extend $\gamma_{i,i+1}$ to a retraction from the mapping cylinder of α_{i+1} onto X_{i+1}. Combining these, we can extend γ to a retraction of T_X onto X.

Lemma 3.1 *If the spaces X_i are ANRs and the maps α_i are ϵ_{i-1}–homotopy equivalences in the system metric with $\sum \epsilon_i < \infty$, then X is an ANR. Moreover, X is $\big(3 \sum_{j=i+1}^{\infty} \epsilon_j + \epsilon_i\big)$–equivalent to X_i.*

Proof By Chapman–Siebenmann [6], T_X is an ANR. Since T_X retracts to X, X is also an ANR. The second assertion follows from Lemma 2.5 (ii). Given $\delta > 0$, X is δ–equivalent to X_k for large k. The maps are given by pushing down the

mapping cylinder and by retracting back into X. Choosing k large, this implies that X is $\left(3\sum_{j=i+1}^{k}\epsilon_j + \epsilon_i + 3\delta\right)$–equivalent to X_i for every $\delta > 0$. The result follows. $\square$

Remark A similar argument shows that if each bonding map α_i is a UV^k–map, then each projection map $X \to X_i$ is a UV^k–map.

We will find it useful to construct new inverse systems by passing to subsequences. Let $(X_i, \alpha_i)_{i=1}^{\infty}$ be an inverse system and let $\{i_j\}_{j=1}^{\infty}$ be a strictly increasing sequence of positive integers. For each j, let ρ'_{i_j} be the metric on X_{i_j} induced by the system metric d on Z_X. The sequence $\left(X_{i_j}, \alpha_{i_j,i_{j-1}}\right)_{j=1}^{\infty}$ forms an inverse system of metric spaces and continuous maps with respect to the metrics ρ'_{i_j}. The system metric d' for the subsystem is defined using the metric ρ'_{i_j} on X_{i_j}, for all j.

Lemma 3.2 *If the maps α_i are ϵ_{i-1}–equivalences in the system metric d, then the bonding maps α_{i_{j+1},i_j} of the subsystem are $\left(3\sum_{k=i_j+1}^{i_{j+1}-1}\epsilon_k + \epsilon_{i_j}\right)$–equivalences with respect to the metric d'.*

Proof Notice that $d'(x,y) \le d(x,y)$, for any $x, y \in \coprod X_{i_j}$, since the bonding maps of any inverse system are 1–Lipschitz in the system metric. Hence, the result follows from Lemma 2.5 (ii). $\square$

4 Controlled Poincaré duality

Let $f\colon E \to B$ be a continuous map from a topological space to a compact connected metric space. Let $\mathcal{E}(f) = \{(e, \omega) \in E \times B^I \,|\, f(e) = \omega(0)\}$. Given $b \in B$, let ω_b denote the constant path at b. We can identify E with a subspace of $\mathcal{E}(f)$ via the map $e \mapsto \{(e, \omega_{f(e)})\}$. There is a projection $\mathcal{P}_f\colon \mathcal{E}(f) \to B$ given by $(e, \omega) \to \omega(1)$ and the diagram

$$
\begin{array}{ccc}
E & \lhook\joinrel\longrightarrow & \mathcal{E}(f) \\
& \searrow{\scriptstyle f} \quad \swarrow{\scriptstyle \mathcal{P}_f} & \\
& B &
\end{array}
$$

is commutative. $\mathcal{P}_f$ is a Hurewicz fibration known as the *path fibration* associated to f. The fiber of $\mathcal{P}_f$ is called the *homotopy fiber* of f.

Definition 4.1 $f\colon E \to B$ is an ϵ–*fibration* if there is a retraction $r\colon \mathcal{E}(f) \to E$ such that $d(f \circ r, \mathcal{P}_f) < \epsilon$. The map $f\colon E \to B$ is an *approximate fibration* if it is an ϵ–fibration for every $\epsilon > 0$. More generally, if $q\colon B \to Z$ is a map to a metric space Z, we say that $f\colon E \to B$ is an ϵ–*fibration over* Z if $d(q \circ f \circ r, q \circ \mathcal{P}_f) < \epsilon$.

Lemma 4.2　*A map $f\colon E \to B$ is an ϵ–fibration over the metric space Z if and only if for any commutative diagram*

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{\ \alpha_0\ } & E \\
\downarrow & \nearrow^{\bar{\alpha}} & \downarrow{\scriptstyle f} \\
X \times I & \xrightarrow{\ \alpha\ } & B \xrightarrow{\ q\ } Z
\end{array}
$$

there is a map $\bar{\alpha}\colon X \times I \to E$ such that $\bar{\alpha}|_{X\times\{0\}} = \alpha_0$ and $d(q \circ \alpha, q \circ f \circ \bar{\alpha}) < \epsilon$.

Proof　($\Rightarrow$) There is a lift $\alpha'\colon X \times I \to \mathcal{E}(f)$ such that $\mathcal{P}_f \circ \alpha' = \alpha$ and $\alpha'|_{X\times\{0\}} = \alpha_0$. Let $\bar{\alpha} = r \circ \alpha'$. Then,

$$
d\left(q \circ f \circ \bar{\alpha}, q \circ \alpha\right) = d(q \circ f \circ r \circ \alpha', q \circ \mathcal{P}_f \circ \alpha') \leq d(q \circ f \circ r, q \circ \mathcal{P}_f) < \epsilon.
$$

($\Leftarrow$) Consider the diagram

$$
\begin{array}{ccc}
\mathcal{E}(f) \times \{0\} & \xrightarrow{\ \alpha_0\ } & E \\
\downarrow & \nearrow^{\bar{\alpha}} & \downarrow{\scriptstyle f} \\
\mathcal{E}(f) \times I & \xrightarrow{\ \alpha\ } & B \xrightarrow{\ q\ } Z,
\end{array}
$$

where $\alpha(e, \omega, t) = \omega(t)$, $\alpha_0(e, \omega) = (e, \omega_{f(e)})$ and $\bar{\alpha}$ is an ϵ–lift of α over Z. Let $r(e, \omega) = \bar{\alpha}(e, \omega, 1)$. The map r is a retraction because we can arrange as in Dugundji [8] that constant paths lift to constant paths. Since $\alpha(e, \omega, 1) = \mathcal{P}_f(e, \omega)$, it follows that $d(q \circ f \circ r, q \circ \mathcal{P}_f) < \epsilon$.　　□

Let X be a compact ANR. A pair (N, X) is an *abstract mapping cylinder neighborhood* of X if N is a compact topological manifold containing X in its interior, and there is a UV^1 map $p\colon \partial N \to X$ such that (N, X) is homeomorphic to (C_p, X), where C_p is the mapping cylinder of p.

Definition 4.3　A finite-dimensional ANR X is an ϵ–*Poincaré duality space* if X has an abstract mapping cylinder neighborhood (N, X) such that $p\colon \partial N \to X$ is an ϵ–fibration with homotopy fiber homotopy equivalent to S^k, for some $k \geq 2$. If $p\colon \partial N \to X$ is an approximate fibration, then X is said to be a *controlled Poincaré duality space over itself*.

Remark　The definition given here differs from the one given in [4] but is equivalent to it. That approximate spherical fibration (the definition we use here) follows from the chain duality (assumed in [4]) is proven on page 446–448 of that paper. On the other hand, the approximate chain level Poincaré duality definition follows directly from the

current definition by combining Thom isomorphism with Poincaré duality. The Thom isomorphism theorem for the spherical fibration that we assume identifies the cochain complex of the space with that of its mapping cylinder neighborhood (up to a shift by codimension), and the latter is identified, by Poincaré duality, with the chain complex of the neighborhood, which is homotopy equivalent to that of the original space.

Proposition 4.4 (Daverman–Husch [7]) *Let X be a finite-dimensional compact ANR. If X is a controlled Poincaré duality space over itself, then X is a homology manifold.*

The converse statement is also valid, but will not be needed in this paper.

The next ingredient is a slight rewording of [4, Propostion 4.6]. The only difference between the two versions is that this version is an "$\epsilon - \delta$" statement while the previous version is a "$T\epsilon - \epsilon$" statement for some constant T. This change is needed because the control space in the present version is a compact ENR rather than a finite polyhedron.

Proposition 4.5 (Bryant et al [4]) *Let X be a compact ENR and let n be given. For every $\epsilon > 0$ there is a $\delta > 0$ so that if*

  (i) *$(M_1, \partial M_1)$ and $(M_2, \partial M_2)$ are orientable n–manifolds.*

  (ii) *$p_1\colon M_1 \to X$ and $p_2\colon M_2 \to X$ are UV^1–maps with $p_i|\partial M_i$ a UV^0–map for $i = 1, 2$.*

  (iii) *$h\colon \partial M_1 \to \partial M_2$ is an orientation-preserving δ–equivalence over X with $p_2 \circ h$ δ– close to p_1.*

then there is a UV^1–map $p\colon M_1 \cup_h M_2 \to X$ that is an ϵ–Poincaré duality space over X. Moreover, p is ϵ–close to p_i for $i = 1, 2$.

5   Excellent metric spaces

Definition 5.1 Let B be a compact metric space. A *pleasant ϵ–controlled 2–patch space over B* is a pair (X, p), where $p\colon X \to B$ is a UV^1 map and $X = S \cup_f T$ is a space obtained by gluing together two compact n–manifolds S and T (the patches of X) by a UV^1 ϵ–homotopy equivalence $f\colon \partial S \to \partial T$ (over B) of their boundaries. We also require that the maps $p|T$, $p|\partial T$ and the composition $S \to X = S \cup_f T \to B$ be UV^1. Finally, we require that every map of a sphere S^k into X be ϵ–homotopic to a map $S^k \to T$, when $k \leq [n/2]$.

Definition 5.2 A compact metric space X is *excellent* if there exist a sequence $\{\epsilon_i\}$ with $\epsilon_i > 0$ and $\sum \epsilon_i < \infty$, and an inverse system (X_i, α_i) such that

(i) $X = \varprojlim(X_i, \alpha_i)$, where each (X_i, α_i) is a pleasant ϵ_i–controlled 2–patch space $X_i = S_i \cup_{f_i} T_i$ over X_{i-1};

(ii) Each α_i is a UV^1 ϵ_{i-1}–equivalence.

Here, all sizes are measured using the system metric d.

Lemma 5.3 *If M^n and N^n are compact manifolds with boundary, $n \geq 5$, and $f: \partial M \to \partial N$ is a UV^1 map, then $X = M \cup_f N$ has the disjoint disks property.*

Proof Let $\alpha_1, \alpha_2: D^2 \to X$ be continuous maps. The singular set of X has a neighborhood homeomorphic to $\partial M \times [-1, 0] \cup_f \partial N \times [0, 1]$. By making the image of the disks transverse to $\partial M \times \{-\epsilon\}$ and pushing along the product structure in the positive direction, one may arrange that the intersection K_i of $\alpha_i(D^2)$ and the singular set is 1–dimensional, $i = 1, 2$. Pushing K_1 and K_2 apart in $\partial N \times \{0\}$, use the lifting property of UV^1 maps and the estimated homotopy extension theorem [4] to extend this homotopy to obtain small deformations $\bar{\alpha}_1$ and $\bar{\alpha}_2$ of α_1 and α_2, respectively, with the property that $\bar{\alpha}_1(D^2) \cap \bar{\alpha}_2(N_2) = \varnothing$ and $\bar{\alpha}_2(D^2) \cap \bar{\alpha}_1(N_1) = \varnothing$, where N_i is a small regular neighborhood of K_i in D^2, $i = 1, 2$. Now, using the DDP on the complement of the singular set of X gives us a small separation of the disks in X. □

Proposition 5.4 *If X is an excellent metric space of dimension $n \geq 5$, then X is an ANR homology n–manifold with the disjoint disks property.*

Proof Let $\{X_i, \alpha_i\}$ be an inverse system of 2–patch spaces exhibiting X as an excellent metric space, where $\alpha_{i+1}: X_{i+1} \to X_i$ an ϵ_i–equivalence. By Lemma 3.1, $X = \varprojlim X_i$ is an ANR. It is easy to check that the inverse limit of ANRs with the DDP and $\overleftarrow{UV^1}$ maps has the DDP. Hence, X has the DDP since, by Lemma 5.3, each X_i has the DDP. To complete the proof of the proposition, we need to show that X is a homology manifold. The argument will be similar to the one presented in [4, Section 5].

Embed T_X tamely in a high-dimensional euclidean space and let W_1 be a mapping cylinder neighborhood. For each i, let T_{X_i} be the part of T_X between X_i and X, and let W_i be a small mapping cylinder neighborhood of T_{X_i} with $W_i \subset W_{i-1}$. For any $i > 1$, the region between W_1 and W_i has a product structure. Let $\{\delta_i, i \geq 1\}$ be a sequence of positive real numbers with $\sum \delta_i < \infty$. If $i_2 < i_3$ are large enough positive integers, by the thin h–cobordism theorem (see Quinn [18]), the region between W_{i_2} and W_{i_3} has a δ_1–controlled product structure over X_1 since $\alpha_{i_3, i_2}: X_{i_3} \to X_{i_2}$ is a fine

equivalence over X_1. Arguing inductively and passing to an appropriate subsequence, one can assume that the region V_i between W_i and W_{i+1} has a δ_i–controlled product structure over X_{i-1}, for $i > 1$. Deforming $W_1 - \text{int}(W_i)$ to ∂W_i along the product structures on $V_1, \ldots, V_{i-1}$ and composing with the regular neighborhood projection $W_i \to X_i$, we obtain a spherical fine approximate fibration structure on $\rho_i' \colon \partial W_1 \to X_i$ that induces a mapping cylinder structure on the neighborhood W_1 of X_i.

Let $\rho_i \colon \partial W_1 \to X$ denote the composition

$$\partial W_1 \xrightarrow{\rho_i'} X_i \xrightarrow{\gamma_i} X.$$

Since $\sum \delta_i < \infty$, the sequence ρ_i converges to a map $\rho \colon \partial W_1 \to X$ that induces a controlled Poincaré duality structure on X, since, by [4, Proposition 4.6], each X_i is a fine Poincaré space over X_{i-1}. Hence, by Proposition 4.4, X is a homology manifold. □
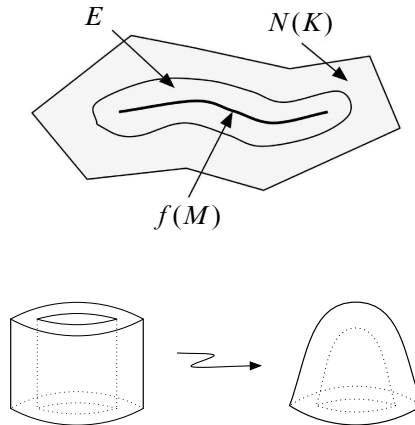
6 Constructing UV^k maps

A deformation theorem of Bestvina and Walsh states that, below middle and adjacent dimensions, $(k+1)$–connected mappings of manifolds to polyhedra can be deformed to UV^k mappings. The proof of our main result uses a controlled analogue of this result. We will first present an alternative proof of the Bestvina–Walsh theorem and then show how the arguments can be modified to yield a controlled version. Kawamura [15] remarks that this controlled theorem can also be proven using the techniques of Bestvina and Walsh.

Definition 6.1 A map $f \colon X \to Y$ is k–*connected* if $f_* \colon \pi_\ell(X) \to \pi_\ell(Y)$ is an isomorphism for $\ell < k$ and an epimorphism for $\ell = k$. This is the same as saying that $\pi_\ell(f) = \pi_\ell(C_f, X) = \{0\}$, for $0 \le \ell \le k$, where C_f is the mapping cylinder of f.

Theorem 6.2 (Bestvina and Walsh [1]) *Let M^m be a compact manifold and K a polyhedron. If $f \colon M \to K$ is a $(k+1)$–connected map and $f|\partial M$ is UV^k, then f is homotopic rel (∂M) to a UV^k map, provided that $k \le \left[\frac{m-3}{2}\right]$.*

Other results of this type are due to Keldyš, Anderson, Wilson, Walsh, Černavskii and Ferry. We begin the proof by recalling a theorem of Černavskii.

Theorem 6.3 (Černavskii [21]) *If M^m is a manifold and $k \le \left[\frac{m-3}{2}\right]$, then there is a UV^k map $p \colon M \to M \times I$. Moreover, for any $\epsilon > 0$, we can choose p so that $\text{proj} \circ p$ is ϵ–close to $1_M \colon M \to M$.*

Iterating Černavskii's construction, we obtain UV^k maps of M onto $M \times I^\ell$, for any ℓ, and also onto any disk bundle E over M. This last follows from writing E as a direct summand of a trivial bundle, mapping M onto the trivial bundle and then projecting back to E.

Proof of the absolute case of the Bestvina–Walsh theorem Let $f\colon M \to K$ be as in the statement of the theorem and let $N(K)$ be a regular neighborhood of K in some euclidean space $\mathbb{R}^n$, $n \geq 2m + 3$. Compose f with the inclusion $K \hookrightarrow N(K)$ to obtain a map $g\colon M \to N(K)$. After a homotopy, we may assume that g is an embedding and that $g(M)$ has a normal disk bundle E in $N(K)$ (see Hirsch [13]). Using Černavskii's process, construct a UV^k map $\phi\colon M \to E$. Since compositions of UV^k maps are UV^k, we will be done if we can construct UV^k map $\psi\colon E \to N(K)$, because composing $\psi \circ \phi$ with the cell-like regular neighborhood collapse $N(K) \to K$ will give a UV^k map $M \to K$ homotopic to f.

By Lemma 6.4 below, the inclusion $\partial E \to N(K) - \mathring{E}$ is $(k+1)$–connected. The usual handle-trading lemma (see Rourke–Sanderson [20, Lemmas 6.14–6.16]) implies that we can find a handle decomposition

$$N(K) - \mathring{E} \cong \partial E \times I \cup \{(k+2) - \text{handles}\} \cup \{(k+3) - \text{handles}\} \cup \ldots.$$

To complete the proof of the theorem, we use the fact that there is a simple UV^r map from a manifold V^{n+1} to the same manifold with an $(r+2)$–handle attached to the boundary. Write V as $V \cup (S^{r+1} \times D^{n-r-1} \times I)$ and collapse $S^{r+1} \times D^{n-r-1} \times I$ to $V \cup (r+2)$–handle by a map whose only non-trivial point-inverses are spheres of dimension $r+1$.

Composing such maps for the handles of the decomposition of $N(K) - \mathring{E}$ given above, we obtain a UV^k map $E \to N(K)$. □

Lemma 6.4 *The inclusion $\partial E \to N(K) - \mathring{E}$ is $(k+1)$–connected.*

Proof Let $\alpha \colon (D^s, S^{s-1}) \to (N(K) - \mathring{E}, \partial E)$ be a map, $0 \le s \le k+1$. Including in $(N(K), E)$ and using the $(k+1)$–connectivity, we obtain a homotopy $\bar{\alpha}_t \colon (D^s, S^{s-1}) \to (N(K), E)$ so that $\bar{\alpha}_0 = \alpha$, $\bar{\alpha}_1(D^s, S^{s-1}) \subseteq E$ and $\bar{\alpha}_t | S^{s-1} = \alpha | S^{s-1}$ for all t. By general position, we may assume that $\bar{\alpha}_t(D^s) \cap f(M) = \varnothing$ for all t. Composing $\bar{\alpha}$ with the retraction $r \colon N(K) - f(M) \to N(K) - \mathring{E}$ gives a homotopy which shows that $[\alpha]$ is trivial in $\pi_s(N(K) - \mathring{E}, \partial E)$. □

We now state our controlled analogue of the Bestvina–Walsh theorem.

Theorem 6.5 *For any non-negative integers m, k and d satisfying $k \le \left[\frac{m-3}{2}\right]$, there is a constant $C(m, k, d) > 0$ such that if M^m is a compact connected manifold and K is a connected d–dimensional polyhedron, then every map $f \colon M \to K$ that is $UV^k(\delta)$ over some metric space B is $C(m, k, d) \cdot \delta$–homotopic over B to a UV^k map.*

Proof Suppose that $f \colon M \to K$ is a $UV^k(\delta)$ map over B. Let $N'(K)$ be a regular neighborhood of K in $\mathbb{R}^{2d+1}$. Thicken $N'(K)$ by taking a product with I^{2m+1} to obtain a regular neighborhood $N(K)$ of K in $\mathbb{R}^{2m+2d+2}$ and a projection $c \colon N(K) \to K$. We abuse notation and refer to the composition

$$M \xrightarrow{f} K \hookrightarrow N(K)$$

as f. We can perturb f along the I^{2m+1}–coordinate to an embedding f'. This ensures that $c \circ f' = f$. Let E be a disk bundle neighborhood of $f'(M)$ in $N(K)$.

Claim *If E is chosen to be a thin enough neighborhood of $f'(M)$, then $(N(K) - \mathring{E}, \partial E)$ is $(\delta, k+1)$–connected over B in the following sense: if $\alpha \colon (D^{s+1}, S^s) \to (N(K) - \mathring{E}, \partial E)$ is a map and $0 \le s \le k$, then there is a homotopy α_t starting with α and ending with a map $\alpha_1 \colon (D^{s+1}, S^s) \to (\partial E, \partial E) \subset (N(K) - \mathring{E}, \partial E)$ such that the tracks of the homotopy have diameter $< \delta$ when projected to B.*

To prove the claim, first note that $f \colon M \to N(K)$ is $UV^k(\delta)$ over B. Given a polyhedral pair (P, Q) and maps $\beta_0 \colon Q \to M$ and $\beta \colon P \to N(K)$ with the property that $f' \circ \beta_0 = \beta$, consider the composition $c \circ \beta$. By the $UV^k(\delta)$ property , there is a map $\bar{\beta} \colon P \to M$ that δ–lifts $c \circ \beta$. Since $c \colon N(K) \to K$ is UV^ℓ for every ℓ, we can lift the homotopy from K to $N(K)$, completing the first part of the argument.

The rest of the proof of the claim follows the proof of Lemma 6.4. Given a map $\alpha\colon (D^{s+1}, S^s) \to (N(K) - \overset{\circ}{E}, \partial E)$, construct a nullhomotopy in $(N(K), E)$. Use general position to move the homotopy off of $f'(M)$ and then retract the nullhomotopy to $(N(K) - \overset{\circ}{E}, \partial E)$. This establishes the claim.

Continuing with the proof of the theorem, we choose a fine handlebody structure on $(N(K) - \overset{\circ}{E}, \partial E)$ and trade handles to eliminate all handles of index $\leq k+1$. By Quinn [18, Theorem 6.1], this can be accomplished at the cost of raising the size of the remaining handles to at most $D(k) \cdot \delta$ as measured in B, where $D(k)$ is a constant. We now map M to E, and E onto $N(K)$ as in the proof of Theorem 6.2. This relaxes the control to $C(m, k, d) \cdot \delta$. $\square$

Remarks

(i) This argument works equally well if the target space is a finite-dimensional ANR X^d. Embed X into $\mathbb{R}^{2d+1}$, take a mapping cylinder neighborhood $N(X)$ of X, and proceed as above. The argument also extends to infinite-dimensional compact ANRs by crossing with the Hilbert cube and triangulating.

(ii) We can remove the dependency of the constant $C(m, k, d)$ on m and d. Take a fine subdivision of $N(K) - \overset{\circ}{E}$ and obtain a small handlebody decomposition by taking regular neighborhoods of vertices in the second derived. Given $\mu > 0$, we can find a UV^k map from $\partial E \times I \cup \{0 - \text{handles}\} \cup \ldots \cup \{(k + 2) - \text{handles}\}$ to $N(K)$ that is μ–close to the inclusion over K. Since handle trading through dimension $k+1$ only reorganizes the handle structure on $\partial E \times I \cup \{0 - \text{handles}\} \cup \ldots \cup \{(k + 2) - \text{handles}\}$, we can trade handles and obtain a UV^k map from $\partial E \times I$ onto $\partial E \times I \cup \{0 - \text{handles}\} \cup \ldots \cup \{(k + 2) - \text{handles}\}$ whose control depends on δ and k. The composition is a UV^k map whose control only depends on k and δ.

(iii) There is a relative version of Theorem 6.5 which will be important in the sequel. If $f\colon M \to K$ is $UV^k(\delta)$ and $f|\partial M\colon \partial M \to K$ is UV^k, then f is controlled homotopic to a UV^k–map rel boundary. To see this, attach copies of $\partial M \times [0, \infty)$ and $K \times [0, \infty)$ to M and K and extend levelwise using the restriction of f. Let the metrics grow at $t \to \infty$. Applying the proper version of Theorem 6.5 to this situation, we get a UV^k approximation to f that can be extended by $f|$ to a UV^k–map $M \cup \partial M \times [0, \infty] \to K \times [0, \infty]$ composing with the projection to K gives the desired map.

We close this section with an application of Theorem 6.5. We show that if M^n is a compact manifold, K is a polyhedron, and $f\colon M \to K$ is a map whose homotopy fiber has finite skeleta (ie, is homotopy equivalent to a CW complex with finite n

skeleton, for every n), then f is homotopic to a map with the approximate homotopy lifting property for polyhedra of dimension $\leq \left[\frac{n-3}{2}\right]$.

Definition 6.6 Let B be a metric space. A map $p\colon E \to B$ has the *approximate homotopy lifting property (AHLP)* with respect to a compact space Z if for any homotopy $f\colon Z \times I \to B$, map $F_0\colon Z \to E$ with the property that $p \circ F_0 = f|Z \times \{0\}$, and $\epsilon > 0$, there is a map $F\colon Z \times I \to E$ such that $F_0 = F|Z \times \{0\}$ and $d(p \circ F(z,t), f(z,t)) < \epsilon$, for each $(z,t) \in Z \times I$.

Theorem 6.7 *Let M be a compact topological n–manifold and K a compact polyhedron. If $f\colon M \to K$ is a map whose homotopy fiber has finite skeleta, then f is homotopic to a map with the approximate homotopy lifting property with respect to compact polyhedra of dimension $\leq \left[\frac{n-3}{2}\right]$.*

Proof Let V be a mapping cylinder neighborhood of M in some large euclidean space with projection $p\colon V \to M$. The homotopy fiber of the composition $f \circ p\colon V \to K$ has finite skeleta, since p is cell-like. By Ferry [11, Theorem 2'], there is a polyhedron V', a $CE-PL$ map $c\colon V' \to V$ and a PL map $f'\colon V' \to K$ such that f' has the $AHLP$ for compacta of dimension $\leq \left[\frac{n-3}{2}\right]$ and $f \circ p \circ c \simeq f'$.

Let $\gamma'\colon M \to V'$ be a homotopy inverse to $p \circ c$. By Theorem 6.5, γ' is homotopic to a UV^k map γ, where $k = \left[\frac{n-3}{2}\right]$. Notice that since $p \circ c$ is CE, we can choose γ so that the homotopy $p \circ c \circ \gamma \simeq 1_M$ has tracks whose diameters are as small as we wish.
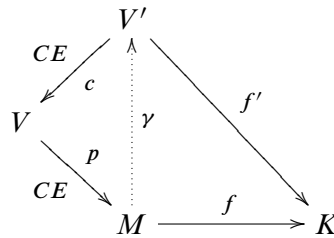
The composition $f' \circ \gamma$ is homotopic to f, since $f' \circ \gamma \simeq (f \circ p \circ c) \circ \gamma = f \circ (p \circ c \circ \gamma) \simeq f \circ 1_M = f$. Moreover, $f' \circ \gamma$ has the desired approximate homotopy lifting property for polyhedra of dimension $\leq k$, since γ is UV^k and f' has the $AHLP$ for compacta of dimension $\leq k$. $\qquad\square$

7 The UV^1–approximation property

Definition 7.1 (i) A space X is said to have the UV^1- *approximation property* if for every $\epsilon > 0$, there is a $\delta > 0$ such that every map $f\colon X \to K$ to a polyhedron K that is $UV^1(\delta)$ over some metric space B is ϵ–homotopic over B to a UV^1 map.

(ii)   A space X is said to have the *linear UV^1– approximation property* if there is a constant $C_X > 0$ such that every map $f\colon X \to K$ to a polyhedron K that is $UV^1(\delta)$ over some metric space B is $C_X \cdot \delta$–homotopic over B to a UV^1 map.

In dimensions ≥ 6, we will characterize excellent metric spaces as ANR homology n–manifolds with the UV^1–approximation property.

Lemma 7.2   *Let X^n be a compact metric ANR homology n–manifold X, $n \geq 6$. Given $\epsilon > 0$, there is a pleasant ϵ–controlled 2–patch space $\phi\colon X_\epsilon \to X$ such that ϕ is an ϵ–homotopy equivalence with respect to the control map $1_X\colon X \to X$.*

Proof   This is essentially contained in [4]. Start with a degree-one normal map $\psi\colon M \to X$, where M is a topological manifold. The existence of ψ is guaranteed by Ferry–Pedersen [12]. There is a $\mu_0 > 0$ such that, for any $0 < \mu < \mu_0$, the obstruction to doing surgery on ψ to a μ–equivalence is an element $\sigma = \sigma(\psi) \in H_n(X, \mathbb{L}_*(e))$ (see Pedersen–Quinn–Ranicki [17] and Ferry [10]). After surgery below middle dimension, using Theorem 6.5, we may assume that ψ is UV^1.

Take a fine triangulation of M and let T be a very thin regular neighborhood of the 2–skeleton of M in this triangulation. Let C be the closure of the complement of T in M.
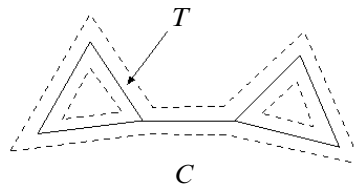
Using the controlled Bestvina–Walsh theorem, after a small homotopy, we may also assume that $\psi|T$, $\psi|C$ and $\psi|\partial T$ are all UV^1 maps.

Using the element $-\sigma \in H_*(X, \mathbb{L}(e))$, do Wall realization on the identity map of $V_0 = \partial T = \partial C$. This produces a cobordism (V, V_0, V_1) and a degree-one normal map $g\colon V \to V_0 \times I$ such that

  (i) $g|V_0 = 1_{V_0}$;

  (ii) $f = g|V_1$ is a fine homotopy equivalence from V_1 to V_0 (the control can be as good as we like over X and we may assume that this map is UV^1);

  (iii)   The controlled surgery obstruction of g relative to the boundary is $-\sigma$.

Form the space X'_ϵ by pasting V to C along V_0 and to T using the controlled equivalence $f = g|V_1: V_1 \to V_0 = \partial T$. The map $\psi: M \to X$ "extends" over V to a map $\psi': X'_\epsilon \to X$ with trivial surgery obstruction. Do controlled surgery on ψ' to obtain a fine homotopy equivalence $\phi: X_\epsilon \to X$. Notice that since all surgeries are performed away from T, X_ϵ is a space of the form $S \cup_f T$, where S is a compact manifold. The degree of control for the equivalence ϕ depends on two things – the control on the homotopy equivalence $f = g|V_1$ and the fineness of the initial triangulation of M. The regular neighborhood T of the 2–skeleton and the succeeding UV^1–maps should be constructed so that we can push codimension three polyhedra in M and X'_ϵ off of T by homotopies that have small tracks in X. The map $\phi: X_\epsilon \to X$ is already UV^1 when restricted to T and ∂T. After a small deformation, we may assume that it is an ϵ–equivalence that restricts to UV^1 maps on the patches of X_ϵ. $\square$

Theorem 7.3 *Let X be an n–dimensional compact metric space, $n \geq 6$. X is excellent if and only if X is an ANR homology n–manifold with the UV^1–approximation property.*

Proof ($\Rightarrow$) By Proposition 5.4, X is an ANR homology n–manifold. Hence, it suffices to show that X has the UV^1–approximation property. Let $f: X \to K$ be a $UV^1(\delta)$ map over B, for some $\delta > 0$. Write $X = \varprojlim(X_i, \alpha_i)$ as in the definition of excellent metric spaces, and let $p_i: X \to X_i$ denote the projection onto X_i and $\gamma_i: X_i \to X$ denote the restriction to X_i of the retraction $\gamma: T_X \to X$ constructed in Section 3.

For sufficiently large i, $f \circ \gamma_i$ is a $UV^1(\delta)$ map, so $f' = f \circ \gamma_i \circ \alpha_{i+1}: X_{i+1} \to K$ is $UV^1(\delta)$ as well. Write $X_{i+1} = S_{i+1} \cup_{f_{i+1}} T_{i+1}$ as in the definition of excellent metric spaces. Since $\alpha_{i+1}|\partial T_{i+1}$ is UV^1, $f \circ \gamma_i \circ \alpha_{i+1}|\partial T_{i+1}$ is $UV^1(\delta)$ and can be $C(1)\delta$–approximated by a UV^1 map. By the estimated homotopy extension theorem (see [4]), this homotopy can be extended to a $C(1)\delta$–homotopy of the map $f \circ \gamma_i \circ \alpha_i$ to a map $f': X_{i+1} \to K$ that is $UV^1((2C(1)+1)\delta)$ and restricts to a UV^1 map on ∂T_{i+1}. Using the relative version of the controlled Bestvina–Walsh theorem, we can $(2C^2(1)+C(1))\delta$–deform f' over the patches of X_{i+1} to a UV^1 map $f'': X_{i+1} \to K$. Notice that $d(f \circ \gamma_i \circ \alpha_{i+1}, f'') \leq (2C^2(1)+2C(1))\delta$. Here, $C(1)$ is the constant for UV^1 approximations as in the remark following the proof of Theorem 6.5.

Let $\bar{f}$ denote the UV^1 map $\bar{f} = f'' \circ p_{i+1}$. We claim that, for i large enough, $d(f, \bar{f}) < (2C^2(1)+2C(1)+1)\delta$. Indeed, first observe that $d(f, f \circ \gamma_i \circ \alpha_{i+1} \circ p_{i+1}) <$

δ, for i large enough. Then,

$$d(f, f'' \circ p_{i+1}) \leq d(f, f \circ \gamma_i \circ \alpha_{i+1} \circ p_{i+1}) + d(f \circ \gamma_i \circ \alpha_{i+1} \circ p_{i+1}, f'' \circ p_{i+1})$$
$$\leq \delta + d(f \circ \gamma_i \circ \alpha_{i+1}, f'')$$
$$\leq \delta + (2C^2(1) + 2C(1))\delta = (2C^2(1) + 2C(1) + 1)\delta.$$

The map $\bar{f}$ is the desired UV^1 approximation to f.

($\Leftarrow$) By Lemma 7.2 and Lemma 2.5, and the assumption that X has the UV^1–approximation property, given any $\epsilon > 0$, we can construct a pleasant ϵ–controlled 2–patch space X_ϵ and a UV^1 ϵ–equivalence $f_\epsilon\colon X_\epsilon \to X$ with UV^1 ϵ–inverse $g_\epsilon\colon X \to X_\epsilon$. We may assume that f_ϵ restricts to UV^1 maps on the patches of X_ϵ. Thus, given a sequence $\{\epsilon_i\}$, we can construct an inverse system

$$X \xleftarrow{f_1} X_1 \xleftarrow{g_1} X \xleftarrow{f_2} X_2 \xleftarrow{g_2} X \xleftarrow{f_3} X_3 \xleftarrow{g_3} X \longleftarrow \cdots$$

such that the f_is and g_is are ϵ_i–homotopy inverses in the system metric (with properties similar to f_ϵ and g_ϵ). By Brown [2, Theorem 5.1], the maps f_i and g_i can be chosen so that the inverse limit of the system is homeomorphic to X, since each $f_i \circ g_i$ is finely homotopic to $1_X\colon X \to X$ in the system metric. Passing to the subsequence

$$X_1 \xleftarrow{g_1 \circ f_2} X_2 \xleftarrow{g_2 \circ f_3} X_3 \xleftarrow{g_3 \circ f_4} X_4 \longleftarrow \cdots$$

exhibits X as an excellent metric space, provided that $\sum \epsilon_i < \infty$. $\qquad\square$

The following results are consequences of the proof of Theorem 7.3.

Corollary 7.4 *If a compact ANR homology n–manifold, $n \geq 6$, has the UV^1–approximation property, then it has the linear UV^1–approximation property.*

Corollary 7.5 *If X is an excellent compact metric space, then for any sequence $\{\epsilon_i\}$ with $\sum \epsilon_i < \infty$, X can be written as the inverse limit of an inverse sequence $\varprojlim(X_i, \alpha_i)$ that exhibits X as an excellent metric space, where each X_i is a pleasant ϵ_{i-1}–controlled 2–patch space over X_{i-1}.*

8 Some controlled topology

We begin with a control improvement theorem for maps defined on pleasant patch spaces. This result improves control in two ways, by lifting control over X to control over B, and by improving the control from size δ to size μ.

Theorem 8.1 *Suppose that $n \geq 6$ and that we are given a compact ENR X. Then for every $\epsilon > 0$ there is a $\delta > 0$ so that for every $\mu > 0$, if we are given a compact ENR B and a UV^1 homotopy equivalence $q\colon B \to X$, there exists η_1, such that if we are given a diagram*

$$
\begin{array}{ccc}
B & \xleftarrow{\ p_1\ } & X_1 \\
{\scriptstyle q}\big\downarrow & \swarrow {\scriptstyle p} & \\
X & &
\end{array}
$$

where $p_1\colon X_1 \to B$ is an n–dimensional η_1–Poincaré duality space over B and $p = q \circ p_1$, then there exists η_2 so that if $f\colon X_2 \to X_1$ is an n–dimensional pleasant η_2–controlled 2–patch space over X_1 and a δ–homotopy equivalence over X, then f is ϵ–homotopic over X to a map f_μ that is a UV^1 μ–homotopy equivalence over B. Moreover, $f_\mu\colon X_2 \to X_1$ is a pleasant n–dimensional η_2–controlled 2–patch space over X_1

$$
\begin{array}{ccccc}
B & \xleftarrow{\ p_1\ } & X_1 & \xleftarrow{\ f\ } & X_2 \\
{\scriptstyle q}\big\downarrow & \swarrow {\scriptstyle p} & & & \\
X & & & &
\end{array}
$$

Proof By the stability theorem for controlled surgery parameterized by UV^1 maps, [10; 17], there is a well-defined surgery obstruction $\sigma(f) \in H_n(B; \mathbb{L})$ for sufficiently small η_1 and η_2. The vanishing of $\sigma(f)$ implies that f is normally cobordant to a μ–equivalence over B. This is proven in [10; 17] for the case in which X_2 is a manifold and the modification for 2–patch spaces such as X_2 was discussed in [4]. The basic idea is that surgery involves manipulations up to the middle dimension and in a space such as X_2 all spheres can be pushed off of the singular set by small moves. At this stage of the construction, the constants η_i depend only on B and μ.

Since $f\colon X_2 \to X_1$ is a δ–homotopy equivalence over X, its controlled surgery obstruction is zero in $H_n(X; \mathbb{L})$, provided that δ is small enough, depending on B, q, and X. Since q is a UV^1 homotopy equivalence, naturality of surgery obstructions implies that $\sigma(f) \in H_n(B; \mathbb{L}) \cong H_n(X; \mathbb{L})$ is also zero. Hence, we can perform surgery on f away from the singular set to obtain a (singular) normal bordism

$$F\colon (V_1, X_2, X_2') \to (X_1 \times I, X_1 \times \{0\}, X_1 \times \{1\}),$$

where $F|_{X_2} = f$ and where $f_1 = F|_{X_2'}\colon X_2' \to X_1$ is a μ–homotopy equivalence over B.

We may assume that μ is small enough so that both f and f_1 are δ–homotopy equivalences over X. Let $\sigma \in H_{n+1}(X; \mathbb{L})$ be the controlled surgery obstruction of

F rel $X_2 \sqcup X_2'$. Using the stability of simply-connected controlled surgery groups and the Wall realization procedure [22], we can construct a normal bordism

$$G \colon (V_2, X_2', X_2'') \to (X_1 \times I, X_1 \times \{1\}, X_1 \times \{2\}),$$

such that $G|_{X_2'} = f_1$, $f_2' = G|_{X_2''} \colon X_2'' \to X_1$ is a μ–equivalence over B, and $\sigma(G) = -\sigma$ as an element of $H_{n+1}(B; \mathbb{L})$. Stacking V_1 and V_2 we obtain a normal map

$$H \colon (V, X_2, X_2'') \to (X_1 \times I, X_1 \times \{0\}, X_1 \times \{2\}),$$

where $H|_{X_2} = f$ and $H_{X_2''} = f_2$ is a μ–equivalence over B, and $\sigma(H) \in H_{n+1}(X; \mathbb{L})$ is zero.

Notice that all constructions were performed away from the singularities of X_2 so that we may assume that V is a product on a small neighborhood N of the singular set of X_2. We may also assume that $M = X_2 \setminus \overset{\circ}{N}$ is a compact manifold. Since $0 = \sigma(H) \in H_{n+1}(X; \mathbb{L})$, we can do surgery on H away from $N \times I$ rel $X_2 \sqcup X_2''$ and assume that V is a finely controlled h–cobordism over X rel $\partial M \times I$. A simple general position argument shows that $V \setminus (\overset{\circ}{N})$ is also a controlled h–cobordism over X, which, by the thin h–cobordism theorem [18], admits an ϵ–product structure rel $\partial M \times I$ if δ is small enough. Gluing $N \times I$ back in, we obtain an ϵ–deformation over X of the map f to a μ–homotopy equivalence over B as desired. $\square$

9   Resolution of singularities

Theorem 9.1   *If X is a compact ANR homology n–manifold, $n \geq 6$, then there exist an excellent ANR homology n–manifold Y and a cell-like map $\phi \colon Y \to X$.*

Proof   Let X be a compact ENR homology manifold of dimension ≥ 6. Our goal is to construct a diagram
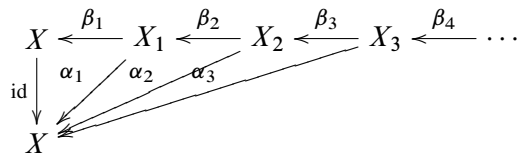
so that for $i \geq 2$ there are sequences $\{\epsilon_i\}$, $\{\tau_i\}$, $\{\gamma_i\}$ of positive numbers so that the following hold.

(i)   Each β_i is UV^1 ϵ_{i-1}–equivalence over X_{i-2} in the system metric. This guarantees that each β_i is an $\epsilon_{i-1} + 1/2^{i-1}$ over X_{i-1}.

(ii) Each X_i is a pleasant ϵ_i–controlled 2–patch space over X_{i-1}.

(iii) $d(\alpha_{i-1} \circ \beta_i, \alpha_i) < \tau_{i-1}$.

(iv)   Each α_i is a UV^1 γ_i–equivalence.

(v) $\sum \epsilon_i < \infty$, $\sum \tau_i < \infty$, and $\lim \gamma_i = 0$.

This will prove the theorem. The inverse limit $Z = \varprojlim\{X_i, \beta_i\}$ is an excellent metric space, so by Proposition 5.4 , Z is an ANR homology manifold with the disjoint disk property. For each i, there is a projection $p_i \colon Z \to X_i$. By condition (iii), the maps $\alpha_i \circ p_i$ converge to a map $\alpha \colon Z \to X$. By condition (iv), α is an ϵ–equivalence for every $\epsilon > 0$, so by Lemma 2.3, α is cell-like. Let $\{\epsilon_i\}$ be any sequence of positive numbers with $\sum \epsilon_i < \infty$.

Letting $\epsilon = \epsilon_1$ and $\mu = \epsilon_2$ in Theorem 8.1 [1], there is a $\delta_1 > 0$ so that there exists η_1, such that if we are given a diagram

$$
\begin{array}{ccc}
X & \xleftarrow{\;\beta_1\;} & X_1 \\
{\scriptstyle \mathrm{id}}\Big\downarrow & \swarrow{\scriptstyle \alpha_1 = \beta_1} & \\
X & &
\end{array}
$$

where $\beta_1 \colon X_1 \to X$ is an n–dimensional η_1–Poincaré duality space over X, then there exists η_2' so that if $f_1 \colon X_2 \to X_1$ is an n–dimensional η_2'–controlled 2–patch space over X_1 and a δ_1–equivalence over X, then f_1 is ϵ_1–homotopic over X to a UV^1 ϵ_2–homotopy equivalence over X.

$$
\begin{array}{ccccc}
X & \xleftarrow{\;\beta_1\;} & X_1 & \xleftarrow{\;f_1\;} & X_2 \\
{\scriptstyle \mathrm{id}}\Big\downarrow & \swarrow{\scriptstyle \alpha_1 = \beta_1} & & & \\
X & & & &
\end{array}
$$

We choose X_1 and $\beta_1 \colon X_1 \to X$ so that β_1 is a UV^1 δ_1–equivalence and so that X_1 is η_1–Poincaré over X. We set $\alpha_1 = \beta_1$. Here, we may assume that $\delta_1 < \epsilon_0$. In choosing this and future δ_i, we should take care that $\sum \delta_i < \infty$.

Letting $\epsilon = \epsilon_2$ and $\mu = \epsilon_3$ in Theorem 8.1, there is a $\delta_2 > 0$ so that there exists η_2'', such that if we are given a diagram

$$
\begin{array}{ccc}
X_1 & \xleftarrow{\;\beta_2\;} & X_2 \\
{\scriptstyle \alpha_1}\Big\downarrow & \swarrow{\scriptstyle \alpha_1 \circ \beta_2} & \\
X & &
\end{array}
$$

---

[1] This case of Theorem 8.1 is degenerate, but applying the theorem syntactically helps to establish the pattern of our induction, especially in choosing the constants η_1.

where $\beta_2\colon X_2 \to X_1$ is an n–dimensional η_2''–Poincaré duality space over X_1, then there exists η_3' so that if $f_2\colon X_3 \to X_2$ is an n–dimensional η_3'–controlled 2–patch space over X_2 and a δ_2–equivalence over X, then f_2 is ϵ_2–homotopic over X to a UV^1 ϵ_3–homotopy equivalence over X_1.

$$X_1 \xleftarrow{\ \beta_2\ } X_2 \xleftarrow{\ f_2\ } X_3$$

$$\alpha_1 \Big\downarrow \quad \swarrow \alpha_1 \circ \beta_2$$

$$X$$

We set $\eta_2 = \min(\eta_2', \eta_2'')$. We have now collected enough data to proceed with the construction of X_2, α_2, and β_2. Our construction is a modification of the construction in Lemma 7.2.

Start with a degree-one normal map $\psi\colon M \to X$, where M is a topological manifold and do surgery below the middle dimension on ψ. There is a $\omega_0 > 0$ such that, for any $0 < \omega < \omega_0$, the obstruction to doing surgery on ψ to a ω–equivalence is an element $\sigma = \sigma(\psi) \in H_n(X, \mathbb{L}_*(e))$.

$$X_1$$
$$\alpha_1 \Big\downarrow \quad \nwarrow \psi_1 = \bar\alpha_1 \circ \psi$$
$$X \xleftarrow{\ \psi\ } M$$

Let $\bar\alpha_1$ be a δ_1–inverse for α_1 and let $\psi_1 = \bar\alpha_1 \circ \psi$. The map ψ_1 is homotopic to a UV^1–map, so we replace ψ_1 by a UV^1–map and replace ψ by $\alpha_1 \circ \psi_1$.

Take a fine triangulation of M and let T be a very thin regular neighborhood of the 2–skeleton of M in this triangulation. Let C be the closure of the complement of T in M.
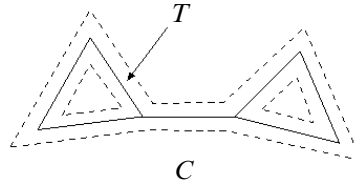
Using the controlled Bestvina–Walsh theorem, after a small homotopy, we may also assume that $\psi|T$, $\psi|C$ and $\psi|\partial T$ are all UV^1 maps. Since α_1 is a homotopy equivalence, there is an element $\sigma' \in H_*(X_1, \mathbb{L}(e))$ such that $\alpha_{1*}(\sigma') = \sigma \in H_*(X, \mathbb{L}(e))$.

Using the element $-\sigma' \in H_*(X_1, \mathbb{L}(e))$, do Wall realization on the identity map of $V_0 = \partial T = \partial C$. Proceed as in the proof of Lemma 7.2 to produce a 2–patch space X_2' with a UV^1–map $\psi'\colon X_2' \to X_1$ such that the controlled surgery obstruction of $\alpha_1 \circ \psi'$

is zero. We can arrange that X_2' is a η_2–Poincaré duality space over X_1. Do surgery on $\alpha_1 \circ \psi': X_2' \to X$ to obtain a 2–patch space X_2 and an equivalence $\alpha_2: X_2 \to X$ that is at least a δ_2–equivalence and is, in any case, so fine that $\bar{\alpha}_1 \circ \alpha_2: X_2 \to X_1$ is a δ_1–equivalence over X. We have a diagram

$$X \xleftarrow{\beta_1} X_1 \xleftarrow{f_1} X_2$$

where $f_1 = \bar{\alpha}_1 \circ \alpha_2$. The surgeries that produced X_2 from X_2' were performed on the complement of the set T, so by estimated homotopy extension, we can arrange that $f_1 = \psi'$ on a small neighborhood of T. Since the Poincaré duality of X_2 over X_1 depends only on the degree of control of the gluing map on the singular set, X_2 is an η_2–Poincaré duality space over X_1.

By our choices of δ_1 and η_2, the map f_1 is ϵ_1–homotopic over X to a UV^1–map $\beta_2: X_2 \to X_1$ that is an ϵ_2–equivalence over X. [2]

Letting $\epsilon = \epsilon_3$ and $\mu = \epsilon_4$ in Theorem 8.1, there is a $\delta_3 > 0$ so that there exists η_3'', such that if we are given a diagram

$$X_2 \xleftarrow{\beta_3} X_3$$

where $\beta_3: X_3 \to X_2$ is an n–dimensional η_3''–Poincaré duality space over X_2, then there exists η_4' so that if $f_3: X_4 \to X_3$ is an n–dimensional η_4'–controlled 2–patch space over X_3 and a δ_3–equivalence over X, then f_3 is ϵ_3–homotopic over X to a UV^1 ϵ_4–homotopy equivalence over X_2.

$$X_2 \xleftarrow{\beta_3} X_3 \xleftarrow{f_3} X_4$$

We set $\eta_3 = \min(\eta_3', \eta_3'')$. As in the previous stage of the construction, we obtain a 2–patch space X_3 with an at least δ_3 fine UV^1 homotopy equivalence $\alpha_3: X_3 \to X$ and a UV^1 map $f_2: X_3 \to X_2$ so that f_2 is a δ_2–equivalence over X and so that

---

[2] Again, this is a degenerate case. The next stage of the construction represents the general inductive step.

X_3 is η_3–Poincaré over X_2. It follows that f_2 is ϵ_2–homotopic over X to a UV^1 map $\beta_3\colon X_3 \to X_2$ that is an ϵ_3–equivalence over X_1. The composition $\alpha_2 \circ \beta_3$ is $(\epsilon_2 + \delta_2)$–close to α_3. The rest of the construction follows this same pattern. $\square$

The proof of Theorem B is similar to the proof of Theorem 9.1. If the total surgery obstruction of $Y \to Z$ is zero, then there is a degree one normal map $f\colon M \to Y$ so that the controlled surgery obstruction of f lies in the image of $H_n(Y, \mathbb{L})$ in $H_n(Z, \mathbb{L})$. This obstruction dies if and only if f is normally bordant to an ϵ–equivalence over Z for sufficiently small ϵ. As in the proof of Theorem 9.1, we can kill this obstruction by replacing M by a controlled 2–patch space Y_1. Iterating this process as in the proof of Theorem 9.1 gives us a diagram

$$Y \xleftarrow{\;\beta_1\;} X_1 \xleftarrow{\;\beta_2\;} X_2 \xleftarrow{\;\beta_3\;} X_3 \xleftarrow{\;\beta_4\;} \cdots$$

where the maps α_i and β_i satisfy conditions (i)–(v) in the proof of Theorem 9.1. In the limit, we obtain a DDPANR homology manifold X together with a map $g\colon X \to Y$ that is an ϵ equivalence for every $\epsilon > 0$. A related result for topological manifolds, was proven by Ferry and Pedersen [12, page 221].

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