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LIMITING DISTRIBUTION OF ELLIPTIC HOMOGENIZATION ERROR WITH PERIODIC DIFFUSION AND RANDOM POTENTIAL

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We study the limiting probability distribution of the homogenization error for second order elliptic equations in divergence form with highly oscillatory periodic conductivity coefficients and highly oscillatory stochastic potential. The effective conductivity coefficients are the same as those of the standard periodic homogenization, and the effective potential is given by the mean. We show that the limiting distribution of the random part of the homogenization error, as random elements in proper Hilbert spaces, is Gaussian and can be characterized by the homogenized Green's function, the homogenized solution and the statistics of the random potential. This generalizes previous results in the setting with slowly varying diffusion coefficients, and the current setting with fast oscillations in the differential operator requires new methods to prove compactness of the probability distributions of the random fluctuation.

1. Introduction

In this article we study the limiting distribution, in certain Hilbert spaces, of the homogenization error for second order elliptic equations in divergence form with highly oscillatory periodic diffusion coefficients and highly oscillatory random potential.

More precisely, we consider the following Dirichlet problem on an open bounded subset $D \subset \mathbb{R}^n$, with homogeneous boundary condition and a source term $f \in L^2(D)$,

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\Big(a_{ij}\Big(\dfrac{x}{\varepsilon}\Big)\dfrac{\partial u^\varepsilon}{\partial x_j}(x,\omega)\Big) + q\Big(\dfrac{x}{\varepsilon},\omega\Big)u^\varepsilon(x,\omega) = f(x), & x \in D, \\ u^\varepsilon(x) = 0, & x \in \partial D. \end{cases} \tag{1-1}$$

The conductivity coefficients $\big(a_{ij}\big(\frac{\cdot}{\varepsilon}\big)\big)$ and the potential $q\big(\frac{\cdot}{\varepsilon},\omega\big)$ are highly oscillatory in space, and $0 < \varepsilon \ll 1$ indicates the small scale on which these coefficients oscillate. We assume that the conductivity coefficients are deterministic and periodic, and the potential is a stationary random field on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. More precise assumptions are given in Section 2. It is well known that, under mild assumptions like stationary ergodicity of $q(x,\omega)$, the equation above homogenizes; i.e., u^ε converges, almost surely in Ω, weakly in $H^1(D)$ and strongly in $L^2(D)$ to the solution of the deterministic homogenized problem

$$\begin{cases} -\bar{a}_{ij}\dfrac{\partial^2 u}{\partial x_i\,\partial x_j}(x) + \bar{q}u(x) = f(x), & x \in D, \\ u(x) = 0, & x \in \partial D. \end{cases} \tag{1-2}$$

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Here, the effective conductivity coefficients $(\bar{a}_{ij})$ are constants defined by

$$\bar{a}_{ij} = \int_{\mathbb{T}^d} a_{ik}(y)\Big(\delta_{kj} + \frac{\partial \chi^k}{\partial x_j}(y)\Big)\, dy, \tag{1-3}$$

where $\mathbb{T}^d = [0,1]^d$ denotes the unit cell and the correctors χ^k, with $k = 1, \dots, d$, are given by the unique solution of the corrector equation

$$-\frac{\partial}{\partial x_i}\Big(a_{ij}(y)\Big(e_k + \frac{\partial \chi^k}{\partial x_j}(y)\Big)\Big) = 0 \quad \text{on } \mathbb{T}^d, \tag{1-4}$$

with the normalization condition $\int_{\mathbb{T}^d} \chi^k\, dy = 0$; e_k above is the k-th standard unit basis vector of $\mathbb{R}^d$. We note that this formula for $(\bar{a}_{ij})$ is exactly the classic periodic homogenization formula for effective conductivity. The effective potential $\bar{q}$ in (1-2) is given by the constant

$$\bar{q} = \mathbb{E} q(0,\omega), \tag{1-5}$$

where $\mathbb{E}$ denotes the mathematical mean with respect to $\mathbb{P}$.

In this paper we study the law (probability distribution) of the homogenization error $u^\varepsilon - u$, viewed as random elements in certain Hilbert spaces. We split this error into two parts: $\mathbb{E}u^\varepsilon - u$ and $u^\varepsilon - \mathbb{E}u^\varepsilon$. In view of the deterministic oscillations in the diffusion coefficients, we expect that the periodic homogenization error, in the replacement of $\big(a^{ij}\big(\frac{\cdot}{\varepsilon}\big)\big)$ to $(\bar{a}_{ij})$, makes significant contributions to the deterministic error $\mathbb{E}u^\varepsilon - u$. Indeed, we show later that this error is essentially of order $O(\varepsilon)$, the same as periodic homogenization. On the other hand, the effect of the random potential $q\big(\frac{\cdot}{\varepsilon}, \omega\big)$ becomes visible in the random fluctuation $u^\varepsilon - \mathbb{E}u^\varepsilon$, in which the (large) mean is removed. We are interested in characterizing the size and the law of this random fluctuation, and the answers depend on finer information of the random potential q, such as the decay rate of the correlations in q and higher-order moments of q; see Section 2 for notations and definitions.

We find that, when $q(x,\omega)$ has short-range correlations, the random fluctuation $u^\varepsilon - \mathbb{E}u^\varepsilon$ scales like $\varepsilon^{d/2 \wedge 2}$ in the $L^1(\Omega, L^2(D))$-norm, and scales like $\varepsilon^{d/2}$ when integrated against a test function. Moreover, the law of the scaled random fluctuation $\varepsilon^{-d/2}(u^\varepsilon - \mathbb{E}u^\varepsilon)$ in $L^2(D)$ for $d = 2, 3$ and in $H^{-1}(D)$ for $d = 4, 5$ converges to Gaussian distributions as follows (see Theorem 2.4 for details):

$$\frac{u^\varepsilon - \mathbb{E}u^\varepsilon}{\sqrt{\varepsilon^d}} \xrightarrow{\text{distribution}} \sigma \int_D G(x,y) u(y)\, dW(y).$$

Here, $W(y)$ is the standard multiparameter Wiener process, and hence the law of the right-hand side above defines a Gaussian probability measure on $L^2(D)$ or $H^{-1}(D)$. This Gaussian distribution is determined by $G(x,y)$, which is the Green's function associated to the homogenized problem (1-2), $u(y)$, which is the homogenized solution, and σ, which is some statistical parameter of the random potential $q(x,\omega)$.

We also consider the case when $q(x,\omega) = \Phi(g(x,\omega))$ is constructed as a function of a Gaussian random field $g(x,\omega)$, and g has long-range correlations that decay like $|x|^{-\alpha}$, with $0 < \alpha < d$. Then the random fluctuation scales like $\varepsilon^{\alpha/2 \wedge 2}$ in the $L^1(\Omega, L^2(D))$-norm, and scales like $\varepsilon^{\alpha/2}$ when integrated against a test function. Moreover, the law of the scaled random fluctuation $\varepsilon^{-\alpha/2}(u^\varepsilon - \mathbb{E}u^\varepsilon)$ in $L^2(D)$ for $d = 2, 3$

and in $H^{-1}(D)$ for $d = 4, 5$ converges to a Gaussian distribution that can be written as a stochastic integral as above, but with dW replaced by $\dot{W}^\alpha\, dy$, where $\dot{W}^\alpha$ is a centered Gaussian random field with correlation function $|x - y|^{-\alpha}$, and σ is replaced by some other statistical parameter; see Theorem 6.2 for details.

The study of the limiting distribution of the homogenization error goes back to [Figari et al. 1982], where the Laplace operator with a random potential formed by Poisson bumps was considered. General random potential with short-range correlations was considered recently in [Bal 2008], and in [Bal and Jing 2010; 2011] for other nonoscillatory differential operators with random potential. Long-range correlated random potential was considered in [Bal et al. 2012]. When oscillatory differential operators were considered, the limiting distribution of homogenization error was obtained in [Bourgeat and Piatnitski 1999] for short-range correlated elliptic coefficients, and in [Bal et al. 2008] for the long-range correlated case, all in the one-dimensional setting. The main results of this paper show that the general framework developed in [Bal 2008; Bal and Jing 2011; Bal et al. 2012], in order to characterize the random fluctuation caused by the random potential, applies even when there are oscillations in the differential operators, as long as these oscillations are not statistically related to those of the random potential.

Our approach is as follows: we introduce an auxiliary problem with periodic diffusion coefficients and homogenized potential; let v^ε be the solution. Then the deterministic homogenization error $\mathbb{E}u^\varepsilon - u$ is essentially characterized by $v^\varepsilon - u$, which amounts to classical periodic homogenization theory. The random fluctuation $u^\varepsilon - \mathbb{E}u^\varepsilon$ is then the same as $(u^\varepsilon - v^\varepsilon) - \mathbb{E}(u^\varepsilon - v^\varepsilon)$, which can be represented as a truncated Neumann series. The first term X^ε in this series contributes to the limiting distribution. By Prohorov's theorem, we need to show that the probability measures of $\{X^\varepsilon\}$ are tight in the proper Hilbert space, and that their characteristic functions converge. The latter is essentially the convergence in distribution of the integration of X^ε against test functions; in view of the uniform-in-ε estimates of the Green's functions associated to the oscillatory diffusion, this step is the same as the earlier setting with nonoscillatory diffusion. The role of oscillations in the diffusion, however, becomes prominent in the step of proving tightness of the measures of $\{X^\varepsilon\}$. The simple and natural method used in [Bal et al. 2012] fails completely; see Section 7 for details. New ideas are needed: we obtain tightness of the measures of $\{X^\varepsilon\}$ in $L^2(D)$ by controlling the mean square of the H^s-norm of $\{X^\varepsilon\}$ for some $0 < s < \frac{1}{2}$; similarly, we get tightness in $H^{-1}(D)$ by controlling the mean square of the H^{-s}-norm with $\frac{1}{2} < s < 1$. The constraints on the spatial dimension d arise naturally in the proof of such controls.

Our analysis relies on uniform estimates of the Green's function associated to the periodic homogenization problem; we refer to [Avellaneda and Lin 1987; 1991] for the classical results, and to [Kenig et al. 2012; 2014] for recent development in this direction. We refer to [Armstrong and Smart 2014; Armstrong et al. 2015; Marahrens and Otto 2015; Gloria and Otto 2014] for recent results on uniform estimates of the Green's function for equations with highly oscillatory random diffusion coefficients in spatial dimension higher than one. We remark also that in the random setting, the limiting distribution of the corrector function and that of the full random fluctuation $u^\varepsilon - \mathbb{E}u^\varepsilon$, in negative Hölder space, were obtained in [Mourrat and Nolen 2015] and [Gu and Mourrat 2015] respectively, in the discrete setting; see also [Mourrat and Otto 2014]. Such results are apparently more challenging to obtain, and the proofs require delicate calculus in the (infinite-dimensional) probability space.

The rest of this paper is organized as follows: In Section 2 we make precise the main assumptions on the parameters of the homogenization problem, in particular on the properties of the random potential, and state the main results in the short-range correlation setting. Homogenization of (1-1) and some useful results on periodic homogenization theory are recalled in Section 3. Sections 4 and 5 are devoted to the proofs of the main results, where we characterize how the random fluctuation scales in the energy norm and in the weak topology, and determine the limiting distribution of the scaled fluctuation. We present new methods to prove the tightness of the probability measures of the random fluctuations. In Section 6, we state and prove the corresponding results in the long-range correlation setting. We make some comments and further discussions in Section 7 and prove some technical results, such as tightness criteria for probability measures, in the Appendix.

2. Assumptions, preliminaries and main results

2A. ***Assumptions on the coefficients.*** Throughout this paper, we assume that the domain D in (1-1) is an open bounded set of $\mathbb{R}^d$ with $C^{1,1}$-boundary. The coefficients $a_{ij}\big(\frac{x}{\varepsilon}\big)$ and $q\big(\frac{x}{\varepsilon},\omega\big)$ are the scaled versions of $a_{ij}(x)$ and $q(x,\omega)$. We make the following main assumptions on a_{ij} and q.

Periodic diffusion coefficients. For the functions (a_{ij}), we assume:

(A1) (periodicity) The function $A := (a_{ij}) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is periodic. That is, for all $x \in \mathbb{R}^d$, $k \in \mathbb{Z}^d$ and $i, j = 1, 2, \dots, d$, we have

$$a_{ij}(x+k) = a_{ij}(x). \tag{2-1}$$

(A2) (uniform ellipticity) For all $y \in \mathbb{T}^d$, the matrix $A(y) = (a_{ij}(y))$ is uniformly elliptic in the sense that, for all $\xi \in \mathbb{R}^d$, one has

$$\lambda|\xi|^2 \le \xi^T A(y)\xi = \sum_{i,j=1}^{d} \xi_i a_{ij}(y)\xi_j \le \Lambda|\xi|^2. \tag{2-2}$$

(A3) (smoothness) For some γ, M with $\gamma \in (0,1]$ and $M > 0$, one has

$$\|A\|_{C^\gamma(\mathbb{T}^d)} \le M. \tag{2-3}$$

We henceforth refer to the above assumptions together as (A).

Random potential. For the random field $q(x,\omega)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we assume:

(P) (stationarity and ergodicity) There exists an ergodic group of $\mathbb{P}$-preserving transformations $(\tau_x)_{x\in\mathbb{R}^d}$ on Ω, where ergodicity means that $E \in \mathcal{F}$ and

$$\tau_x E = E \quad \text{for all } x \in \mathbb{R}^d$$

imply that $\mathbb{P}(E) \in \{0,1\}$. The random potential $q(y,\omega)$ is given by $\tilde{q}(\tau_y\omega)$, where $\tilde{q} : \Omega \to \mathbb{R}$ is a random variable satisfying

$$0 \le \tilde{q}(\omega) \le M \quad \text{for all } \omega \in \Omega. \tag{2-4}$$

Further assumptions on q. The above assumptions are sufficient for proving the homogenization result. However, to estimate the size of the homogenization error and to characterize the limiting distribution of the random fluctuation, more assumptions on the random field $q(\cdot, \omega)$ are necessary.

To simplify notations, we write in the sequel

$$q(x,\omega) = \bar{q} + \nu(x,\omega),$$

where $\bar{q}$ is the mean of q and ν is the fluctuation. Note that $\bar{q}$ is a deterministic constant and ν is a mean zero stationary ergodic random field. The autocorrelation function $R(x)$ of q (and hence ν) is defined as

$$R(x) = \mathbb{E}\big(\nu(x+y,\omega)\nu(y,\omega)\big), \quad \sigma^2 := \int_{\mathbb{R}^d} R(x)\,dx. \tag{2-5}$$

By Bochner's theorem, $R(x)$ is a positive definite function and $\sigma^2 \geq 0$. We assume that $\sigma > 0$. When R is integrable on $\mathbb{R}^d$, i.e., $\sigma^2 < \infty$, we say that q has short-range correlations; we say q has long-range correlations if otherwise. We state and prove the main results in the setting where q has short-range correlations, and mention the corresponding results for the long-range correlation setting in Section 6.

Short-range correlated random fields. In this case, we make an assumption on the rate of decay of the correlation function. We denote by $\mathcal{C}$ the set of compact sets in $\mathbb{R}^d$, and for two sets K_1, K_2 in $\mathcal{C}$, the distance $d(K_1, K_2)$ is defined to be

$$d(K_1,K_2) = \min_{x\in K_1, y\in K_2} |x-y|.$$

Given any compact set $K \subset \mathcal{C}$, we denote by $\mathcal{F}_K$ the σ-algebra generated by the random variables $\{q(x) : x \in K\}$. We define the "maximal correlation coefficient" ϱ of q as follows: for each $r > 0$, $\varrho(r)$ is the smallest value such that the bound

$$\mathbb{E}\big(\varphi_1(q)\varphi_2(q)\big) \leq \varrho(r)\sqrt{\mathbb{E}(\varphi_1^2(q))\,\mathbb{E}(\varphi_2^2(q))} \tag{2-6}$$

holds for any two compact sets $K_1, K_2 \in \mathcal{C}$ such that $d(K_1, K_2) \geq r$ and for any two random variables of the form $\varphi_i(q)$, with $i = 1, 2$, such that $\varphi_i(q)$ is $\mathcal{F}_{K_i}$-measurable and $\mathbb{E}\varphi_i(q) = 0$. We assume that

(S) The maximal correlation function satisfies $\varrho^{1/2} \in L^1(\mathbb{R}_+, r^{d-1}dr)$; that is,

$$\int_0^\infty \varrho^{\frac{1}{2}}(r) r^{d-1}\,dr < \infty.$$

Assumptions on the mixing coefficient ϱ of random media have been used in [Bal 2008; Bal and Jing 2011; Hairer et al. 2013]; we refer to these papers for explicit examples of random fields satisfying the assumptions. We note that the autocorrelation function $R(x)$ can be bounded by ϱ. For any $x \in \mathbb{R}^d$,

$$|R(x)| = \big|\mathbb{E}(q(x)-\mathbb{E}q)(q(0)-\mathbb{E}q)\big| \leq \varrho(|x|)\operatorname{Var}(q).$$

By (2-4), q, and hence its variance, is bounded. In view of (S) and the fact that one can assume $\varrho \in [0, 1]$ (hence $\varrho \leq \sqrt{\varrho}$), we find that R is integrable. Therefore, (S) implies that $q(x,\omega)$ has short-range correlations. In fact, (S) is a much stronger assumption, and not necessary for the main results of this paper to hold. In Section 7, we will provide alternative and less restrictive assumptions that are sufficient.

However, using the assumption (S) and Lemma 4.3 below, we can simplify significantly certain fourth-order moment estimates of the random potential $\nu(x,\omega)$; such estimates appear often in the study of the limiting distribution of the homogenization error.

Notations. Throughout the paper, by *universal parameters* we refer to λ, Λ, γ and M in the assumptions (A), the autocorrelation function R, σ^2, and the mixing coefficients ϱ, the domain D and its boundary ∂D, and the dimension d. If a constant C depends only on these parameters, we say either C depends on universal parameters or C is a universal constant. For the random potential $\nu(x,\omega)$ and the functions $\varrho(x)$, $R(x)$, etc. which are related to ν, we use $\nu^\varepsilon, \varrho^\varepsilon, R^\varepsilon$, etc. to denote the scaled versions. For instance, $\nu^\varepsilon(x,\omega)$ is shorthand notation for $\nu\big(\frac{x}{\varepsilon}\big)$. We use the notation $H^s(K)$, with $s \ge 0$, for the Sobolev or the fractional Sobolev space $W^{s,2}(K)$ on some domain $K \subset \mathbb{R}^d$; when K is bounded, we use $H^s_0(K)$ for the subspace that consists of functions having trace zero at ∂K; note that $H^s_0(\mathbb{R}^d) = H^s(\mathbb{R}^d)$. We denote by $H^{-s}(K)$, with $s > 0$, the dual space $(H^s_0(K))'$. For any Hilbert space $\mathcal{H}$, we denote the inner product in $\mathcal{H}$ by $(\cdot\,,\cdot)_{\mathcal{H}}$; when $\mathcal{H} = L^2(D)$, we very often omit the subscript and write $(\cdot\,,\cdot)$ instead. We use $\langle f, g\rangle$ whenever the formal integral $\int_D fg$ makes sense. We typically use $\mathbb{1}_A$ for the indication function of a set $A \subset \mathbb{R}^d$, or if A is a statement, the indication function of A being true. Finally, for two real numbers a and b, we use $a \wedge b$ as a shorthand notation for $\min\{a,b\}$, and $a \vee b$ means $\max\{a,b\}$.

2B. ***Probability distribution on functional spaces.*** We view the random fluctuation $u^\varepsilon - \mathbb{E}u^\varepsilon$ in the homogenization error as random elements in certain functional spaces, and aim to find the limit of its law in that space. It turns out that the choice of functional spaces depends on the spatial dimension d.

When $d = 1$, one can choose the space $C(D)$ of continuous functions. In fact, convergence in distribution in $C(D)$ was proved in [Bal 2008] for random diffusion coefficient $a(x,\omega)$ with random potential $q(x,\omega)$, both having short-range correlations. In this paper, we prove that for $d = 2,3$, the space can be chosen as $L^2(D)$ and for $d = 4,5$, the space can be chosen as $H^{-1}(D)$. Note that both choices are Hilbert spaces. We recall some facts concerning weak convergence of probability measures on Hilbert spaces. We refer to the books of Billingsley [1999] and Parthasarathy [1967] for more details.

Probability distributions on a Hilbert space. Let $\mathcal{H}$ be a separable Hilbert space, and let $X(\omega)$ be an $\mathcal{H}$-valued random element on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then X determines a probability measure P^X on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$, where $\mathcal{B}(\mathcal{H})$ denotes the Borel σ-algebra generated by open sets in $\mathcal{H}$, by

$$P^X(\mathcal{S}) = \mathbb{P}(X \in \mathcal{S}) \quad \text{for any } \mathcal{S} \in \mathcal{B}(\mathcal{H}). \tag{2-7}$$

We say a family $\{X^\varepsilon\}_{\varepsilon\in(0,1)}$ of random elements in $\mathcal{H}$ converges in probability distribution (or in law), as $\varepsilon \to 0$, to another random element X on $\mathcal{H}$, if the probability measures P^{X^ε} converge weakly to P^X; i.e., for any real bounded continuous functional $f : \mathcal{H} \to \mathbb{R}$,

$$\int_{\mathcal{H}} f(g)\, dP^{X^\varepsilon}(g) \to \int_{\mathcal{H}} f(g)\, dP^X(g).$$

In particular, any probability measure P on a separable Hilbert space $\mathcal{H}$ is determined by its characteristic function $\phi^P : \mathcal{H} \to \mathbb{C}$,

$$\phi^P(h) = \int_{\mathcal{H}} e^{i(h,g)_{\mathcal{H}}}\, dP(g). \tag{2-8}$$

Moreover, the following result holds:

Theorem 2.1 [Parthasarathy 1967, Chapter VI, Lemma 2.1]. *Let $\{X^\varepsilon\}_{\varepsilon\in(0,1)}$ and X be random elements in $\mathcal{H}$, possibly defined on different probability spaces. Then X^ε converges to X in law in $\mathcal{H}$, as $\varepsilon \to 0$, if the family of probability measures $\{P^{X^\varepsilon}\}_{\varepsilon\in(0,1)}$ is tight and for any $h \in \mathcal{H}$,*

$$\lim_{\varepsilon\to 0} \phi^{P^{X^\varepsilon}}(h) = \phi^{P^X}(h). \tag{2-9}$$

Remark 2.2. Let $\mathcal{H} = L^2(D)$, which is a separable Hilbert space, and let X be a random element in $L^2(D)$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The characteristic function of P^X can be calculated as follows: for any $h \in L^2(D)$,

$$\phi^{P^X}(h) = \int_{\mathbb{R}} e^{iz}\, dP^X\big(\{(h,g) > z\}\big) = \int_{\mathbb{R}} e^{iz}\, d\mathbb{P}\big(\{(h, X(\omega)) > z\}\big) = \mathbb{E}\, e^{i(h,X)}. \tag{2-10}$$

Therefore, to prove that X^ε converges in distribution to X as L^2-paths, it suffices to show that $\{P^{X^\varepsilon}\}$ is tight and that for any $h \in L^2(D)$,

$$(h, X^\varepsilon) \xrightarrow{\text{distribution}} (h, X); \tag{2-11}$$

that is, the random variables (h, X^ε) converge in distribution to the random variable (h, X).

In Theorem A.1 in the Appendix, we provide a tightness criterion for $\{P^{X^\varepsilon}\}$ on $L^2(D)$, with the assumption that $\{X^\varepsilon(\cdot,\omega)\}$ is in $H_0^s(D)$ for certain $s > 0$. The criterion is sufficient but by no means necessary. Nevertheless, it is very handy for our analysis since the random fields X^ε that we are dealing with come from solutions of (1-1), and hence are naturally in $H_0^s(D)$.

2C. *Main results.* We now state the main results of the paper under the assumption that $q(x,\omega)$ has *short-range correlations*. Analogous results for the long-range correlation setting will be presented in Section 6.

The first main theorem concerns how the homogenization error scales.

Theorem 2.3. *Let $D \subset \mathbb{R}^d$ be an open bounded $C^{1,1}$ domain, u^ε and u be the solutions to (1-1) and (1-2) respectively. Suppose that (A), (P) and (S) hold, $f \in L^2(D)$ and $2 \le d \le 7$. Then, there exists positive constant C, depending only on the universal parameters, such that*

$$E\,\|u^\varepsilon - u\|_{L^2} \le C\varepsilon \|f\|_{L^2}. \tag{2-12}$$

Moreover,

$$E\,\|u^\varepsilon - \mathbb{E}u^\varepsilon\|_{L^2} \le \begin{cases} C\varepsilon^{2\wedge\frac{d}{2}}\|f\|_{L^2} & \text{if } d \ne 4,\\ C\varepsilon^2|\log\varepsilon|^{\frac12}\|f\|_{L^2} & \text{if } d = 4.\end{cases} \tag{2-13}$$

Furthermore, for any $\varphi \in L^2(D)$,

$$\mathbb{E}\,\big|(u^\varepsilon - \mathbb{E}u^\varepsilon, \varphi)_{L^2}\big| \le C\varepsilon^{\frac{d}{2}}\|\varphi\|_{L^2}\|f\|_{L^2}. \tag{2-14}$$

This theorem provides $L^1(\Omega, L^2(D))$-estimates of $u^\varepsilon - u$ and its random part, and its proof is detailed in Section 4. We note that the size of the full homogenization error is much larger than that of its random part. This is because the oscillations in the diffusion coefficients cause some deterministic fluctuation in the solution of size $O(\varepsilon)$, as in standard periodic homogenization. The additional random fluctuation caused by the short-range correlated random potential scales like $\varepsilon^{(d\wedge 4)/2}$ in the energy norm, and scales like $\varepsilon^{d/2}$ in the weak topology. These results agree with the case of nonoscillatory diffusion coefficients; see [Bal 2008; Bal and Jing 2011]. The next result exhibits the limiting law of the rescaled random fluctuation $\varepsilon^{-d/2}(u^\varepsilon - \mathbb{E}u^\varepsilon)$.

Theorem 2.4. *Suppose the assumptions in Theorem 2.3 hold. Let σ be defined as in (2-5) and $G(x,y)$ be the Green's function of (1-2). Let $W(y)$ denote the standard d-parameter Wiener process. Then*

(i) *For $d = 2, 3$, as $\varepsilon \to 0$,*

$$\frac{u^\varepsilon - \mathbb{E}u^\varepsilon}{\sqrt{\varepsilon^d}} \xrightarrow{\text{distribution}} \sigma \int_D G(x,y)u(y)\,dW(y) \quad \text{in } L^2(D). \tag{2-15}$$

(ii) *For $d = 4, 5$, as $\varepsilon \to 0$, the above holds as convergence in law in $H^{-1}(D)$.*

The proof of item (i) above can be found on page 217 and that of item (ii) is on page 220.

Remark 2.5. The integral on the right-hand side of (2-15) is understood, for each fixed x, as a Wiener integral in y with respect to the multiparameter Wiener process $W(y)$. Let X denote the result. For $d = 2, 3$, because the Green's function $G(x,y)$ is square integrable, X is a random element in $L^2(D)$. For $d = 4, 5$, X is understood through the Fourier transform of its distribution: given $h^* \in H^{-1}(D)$, $\phi^{P^X}(h^*)$ is defined to be $\boldsymbol{E}\exp\bigl(i\sigma\int_D \langle G(\cdot\,,y), h^*(\cdot)\rangle u(y)\,dW(y)\bigr)$, where $\boldsymbol{E}$ is the expectation with respect to the law of W.

Remark 2.6. We expect that the scaling factor for the random fluctuation, with respect to the weak topology, should be $\varepsilon^{-d/2}$ in all dimensions. More precisely, for any $\varphi \in L^2(D)$, we expect that $\varepsilon^{-d/2}(u^\varepsilon - \mathbb{E}u^\varepsilon, \varphi)$ should converge in distribution for all dimensions. However, in this paper we control this term only for $d \le 7$. This constraint is not intrinsic, and is mainly due to the fact that we stopped at second order iteration in the series expansion (4-11). In fact, if higher- (than six or more) order moments of the random field are under control, we can iterate as many times as we need in (4-11) until the last term is small, and use higher-order moments to estimate the terms in between; see Remark 4.6 below.

The spatial dimension plays an intrinsic role on the choice of topology that one should use for the limiting distribution of the random fluctuations. Indeed, for the term $X^\varepsilon = -\varepsilon^{-d/2}\mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon$ to converge in law in $L^2(D)$, it is necessary that $\mathbb{E}\|X^\varepsilon\|_{L^2}^2$ is controlled uniformly in ε. In view of the singularity of the Green's function, namely, of order $|x-y|^{-d+2}$ near the diagonal, we expect to control $\mathbb{E}\|X^\varepsilon\|_{L^2}^2$ only for $d \le 3$, and similarly, we expect to have convergence in law in H^{-1} only for $d < 6$. Nevertheless, we expect that convergence in law in $H^{-k}(D)$, for certain $k > 0$ increasing with respect to d, could be proved, provided more controls on the random field are available.

Finally, we remark that other topologies, e.g., those in [Bal et al. 2012; Gu and Mourrat 2015], can be considered for the law of the random fluctuation as well. In particular, tightness criteria in the Hölder space C^α, with α possibly negative, were established in [Mourrat 2015]. By a formal scaling argument, the short-range noise ν_ε belongs to the Hölder class C^{0-} and the Green's function is in C^{2-d}. The convergence of X^ε, which is essentially a convolution of the Green's function with the noise and then divided by $\varepsilon^{d/2}$, should take place in C^α, for $\alpha < -\frac{d}{2} + 2$. In fact, this agrees with the constraint that convergence in L^2 can be expected only for $d < 4$, and convergence in H^{-1} for $d < 6$. It would be interesting to pursue this direction of studies further.

3. Homogenization and periodic error estimates

The following homogenization result for (1-1), without the random potential $q^\varepsilon(x,\omega)$, is well known. The effect of the presence of q^ε turns out to be minor for homogenization; nevertheless, we include a proof here for the sake of completeness.

Theorem 3.1. *Assume (A1), (A2) and (P) hold. Then there exists $\Omega_1 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_1) = 1$, and for all $\omega \in \Omega_1$, the solution u^ε of (1-1) converges to the solution u of (1-2) weakly in $H^1(D)$ and strongly in $L^2(D)$ for any $f \in H^{-1}(D)$.*

Let $\mathcal{L}_\varepsilon$ denote the differential operator

$$-\frac{\partial}{\partial x_i}\Big(a_{ij}\Big(\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial x_j}\Big) + \bar{q}, \tag{3-1}$$

and let $\mathcal{L}^{\varepsilon,\omega}$ be the differential operator $\mathcal{L}_\varepsilon + \nu(\frac{x}{\varepsilon},\omega)$. We remark that $\mathcal{L}_\varepsilon$ has highly oscillatory but deterministic coefficients while $\mathcal{L}^{\varepsilon,\omega}$ has, in addition, a highly oscillatory and random potential. Let $\mathcal{G}^{\varepsilon,\omega}$ and $\mathcal{G}_\varepsilon$ be the solution operator of the Dirichlet boundary problems associated to $\mathcal{L}^{\varepsilon,\omega}$ and $\mathcal{L}_\varepsilon$. Owing to the conditions (2-2) and (2-4), $\mathcal{G}^{\varepsilon,\omega}$ is well-defined for any $\omega \in \Omega$. Moreover, we have the standard estimate, for any $\omega \in \Omega$ and $\varepsilon > 0$,

$$\|\mathcal{G}^{\varepsilon,\omega} f\|_{H^1(D)} \le C\|f\|_{H^{-1}(D)}, \tag{3-2}$$

with some constant C that depends on the universal parameters, and neither on ω nor ε. By the same token, $\mathcal{G}_\varepsilon$ is well-defined and shares the same estimate above.

Proof of Theorem 3.1. Step 1: For each $\omega \in \Omega$, the solution u^ε of (1-1) is given by $\mathcal{G}^{\varepsilon,\omega} f$, which satisfies the standard estimates

$$\|u^\varepsilon\|_{H^1(D)} + \|A^\varepsilon \nabla u^\varepsilon\|_{L^2(D)} + \|q^\varepsilon(x,\omega)u^\varepsilon\|_{L^2(D)} \le C,$$

where C depends the universal parameters and f and is uniform in ε and ω. As a result, due to the compact embeddings $H^1(D) \hookrightarrow L^2(D) \hookrightarrow H^{-1}(D)$, through a subsequence $\varepsilon_j(\omega) \to 0$, which by an abuse of notation is still denoted by ε, we have

$$\begin{aligned}
&\nabla u^\varepsilon(\cdot,\omega) \xrightarrow[\varepsilon\to 0]{L^2} \nabla v(\cdot,\omega), \quad && A\Big(\frac{\cdot}{\varepsilon}\Big)\nabla u^\varepsilon(\cdot,\omega) \xrightarrow[\varepsilon\to 0]{L^2} \xi(\cdot,\omega),\\
&u^\varepsilon(\cdot,\omega) \xrightarrow[\varepsilon\to 0]{L^2} v(\cdot,\omega), && q\Big(\frac{\cdot}{\varepsilon},\omega\Big)u^\varepsilon(\cdot,\omega) \xrightarrow[\varepsilon\to 0]{H^{-1}} p(\cdot,\omega)
\end{aligned} \tag{3-3}$$

for some function $v(\cdot,\omega) \in H^1(D)$ and some vector-valued function $\xi(\cdot,\omega) \in [L^2(D)]^d$.

Step 2: Recall that $\{\chi^k\}_{k=1}^d$ are the correctors defined in (1-4), and we can extend them periodically to functions defined on $\mathbb{R}^d$. Since $A(y)(e_k + \nabla\chi^k(y))$ is periodic, we have that

$$A\Big(\frac{x}{\varepsilon}\Big)\Big(e_k + (\nabla\chi^k)\Big(\frac{x}{\varepsilon}\Big)\Big) \xrightarrow{L^2} \int_{\mathbb{T}^d} A(y)\big(e_k(y) + \nabla\chi^k(y)\big)\,dy = \bar{A}e_k. \tag{3-4}$$

For the same reason and the fact $\int_{\mathbb{T}^d} \nabla \chi^k \, dy = 0$, we have

$$e_k + (\nabla \chi^k)\Big(\frac{x}{\varepsilon}\Big) \xrightarrow{L^2} \int_{\mathbb{T}^d} e_k + \nabla \chi^k(y)\, dy = e_k. \tag{3-5}$$

Now fix an arbitrary function $\varphi \in C_0^\infty(D)$. For each fixed $\omega \in \Omega$, let $\varepsilon(\omega) \to 0$ be the subsequence in Step 1. Consider the integral

$$\int_D A\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon(x,\omega) \cdot \nabla\Big\{x_k + \varepsilon \chi^k\Big(\frac{x}{\varepsilon}\Big)\Big\}\varphi(x)\, dx.$$

On one hand, in view of the third item in (3-3), (3-5), and the facts that $\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = -f + q^\varepsilon u^\varepsilon$ converges in H^{-1} (to $-f + p(\cdot\,,\omega)$, where p is defined in (3-3)) and that $e_k + (\nabla \chi^k)(x/\varepsilon)$ is curl-free, by the div–curl lemma [Jikov et al. 1994, Lemma 1.1], the above integral satisfies

$$\lim_{\varepsilon \to 0} \int_D A\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon(x,\omega) \cdot \nabla\Big\{x_k + \varepsilon \chi^k\Big(\frac{x}{\varepsilon}\Big)\Big\}\varphi(x)\, dx = \int_D \xi(x,\omega) \cdot e_k \varphi(x)\, dx.$$

On the other hand, in view of the first item in (3-3), (3-4), and the facts that $\operatorname{div}(A^\varepsilon(e_k + \nabla \chi^k(x/\varepsilon)))$ converges in H^{-1} (they are all equal to zero) and that ∇u^ε is curl-free, by the div–curl lemma, we have

$$\lim_{\varepsilon \to 0} \int_D A\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon(x,\omega) \cdot \nabla\Big\{x_k + \varepsilon \chi^k\Big(\frac{x}{\varepsilon}\Big)\Big\}\varphi(x)\, dx = \int_D \nabla v \bar{A} \cdot e_k \varphi(x)\, dx.$$

The two limits above must be equal, and it follows that $\xi(\cdot\,,\omega) = \bar{A}\nabla v(\cdot\,,\omega)$ in distribution.

Step 3: Recall that the stationary random potential $q(x,\omega)$ can be written as $\tilde{q}(\tau_x \omega)$, where $\tilde{q}$ is an essentially bounded random variable on Ω. By the Birkhoff ergodic theorem [Jikov et al. 1994, Theorem 7.2], there exists $\Omega_1 \in \mathcal{F}$ with $\mathbb{P}(\Omega_1) = 1$, and for each $\omega \in \Omega_1$,

$$q\Big(\frac{x}{\varepsilon},\omega\Big) = \tilde{q}(\tau_{\frac{x}{\varepsilon}}\omega) \xrightarrow{L^\alpha_{\mathrm{loc}}(\mathbb{R}^d)} \bar{q} = \mathbb{E} q(0,\omega) \tag{3-6}$$

for any $\alpha \in (1,\infty)$. From the weak formulation of u^ε, for any $\omega \in \Omega_1$ and for any $\varphi \in C_0^\infty(D)$, we have

$$\int_D A\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon(x,\omega) \cdot \nabla \varphi(x)\, dx + \int_D q\Big(\frac{x}{\varepsilon},\omega\Big) u^\varepsilon(x)\varphi(x)\, dx = \int_D f(x)\varphi(x)\, dx.$$

Passing to the limit along the subsequence $\varepsilon(\omega)$ found in Step 1, we have

$$\int_D \bar{A}\nabla v \cdot \nabla \varphi + \int_D \bar{q} v(x)\varphi(x)\, dx = \int_D f(x)\varphi(x)\, dx - \lim_{\varepsilon \to 0} \int_D q\Big(\frac{x}{\varepsilon},\omega\Big)(u^\varepsilon - v)\varphi(x)\, dx.$$

The first term on the left follows from (3-3) and the fact that $\xi = \bar{A}\nabla v$; the second term on the left is due to (3-6). Finally, the last term on the right-hand side is zero since q is uniformly bounded and $u^\varepsilon - v$ converges to zero strongly in $L^2(D)$. Consequently, the above limit shows that v solves the homogenized equation (1-2). By uniqueness of the homogenized problem, we must have that $v = u$ and v is deterministic.

Finally, for each $\omega \in \Omega_1$, by the weak compactness in $H^1(D)$ and the uniqueness of the possible limit, the whole sequence u^ε converges to u. This proves the homogenization theorem. □

Remark 3.2. We remark that the same proof works in the case when (a_{ij}) is not symmetric; indeed, it suffices to replace χ^k above by the solution of the adjoint corrector equation. The same idea of proof can also be carried out in the case when (a_{ij}) are stationary ergodic random fields; indeed, the corrector equation in that case is much more involved but, by now, its solution and analogs of (3-4) and (3-5), are well known.

3A. ***Decomposition of the homogenization error.*** To separate the fluctuations in the homogenization error $u^\varepsilon - u$ that are due to the periodic oscillations in the diffusion coefficients from those due to the random potential, we introduce the function v^ε which solves the following deterministic problem:

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\Big(a_{ij}\Big(\dfrac{x}{\varepsilon}\Big)\dfrac{\partial}{\partial x_j}v^\varepsilon(x,\omega)\Big) + \bar{q}v^\varepsilon(x,\omega) = f(x), & x \in D, \\ v^\varepsilon(x) = 0, & x \in \partial D. \end{cases} \tag{3-7}$$

Here, the potential field is already homogenized, and we expect that $v^\varepsilon - u$ filters out the effect of the random potential. The problem above is well-posed and its solution v^ε is given by $\mathcal{G}_\varepsilon f$.

The standard periodic homogenization theory yields that v^ε converges weakly in $H^1(D)$ and strongly in $L^2(D)$ to u for any $f \in H^{-1}(D)$. Using this function, we can write the homogenization error for (1-1) as

$$u^\varepsilon - u = (u^\varepsilon - v^\varepsilon) + (v^\varepsilon - u). \tag{3-8}$$

The deterministic part of the homogenization error is

$$\mathbb{E}u^\varepsilon - u = \mathbb{E}(u^\varepsilon - v^\varepsilon) + (v^\varepsilon - u), \tag{3-9}$$

and the random fluctuation part of the homogenization error is

$$u^\varepsilon - \mathbb{E}u^\varepsilon = (u^\varepsilon - v^\varepsilon) - \mathbb{E}(u^\varepsilon - v^\varepsilon). \tag{3-10}$$

The deterministic part of the homogenization error hence contains two parts, the mean of $u^\varepsilon - v^\varepsilon$ and the periodic homogenization error $v^\varepsilon - u$. Estimates for the second part amount to the convergence rate of periodic homogenization, and we recall some of the well-known results below, together with uniform-in-ε estimates of the Green's function associated to $\mathcal{G}_\varepsilon$. We postpone the estimates for $\mathbb{E}(u^\varepsilon - v^\varepsilon)$ to the next section.

Theorem 3.3 (estimates in periodic homogenization). *Let $D \subset \mathbb{R}^d$ be an open bounded $C^{1,1}$-domain, and v^ε and u be the solutions to (3-7) and (1-2) respectively. Let $G_\varepsilon(x,y)$, with $x, y \in D$, be the Green's function associated to the Dirichlet problem of (3-7). Assume (A) holds. Then there exists positive constant C, depending only on the universal parameters, such that*

(i) *for any $f \in L^2(D)$, we have $\|v^\varepsilon - u\|_{L^2} \le C\varepsilon\|f\|_{L^2}$,*

(ii) *for $d \ge 2$ and for any $x, y \in D$, $x \ne y$, we have that $G_\varepsilon(x,y)$ satisfies*

$$|G_\varepsilon(x,y)| \le \begin{cases} C|x-y|^{2-d} & \text{if } d \ne 2, \\ C\big(1 + \big|\log|x-y|\big|\big) & \text{if } d = 2, \end{cases} \tag{3-11}$$

and

$$|\nabla G_\varepsilon(x,y)| \le C|x-y|^{1-d}. \tag{3-12}$$

The $O(\varepsilon)$-error estimates in L^2 were proved in [Moskow and Vogelius 1997] for $d = 2$, and in [Griso 2006] for general $C^{1,1}$-domains; see also [Kenig et al. 2012]. The uniform-in-ε estimates on the Green's function and its gradient can be found, e.g., in [Avellaneda and Lin 1987; 1991; 2015]. In particular, (3-11) was proved in [Avellaneda and Lin 1987, Theorem 13]; the estimate (3-12) follows from an interior Lipschitz estimate, e.g., [Avellaneda and Lin 1987, Lemma 16], if the distance between x and y is smaller compared with their distance from the boundary, and it follows from a boundary Lipschitz estimate, e.g., [Avellaneda and Lin 1987, Lemma 20], if otherwise; see also the proof of [Armstrong and Shen 2015, Theorem 1.1].

The homogeneities in these bounds are the same as those for the Green's function associated to constant coefficient equations, namely the Laplace equations. The striking fact that these bounds still hold for oscillatory equations is due to the fact that the problem (3-7) homogenizes to constant (smooth) coefficient equations. Periodicity or other structural assumptions on the coefficients are crucial. We remark also that it is to obtain such pointwise estimates that the Hölder regularity of the diffusion matrix, i.e., assumption (A3), is needed.

4. Estimates for the homogenization error

In this section, we estimate the size of the homogenization error $u^\varepsilon - u$. In view of the decomposition (3-8), (3-9), (3-10) and the error estimates in Theorem 3.3, it suffices to focus on the intermediate homogenization error $u^\varepsilon - v^\varepsilon$, with $v^\varepsilon = \mathcal{G}_\varepsilon f$ defined in (3-7).

We introduce the function w^ε which solves

$$\mathcal{L}_\varepsilon w^\varepsilon = -\nu_\varepsilon v^\varepsilon, \tag{4-1}$$

with homogeneous Dirichlet boundary condition. With the notations $\mathcal{G}_\varepsilon$ and $\mathcal{G}^{\varepsilon,\omega}$ introduced earlier, w^ε is given by $-\mathcal{G}_\varepsilon \nu_\varepsilon v^\varepsilon$. It follows that

$$\mathcal{L}^{\varepsilon,\omega}(u^\varepsilon - v^\varepsilon - w^\varepsilon) = -\nu_\varepsilon w^\varepsilon,$$

and $u^\varepsilon - v^\varepsilon - w^\varepsilon$ vanishes at the boundary. Hence we have $u^\varepsilon - v^\varepsilon - w^\varepsilon = -\mathcal{G}^{\varepsilon,\omega}\nu_\varepsilon w^\varepsilon$. Due to the assumptions (A), $\mathcal{G}^{\varepsilon,\omega}$ is uniformly (in ε and ω) bounded as a linear operator $L^2 \to L^2$; we have

$$\|u^\varepsilon - v^\varepsilon\|_{L^2} \le C\|w^\varepsilon\|_{L^2}. \tag{4-2}$$

An estimate of $u^\varepsilon - v^\varepsilon$ thus follows from this result:

Lemma 4.1. *Let $v^\varepsilon = \mathcal{G}_\varepsilon f$ and w^ε be as above. Under the same conditions of Theorem 2.3, there exists a universal constant C and*

$$\mathbb{E}\|w^\varepsilon\|^2_{L^2(D)} \le \begin{cases} C\varepsilon^{d\wedge 4}\|f\|^2_{L^2} & \text{if } d \ne 4, \\ C\varepsilon^4|\log\varepsilon|\,\|f\|^2_{L^2} & \text{if } d = 4. \end{cases} \tag{4-3}$$

Proof. Using the Green's function G_ε, we write

$$w^\varepsilon(x,\omega) = \int_D G_\varepsilon(x,y)\nu\Big(\frac{y}{\varepsilon}\Big)v^\varepsilon(y)\,dy. \tag{4-4}$$

The L^2-energy of w^ε is then

$$\|w^\varepsilon(\cdot,\omega)\|^2_{L^2} = \int_{D^3} G_\varepsilon(x,y)G_\varepsilon(x,z)\nu\Big(\frac{y}{\varepsilon}\Big)\nu\Big(\frac{z}{\varepsilon}\Big)v^\varepsilon(y)v^\varepsilon(z)\,dy\,dz\,dx.$$

Taking the expectation and using the definition of the autocorrelation function R of q, we have

$$\mathbb{E}\|w^\varepsilon(\cdot,\omega)\|_{L^2}^2 = \int_{D^3} G_\varepsilon(x,y)G_\varepsilon(x,z)R\Big(\frac{y-z}{\varepsilon}\Big)v^\varepsilon(y)v^\varepsilon(z)\,dy\,dz\,dx. \tag{4-5}$$

Integrate over x first. Apply the uniform estimates (3-11) and the fact (see, e.g., Lemma A.1 of [Bal and Jing 2011]): for any $y \neq z$, $0 < \alpha, \beta < d$,

$$\int_D \frac{dx}{|x-y|^{d-\alpha}|x-z|^{d-\beta}} \le \begin{cases} C & \text{if } \alpha+\beta > d, \\ C\big(1+\big|\log|y-z|\big|\big) & \text{if } \alpha+\beta = d, \\ C|y-z|^{\alpha+\beta-d} & \text{if } \alpha+\beta < d. \end{cases} \tag{4-6}$$

We get

$$\int_D |G_\varepsilon(x,y)G_\varepsilon(x,z)|\,dx \le \begin{cases} C|y-z|^{-((d-4)\wedge 0)} & \text{if } d \neq 4, \\ C(1+\log|y-z|) & \text{if } d = 4. \end{cases} \tag{4-7}$$

Hence, if $d \ge 2$ and $d \neq 4$,

$$\mathbb{E}\|w^\varepsilon(\cdot,\omega)\|_{L^2}^2 \le C\int_{D^2} \frac{|v^\varepsilon(y)v^\varepsilon(z)|}{|y-z|^{(d-4)\vee 0}}\left|R\Big(\frac{y-z}{\varepsilon}\Big)\right|\,dy\,dz\,dx.$$

When $d=4$, the term $(|y-z|^{(d-4)\vee 0})^{-1}$ should be replaced by $1+\big|\log|y-z|\big|$. In any case, the above yields a bound of the form

$$\mathbb{E}\|w^\varepsilon(\cdot,\omega)\|_{L^2}^2 \le C\int_{\mathbb{R}^d} |\tilde v^\varepsilon(y)|(K^\varepsilon * \tilde v^\varepsilon)(y)\,dy. \tag{4-8}$$

Here, $\tilde v^\varepsilon = v^\varepsilon \mathbb{1}_D$ and $\mathbb{1}_D$ denotes the indicator function of the set D, $K^\varepsilon(y) = R\big(\frac{y}{\varepsilon}\big)|y|^{(4-d)\wedge 0}\mathbb{1}_{B_\rho}(y)$ if $d \neq 4$ and $K^\varepsilon(y) = R\big(\frac{y}{\varepsilon}\big)\big(1+\mathbb{1}_{B_\rho}(y)\big|\log|y|\big|\big)$ if $d=4$. Here, B_ρ is the ball centered at zero with radius ρ and ρ is the diameter of D. We check that, when $d \neq 4$,

$$\|K^\varepsilon(y)\|_{L^1} \le \int_{\mathbb{R}^d}\left|R\Big(\frac{y}{\varepsilon}\Big)\right|\frac{1}{|y|^{(d-4)\vee 0}}\,dy = \frac{\varepsilon^d}{\varepsilon^{(d-4)\vee 0}}\int_{\mathbb{R}^d}\frac{|R(y)|}{|y|^{(d-4)\vee 0}} = C\varepsilon^{d\wedge 4}, \tag{4-9}$$

where in the last inequality we used $R \in L^\infty \cap L^1(\mathbb{R}^d)$. Similarly, when $d=4$,

$$\|K^\varepsilon(y)\|_{L^1} = \int_{B_\rho}\left|R\Big(\frac{y}{\varepsilon}\Big)\right|\big(1+\big|\log|y|\big|\big)\,dy = \varepsilon^4\int_{B_{\rho/\varepsilon}}|R(y)|\big(1+\big|\log|\varepsilon y|\big|\big) \le C\varepsilon^4|\log\varepsilon|. \tag{4-10}$$

To get the last inequality, we evaluate the integral on B_1 and $B_{\rho/\varepsilon}\setminus B_1$, and bound $\big|\log|\varepsilon y|\big|$ by $\big|\log|\varepsilon\rho|\big|$ for the second part. Applying Hölder's and then Young's inequalities to (4-8), we get

$$\mathbb{E}\|w^\varepsilon\|_{L^2}^2 \le C\|K^\varepsilon\|_{L^1}\|v^\varepsilon\|_{L^2}^2 \le C\|K^\varepsilon\|_{L^1}\|f\|_{L^2}^2.$$

Combining this with the estimates in (4-9) and (4-10), we complete the proof of the lemma. □

4A. ***Scaling of the energy in the random fluctuation.*** Now we estimate the $L^2(D)$-norm (the energy) of the random fluctuation $u^\varepsilon - \mathbb{E}u^\varepsilon$ which, in view of (3-10), is the same as the fluctuation $u^\varepsilon - v^\varepsilon - \mathbb{E}(u^\varepsilon - v^\varepsilon)$.

Using the first-order corrector w^ε defined by (4-1), and following the approach of [Bal 2008; Bal and Jing 2011], we can derive an expansion formula for $u^\varepsilon - v^\varepsilon$ as follows. Rewrite the equations (1-1) and (3-7) as

$$\mathcal{L}_\varepsilon u^\varepsilon = f - \nu_\varepsilon u^\varepsilon, \qquad \mathcal{L}^\varepsilon v^\varepsilon = f.$$

Then it follows that $u^\varepsilon - v^\varepsilon = -\mathcal{G}_\varepsilon \nu_\varepsilon u^\varepsilon = -\mathcal{G}_\varepsilon \nu_\varepsilon v^\varepsilon - \mathcal{G}_\varepsilon \nu_\varepsilon (u^\varepsilon - v^\varepsilon)$. Iterate this relation another time; we get the truncated Neumann series

$$u^\varepsilon - v^\varepsilon = -\mathcal{G}_\varepsilon \nu_\varepsilon v^\varepsilon + \mathcal{G}_\varepsilon \nu_\varepsilon \mathcal{G}_\varepsilon \nu_\varepsilon v^\varepsilon + \mathcal{G}_\varepsilon \nu_\varepsilon \mathcal{G}_\varepsilon \nu_\varepsilon (u^\varepsilon - v^\varepsilon). \tag{4-11}$$

In particular, the fluctuations in $u^\varepsilon - v^\varepsilon$ can be written as

$$u^\varepsilon - \mathbb{E}u^\varepsilon = -\mathcal{G}_\varepsilon \nu_\varepsilon v^\varepsilon + \big(\mathcal{G}_\varepsilon \nu_\varepsilon \mathcal{G}_\varepsilon \nu_\varepsilon v^\varepsilon - \mathbb{E}\mathcal{G}_\varepsilon \nu_\varepsilon \mathcal{G}_\varepsilon \nu_\varepsilon v^\varepsilon\big) + \big(\mathcal{G}_\varepsilon \nu_\varepsilon \mathcal{G}_\varepsilon \nu_\varepsilon (u^\varepsilon - v^\varepsilon) - \mathbb{E}\mathcal{G}_\varepsilon \nu_\varepsilon \mathcal{G}_\varepsilon \nu_\varepsilon (u^\varepsilon - v^\varepsilon)\big).$$

The first term above is exactly w^ε, which has mean zero and its energy was estimated in Lemma 4.1. The next lemma provides an estimate for the energy of the second term in the above expansion.

Lemma 4.2. *Suppose that the assumptions of Theorem 2.3 are satisfied. Then there exists a constant $C > 0$, depending only on the universal parameters and f, such that*

$$\mathbb{E}\big\|\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon - \mathbb{E}\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon\big\|^2_{L^2(D)} \le \begin{cases} C\varepsilon^{2d} & \text{if } d = 2, 3, \\ C\varepsilon^8 |\log \varepsilon|^2 & \text{if } d = 4, \\ C\varepsilon^8 & \text{if } 5 \le d \le 7. \end{cases} \tag{4-12}$$

Let I_2^ε denote the left-hand side of (4-12); it has the expression

$$\begin{aligned} I_2^\varepsilon &= \mathbb{E}\int_D \bigg(\int_{D^2} G_\varepsilon(x,y) G_\varepsilon(y,z)\big(\nu^\varepsilon(y)\nu^\varepsilon(z) - \mathbb{E}\nu^\varepsilon(y)\nu^\varepsilon(z)\big) v^\varepsilon(z)\,dz\,dy\bigg)^2 dx \\ &= \int_{D^5} G_\varepsilon(x,y) G_\varepsilon(x,y') G_\varepsilon(y,z) G_\varepsilon(y',z') v^\varepsilon(z) v^\varepsilon(z') \\ &\qquad \bigg(\mathbb{E}\bigg(\nu\Big(\frac{y}{\varepsilon}\Big)\nu\Big(\frac{y'}{\varepsilon}\Big)\nu\Big(\frac{z}{\varepsilon}\Big)\nu\Big(\frac{z'}{\varepsilon}\Big)\bigg) - R\Big(\frac{y-z}{\varepsilon}\Big) R\Big(\frac{y'-z'}{\varepsilon}\Big)\bigg)\,dz'\,dy'\,dz\,dy\,dx. \end{aligned}$$

It is then evident that we need to estimate certain fourth-order moments of $\nu(x,\omega)$, namely, the function

$$\Psi_\nu(x,y,t,s) := \mathbb{E}\,\nu(x)\nu(y)\nu(t)\nu(s) - \big(\mathbb{E}\,\nu(x)\nu(y)\big)\big(\mathbb{E}\,\nu(t)\nu(s)\big). \tag{4-13}$$

Were ν a Gaussian random field, its fourth-order moments would decompose as a sum of products of pairs of R. This nice property does not hold for general random fields; however, the following estimate for ρ-mixing random fields provides almost the same convenience.

Lemma 4.3. *Suppose $\nu(x,\omega)$ is a random field with maximal correlation function ϱ defined as in (2-6). Then*

$$|\Psi_\nu(x,t,y,s)| \le \vartheta(|x-t|)\vartheta(|y-s|) + \vartheta(|x-s|)\vartheta(|y-t|), \tag{4-14}$$

where $\vartheta(r) = (K\varrho(r/3))^{1/2}$, with $K = 4\|\nu\|_{L^\infty(\Omega\times D)}$.

We refer to [Hairer et al. 2013] for the proof of this lemma. Estimates of this type based on mixing properties already appeared in [Bal 2008]. We refer to [Bal and Jing 2011] for an alternative way to control terms like Ψ_ν, and to Section 7 for some comments on the connection of condition (S) with the lemma above.

Proof of Lemma 4.2. Integrate over x in the expression of I_2^ε, and apply the estimates (3-11), (4-7) and (4-14). We find, for $d \ge 3$,

$$I_2^\varepsilon \le C\Bigg(\int_{D^4} \frac{\big(1+\mathbb{1}_{d=4}\big|\log|y-y'|\big|\big)|v^\varepsilon(z)v^\varepsilon(z')|}{|y-y'|^{(d-4)\vee 0}|y-z|^{d-2}|y'-z'|^{d-2}}\,\vartheta\Big(\frac{y-y'}{\varepsilon}\Big)\vartheta\Big(\frac{z-z'}{\varepsilon}\Big)\,dz'\,dy'\,dz\,dy$$
$$+\int_{D^4} \frac{\big(1+\mathbb{1}_{d=4}\big|\log|y-y'|\big|\big)|v^\varepsilon(z)v^\varepsilon(z')|}{|y-y'|^{(d-4)\vee 0}|y-z|^{d-2}|y'-z'|^{d-2}}\,\vartheta\Big(\frac{y-z'}{\varepsilon}\Big)\vartheta\Big(\frac{z-y'}{\varepsilon}\Big)\,dz'\,dy'\,dz\,dy\Bigg).$$

For $d = 2$, the terms $|y-z|^{-(d-2)}$ and $|y'-z'|^{-(d-2)}$ above should be replaced by $1+\big|\log|y-z|\big|$ and $1+\big|\log|y'-z'|\big|$ respectively. Let I_{21}^ε and I_{22}^ε denote the two terms on the right-hand side of the estimate above. In the following, we set ρ to be the diameter of D.

Estimate of I_{21}^ε. We use the change of variables

$$\frac{y-y'}{\varepsilon}\mapsto y,\qquad \frac{z-z'}{\varepsilon}\mapsto z,\qquad y'-z'\mapsto y',\qquad z'\mapsto z'.$$

Then the integral in I_{21}^ε becomes, for $d \ge 3$,

$$\frac{C\varepsilon^{2d}}{\varepsilon^{(d-4)\vee 0}}\int_{B^2_{\rho/\varepsilon}} dy\,dz\int_{B_\rho} dy'\int_D dz'\,\frac{\big(1+\mathbb{1}_{d=4}\big|\log|\varepsilon y|\big|\big)|v^\varepsilon(z')v^\varepsilon(z'+\varepsilon z)|}{|y|^{(d-4)\vee 0}|y'+\varepsilon(y-z)|^{d-2}|y'|^{d-2}}\vartheta(y)\vartheta(z).$$

We integrate over y' first and apply (4-6), then integrate over z' and obtain

$$I_{21}^\varepsilon \le C\|v^\varepsilon\|^2_{L^2}\varepsilon^{2d-2(d-4)\vee 0}\int_{B^2_{\rho/\varepsilon}} \frac{\big(1+\mathbb{1}_{d=4}\big|\log|\varepsilon y|\big|\big)\big(1+\mathbb{1}_{d=4}\big|\log|\varepsilon(y-z)|\big|\big)\vartheta(y)\vartheta(z)}{|y|^{(d-4)\vee 0}|y-z|^{(d-4)\vee 0}}\,dy\,dz.$$

When $d = 3$, the integral above is bounded because $\vartheta \in L^1(\mathbb{R}^d)$ thanks to assumption (S), and we have $I_{21}^\varepsilon \le C\varepsilon^{2d}$. When $d = 2$, the situation is similar; after the integral over y', there is again no singularity in the denominator. Hence, $I_{21}^\varepsilon \le C\varepsilon^{2d}$.

When $d \ge 5$, by the Hardy–Littlewood–Sobolev inequality [Lieb and Loss 2001, Theorem 4.3], we have, for $p, r \in (1,\infty)$,

$$\int_{\mathbb{R}^{2d}} \frac{(\vartheta(y)/|y|^{d-4})\vartheta(z)}{|y-z|^{d-4}}\,dy\,dz \le C\left\|\frac{\vartheta(y)}{|y|^{d-4}}\right\|_{L^p(\mathbb{R}^d)}\|\vartheta\|_{L^r(\mathbb{R}^d)},\qquad \frac{1}{p}+\frac{d-4}{d}+\frac{1}{r}=2.$$

Take $p = d/(4+\delta)$ and $r = d/(d-\delta)$ for any $(d-8)\vee 0 < \delta < d-4$. Then because $\vartheta \in L^\infty\cap L^1(\mathbb{R}^d)$ and $|y|^{4-d} \in L^p(B_1)$, the above is finite and we have $I_{21}^\varepsilon \le C\varepsilon^8$.

When $d = 4$, we need to control the integral

$$\int_{B^2_{\rho/\varepsilon}} \big(1+\big|\log|\varepsilon y|\big|\big)\big(1+\big|\log|\varepsilon(y-z)|\big|\big)\vartheta(y)\vartheta(z)\,dy\,dz,$$

where $D^* = \{y - y' - z + z' : y, y', z, z' \in D\}$ is some bounded region formed by certain combinations of points in D. As a result, the logarithmic terms are bounded away from the poles. Hence, the above integral is bounded by $O(|\log \varepsilon|^2)$, and $I_{21}^\varepsilon \le C\varepsilon^8 |\log \varepsilon|^2$.

Estimate of I_{22}^ε. We apply the change of variables

$$\frac{y - z'}{\varepsilon} \mapsto y, \quad \frac{y' - z}{\varepsilon} \mapsto y', \quad z - z' \mapsto z, \quad z' \mapsto z'.$$

Then the integral in I_{22}^ε becomes, for $d = 2$,

$$C\varepsilon^{2d} \int_{B^2_{\rho/\varepsilon}} dy\, dy' \int_D dz' \int_{B_\rho} dz \big|v^\varepsilon(z') v^\varepsilon(z' + z)\big| \big(1 + \big|\log|z - \varepsilon y|\big|\big)(1 + \log|z - \varepsilon y'|) \vartheta(y)\vartheta(y').$$

Integrate over z', z and then over y' and y. We find that $I_{22}^\varepsilon \le C\varepsilon^{2d}$. For $d \ge 3$, the same change of variables transforms I_{22}^ε to

$$C\varepsilon^{2d} \int_{B^2_{\rho/\varepsilon}} dy\, dy' \int_{B_\rho} dz \int_D dz' \frac{\big(1 + \mathbb{1}_{d=4}\big|\log|z - \varepsilon(y - y')|\big|\big) |v^\varepsilon(z') v^\varepsilon(z' + z)|}{|z - \varepsilon(y - y')|^{(d-4)\vee 0} |z - \varepsilon y|^{d-2} |z + \varepsilon y'|^{d-2}} \vartheta(y)\vartheta(y').$$

After an integration over z', we only need to control

$$C\varepsilon^{2d} \int_{B^2_{\rho/\varepsilon}} dy\, dy' \int_{B_\rho} dz \frac{\big(1 + \mathbb{1}_{d=4}\big|\log|z - \varepsilon(y - y')|\big|\big)}{|z - \varepsilon(y - y')|^{(d-4)\vee 0} |z - \varepsilon y|^{d-2} |z + \varepsilon y'|^{d-2}} \vartheta(y)\vartheta(y').$$

When $d = 3$, an integration over z removes the singularities in the denominator. Then integrating over y and y' yields that $I_{22}^\varepsilon \le C\varepsilon^{2d}$.

When $d \ge 5$, we need to control the integral; after another change of variables, $\varepsilon^{-1} z - (y - y') \mapsto z$ and $-y \mapsto y$, we have

$$\int_{\mathbb{R}^{3d}} dy\, dy'\, dz \frac{C\varepsilon^8 \vartheta(y)\vartheta(y')}{|z|^{d-4} |z - y|^{d-2} |z - y'|^{d-2}} = \int_{\mathbb{R}^d} dz \frac{C\varepsilon^8 |K(z)|^2}{|z|^{d-4}},$$

with $K(z) = (|y|^{-(d-2)} * \vartheta)(z)$. Since $\vartheta \in L^1 \cap L^\infty(\mathbb{R}^d)$, we have

$$|K(z)| = \int_{B_1(z)} \frac{\vartheta(y)\, dy}{|z - y|^{d-2}} + \int_{\mathbb{R}^d \setminus B_1(z)} \frac{\vartheta(y)\, dy}{|z - y|^{d-2}} \le \int_{B_1(z)} \frac{\|\vartheta\|_{L^\infty}\, dy}{|y - z|^{d-2}} + \int_{\mathbb{R}^d \setminus B_1(z)} \vartheta(y)\, dy \le C.$$

Moreover, by the Hardy–Littlewood–Sobolev inequality, we have that

$$\|K\|_{L^2(\mathbb{R}^d)} = \big\| |y|^{-(d-2)} * \vartheta(y) \big\|_{L^2(\mathbb{R}^d)} \le C \|\vartheta\|_{L^{2d/(d+4)}(\mathbb{R}^d)} \le C \|\vartheta\|_{L^\infty}^{\frac{d-4}{2d}} \|\vartheta\|_{L^1}^{\frac{d+4}{2d}}.$$

Now we show that $K \in L^\infty \cap L^2(\mathbb{R}^d)$. It follows that the integral to be controlled is finite and we have $I_{22}^\varepsilon \le C\varepsilon^8$.

When $d = 4$, after the same change of variables as in the case of $d \ge 5$, we are left to control

$$\varepsilon^8 \int_{B^2_{\rho/\varepsilon}} dy\, dy' \int_{B_{3\rho/\varepsilon}} dz \frac{\big(1 + \big|\log|\varepsilon z|\big|\big) \vartheta(y)\vartheta(y')\, dy\, dy'\, dz}{|z - y|^2 |z - y'|^2} = \varepsilon^8 \int_{B_{3\rho/\varepsilon}} \big(1 + \big|\log|\varepsilon z|\big|\big) (K(z))^2\, dz,$$

where $K(z) = (\mathbb{1}_{B_{\rho/\varepsilon}}(y)|y|^{-2} * \vartheta)(z)$. We verify again that $K \in L^\infty \cap L^{2-\delta}(\mathbb{R}^d)$ for any $\delta \in (0,1)$. Estimate the integral again by breaking it into pieces inside and outside B_1; we find $I_{22}^\varepsilon \le C\varepsilon^8|\log\varepsilon|$.

Combining these estimates above, we have proved (4-12). □

Moving on to the last term in the series (4-11), we observe that it cannot be controlled in the same manner as above. Indeed, the term $u^\varepsilon - v^\varepsilon$ is random and depends on $\nu(x,\omega)$ in a nonlinear way. As a result, when we move the expectation into the integral representation, like in step (4-5), we cannot get a simple closed form in terms of R.

We hence choose not to address the interaction between $u^\varepsilon - v^\varepsilon$ and the random fluctuation ν^ε in the potential directly. Instead, by an application of Minkowski's inequality, we have

$$\big\|\mathbb{E}\,\mathcal{G}_\varepsilon\nu^\varepsilon\mathcal{G}_\varepsilon\nu^\varepsilon(u^\varepsilon-u)\big\|_{L^2(D)} \le \mathbb{E}\big\|\mathcal{G}_\varepsilon\nu^\varepsilon\mathcal{G}_\varepsilon\nu^\varepsilon(u^\varepsilon-u)\big\|_{L^2(D)}.$$

Thus, we use the trivial bound on the $L^1(\Omega, L^2(D))$-norm of the fluctuations in $\mathcal{G}_\varepsilon\nu^\varepsilon\mathcal{G}_\varepsilon\nu^\varepsilon(u^\varepsilon-u)$:

$$r_2^\varepsilon := \mathbb{E}\big\|\mathcal{G}_\varepsilon\nu^\varepsilon\mathcal{G}_\varepsilon\nu^\varepsilon(u^\varepsilon-u) - \big(\mathbb{E}\,\mathcal{G}_\varepsilon\nu^\varepsilon\mathcal{G}_\varepsilon\nu^\varepsilon(u^\varepsilon-u)\big)\big\|_{L^2(D)} \le 2\,\mathbb{E}\big\|\mathcal{G}_\varepsilon\nu^\varepsilon\mathcal{G}_\varepsilon\nu^\varepsilon(u^\varepsilon-u)\big\|_{L^2(D)},$$

and only control the energy of the last term in (4-11) itself, as contrast to its variance.

Lemma 4.4. *Suppose that the assumptions of Theorem 2.3 are satisfied. Then there exists some constant C, depending only on the universal parameters and f, such that*

$$\mathbb{E}\big\|\mathcal{G}_\varepsilon\nu^\varepsilon\mathcal{G}_\varepsilon\nu^\varepsilon(u^\varepsilon-v^\varepsilon)\big\|_{L^2(D)} \le \begin{cases} C\varepsilon^d & \text{if } d=2,3,\\ C\varepsilon^4|\log\varepsilon|^{\frac{3}{2}} & \text{if } d=4,\\ C\varepsilon^{6-\frac{d}{2}} & \text{if } d\ge 5.\end{cases} \tag{4-15}$$

To prove this result, we estimate the operator norm of $\mathcal{G}_\varepsilon\nu^\varepsilon\mathcal{G}_\varepsilon$, which is random since ν^ε depends on ω, and combine it with the control of $u^\varepsilon - v^\varepsilon$, which was obtained earlier.

Lemma 4.5 (mean value of the operator norm $\|\mathcal{G}_\varepsilon\nu^\varepsilon\mathcal{G}_\varepsilon\|_{L^2\to L^2}$). *Under the same assumptions of Theorem 2.3, there exists some universal constant C such that*

$$\mathbb{E}\,\|\mathcal{G}_\varepsilon\nu^\varepsilon\mathcal{G}_\varepsilon\|_{L^2\to L^2}^2 \le \begin{cases} C\varepsilon^d & \text{if } d=2,3,\\ C\varepsilon^4|\log\varepsilon|^2 & \text{if } d=4,\\ C\varepsilon^{8-d} & \text{if } d\ge 5.\end{cases} \tag{4-16}$$

Proof. For any $h \in L^2(D)$, we have

$$\|\mathcal{G}_\varepsilon\nu^\varepsilon\mathcal{G}_\varepsilon h\|_{L^2}^2 = \int_D\bigg(\int_{D^2} G_\varepsilon(x,y)\nu^\varepsilon(y)G_\varepsilon(y,z)h(z)\,dz\,dy\bigg)^2\,dx.$$

Note that for almost every fixed $x \in D$,

$$\bigg|\int_{D^2} G_\varepsilon(x,y)\nu^\varepsilon(y)G_\varepsilon(y,z)h(z)\,dz\,dy\bigg| \le \|h\|_{L^2}\bigg\|\int_D G_\varepsilon(x,y)\nu^\varepsilon(y)G_\varepsilon(y,\cdot)\,dy\bigg\|_{L^2}.$$

It then follows that

$$\|\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon\|^2_{L^2\to L^2}(\omega) \le \int_{D^2}\left(\int_D G_\varepsilon(x,y)\nu^\varepsilon(y,\omega)G_\varepsilon(y,z)\,dy\right)^2 dz\,dx.$$

Taking the expectation, we find

$$\mathbb{E}\,\|\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon\|^2_{L^2\to L^2} \le \int_{D^4} G_\varepsilon(x,y)G_\varepsilon(x,\eta)R\left(\frac{y-\eta}{\varepsilon}\right)G_\varepsilon(y,z)G_\varepsilon(\eta,z)\,dy\,d\eta\,dz\,dx.$$

Integrate over z- and x-variables first. Using (3-11) and (4-6), we find that the integrals over x- and z-variables are estimated as in (4-7). Then we have

$$\mathbb{E}\,\|\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon\|^2_{L^2\to L^2} \le C\int_{D^2}\left(\frac{1+\mathbb{1}_{d=4}\big|\log|y-\eta|\big|}{|y-\eta|^{(d-4)\vee 0}}\right)^2\left|R\left(\frac{y-\eta}{\varepsilon}\right)\right|dy\,d\eta.$$

Change variables in the above integral and carry out the analysis as before. We find that (4-16) holds. Note that the estimates become useless for $d \ge 8$. □

Proof of Lemma 4.4. For each $\omega \in \Omega$, we have

$$\big\|\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon \nu^\varepsilon (u^\varepsilon - v^\varepsilon)\big\|_{L^2} \le M\,\|\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon\|_{L^2\to L^2}\,\|u^\varepsilon - v^\varepsilon\|_{L^2},$$

where M is the uniform bound on the random potential in (2-4). Take the expectation and then the desired estimate follows from (4-16), (4-2) and (4-3). □

4B. ***Scaling factor of the random fluctuations in the weak topology.*** In this section we aim to find the correct scaling factor such that the random fluctuation $u^\varepsilon - \mathbb{E}\,u^\varepsilon$, normalized properly according to this factor, converges with respect to the weak topology. For that purpose, we fix an arbitrary $\varphi \in L^2(D)$ with unit norm, and estimate $\mathbb{E}(u^\varepsilon - \mathbb{E}\,u^\varepsilon, \varphi)^2$.

Using the series expansion formula (4-11), we have

$$\begin{aligned}(u^\varepsilon - \mathbb{E}\,u^\varepsilon, \varphi) = -(\mathcal{G}_\varepsilon \nu_\varepsilon v^\varepsilon, \varphi) + \big(\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon - \mathbb{E}(\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon), \varphi\big)&\\ + \big(\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon \nu^\varepsilon (u^\varepsilon - v^\varepsilon) - \mathbb{E}(\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon \nu^\varepsilon (u^\varepsilon - v^\varepsilon)), \varphi\big)&.\end{aligned}$$

Since the operators $\mathcal{G}_\varepsilon$ and $\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon$ are self-adjoint on $L^2(D)$, we can move them to φ. Set $\psi^\varepsilon = \mathcal{G}_\varepsilon \varphi$. The above expression becomes

$$\begin{aligned}(u^\varepsilon - \mathbb{E}u^\varepsilon, \varphi) &= -(\nu^\varepsilon v^\varepsilon, \psi^\varepsilon) + \big((\nu^\varepsilon \mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon, \psi^\varepsilon) - \mathbb{E}(\nu^\varepsilon \mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon, \psi^\varepsilon)\big)\\ &\qquad + \big((\nu^\varepsilon (u^\varepsilon - v^\varepsilon), \mathcal{G}_\varepsilon \nu^\varepsilon \psi^\varepsilon) - \mathbb{E}(\nu^\varepsilon (u^\varepsilon - v^\varepsilon), \mathcal{G}_\varepsilon \nu^\varepsilon \psi^\varepsilon)\big)\\ &:= I_1^\varepsilon + (I_2^\varepsilon - \mathbb{E}\,I_2^\varepsilon) + (I_3^\varepsilon - \mathbb{E}\,I_3^\varepsilon).\end{aligned} \tag{4-17}$$

The aim now is to control the variances of I_j^ε, with $j = 1, 2, 3$.

Estimate for I_1^ε. For I_1^ε, which is mean-zero, we have

$$\begin{aligned}\mathbb{E}(I_1^\varepsilon)^2 &= \mathbb{E}\Big(\int_D v^\varepsilon(x)v^\varepsilon(x)\psi^\varepsilon(x)\,dx\Big)^2 = \int_{D^2} R\Big(\frac{x-y}{\varepsilon}\Big)v^\varepsilon(x)v^\varepsilon(y)\psi^\varepsilon(x)\psi^\varepsilon(y)\,dx\,dy\\ &= \int_{\mathbb{R}^d}\big[R^\varepsilon * \big(v^\varepsilon(y)\psi^\varepsilon(y)\mathbb{1}_D(y)\big)\big](x)\,v^\varepsilon(x)\psi^\varepsilon(x)\mathbb{1}_D(x)\,dx\\ &\le C\|R^\varepsilon\|_{L^1(\mathbb{R}^d)}\|v^\varepsilon\psi^\varepsilon\mathbb{1}_D\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Here, $R^\varepsilon(y) = R\big(\frac{y}{\varepsilon}\big)$ is a shorthand notation. To obtain the last inequality, we applied Hölder's and Young's inequalities. Note that $\|R^\varepsilon\|_{L^1(\mathbb{R}^d)} = \varepsilon^d\|R\|_{L^1(\mathbb{R}^d)}$. Note also that $f, \varphi \in L^2(D)$ implies that $v^\varepsilon, \psi^\varepsilon \in H^2(D)$, which is embedded in $L^4(D)$ for all $2 \le d \le 7$. As a result, we conclude that $\mathbb{E}\,|I_1^\varepsilon| \le C\varepsilon^{d/2}$.

Estimate for $\mathrm{Var}(I_2^\varepsilon)$. Before calculating the variance of I_2^ε, we first check that $\|I_2^\varepsilon\|_{L^2(\Omega)}$ can have size larger than $\varepsilon^{d/2}$ for $d \ge 4$. By direct computation, for $d \ge 3$,

$$\begin{aligned}\mathbb{E}(I_2^\varepsilon)^2 &= \int_{D^4} R\Big(\frac{x-y}{\varepsilon}\Big)R\Big(\frac{x'-y'}{\varepsilon}\Big)G_\varepsilon(x,y)G_\varepsilon(x',y')v^\varepsilon(y)v^\varepsilon(y')\psi^\varepsilon(x)\psi^\varepsilon(x')\,dx'\,dy'\,dx\,dy\\ &\lesssim \int_{D^4}\Big|R\Big(\frac{x-y}{\varepsilon}\Big)R\Big(\frac{x'-y'}{\varepsilon}\Big)\Big|\frac{|v^\varepsilon(y)v^\varepsilon(y')\psi^\varepsilon(x)\psi^\varepsilon(x')|}{|x-y|^{d-2}|x'-y'|^{d-2}}\,dy'\,dx'\,dy\,dx.\end{aligned}\tag{4-18}$$

For $d = 2$, the last integral above should be replaced by

$$\int_{D^4}\Big|R\Big(\frac{x-y}{\varepsilon}\Big)R\Big(\frac{x'-y'}{\varepsilon}\Big)v^\varepsilon(y)v^\varepsilon(y')\psi^\varepsilon(x)\psi^\varepsilon(x')\big(1+\big|\log|x-y|\big|\big)\big(1+\big|\log|x'-y'|\big|\big)\Big|\,dy'\,dx'\,dy\,dx.$$

After the change of variables

$$\frac{x-y}{\varepsilon}\mapsto x,\qquad \frac{x'-y'}{\varepsilon}\mapsto x',\qquad y\to y,\qquad y'\to y',$$

the integral to be controlled, for $d \ge 3$, becomes

$$\varepsilon^4\int_{B^2_{\rho/\varepsilon}}\int_{D^2}|R(x)R(x')|\frac{\big|v^\varepsilon(y)\psi^\varepsilon(y+\varepsilon x)v^\varepsilon(y')\psi^\varepsilon(y'+\varepsilon x')\big|}{|x|^{d-2}|x'|^{d-2}}\,dy'\,dy\,dx'\,dx.$$

Integrating over y and y' and then over x and x', we find that the integral above is finite. Hence, $\mathbb{E}(I_2^\varepsilon)^2$ is of order ε^4 when $d \ge 3$. When $d = 2$, the change of variables in the logarithmic functions yields the term $|\log\varepsilon|^2$, and we have $\mathbb{E}(I_2^\varepsilon)^2$ is of order $\varepsilon^4|\log\varepsilon|^2$. This shows that the second term in (4-11), i.e., I_2^ε, is larger than or comparable to $\varepsilon^{d/2}$ for $d \ge 4$.

We show next that the variance of I_2^ε, however, is smaller than $\varepsilon^{d/2}$ in all dimensions. Using the definition of Ψ_ν in (4-13) and the estimate in Lemma 4.3, we bound $\mathbb{E}(I_2^\varepsilon - \mathbb{E} I_2^\varepsilon)^2 = \mathrm{Var}(I_2^\varepsilon)$, for $d \geq 3$, by

$$\begin{aligned}
\mathrm{Var}(I_2^\varepsilon) &= \int_{D^4} \Psi_\nu\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x'}{\varepsilon}, \frac{y'}{\varepsilon}\Big) G_\varepsilon(x,y) G_\varepsilon(x',y') v^\varepsilon(y) v^\varepsilon(y') \psi^\varepsilon(x) \psi^\varepsilon(x')\, dx'\, dy'\, dx\, dy \\
&\leq C \int_{D^4} \vartheta\Big(\frac{x-x'}{\varepsilon}\Big) \vartheta\Big(\frac{y-y'}{\varepsilon}\Big) \frac{|v^\varepsilon(y) v^\varepsilon(y') \psi^\varepsilon(x) \psi^\varepsilon(x')|}{|x-y|^{d-2} |x'-y'|^{d-2}}\, dx'\, dy'\, dx\, dy \\
&\qquad + C \int_{D^4} \vartheta\Big(\frac{x-y'}{\varepsilon}\Big) \vartheta\Big(\frac{y-x'}{\varepsilon}\Big) \frac{|v^\varepsilon(y) v^\varepsilon(y') \psi^\varepsilon(x) \psi^\varepsilon(x')|}{|x-y|^{d-2} |x'-y'|^{d-2}}\, dx'\, dy'\, dx\, dy.
\end{aligned}$$

The second integral above is essentially the same with the first one if we interchange x' and y'. Hence, we focus only on the first one. After the change of variables

$$\frac{x-x'}{\varepsilon} \mapsto x, \quad \frac{y-y'}{\varepsilon} \mapsto y, \quad y - x' \mapsto x', \quad y' \mapsto y',$$

the first integral becomes

$$C\varepsilon^{2d} \int_{\mathbb{R}^{2d}} dx\, dy \int_{B_\rho} dx' \int_D dy' \vartheta(x) \vartheta(y) \frac{|v^\varepsilon(y'+\varepsilon y) v^\varepsilon(y') \psi^\varepsilon(y'-x'+\varepsilon x+\varepsilon y) \psi^\varepsilon(y'-x'+\varepsilon y)|}{|x'-\varepsilon x|^{d-2} |x'-\varepsilon y|^{d-2}}.$$

Integrate over y' first and use the fact that $\|v^\varepsilon\|_{L^4} \leq C$ and $\|\psi^\varepsilon\|_{L^4(D)} \leq C$. Then the above integral is bounded by

$$\int_{\mathbb{R}^{2d}} dx\, dy \int_{B_\rho} \vartheta(x) \vartheta(y) \frac{C\varepsilon^{2d}\, dx'}{|x'-\varepsilon x|^{d-2} |x'-\varepsilon y|^{d-2}} \leq \int_{\mathbb{R}^{3d}} \frac{C\varepsilon^{2d-(d-4)\vee 0} \vartheta(x) \vartheta(y)\, dx'\, dx\, dy}{|x-y|^{(d-4)\vee 0}}$$

for $d \neq 4$, where we integrated over x' and used (4-6) to have the inequality. The resulting integral is clearly finite. Hence we conclude that $\mathrm{Var}(I_2^\varepsilon) \leq C\varepsilon^{2d}$ for $d = 3$ and that it is of order ε^{d+4} for $d \geq 5$.

When $d = 2$, there is only logarithmic singularity to start with in the expression of $\mathrm{Var}(I_2^\varepsilon)$, and we find $\mathrm{Var}(I_2^\varepsilon) \leq C\varepsilon^{2d}$.

When $d = 4$, the integral that remains after we integrate over x' has a term of the form

$$\big(\log|\varepsilon(x-y)|\big) \mathbb{1}_{\varepsilon(x-y) \in B_{2\rho}}.$$

It follows then that $\mathrm{Var}(I_2^\varepsilon) \leq C\varepsilon^8 |\log \varepsilon|$.

To summarize, for $d \geq 2$, we have $\mathbb{E}|I_2^\varepsilon - \mathbb{E}\, I_2^\varepsilon| \ll \mathbb{E}|I_1^\varepsilon|$. That is, when the series expansion is integrated against test functions and the mean is removed, the second term is much smaller than the leading term.

Estimate for I_3^ε. For the last term, we control it by the crude estimate $\mathbb{E}(I_3^\varepsilon - \mathbb{E}\, I_3^\varepsilon)^2 \leq 2\mathbb{E}(I_3^\varepsilon)^2$. From the expression $I_3^\varepsilon = (\nu^\varepsilon(u^\varepsilon - v^\varepsilon), \mathcal{G}_\varepsilon \nu^\varepsilon \psi^\varepsilon)$, we have

$$\mathbb{E}|I_3^\varepsilon| \leq \mathbb{E}\big(\|\nu^\varepsilon\|_{L^\infty} \|u^\varepsilon - v^\varepsilon\|_{L^2} \|\mathcal{G}_\varepsilon \nu^\varepsilon \psi^\varepsilon\|_{L^2}\big) \leq C\big(\mathbb{E}\|u^\varepsilon - v^\varepsilon\|_{L^2}^2\, \mathbb{E}\|\mathcal{G}_\varepsilon \nu^\varepsilon \psi^\varepsilon\|_{L^2}^2\big)^{\frac{1}{2}}$$

since $\mathcal{G}_\varepsilon \nu^\varepsilon \psi^\varepsilon$ is exactly of the form of w^ε defined in (4-1). Owing to (4-2) and Lemma 4.1, we conclude that $\mathbb{E}\,|I_3^\varepsilon|$ is of order ε^d for $d = 2, 3$, of order $\varepsilon^4 |\log \varepsilon|$ for $d = 4$, and of order ε^4 for $d \geq 5$. Hence, for all $2 \le d \le 7$, the truncation term in the Neumann series, with respect to the weak topology, has a scaling factor that is smaller than that of the leading term (which is of order $\varepsilon^{d/2}$).

Remark 4.6. We find that for $2 \le d \le 7$, the random fluctuation $u^\varepsilon - \mathbb{E}\, u^\varepsilon$ scales like $\varepsilon^{d/2}$ when integrated against test functions, and the leading term is the dominating one. We do not expect the dimension constraint $d \le 7$ to be intrinsic. Firstly, it is related to the fact that we stopped at the second-order iteration in the Neumann series, and had to control the last term by the crude estimate given by the Minkowski inequality (not taking advantage of removing the mean). Secondly, it is also needed when we claim that $\psi^\varepsilon = \mathcal{G}_\varepsilon \nu^\varepsilon \varphi$ is in $L^4(D)$. In general, if we assume a stronger condition, namely $f \in C(\overline{D})$, then v^ε is always bounded, and we only need $\psi^\varepsilon \in L^2(D)$, which holds in all dimensions if $\varphi \in L^2(D)$.

We conclude this section by collecting the facts obtained above to give a proof of Theorem 2.3.

Proof of Theorem 2.3. Let v^ε be as defined in (3-7). In view of (3-10) and the Minkowski inequality, we have

$$\mathbb{E}\,\|u^\varepsilon - \mathbb{E}\, u^\varepsilon\|_{L^2}^2 \le \mathbb{E}\big(2\|u^\varepsilon - v^\varepsilon\|_{L^2}^2 + 2\,\|\mathbb{E}(u^\varepsilon - v^\varepsilon)\|_{L^2}^2\big) \le 4\,\mathbb{E}\,\|u^\varepsilon - v^\varepsilon\|_{L^2}^2.$$

Owing to (4-2) and Lemma 4.1, we have (2-13).

In view of (3-8), (4-2), Lemma 4.1 and Theorem 3.3, we have

$$\mathbb{E}\,\|u^\varepsilon - u\|_{L^2} \le \mathbb{E}\,\|u^\varepsilon - v^\varepsilon\|_{L^2} + \|v^\varepsilon - u\|_{L^2} \le C\varepsilon \|f\|_{L^2}.$$

This proves (2-12).

Finally, to estimate $\mathbb{E}|(u^\varepsilon - \mathbb{E} u^\varepsilon, \varphi)_{L^2}|$ for an arbitrary field $\varphi \in L^2(D)$, without loss of generality we can assume $\|\varphi\|_{L^2} = 1$. Then this term is precisely what was studied immediately above. With I_j^ε, where $j = 1, 2, 3$, defined earlier, we have showed that for $2 \le d \le 7$, we have $\mathbb{E}\,|\sum_{j=1}^3 (I_j^\varepsilon - \mathbb{E}\, I_j^\varepsilon)| \le C\varepsilon^{d/2}$, which is precisely (2-14). ∎

5. Limiting distribution of the random fluctuation

In this section, we study the limiting distribution of the scaled random fluctuation, $\varepsilon^{-d/2}(u^\varepsilon - \mathbb{E}\, u^\varepsilon)$, in functional spaces. As mentioned earlier, the choice of space depends on dimension. When $d = 1$, convergence in law in $C(D)$ of the random fluctuation was proved in [Bourgeat and Piatnitski 1999; Bal 2008]. We prove Theorem 2.4 below, which establishes convergence in law of the random fluctuation in $L^2(D)$ for $d = 2, 3$ and in $H^{-1}(D)$ for $d = 4, 5$.

Multiplying $\varepsilon^{-d/2}$ to the series expansion (4-11), we obtain the following expression for the scaled random fluctuation:

$$-\frac{\mathcal{G}_\varepsilon \nu_\varepsilon v^\varepsilon}{\sqrt{\varepsilon^d}} + \frac{\mathcal{G}_\varepsilon \nu_\varepsilon \mathcal{G}_\varepsilon \nu_\varepsilon v^\varepsilon - \mathbb{E}\mathcal{G}_\varepsilon \nu_\varepsilon \mathcal{G}_\varepsilon \nu_\varepsilon v^\varepsilon}{\sqrt{\varepsilon^d}} + \frac{\mathcal{G}_\varepsilon \nu_\varepsilon \mathcal{G}_\varepsilon \nu_\varepsilon (u^\varepsilon - v^\varepsilon) - \mathbb{E}\mathcal{G}_\varepsilon \nu_\varepsilon \mathcal{G}_\varepsilon \nu_\varepsilon (u^\varepsilon - v^\varepsilon)}{\sqrt{\varepsilon^d}}. \tag{5-1}$$

Our strategy, as in [Bal 2008; Bal and Jing 2011], is to prove that the leading term $X^\varepsilon = -\varepsilon^{-d/2}\mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon$ contributes and converges in law to the right distribution depicted by Theorem 2.4, and show that the other terms converge in stronger mode to the zero function and hence have no contribution to the limiting law.

At the purely formal level, all these steps are the same as in the setting of nonoscillatory diffusion coefficients. Indeed, we already established controls for the second and last terms above in the previous section. Moreover, the ε-dependence in $\mathcal{G}_\varepsilon$ and v^ε is not a problem, as we will see later, for the convergence of the characteristic functions of X^ε, thanks to the fact that $\mathcal{G}_\varepsilon\varphi \to \mathcal{G}\varphi$ in L^2 for any $\varphi \in H^{-1}(D)$. This dependence, however, does impose difficulty on showing the tightness of the measures of $\{X^\varepsilon\}_\varepsilon$. As discussed in Section 7, the old approach for tightness in [Bal et al. 2012] fails and new ideas are needed.

Our new approach is to use some nonoptimal but convenient tightness criteria, described in Theorems A.1 and A.2, for probability measures on $H^k(D)$ that are induced by processes in $H^{k+s}(D)$, with $k=-1,0$ and $s>0$. Since we do need s to be fractional in $(0,1)$, we recall some definitions regarding fractional Sobolev spaces; see [Di Nezza et al. 2012] for reference. Given an open set $K \subset \mathbb{R}^d$, the fractional Sobolev space $H_0^s(K)$, for $s \in (0,1)$, is the closure of $C_0^\infty(K)$ in the norm

$$\|u\|_{H^s(K)}^2 := \|u\|_{L^2(K)}^2 + \int_{K^2} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,dx\,dy.$$

When $K=\mathbb{R}^d$, an equivalent norm for $u \in H^s(\mathbb{R}^d)$ is

$$\|u\|_{H^s(\mathbb{R}^d)}^2 := \int_{\mathbb{R}^d} (1+|\xi|^2)^s\,|\mathscr{F}u|^2(\xi)\,d\xi. \tag{5-2}$$

Moreover, for $s \in (0,1)$, the space $H^{-s}(K)$ is defined to be the dual space $(H_0^s(K))'$, and in particular when $K=\mathbb{R}^d$, the above norm for $H^{-s}(\mathbb{R}^d)$ is still valid.

5A. *Limiting distribution in $L^2(D)$ for dimensions two and three.* For $d=2,3$, we prove that the leading term X^ε in (4-11) converges in law in $L^2(D)$ and show that the other terms vanish in the limit. The next lemma, together with Theorem A.1, yields tightness of X^ε, which is the key step.

Lemma 5.1. *Suppose that the conditions of Theorem 2.3 are satisfied. Assume further that $d=2,3$. Then for any $s\in\left(0,\frac{1}{2}\right)$, there exists a constant C, depending only on the universal parameters and s, such that*

$$\mathbb{E}\,\|\varepsilon^{-\frac{d}{2}}\mathcal{G}_\varepsilon v^\varepsilon v^\varepsilon\|_{H^s}^2 \le C. \tag{5-3}$$

Proof. For each fixed $\omega \in \Omega$ and $\varepsilon>0$, we know that $\varepsilon^{-d/2}\mathcal{G}_\varepsilon v^\varepsilon v^\varepsilon$ belongs to $H_0^1(D)$ and hence also to $H_0^s(D)$ for any $s \in (0,1)$. In particular, its H^s-seminorm has the expression

$$[\varepsilon^{-\frac{d}{2}}\mathcal{G}_\varepsilon v^\varepsilon v^\varepsilon]_{H^s(D)}^2 = \frac{1}{\varepsilon^d}\int_{D^2} \frac{\left|(\mathcal{G}_\varepsilon v^\varepsilon v^\varepsilon)(x) - (\mathcal{G}_\varepsilon v^\varepsilon v^\varepsilon)(y)\right|^2}{|x-y|^{d+2s}}\,dy\,dx.$$

Taking the expectation and using the L^4-bounds of v^ε, we have

$$\mathbb{E}[\varepsilon^{-\frac{d}{2}}\mathcal{G}_\varepsilon v^\varepsilon v^\varepsilon]_{H^s(D)}^2 \le \frac{C}{\varepsilon^d}\int_{D^4} \frac{\left|\big(G_\varepsilon(x,z)-G_\varepsilon(y,z)\big)\big(G_\varepsilon(x,\xi)-G_\varepsilon(y,\xi)\big)\right|}{|x-y|^{d+2s}}\left|R\left(\frac{z-\xi}{\varepsilon}\right)\right|d\xi\,dz\,dy\,dx.$$

We claim: there exists C, depending only on the universal parameters and s, such that for all $\xi, z \in D$,

$$\int_{D^2} \frac{\left|\big(G_\varepsilon(x,z)-G_\varepsilon(y,z)\big)\big(G_\varepsilon(x,\xi)-G_\varepsilon(y,\xi)\big)\right|}{|x-y|^{d+2s}}\,dy\,dx \le C. \tag{5-4}$$

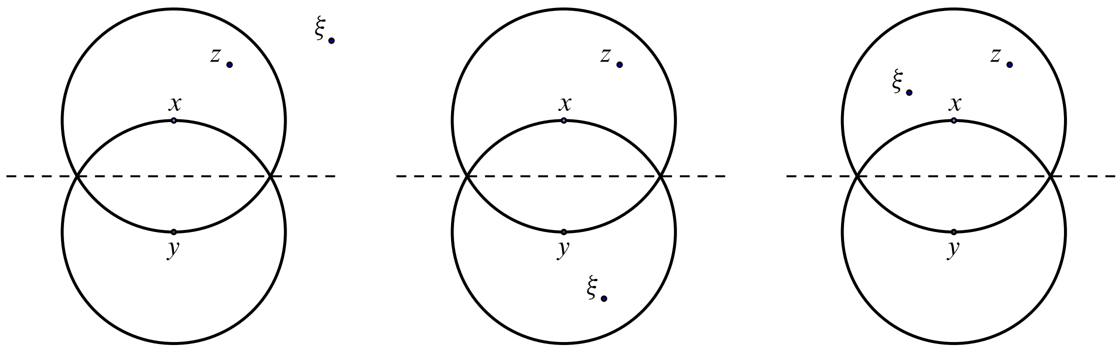

Figure 1. Decomposition criteria of the domain of integration based on the relative position between four points. Left: $(x, y) \in D_1^2$; middle: $(x, y) \in D_2^2$; right: $(x, y) \in D_3^2$.

We decompose the integration region D^2 into three parts D_j^2, with $j = 1, 2, 3$, as follows: in D_1^2, one of the points in $\{z, \xi\}$, namely ξ without loss of generality, lies outside $B_\rho(x) \cup B_\rho(y)$, where $\rho = |x - y|$; in D_2^2, one of the points, namely z without loss of generality, lies in $B_\rho(x)$ and satisfies $|z - x| \le |z - y|$ and at the same time $\xi \in B_\rho(y)$ and $|\xi - y| \le |\xi - x|$; in D_3^2, we have that ξ and z cluster around one of the points in $\{x, y\}$; without loss of generality, assume this point is x, so $z, \xi \in B_\rho(x) \cap \{\eta : |\eta - x| < |\eta - y|\}$. In Figure 1, the relative positions between $\{x, y, z, \xi\}$ are illustrated for each case.

Let I_j be the integral over D_j^2 of the integrand in (5-4). We estimate I_j, with $j = 1, 2, 3$, separately and we focus on the case of $d = 3$. It is clear that when $d = 2$, the only change is that the Green's function has logarithmic bound, and the analysis below can be adapted.

On D_1^2, without loss of generality, we assume that $|z - x| \le |z - y|$ (if otherwise, we would switch the roles of x and y). Hence $|G_\varepsilon(x, z) - G_\varepsilon(y, z)| \le C|x - z|^{2-d}$. By the mean value theorem,

$$|G_\varepsilon(x, \xi) - G_\varepsilon(y, \xi)| \le |\nabla G_\varepsilon(\eta, \xi)||x - y| \quad \text{for some } \eta \text{ between } x \text{ and } y.$$

By the gradient bound (3-12) and the fact that $|\eta - \xi| \ge |y - \xi|/2$, we have

$$|\nabla G_\varepsilon(\eta, \xi)| \le \frac{C}{|\eta - \xi|^{d-1}} \le \frac{C}{|y - \xi|^{d-1}}, \quad \text{hence} \quad |G_\varepsilon(x, \xi) - G_\varepsilon(y, \xi)| \le \frac{C|x - y|}{|y - \xi|^{d-1}}.$$

As a result, we have,

$$I_1 \le \int_{D_1^2} \frac{C}{|x - y|^{d+2s-1}} \frac{1}{|x - z|^{d-2}} \frac{1}{|y - \xi|^{d-1}} \, dx \, dy.$$

Integrate over x first and then over y, using (4-6) in each step; we find that as long as $0 < s < \frac{1}{2}$, we have $I_1 \le C$ for some C that only depends on the universal parameters and s.

On D_2^2, we have $|G_\varepsilon(x, z) - G_\varepsilon(y, z)| \le C|x - z|^{2-d}$ and $|G_\varepsilon(x, \xi) - G_\varepsilon(y, \xi)| \le C|y - \xi|^{2-d}$. At the same time, $|x - z| \le |x - y|$ and $|y - \xi| \le |x - y|$, so we may split the singularity into the integrals over x and y so that each of them is essentially not singular. That is,

$$I_2 \le \int_{D_2^2} \frac{C}{|x - z|^{\frac{d}{2}+s} |y - \xi|^{\frac{d}{2}+s}} \frac{1}{|x - z|^{d-2}} \frac{1}{|y - \xi|^{d-2}} \, dx \, dy.$$

We note that the integral above can be separated, and as long as $0 < s < 2 - \frac{d}{2} = \frac{1}{2}$, each integral is finite and hence $I_2 \le C$.

On D_3^2, we assume without loss of generality that z and ξ cluster around x. Then we have

$$|G_\varepsilon(x,\eta) - G_\varepsilon(y,\eta)| \le C|x-\eta|^{2-d}$$

for $\eta \in \{z, \xi\}$. At the same time, $|x-y| > |y-z|$. As a result, we have

$$I_3 \le \int_{D_3^2} \frac{C}{|y-z|^{d-\tau}|x-z|^{2s+\tau}} \frac{1}{|x-z|^{d-2}} \frac{1}{|x-\xi|^{d-2}}\, dx\, dy.$$

We choose $\tau > 0$ so the integral over y is uniformly bounded. The integral over x is also bounded as long as $2s + \tau < (4-d) \wedge 2 = 1$, and we have $I_3 \le C$. We note that for any $s \in \left(0, \frac{1}{2}\right)$, there exists $\tau \in (0, 1-2s)$ satisfying the constraint $2s + \tau < 1$.

The above bounds are uniform in δ. Therefore, taking the limit $\delta \to 0$, we prove (5-4). Integrate over z and ξ in the integral expression of $\mathbb{E}[\varepsilon^{-d/2}\mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon]_{H^s}^2$; in particular, integrating $R(\cdot/\varepsilon)$ yields a factor of ε^d that cancels the one in the denominator. We conclude that $\mathbb{E}[\varepsilon^{-d/2}\mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon]_{H^s}^2 \le C$ for each fixed $s \in \left(0, \frac{1}{2}\right)$. Combining this with $E\,\|\varepsilon^{-d/2}\mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon\|_{L^2}^2 \le C$, which is due to (4-1) for $d = 2, 3$, we prove (5-3). ☐

Remark 5.2. The key step in the proof above is to derive (5-4), which concerns only the Green's function G_ε and hence is obtained from a purely deterministic argument. Indeed, the scaling factor $\varepsilon^{-d/2}$ plays a role only afterward when we integrate against R^ε, and it disappears in the final estimate because it is the right scaling for integrals of R^ε. In Section 6, where we consider the case of long-range correlated random potential $q(x,\omega)$, the scaling in X^ε will be different, but the tightness of (the measures of) X^ε, with the right scaling, is obtained in the same way as above.

Next we address the convergence of the characteristic function of the measure of X^ε. In view of Theorem 2.1, this amounts to proving this:

Lemma 5.3. *Assume (S) holds. For any fixed $\varphi \in L^2(D)$, we have*

$$-\frac{1}{\sqrt{\varepsilon^d}}(\mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon, \varphi)_{L^2} \xrightarrow{\text{distribution}} \mathcal{N}(0, \sigma_\varphi^2), \quad \text{where } \sigma_\varphi^2 := \sigma^2 \int_D u^2(x)(\mathcal{G}\varphi)^2(x)\, dx. \tag{5-5}$$

Proof. Moving the operator $\mathcal{G}_\varepsilon$ to φ, we have

$$-\frac{1}{\sqrt{\varepsilon^d}}(\mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon, \varphi)_{L^2} = -\frac{1}{\sqrt{\varepsilon^d}} \int_D \nu\left(\frac{x}{\varepsilon}\right) v^\varepsilon(x) \psi^\varepsilon(x)\, dx,$$

where $\psi^\varepsilon = \mathcal{G}_\varepsilon \varphi$. Let $I_1^\varepsilon[\varphi]$ denote the random variable above. Set $\psi = \mathcal{G}\varphi$ and introduce

$$J_1^\varepsilon[\varphi] := -\frac{1}{\sqrt{\varepsilon^d}} \int_D \nu\left(\frac{x}{\varepsilon}\right) u(x)\psi(x)\, dx. \tag{5-6}$$

Since $\nu(x,\omega)$ is a stationary ergodic random field that has short-range correlation, $u\psi \in L^2(D)$, we apply the well-known functional central limit theorem (see, e.g., [Bal 2008, Theorem 3.8]) and obtain

$$J_1^\varepsilon[\varphi] \xrightarrow{\text{distribution}} I_1[\varphi] := \sigma\left(\int_D G(x,y)u(y)\,dW(y),\varphi\right)_{L^2} \sim \mathcal{N}\left(0,\sigma^2\int_D (u(y)\psi(y))^2\,dy\right). \quad (5\text{-}7)$$

The last relation $\sim$ above means equal in law. We note that

$$\mathbb{E}\left|J_1^\varepsilon[\varphi]-I_1^\varepsilon[\varphi]\right|^2 = \frac{1}{\varepsilon^d}\mathbb{E}\left(\int_D \nu\left(\frac{x}{\varepsilon}\right)(v^\varepsilon\psi^\varepsilon - u\psi)\,dx\right)^2 \le C\|v^\varepsilon\psi^\varepsilon-u\psi\|_{L^2}^2,$$

and from periodic homogenization theory, we have $v^\varepsilon \to u$ in L^2, $\psi^\varepsilon\to\psi$ in L^2, as $\varepsilon\to 0$; moreover, v^ε and ψ^ε are bounded in L^∞ since $H^2(D)$ is embedded in $L^\infty(D)$ for $d=2,3$. As a consequence, the right-hand side above converges to zero as $\varepsilon\to 0$. As a result,

$$I_1^\varepsilon[\varphi] = J_1^\varepsilon[\varphi] + (I_1^\varepsilon[\varphi]-J_1^\varepsilon[\varphi])$$

is the sum of a term that converges in distribution to $I_1[\varphi]$ and a term that converges to zero in $L^2(\Omega)$. The desired result follows immediately. ☐

Finally, we collect the facts obtained above to give a proof of Theorem 2.4(i).

Proof of Theorem 2.4(i). Owing to Lemma 4.2 and Lemma 4.4, for $d=2,3$, we have

$$\mathbb{E}\left\|\varepsilon^{-\frac{d}{2}}\left(\mathcal{G}_\varepsilon\nu_\varepsilon\mathcal{G}_\varepsilon\nu_\varepsilon v^\varepsilon - \mathbb{E}\mathcal{G}_\varepsilon\nu_\varepsilon\mathcal{G}_\varepsilon\nu_\varepsilon v^\varepsilon\right) + \varepsilon^{-\frac{d}{2}}\left(\mathcal{G}_\varepsilon\nu_\varepsilon\mathcal{G}_\varepsilon\nu_\varepsilon(u^\varepsilon-v^\varepsilon) - \mathbb{E}\mathcal{G}_\varepsilon\nu_\varepsilon\mathcal{G}_\varepsilon\nu_\varepsilon(u^\varepsilon-v^\varepsilon)\right)\right\|_{L^2} \le \varepsilon^{\frac{d}{2}}.$$

By Chebyshev's inequality, these two terms, as random elements of $L^2(D)$, converge in probability to the zero function. It follows that the limiting distribution of $\varepsilon^{-d/2}(u^\varepsilon - \mathbb{E}\,u^\varepsilon)$ is given by that of the leading term $X^\varepsilon(\omega) := -\varepsilon^{-d/2}\mathcal{G}_\varepsilon\nu^\varepsilon v^\varepsilon$.

Let X be the right-hand side of (2-15). It is a random element of $L^2(D)$ defined on some probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ on which the Wiener process $W(y,\tilde{\omega})$ is defined. Let the distribution of X be P^X and its characteristic function be ϕ^{P^X}. We note that, for any $\varphi\in L^2(D)$, the inner product (X,φ) has Gaussian distribution $\mathcal{N}(0,\sigma_\varphi^2)$, with σ_φ^2 defined in (5-5). Indeed,

$$\mathbb{E}^{P^X}(X,\varphi) = \sigma\,\mathbb{E}^{P^X}\int_D\left(\int_D G(x,y)\varphi(x)\,dx\right)u(y)\,dW(y) = 0,$$

and

$$\mathbb{E}^{P^X}(X,\varphi)^2 = \sigma^2\,\mathbb{E}^{P^X}\left(\int_D\left(\int_D G(x,y)\varphi(x)\,dx\right)u(y)\,dW(y)\right)^2 = \sigma^2\int_D(\mathcal{G}\varphi)^2u^2\,dy.$$

This shows $(X,\varphi)\sim\mathcal{N}(0,\sigma_\varphi^2)$ in law. By Lemma 5.3, for any fixed $\varphi\in L^2(D)$, the random variable (X^ε,φ) converges in distribution to $\mathcal{N}(0,\sigma_\varphi^2)$. This shows that, as mentioned in Remark 2.2, the characteristic function of the law of X^ε converges to that of X. In view of Lemma 5.1 and Theorem A.1, the distribution of $\{X^\varepsilon\}_{\varepsilon\in(0,1)}$ in $L^2(D)$ is tight as well. Consequently, by applying Theorem 2.1, we complete the proof of Theorem 2.4(i). ☐

5B. ***Limiting distribution in $H^{-1}(D)$ for dimensions four and five.*** For dimension $d \geq 4$, we do not expect $\varepsilon^{-d/2}(u^\varepsilon - \mathbb{E}\, u^\varepsilon)$ to converge in distribution in $L^2(D)$, because as shown in (2-13), the fluctuations scale like $\varepsilon^2|\log\varepsilon|^{1/2}$ for $d = 4$, and scale like ε^2 for $d \geq 5$. In both cases, the scaling is much stronger than $\varepsilon^{d/2}$. Nevertheless, we prove that convergence in law in $H^{-1}(D)$ holds.

As before, the key step is to show that the probability measure in $H^{-1}(D)$ of the scaled leading term $\{X^\varepsilon\} := -\varepsilon^{-d/2}\mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon$ in the expansion (4-11) is tight, and to show that the characteristic function of this measure converges.

Let us first address the characteristic function ϕ^{X^ε}. We note that $L^2(D)$ is naturally embedded to $H^{-1}(D)$. For any $f \in L^2$, the linear form $L_f : H_0^1(D) \to \mathbb{R}$ given by $L_f(\psi) = (f, \psi)$ is clearly an element of $H^{-1}(D)$, and

$$\|L_f\|_{H^{-1}(D)} = \sup_{\psi \in H_0^1(D),\ \|\psi\|_{H^1} \leq 1} L_f(\psi) \leq \|f\|_{L^2}.$$

We henceforth identify $L_f \in H^{-1}(D)$ with f when $f \in L^2(D)$. For any $\ell \in H^{-1}(D)$, let l be the element in $H_0^1(D)$ that is related to ℓ by a Riesz representation. Then we have

$$(f, \ell)_{H^{-1}(D)} = L_f(l) = (f, l).$$

That is, the $H^{-1}(D)$ inner product of $f \in L^2(D)$ with ℓ is the same as the L^2 inner product of f with l. As a result, Remark 2.2 applies for distribution on $H^{-1}(D)$: to show $\phi^{P^{X^\varepsilon}}$ converges to ϕ^{P^X} as characteristic functions of distributions in $H^{-1}(D)$, it suffices to prove $(X^\varepsilon, h) \to (X, h)$ in distribution as random variables for each fixed $h \in L^2(D)$.

Now we address the tightness of the measures of $\{X^\varepsilon\}$. Our strategy is to control the mean of $\|X^\varepsilon\|_{H^{-s}(D)}$ for some $s \in (0, 1)$ and then apply Theorem A.2. To this purpose, we first observe that $X^\varepsilon \in L^2(D)$ and hence $X^\varepsilon \in H^{-s}(D)$ if we set

$$X^\varepsilon : H_0^s(D) \to \mathbb{R} \quad \text{by} \quad X^\varepsilon(h) = \int_D X^\varepsilon(x) h(x)\, dx. \tag{5-8}$$

For any $h \in H_0^s(D)$, the above clearly defines a continuous linear functional. Moreover, if we identify the function X^ε with its extension to $\mathbb{R}^d$ by zero outside D, the above also defines an element in $H^{-s}(\mathbb{R}^d)$. Since ∂D is regular (as a matter of fact, a $C^{0,1}$-boundary is sufficient), any $h \in H_0^s(D)$ can be extended continuously to $Eh \in H^s(\mathbb{R}^d)$, which satisfies $\|Eh\|_{H^s(\mathbb{R}^d)} \leq T\|h\|_{H^s(D)}$; see [Di Nezza et al. 2012, Theorem 5.4]; by duality, $H^{-s}(\mathbb{R}^d)$ is continuously embedded in $H^{-s}(D)$. If fact, we have

$$\begin{aligned} \|X^\varepsilon\|_{H^{-s}(D)} &:= \sup_{w \in H_0^s(D), \|w\|_{H^s(D)} \leq 1} (X^\varepsilon, w)_{L^2} = \sup_{w \in H^s(D), \|w\|_{H_0^s(D)} \leq 1} (X^\varepsilon, Ew)_{L^2} \\ &\leq \sup_{v \in H^s(\mathbb{R}^d), \|v\|_{H^s(\mathbb{R}^d)} \leq T} (X^\varepsilon, v)_{L^2} \leq T\|X^\varepsilon\|_{H^{-s}(\mathbb{R}^d)}. \end{aligned} \tag{5-9}$$

We note that $\|X^\varepsilon\|_{H^{-s}(\mathbb{R}^d)}$ can be calculated using the formula (5-2).

Consider, for each fixed $y \in D$, the Green's function $G_\varepsilon(\cdot\,, y)$ for the Dirichlet problem (3-7). Extend $G_\varepsilon(\cdot\,, y)$ to $\mathbb{R}^d$ by zero outside D, and let G_ε^y denote the extended function. Then G_ε^y defines naturally a linear form on $H^s(\mathbb{R}^d)$ by

$$\begin{aligned} G^y_\varepsilon &: H^s(\mathbb{R}^d) \to \mathbb{R}, \\ & h \mapsto G^y_\varepsilon(h) := \int_{\mathbb{R}^d} G^y_\varepsilon(x)h(x)\,dx = \int_D G_\varepsilon(x,y)h(x)\,dx, \end{aligned} \tag{5-10}$$

provided the integral is finite. Since $\mathcal{G}_\varepsilon$ is self-adjoint and by the Green's function representation, $w(y) := G^y_\varepsilon(h)$ is the solution to the Dirichlet problem $\mathcal{L}_\varepsilon w = h$ on D, with zero boundary condition. Note that the restriction of h on D is in $H^s(D)$. Invoking elliptic regularity, we find that w is bounded in $H^{s+2}(D)$. Let $s \in (0,1)$ if $d = 4$ and $s \in \left(\frac{1}{2},1\right)$ if $d = 5$; then by the embedding theorem of fractional Sobolev spaces, $H^{s+2}(D) \subset C^{0,\alpha}(D)$ with $\alpha = s+2-\frac{d}{2} \in (0,1)$; see [Grisvard 1985, Theorem 1.4.4.1]. As a result, $|G^y_\varepsilon(h)| \le C\|h\|_{H^s}$, where C only depends on the universal constants and the index s. We hence proved the following fact:

Lemma 5.4. *Assume (A) holds and $\bar q \ge 0$. Identify $G_\varepsilon(\,\cdot\,,y)$, for each fixed $y \in D$, with the element in $H^{-s}(\mathbb{R}^d)$ defined above. Suppose $s \in (0,1)$ for $d = 4$ and $s \in \left(\frac{1}{2},1\right)$ for $d = 5$. Then there exists $C > 0$, depending only on universal parameters and s, such that*

$$\|G_\varepsilon(\,\cdot\,,y)\|_{H^{-s}(\mathbb{R}^d)} \le C. \tag{5-11}$$

Using this fact and the Fourier transform formula for the $H^{-s}(\mathbb{R}^d)$-norm, we can prove the following control of $\|X^\varepsilon\|_{H^{-s}(\mathbb{R}^d)}$ which, together with Theorem A.2, yields the tightness of $\{X^\varepsilon\}$.

Lemma 5.5. *Suppose that the conditions of Theorem 2.3 are satisfied. Assume further that $d = 4, 5$. Let $s \in (0,1)$ if $d = 4$ and $s \in \left(\frac{1}{2},1\right)$ if $d = 5$. Then there exists a constant $C > 0$, depending only on the universal parameters and on s, such that*

$$\mathbb{E}\|X^\varepsilon\|^2_{H^{-s}(D)} \le C. \tag{5-12}$$

Proof. We identify X^ε with the element in $H^{-s}(D) \subset H^{-s}(\mathbb{R}^d)$ defined earlier. In view of (5-9), we have

$$\|X^\varepsilon\|^2_{H^{-s}(D)} \le C\|X^\varepsilon\|_{H^{-s}(\mathbb{R}^d)} = \frac{1}{\varepsilon^d}\int_{\mathbb{R}^d} |\mathcal{F}X^\varepsilon(\xi)|^2(1+|\xi|^2)^{-s}\,d\xi,$$

where $\mathcal{F}X^\varepsilon$ denotes the Fourier transform of the (extended) function X^ε. Using the integral representation of X^ε, we rewrite the above as

$$\|X^\varepsilon\|^2_{H^{-s}} = \frac{1}{\varepsilon^d}\int_{\mathbb{R}^d} d\xi\,(1+|\xi|^2)^{-s}\int_{\mathbb{R}^{2d}} dx\,dy \int_{D^2} dz\,dt\, e^{i\xi\cdot(x-y)}G_\varepsilon(x,z)G_\varepsilon(y,t)\nu^\varepsilon(z)v^\varepsilon(z)\nu^\varepsilon(t)v^\varepsilon(t),$$

where the Green's functions are extended by zero to $\mathbb{R}^d$ for their first variables. Take the expectation in this formula; we have

$$\mathbb{E}\|X^\varepsilon\|^2_{H^{-s}} = \frac{1}{\varepsilon^d}\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^{3d}} \frac{e^{i\xi\cdot(x-y)}G_\varepsilon(x,z)G_\varepsilon(y,t)}{(1+|\xi|^2)^s}\,dx\,dy\,d\xi\right) R\left(\frac{z-t}{\varepsilon}\right)v^\varepsilon(z)v^\varepsilon(t)\,dt\,dz.$$

We claim that for any z and t in D,

$$\left|\int_{\mathbb{R}^{3d}} \frac{e^{i\xi\cdot(x-y)}G_\varepsilon(x,z)G_\varepsilon(y,t)}{(1+|\xi|^2)^s}\,dx\,dy\,d\xi\right| \le C. \tag{5-13}$$

Indeed, we recognize the quantity inside the absolute value sign to be

$$\int_{\mathbb{R}^d} \frac{\mathscr{F}G_\varepsilon^z(\xi)\,\overline{\mathscr{F}G_\varepsilon^t(\xi)}}{(1+|\xi|^2)^s}\,d\xi \le \left(\int_{\mathbb{R}^d} |\mathscr{F}G_\varepsilon^z(\xi)|^2(1+|\xi|^2)^{-s}\right)^{\frac12} \left(\int_{\mathbb{R}^d} |\mathscr{F}G_\varepsilon^t(\xi)|^2(1+|\xi|^2)^{-s}\right)^{\frac12}.$$

The term on the right-hand side is precisely $\|G_\varepsilon^z\|_{H^{-s}(\mathbb{R}^d)}\|G_\varepsilon^t\|_{H^{-s}(\mathbb{R}^d)}$. In view of Lemma 5.4, we can apply (5-11) to get an upper bound for the quantity above and prove (5-11). Then (5-13) follows, which in turn completes the proof. □

Finally, we conclude this section by collecting the facts above and proving Theorem 2.4(ii).

Proof of Theorem 2.4(ii). Step 1: Limiting distribution of the leading term. In view of Theorem A.1 and Lemma 5.5, the probability measures on $H^{-1}(D)$ induced by $\{X^\varepsilon\}$ are tight. To check the limit of the characteristic functions of $\{P^{X^\varepsilon}\}$, it suffices to prove (2-11). This is done in Lemma 5.3. By Theorem 2.1, we conclude that $X^\varepsilon \to X$ in distribution on $H^{-1}(D)$, where X is defined to be the right-hand side of (2-15).

Step 2: Convergence to zero of the higher-order terms. By Lemma 4.2, and $d = 4, 5$, we see that the second term in $u^\varepsilon - \mathbb{E}\,u^\varepsilon$, i.e., $\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon - \mathbb{E}\,\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon \nu^\varepsilon v^\varepsilon$, converges in $L^2(\Omega, L^2(D))$ and hence in $L^2(\Omega, H^{-1}(D))$ to the zero function. Similarly, the remainder term $\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon \nu^\varepsilon(u^\varepsilon - v^\varepsilon) - \mathbb{E}\mathcal{G}_\varepsilon \nu^\varepsilon \mathcal{G}_\varepsilon \nu^\varepsilon(u^\varepsilon - v^\varepsilon)$ converges to the zero function in $L^1(\Omega, H^{-1}(D))$. These convergence results are stronger than the mode of convergence in distribution in $H^{-1}(D)$. The proof of Theorem 2.4(ii) is thus complete. □

6. The long-range correlated setting

In this section, we consider the setting where $q(x,\omega)$ has long-range correlations. In this setting, the general central limit theorem (Lemma 5.3) does not hold, and we hence restrict to the special case where q is constructed as a function of Gaussian random fields. Limiting theorems in the spirit of Lemma 5.3 are then obtained from Gaussian computations; see, e.g., [Bal et al. 2008; 2012].

Long-range correlated potentials constructed from Gaussian fields. Let $q(x,\omega) = \bar q + \nu(x,\omega)$ with $\bar q$ a nonnegative constant; we assume:

(L1) $\nu(x,\omega) = \Phi(g(x))$, and $g(x,\omega)$ is a centered stationary Gaussian random field with unit variance. Furthermore, the correlation function of $g(x,\omega)$ has heavy tail. That is, for some positive constant κ_g and some real number $\alpha \in (0,d)$,

$$R_g(x) := \mathbb{E}\big\{g(y,\omega)g(y+x,\omega)\big\} \sim \kappa_g |x|^{-\alpha} \quad \text{as } |x| \to \infty. \tag{6-1}$$

(L2) The function $\Phi : \mathbb{R} \to \mathbb{R}$ satisfies $-\bar q \le \Phi \le M - \bar q$, and has Hermite rank one, i.e.,

$$\int_{\mathbb{R}} \Phi(s) e^{-\frac{s^2}{2}}\,ds = 0, \qquad V_1 := \int_{\mathbb{R}} s\Phi(s) e^{-\frac{s^2}{2}}\,ds \neq 0. \tag{6-2}$$

(L3) The Fourier transform $\widehat{\Phi}$ of the function Φ satisfies

$$\int_{\mathbb{R}} |\widehat{\Phi}(\xi)|\big(1+|\xi|^3\big) < \infty. \tag{6-3}$$

We henceforth refer to the above conditions together as (L).

The assumption (L2) makes $\nu(x,\omega) = \Phi(g(x,\omega))$ mean zero, and the bounds on Φ ensure that $0 \le q(x,\omega) \le M$, which is (2-4). From the above construction, we check that $\nu(x,\omega)$ is stationary ergodic and has a long-range correlation function that decays like $\kappa|x|^{-\alpha}$, where $\kappa = V_1^2\kappa_g$; see Lemma A.3 for the details. Assumption (L3) allows one to derive a (nonasymptotic) estimate (see Lemma A.4 in the Appendix) for the fourth-order moments of $\nu(x,\omega)$. Universal constants in the long-range correlation setting may depend on α, R_g, κ_g, Φ and κ.

For the scaling of the homogenization error, we have the following analogue of Theorem 2.3. We focus on the error $u^\varepsilon - \mathbb{E}u^\varepsilon$ because, as seen earlier, the main contribution to the deterministic error $\mathbb{E}u^\varepsilon - u$ comes from the periodic oscillation in the diffusion coefficients, and Theorem 3.3(i) holds independent of the correlation length of $\nu(x,\omega)$.

Theorem 6.1. *Let $D \subset \mathbb{R}^d$ be an open bounded $C^{1,1}$-domain, u^ε and u be the solutions to (1-1) and (1-2) respectively. Suppose that (A), (P) and (L) hold and $f \in L^2(D)$. Then, there exists positive constant C, which depends only on the universal parameters, such that if $2 \le d \le 5$ and $0 < \alpha < d$ or $6 \le d \le 7$ and $0 < \alpha < 6$,*

$$\mathbb{E}\|u^\varepsilon - \mathbb{E}u^\varepsilon\|_{L^2} \le \begin{cases} C\varepsilon^{\frac{\alpha}{2}\wedge 2}\|f\|_{L^2} & \text{if } d \ne 4, \\ C\varepsilon^{\frac{\alpha}{2}}\|f\|_{L^2} & \text{if } d = 4. \end{cases} \tag{6-4}$$

Moreover, for any $\varphi \in L^2(D)$, with $2 \le d \le 7$ and $0 < \alpha < d$,

$$\mathbb{E}\Big|(u^\varepsilon - \mathbb{E}u^\varepsilon, \varphi)_{L^2}\Big| \le C\varepsilon^{\frac{\alpha}{2}}\|\varphi\|_{L^2}\|f\|_{L^2}. \tag{6-5}$$

This result shows that the random fluctuation $u^\varepsilon - \mathbb{E}u^\varepsilon$ caused by the long-range correlated random potential scales like $\varepsilon^{(\alpha\wedge 4)/2}$ in the energy norm, and scales like $\varepsilon^{\alpha/2}$ with respect to the weak topology. Since $\alpha < d$, we note that the random fluctuation in this setting is larger than the case of short-range correlated potential. We mention that if $\alpha < 2$, then the random fluctuations dominate the deterministic fluctuation caused by the periodic diffusion.

The next result exhibits the limiting law of the rescaled random fluctuation $\varepsilon^{-\alpha/2}(u^\varepsilon - \mathbb{E}u^\varepsilon)$. In the presentation, we define formally $W^\alpha(dy)$ as $\dot W^\alpha(y)\,dy$; here $\dot W^\alpha(y)$ is a centered stationary Gaussian random field with covariance function $\boldsymbol{E}(\dot W^\alpha(x)\dot W^\alpha(y)) = \kappa|x-y|^{-\alpha}$, where $\boldsymbol{E}$ denotes the expectation with respect to the distribution of $\dot W^\alpha$. Here, $\kappa = \kappa_g V_1^2 > 0$, where κ_g and V_1 are defined in (6-1) and (6-2).

Theorem 6.2. *Suppose that the assumptions in Theorem 6.1 hold. Let κ be defined as in (6-2) and $G(x,y)$ be the Green's function of (1-2). Let $W^\alpha(dy)$ be defined as above. Then*

(i) *For $d = 2, 3$, as $\varepsilon \to 0$,*

$$\frac{u_\varepsilon - \mathbb{E}\{u_\varepsilon\}}{\sqrt{\varepsilon^\alpha}} \xrightarrow[\varepsilon\to 0]{\text{distribution}} \sqrt{\kappa}\int_D G(x,y)u(y)W^\alpha(dy) \quad \text{in } L^2(D). \tag{6-6}$$

(2) *For $d = 4, 5$, as $\varepsilon \to 0$, the above holds as convergence in law in $H^{-1}(D)$.*

Remark 6.3. The right-hand side of (6-6) is an integral with respect to the multiparameter Gaussian random processes W^α; we refer to [Khoshnevisan 2002] for the theory. Let X denote the result of the integral. When $d \le 4$, $\alpha < 4$, the Green's function $G(x,\cdot)$ is in $L^{d/(d-\alpha/2)}$ and X is a random element in $L^2(D)$. In general, X is understood through the Fourier transform of its distribution. Given $h^* \in H^{-1}(D)$, the function $\phi^{P^X}(h^*)$ is defined to be $\boldsymbol{E} \exp\big(i\sqrt{\kappa}\int_D \langle G(\cdot\,,y), h^*(\cdot)\rangle u(y) W^\alpha(dy)\big)$. In particular, for any fixed positive integer N and functions $\{\varphi_i : 1 \le i \le N\}$ in $L^2(D)$, the random variables $I_i := \langle X, \varphi_i\rangle = \sqrt{\kappa}\int_D \langle G(\cdot\,,y), \varphi_i(\cdot)\rangle u(y) W^\alpha(dy)$, with $i = 1,\dots,N$, are joint Gaussian, centered and have covariance matrix $\Sigma_{ij} := \boldsymbol{E}(I_i I_j)$ given by

$$\Sigma_{ij} := \kappa \int_{D^2} \frac{(u\mathcal{G}\varphi_i)(y)(u\mathcal{G}\varphi_j)(z)}{|y-z|^\alpha}\, dy\, dz. \tag{6-7}$$

We will not present the proofs of the results above here, but they can be found in a longer version of this paper [Jing 2015]. The proofs are again based on the expansion formulas (4-11) and (4-17): the leading term has mean zero and contributes to the limiting law; the other terms have larger mean but smaller variance and, after the mean is removed, do not contribute to the limiting law. The main difference in the analysis of the long-range correlation setting is as follows. Firstly, to estimate integrals of $R^\varepsilon(x)$, because $R(x)$ is not integrable, we cannot expect to gain a factor of ε^d by scaling the variable in R^ε. Instead, we gain a factor of ε^α by using the asymptotic of R^ε outside a $(T\varepsilon)$-ball; see Lemma A.3. Secondly, to control fourth-order moments of ν, Lemma 4.3 is no longer useful and we use the estimate in Lemma A.4 instead. In fact, this estimate is less restrictive and, even in the short-range correlation setting, it could be used to replace $\varrho^{1/2}$ in the stronger assumption (S) by ϱ. Last but not least, as mentioned earlier, general central limit theorems, e.g., Lemma 5.3, are not available for the limiting law of the first term in (4-17), and we need to appeal to limit theorems that are special for functions of Gaussian processes.

7. Further discussions

7A. ***An alternative condition for (S).*** In the short-range correlation setting for $\nu(x,\omega)$, we assumed the condition (S). Upon applying Lemma 4.3, we can bound the (partial) fourth-order moment Ψ_ν by the sum of two terms, each consisting of the product of a pair of functions $\vartheta \in L^\infty \cap L^1(\mathbb{R}^d)$. However, as remarked earlier, (S) essentially requires the mixing coefficient $\varrho(r)$, and hence $R(|x|)$, to behave like $o(r^{-2d})$ at infinity, which is much stronger than $R(x)$ being integrable.

We remark that (S) is assumed mainly to simplify the presentation and the $o(r^{-2d})$ decay of ϱ is not necessary. In fact, an alternative assumption used in [Bal and Jing 2011] to control fourth-order moments is: there exists $\vartheta : \mathbb{R}^d \to \mathbb{R}_+$ in $L^1 \cap L^\infty(\mathbb{R}^d)$ such that (A-6) holds. This is clearly a much more general assumption, and it is satisfied if $\nu(x,\omega) = \Phi(g(x,\omega))$, with Φ satisfying (L2) and (L3), and $g(x,\omega)$ a centered stationary Gaussian random field with correlation function $R_g = o(|x|^{-d})$ as $|x| \to \infty$.

The conclusions of Theorem 2.4 still hold if (S) is replaced by the above alternative assumption. Indeed, we only need to modify the control of Ψ_ν in the proof of Lemma 4.2 and in Section 4B. For instance, in the first inequality in the proof of Lemma 4.2, we have more but finitely many integrals instead of two on the right-hand side. Nevertheless, in all of these integrals, at most one of the functions ϑ has the same variable as one of the Green's function, and all of them can be controlled. We refer to [Jing 2015, Section 6] for the details.

7B. ***Comparison with the case of nonoscillatory diffusion.*** The main results of this paper show that the framework developed in [Bal 2008; Bal and Jing 2011; Bal et al. 2012], in the setting of a nonoscillatory differential operator with oscillatory random potential, applies even when the differential operator is also oscillatory, as long as we have uniform-in-ε control of the Green's functions and their gradients, i.e., (3-11) and (3-12), and provided that there is no random correlation between the diffusion coefficients and the potential. At the formal level, there is no difference in the proof, and the usual strategy using (truncated) series expansion applies. However, the role played by the oscillatory diffusion coefficients becomes prominent in getting the tightness of the measures of $\{X^\varepsilon\}$.

Let us recall the previous method used for tightness in the setting of a nonoscillatory differential operator. Set

$$\mathcal{L} := -\sum_{i,j=1}^{d} \bar{a}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \bar{q},$$

and consider the Dirichlet problem $(\mathcal{L} + \nu^\varepsilon)u^\varepsilon = f$ in D with zero boundary condition. Then u^ε homogenizes to u, the solution of (1-2). As in [Bal et al. 2012], the limiting distribution of $\varepsilon^{-d/2}(u^\varepsilon - \mathbb{E}u^\varepsilon)$, say, in the short-range correlation setting, is characterized by that of $X^\varepsilon = -\varepsilon^{-d/2}\mathcal{G}\nu^\varepsilon u$. To prove tightness of the measures $\{X^\varepsilon\}$ in $L^2(D)$, the strategy of [Bal et al. 2012] is to use the spectral representation of $L^2(D)$. Note that $\mathcal{L}$ is formally self-adjoint and its inverse, i.e., $\mathcal{G}$, is compact on $L^2(D)$. Hence, $\mathcal{L}$ admits real eigenvalues $\{\lambda_k\}_{k=1}^\infty$ such that,

$$0 \le \bar{q} < \lambda_1 \le \lambda_2 \le \cdots, \quad \lambda_k \to \infty \quad \text{as} \quad k \to \infty,$$

and eigenfunctions $\{\phi_k\}_{k=1}^\infty$, with $\|\phi_k\|_{L^2} = 1$, such that

$$\begin{cases} \mathcal{L}\phi_k = \lambda_k \phi_k & \text{in } D, \\ \phi_k = 0 & \text{on } \partial D. \end{cases}$$

Moreover, $\{\phi_k\}$ form an orthonormal basis of $L^2(D)$ and we have the following representation of the space $\mathcal{H}^0(D) = L^2(D)$ and the Sobolev space $\mathcal{H}^1(D) = H_0^1(D)$; see [Evans 1998, Section 6.5]: for $s = 0, 1$,

$$\mathcal{H}^s(D) = \overline{\left\{ f \in C^\infty(D) : \sum_{k=1}^\infty (f, \phi_k)_{L^2}^2 \lambda_k^s < \infty \right\}} \quad \text{and} \quad \|v\|_{\mathcal{H}^s}^2 := \sum_{k=1}^\infty (f, \phi_k)_{L^2}^2 \lambda_k^s. \tag{7-1}$$

A natural criterion for tightness of (the measures of) $\{X^\varepsilon\}$ is that their measures do not scatter to higher and higher modes. More precisely, let P_N denote the projection operator in $L^2(D)$ to the space $W_N := \operatorname{span}\{\phi_1, \dots, \phi_N\}$ spanned by the first N modes. Then $\{X^\varepsilon\}$ is tight if $\mathbb{E}\|X^\varepsilon\|_{L^2} \le C$ and

$$\lim_{N\to\infty} \sup_{\varepsilon\in(0,1)} \mathbb{E}\|X^\varepsilon - P_N X^\varepsilon\|_{L^2} = 0. \tag{7-2}$$

Using the representation formula in (7-1), and the fact that $\mathcal{G}\phi_k = (\lambda_k)^{-1}\phi_k$, we have

$$\mathbb{E}\|X^\varepsilon - P_N X^\varepsilon\|_{L^2}^2 = \frac{1}{\varepsilon^d} \sum_{k=N+1}^\infty \mathbb{E}(\mathcal{G}\nu^\varepsilon u, \phi_k)^2 = \frac{1}{\varepsilon^d} \sum_{k=N+1}^\infty \frac{1}{\lambda_k^2} \mathbb{E}(\nu^\varepsilon u, \phi_k)^2.$$

As in Section 4B, we have $\sup_{\varepsilon\in(0,1)}\sup_k \mathbb{E}(\nu^\varepsilon u,\phi^k)^2 \le C$. In view of Weyl's asymptotic formula for the eigenvalues, $\lambda_k \approx k^{2/d}$ for k large, we conclude that

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}\|X^\varepsilon - P_N X^\varepsilon\|_{L^2}^2 \lesssim \sum_{k=N+1}^{\infty} \frac{1}{\lambda_k^2} \lesssim \sum_{k=N+1}^{\infty} \frac{1}{k^{\frac{4}{d}}}.$$

Hence, for $d = 2, 3$, we obtain tightness of $\{X^\varepsilon\}$ for free, as byproduct of the analysis in Section 4B.

In the setting of this paper, $\mathcal{L}$ above is replaced by $\mathcal{L}_\varepsilon$, defined in (3-1). The above approach for tightness fails completely. On the one hand, if we replace the eigenpairs $(\lambda_k,\phi_k)_k$ by $(\lambda_k^\varepsilon,\phi_k^\varepsilon)_k$, where the latter solve the eigenvalue problems associated to $\mathcal{L}_\varepsilon$, then instead of (7-2), we obtain

$$\lim_{N\to\infty}\sup_{\varepsilon\in(0,1)} \mathbb{E}\|X^\varepsilon - P_N^\varepsilon X^\varepsilon\|_{L^2} = 0,$$

where P_N^ε is the projection to $W_N^\varepsilon := \operatorname{span}\{\phi_1^\varepsilon,\dots,\phi_N^\varepsilon\}$. This is useless because, a priori, the basis $(\phi_k^\varepsilon)_k$ changes with ε, and it is not clear that the union (over $\varepsilon\in(0,1)$) of unit balls in W_N^ε is still compact for all N. On the other hand, if we fix a spectral representation, say, using $(\lambda_k,\phi_k)_k$ defined before, then we no longer have the relation $\mathcal{G}_\varepsilon\phi_k = (\lambda_k)^{-1}\phi_k$. It is not difficult to check that $\|\nabla\mathcal{G}_\varepsilon\phi_k\|_{L^2} \approx 1/\sqrt{\lambda_k}$ and this estimate is sharp. An application of the Poincaré inequality yields that $\|\mathcal{G}_\varepsilon\phi_k\|_{L^2} \le C/\sqrt{\lambda_k} \approx k^{-1/d}$, with C uniform in ε and k. It is not clear at all how to improve this estimate. Consequently, in view of the estimate on I_1^ε in Section 4B, we have

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}\|X^\varepsilon - P_N X^\varepsilon\|_{L^2}^2 = \frac{1}{\varepsilon^d}\sum_{k=N+1}^{\infty} \mathbb{E}(\nu^\varepsilon u,\mathcal{G}_\varepsilon\phi_k)^2 \le \sum_{k=N+1}^{\infty} C\|\mathcal{G}_\varepsilon\phi_k\|_{L^2}^2 \approx \sum_{k=N+1}^{\infty}\frac{1}{\lambda_k} \approx \sum_{k=N+1}^{\infty}\frac{1}{k^{\frac{2}{d}}}.$$

This fails to show (7-2) or the tightness of $\{X^\varepsilon\}$, even for $d = 2$.

In view of the analysis above, we find that the above approach for tightness, which is natural for nonoscillatory differential operators, fails completely in the presence of fast oscillations in the diffusion coefficients. The new approach used in Section 5 is necessary and more stable.

Appendix: Some technical results

Tightness criteria for probability measures in functional spaces. We first present a tightness criterion for the probability measures $\{P^{X^\varepsilon}\}_{\varepsilon\in(0,1)}$ on $L^2(D)$ induced by $\{X^\varepsilon(\cdot,\omega)\}$ that are random elements in $H_0^s(D)\subset L^2(D)$, with $s\in(0,1]$.

Theorem A.1 (tightness in $L^2(D)$). *Let $\{X^\varepsilon(\cdot,\omega)\}_{\varepsilon\in(0,1)}$ be a family of random fields on the probability space $(\Omega,\mathcal{F},\mathbb{P})$, with $X^\varepsilon(\cdot,\omega)\in H_0^s(D)$ for some $0<s\le 1$, for each fixed $\varepsilon\in(0,1)$ and $\omega\in\Omega$. Suppose there exists $C>0$, independent of ε and ω, such that*

$$\mathbb{E}\|X^\varepsilon\|_{H^s} \le C. \tag{A-1}$$

Then the family of probability measures $\{P^{X^\varepsilon}\}_{\varepsilon\in(0,1)}$ on $L^2(D)$ is tight.

Proof. By assumption, P^{X^ε} concentrates on the subspace $H_0^s(D)$. For any fixed $\delta > 0$, set $M_\delta = C\delta^{-1}$ and define

$$\mathcal{A}_\delta = \{f \in H_0^s(D) : \|f\|_{H^s} \le M_\delta\}.$$

Clearly, $\mathcal{A}_\delta$ is closed and bounded in $H_0^s(D)$. In light of the fact that the embedding $H_0^s(D) \hookrightarrow L^2(D)$ is compact [Palatucci et al. 2013], we note that $\mathcal{A}_\delta$ is a compact set of $L^2(D)$. Now for any fixed $\varepsilon \in (0,1)$, applying Chebyshev's inequality, we find

$$\begin{aligned} P^{X^\varepsilon}(\mathcal{A}_\delta) &= \mathbb{P}\big(\{X^\varepsilon \in H_0^s(D),\ \|X^\varepsilon\|_{H^s} \le M_\delta\}\big) = 1 - \mathbb{P}(\{\|X^\varepsilon\|_{H^s} > M_\delta\}) \\ &\ge 1 - \frac{\mathbb{E}\|X^\varepsilon\|_{H^s}}{M_\delta} \ge 1 - \frac{C}{M_\delta} = 1 - \delta. \end{aligned}$$

Since δ and ε are arbitrary, the above shows that $\{P^{X^\varepsilon}\}_{\varepsilon\in(0,1)}$ is tight. □

Next we give a similar tightness criterion for probability measures $\{P^{X^\varepsilon}\}_{\varepsilon\in(0,1)}$ on $H^{-1}(D)$ induced by $\{X^\varepsilon(\cdot,\omega)\}$ which belong to a smoother space.

Theorem A.2 (tightness in $H^{-1}(D)$). *Let $\{X^\varepsilon(\cdot,\omega)\}_{\varepsilon\in(0,1)}$ be a family of random fields on the probability space $(\Omega,\mathcal{F},\mathbb{P})$, with $X^\varepsilon(\cdot,\omega) \in H^{-s}(D)$ for some $0 \le s < 1$, for each fixed $\varepsilon \in (0,1)$ and $\omega \in \Omega$. Suppose there exists a constant $C > 0$, independent of ε and ω, such that*

$$\mathbb{E}\|X^\varepsilon\|_{H^{-s}} \le C. \tag{A-2}$$

Then the probability measures $\{P^{X^\varepsilon}\}_{\varepsilon\in(0,1)}$ on $H^{-1}(D)$ are tight.

Proof. Since D is a bounded open set with regular boundary, the embedding $H_0^1(D) \hookrightarrow H_0^s(D)$, for any $0 \le s < 1$, is compact [Grisvard 1985, Theorem 1.4.3.2]. By duality, the embedding $H^{-s}(D) \hookrightarrow H^{-1}(D)$ is also compact. The rest of the proof is exactly the same as in the proof of the theorem above. □

Functions of long-range correlated Gaussian random fields. Here we record some results for the random potential $\nu(x,\omega) = \Phi(g(x,\omega))$ that is constructed in (L). In particular, we express the asymptotic behavior of its correlation function $R(x)$, and derive a (partial) fourth-order moment for ν.

Autocorrelation function of the long-range model.

Lemma A.3. *Assume (L1) and (L2) hold and let $\nu(x,\omega)$ be as constructed there. Set $V_1 = \mathbb{E}\{g_0\Phi(g_0)\}$, where g_x is the underlying Gaussian random field in (L). Then there exist constants $T, C > 0$, depending only on the universal parameters, such that the autocorrelation function $R(x)$ of q satisfies*

$$|R(x) - V_1^2 R_g(x)| \le C R_g^2(x) \quad \textit{for all } |x| \ge T, \tag{A-3}$$

where R_g is the correlation function of g. Further,

$$\big|\mathbb{E}\{g(y)q(y+x)\} - V_1 R_g(x)\big| \le C R_g^2(x) \quad \textit{for all } |x| \ge T. \tag{A-4}$$

The proof of this result can be found in [Bal et al. 2008; 2012]. It says that $\nu(x,\omega)$ inherits the heavy tail from the underlying Gaussian random field. The next result describes estimates on the integrals of R, possibly against some potential function that has singularity at the origin.

Fourth-order moments of $\nu(x,\omega)$. Finally, we present a nonasymptotic estimate for the four-moments of $\nu(x,\omega)$ constructed in (L1) and (L2), with the additional assumption (L3). In the following, we denote by $\mathcal{U}$ the collections of two pairs of unordered numbers in the set $\{1,2,3,4\}$,

$$\mathcal{U} := \big\{p = \{(p(1), p(2)), (p(3), p(4))\} : p(i) \in \{1,2,3,4\},\ p(1) \neq p(2),\ p(3) \neq p(4)\big\}. \tag{A-5}$$

As members in a set, the pairs $(p(1), p(2))$ and $(p(3), p(4))$ are required to be distinct; however, the two pairs can have one common index. There are three elements in $\mathcal{U}$ that collect all four numbers. They are precisely $\{(1,2),(3,4)\}$, $\{(1,3),(2,4)\}$ and $\{(1,4),(2,3)\}$. Let $\mathcal{U}_*$ denote the subset formed by these three elements, and let $\mathcal{U}^*$ be its complement.

Lemma A.4. *Assume (L) holds and let $\nu(x,\omega)$ be as constructed there. Then there exists $\vartheta : \mathbb{R}^d \to \mathbb{R}_+$, bounded and satisfying $\vartheta(x) \sim |x|^{-\alpha}$ as $|x| \to \infty$, and some $C > 0$, depending only on the universal parameters, such that for any four points $\{x_i \in \mathbb{R}^d : 1 \le i \le 4\}$,*

$$\bigg|\mathbb{E}\prod_{i=1}^{4}\nu(x_i) - \sum_{p\in\mathcal{U}_*} R(x_{p(1)}-x_{p(2)})R(x_{p(3)}-x_{p(4)})\bigg| \le C\sum_{p\in\mathcal{U}^*}\vartheta(x_{p(1)}-x_{p(2)})\vartheta(x_{p(3)}-x_{p(4)}). \tag{A-6}$$

We refer to [Bal and Jing 2011, Proposition 4.1] for the proof of this result. In particular, ϑ above can be chosen as the autocorrelation function $R(x)$ of $\nu(x,\omega)$. As discussed earlier, (A-6) can be viewed as an alternative for the estimates in Lemma 4.3.

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