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BLOW-UP OF A CRITICAL SOBOLEV NORM FOR ENERGY-SUBCRITICAL AND ENERGY-SUPERCRITICAL WAVE EQUATIONS

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We consider a wave equation in three space dimensions, with a power-like nonlinearity which is either focusing or defocusing. The exponent is greater than 3 (conformally supercritical) and not equal to 5 (not energy-critical). We prove that for any radial solution which does not scatter to a linear solution, an adapted scale-invariant Sobolev norm goes to infinity at the maximal time of existence. The proof uses a conserved generalized energy for the radial linear wave equation, new Strichartz estimates adapted to this generalized energy, and a bound from below of the generalized energy of any nonzero solution outside wave cones. It relies heavily on the fact that the equation does not have any nontrivial stationary solution. Our work yields a qualitative improvement on previous results on energy-subcritical and energy-supercritical wave equations, with a unified proof.

1. Introduction

1A. ***Motivation and background.*** Consider the semilinear wave equation in 1+3 dimensions

$$(\partial_t^2-\Delta)u=\iota|u|^{2m}u, \tag{1-1}$$

with initial data

$$u(0,x)=u_0(x),\quad \partial_t u(0,x)=u_1(x), \tag{1-2}$$

where $x\in\mathbb{R}^3$ and $t\in\mathbb{R}$. The parameters $m>1$ and $\iota\in\{\pm1\}$ are fixed. The equation is *focusing* when $\iota=1$ and *defocusing* when $\iota=-1$. It has the following scaling invariance: if $u(t,x)$ is a solution of (1-1) and $\lambda>0$, then $\lambda^{\frac{1}{m}}u(\lambda t,\lambda x)$ is also a solution. It is well-posed in the scale-invariant Sobolev space $\dot{\mathcal{H}}^{s_c}:=\dot H^{s_c}(\mathbb{R}^3)\times\dot H^{s_c-1}(\mathbb{R}^3)$, where $s_c=\frac{3}{2}-\frac{1}{m}$ is the critical Sobolev exponent. Equation (1-1) is *energy-subcritical* if $s_c<1$ (equivalently $m<2$), *energy-critical* if $s_c=1$ ($m=2$) and *energy-supercritical* if $s_c>1$ ($m>2$).

The dynamics of (1-1) depend in a crucial way on the value of m and the sign of ι.

The energy-critical case $m=2$ is particular. The conserved energy

$$E(\vec u(t))=\frac12\int|\nabla u(t,x)|^2\,dx+\frac12\int(\partial_t u(t,x))^2\,dx-\frac{\iota}{2m+2}\int|u(t,x)|^{2m+2}\,dx$$

is well-defined in $\dot{\mathcal{H}}^{s_c}=\dot{\mathcal{H}}^1=\dot H^1\times L^2$. When the nonlinearity is defocusing, the conservation of the energy implies that all solutions are bounded in $\dot{\mathcal{H}}^1$. It was proved in the 90s that all solutions are global

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and scatter to a linear solution in the energy space, i.e., that there exists a solution u_{L} of the linear wave equation

$$(\partial_t^2 - \Delta)u_{\mathrm{L}} = 0, \ (t,x) \in \mathbb{R} \times \mathbb{R}^3, \tag{1-3}$$

with initial data in $\dot{\mathcal{H}}^1$, such that

$$\lim_{t\to+\infty} \|\vec{u}(t) - \vec{u}_{\mathrm{L}}(t)\|_{\dot{\mathcal{H}}^1} = 0; \tag{1-4}$$

see [Grillakis 1990; 1992; Ginibre et al. 1992; Shatah and Struwe 1993; 1994; Kapitanski 1994; Ginibre and Velo 1995; Nakanishi 1999; Bahouri and Shatah 1998]. In the focusing case, there exist solutions that do not scatter. Indeed, there exist solutions of (1-1) that blow up in finite time with a *type I* behavior; i.e., there are solutions u such that

$$\lim_{t\to T_+(u)} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^1} = +\infty,$$

where $T_+(u)$ is the maximal time of existence of u. Furthermore, the equation also admits stationary solutions and more generally traveling waves. It was proved in [Duyckaerts et al. 2013] that any radial solution that does not scatter and is not a type I blow-up solution decouples asymptotically as a sum of rescaled stationary solutions and a dispersive term. This includes global nonscattering solutions (see [Krieger and Schlag 2007; Donninger and Krieger 2013], and also [Martel and Merle 2016; Jendrej 2016] in higher space dimensions, for examples of such solutions) and solutions that blow up in finite time but remain bounded in the energy space, called *type II blow-up* solutions (see, e.g., [Krieger et al. 2009; Krieger and Schlag 2014a] and, in higher dimensions [Hillairet and Raphaël 2012; Jendrej 2017]).

The case $m \neq 2$ is quite different. It is known that stationary solutions do not exist in the critical Sobolev space, even for focusing nonlinearity, see, e.g., [Joseph and Lundgren 1973; Farina 2007, Theorem 2], and it is conjectured that any solution that does not satisfy

$$\lim_{t\to T_+(u)} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^{s_c}} = +\infty \tag{1-5}$$

is global and scatters to a linear solution for positive times. A slightly weaker version of this result was proved in many works; namely, if the solution does not scatter, then

$$\limsup_{t\to T_+(u)} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^{s_c}} = +\infty. \tag{1-6}$$

See [Kenig and Merle 2011; Duyckaerts et al. 2014] for the radial case, $m > 2$, [Shen 2013; Rodriguez 2017] for the radial case, $1 < m < 2$, [Killip and Visan 2011] for the defocusing nonradial case, $m > 2$, [Dodson and Lawrie 2015] for the radial case, $m = 1$, and also [Killip et al. 2014] for the nonradial defocusing case, $1 \leq m < 2$, where (1-6) is proved for finite time blow-up solutions with initial data in the energy space.

Note that none of the preceding works excludes the existence of a nonscattering solution of (1-1) such that

$$\limsup_{t\to T_+(u)} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^{s_c}} = +\infty \quad \text{and} \quad \liminf_{t\to T_+(u)} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^{s_c}} < \infty.$$

In [Duyckaerts and Roy 2015], this type of solution was ruled out in the case $m > 2$: for any radial nonscattering solution of the equation, the critical Sobolev norm goes to infinity as $t \to T_+(u)$.

It is interesting to compare the theorems cited above with analogous ones for other equations, and in particular for the nonlinear Schrödinger equation

$$i\partial_t v - \Delta v = \iota |v|^{2m} v. \tag{1-7}$$

For the defocusing equation ($\iota = -1$), the fact that the bound of a critical norm implies scattering is known in the cubic case in three space dimensions [Kenig and Merle 2010] and in energy-supercritical cases in large space dimensions [Killip and Visan 2010]. Merle and Raphaël [2008] considered the focusing equation (1-7) with $\iota = 1$ and an L^2 supercritical (i.e., pseudoconformally supercritical), energy subcritical nonlinearity, that is, $\frac{2}{3} < m < 2$ when the number of space dimensions is three. This condition is the analogue of the condition $1 < m < 2$ (conformally supercritical and energy subcritical power) for the wave equation. They proved that if u is radial with initial data in the intersection of $\dot{H}^1$ and the critical Sobolev space, and if $T_+(v)$ is finite, then

$$\|v(t)\|_{L^{3m}} \geq \frac{1}{C} |\log(T_+(v) - t)|^{\alpha}$$

for some constant $\alpha > 0$. Note that in this case there exists a global, bounded, nonscattering solution. The space L^{3m} is scale-invariant and strictly larger than the critical Sobolev space. Analogous results are known for Navier–Stokes equations; see [Iskauriaza et al. 2003; Kenig and Koch 2011; Seregin 2012; Gallagher et al. 2013; 2016]. For example, it is proved in [Seregin 2012] that the scale-invariant L^3 norm of a solution blowing-up in finite time goes to infinity at the blow-up time.

Going back to (1-1) with $m \neq 2$, many questions remain open:

- Is it true that all nonscattering solutions of (1-1) satisfy (1-5) in the nonradial case, or if $1 < m < 2$?
- Can one lower the regularity of the scale-invariant norm used in (1-5), as in the case of nonlinear Schrödinger and Navier–Stokes equations?
- Is it possible to give an explicit lower-bound of the critical norm, in the spirit of [Merle and Raphaël 2008]?

In this article, we give a partial answer to the first two questions in the radial case. This is based on a new well-posedness theory for (1-1), in a scale-invariant weighted Sobolev space $\mathcal{L}^m$ which is not Hilbertian, but is related to a conserved quantity of the linear wave equation and is compatible with the finite speed of propagation.

1B. ***Strichartz estimates and local well-posedness.*** Consider the following norm for radial functions (u_0, u_1) on $\mathbb{R}^3$:

$$\|(u_0, u_1)\|_{\mathcal{L}^m} = \left(\int_0^{+\infty} (|r\partial_r u_0|^m + |ru_1|^m)\, dr \right)^{\frac{1}{m}},$$

and define the space $\mathcal{L}^m$ as the closure of radial, smooth, compactly supported functions for this norm. Note that $\mathcal{L}^2$ is exactly[1] $\dot{\mathcal{H}}^1_{\mathrm{rad}}$. The $\mathcal{L}^m$ norm was introduced in [Duyckaerts and Roy 2015], in the case $m > 2$, as a scale-invariant substitute to the energy norm $\dot{H}^1 \times L^2$ norm. Let us mention that $\dot{\mathcal{H}}^{s_c}_{\mathrm{rad}} \subset \mathcal{L}^m$ if $m > 2$, and $\mathcal{L}^m \subset \dot{\mathcal{H}}^{s_c}_{\mathrm{rad}}$ if $1 < m < 2$ (see Proposition 2.2 below). It was observed in [Duyckaerts and Roy 2015] that the $\mathcal{L}^m$ norm is almost conserved for solutions of the linear wave equation: we will indeed introduce in Section 2 a *conserved* quantity (the generalized energy) which is equivalent to this norm. We first prove Strichartz estimates for the linear wave equation. If I is a real interval, we denote by $S(I)$ the space defined by the norm

$$\|f\|_{S(I)} = \left(\int_I \left(\int_0^{+\infty} |f(t,r)|^{(2m+1)m} r^m \, dr\right)^{\frac{1}{m}} dt\right)^{\frac{1}{2m+1}}.$$

Theorem 1. *Let v be a solution of the linear wave equation*

$$\partial_t^2 v - \Delta v = 0, \quad (v, \partial_t v)_{\restriction t=0} = (v_0, v_1) \in \mathcal{L}^m.$$

Then $v \in S(\mathbb{R})$ and

$$\|v\|_{S(\mathbb{R})} \le C \|(v_0, v_1)\|_{\mathcal{L}^m}.$$

Note that Theorem 1 generalizes, in the radial case, the L^5L^{10} Strichartz/Sobolev estimate for finite-energy solutions of the linear wave equation to the case $m \neq 2$. Let us mention that we prove more general Strichartz estimates, including estimates for the nonhomogeneous wave equation (see Section 2B for the details). As a consequence, we obtain local well-posedness in $\mathcal{L}^m$ for (1-1):

Theorem 2. *For $m > 1$, (1-1) is locally well-posed in $\mathcal{L}^m$. For any initial data (u_0, u_1) in $\mathcal{L}^m$, there exists a unique solution u of (1-1), (1-2) defined on a maximal interval of existence $I_{\max}(u) = (T_-(u), T_+(u))$ such that $\vec{u} \in C^0(I_{\max}(u), \mathcal{L}^m)$ and for all compact intervals $J \Subset I_{\max}(u)$, we have $u \in S(J)$. Furthermore,*

$$T_+(u) < \infty \implies \|u\|_{S([0,T_+(u)))} = +\infty.$$

We obtain Theorem 1 and the other generalized Strichartz estimates of Section 2B by interpolating between the known generalized Strichartz estimates of [Ginibre and Velo 1995], see also [Lindblad and Sogge 1995], in correspondence to the case $m = 2$, and Strichartz-type estimates obtained by a new method, based on the continuity of the Hardy–Littlewood maximal function from L^1 to L^1_w (see Section 2B).

We also construct a profile decomposition for sequences of functions that are bounded in $\mathcal{L}^m$, which is adapted to (1-1), in the spirit of the one of [Bahouri and Gérard 1999] which corresponds to the case $m = 2$. This construction is based on a refined Sobolev embedding due to Chamorro [2011]. The fact that $\mathcal{L}^m$ is not a Hilbert space yields a new technical difficulty, namely that the usual Pythagorean expansion of the norm does not seem to be valid and must be replaced by a weaker statement, closer to Bessel's inequality than to the Pythagorean theorem. We refer to [Solimini 1995; Jaffard 1999] for other non-Hilbertian profile decompositions where this type of inequality also appears.

The definition of the space $\mathcal{L}^m$ does not involve any fractional derivatives and is technically easier to handle than the space $\dot{\mathcal{H}}^{s_c}$ with $m \neq 2$, where the latter are all defined by norms that are not compatible

[1] Throughout the article, the index rad denotes the subspace of radial elements of a given space of distributions on $\mathbb{R}^3$.

with finite speed of propagation. We hope that the Strichartz estimates and profile decomposition proved in this article will find applications for nonlinear wave equations apart from (1-1).

1C. ***Blow-up of the critical Sobolev norm for the nonlinear equation.*** Our second result is that the dichotomy proved in [Duyckaerts and Roy 2015] remains valid in $\mathcal{L}^m$, as long as $m \neq 2$:

Theorem 3. *Assume $m > 1$ and $m \neq 2$. Let u be a radial solution of* (1-1), (1-2), *with $(u_0, u_1) \in \mathcal{L}^m$ and maximal positive time of existence T_+. Then one of the following holds*:

(1) $\lim_{t \to T_+(u)} \|\vec{u}(t)\|_{\mathcal{L}^m} = +\infty$.

(2) $T_+(u) = +\infty$ *and u scatters forward in time to a linear solution*; *i.e.*, *there exists a solution u_{L} of* (1-3), *with initial data $\mathcal{L}^m$*, *such that*

$$\lim_{t \to +\infty} \|\vec{u}(t) - \vec{u}_{\mathrm{L}}(t)\|_{\mathcal{L}^m} = 0.$$

In the energy-supercritical case $m > 2$, Theorem 3 improves the result of [Duyckaerts and Roy 2015] since $\dot{\mathcal{H}}^{s_c}$ is continuously embedded into $\mathcal{L}^m$. In the case $1 < m < 2$, we know $\mathcal{L}^m$ is continuously embedded into $\dot{\mathcal{H}}^{s_c}$ and Theorem 3 is not strictly stronger than the result of [Shen 2013]. However, Theorem 3 is also new, since it says that as least some scale-invariant norm of u must go to infinity as t goes to $T_+(u)$. It is very natural to conjecture that the $\dot{\mathcal{H}}^{s_c}$ norm of the solution also goes to infinity, but this is still an open question.

Once the Strichartz estimates, well-posed theory and profile decomposition in $\mathcal{L}^m$ are known, the proof of Theorem 3 (sketched in Sections 4, 5 and 6) is very close to the proof of the corresponding result in [Duyckaerts and Roy 2015], with some simplifications due to the use of the space $\mathcal{L}^m$ instead of $\dot{\mathcal{H}}^{s_c}$ throughout the proof. As in [loc. cit.], we use the *channels of energy method* initiated in [Duyckaerts et al. 2011], and the main ingredient of the proof is an exterior energy estimate for radial solutions of the linear wave equation for the $\mathcal{L}^m$-energy, which generalizes the exterior energy estimate used in [Duyckaerts et al. 2011; 2013; 2014].

According to Theorem 3, there are three potential types of dynamics for (1-1): scattering, finite time blow-up solutions such that the critical norm goes to infinity at the blow-up time, and global solutions such that the critical norm goes to infinity as t goes to infinity. Only two of these dynamics are known to exist: scattering (for both focusing and defocusing nonlinearities) and finite time blow-up (for focusing nonlinearity only). Indeed, in the focusing case, it is possible to construct blow-up solutions with smooth, compactly supported initial data using finite speed of propagation and the ordinary differential equation $y'' = |y|^{2m} y$. Another type of blow-up solution was constructed by C. Collot [2014] for some energy-supercritical nonlinearity in large space dimension: in this case the scale-invariant Sobolev norms blow up logarithmically.

It is natural to conjecture that all solutions in $\mathcal{L}^m$ are global in the defocusing case. For $m < 2$, this follows from conservation of the energy if the data is assumed to be in $\dot{\mathcal{H}}^1$, and only the case of low-regularity solution is open. For supercritical nonlinearity $m > 2$, it is a very delicate issue even for smooth initial data, as the recent construction by T. Tao [2016] of a finite time blow-up solution for a defocusing *system*[2] of energy supercritical wave equation suggests.

The existence of global solutions blowing-up at infinity with initial data in $\mathcal{L}^m$ (or $\dot{\mathcal{H}}^{s_c}$) is also completely open. We refer to [Krieger and Schlag 2014b; Luk et al. 2016, Appendix A] for two different constructions of global, smooth, nonscattering solutions in the case $m = 3$. The initial data of these solutions do not belong either to the critical Sobolev spaces $\dot{\mathcal{H}}^{\frac{7}{6}}$ or to the $\mathcal{L}^3$ space, but are, however, in all spaces $\dot{\mathcal{H}}^s$, $s > \frac{7}{6}$. These constructions and Theorem 3 seem to suggest that any global solution with initial data decaying sufficiently at infinity actually scatters, but we do not know of any rigorous result in this direction.

Let us finally mention [Beceanu and Soffer 2017] on (1-1) with supercritical nonlinearity $m > 2$, where global existence is proved for a class of outgoing initial data.

The outline of the paper is as follows: in Section 2, we prove the Strichartz estimate for the linear wave equation and deduce the Cauchy theory for (1-1). In Section 3, we construct the profile decomposition. In Section 4, we prove the exterior energy property for nonzero solutions of (1-1), which is the core of the proof of Theorem 3. In Section 5, we introduce the radiation term (i.e., the dispersive part) of a solution which is bounded in the critical space for a sequence of times. In Section 6, we conclude the proof.

Notation. If a and b are two positive quantities we write $a \lesssim b$ when there exists a constant $C > 0$ such that $a \le Cb$, where the constant will be clear from the context. When the constant depends on some other quantity M, we emphasize the dependence by writing $a \lesssim_M b$. We will write $a \approx b$ when we have both $a \lesssim b$ and $b \lesssim a$. We will write $a \ll b$ or $a \gg b$ if there exists a sufficiently large constant $C > 0$ such that $Ca \le b$ or $a \ge Cb$ respectively. We use $\mathcal{S}(\mathbb{R}^d)$ to denote the Schwartz class of functions on the Euclidean space $\mathbb{R}^d$.

If f is a radial function depending on t and $r := |x|$, let

$$\vec{f} := (f, \partial_t f) \quad \text{and} \quad [f]_\pm(t,r) = (\partial_r \pm \partial_t)(rf).$$

Given $s \ge 0$ and n a positive integer, we define

$$\dot{\mathcal{H}}^s(\mathbb{R}^n) := \dot{H}^s(\mathbb{R}^n) \times \dot{H}^{s-1}(\mathbb{R}^n),$$

where $\dot{H}^s$ denotes the standard homogeneous Sobolev space. We let $L^p_t(I, L^q_x)$ be the space of measurable functions f on $I \times \mathbb{R}^3$ such that

$$\|f\|_{L^p_t(I,L^q_x)} = \left(\int_I \left(\int_{\mathbb{R}^3} |f(t,x)|^q \, dx\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}} < \infty.$$

Unless specified, the functional spaces (L^p, $\dot{H}^s$, etc...) are spaces of functions or distributions on $\mathbb{R}^3$ with the Lebesgue measure. On a measurable space $(\Omega, d\mu)$ where μ is positive, the weak L^q quasinorm of a function f is defined as

$$\|f\|_{L^q_w} := \sup_{\lambda>0} \lambda \left(\mu\{x \in \Omega : |f(x)| > \lambda\}\right)^{\frac{1}{q}}.$$

[2] The unknown u is $\mathbb{R}^{40}$-valued.

We shall also use the weighted Lebesgue norm $L^q(\mathbb{R}^n, \omega)$, defined as

$$\|f\|_{L^q(\mathbb{R}^n,\omega)} := \left(\int_{\mathbb{R}^n} |f(x)|^q \omega(x)\, dx\right)^{\frac{1}{q}}$$

for some measurable function $\omega(x)$ as a weight. For $q > 1$, we use $q' = \frac{q}{q-1}$ to mean its Lebesgue conjugate.

We denote by $\mathcal{T}_R$ the operator

$$f \mapsto \mathcal{T}_R(f) := \begin{cases} f(R), & |x| \le R, \\ f(|x|), & |x| \ge R. \end{cases}$$

Let $S_{\mathrm{L}}(t)$ denote the linear propagator; i.e.,

$$S_{\mathrm{L}}(t)(w_0, w_1) := \cos(tD) w_0 + \frac{\sin(tD)}{D} w_1, \quad D = \sqrt{-\Delta}.$$

If u is a function of t and r, we will denote by $F(\partial_{r,t}u)$ the sum $F(\partial_r u) + F(\partial_t u)$; for example, $|\partial_{t,r}u|^m := |\partial_t u|^m + |\partial_r u|^m$.

2. Strichartz estimates and local well-posedness

2A. *Preliminaries.* For $m > 1$, we denote by $\dot{\mathcal{W}}^{1,m}$ the closure of $C^\infty_{0,\mathrm{rad}}$ for the norm $\|\cdot\|_{\dot{\mathcal{W}}^{1,m}}$ defined by

$$\|\varphi\|_{\dot{\mathcal{W}}^{1,m}} := \left(\int_0^{+\infty} |\partial_r \varphi(r)|^m r^m\, dr\right)^{\frac{1}{m}}.$$

Proposition 2.1. *We have $f \in \dot{\mathcal{W}}^{1,m}$ if and only if $f(r) \in C^0_{\mathrm{rad}}((0, +\infty))$ satisfies the conditions*

$$\int_0^{+\infty} |r \partial_r f(r)|^m\, dr < +\infty, \tag{2-1}$$

$$\lim_{r\to 0} r^{\frac{1}{m}} f(r) = \lim_{r\to\infty} r^{\frac{1}{m}} f(r) = 0. \tag{2-2}$$

The proof is given in the Appendix.

We denote by $\mathcal{L}^m$ the closure of $(C^\infty_{0,\mathrm{rad}})^2$ for the norm $\|\cdot\|_{\mathcal{L}^m}$,

$$\|(u_0, u_1)\|_{\mathcal{L}^m} := \|u_0\|_{\dot{\mathcal{W}}^{1,m}} + \left(\int_0^{+\infty} |u_1(r)|^m r^m\, dr\right)^{\frac{1}{m}}.$$

Then:

Proposition 2.2. (1) *If $m > 2$ and $(u_0, u_1) \in \dot{\mathcal{H}}^{s_c}$, then $(u_0, u_1) \in \mathcal{L}^m$ and*

$$\|(u_0, u_1)\|_{\mathcal{L}^m} \lesssim \|(u_0, u_1)\|_{\dot{\mathcal{H}}^{s_c}}.$$

(2) *If $1 < m < 2$ and $(u_0, u_1) \in \mathcal{L}^m$, then $(u_0, u_1) \in \dot{\mathcal{H}}^{s_c}$ and*

$$\|(u_0, u_1)\|_{\dot{\mathcal{H}}^{s_c}} \lesssim \|(u_0, u_1)\|_{\mathcal{L}^m}.$$

(3) *If* $u_0 \in \dot{\mathcal{W}}^{1,m}$, *then* $u_0 \in L^{3m}(\mathbb{R}^3)$ *and*

$$\|u_0\|_{L^{3m}} \lesssim \|u_0\|_{\dot{\mathcal{W}}^{1,m}}.$$

(4) *If* $u_0 \in \dot{\mathcal{W}}^{1,m}$, *and* $R > 0$, *then*

$$R|u_0(R)|^m + \int_R^{+\infty} |\partial_r(ru_0)|^m \, dr \approx \int_R^{+\infty} |\partial_r u_0|^m r^m \, dr,$$

where the implicit constant does not depend on R.

Proof. For the proofs of properties (1), (3), (4), see [Kenig and Merle 2011, Lemma 3.2; Duyckaerts and Roy 2015, Lemmas 3.2 and 3.3]. We prove (2) by duality from (1). Assume $m \in (1,2)$ and let m' be the Lebesgue dual exponent of m. Let $(u_0, u_1) \in \mathcal{L}^m$ and $\varphi, \psi \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^3)$. Note that

$$\int_0^\infty r^2 \partial_r u_0 \partial_r \varphi \, dr = \int_0^\infty \partial_r(ru_0)\partial_r(r\varphi)\, dr.$$

By Hölder's inequality and (1),

$$\left|\int_0^\infty r^2 \partial_r u_0 \partial_r \varphi \, dr\right| + \left|\int_0^\infty r^2 u_1 \psi \, dr\right| \le \|(u_0,u_1)\|_{\mathcal{L}^m} \|(\varphi,\psi)\|_{\mathcal{L}^{m'}} \le \|(u_0,u_1)\|_{\mathcal{L}^m} \|(\varphi,\psi)\|_{\dot{\mathcal{H}}^{1/2+1/m}}.$$

This yields the announced result. □

Let $v(t,x)$ be a solution to the Cauchy problem

$$(\partial_t^2 - \Delta)v(t,x) = 0, \quad (v, \partial_t v)|_{t=0} = (v_0, v_1), \quad t \in \mathbb{R},\ x \in \mathbb{R}^3, \tag{2-3}$$

where the initial data is in $\mathcal{L}^m$. Define $r = |x|$ and set

$$F(\sigma) = \frac{1}{2}\sigma\, v_0(|\sigma|) + \frac{1}{2}\int_0^{|\sigma|} r\, v_1(r)\, dr. \tag{2-4}$$

An explicit computation, using

$$(\partial_t^2 - \partial_r^2)(rv) = 0 \tag{2-5}$$

yields

$$v(t,r) = \frac{1}{r}\big(F(t+r) - F(t-r)\big). \tag{2-6}$$

We have

$$[v]_+(t,r) = (\partial_r + \partial_t)(rv) = 2\dot{F}(t+r), \quad [v]_-(t,r) = (\partial_r - \partial_t)(rv) = 2\dot{F}(t-r). \tag{2-7}$$

If $(v_0, v_1) \in \mathcal{L}^m$, we define

$$E_m(v_0,v_1) = \int_0^{+\infty} \big(|\partial_r(rv_0) + rv_1|^m + |\partial_r(rv_0) - rv_1|^m\big)\, dr,$$

so that

$$E_m(\vec{v}(t)) = \int_0^{+\infty} \big|[v]_+(t,r)\big|^m \, dr + \int_0^{+\infty} \big|[v]_-(t,r)\big|^m \, dr.$$

Proposition 2.3. *Assume* $1 < m < +\infty$. *Let* $(v_0, v_1) \in \mathcal{L}^m$ *and* $v(t,r)$ *be given by* (2-3).

(1) Equivalence of energy and $\mathcal{L}^m$ norm.
$$\|(v_0, v_1)\|_{\mathcal{L}^m}^m \approx \int_0^{+\infty} |\partial_r(rv_0)|^m \, dr + \int_0^{+\infty} |rv_1|^m \, dr \approx E_m(v_0, v_1).$$

(2) Energy conservation. $E_m(\vec{v})$ *is independent of time. We call* E_m *the* $\mathcal{L}^m$*-modified energy for* (1-3).

(3) Exterior energy bound. *If* $R > 0$, *the following holds for all* $t \geq 0$ *or for all* $t \leq 0$:
$$\int_R^{+\infty} |\partial_r(rv_0)|^m + |rv_1|^m \, dr \lesssim \int_{R+|t|}^{+\infty} |\partial_r(rv)|^m + |\partial_t(rv)|^m \, dr.$$

Property (2) follows from direct computation, and the formula (2-5). Let us mention that the notation E_m has a slightly different meaning in [Duyckaerts and Roy 2015].

Remark 2.4. Note that
$$E_2(v(t)) = \int_{\mathbb{R}^3} |\nabla v(t,x)|^2 \, dx + \int_{\mathbb{R}^3} |\partial_t v(t,x)|^2 \, dx, \tag{2-8}$$
which coincides (up to a constant) with the standard energy functional for (2-3). Moreover, from (2-6) we know for any $m \in (1, +\infty)$, there exists $C_m > 0$ such that
$$C_m^{-1} \|\vec{v}(0)\|_{\mathcal{L}^m} \leq \|\vec{v}(t)\|_{\mathcal{L}^m} \leq C_m \|\vec{v}(0)\|_{\mathcal{L}^m} \quad \text{for all } t. \tag{2-9}$$
Thus $\|\vec{v}(t)\|_{\mathcal{L}^m}$ enjoys the pseudoconservation law, namely (2-9), and extends the classical energy to the general case $m > 1$.

From the conservation of the energy, we deduce the following energy estimate for the equation with a right-hand side.

Corollary 2.5. *Consider the problem*
$$(\partial_t^2 - \Delta) u(t,x) = f(t,x), \quad (u, \partial_t u)|_{t=0} = (u_0, u_1), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^3, \tag{2-10}$$
with $(u_0, u_1) \in \mathcal{L}^m$ *for a fixed* $m > 1$, *and* f *radial. Then we have the following inequality as long as the right-hand side is finite*:
$$\sup_{t \in \mathbb{R}} \left(\int_0^{\infty} \big[|\partial_r(ru)|^m(t) + |\partial_t(ru)|^m(t) \big] \, dr \right)^{\frac{1}{m}} \leq C \left(\|(u_0, u_1)\|_{\mathcal{L}^m} + \int_{-\infty}^{+\infty} \left(\int_0^{\infty} |rf(t,r)|^m \, dr \right)^{\frac{1}{m}} dt \right) \tag{2-11}$$

Proof. Write $u(t,r) = u_{\mathrm{L}}(t,r) + u_{\mathcal{N}}(t,r)$ with
$$u_{\mathrm{L}}(t,r) = S_{\mathrm{L}}(t)(u_0, u_1), \quad u_{\mathcal{N}}(t) = \int_0^t \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} f(s) \, ds.$$
The bound for $\|\vec{u}_{\mathrm{L}}\|_{\mathcal{L}^m}$ follows from (2-9) and the conservation of the $\mathcal{L}^m$ modified energy. Moreover,
$$\|\vec{u}_{\mathcal{N}}(t,r)\|_{\mathcal{L}^m} \leq \int_0^t \left\| \left(\frac{\sin\big((t-s)\sqrt{-\Delta}\big)}{\sqrt{-\Delta}} f(s), \cos\big((t-s)\sqrt{-\Delta}\big) f(s) \right) \right\|_{\mathcal{L}^m} ds,$$
and the estimate on $u_{\mathcal{N}}$ follows again from (2-9) and the conservation of the $\mathcal{L}^m$-modified energy. ☐

2B. ***Strichartz estimates in weighted Sobolev spaces.*** Let Ω be a measurable subset of $\mathbb{R}_t \times (0,+\infty)$ of the form $\Omega = \bigcup_{t\in\mathbb{R}}\{t\}\times J_t$, where for all t, we have J_t is a measurable subset of $(0,+\infty)$. If f is a measurable function on Ω, we let

$$\|f\|_{S(\Omega)} = \left(\int_{\mathbb{R}}\left(\int_{J_t} |f(t,r)|^{(2m+1)m} r^m\,dr\right)^{\frac{1}{m}} dt\right)^{\frac{1}{2m+1}}.$$

If $\Omega = I\times(0,+\infty)$, where I is a time interval, we will set $S(\Omega)=S(I)$ to lighten notation:

$$\|f\|_{S(I)} = \left(\int_I\left(\int_0^{+\infty} |f(t,r)|^{(2m+1)m} r^m\,dr\right)^{\frac{1}{m}} dt\right)^{\frac{1}{2m+1}}.$$

In this subsection we prove the following Strichartz estimate:

Proposition 2.6. *Let $m>1$ and assume $v(t,x)$ is the solution of the Cauchy problem (2-3) with radial initial data $(v_0,v_1)\in\mathcal{L}^m$. Then there exists a constant C such that*

$$\|v\|_{S(\mathbb{R})} \le C\|\vec{v}(0)\|_{\mathcal{L}^m}. \tag{2-12}$$

We also have its analogue for the inhomogeneous part:

Proposition 2.7. *Let $m>1$ and $u(t,r)$ be the solution of (2-10) with $\vec{u}(0)=(0,0)$. Assume*

$$\|f\|_{L^1_tL^m_x(r^m\,dr)} := \int_{-\infty}^{+\infty}\left(\int_0^{+\infty} |f(t,r)|^m r^m\,dr\right)^{\frac{1}{m}} dt < \infty.$$

Then we have

$$\|u\|_{S(\mathbb{R})} \le C\|f\|_{L^1_tL^m_x(r^m dr)}. \tag{2-13}$$

We start by proving auxiliary symmetric Strichartz-type estimates in Section 2B1, using the weak continuity in L^1 of the Hardy–Littlewood maximal function. In Section 2B2 we will interpolate these estimates with standard Strichartz inequalities to obtain the key estimates (2-12) and (2-13).

2B1. *A family of symmetric Strichartz estimates.* With the explicit expression (2-6), we are ready to deduce a crucial estimate for the linear wave equation (2-3) with $\vec{v}(0)\in\mathcal{L}^m$.

Proposition 2.8. *Let $v(t,x)=S_{\mathrm{L}}(t)(v_0,v_1)$ be a radial solution of (2-3). Then for any $m\in(1,+\infty)$ and $\alpha\in(1,+\infty)$, there is a constant C such that the following* a priori *estimate is valid*:

$$\left(\int_{\mathbb{R}}\int_0^{+\infty} |v(t,r)|^{\alpha m} r^{\alpha-2}\,dr\,dt\right)^{\frac{1}{\alpha m}} \le C\|\vec{v}(0)\|_{\mathcal{L}^m}. \tag{2-14}$$

Proof. We assume $v_1\equiv 0$ first. Then from (2-4) and the fundamental theorem of calculus,

$$v(t,r) = \frac{1}{2r}\int_{t-r}^{t+r} \partial_s(s\,v_0(|s|))\,ds, \quad r=|x|. \tag{2-15}$$

Let us consider the operator

$$\mathcal{T}:\ G(s)\mapsto \frac{1}{2r}\int_{t-r}^{t+r} G(s)\,ds. \tag{2-16}$$

First, it is clear that

$$\sup_{(t,r)\in\mathbb{R}\times\mathbb{R}_+} |\mathcal{T}G(t,r)| \le \|G\|_{L^\infty(\mathbb{R};\,ds)}. \tag{2-17}$$

Next, we demonstrate the weak-type estimate

$$\|\mathcal{T}G\|_{L^\alpha_w(\mathbb{R}\times\mathbb{R}_+;\,r^{\alpha-2}drdt)} \le C\|G\|_{L^1(\mathbb{R};\,ds)}, \tag{2-18}$$

or equivalently, there is $C>0$ such that for any $\lambda>0$ we have

$$\iint_{\mathcal{E}_\lambda} r^{\alpha-2}drdt \le C\left(\frac{\|G\|_{L^1}}{\lambda}\right)^{\alpha}, \tag{2-19}$$

where $\mathcal{E}_\lambda=\{(t,r)\in\mathbb{R}\times\mathbb{R}_+ : |\mathcal{T}G(t,r)|>\lambda\}$.

Given this, we have, interpolating between (2-17) and (2-18),

$$\left(\int_{\mathbb{R}}\int_0^{+\infty} |\mathcal{T}G(t,r)|^{\alpha m} r^{\alpha-2}\,dr\,dt\right)^{\frac{1}{\alpha}} \le C\int_{\mathbb{R}} |G(s)|^m\,ds; \tag{2-20}$$

see Theorem 5.3.2 in [Bergh and Löfström 1976]. The estimate (2-14) with $v_1\equiv 0$ now follows by using (2-20) with

$$G(s)=\partial_s(s\,v_0(|s|)).$$

To show (2-19), one observes that on $\mathcal{E}_\lambda$,

$$0<r<\frac{\|G\|_{L^1}}{\lambda}\quad\text{and}\quad (\mathcal{M}G)(t)>\lambda,$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal function. Therefore, we can bound the left-hand side of (2-19) as follows:

$$\int_0^{\frac{1}{2\lambda}\|G\|_{L^1}} r^{\alpha-2}\,dr\int_{\{t\in\mathbb{R}|(\mathcal{M}G)(t)>\lambda\}} dt \le C\left(\frac{\|G\|_{L^1}}{\lambda}\right)^{\alpha}, \tag{2-21}$$

where we have used the weak estimate $\mathcal{M}: L^1(\mathbb{R})\to L^1_w(\mathbb{R})$.

The case $v_0\equiv 0$ follows from the same argument. Indeed, in this case we have

$$v(t,r)=\frac{1}{2r}\int_{t-r}^{t+r} sv_1(|s|)\,ds. \tag{2-22}$$

Letting $G(s)=sv_1(|s|)$ and applying (2-20) we are done. □

Let $u(t,x)$ be a solution to the nonhomogeneous Cauchy problem (2-10), where $f(t,x)$ is radial in the space variable and locally integrable. If we set

$$g(t,\rho)=\rho f(t,|\rho|), \tag{2-23}$$

then we have

$$u(t,r)=\frac{1}{2r}\int_0^t\int_{\tau-r}^{\tau+r} g(t-\tau,\,\sigma)\,d\sigma\,d\tau. \tag{2-24}$$

After a change of variables, we obtain

$$u(t,r)=\frac{1}{2r}\int_{t-r}^{t+r} G(t,\rho)\,d\rho, \tag{2-25}$$

with

$$G(t,\rho)=\int_0^t g(s,\rho-s)\,ds.$$

A proof very close to the one of Proposition 2.8 yields symmetric Strichartz estimates for the nonhomogeneous equation:

Proposition 2.9. *Let $u(t,x)$ be a radial solution of the problem (2-10) with initial data $\vec{u}(0)=(0,0)$. Then for any $m\in(1,+\infty)$ and $\alpha\in(1,+\infty)$ there is a constant C such that we have*

$$\left(\int_{\mathbb{R}}\int_0^{+\infty} |u(t,r)|^{\alpha m} r^{\alpha-2}\,dr\,dt\right)^{\frac{1}{\alpha m}} \le C\int_{\mathbb{R}}\left(\int_0^{+\infty} |rf(t,r)|^m\,dr\right)^{\frac{1}{m}} dt. \tag{2-26}$$

Proof. In view of (2-25), we have

$$|u(t,r)|\le \mathcal{T}\widetilde{G}(t,r),$$

where $\mathcal{T}$ is defined as in (2-16) and

$$\widetilde{G}(\rho)=\int_{-\infty}^{+\infty} |g(s,\rho-s)|\,ds,$$

with g given by (2-23). Noting that $m>1$, we obtain (2-26) by using (2-20) and Minkowski's inequality. □

Remark 2.10. Notice that from (2-15) and (2-22), one may deduce the following end-point Strichartz estimate for linear wave equations in three dimensions with radial initial data

$$\|S_{\mathrm{L}}(t)(v_0,v_1)\|_{L^2(\mathbb{R}_t,\,L^\infty(\mathbb{R}_x^3))}\le C\left(\|v_0\|_{\dot{H}^1(\mathbb{R}^3)}+\|v_1\|_{L^2(\mathbb{R}^3)}\right), \tag{2-27}$$

where $(v_0,v_1)\in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^3)\times L^2_{\mathrm{rad}}(\mathbb{R}^3)$. In fact, we may assume without loss of generality that (v_0,v_1) belongs to the Schwartz class. Then (2-27) follows from (2-15) and (2-22) by using the L^2-boundedness of the Hardy–Littlewood maximal function and integration by parts.

2B2. *Proof of the key Strichartz inequality.* We prove here Propositions 2.6 and 2.7. Let us first recall the following classical Strichartz estimates for wave equations; see [Ginibre and Velo 1995].

Theorem 2.11. *Consider $v(t,x)$, the solution of the linear Cauchy problem*

$$\begin{cases} (\partial_t^2-\Delta)v=h(t,x), & (x,t)\in\mathbb{R}^3\times\mathbb{R},\\ \quad v|_{t=0}=v_0\in \dot{H}^1(\mathbb{R}^3), & \\ \partial_t v|_{t=0}=v_1\in L^2(\mathbb{R}^3), & \end{cases} \tag{2-28}$$

so that

$$v(t)=S_{\mathrm{L}}(t)(v_0,v_1)+\int_0^t \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}h(s)\,ds.$$

Let $2 \le q, \sigma \le \infty$ and let the following conditions be satisfied:

$$\frac{1}{q}+\frac{1}{\sigma}\le\frac{1}{2},\quad (q,\sigma)\neq(2,\infty),\quad \frac{1}{q}+\frac{3}{\sigma}=\frac{1}{2}.$$

Then there exists $C > 0$ such that v satisfies the estimate

$$\|v\|_{L^q(\mathbb{R},L^\sigma(\mathbb{R}^3))}\le C\big(\|v_0\|_{\dot H^1(\mathbb{R}^3)}+\|v_1\|_{L^2(\mathbb{R}^3)}+\|h\|_{L^1(\mathbb{R};L^2(\mathbb{R}^3))}\big). \tag{2-29}$$

We are now ready to prove Proposition 2.6

Proof. Since (2-12) is classical when $m = 2$, it suffices to consider below the cases for $m > 2$ and $1 < m < 2$ separately.

If $m > 2$, we define $m^* = 2m$ and take $\alpha = \frac{4}{3}(2m+1)$. Then we have from (2-14)

$$\Big(\int_{-\infty}^{+\infty}\int_0^{+\infty}|v(t,r)\,r^{\gamma_1}|^{\alpha m^*}r^{\gamma_2}\,dr\,dt\Big)^{\frac{1}{\alpha m^*}}\le C\|\vec v(0)\|_{\mathcal{L}^{m^*}}, \tag{2-30}$$

where

$$\gamma_1=\frac{5m-2}{5m(2m+1)},\quad \gamma_2=\frac{2}{5},$$

so that $\gamma_1\alpha m^*+\gamma_2=\alpha-2$. Let

$$q=\frac{8m(2m+1)}{8m^2-11m+6},\quad \sigma=\frac{8m(2m+1)}{5m-2}.$$

Then (2-29) yields

$$\Big(\int_{-\infty}^{+\infty}\Big(\int_0^{+\infty}|v(t,r)\,r^{\gamma_1}|^{\sigma}r^{\gamma_2}\,dr\Big)^{\frac{q}{\sigma}}dt\Big)^{\frac{1}{q}}\le C\|\vec v(0)\|_{\mathcal{L}^2}. \tag{2-31}$$

In view of

$$\frac{1}{m}=\frac{\theta}{2}+\frac{1-\theta}{m^*},\quad \frac{1}{2m+1}=\frac{\theta}{q}+\frac{1-\theta}{\alpha m^*},\quad \frac{1}{m(2m+1)}=\frac{\theta}{\sigma}+\frac{1-\theta}{\alpha m^*},\quad \theta=\frac{1}{m-1},$$

and the fact that $\gamma_1 m(2m+1)+\gamma_2=m$, we obtain (2-12) by interpolating (2-30) and (2-31); see Theorem 5.1.2 in [Bergh and Löfström 1976].

If $1 < m < 2$, we set

$$m^*=\frac{m+1}{2},\quad \alpha=\frac{8(2m+1)}{3m+5},\quad \theta=\frac{2(m-1)}{m(3-m)},$$
$$q=\frac{8(2m+1)}{10-m},\quad \sigma=\frac{8(2m+1)}{3m-2},$$
$$\gamma_1=\frac{3m-2}{6m^2+11m+4}=\frac{3m-2}{(2m+1)(3m+4)},\quad \gamma_2=\frac{6m}{3m+4}.$$

One can verify that (2-30) and (2-31) along with the interpolation relations as in the first case remain valid. □

Using the same argument as above and (2-26), we obtain Proposition 2.7.

We conclude this subsection with some additional Strichartz-type estimates that will be useful in the construction of the profile decomposition in Section 3 and follow from Proposition 2.8 and (2-27).

Proposition 2.12. *Assume $m > 2$ and $v(t,x)$ is the solution of the Cauchy problem (2-3) with radial initial data $(v_0, v_1) \in \mathcal{L}^m$. Let*

$$a = \frac{2m(m-1)(m+2)}{m^2+3m-2}, \qquad b = \frac{2m(m-1)(m+2)}{m-2}.$$

Then there exists a constant C such that

$$\left(\int_{-\infty}^{+\infty}\left(\int_0^{+\infty} |v(t,r)|^b r^m \, dr\right)^{\frac{a}{b}} dt\right)^{\frac{1}{a}} \leq C\|\vec{v}(0)\|_{\mathcal{L}^m}. \tag{2-32}$$

Proof. Indeed, from (2-14), we have

$$\left(\int_{-\infty}^{+\infty}\int_0^{+\infty} |v(t,r)|^{2m(m+2)} r^m \, dr\, dt\right)^{\frac{1}{2m(m+2)}} \leq C\|\vec{v}(0)\|_{\mathcal{L}^{2m}}. \tag{2-33}$$

Interpolating (2-33) with (2-27), we are done. □

The choice of (a,b) above is not suitable in the case $m < 2$, where we will use the following estimates:

Proposition 2.13. *Assume $1 < m < 2$ and $v(t,x)$ is the solution of the Cauchy problem (2-3) with radial initial data $(v_0, v_1) \in \mathcal{L}^m$. Let*

$$a = \frac{m(m+2)(3-m)}{m^2-m+2}, \qquad b = \frac{m(m+2)(3-m)}{2(2-m)}.$$

Then there exists a constant C such that

$$\left(\int_{-\infty}^{+\infty}\left(\int_0^{+\infty} |v(t,r)|^b r^m \, dr\right)^{\frac{a}{b}} dt\right)^{\frac{1}{a}} \leq C\|\vec{v}(0)\|_{\mathcal{L}^m}. \tag{2-34}$$

Proof. Let $m^* = \frac{m+1}{2}$. From (2-14), we have

$$\left(\int_{-\infty}^{+\infty}\int_0^{+\infty} |v(t,r)|^{m^*(m+2)} r^m \, dr\, dt\right)^{\frac{1}{(m+2)m^*}} \leq C\|\vec{v}(0)\|_{\mathcal{L}^{m^*}}. \tag{2-35}$$

Interpolating (2-35) with (2-27), we are done. □

Remark 2.14. In both propositions, we have $m < a < 2m+1 < \frac{b}{m} < \infty$.

Remark 2.15. The interpolations we used in the above two propositions are based on the complex method. In fact, we used Theorems 5.1.1 and 5.1.2 in [Bergh and Löfström 1976].

Remark 2.16. Notice that when $m = 2$, we have $(a,b) = (2,\infty)$ coincides with the end-point Strichartz estimate (2-27).

2C. ***Local well-posedness.*** Consider here the Cauchy problem for the nonlinear wave equations (1-1), (1-2), with $(u_0, u_1) \in \mathcal{L}^m$, $m > 1$. In this subsection, we prove the following small-data well-posedness statement, which implies Theorem 2:

Proposition 2.17. *There exists $\delta_0 > 0$ such that if $0 \in I \subset \mathbb{R}$ is an interval and*

$$\|S_{\mathrm{L}}(t)(u_0, u_1)\|_{S(I)} = \delta \le \delta_0, \tag{2-36}$$

then there exists a unique solution $u \in S(I)$ to the Cauchy problem (1-1), (1-2) for $t \in I$ such that $\vec{u} \in C^0(I, \mathcal{L}^m)$. Moreover,

$$\|u\|_{S(I)} \le 2\delta \tag{2-37}$$

and we have

$$\sup_{t \in I} \|\vec{u}(t)\|_{\mathcal{L}^m} \le C_m\big(\|(u_0, u_1)\|_{\mathcal{L}^m} + \delta^{2m+1}\big). \tag{2-38}$$

Remark 2.18. From the assumption on the initial data and the Strichartz-type inequality (2-12), we see that for each $(u_0, u_1) \in \mathcal{L}^m$ and $\delta > 0$, there is an interval $I = I(u_0, u_1, \delta)$ such that (2-36) holds. Using this observation and standard arguments, it is easy to construct from Proposition 2.17 a maximal solution of (1-1), (1-2) that satisfies the conclusion of Theorem 2.

Proof. Let C_0 be the constant in the estimates (2-12) and (2-13) . Consider

$$\mathfrak{X} = \{v \text{ on } \mathbb{R} \times \mathbb{R}^3 \mid v(t, x) = v(t, |x|),\ \|v\|_{S(I)} \le 2\delta\},$$

where

$$0 < \delta < \min\big(C_0^{-\frac{1}{p-1}} 2^{-\frac{p}{p-1}}, 2^{-\frac{p+2}{p-1}} (pC_0)^{-\frac{1}{p-1}}\big), \quad p = 2m + 1.$$

Define

$$\Phi_{(u_0,u_1)}(v) = S_{\mathrm{L}}(t)(u_0, u_1) + \iota \int_0^t \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} |v|^{2m} v(s)\, ds. \tag{2-39}$$

If $v, w \in \mathfrak{X}$, we have from (2-13)

$$\|\Phi_{(u_0,u_1)}(v)\|_{S(I)} \le \delta + C_0(2\delta)^p \le 2\delta,$$

and by the Hölder inequality

$$\begin{aligned} \big\|\Phi_{(u_0,u_1)}(v) - \Phi_{(u_0,v_0)}(w)\big\|_{S(I)} &\le 2pC_0\big(\|v\|_{S(I)}^{p-1} + \|w\|_{S(I)}^{p-1}\big)\|v - w\|_{S(I)} \\ &\le 4\, p\, C_0(2\delta)^{p-1}\|v - w\|_{S(I)} \\ &\le \tfrac{1}{2}\|v - w\|_{S(I)} \end{aligned}$$

for all $v, w \in \mathfrak{X}$. Thus, there exists a unique fixed point $u \in \mathfrak{X}$ such that

$$u = \Phi_{u_0,u_1}(u).$$

Note that (2-37) follows from the construction and (2-38) follows from the energy estimates and (2-37). □

2D. ***Exterior long-time perturbation theory.*** We conclude this section by a long-time perturbation theory result for (1-1) with initial data in $\mathcal{L}^m$. Taking into account the finite speed of propagation, we will give a statement that works as well when the estimates are restricted to the exterior $\{r > A + |t|\}$ of a wave cone. This generalization will be very useful when using the channels of energy arguments.

Lemma 2.19. *Let $M > 0$. There exist $\varepsilon_M > 0$, $C_M > 0$ with the following properties. Let $T \in (0, +\infty]$, $u, \tilde{u} \in S((0, T))$ such that $\vec{u}, \vec{\tilde{u}} \in C^0([0, T), \mathcal{L}^m)$. Assume that u is a solution of (1-1), (1-2) on $[0, T)$ and that*[3]

$$\begin{cases} \partial_t^2 \tilde{u} - \Delta \tilde{u} = \iota \mathbb{1}_{\{r \geq (A+|t|)_+\}} |\tilde{u}|^{2m} \tilde{u} + e, \\ \tilde{u}_{\restriction t=0} = (\tilde{u}_0, \tilde{u}_1), \end{cases} \tag{2-40}$$

where $e \in L^1_t L^m_x(r^m\, dr)$, $A \in \mathbb{R} \cup \{-\infty\}$. Let

$$R_{\mathrm{L}}(t) = S_{\mathrm{L}}(t)((u_0, u_1) - (\tilde{u}_0, \tilde{u}_1)).$$

Assume

$$\|\tilde{u}\|_{S(\{t \in (0,T),\, r \geq (A+|t|)_+\})} \leq M, \tag{2-41}$$

$$\int_0^T \left(\int_{(A+|t|)_+}^{+\infty} |r\, e|^m\, dr \right)^{\frac{1}{m}} dt + \|R_{\mathrm{L}}\|_{S(\{t \in [0,T),\, r \geq (A+|t|)_+\})} = \varepsilon \leq \varepsilon_M. \tag{2-42}$$

Then $u(t) = \tilde{u}(t) + R_{\mathrm{L}}(t) + \epsilon(t)$ with

$$\|\epsilon\|_{S(\{t[0,T),\, r \geq (A+|t|)_+\})} + \sup_{t \in [0,T)} \int_{(A+|t|)_+} |r \partial_{t,r} \epsilon|^m\, dr \leq C_M \varepsilon.$$

In the lemma, we have set $(A + |t|)_+ = \max(0, A + |t|)$. By convention, if $A = -\infty$, this quantity equals 0 for all t. Note that the case $A = -\infty$ corresponds to the usual long-time perturbation theory statement;[4] see, e.g., [Tao and Visan 2005].

Sketch of the proof. We let, for $t \in [0, T)$,

$$\mathfrak{E}(t) = \left(\int_{(A+|t|)_+}^{+\infty} |\epsilon(t, r)|^{(2m+1)m} r^m\, dr \right)^{\frac{1}{(2m+1)m}},$$

$$\widetilde{\mathfrak{U}}(t) = \left(\int_{(A+|t|)_+}^{+\infty} |\tilde{u}(t, r)|^{(2m+1)m} r^m\, dr \right)^{\frac{1}{(2m+1)m}},$$

$$\mathfrak{R}(t) = \left(\int_{(A+|t|)_+}^{+\infty} |R_{\mathrm{L}}(t, r)|^{(2m+1)m} r^m\, dr \right)^{\frac{1}{(2m+1)m}}.$$

By the assumptions (2-41), (2-42),

$$\|\widetilde{\mathfrak{U}}\|_{L^{2m+1}(0,T)} \leq M, \quad \|\mathfrak{R}\|_{L^{2m+1}(0,T)} \leq \varepsilon.$$

[3] in the sense that $\tilde{u}$ satisfies the usual integral equation

[4] Traditionally the "linear part" of the solution $R_{\mathrm{L}}(t)$ is incorporated into $\tilde{u}$. For convenience we preferred to distinguish between these two components.

Since

$$(\partial_t^2 - \Delta)\epsilon = \iota(|u|^{2m}u - |\tilde{u}|^{2m}\tilde{u}) + e,$$

we obtain by (2-11), Strichartz estimates and finite speed of propagation that for all $\theta \in [0, T)$,

$$\sup_{t\in[0,\theta]}\left[\left(\int_{(A+|t|)_+}^{+\infty} |r\partial_{t,r}\epsilon|^m \,dr\right)^{\frac{1}{m}} + \|\vec{\epsilon}(t)\|_{\mathcal{L}^m} + \|\mathfrak{E}(t)\|_{L^{2m+1}}\right] \\ \leq C\int_0^\theta \left(\int_{(A+|t|)_+}^{+\infty} \left(\left||\tilde{u}|^{2m}\tilde{u} - |u|^{2m}u\right|^m + |e|^m\right)r^m \,dr\right)^{\frac{1}{m}} dt. \tag{2-43}$$

We have

$$\int_0^\theta \left(\int_{(A+|t|)_+}^{+\infty} |e|^m r^m \,dr\right)^{\frac{1}{m}} dt \leq \varepsilon$$

and, using Hölder's inequality

$$\begin{aligned}\int_0^\theta &\left(\int_{(A+|t|)_+}^{+\infty} \left||\tilde{u}|^{2m}\tilde{u} - |u|^{2m}u\right|^m r^m \,dr\right)^{\frac{1}{m}} dt \\ &\lesssim \int_0^\theta (\mathfrak{E}(t) + \mathfrak{R}(t))(\widetilde{\mathfrak{U}}(t)^{2m} + \mathfrak{R}(t)^{2m} + \mathfrak{E}(t)^{2m})\,dt \\ &\leq C\left(\|\mathfrak{E}\|^{2m+1}_{L^{2m+1}(0,\theta)} + \|\mathfrak{R}\|^{2m+1}_{L^{2m+1}(0,\theta)} + \int_0^\theta \mathfrak{R}(t)\widetilde{\mathfrak{U}}(t)^{2m}\,dt + \int_0^\theta \mathfrak{E}(t)\widetilde{\mathfrak{U}}(t)^{2m}\,dt\right) \\ &\leq C\left(\|\mathfrak{E}\|^{2m+1}_{L^{2m+1}(0,\theta)} + \varepsilon^{2m+1} + M^{2m}\varepsilon + \int_0^\theta \mathfrak{E}(t)\widetilde{\mathfrak{U}}(t)^{2m}\,dt\right).\end{aligned}$$

Collecting the above, we obtain, for all $\theta \in [0, T)$,

$$\|\mathfrak{E}\|_{L^{2m+1}(0,\theta)} \leq C\left(\varepsilon + \varepsilon^{2m+1} + M^{2m}\varepsilon + \|\mathfrak{E}\|^{2m+1}_{L^{2m+1}(0,\theta)} + \int_0^\theta \mathfrak{E}(t)\widetilde{\mathfrak{U}}(t)^{2m}\,dt\right).$$

This is a Grönwall-type inequality classical in this context. Using, e.g., Lemma 8.1 in [Fang et al. 2011], we deduce that for all $\theta \in [0, T)$,

$$\|\mathfrak{E}\|_{L^{2m+1}(0,\theta)} \leq C\left(\varepsilon + \varepsilon^{2m+1} + M^{2m}\varepsilon + \|\mathfrak{E}\|^{2m+1}_{L^{2m+1}(0,\theta)}\right)\Phi(CM^{2m}),$$

where $\Phi(s) = 2\Gamma(3 + 2s)$, and Γ is the usual Gamma function. Using a standard bootstrap argument, we deduce, assuming that $\varepsilon \leq \varepsilon_M$ for some small ε_M,

$$\|\mathfrak{E}\|_{L^{2m+1}(0,\theta)} \leq C_M\varepsilon,$$

and going back to (2-43) and the computations that follow this inequality, we obtain also the desired bound on the $\mathcal{L}^m$ norm of ϵ. □

3. Profile decomposition

3A. *Linear profile decomposition.* The main result of this section is the following:

Theorem 3.1. *Let $(u_{\mathrm{L},n})_n$ be a sequence of radial solutions of (1-3) such that $(\vec{u}_{\mathrm{L},n}(0))_n$ is bounded in $\mathcal{L}^m$. Then there exists a subsequence of $(u_{\mathrm{L},n})_n$ (still denoted by $(u_{\mathrm{L},n})_n$) and, for all $j \geq 1$, a solution U_{L}^j of (1-3) with initial data (U_0^j, U_1^j) in $\mathcal{L}^m$ and sequences $(\lambda_{j,n})_n \in (0,\infty)^{\mathbb{N}}$, $(t_{j,n})_n \in \mathbb{R}^{\mathbb{N}}$ such that the following properties hold*:

- Pseudo-orthogonality. *For all $j, k \geq 1$, one has*
$$j \neq k \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{\lambda_{j,n}}{\lambda_{k,n}} + \frac{\lambda_{k,n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{k,n}|}{\lambda_{j,n}} = +\infty. \tag{3-1}$$
- Weak convergence. *For all $j \geq 1$,*
$$\left(\lambda_{j,n}^{\frac{1}{m}} u_{\mathrm{L},n}(t_{j,n}, \lambda_{j,n}\,\cdot), \lambda_{j,n}^{\frac{1}{m}+1} \partial_t u_{\mathrm{L},n}(t_{j,n}, \lambda_{j,n}\,\cdot)\right) \xrightarrow[n\to\infty]{} (U_0^j, U_1^j), \tag{3-2}$$
weakly in $\mathcal{L}^m$.
- Bessel-type inequality. *For all $J \geq 1$,*
$$\lim_{n\to\infty} E_m(u_{0,n}, u_{1,n}) - \sum_{j=1}^{J} E_m(\vec{U}_{\mathrm{L}}^j(0)) \geq 0. \tag{3-3}$$
- Vanishing in the dispersive norm.
$$\lim_{J\to\infty} \lim_{n\to\infty} \|w_n^J\|_{S(\mathbb{R})} = 0, \tag{3-4}$$

In the above, we have taken
$$w_n^J(t,x) = u_{\mathrm{L},n}(t,x) - \sum_{j=1}^{J} U_{\mathrm{L},n}^j(t,x), \tag{3-5}$$
$$U_{\mathrm{L},n}^j(t,x) = \frac{1}{\lambda_{j,n}^{\frac{1}{m}}} U_{\mathrm{L}}^j\left(\frac{t - t_{j,n}}{\lambda_{j,n}}, \frac{x}{\lambda_{j,n}}\right). \tag{3-6}$$

Theorem 3.1 generalizes (in the radial setting) the profile decomposition of [Bahouri and Gérard 1999] to sequences that are bounded in $\mathcal{L}^m$ instead of the classical energy space. The only difference between the two decompositions is the fact that the Pythagorean expansion proved in that paper is replaced by the weaker property (3-3). One cannot hope, in this context, to have an exact Pythagorean expansion; see the example on p. 387 of [Jaffard 1999].

The proof of Theorem 3.1 is based on the following two propositions, which we will prove in Sections 3B and 3C respectively.

Proposition 3.2. *Let $(u_{\mathrm{L},n})_n$ be a sequence of radial solutions to the linear wave equation and set $(u_{0,n}, u_{1,n}) = \vec{u}_{\mathrm{L},n}(0)$. Assume for $m \in (1,+\infty)$, the sequence $(\vec{u}_{\mathrm{L},n}(0))_n$ is bounded in $\mathcal{L}^m$ and that for* *all sequences* $(\lambda_n)_n \in (0,\infty)^{\mathbb{N}}$ *and* $(t_n)_n \in \mathbb{R}^{\mathbb{N}}$,

$$\left(\frac{1}{\lambda_n^{\frac{1}{m}}} u_{\mathrm{L},n}\left(\frac{-t_n}{\lambda_n}, \frac{\cdot}{\lambda_n}\right), \frac{1}{\lambda_n^{1+\frac{1}{m}}} \partial_t u_{\mathrm{L},n}\left(\frac{-t_n}{\lambda_n}, \frac{\cdot}{\lambda_n}\right)\right)_n \tag{3-7}$$

converges weakly to 0 *in* $\mathcal{L}^m$ *as* $n \to +\infty$. *Then*

$$\lim_{n\to+\infty} \|u_{\mathrm{L},n}\|_{S(\mathbb{R})} = 0. \tag{3-8}$$

Proposition 3.3. *Let* $J \geq 1$ *and* $(U_{\mathrm{L}}^j)_{j=1,\dots,J}$ *be solutions of the linear wave equations with initial data in* $\mathcal{L}^m$. *For all* $j = 1, \dots, J$, *we let* $(\lambda_{j,n})_n \in (0,\infty)^{\mathbb{N}}$ *and* $(t_{j,n})_n \in \mathbb{R}^{\mathbb{N}}$ *be sequences of parameters that satisfy the pseudo-orthogonality property* (3-1). *Let* $(u_{\mathrm{L},n})$ *be a sequence of solutions of the linear wave equation with initial data in* $\mathcal{L}^m$. *Let* w_n^J *be defined by* (3-5), (3-6) *and assume that for all* $j \in \{1, \dots, J\}$,

$$\left(\lambda_{j,n}^{\frac{1}{m}} w_n^J(t_{j,n}, \lambda_{j,n}\,\cdot), \lambda_{j,n}^{\frac{1}{m}+1} \partial_t w_n^J(t_{j,n}, \lambda_{j,n}\,\cdot)\right) \xrightarrow[n\to\infty]{} 0 \quad \textit{weakly in } \mathcal{L}^m. \tag{3-9}$$

Then the Bessel-type inequality (3-3) *holds.*

Proof of the theorem. The proof of Theorem 3.1, assuming Proposition 3.2 and 3.3, is quite standard, at least in the Hilbertian setting. We give it for the sake of completeness. We mainly need to check that it is harmless that we have only a Bessel-type inequality (3-3) in the $\mathcal{L}^m$ setting, which is not Hilbertian, instead of a more precise Pythagorean expansion.

We construct the profiles U_{L}^j and the parameters $\lambda_{j,n}$, $t_{j,n}$ by induction.

Let $J \geq 1$ and assume that for $1 \leq j \leq J-1$, we have constructed profiles U_{L}^j such that (3-1) and (3-2) hold after extraction of a subsequence in n (if $J = 1$ we do not assume anything and set $w_n^0 = u_{\mathrm{L},n}$). Note that this implies (3-3) by Proposition 3.3. Let $\mathcal{A}_J$ be the set of $(U_0, U_1) \in \mathcal{L}^m$ such that there exist sequences $(\lambda_n)_n$, $(t_n)_n$ of parameters such that, after extraction of a subsequence,

$$\left(\lambda_n^{\frac{1}{m}} w_n^{J-1}(t_n, \lambda_n\,\cdot), \lambda_n^{\frac{1}{m}+1} \partial_t w_n^{J-1}(t_n, \lambda_n\,\cdot)\right) \xrightarrow[n\to\infty]{} (U_0, U_1)$$

weakly in $\mathcal{L}^m$, where w_n^{J-1} is defined by (3-5). We distinguish two cases.

Case 1: $\mathcal{A}_J = \{(0,0)\}$. In this case we stop the process and let $U_{\mathrm{L}}^j = 0$ for all $j \geq J$.

Case 2: There exists a nonzero element in $\mathcal{A}_J$. In this case, we choose $(U_0^J, U_1^J) \in \mathcal{A}_J$ such that

$$E_m(U_0^J, U_1^J) \geq \tfrac{1}{2} \sup_{(U_0,U_1)\in\mathcal{A}_J} E_m(U_0, U_1), \tag{3-10}$$

and we choose sequences $(\lambda_{J,n})_n$ and $(t_{J,n})_n$ such that, (after extraction of subsequences in n),

$$\left(\lambda_{J,n}^{\frac{1}{m}} w_n^{J-1}(t_{J,n}, \lambda_{J,n}\,\cdot), \lambda_{J,n}^{\frac{1}{m}+1} \partial_t w_n^{J-1}(t_{J,n}, \lambda_{J,n}\,\cdot)\right) \xrightarrow[n\to\infty]{} (U_0^J, U_1^J) \tag{3-11}$$

weakly in $\mathcal{L}^m$. Note that (3-2) holds for $j = J$ thanks to (3-11). Furthermore, (3-1) for $j \in \{1, \dots, J-1\}$, $k = J$ follows from (3-2) (for $j \in \{1, \dots, J-1\}$), (3-11) and the fact that $(U_0^J, U_1^J) \neq (0,0)$. Finally, as already observed, (3-3) is a consequence of (3-1), (3-2) and Proposition 3.3.

If there exists a $J \geq 1$ such that Case 1 above holds, then we are done: indeed, in this case, w_n^J does not depend on J for large n, and (3-4) is an immediate consequence of the definition of $\mathcal{A}_J$ and Proposition 3.2.

Next assume that Case 2 holds for all $J \geq 1$. Using a diagonal extraction argument, we obtain, for all $j \geq 1$, profiles U_{L}^j, and sequences of parameters $(\lambda_n^j)_n$ and $(t_n^j)_n$ such that (3-1), (3-2) and (3-3) hold for all j, k, J. It remains to prove (3-4). In view of Proposition 3.2, it is sufficient to prove

$$\lim_{J\to\infty} \sup_{(A_0,A_1)\in\mathcal{A}_J} \|(A_0, A_1)\|_{\mathcal{L}^m} = 0.$$

This follows from (3-10), the equivalence between $E_m^{\frac{1}{m}}$ and the $\mathcal{L}^m$ norm, and the fact that, by (3-3),

$$\lim_{J\to\infty} E_m(U_0^J, U_1^J) = 0. \qquad \square$$

3B. ***Convergence to* 0 *of the Strichartz norm.*** First of all, let us introduce the notation $\dot{B}^s_{\infty,\infty}(\mathbb{R}^d)$ for the homogeneous Besov space on $\mathbb{R}^d$, which is defined as follows. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a radial function, supported in $\left\{\xi \in \mathbb{R}^d : \frac{1}{2} \leq |\xi| \leq 2\right\}$ and such that

$$\sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.$$

We denote by $\dot{\Delta}_j$ the Littlewood–Paley projector

$$\dot{\Delta}_j f(x) = \left(\psi(2^{-j}\cdot)\hat{f}(\cdot)\right)^\vee(x), \quad j \in \mathbb{Z},$$

where

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\, dx$$

is the Fourier transform on $\mathbb{R}^d$ and we use

$$g^\vee(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(\xi) e^{ix\cdot\xi}\, d\xi$$

to denote the inverse Fourier transform. For a tempered distribution f on $\mathbb{R}^d$, we set

$$\|f\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^d)} := \sup_{j\in\mathbb{Z}} 2^{js} \left\|\dot{\Delta}_j f\right\|_{L^\infty(\mathbb{R}^d)}.$$

If $\|f\|_{\dot{B}^s_{\infty,\infty}} < +\infty$, we say f belongs to $\dot{B}^s_{\infty,\infty}$.

We have the following refined Sobolev inequality in weighted norms.

Lemma 3.4. *Let $\omega(x) \in A_p$ with $1 < p < +\infty$; i.e.,*

$$\sup_B \left(\frac{1}{|B|}\int_B \omega(x)\, dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-\frac{1}{p-1}}\, dx\right)^{p-1} < +\infty, \tag{3-12}$$

where the supremum is taken over all balls B in $\mathbb{R}^d$. If $\nabla f \in L^p(\mathbb{R}^d, \omega(x)dx)$ and $f \in \dot{B}^{-\beta}_{\infty,\infty}(\mathbb{R}^d)$, then

$$\|f\|_{L^q(\mathbb{R}^d,\omega)} \le C\|\nabla f\|^{\theta}_{L^p(\mathbb{R}^d,\omega)}\|f\|^{1-\theta}_{\dot{B}^{-\beta}_{\infty,\infty}(\mathbb{R}^d)}, \tag{3-13}$$

where $1 < p < q < +\infty$, $\theta = \frac{p}{q}$, $\beta = \frac{\theta}{1-\theta}$.

The refined Sobolev inequality (3-13) in weighted norms was proved in [Chamorro 2011], where the author considered more general situations with the underlying domain $\mathbb{R}^d$ replaced by stratified Lie groups. The above lemma follows immediately since the Euclidean space $\mathbb{R}^d$ with its natural group structure is an example of a stratified Lie group. Notice that $1 \in A_p$, and one recovers the classical result on the refined Sobolev inequalities established first in [Gerard et al. 1997].

With Lemma 3.4 at hand, we are ready to prove the Proposition 3.2.

Proof. Since $((u_{0,n}, u_{1,n}))_n$ is bounded in $\mathcal{L}^m$, there exists $A > 0$ such that

$$\int_0^{+\infty} |r\partial_r u_{0,n}(r)|^m\, dr + \int_0^{+\infty} |ru_{1,n}(r)|^m\, dr \le A < +\infty$$

for all n.

Assuming (3-8) fails, we have for some constant c_0 having the property that $0 < c_0 \le C\, A^{\frac{1}{m}}$, that

$$\limsup_{n\to\infty} \|u_{\mathrm{L},n}\|_{L_t^{2m+1}(\mathbb{R},\, L_x^{m(2m+1)}(\mathbb{R}^3, r^{m-2}))} = c_0, \tag{3-14}$$

where C is the constant in (2-12), (2-32) and (2-34). From (2-32), (2-34) and Hölder's inequality, we know that up to a subsequence, there exists some $\theta \in (0,\ 1)$ such that

$$\lim_{n\to\infty} \|u_{\mathrm{L},n}\|_{L_t^{\infty}(\mathbb{R},\, L_x^{m(m+1)}(\mathbb{R}^3, r^{m-2}))} \ge \left(\frac{c_0}{(CA^{\frac{1}{m}})^{\theta}}\right)^{\frac{1}{1-\theta}}. \tag{3-15}$$

For $m > 1$, we denote by $[m]$ the greatest integer less than or equal to m and by $\{m\} := m - [m]$ the fractional part of m. Notice that $\{m\} \in [0, 1)$ and $\{m\} = 0$ if and only if $m \in \mathbb{N}$.

Let $d = [m] + 1$ and $\omega(x) = |x|^{\gamma}$ with $\gamma = \{m\}$, $x \in \mathbb{R}^d$. It is easy to see that $\omega \in A_m$, see for example [Grafakos 2014], and we have the following refined Sobolev inequality in view of Lemma 3.4:

$$\|f\|_{L^{m(m+1)}(\mathbb{R}^d,|x|^{\gamma})} \le C_0\|\nabla f\|^{\frac{1}{m+1}}_{L^m(\mathbb{R}^d,|x|^{\gamma})}\|f\|^{\frac{m}{m+1}}_{\dot{B}^{-1/m}_{\infty,\infty}(\mathbb{R}^d)}. \tag{3-16}$$

If we apply (3-16) to functions $u_{\mathrm{L},n}(t, |x|)$ with respect to the spatial variable $x \in \mathbb{R}^{[m]+1}$, we obtain by transferring the formula into polar coordinates

$$\int_0^{+\infty} |u_{\mathrm{L},n}(t,r)|^{m(m+1)} r^m\, dr \le C_0^{m(m+1)} \int_0^{+\infty} |r\partial_r u_{\mathrm{L},n}(t,r)|^m\, dr \\ \times \sup_{j\in\mathbb{Z}}\ \sup_{x\in\mathbb{R}^{[m]+1}} \left(2^{-\frac{j}{m}} \int_{\mathbb{R}^{[m]+1}} \psi^{\vee}(y)\, u_{\mathrm{L},n}(t, |x - 2^{-j}y|)\, dy\right)^{m^2}. \tag{3-17}$$

In view of the conservation of the $\mathcal{L}^m$-energy, and the fact that the norms $\|\cdot\|_{\mathcal{L}^m}$ and $(E_m)^{\frac{1}{m}}$ are equivalent, there exists some $N > 0$ such that if $n \geq N$

$$\sup_{t\in\mathbb{R}} \sup_{j\in\mathbb{Z}} \sup_{x\in\mathbb{R}^{[m]+1}} \left| \int_{\mathbb{R}^{[m]+1}} 2^{-\frac{j}{m}} u_{\mathrm{L},n}(t, |x-2^{-j}y|)\, \psi^{\vee}(y)\, dy \right| \geq \delta_0, \tag{3-18}$$

where

$$\delta_0 = \tfrac{1}{2} c_0^{\frac{m+1}{(1-\theta)m}} (C^{\frac{\theta}{1-\theta}} C_0)^{-\frac{m+1}{m}} C_m^{-\frac{1}{m}} A^{-\frac{m+1}{m^2}\left(\frac{\theta}{1-\theta}+\frac{1}{m+1}\right)} > 0,$$

and C_m is the constant in (2-9).

As a result of (3-18), we have a family of $(t_n^0)_n$ in $\mathbb{R}^{\mathbb{N}}$, a sequence of $(j_n)_n \in \mathbb{Z}^{\mathbb{N}}$ and $(x_n)_n$ in $(\mathbb{R}^{[m]+1})^{\mathbb{N}}$such that

$$\left| \int_{\mathbb{R}^{[m]+1}} 2^{-\frac{j_n}{m}} u_{\mathrm{L},n}(t_n^0, |x_n - 2^{-j_n} y|) \psi^{\vee}(y)\, dy \right| \geq \frac{\delta_0}{2}, \quad n \geq N.$$

Setting $\varphi(\cdot) = \psi^{\vee}(\cdot)$, $\lambda_n = 2^{j_n}$, $t_n = -t_n^0 \lambda_n$, and $y_n = \lambda_n x_n$, we will obtain a contradiction by letting $n \to \infty$ provided, up to some subsequences,

$$\int_{\mathbb{R}^{[m]+1}} \frac{1}{\lambda_n^{\frac{1}{m}}} u_{\mathrm{L},n}\left(\frac{-t_n}{\lambda_n}, \frac{|y-y_n|}{\lambda_n} \right) \varphi(y)\, dy \to 0, \quad n \to +\infty. \tag{3-19}$$

To prove this, we divide the argument into two cases.

Case 1: $\limsup_{n\to\infty} |y_n| = +\infty$. Up to a subsequence, we may assume

$$0 < |y_1| \ll |y_2| \ll \cdots \ll |y_n| \ll |y_{n+1}| \cdots \to +\infty, \quad n \to +\infty. \tag{3-20}$$

Define

$$V_n(y) = \frac{1}{\lambda_n^{\frac{1}{m}}} u_{\mathrm{L},n}\left(-\frac{t_n}{\lambda_n}, \frac{|y|}{\lambda_n} \right).$$

Note that V_n is a radial function on $\mathbb{R}^{[m]+1}$. Then from the radial Sobolev embedding (see (4) in Proposition 2.2), we have

$$|V_n(y)| \leq \frac{1}{|y|^{\frac{1}{m}}} \left(\int_0^{+\infty} \left| r\partial_r u_{\mathrm{L},n}\left(-\frac{t_n}{\lambda_n}, r \right) \right|^m dr \right)^{\frac{1}{m}} \leq C_m \left(\frac{A}{|y|} \right)^{\frac{1}{m}} \tag{3-21}$$

for all n. As a consequence, (3-19) is bounded by

$$c_n := \int_{\mathbb{R}^{[m]+1}} |y-y_n|^{-\frac{1}{m}} |\varphi(y)|\, dy, \tag{3-22}$$

and it suffices to show

$$\lim_{n\to+\infty} c_n = 0. \tag{3-23}$$

We write

$$c_n = \int_{|y-y_n|\leq 1} |y-y_n|^{-\frac{1}{m}} |\varphi(y)|\, dy + \int_{|y-y_n|\geq 1} |y-y_n|^{-\frac{1}{m}} |\varphi(y)|\, dy.$$

The first term is bounded by

$$\Big(\sup_{|y-y_n|\le 1}|\varphi(y)|\Big)\int_{|z|\le 1}|z|^{-\frac{1}{m}}\underset{n\to\infty}{\longrightarrow}0,$$

while the second one goes to zero by dominated convergence. Hence (3-23).

Case 2: There exists $c>0$ such that $|y_n|\le c<+\infty$ for all n. We have, up to some subsequences, $y_n\to y_*$ as $n\to\infty$, where $y_*\in\mathbb{R}^{[m]+1}$ such that $|y_*|\le c$. Setting $\tau_n\varphi(\cdot)=\varphi(\cdot+y_n)$ and $\tau_*\varphi(\cdot)=\varphi(\cdot+y_*)$, we have

$$\tau_n\varphi\to\tau_*\varphi,\quad n\to+\infty,\quad \text{in } \mathcal{S}(\mathbb{R}^{[m]+1}). \tag{3-24}$$

From the condition that (3-7) converges weakly to zero in $\mathcal{L}^m$, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^{[m]+1}}V_n(x)\,\tau_*\varphi(x)\,dx=0.$$

In fact, considered as a function on $\mathbb{R}^3$, we have, by (3) in Proposition 2.2,

$$V_n\underset{n\to\infty}{\longrightarrow}0\quad\text{weakly in } L^{3m}(\mathbb{R}^3).$$

Furthermore,

$$\begin{aligned}\int_{\mathbb{R}^{[m]+1}}V_n(x)\,\tau_*\varphi(x)\,dx&=\int_0^{+\infty}\int_{S^{[m]}}\tau_*\varphi(r\omega)\,d\sigma(\omega)\,V_n(r)r^{[m]}\,dr\\&=\int_0^{+\infty}\underbrace{\Big(\int_{S^{[m]}}\tau_*\varphi(r\omega)\,d\sigma(\omega)\,r^{[m]-2}\Big)}_{:=\Psi(r)}V_n(r)r^2\,dr\underset{n\to\infty}{\longrightarrow}0,\end{aligned}$$

since $\Psi(r)$ can be considered as a radial function in $L^{(3m)'}(\mathbb{R}^3)$ for $1<m<+\infty$. On the other hand, we have by the fundamental theorem of calculus and integration by parts

$$\int_{\mathbb{R}^{[m]+1}}V_n(|y|)\,(\tau_n\varphi(y)-\tau_*\varphi(y))\,dy=\int_0^1\int_{\mathbb{R}^{[m]+1}}\big\langle\nabla V_n(y),(y_*-y_n)\varphi(y+s(y_n-y_*)+y_*)\big\rangle\,dy\,ds.$$

After using Hölder's inequality and the energy estimate, we see the term on the right-hand side is bounded by

$$C_mA^{\frac{1}{m}}|y_n-y_*|\int_0^1\Big(\int_{\mathbb{R}^{[m]+1}}\big|\varphi(y+s(y_n-y_*)+y_*)\big|^{\frac{m}{m-1}}|y|^{-\frac{m-[m]}{m-1}}\,dy\Big)^{\frac{m-1}{m}}\,ds.$$

Notice that $\varphi\in\mathcal{S}(\mathbb{R}^{[m]+1})$, $|y_*|\le c$ and $|y|^{-\frac{m-[m]}{m-1}}$ is integrable near the origin of $\mathbb{R}^{[m]+1}$ when $m>1$. We have

$$\lim_{n\to\infty}\int_{\mathbb{R}^{[m]+1}}V_n(y)\,\big(\tau_n\varphi(y)-\tau_*\varphi(y)\big)\,dy=0. \qquad\square$$

3C. *Bessel-type inequality.* In this subsection we prove Proposition 3.3.

We let $\{u_{L,n}\}_{n\in\mathbb{N}}$ and, for $1\le j\le J$, let U_L^j and $(\lambda_{j,n},t_{j,n})_n$ be as in Proposition 3.3, and define $U_{L,n}^j$ by (3-6) and w_n^J by (3-5).

First of all, we have the explicit formula for $[U_{\mathrm{L}}^j]_\pm(t,r)$

$$[U_{\mathrm{L}}^j]_+(t,r) = 2\dot{F}^j(t+r), \quad [U_{\mathrm{L}}^j]_-(t,r) = 2\dot{F}^j(t-r), \quad j \geq 1, \tag{3-25}$$

with

$$F^j(\sigma) = \frac{1}{2}\sigma U_0^j(|\sigma|) + \frac{1}{2}\int_0^{|\sigma|} \varrho\, U_1^j(\varrho)\, d\varrho.$$

In view of (2-7), one easily verifies that

$$[U_{\mathrm{L},n}^j]_\pm(t,r) = \frac{1}{\lambda_{j,n}^{\frac{1}{m}}}[U_{\mathrm{L}}^j]_\pm\left(\frac{t-t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}}\right).$$

Up to subsequences, we may assume, after translating in time and rescaling U_{L}^j if necessary,

$$j \geq 1, \quad \lim_{n\to\infty} -\frac{t_{j,n}}{\lambda_{j,n}} = \pm\infty \qquad \text{or} \qquad \text{for all } n, \quad t_{j,n} = 0. \tag{3-26}$$

Step 1: decoupling of linear profiles. In this step, we prove

$$\lim_{n\to+\infty} E_m\left(\sum_{j=1}^J \vec{U}_{\mathrm{L},n}^j(0)\right) = \sum_{j=1}^J E_m(\vec{U}_{\mathrm{L}}^j(0)). \tag{3-27}$$

Recall that for any solution u of the linear wave equation, we have

$$E_m(\vec{u}(0)) = E_m(\vec{u}(t)) = \sum_\pm \int_0^{+\infty} \left|[u]_\pm(t,r)\right|^m dr,$$

where $[u]_\pm$ is defined in (2-7). Hence (for constants $C > 0$ that depend on J and m, but not on n)

$$\begin{aligned}
\left|E_m\left(\sum_{j=1}^J U_{\mathrm{L},n}^j(0)\right) - \sum_{j=1}^J E_m(U_{\mathrm{L}}^j(0))\right| &= \left|E_m\left(\sum_{j=1}^J U_{\mathrm{L},n}^j(0)\right) - \sum_{j=1}^J E_m(U_{\mathrm{L},n}^j(0))\right| \\
&\leq C \sum_{\substack{j\neq k\\ \pm}} \int_0^{+\infty} \left|[U_{\mathrm{L},n}^j]_\pm(0,r)\right|^{m-1} \left|[U_{\mathrm{L},n}^k]_\pm(0,r)\right| dr \\
&\leq C \sum_{\substack{j\neq k\\ \pm}} \underbrace{\int_0^{+\infty} \left|\frac{1}{\lambda_{j,n}^{\frac{1}{m}}}\dot{F}^j\left(\frac{-t_{j,n}\pm r}{\lambda_{j,n}}\right)\right|^{m-1} \left|\frac{1}{\lambda_{k,n}^{\frac{1}{m}}}\dot{F}^k\left(\frac{-t_{k,n}\pm r}{\lambda_{k,n}}\right)\right| dr}_{I_{j,k,n}^\pm}.
\end{aligned}$$

We are thus reduced to proving that each of the terms $I_{j,k,n}^\pm$ $(j \neq k)$ goes to 0 as n goes to infinity. By density we may assume

$$U_0^j,\ U_1^j,\ U_0^k,\ U_1^k \in C_0^\infty,$$

and thus $\dot{F}^j, \dot{F}^k \in C_0^\infty$. We will only consider $I_{j,k,n}^+$, whereas the proof for $I_{j,k,n}^-$ is the same. Extracting subsequences and arguing by contradiction, we can distinguish without loss of generality between the following three cases.

Case 1: We assume $\lim_{n\to\infty}\frac{\lambda_{k,n}}{\lambda_{j,n}}=0$. By the change of variable $s=\frac{-t_{k,n}+r}{\lambda_{k,n}}$, we obtain

$$I^{+}_{j,k,n}=\int_{-\frac{t_{k,n}}{\lambda_{k,n}}}^{+\infty}\left(\frac{\lambda_{k,n}}{\lambda_{j,n}}\right)^{1-\frac{1}{m}}\left|\dot F^{j}\left(\frac{\lambda_{k,n}s+t_{k,n}-t_{j,n}}{\lambda_{j,n}}\right)\right|^{m-1}|\dot F^{k}(s)|\,ds\lesssim\left(\frac{\lambda_{k,n}}{\lambda_{j,n}}\right)^{1-\frac{1}{m}},\tag{3-28}$$

where we have used that $\dot F^{j}$ and $\dot F^{k}$ are bounded and compactly supported. Since $\frac{\lambda_{k,n}}{\lambda_{j,n}}$ goes to 0 as n goes to infinity, we are done.

Case 2: We assume $\lim_{n\to\infty}\frac{\lambda_{j,n}}{\lambda_{k,n}}=0$. We argue similarly by using the change of variable $s=\frac{-t_{j,n}+r}{\lambda_{j,n}}$.

Case 3: We assume that the sequence $\left(\frac{\lambda_{j,n}}{\lambda_{k,n}}+\frac{\lambda_{k,n}}{\lambda_{j,n}}\right)_n$ is bounded. We use as in Case 1 the change of variable $s=\frac{-t_{k,n}+r}{\lambda_{k,n}}$. By the pseudo-orthogonality condition (3-1) we see that

$$\lim_{n\to\infty}\frac{|t_{j,n}-t_{k,n}|}{\lambda_{j,n}}=+\infty,$$

and thus, as a consequence of the first line of (3-28), $I^{+}_{j,k,n}$ is 0 for large n, which concludes Step 1.

Step 2: end of the proof. For $1<m<+\infty$, we introduce the notation

$$\begin{aligned}\Phi^{j}_{n,0}(r)&=\frac{1}{2r}\sum_{\pm}\int_0^r\left|[U^{j}_{\mathrm{L},n}]_{\pm}(0,s)\right|^{m-2}[U^{j}_{\mathrm{L},n}]_{\pm}(0,s)\,ds,\\ \Phi^{j}_{n,1}(r)&=\frac{1}{2r}\sum_{\pm}\pm\left|[U^{j}_{\mathrm{L},n}]_{\pm}(0,r)\right|^{m-2}[U^{j}_{\mathrm{L},n}]_{\pm}(0,r),\end{aligned}$$

and let $\Phi^{j}_{n,\mathrm{L}}(t)$ be the solution of the linear wave equations with initial data $(\Phi^{j}_{n,0},\Phi^{j}_{n,1})\in\mathcal{L}^{m'}$, where $m'=\frac{m}{m-1}$. Then we have

$$[\Phi^{j}_{n,\mathrm{L}}]_{\pm}(0,r)=\left|[U^{j}_{\mathrm{L},n}]_{\pm}(0,r)\right|^{m-2}[U^{j}_{\mathrm{L},n}]_{\pm}(0,r),$$

and note that

$$E_m(\vec U^{j}_{\mathrm{L}}(0))=E_m(\vec U^{j}_{\mathrm{L},n}(0))=\int_0^{+\infty}\sum_{\pm}[\Phi^{j}_{n,\mathrm{L}}]_{\pm}(0)[U^{j}_{\mathrm{L},n}]_{\pm}(0)\,dr.\tag{3-29}$$

From the weak convergence condition satisfied by the remainder term w^{J}_{n}, we have by time translation and changing variables

$$\begin{aligned}\int_0^{+\infty}&\left([\Phi^{j}_{n,\mathrm{L}}]_{+}(0,r)[w^{J}_{n}]_{+}(0,r)+[\Phi^{j}_{n,\mathrm{L}}]_{-}(0,r)[w^{J}_{n}]_{-}(0,r)\right)dr\\ &=\int_0^{+\infty}\left|[U^{j}_{\mathrm{L}}]_{+}(0,r)\right|^{m-2}[U^{j}_{\mathrm{L}}]_{+}(0,r)\,\lambda_{j,n}^{\frac{1}{m}}[w^{J}_{n}]_{+}(t_{j,n},\lambda_{j,n}r)\,dr\\ &\qquad+\int_0^{+\infty}\left|[U^{j}_{\mathrm{L}}]_{-}(0,r)\right|^{m-2}[U^{j}_{\mathrm{L}}]_{-}(0,r)\,\lambda_{j,n}^{\frac{1}{m}}[w^{J}_{n}]_{-}(t_{j,n},\lambda_{j,n}r)\,dr,\end{aligned}$$

which goes to zero as $n\to+\infty$ for $1\le j\le J$. Furthermore,

$$\int_0^{+\infty}\left|[\Phi^{j}_{n,\mathrm{L}}]_{\pm}(0,r)[U^{k}_{\mathrm{L},n}]_{\pm}(0,r)\right|dr=\int_0^{+\infty}\left|[U^{j}_{\mathrm{L},n}]_{\pm}(0,r)\right|^{m-1}\left|[U^{k}_{\mathrm{L},n}]_{\pm}(0,r)\right|dr,$$

and, by Step 1, this goes to 0 as n goes to infinity if $j \neq k$. Hence from (3-29), we have

$$\sum_{j=1}^{J} E_m(\vec{U}_{\mathrm{L}}^j(0)) \\ = \lim_{n\to+\infty}\left[\int_0^{+\infty}[u_{\mathrm{L},n}]_+(0,r)\left(\sum_{j=1}^{J}[\Phi_{n,\mathrm{L}}^j]_+(0,r)\right)dr + \int_0^{+\infty}[u_{\mathrm{L},n}]_-(0,r)\left(\sum_{j=1}^{J}[\Phi_{n,\mathrm{L}}^j]_-(0,r)\right)dr\right],$$

which is bounded after using Hölder's inequality by

$$\left[\lim_{n\to+\infty} E_{m'}\left(\sum_{j=1}^{J}\vec{\Phi}_{n,\mathrm{L}}^j(0,r)\right)\right]^{\frac{1}{m}'}\left[\limsup_{n\to+\infty} E_m(\vec{u}_{\mathrm{L},n}(0))\right]^{\frac{1}{m}}.$$

Furthermore, by the decoupling property proved in Step 1 we obtain

$$\lim_{n\to+\infty} E_{m'}\left(\sum_{j=1}^{J}\vec{\Phi}_{n,\mathrm{L}}^j(0,r)\right) = \sum_{j=1}^{J} E_{m'}(\vec{\Phi}_{n,\mathrm{L}}^j(0)) = \sum_{j=1}^{J} E_m(\vec{U}_{\mathrm{L}}^j(0))$$

and this concludes the result.

3D. ***Approximation by sum of profiles.*** We next write a lemma approximating a nonlinear solution by a sum of profiles outside a wave cone. This type of approximation is only available in space-time slabs where the S norm of all the profiles remain finite. To satisfy this assumption, we will work outside a sufficiently large wave cone.

Let $\{(u_{0,n}, u_{1,n})\}_n$ be a sequence of functions in $\mathcal{L}^m$ that has a profile decomposition with profiles (U_0^j, U_1^j) and parameters $(\lambda_{j,n}, t_{j,n})_n$, $j \geq 1$. Extracting subsequences and time-translating the profiles, we can assume that for all $j \geq 1$ one of the following holds:

$$\lim_{n\to\infty} -\frac{t_{j,n}}{\lambda_{j,n}} \in \{\pm\infty\} \quad \text{or} \tag{3-30}$$

$$\text{for all } n, \quad t_{j,n} = 0. \tag{3-31}$$

We will denote by $\mathcal{J}_\infty$ the set of indices j such that (3-30) holds and by $\mathcal{J}_0$ the set of indices such that (3-31) holds. We assume:

(1) There exist $j_0 \geq 1$, $A > 0$ and a global solution U^{j_0} of

$$\begin{cases} \partial_t^2 U^{j_0} - \Delta U^{j_0} = \iota |U^{j_0}|^{2m} U^{j_0} \mathbb{1}_{\{r\geq|t|+A\}}, \\ \vec{U}^{j_0}(0,r) = \vec{U}_{\mathrm{L}}^{j_0}(0,r), \quad r \geq A, \end{cases}$$

such that $\vec{U}^{j_0}(0) \in \mathcal{L}^m$ and $\|U^{j_0}\|_{S(\{r\geq|t|+A\})} < \infty$.

(2) If $j \in \mathcal{J}_0 \setminus \{j_0\}$, then the solution of (1-1) with initial $\vec{U}_{\mathrm{L}}^j(0)$ scatters in both time directions or

$$\lim_{n\to\infty}\frac{\lambda_{j,n}}{\lambda_{j_0,n}} = 0.$$

For $j \geq 1$, we define U^j as follows:

- U^{j_0} is defined as in point (1) above.
- If $j \in \mathcal{J}_0$ and $\lim_{n\to\infty}\frac{\lambda_{j,n}}{\lambda_{j_0,n}} = 0$, then U^j is the solution of (1-1) with initial data $\vec{U}_{\mathrm{L}}^j(0)$.

- If $j \in \mathcal{J}_0$ and $\lim_{n\to\infty} \frac{\lambda_{j,n}}{\lambda_{j_{0,n}}} = \infty$, then $U^j = 0$.
- If $j \in \mathcal{J}_\infty$, then $U^j = U_{\mathrm{L}}^j$.

We let U_n^j be the corresponding modulated profiles:

$$U_n^j(t,x) = \frac{1}{\lambda_{j,n}^{\frac{1}{m}}} U^j\left(\frac{t-t_{j,n}}{\lambda_{j,n}}, \frac{x}{\lambda_{j,n}}\right).$$

Lemma 3.5. *Assume that points (1) and (2) above hold, let u_n be the solution of (1-1) with initial data $(u_{0,n}, u_{1,n})$, and I_n be its maximal interval of existence. Then*

$$u_n(t,x) = \sum_{j=1}^{J} U_n^j(t,x) + w_n^J(t,x) + \varepsilon_n^J(t,x),$$

where

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left(\|\varepsilon_n^J\|_{S(\{t\in I_n,\, r\geq A\lambda_{j_0,n}+|t|\})} + \sup_{t\in I_n} \int_{A\lambda_{j_0,n}+|t|}^{+\infty} |r\,\partial_{t,r}\vec{\varepsilon}_n^J(t,r)|^m\, dr \right) = 0.$$

Proof. This follows from Lemma 2.19 with

$$\tilde{u}_n = \sum_{j\in\mathcal{J}_0} U_n^j.$$

We omit the details of the proof that are by now standard; see, e.g., the proof of the main theorem in [Bahouri and Gérard 1999]. □

3E. *Exterior energy of a sum of profiles.*

Proposition 3.6. *Let $\{(u_{0,n}, u_{1,n})\}_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{L}^m$ that has a profile decomposition with profiles $\{U_{\mathrm{L}}^j\}_{j\geq 1}$ and parameters $\{(t_{j,n}, \lambda_{j,n})_n\}_{j\geq 1}$. Let $\{(\theta_n, \rho_n, \sigma_n)\}_{n\in\mathbb{N}}$ be a sequence such that $0 \leq \rho_n < \sigma_n \leq \infty$, $\theta_n \in \mathbb{R}$. Let $k \geq 1$. Then, extracting a subsequence if necessary*

$$o_n(1) + \int_{\rho_n}^{\sigma_n} |r\,\partial_{r,t} u_{\mathrm{L},n}(\theta_n, r)|^m\, dr \geq \int_{\rho_n}^{\sigma_n} |r\,\partial_{r,t} U_{\mathrm{L},n}^k(\theta_n, r)|^m\, dr, \tag{3-32}$$

where $\lim_n o_n(1) = 0$, $u_{\mathrm{L},n}$ is the solution of the linear wave equation with initial data $(u_{0,n}, u_{1,n})$ and $U_{\mathrm{L},n}^k$ is defined in (3-6).

See [Duyckaerts and Roy 2015, Proposition 3.12] for the proof.

4. Exterior energy for solutions of the nonlinear equation

4A. *Preliminaries on singular stationary solutions.* We recall from [Duyckaerts et al. 2014; Duyckaerts and Roy 2015; Shen 2013] the following result on existence of stationary solutions for (1-1).

Proposition 4.1. *Let $\ell \in \mathbb{R}\setminus\{0\}$. Assume $m > 1$, $m \neq 2$. There exists $R_\ell \geq 0$ and a maximal radial C^2 solution Z_ℓ of*

$$\Delta Z_\ell + \iota |Z_\ell|^{2m} Z_\ell = 0 \quad \text{on } \mathbb{R}^3 \cap \{|x| > R_\ell\} \tag{4-1}$$

such that

$$|rZ_\ell(r) - \ell| + |r^2 Z_\ell'(r) + \ell| \lesssim \frac{1}{r^{2m-2}}, \quad r \gg 1. \tag{4-2}$$

Furthermore,

- *if* $\iota = +1$ *(focusing nonlinearity), then* $R_\ell = 0$ *and* $Z_\ell \notin L^{3m}(\mathbb{R}^3)$,
- *if* $\iota = -1$ *(defocusing nonlinearity), then* $R_\ell > 0$ *and*

$$\lim_{r \to R_\ell} |Z_\ell(r)| = +\infty. \tag{4-3}$$

Remark 4.2. We will construct Z_1 and let

$$Z_\ell = \frac{\pm 1}{|\ell|^{\frac{1}{m-1}}} Z_1\left(\frac{r}{|\ell|^{\frac{m}{m-1}}}\right)$$

(where $\pm$ is the sign of ℓ), which will satisfy the conclusion of Proposition 4.1 for all $\ell \in \mathbb{R} \setminus \{0\}$. In particular,

$$R_\ell = R_1 |\ell|^{\frac{m}{m-1}}.$$

Let us mention that the uniqueness of Z_ℓ can be proved by elementary arguments. However, it will follow from Proposition 4.3 and we will not prove it here.

Proof. The proof is essentially contained in [Duyckaerts et al. 2014; Shen 2013] (focusing case for $m > 2$ and $m \in (1, 2)$ respectively) and [Duyckaerts and Roy 2015] (defocusing case for $m > 2$). We give a sketch for the sake of completeness.

We assume $\ell = 1$ (see Remark 4.2).

Existence for large r. Letting $g = rZ_1$, we see that the equation on Z_1 is equivalent to

$$g''(r) = -\frac{\iota}{r^{2m}} |g(r)|^{2m} g(r). \tag{4-4}$$

It is sufficient to find a fixed point for the operator A defined by

$$A(g) = 1 - \int_r^\infty \int_s^\infty \frac{\iota}{\sigma^{2m}} |g(\sigma)|^{2m} g(\sigma)\, d\sigma\, ds$$

in the ball

$$B = \{g \in C^0([r_0, +), \mathbb{R}) : d(g, 1) \le M\},$$

where r_0 and M are two large parameters and

$$d(g, h) := \sup_{r \ge r_0} (r^{2m-2} |g(r) - h(r)|).$$

Noting that (B, d) is a complete metric space, it is easy to prove that A is a contraction on B assuming $M \gg 1$ and $r_0 \gg 1$ (depending on M), and thus that A has a fixed point g_1. The fact that $Z_1 := \frac{1}{r} g_1$ satisfies the estimates (4-2) follows easily. Let $R_1 \ge 0$ such that $(R_1, +\infty)$ is the maximal interval of existence of g_1 as a solution of the ordinary differential equation.

Focusing case. We next assume $\iota = 1$ and prove that $R_1 = 0$ and $Z_\ell \notin L^{3m}$. Let

$$G(r) = \frac{1}{2} g'(r)^2 + \frac{1}{(2m+2) r^{2m}} |g(r)|^{2m+2}.$$

By (4-4), if $r \in (R_1, +\infty)$,

$$G'(r) = -\frac{m}{(m+1)r^{2m+1}} |g(r)|^{2m+2}.$$

Hence

$$|G'(r)| \leq \frac{C}{r} G(r).$$

This proves that G is bounded on $(R_1, +\infty)$ if $R_1 > 0$, a contradiction with the standard ODE blow-up criterion. Thus $R_1 = 0$.

The fact that $Z_1 \notin L^{3m}(\mathbb{R}^3)$ is nontrivial but classical. Assume by contradiction that $Z_1 \in L^{3m}$. Then one can prove, see [Duyckaerts et al. 2014], that Z_1 is a solution in the distributional sense on $\mathbb{R}^3$ of

$$-\Delta Z_1 = |Z_1|^{2m} Z_1.$$

Noting that $|Z_1|^{2m} \in L^{\frac{3}{2}}$, one can use [Trudinger 1968] to prove that $Z_1 \in L^\infty$, and thus, by elliptic regularity, that Z_1 is C^2 on $\mathbb{R}^3$. To deduce a contradiction, we introduce, as in [Shen 2013], the function $v(r) = r^{\frac{1}{m}} Z_1$. It is easy to check, using (4-2), for the limits at infinity and the fact that Z_1 is C^2 for the limit at 0, that

$$\lim_{r\to 0^+} v(r) = \lim_{r\to 0^+} rv'(r) = \lim_{r\to +\infty} v(r) = \lim_{r\to +\infty} rv'(r) = 0.$$

Furthermore,

$$v'' + \frac{2}{r}\left(1 - \frac{1}{m}\right)v' + \frac{1}{r^2}\left(\frac{1}{m^2} - \frac{1}{m}\right)v + \frac{1}{r^2}|v|^{2m} v = 0.$$

Integrating the identity

$$\frac{d}{dr}\left(r^2 \frac{|v'(r)|^2}{2} - \frac{m-1}{2m^2} v^2(r) + \frac{|v(r)|^{2m+2}}{2m+2}\right) = \frac{2-m}{m} r|v'(r)|^2 \tag{4-5}$$

between 0 and $+\infty$, one sees that v must be a constant, a contradiction with the construction of Z_1. Note that we have used in this last step that the constant $\frac{2-m}{m}$ in the right-hand side of the identity (4-5) is nonzero, i.e., $m \neq 2$.

Defocusing case. Assume $\iota = -1$. We prove that $R_1 > 0$ by contradiction. Assume $R_1 = 0$ and let

$$h(s) := Z_\ell\left(\frac{1}{s}\right).$$

Then

$$h''(s) = \frac{1}{s^4} |h(s)|^{2m} h(s)$$

and by (4-2),

$$\lim_{s\to 0^+} \frac{h(s)}{s} = \lim_{s\to 0^+} h'(s) = 1.$$

By a classical ODE argument, see [Duyckaerts and Roy 2015] for the details, one can prove that h blows up in finite time, a contradiction. This proves that $R_1 > 0$. The condition (4-3) follows from the standard ODE blow-up criterion. □

4B. ***Statement.*** One of the main ingredients of the proof of Theorem 3 is a bound from below of the exterior $\mathcal{L}^m$-energy for nonzero, $\mathcal{L}^m$ solutions of (1-1). It is similar to [Duyckaerts et al. 2013, Propostions 2.1 and 2.2] and [Duyckaerts and Roy 2015, Propositions 4.1 and 4.2]. The statements in these articles are divided between two cases, whether the support of $(u_0, u_1) - (Z_\ell, 0)$ is compact for all $\ell \neq 0$ or not. We give below a unified statement.

If $(u_0, u_1) \in \mathcal{L}^m$ and $A > 0$ we will denote by $\mathcal{T}_A(u_0, u_1)$ the element of $\mathcal{L}^m$ defined by

$$\mathcal{T}_A(u_0, u_1)(r) = (u_0, u_1)(r) \quad \text{if } r > A, \tag{4-6}$$

$$\mathcal{T}_A(u_0, u_1)(r) = (u_0(A), 0) \quad \text{if } r \leq A. \tag{4-7}$$

We note that

$$\|\mathcal{T}_A(u_0, u_1)\|_{\mathcal{L}^m}^m = \int_A^{+\infty} \left(|\partial_r u_0(r) ht|^m + |u_1(r)|^m\right) r^m \, dr. \tag{4-8}$$

We denote by ess supp the essential support of a function defined on a domain D of $\mathbb{R}^3$:

$$\operatorname{ess\,supp}(f) = D \setminus \bigcup \{\Omega \subset D \mid \Omega \text{ is open and } f = 0 \text{ a.e. in } \Omega\}.$$

Recall from Proposition 4.1 the definition of Z_1 and R_1.

Proposition 4.3. *Let u be a radial solution of* (1-1) *with $(u_0, u_1) \in \mathcal{L}^m$. Assume that (u_0, u_1) is not identically* 0. *Then there exist $A > 0$, $\eta > 0$ such that, if $(\tilde{u}_0, \tilde{u}_1) = \mathcal{T}_A(u_0, u_1)$, and $\tilde{u}$ is the solution of*

$$\partial_t^2 \tilde{u} - \Delta \tilde{u} = \iota |\tilde{u}|^{2m} \tilde{u} \mathbb{1}_{\{r \geq A + |t|\}} \tag{4-9}$$

with initial data $(\tilde{u}_0, \tilde{u}_1)$, then $\tilde{u}$ is global, scatters in $\mathcal{L}^m$ and the following holds for all $t \geq 0$ or for all $t \leq 0$:

$$\int_{A+|t|}^{+\infty} |\partial_r \tilde{u}(r)|^m r^m \, dr + \int_{A+|t|}^{+\infty} |\partial_t \tilde{u}(r)|^m r^m \, dr \geq \eta. \tag{4-10}$$

The proof of Proposition 4.3 is very close to the proofs of the analogous propositions in [Duyckaerts et al. 2014; Duyckaerts and Roy 2015]. We give a sketch of proof for the sake of completeness.

4C. ***Sketch of proof of Proposition 4.3.*** We argue by contradiction, assuming that for all $A > 0$ the solution $\tilde{u}$ of (4-9) with initial data $\mathcal{T}_A(u_0, u_1)$ is not a scattering solution, or is scattering and satisfies

$$\liminf_{t \to \pm\infty} \int_{A+|t|} |\partial_{t,r} \tilde{u}(t, r)|^m r^m \, dr = 0. \tag{4-11}$$

We let

$$v(r) = r u(r), \quad v_0(r) = r u_0(r), \quad v_1(r) = r u_1(r).$$

Step 1: In this step we prove that there exists $\varepsilon_0 > 0$ such that, if $A > 0$ is such that

$$\int_A^{+\infty} (|\partial_r u_0|^m + |u_1|^m) r^m \, dr = \varepsilon \leq \varepsilon_0, \tag{4-12}$$

then

$$\int_A^{+\infty} |\partial_r v_0|^m + |v_1|^m \, dr \le \frac{C}{A^{(2m+1)(m-1)}} |v_0(A)|^{m(2m+1)}, \tag{4-13}$$

$$\text{for all } B \in [A, 2A], \quad |v_0(B) - v_0(A)| \le C A^{2-2m} |v_0(A)|^{2m+1} \le C\varepsilon^2 |v_0(A)|. \tag{4-14}$$

We first assume (4-13) and prove (4-14). By the Hölder inequality and (4-13) we have

$$|v_0(B) - v_0(A)| \le \int_A^{2A} |\partial_r v_0(r)| \, dr \le A^{\frac{m-1}{m}} \left(\int_A^{2A} |\partial_r v_0|^m \, dr \right)^{\frac{1}{m}} \le C A^{2-2m} |v_0(A)|^{2m+1}. \tag{4-15}$$

Furthermore, by (4-12) and (4) in Proposition 2.2,

$$\frac{1}{A^{m-1}} |v_0(A)|^m = A |u_0(A)|^m \lesssim \varepsilon,$$

which yields

$$|v_0(A)|^{2m} \lesssim \varepsilon^2 A^{2m-2}.$$

Combining with (4-15), we obtain the second inequality of (4-14).

We next prove (4-13). Let

$$(\tilde{u}_0, \tilde{u}_1) = \mathcal{T}_A(u_0, u_1).$$

Let $\tilde{u}$ and $\tilde{u}_{\mathrm{L}}$ be the solutions of the nonlinear wave equation (1-1) and the linear wave equation (1-3), respectively, with initial data $(\tilde{u}_0, \tilde{u}_1)$. By the small data theory, $\tilde{u}$ is global and

$$\sup_{t \in \mathbb{R}} \| \vec{\tilde{u}}(t) - \vec{\tilde{u}}_{\mathrm{L}}(t) \|_{\mathcal{L}^m} \le C \varepsilon^{2m+1}. \tag{4-16}$$

Using the exterior energy property (3) in Proposition 2.3, we have that the following holds for all $t \ge 0$ or for all $t \le 0$:

$$\int_A^{+\infty} (|\partial_r(v_0)|^m + |v_1|^m) \, dr \le C \int_{A+|t|}^{+\infty} |\partial_{r,t}(r\tilde{u}_{\mathrm{L}})(t,r)|^m \, dr \le C \int_{A+|t|}^{+\infty} |\partial_{r,t} \tilde{u}_{\mathrm{L}}(t,r)|^m r^m \, dr.$$

Using (4-16), we obtain that the following holds for all $t \ge 0$ or for all $t \le 0$:

$$\int_A^{+\infty} (|\partial_r(v_0)|^m + |v_1|^m) \, dr \le C \left(\int_{A+|t|}^{+\infty} |\partial_{r,t} \tilde{u}(t,r)|^m \, dr + \varepsilon^{(2m+1)m} \right). \tag{4-17}$$

Using (4-11) and the definition (4-12) of ε, and letting $t \to +\infty$ or $t \to -\infty$, we obtain

$$\frac{1}{C} \int_A^{+\infty} (|\partial_r v_0|^m + |v_1|^m) \, dr \le \left(\int_A^{+\infty} (|\partial_r u_0|^m + |u_1|^m) r^m \, dr \right)^{2m+1}.$$

By (4) in Proposition 2.2, and since $A|u_0(A)|^m = \frac{1}{A^{m-1}} |v_0(A)|^m$,

$$\int_A^{+\infty} (|\partial_r v_0|^m + |v_1|^m) \, dr \le C \left(\int_A^{+\infty} (|\partial_r v_0|^m + |v_1|^m) \, dr + \frac{1}{A^{m-1}} |v_0(A)|^m \right)^{2m+1}.$$

Since $\int_A^{+\infty} (|\partial_r v_0|^m + |v_1|^m) \, dr$ is small, we deduce (4-13).

Step 2: We prove that there exists $\ell \in \mathbb{R} \setminus 0$ such that

$$\lim_{r\to\infty} v_0(r) = \ell, \tag{4-18}$$

and that there exists a constant $M > 0$ (depending on u) such that

$$|v_0(r) - \ell| \leq \frac{M}{r^{2m-2}} \tag{4-19}$$

for large r.

Let $\varepsilon > 0$ and fix A_0 such that

$$\int_{A_0}^{+\infty} (|\partial_r u_0|^m + |u_1|^m) r^m \, dr = \varepsilon \leq \varepsilon_0, \tag{4-20}$$

where ε_0 is given by Step 1. By (4-14),

$$\text{for all } k \geq 0, \quad |v_0(2^{k+1}A_0)| \leq (1 + C\varepsilon^2)(|v_0(2^k A_0)|).$$

Hence, by a straightforward induction,

$$\text{for all } k \geq 0, \quad |v_0(2^{k+1}A_0)| \leq (1 + C\varepsilon^2)^k |v_0(A_0)|.$$

Using (4-14) again, we deduce

$$\left|v_0(2^{k+1}A_0) - v_0(2^k A_0)\right| \leq C(2^k A_0)^{2-2m}(1 + C\varepsilon^2)^{k(2m+1)} |v_0(A_0)|^{2m+1}. \tag{4-21}$$

Choosing ε small enough (so that $2^{2-2m}(1 + C\varepsilon^2)^{2m+1} < 1$), we see that

$$\sum_{k\geq 1} \left|v_0(2^{k+1}A_0) - v_0(2^k A_0)\right| < \infty,$$

and thus that $v_0(2^k A_0)$ has a limit ℓ as $k \to +\infty$. Using (4-14) again, we deduce

$$\lim_{r\to\infty} |v_0(r)| = \ell.$$

Summing (4-21) over all $k \geq 0$, we deduce, using that v_0 is bounded, that there exists a constant $M > 0$, such that $|v_0(A_0) - \ell| \leq M A_0^{2-2m}$ for A_0 large enough. This yields (4-19).

It remains to prove that $\ell \neq 0$. We argue by contradiction. By (4-19), if $\ell = 0$, then

$$|v_0(r)| \leq \frac{M}{r^{2m-2}}.$$

On the other hand, using (4-14) and an easy induction argument, we obtain that for all $\varepsilon > 0$, for all A_0 satisfying (4-20),

$$|v_0(2^k A_0)| \geq (1 - C\varepsilon^2)^k |v_0(A_0)|.$$

Combining with the previous bound, we obtain

$$(1 - C\varepsilon^2)^k |v_0(A_0)| \leq \frac{M}{(2^k A_0)^{2m-2}},$$

a contradiction if ε is chosen small enough unless $v_0(A_0) = 0$. Using (4-13), we see that this would imply $v_0(r) = 0$ and $v_1(r) = 0$ for almost all $r \geq A_0$. Since this is true for any A_0 such that (4-20) holds, an obvious bootstrap argument proves that $(v_0, v_1) = (0, 0)$ almost everywhere, contradicting our assumption.

Step 3: Recall from Proposition 4.1 the definition of R_ℓ. Let, for $r > R_\ell$,

$$(g_0, g_1)(r) := (u_0(r) - Z_\ell(r), u_1(r)), \quad (h_0, h_1)(r) = r(g_0(r), g_1(r)).$$

If $\varepsilon > 0$, we fix $A_\varepsilon > R_\ell$ such that

$$\int_{A_\varepsilon}^{+\infty} |\partial_r Z_\ell|^m r^m \, dr + \|Z_\ell\|^m_{S(\{r \geq A_\varepsilon + |t|\})} \leq \frac{\varepsilon^m}{C}, \tag{4-22}$$

In this step, we prove that for all $\varepsilon > 0$, if $A > A_\varepsilon$ satisfies

$$\int_A^{+\infty} (|\partial_r g_0|^m + |g_1|^m) r^m \, dr < \frac{\varepsilon^m}{C} \tag{4-23}$$

then

$$\int_A^{+\infty} |\partial_r h_0|^m + |h_1|^m \, dr \leq \frac{\varepsilon}{A^{m-1}} |h_0(A)|^m. \tag{4-24}$$

Fix $A > A_\varepsilon$, let $(\tilde{u}_0, \tilde{u}_1) = \mathcal{T}_A(u_0, u_1)$, and let $\tilde{u}$ be the solution of the nonlinear wave equation (1-1) with initial data $(\tilde{u}_0, \tilde{u}_1)$ at $t = 0$. Note that by (4-23) and small data theory, $\tilde{u}$ is global and scatters in both time directions. Note also that by our assumption, $\tilde{u}$ satisfies (4-11).

Define $\tilde{g}$ as the solution to the equation

$$\begin{cases} \partial_t^2 \tilde{g} - \Delta \tilde{g} = \mathbb{1}_{\{r \geq A + |t|\}} (|\tilde{u}|^{2m} \tilde{u} - |Z_\ell|^{2m} Z_\ell), \\ \vec{\tilde{g}}_{\restriction t=0} = \mathcal{T}_A(g_0, g_1), \end{cases} \tag{4-25}$$

and $\tilde{g}_{\mathrm{L}}$ the solution of the free wave equation with the same initial data. Notice that $(\partial_t^2 - \Delta)(\tilde{u} - Z_\ell) = (\partial_t^2 - \Delta)\tilde{g}$ for $r > A + |t|$ and $\vec{\tilde{g}}(0, r) = (\tilde{u}_0 - Z_\ell, \tilde{u}_1)(r)$ for $r > A$. Thus, by finite speed of propagation, $\tilde{g} = \tilde{u} - Z_\ell$ for $r > A + |t|$, and we can rewrite the first equation in (4-25):

$$\partial_t^2 \tilde{g} - \Delta \tilde{g} = \mathbb{1}_{\{r \geq A + |t|\}} \big(|Z_\ell + \tilde{g}|^{2m} (Z_\ell + \tilde{g}) - |Z_\ell|^{2m} Z_\ell \big). \tag{4-26}$$

Using (4-26), Strichartz estimates and the Hölder inequality, we see that for all time intervals I containing 0

$$\|\tilde{g} - \tilde{g}_{\mathrm{L}}\|_{S(I)} + \sup_{t \in I_{\max}(u)} \|\vec{\tilde{g}}(t) - \vec{\tilde{g}}_{\mathrm{L}}(t)\|_{\mathcal{L}^m} \leq C \big(\|Z_\ell\|^{2m}_{S(\{r \geq A + |t|\})} \|\tilde{g}\|_{S(I)} + \|\tilde{g}\|^{2m+1}_{S(I)} \big).$$

By (4-23), (4-22) and a straightforward bootstrap argument, we deduce that for all intervals I with $0 \in I$,

$$\|\tilde{g}\|_{S(I)} \leq C \|\tilde{g}_{\mathrm{L}}\|_{S(I)} \leq C \|\mathcal{T}_A(g_0, g_1)\|_{\mathcal{L}^m} \leq C\varepsilon,$$

and

$$\sup_{t \in \mathbb{R}} \|\vec{\tilde{g}}(t) - \vec{\tilde{g}}_{\mathrm{L}}(t)\|_{\mathcal{L}^m} \leq C \varepsilon^{2m} \|\mathcal{T}_A(g_0, g_1)\|_{\mathcal{L}^m}. \tag{4-27}$$

By the exterior energy property (3) in Proposition 2.3, the following holds for all $t \geq 0$ or for all $t \leq 0$:

$$\int_A^{+\infty} (|h_0|^m + |h_1|^m)\,dr \leq C \int_{A+|t|}^{+\infty} |\partial_{t,r}\tilde{g}_{\mathrm{L}}|^m r^m\,dr$$
$$\leq C\left((\varepsilon^{2m}\|\mathcal{T}_A(g_0,g_1)\|_{\mathcal{L}^m})^m + \int_{A+|t|}^{+\infty} |\partial_{t,r}\tilde{g}|^m r^m\,dr\right),$$

where in the last line we used (4-27).

Letting $t \to \pm\infty$ and using (4-11), we deduce

$$\int_A^{+\infty} |\partial_r h_0|^m + |h_1|^m\,dr \leq C\varepsilon^{2m^2} \int_A^{+\infty} (|\partial_r g_0|^m + |g_1|^m) r^m\,dr.$$

The desired estimate (4-24) follows, taking ε small and using (4) in Proposition 2.2.

Step 4: Fix a small $\varepsilon > 0$ and let A_ε be as in Step 3, i.e., such that (4-22) holds. In this step, we prove that $r \leq A_\varepsilon$ on $\operatorname{ess\,supp}(u_0 - Z_\ell, u_1)$.

Indeed, if not, we obtain from (4-24) that there exists $A > A_\varepsilon$ such that $h_0(A) \neq 0$. Using a similar argument to that in Step 1, we deduce from (4-24) that for all $A \geq A_\varepsilon$ such that (4-23) holds,

$$\text{for all } B \in [A, 2A], \quad |h_0(A) - h_0(B)| \leq C\varepsilon |h_0(A)|. \tag{4-28}$$

If $\operatorname{ess\,supp}(u_0 - Z_\ell, u_1)$ is not bounded, we deduce by (4-24) that $h_0(A) \neq 0$ for all large $A > 0$. If $\varepsilon > 0$ is small enough, we deduce using (4-28) that

$$\lim_{r\to+\infty} r^\alpha h_0(r) = +\infty,$$

where $\alpha \in (0, 2m-2)$ is fixed. Since

$$v_0(r) - \ell = h_0(r) - \ell + rZ_\ell,$$

this contradicts (4-19) in Step 2 and the asymptotic estimate (4-2) of Z_ℓ.

We have proved that $\operatorname{ess\,supp}(u_0 - Z_\ell, u_1)$ is bounded. Using (4-24), (4-28) and a straightforward bootstrap argument, we deduce that $r \leq A_\varepsilon$ on the support of $\operatorname{ess\,supp}(u_0 - Z_\ell, u_1)$.

Step 5: Fix a small $\varepsilon > 0$. We have proved in Step 4 that $(u_0, u_1)(r) = (Z_\ell(r), 0)$ for almost every $r \geq A_\varepsilon$, where A_ε depends only on ℓ. We will prove $(u_0, u_1)(r) = (Z_\ell(r), 0)$ for $r > R_\ell$, a contradiction with Proposition 4.1 since $(u_0, u_1) \in \mathcal{L}^m$.

We argue by contradiction, assuming that there exists $B > R_\ell$ such that $B \in \operatorname{ess\,supp}(u_0 - Z_\ell, u_1)$. Using a similar argument to that in Step 3, but on small time intervals (see, e.g., the proof of Proposition 2.2(a), §2.2.1 in [Duyckaerts et al. 2013]), we prove that the following holds for all $t \geq 0$ or for all $t \leq 0$:

$$B + |t| \in \operatorname{ess\,supp}\big((u(t) - Z_\ell, \partial_t u(t))\big). \tag{4-29}$$

Choose t_0 such that

$$B + |t_0| > A_\varepsilon \quad \text{on } \operatorname{ess\,supp}\big((u(t_0) - Z_\ell, \partial_t u(t_0))\big). \tag{4-30}$$

It is easy to see that u satisfies the following: for all $A > |t_0|$ the solution $\tilde{u}$ of

$$\partial_t^2 \tilde{u} - \Delta \tilde{u} = \iota |\tilde{u}|^{2m} \tilde{u} \mathbb{1}_{\{r \geq A + |t - t_0|\}}$$

with initial data $\mathcal{T}_A(\vec{u}(t_0))$ at $t = t_0$ is not a scattering solution, or is scattering and satisfies

$$\liminf_{t \to \pm\infty} \int_{A+|t-t_0|}^{+\infty} |\partial_{t,r} \tilde{u}(t,r)|^m r^m \, dr = 0.$$

We can then go through Steps 1–4 above, but with initial data at $t = t_0$, and restricting to $r > |t_0|$. Note that by finite speed of propagation, the limit ℓ obtained in Step 2 for $t = 0$ and for $t = t_0$ is the same; i.e.,

$$\lim_{r \to +\infty} r u(t_0, r) = \lim_{r \to +\infty} r u(0, r).$$

By the conclusion of Step 4, we obtain that $r < \max(A_\varepsilon, t_0)$ on $\operatorname{ess\,supp}(\vec{u}(t_0) - Z_\ell)$, contradicting (4-30). □

5. Dispersive term

This section concerns the existence of a "dispersive" component for a solution u of (1-1) that remains bounded in $\mathcal{L}^m$ along a sequence of times. This component is the strong limit of $\vec{u}(t)$, in $\mathcal{L}^m$, outside the origin in the finite time blow-up case (see Section 5A), and a solution of the linear wave equation in the global case (see Section 5B).

5A. *Regular part in the finite time blow-up case.*

Proposition 5.1. *Let u be a radial solution of (1-1), (1-2). Assume*

$$T_+(u) < \infty, \qquad \liminf_{t \to T_+(u)} \|\vec{u}(t)\|_{\mathcal{L}^m} < \infty.$$

Then there exists a solution v of (1-1), defined in a neighborhood of $t = T_+$, such that for all t in $I_{\max}(u) \cap I_{\max}(v)$,

$$\text{for all } r > T_+ - t, \quad \vec{u}(t,r) = \vec{v}(t,r).$$

We omit the proof; see Section 6.3 in [Duyckaerts and Roy 2015] for a very close proof.

5B. *Extraction of the radiation term in the global case.* We prove here:

Proposition 5.2. *Let u be a radial solution of (1-1), (1-2). Assume*

$$T_+(u) = +\infty, \qquad \liminf_{t \to +\infty} \|\vec{u}(t)\|_{\mathcal{L}^m} < \infty.$$

Then there exists a solution v_{L} of the free wave equation (1-3) such that for all $A \in \mathbb{R}$,

$$\lim_{t \to +\infty} \int_{|x| \geq A+|t|} \left(|\partial_t (u - v_{\mathrm{L}})|^m + |\partial_r (u - v_{\mathrm{L}})|^m\right) r^m \, dr = 0. \tag{5-1}$$

The proof relies on the following lemma, which is a consequence of finite speed of propagation, Strichartz estimates and the small data theory. We omit the proof, which is an easy adaptation of the proofs of Claims 2.3 and 2.4 in [Duyckaerts et al. 2016] where the usual energy is replaced by the $\mathcal{L}^m$-energy:

Lemma 5.3. *There exists $\varepsilon_1 > 0$ with the following property. Let u be a solution of (1-1), (1-2) such that $T_+(u) = +\infty$. Let $T \geq 0$ and $A \geq 0$. Assume $\|S_{\mathrm{L}}(\cdot - T)\vec{u}(T)\|_{S(\{|x|\geq A+t,\, t\geq T\})} = \varepsilon' < \varepsilon_1$. Then $\|u\|_{S(\{|x|\geq A+t,\, t\geq T\})} \leq 2\varepsilon'$, and there exists a solution v_{L} of the linear wave equation such that (5-1) holds.*

Proof of Proposition 5.2. See also Section 3.3 in [Duyckaerts et al. 2013].

Step 1: Let $t_n \to +\infty$ such that the sequence $(\vec{u}(t_n))_n$ is bounded in $\mathcal{L}^m$. In this step we prove that there exists $\delta > 0$ such that for large n,

$$\|S_{\mathrm{L}}(\cdot)\vec{u}(t_n)\|_{S(\{|x|\geq(1-\delta)t_n+t,\, t\geq 0\})} < \varepsilon_1, \tag{5-2}$$

where ε_1 is given by Lemma 5.3. We argue by contradiction, assuming (after extraction of subsequences) that there exists a sequence $\delta_n \to 0$ such that

$$\|S_{\mathrm{L}}(\cdot)\vec{u}(t_n)\|_{S(\{|x|\geq(1-\delta_n)t_n+t,\, t\geq 0\})} \geq \varepsilon_1. \tag{5-3}$$

Extracting subsequences again, we can assume that the sequence $(\vec{u}(t_n))_n$ has a profile decomposition with profiles U_{L}^j and parameters $(\lambda_{j,n}, t_{j,n})_n$. Let J be a large integer such that

$$\left\| S_{\mathrm{L}}(\cdot)\left(\vec{u}(t_n) - \sum_{j=1}^{J} \vec{U}_{\mathrm{L},n}^j(0)\right)\right\|_{S(\mathbb{R})} \leq \frac{\varepsilon_1}{2}.$$

A contradiction will follow if we prove (possibly extracting subsequences in n) that for all $j \in \{1, \dots, J\}$,

$$\lim_{n\to\infty} \|S_{\mathrm{L}}(\cdot)\vec{U}_{\mathrm{L},n}^j(0)\|_{S(\{|x|\geq(1-\delta_n)t_n+t,\, t\geq 0\})} = 0. \tag{5-4}$$

We have

$$\|S_{\mathrm{L}}(\cdot)\vec{U}_{\mathrm{L},n}^j(0)\|_{S(\{r\geq(1-\delta_n)t_n+t,\, t\geq 0\})} = \|U_{\mathrm{L}}^j\|_{S(A_{j,n})},$$

where

$$A_{j,n} := \left\{(t,r) \in \mathbb{R}\times(0,\infty) : t \geq -\frac{t_{j,n}}{\lambda_{j,n}} \text{ and } r \geq \frac{(1-\delta_n)t_n}{\lambda_{j,n}} + \left|t + \frac{t_{j,n}}{\lambda_{j,n}}\right|\right\}.$$

As a consequence, we see that we can extract subsequences so that the characteristic function of $A_{j,n}$ goes to 0 pointwise unless $\frac{t_{j,n}}{\lambda_{j,n}}$ and $\frac{t_n}{\lambda_{j,n}}$ are bounded. Time translating the profile U_{L}^j and extracting again, we can assume

$$\lim_{n\to\infty} \frac{t_n}{\lambda_{j,n}} = \tau_0 \in [0,\infty) \quad \text{for all } n,\ t_{j,n} = 0.$$

By finite speed of propagation and the small data theory,

$$\lim_{A\to+\infty} \limsup_{n\to+\infty} \int_{|x|\geq t_n+A} |r\partial_{r,t}u(t_n)|^m \, dr = 0. \tag{5-5}$$

By Proposition 3.6, for all $A\in\mathbb{R}$, we have that for large n,

$$\int_{t_n+A}^{+\infty}|r(\partial_{r,t}u(t_n))|^m\,dr \geq \frac{1}{2}\int_{t_n+A}^{+\infty}|r(\partial_{r,t}U^j_{\mathrm{L,n}}(0))|^m\,dr$$
$$=\frac{1}{2}\int_{\frac{t_n+A}{\lambda_{j,n}}}^{+\infty}|r(\partial_{r,t}U^j_{\mathrm{L}}(0))|^m\,dr \underset{n\to\infty}{\longrightarrow}\frac{1}{2}\int_{\tau_0}^{+\infty}|r(\partial_{r,t}U^j_{\mathrm{L}}(0))|^m\,dr.$$

Combining with (5-5), we see that if U^j_{L} is not identically 0, then τ_0 is strictly positive, and we can rescale the profile U^j_{L} to assume $\tau_0=1$, and $\lambda_{j,n}=t_n$. Using (5-5) we see that ess supp $\vec{U}^j_{\mathrm{L}}(0)$ is included in the unit ball of $\mathbb{R}^3$, which implies

$$\|U^j_{\mathrm{L}}\|_{S(A_{j,n})}=\|U^j_{\mathrm{L}}\|_{S(\{t\geq 0,\, r\geq(1-\delta_n)+t\})}\underset{n\to\infty}{\longrightarrow}0,$$

concluding the proof of (5-4) in this case. Step 1 is complete.

Step 2: By Step 1 and Lemma 5.3, for all $A\in\mathbb{R}$, there exists a solution v^A_{L} of the free wave equation such that

$$\lim_{t\to+\infty}\int_{|x|\geq A+|t|}\big(|\partial_t(u-v^A_{\mathrm{L}})|^m+|\partial_r(u-v^A_{\mathrm{L}})|^m\big)r^m\,dr=0. \tag{5-6}$$

We consider the sequence $t_n\to+\infty$ of Step 1 and assume, extracting a subsequence if necessary, that $\vec{u}(t_n)$ has a profile decomposition $(U^j_{\mathrm{L}},(\lambda_{j,n},t_{j,n})_n)_{j\geq 1}$. Reordering the profiles and rescaling and time-translating U^1_{L} if necessary, we can assume, without loss of generality, that $t_{1,n}=t_n$ and $\lambda_{1,n}=1$ for all n. In other words, $\vec{U}^1_{\mathrm{L}}(0)$ is the weak limit, as n goes to infinity, of $\vec{S}_{\mathrm{L}}(-t_n)\vec{u}(t_n)$. Note that U^1_{L} might be identically 0.

Fix $A\in\mathbb{R}$. Then

$$\vec{u}(t_n)-\vec{v}^A_{\mathrm{L}}(t_n)=\vec{U}^1_{\mathrm{L}}(t_n)-\vec{v}^A_{\mathrm{L}}(t_n)+\sum_{j=2}^{J}\vec{U}^j_{\mathrm{L},n}(0)+\vec{w}^J_n(t_n);$$

i.e., $\vec{u}(t_n)-\vec{v}^A_{\mathrm{L}}(t_n)$ has a profile decomposition $(\tilde{U}^j_{\mathrm{L}},(\lambda_{j,n},t_{j,n})_n)_{j\geq1}$, with $\tilde{U}^j_{\mathrm{L}}=U^j_{\mathrm{L}}$ if $j\geq 2$, and $\tilde{U}^1_{\mathrm{L}}=U^1_{\mathrm{L}}-v^A_{\mathrm{L}}$. By Proposition 3.6,

$$\limsup_{n\to\infty}\int_{r\geq A+t_n}|r\partial_{r,t}(u-v^A_{\mathrm{L}})(t_n)|^m\,dr\geq\limsup_{n\to\infty}\int_{r\geq t_n+A}|r\partial_{r,t}(U^1_{\mathrm{L}}-v^A_{\mathrm{L}})(t_n)|^m,$$

and thus, by (5-6)

$$\lim_{n\to\infty}\int_{r\geq t_n+A}|r\partial_{r,t}(U^1_{\mathrm{L}}-v^A_{\mathrm{L}})(t_n)|^m=0.$$

Using (5-6) again, we obtain

$$\lim_{n\to\infty}\int_{r\geq t_n+A}|r\partial_{r,t}(U^1_{\mathrm{L}}-u)(t_n)|^m=0.$$

This is valid for all $A \in \mathbb{R}$. A simple argument using finite speed of propagation and small data theory yields

$$\lim_{t\to\infty} \int_{r\geq t+A} |r\partial_{r,t}(U_{\mathrm{L}}^1 - u)(t)|^m = 0,$$

concluding the proof of the proposition with $v_{\mathrm{L}} = U_{\mathrm{L}}^1$. □

6. Scattering/blow-up dichotomy

In this section we prove Theorem 3. Let u be a solution of (1-1). Consider the property:

$$\liminf_{t\to T_+(u)} \|\vec{u}(t)\|_{\mathcal{L}^m} < \infty. \tag{6-1}$$

We must prove:

(1) If (6-1) holds then $T_+(u) = +\infty$.

(2) If $T_+(u) = +\infty$ and (6-1) holds, then u scatters to a linear solution in $\mathcal{L}^m$.

The proofs of (1) and (2) are very similar, and are simplified versions of the corresponding proofs in [Duyckaerts and Roy 2015]. We will only sketch the proof of (2) and explain the necessary modification to obtain (1).

6A. ***Proof of scattering.*** Let u be a global solution and let $t_n \to +\infty$ such that $\vec{u}(t_n)$ is bounded. Let v_{L} be the linear component of u, given by Proposition 5.2. Extracting subsequences, we can assume that $(\vec{u}(t_n) - \vec{v}_{\mathrm{L}}(t_n))_n$ has a profile decomposition with profiles U_{L}^j and parameters $(\lambda_{j,n}, t_{j,n})_n$. As before, we denote by $U_{\mathrm{L},n}^j$ the modulated profiles; see (3-6). Extracting subsequences and translating the profiles in time if necessary, one of the following three cases holds.

Case 1: Assume

$$\forall j \geq 1, \quad U_{\mathrm{L}}^j \equiv 0 \quad \text{or} \quad \lim_{n\to\infty} \frac{-t_{j,n}}{\lambda_{j,n}} = -\infty. \tag{6-2}$$

Let $T \gg 1$ such that $\|v_{\mathrm{L}}\|_{S((T,+\infty))} < \frac{\delta_0}{2}$, where δ_0 is given by the small data theory (see Proposition 2.17). By (6-2), for all j,

$$\lim_{n\to\infty} \|U_{\mathrm{L},n}^j\|_{S((T-t_n,0))} = \lim_{n\to\infty} \|U_{\mathrm{L}}^j\|_{S\left(\left(\frac{T-t_n-t_{j,n}}{\lambda_{j,n}}, \frac{-t_{j,n}}{\lambda_{j,n}}\right)\right)} = 0.$$

Thus for large n,

$$\|S_{\mathrm{L}}(\cdot)\vec{u}(t_n)\|_{S((T-t_n,0))} < \delta_0.$$

By Proposition 2.17, for large n,

$$\|u\|_{S((T,t_n))} = \|u(t_n + \cdot)\|_{S((T-t_n,0))} < 2\delta_0.$$

Letting $n \to \infty$, we deduce $\|u\|_{S((T,+\infty))} < 2\delta_0$, and thus u scatters.

Case 2: We assume

$$\forall j \geq 1, \quad U_{\mathrm{L}}^{j} \equiv 0 \quad \text{or} \quad \lim_{n\to\infty} \frac{-t_{j,n}}{\lambda_{j,n}} \in \{\pm\infty\}. \tag{6-3}$$

and

$$\exists j_0 \geq 1, \quad U_{\mathrm{L}}^{j_0} \not\equiv 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{-t_{j_0,n}}{\lambda_{j_0,n}} = +\infty. \tag{6-4}$$

We will use a channel of energy argument based on the following observation, which is a direct consequence of the explicit form of the solution; see (2-4), (2-6):

Claim 6.1. *Let u_{L} be a nonzero solution of the linear wave equation (1-3) with initial data in $\mathcal{L}^m$. Then there exists $A \in \mathbb{R}$ such that*

$$\liminf_{t\to+\infty} \int_{A+t}^{+\infty} r^m |\partial_{r,t} u_{\mathrm{L}}|^m \, dr > 0.$$

If $j \geq 1$, we have

$$\|U_{\mathrm{L},n}^{j}\|_{S(\{t\geq 0,\, r\geq t\})} = \|U_{\mathrm{L}}^{j}\|_{S\left(\{t\geq -\frac{t_{j,n}}{\lambda_{j,n}},\, r\geq t+\frac{t_{j,n}}{\lambda_{j,n}}\}\right)}.$$

Noting that under the assumptions of Case 2,

$$\text{for all } j \geq 1, \quad \mathbb{1}_{\{t\geq -\frac{t_{j,n}}{\lambda_{j,n}},\, r\geq t+\frac{t_{j,n}}{\lambda_{j,n}}\}} \underset{n\to\infty}{\longrightarrow} 0$$

pointwise, otherwise $U_{\mathrm{L}}^{j} \equiv 0$. We obtain

$$\text{for all } j \geq 1, \quad \lim_{n\to\infty} \|U_{\mathrm{L},n}^{j}\|_{S(\{r\geq t\geq 0\})} = 0$$

and thus

$$\lim_{n\to\infty} \|S_{\mathrm{L}}(t)\vec{u}(t_n)\|_{S(\{r\geq t\geq 0\})} = 0.$$

By the small data theory (see Proposition 2.17) and finite speed of propagation

$$\lim_{n\to\infty} \left(\|u(t_n+\cdot)\|_{S(\{r\geq t\geq 0\})} + \sup_{t\geq 0} \int_t^{+\infty} \left|\partial_{t,r}\big(u(t_n+t) - S_{\mathrm{L}}(t)\vec{u}(t_n)\big)\right|^m r^m \, dr \right) = 0. \tag{6-5}$$

Let j_0 be as in (6-4). By Claim 6.1, there exists $A \in \mathbb{R}$ such that

$$\liminf_{t\to+\infty} \int_{\lambda_{j_0,n}A - t_{j_0,n}+t}^{\infty} |r\partial_{t,r} U_{\mathrm{L},n}^{j_0}|^m \, dr > 0.$$

For large n, we have $\lambda_{j_0,n}A - t_{j_0,n} \geq 0$. By Proposition 3.6, we deduce from (6-5) that for large n,

$$\liminf_{t\to+\infty} \int_{\lambda_{j_0,n}A - t_{j_0,n}+t-t_n}^{\infty} r^m |\partial_{t,r}(u(t,r) - v_{\mathrm{L}}(t,r))|^m \, dr > 0,$$

contradicting the definition of v_{L}.

Case 3: In this last case we assume

$$\exists j \geq 1,\ \forall n, \quad t_{j,n} = 0 \quad \text{and} \quad U_{\mathrm{L}}^{j} \not\equiv 0. \tag{6-6}$$

This is the core of the proof, where we use Proposition 4.3, and thus the fact that (1-1) has no nonzero stationary solution in $\mathcal{L}^m$.

We will use Section 3D to approximate u, outside appropriate wave cones, by a sum of profiles. As in Section 3D, we let $\mathcal{J}_0$ be the set of indices j such that $t_{j,n}=0$ for all n and $\mathcal{J}_\infty$ the set of j such that $\frac{t_{j,n}}{\lambda_{j,n}}$ goes to $+\infty$ or $-\infty$. Extracting subsequences and translating the profiles in time if necessary, we can assume $\mathbb{N}\setminus\{0\}=\mathcal{J}_0\cup\mathcal{J}_\infty$. Let $\delta_1>0$ be a small number, smaller than the number given by the small data theory, and such that there exists $j\in\mathcal{J}_0$ with $\|\vec{U}_{\mathrm{L}}^j(0)\|_{\mathcal{L}^m}>\delta_1$. We let $j_0\in\mathcal{J}_0$ such that $\|\vec{U}_{\mathrm{L}}^{j_0}(0)\|_{\mathcal{L}^m}>\delta_1$, and

$$\big(j\in\mathcal{J}_0 \quad\text{and}\quad \|\vec{U}_{\mathrm{L}}^j(0)\|_{\mathcal{L}^m}>\delta_1\big) \quad\Longrightarrow\quad \lim_{n\to\infty}\frac{\lambda_{j_0,n}}{\lambda_{j,n}}=+\infty. \tag{6-7}$$

We note that by Proposition 3.3, there exists a finite number of $j\in\mathcal{J}_0$ with $\|\vec{U}_{\mathrm{L}}^j(0)\|_{\mathcal{L}^m}>\delta_1$, so that, in view of the pseudo-orthogonality property (3-1), j_0 is well-defined. By Proposition 4.3, there exist A, $\eta>0$, $U^{j_0}\in S(\mathbb{R})$ such that $\vec{U}^{j_0}\in C^0(\mathbb{R},\mathcal{L}^m)$,

$$(\partial_t^2-\Delta)U^{j_0}=\iota|U^{j_0}|^{2m}U^{j_0}\mathbb{1}_{\{r\geq A+|t|\}},\qquad \vec{U}^{j_0}(0)=\mathcal{T}_A(\vec{U}_{\mathrm{L}}^{j_0}(0)), \tag{6-8}$$

and the following holds for all $t\geq 0$ or for all $t\leq 0$:

$$\int_{|t|+A}^{+\infty}|r\partial_{t,r}U^{j_0}|^m\,dr\geq\eta. \tag{6-9}$$

Note that $(U_{\mathrm{L}}^j,\lambda_{j,n},t_{j,n})_{j\geq 0}$ with $U_{\mathrm{L}}^0=v_{\mathrm{L}}$ and $\lambda_{0,n}=1,t_{0,n}=t_n$ is a profile decomposition of $\vec{u}(t_n)$. According to Lemma 3.5,

$$u(t+t_n)=v_{\mathrm{L}}(t+t_n)+\sum_{j=1}^J U_n^j(t)+w_n^J(t)+\varepsilon_n^J(t),\qquad t\in[-t_n,+\infty), \tag{6-10}$$

where the modulated profiles U_n^j for $j\neq j_0$ are defined in Section 3D and

$$\limsup_{n\to\infty}\bigg(\|\varepsilon_n^J\|_{S(\{t\in[-t_n,+\infty),\,r>A\lambda_{j_0,n}+|t|\})}+\sup_{t\geq -t_n}\int_{|t|+A\lambda_{j_0,n}}^{+\infty}|r\partial_{t,r}\varepsilon_n^J(t,r)|^m\,dr\bigg)$$

goes to 0 as J goes to infinity. It can be deduced from Proposition 3.6 that for all sequences $(\theta_n)_n$ in $[-t_n,+\infty)$,

$$o_n(1)+\int_{A\lambda_{j_0,n}+|\theta_n|}^{+\infty}\big|r\partial_{r,t}(u-v_{\mathrm{L}})(t_n+\theta_n,r)\big|^m\,dr\geq\int_{A\lambda_{j_0,n}+|\theta_n|}^{+\infty}|r\partial_{r,t}U_n^{j_0}(\theta_n,r)|^m\,dr. \tag{6-11}$$

Indeed, this can be proved by noticing that (6-10) (and its time derivative) at $t=\theta_n$ can be considered as a profile decomposition of the sequence $((\vec{u}-\vec{v}_{\mathrm{L}})(\theta_n+t_n))_n$ and using Proposition 3.6 and finite speed of propagation. We refer to the proof of (3.18) in [Duyckaerts and Roy 2015] for a detailed proof in a very similar setting.

If (6-9) holds for $t\geq 0$, then by (6-11), for large n,

$$\limsup_{t\to+\infty}\int_{t+A\lambda_{j_0,n}}^{+\infty}|r\partial_{t,r}(u(t_n,r)-v_{\mathrm{L}}(t_n,r))|^m\,dr\geq\frac{\eta}{2},$$

contradicting the definition of v_{L}.

If (6-9) holds for $t \leq 0$, we use (6-11) at $\theta_n = -t_n$ together with (6-9) and obtain that for large n

$$\int_{t_n + A\lambda_{j_0,n}}^{+\infty} |r\partial_{t,r}(u(0,r) - v_{\mathrm{L}}(0,r))|^m \, dr \geq \frac{\eta}{2},$$

a contradiction since $\vec{u}(0) \in \mathcal{L}^m$. □

6B. *Proof of global existence.* We argue by contradiction, assuming that (6-1) holds and that $T_+ = T_+(u)$ is finite. Let v be the regular part of u at $t = T_+$, defined by Proposition 5.1. Recall that v is a solution of (1-1) defined in a neighborhood of $T_+(u)$ and such that

$$\text{for all } t \in I_{\max}(u) \cap I_{\max}(v), \text{ for all } r > T_+ - t, \quad \vec{u}(t,r) = \vec{v}(t,r). \tag{6-12}$$

As in Section 6A, we consider a sequence $t_n \to T_+$ such that $\vec{u}(t_n)$ is bounded in $\mathcal{L}^m$, and we assume (extracting subsequences if necessary) that $(\vec{u}(t_n) - \vec{v}(t_n))_n$ has a profile decomposition with profiles U_{L}^j and parameters $(\lambda_{j,n}, t_{j,n})_n$. We distinguish again between three cases.

Case 1: We assume (6-2). By the same proof as in Case 1 of Section 6A, we obtain

$$\lim_{n\to\infty} \left\| S_{\mathrm{L}}(\,\cdot\,)(\vec{u}(t_n) - \vec{v}(t_n)) \right\|_{S((-\infty,0))} = 0.$$

By Lemma 2.19, if $T < T_+(u)$ is in the domain of definition of v, close to $T_+(u)$,

$$\lim_{n\to\infty} \|\vec{u}(t_n)\|_{S((T,t_n))} < \infty,$$

which contradicts the blow-up criterion

$$\|\vec{u}(t_n)\|_{S((T,T_+(u)))} = +\infty,$$

Case 2: We assume (6-3) and (6-4). Fix $j_0 \geq 1$ such that (6-4) holds. Using Claim 6.1 and an argument very similar to the one of Case 2 of Section 6A, we obtain that for large n,

$$\liminf_{t\to T_+(u)} \int_{r \geq A\lambda_{j_0,n} - t_{j_0,n} + t - t_n} |\partial_{t,r}(u-v)(t,r)|^m \, r^m \, dr > 0,$$

where $A \in \mathbb{R}$ is given by Claim 6.1, contradicting (6-12) (since for large n, we have $A\lambda_{j_0,n} - t_{j_0,n} \geq 0$).

Case 3: We assume (6-6). We define $\mathcal{J}_0$, $\mathcal{J}_\infty$ as in Case 3 of Section 6A and choose $j_0 \in \mathcal{J}_0$ such that (6-7) holds. Using Proposition 4.3, we obtain A, $\eta > 0$, and a solution $U^{j_0} \in S(\mathbb{R})$ of (6-8), such that (6-9) holds for all $t \geq 0$ or for all $t \leq 0$. We distinguish two cases.

If (6-9) holds for all $t \geq 0$, then we prove using Lemma 2.19 and Proposition 3.6 that for large n,

$$\liminf_{t\to T_+(u)} \int_{r \geq A\lambda_{j_0,n} + t - t_n} |\partial_{t,r}(u-v)(t,r)|^m \, r^m \, dr > 0,$$

a contradiction with (6-12).

If (6-9) holds for all $t \leq 0$, we let $T \in [0, T_+(u))$ such that T is in the domain of definition of v. Using Lemma 2.19 and Proposition 3.6, we deduce that for large n

$$\int_{r \geq A\lambda_{j_0,n}+t_n-T} |\partial_{t,r}(u-v)(T,r)|^m \, r^m \, dr \geq \frac{\eta}{2},$$

a contradiction for large n, since $\partial_{t,r}(u-v)(T,r)$ is supported in $|x| \leq T_+ - T$. This concludes the sketch of proof.

Appendix: Proof of Proposition 2.1

The "only if" part. First of all, we have a sequence of smooth radial functions $(f_n)_n$ with compact supports such that

$$\int_0^{+\infty} |\partial_r (f - f_n)(r)|^m r^m \, dr \to 0, \quad n \to \infty. \tag{A-1}$$

As a consequence, we clearly have (2-1). Notice that for $0 < r < r' < +\infty$, we have

$$|f(r') - f(r)| \leq \frac{C_m}{r^{\frac{1}{m}}} \left(\int_r^{r'} |s \partial_s f(s)|^m \, ds \right)^{\frac{1}{m}},$$

and this yields that $f(r)$ is continuous.

To see (2-2), we first prove

$$|f(r)| \leq \frac{1}{r^{\frac{1}{m}}} \left(\int_r^{+\infty} |s \partial_s f(s)|^m \, ds \right)^{\frac{1}{m}},$$

and

$$|r f(r)| \leq r^{\frac{m-1}{m}} \left(\int_0^r |\partial_s (s f(s))|^m \, ds \right)^{\frac{1}{m}}.$$

Indeed, if $f \in C_0^\infty((0, +\infty))$, then the preceding inequality follows from the fundamental theorem of calculus and the Hölder inequality. The case of a general function f can be deduced from (A-1). The desired estimate (2-2) is an immediate consequence of these two inequalities.

The "if" part. Given a radial function $f(x)$ on $\mathbb{R}^3$, satisfying the conditions (2-1), (2-2), we are to construct a sequence of smooth radial functions $f_n(x)$ compactly supported in $\mathbb{R}^3$ such that (A-1) holds.

To achieve this, we take a smooth radial function $\varphi(x)$ on $\mathbb{R}^3$ such that $\varphi(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ if $|x| \geq 2$. Let $(\varepsilon_n)_n$ be a sequence of positive numbers, tending to zero as $n \to \infty$. Define

$$f_n(x) = \varphi(\varepsilon_n x) \left(1 - \varphi\left(\frac{x}{\varepsilon_n} \right) \right) (f * \zeta_{\varepsilon_n})(x), \tag{A-2}$$

where $\zeta_\varepsilon(\varrho)$ is the usual approximate delta function supported in $-\frac{\varepsilon}{2} < \varrho < 0$ and $f * \zeta_\varepsilon$ denotes the radial convolution as in [Strauss 1977], namely

$$f * \zeta_\varepsilon(x) = \int_{-\frac{\varepsilon}{2}}^0 \zeta_\varepsilon(\varrho) \, f(|x| - \varrho) \, d\varrho.$$

Then it is clear that $f_n(x)$ is smooth, radial and supported in $\{x \in \mathbb{R}^3 \mid \varepsilon_n \le |x| \le \frac{2}{\varepsilon_n}\}$. We have

$$\partial_r(f(r) - f_n(r)) = -\varepsilon_n(\partial_r\varphi)(\varepsilon_n r)\Big(1-\varphi\Big(\frac{r}{\varepsilon_n}\Big)\Big)f(r) \tag{A-3}$$

$$+\varphi(\varepsilon_n r)\frac{1}{\varepsilon_n}(\partial_r\varphi)\Big(\frac{r}{\varepsilon_n}\Big)f(r) \tag{A-4}$$

$$+\Big[1-\varphi(\varepsilon_n r)\Big(1-\varphi\Big(\frac{r}{\varepsilon_n}\Big)\Big)\Big]\partial_r f(r) \tag{A-5}$$

$$+\partial_r\Big[\varphi(\varepsilon_n r)\Big(1-\varphi\Big(\frac{r}{\varepsilon_n}\Big)\Big)\Big]\int_{-\frac{\varepsilon_n}{2}}^{0}(f(r)-f(r-\varrho))\,\zeta_{\varepsilon_n}(\varrho)\,d\varrho \tag{A-6}$$

$$+\varphi(\varepsilon_n r)\Big(1-\varphi\Big(\frac{r}{\varepsilon_n}\Big)\Big)\int_{-\frac{\varepsilon_n}{2}}^{0}(\partial_r f(r)-\partial_r f(r-\varrho))\,\zeta_{\varepsilon_n}(\varrho)\,d\varrho. \tag{A-7}$$

In view of (2-1), one easily sees that multiplying by r on both sides of the above identity, raising them to the power m and integrating over $(0,+\infty)$, we have the contributions of (A-6), (A-7) go to zero as $n\to\infty$. In fact, this is immediate for (A-7) in view of the boundedness of φ and the fact that ζ_ε is an approximation of the identity. For (A-6), we need to estimate two terms produced correspondingly by the cases when ∂_r hits on $\varphi(\varepsilon_n r)$ and $\varphi\big(\frac{r}{\varepsilon_n}\big)$. In the first case, we use the fundamental theorem of calculus to write

$$f(r)-f(r-\varrho)=\int_0^1 \varrho\,\partial_r f(r-\theta\varrho)\,d\theta.$$

Applying Minkowski's inequality, we are led to estimating

$$\varepsilon_n\int_{-\frac{\varepsilon_n}{2}}^{0}\int_0^1\Big(\int_{\frac{1}{2\varepsilon_n}}^{\frac{4}{\varepsilon_n}}|r\,\partial_r f(r)|^m\,dr\Big)^{\frac{1}{m}}|\varrho\,\zeta_{\varepsilon_n}(\varrho)|\,d\theta\,d\varrho,$$

which is clearly tending to zero as $n\to\infty$. A similar argument applies to the second case. In fact, applying the same trick will lead us to estimating

$$\int_{-\frac{\varepsilon_n}{2}}^{0}\int_0^1\Big(\int_{\frac{\varepsilon_n}{2}}^{2\varepsilon_n}|r\,\partial_r f(r)|^m\,dr\Big)^{\frac{1}{m}}\frac{|\varrho|}{\varepsilon_n}\,\zeta_{\varepsilon_n}(\varrho)\,d\theta\,d\varrho,$$

which tends to zero as $n\to\infty$.

Next, by invoking (2-2), one sees that the contribution from (A-3) is bounded by

$$\Big(\sup_{\frac{1}{\varepsilon_n}\le r\le\frac{2}{\varepsilon_n}} r^{\frac{1}{m}}|f(r)|\Big)^m\cdot\int_{\frac{1}{\varepsilon_n}}^{\frac{2}{\varepsilon_n}}|\varphi'(\varepsilon_n r)|^m\varepsilon_n^m r^{m-1}\,dr\to 0,\quad n\to\infty.$$

Similar argument applies to (A-4) thanks to (2-2). Finally, the contribution of (A-5) is easily seen to be bounded by

$$\int_0^{2\varepsilon_n}|r\,\partial_r f(r)|^m\,dr+\int_{\frac{1}{\varepsilon_n}}^{+\infty}|r\,\partial_r f(r)|^m dr\longrightarrow 0,\quad n\to\infty. \qquad\square$$

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References

[Bahouri and Gérard 1999] H. Bahouri and P. Gérard, "High frequency approximation of solutions to critical nonlinear wave equations", *Amer. J. Math.* **121**:1 (1999), 131–175. MR Zbl

[Bahouri and Shatah 1998] H. Bahouri and J. Shatah, "Decay estimates for the critical semilinear wave equation", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **15**:6 (1998), 783–789. MR Zbl

[Beceanu and Soffer 2017] M. Beceanu and A. Soffer, "Large outgoing solutions to supercritical wave equations", *Int. Math. Res. Not.* (online publication April 2017).

[Bergh and Löfström 1976] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Grundlehren der Mathematischen Wissenschaften **223**, Springer, 1976. MR Zbl

[Chamorro 2011] D. Chamorro, "Improved Sobolev inequalities and Muckenhoupt weights on stratified Lie groups", *J. Math. Anal. Appl.* **377**:2 (2011), 695–709. MR Zbl

[Collot 2014] C. Collot, "Type II blow up manifolds for the energy supercritical wave equation", preprint, 2014. To appear in *Mem. Amer. Math. Soc.* arXiv

[Dodson and Lawrie 2015] B. Dodson and A. Lawrie, "Scattering for the radial 3D cubic wave equation", *Anal. PDE* **8**:2 (2015), 467–497. MR Zbl

[Donninger and Krieger 2013] R. Donninger and J. Krieger, "Nonscattering solutions and blowup at infinity for the critical wave equation", *Math. Ann.* **357**:1 (2013), 89–163. MR Zbl

[Duyckaerts and Roy 2015] T. Duyckaerts and T. Roy, "Blow-up of the critical Sobolev norm for nonscattering radial solutions of supercritical wave equations on $\mathbb{R}^3$", preprint, 2015. To appear in *Bull. Soc. Math. France*. arXiv

[Duyckaerts et al. 2011] T. Duyckaerts, C. Kenig, and F. Merle, "Universality of blow-up profile for small radial type II blow-up solutions of the energy-critical wave equation", *J. Eur. Math. Soc.* (*JEMS*) **13**:3 (2011), 533–599. MR Zbl

[Duyckaerts et al. 2013] T. Duyckaerts, C. Kenig, and F. Merle, "Classification of radial solutions of the focusing, energy-critical wave equation", *Camb. J. Math.* **1**:1 (2013), 75–144. MR Zbl

[Duyckaerts et al. 2014] T. Duyckaerts, C. Kenig, and F. Merle, "Scattering for radial, bounded solutions of focusing supercritical wave equations", *Int. Math. Res. Not.* **2014**:1 (2014), 224–258. MR Zbl

[Duyckaerts et al. 2016] T. Duyckaerts, C. Kenig, and F. Merle, "Scattering profile for global solutions of the energy-critical wave equation", preprint, 2016. To appear in *J. Eur. Math. Soc.* (*JEMS*). arXiv

[Fang et al. 2011] D. Fang, J. Xie, and T. Cazenave, "Scattering for the focusing energy-subcritical nonlinear Schrödinger equation", *Sci. China Math.* **54**:10 (2011), 2037–2062. MR Zbl

[Farina 2007] A. Farina, "On the classification of solutions of the Lane–Emden equation on unbounded domains of $\mathbb{R}^N$", *J. Math. Pures Appl.* (9) **87**:5 (2007), 537–561. MR Zbl

[Gallagher et al. 2013] I. Gallagher, G. S. Koch, and F. Planchon, "A profile decomposition approach to the $L^\infty_t(L^3_x)$ Navier–Stokes regularity criterion", *Math. Ann.* **355**:4 (2013), 1527–1559. MR Zbl

[Gallagher et al. 2016] I. Gallagher, G. S. Koch, and F. Planchon, "Blow-up of critical Besov norms at a potential Navier–Stokes singularity", *Comm. Math. Phys.* **343**:1 (2016), 39–82. MR Zbl

[Gerard et al. 1997] P. Gerard, Y. Meyer, and F. Oru, "Inégalités de Sobolev précisées", pp. Exp. No. IV, 11 in *Séminaire sur les Équations aux Dérivées Partielles, 1996–1997*, École Polytech., Palaiseau, 1997. MR Zbl

[Ginibre and Velo 1995] J. Ginibre and G. Velo, "Generalized Strichartz inequalities for the wave equation", *J. Funct. Anal.* **133**:1 (1995), 50–68. MR Zbl

[Ginibre et al. 1992] J. Ginibre, A. Soffer, and G. Velo, "The global Cauchy problem for the critical nonlinear wave equation", *J. Funct. Anal.* **110**:1 (1992), 96–130. MR Zbl

[Grafakos 2014] L. Grafakos, *Modern Fourier analysis*, 3rd ed., Graduate Texts in Mathematics **250**, Springer, 2014. MR Zbl

[Grillakis 1990] M. G. Grillakis, "Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity", *Ann. of Math.* (2) **132**:3 (1990), 485–509. MR Zbl

[Grillakis 1992] M. G. Grillakis, "Regularity for the wave equation with a critical nonlinearity", *Comm. Pure Appl. Math.* **45**:6 (1992), 749–774. MR Zbl

[Hillairet and Raphaël 2012] M. Hillairet and P. Raphaël, "Smooth type II blow-up solutions to the four-dimensional energy-critical wave equation", *Anal. PDE* **5**:4 (2012), 777–829. MR Zbl

[Iskauriaza et al. 2003] L. Iskauriaza, G. A. Serëgin, and V. Shverak, "$L_{3,\infty}$-solutions of Navier–Stokes equations and backward uniqueness", *Uspekhi Mat. Nauk* **58**:2(350) (2003), 3–44. In Russian; translated in *Russian Math. Surveys* **58**:2 (2003), 211–250. MR

[Jaffard 1999] S. Jaffard, "Analysis of the lack of compactness in the critical Sobolev embeddings", *J. Funct. Anal.* **161**:2 (1999), 384–396. MR Zbl

[Jendrej 2016] J. Jendrej, "Construction of two-bubble solutions for energy-critical wave equations", preprint, 2016. To appear in *Amer. J. Math.* arXiv

[Jendrej 2017] J. Jendrej, "Construction of type II blow-up solutions for the energy-critical wave equation in dimension 5", *J. Funct. Anal.* **272**:3 (2017), 866–917. MR Zbl

[Joseph and Lundgren 1973] D. D. Joseph and T. S. Lundgren, "Quasilinear Dirichlet problems driven by positive sources", *Arch. Rational Mech. Anal.* **49** (1973), 241–269. MR Zbl

[Kapitanski 1994] L. Kapitanski, "Global and unique weak solutions of nonlinear wave equations", *Math. Res. Lett.* **1**:2 (1994), 211–223. MR Zbl

[Kenig and Koch 2011] C. E. Kenig and G. S. Koch, "An alternative approach to regularity for the Navier–Stokes equations in critical spaces", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **28**:2 (2011), 159–187. MR Zbl

[Kenig and Merle 2010] C. E. Kenig and F. Merle, "Scattering for $\dot{H}^{1/2}$ bounded solutions to the cubic, defocusing NLS in 3 dimensions", *Trans. Amer. Math. Soc.* **362**:4 (2010), 1937–1962. MR Zbl

[Kenig and Merle 2011] C. E. Kenig and F. Merle, "Nondispersive radial solutions to energy supercritical non-linear wave equations, with applications", *Amer. J. Math.* **133**:4 (2011), 1029–1065. MR Zbl

[Killip and Visan 2010] R. Killip and M. Visan, "Energy-supercritical NLS: critical $\dot{H}^s$-bounds imply scattering", *Comm. Partial Differential Equations* **35**:6 (2010), 945–987. MR Zbl

[Killip and Visan 2011] R. Killip and M. Visan, "The defocusing energy-supercritical nonlinear wave equation in three space dimensions", *Trans. Amer. Math. Soc.* **363**:7 (2011), 3893–3934. MR Zbl

[Killip et al. 2014] R. Killip, B. Stovall, and M. Visan, "Blowup behaviour for the nonlinear Klein–Gordon equation", *Math. Ann.* **358**:1-2 (2014), 289–350. MR Zbl

[Krieger and Schlag 2007] J. Krieger and W. Schlag, "On the focusing critical semi-linear wave equation", *Amer. J. Math.* **129**:3 (2007), 843–913. MR Zbl

[Krieger and Schlag 2014a] J. Krieger and W. Schlag, "Full range of blow up exponents for the quintic wave equation in three dimensions", *J. Math. Pures Appl.* (9) **101**:6 (2014), 873–900. MR Zbl

[Krieger and Schlag 2014b] J. Krieger and W. Schlag, "Large global solutions for energy supercritical nonlinear wave equations on $\mathbb{R}^{3+1}$", preprint, 2014. To appear in *J. Anal. Math.* arXiv

[Krieger et al. 2009] J. Krieger, W. Schlag, and D. Tataru, "Slow blow-up solutions for the $H^1(\mathbb{R}^3)$ critical focusing semilinear wave equation", *Duke Math. J.* **147**:1 (2009), 1–53. MR Zbl

[Lindblad and Sogge 1995] H. Lindblad and C. D. Sogge, "On existence and scattering with minimal regularity for semilinear wave equations", *J. Funct. Anal.* **130**:2 (1995), 357–426. MR Zbl

[Luk et al. 2016] J. Luk, S.-J. Oh, and S. Yang, "Solutions to the Einstein-scalar-field system in spherical symmetry with large bounded variation norms", preprint, 2016. To appear in *Annals of PDE*. arXiv

[Martel and Merle 2016] Y. Martel and F. Merle, "Construction of multi-solitons for the energy-critical wave equation in dimension 5", *Arch. Ration. Mech. Anal.* **222**:3 (2016), 1113–1160. MR Zbl

[Merle and Raphaël 2008] F. Merle and P. Raphaël, "Blow up of the critical norm for some radial L^2 super critical nonlinear Schrödinger equations", *Amer. J. Math.* **130**:4 (2008), 945–978. MR Zbl

[Nakanishi 1999] K. Nakanishi, "Scattering theory for the nonlinear Klein–Gordon equation with Sobolev critical power", *Internat. Math. Res. Notices* **1999**:1 (1999), 31–60. MR Zbl

[Rodriguez 2017] C. Rodriguez, "Scattering for radial energy-subcritical wave equations in dimensions 4 and 5", *Comm. Partial Differential Equations* **42**:6 (2017), 852–894. MR arXiv

[Seregin 2012] G. Seregin, "A certain necessary condition of potential blow up for Navier–Stokes equations", *Comm. Math. Phys.* **312**:3 (2012), 833–845. MR Zbl

[Shatah and Struwe 1993] J. Shatah and M. Struwe, "Regularity results for nonlinear wave equations", *Ann. of Math.* (2) **138**:3 (1993), 503–518. MR Zbl

[Shatah and Struwe 1994] J. Shatah and M. Struwe, "Well-posedness in the energy space for semilinear wave equations with critical growth", *Internat. Math. Res. Notices* **1994**:7 (1994), 303–309. MR Zbl

[Shen 2013] R. Shen, "On the energy subcritical, nonlinear wave equation in $\mathbb{R}^3$ with radial data", *Anal. PDE* **6**:8 (2013), 1929–1987. MR Zbl

[Solimini 1995] S. Solimini, "A note on compactness-type properties with respect to Lorentz norms of bounded subsets of a Sobolev space", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **12**:3 (1995), 319–337. MR Zbl

[Strauss 1977] W. A. Strauss, "Existence of solitary waves in higher dimensions", *Comm. Math. Phys.* **55**:2 (1977), 149–162. MR Zbl

[Tao 2016] T. Tao, "Finite-time blowup for a supercritical defocusing nonlinear wave system", *Anal. PDE* **9**:8 (2016), 1999–2030. MR Zbl

[Tao and Visan 2005] T. Tao and M. Visan, "Stability of energy-critical nonlinear Schrödinger equations in high dimensions", *Electron. J. Differential Equations* **2005** (2005), art. id. 118. MR Zbl

[Trudinger 1968] N. S. Trudinger, "Remarks concerning the conformal deformation of Riemannian structures on compact manifolds", *Ann. Scuola Norm. Sup. Pisa* (3) **22** (1968), 265–274. MR Zbl

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