ANALYSIS & PDE

Volume 9 No. 1 2016

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ANALYSIS AND PDE
Vol. 9, No. 1, 2016
dx.doi.org/10.2140/apde.2016.9.1

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RADEMACHER FUNCTIONS IN NAKANO SPACES

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The closed span of Rademacher functions is investigated in Nakano spaces $L^{p(\cdot)}$ on [0, 1] equipped with the Lebesgue measure. The main result of this paper states that under some conditions on distribution of the exponent function p the Rademacher functions form in $L^{p(\cdot)}$ a basic sequence equivalent to the unit vector basis in ℓ_2.

1. Introduction

We recall that the Rademacher functions on [0, 1] are defined by $r_k(t) = \operatorname{sign}(\sin 2^k \pi t)$ for every $t \in [0, 1]$ and each $k \in \mathbb{N}$. It is well known that (r_k) is an incomplete orthogonal system of independent random variables. This system plays a prominent role in the modern theory of Banach spaces and operators (see, e.g., [Diestel et al. 1995; Pisier 1986]). Special emphasis in this connection is placed on the study of local theory of Banach spaces and especially on using the notions of (Rademacher) type and cotype, which reflect the interplay between geometry and probability in these spaces. We mention here only a special case of the famous result due to Maurey and Pisier [1976]; it states that a Banach space has type strictly bigger than 1 (resp., finite cotype) if and only if it does not contain ℓ_1^n's (resp., ℓ_∞^n's) uniformly. For more details and a precise quantitative version of this result we refer, for example, to [Diestel et al. 1995, Chapter 14].

Rademacher functions play a significant role in the study of lattice and rearrangement-invariant structures in arbitrary Banach spaces. This research was initiated in the memoir [Johnson et al. 1979] by Johnson, Maurey, Schechtman and Tzafriri. By way of motivation let us also mention a classical result of Rodin and Semenov [1975], which states that the sequence (r_k) is equivalent in a symmetric space X to the unit vector basis in ℓ_2, that is,

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_X \approx \Big(\sum_{k=1}^{\infty} |a_k|^2 \Big)^{1/2}, \quad (a_k) \in \ell_2,$$

if and only if $G \subset X$, where G is the closure of $L^\infty[0, 1]$ in the Orlicz space $L_N[0, 1]$ generated by the function $N(t) = \exp(t^2) - 1$ for all $t \geq 0$. When this condition is satisfied, the span $[r_k]$ of Rademacher functions is complemented in X if and only if $X \subset G'$, where the Köthe dual space G' to G coincides (with equivalence of norms) with the Orlicz space $L_{N_*}[0, 1]$ generated by the Young conjugate N_* which

Astashkin's research was supported by the Ministry of Education and Science of Russian Federation. Mastyło's research was supported by the Foundation for Polish Science (FNP)..
MSC2010: primary 46E30; secondary 46B20, 46B42.
Keywords: Rademacher functions, Nakano spaces, symmetric spaces.

is equivalent at infinity to the function $t \mapsto t \log^{1/2} t$. This was proved independently by Rodin and Semenov [1979] and Lindenstrauss and Tzafriri [1979, pp. 134–138].

It is well known that (r_k) is a symmetric basic sequence in every symmetric space on [0, 1], however this is not true in the case of nonsymmetric Banach function lattices. In particular, this phenomenon takes place, for example, in the space of functions of bounded mean oscillation and as well as in Cesàro function spaces (see [Astashkin et al. 2011; Astashkin and Maligranda 2010]); this motivates searching for conditions under which Rademacher functions form a symmetric or an unconditional basic sequence in Banach function lattices.

The main purpose of this paper is to investigate the behaviour of Rademacher functions in the Nakano function spaces $L^{p(\cdot)}$ on [0, 1]. These spaces (which are also called "variable exponent Lebesgue spaces" in certain parts of the literature) are generalisations of the classical L^p-spaces, where the exponent p is allowed to vary measurably over a set of values in $[1, \infty)$.

Nakano spaces belong to the large family of Musielak–Orlicz spaces, and therefore many their basic properties follow from general results (see [Musielak 1983]). There are several books related to Nakano spaces, which cover some joint material, however, from somewhat different viewpoints. Let us mention [Diening et al. 2011] and [Cruz-Uribe and Fiorenza 2013], in which the authors provide a presentation of fundamentals of Nakano spaces and study whether certain principal results in modern harmonic analysis have natural analogues in the Nakano space setting. In the last decades the investigation on this topic has been also motivated by the modelling the so-called electrorheological fluids and some other applications (see [Cruz-Uribe and Fiorenza 2013], and also the more recent [Cruz-Uribe et al. 2014], where interesting connections between theory of Nakano spaces and strongly hyperbolic systems with time-dependent coefficients were discovered).

It is worth noting that a number of results related to the spaces $L^{p(\cdot)}$ is proved under some smoothness conditions on the exponent function p. Let us recall, as an example, a result of Sharapudinov [1986] which states that the Haar system is a basis in a Nakano space $L^{p(\cdot)}$ provided the exponent function p satisfies the piecewise Dini–Lipschitz condition with exponent $\alpha \geq 1$ (see also the above-cited [Diening et al. 2011; Cruz-Uribe and Fiorenza 2013]). In contrast to that in this paper we impose conditions upon distribution of p and investigate the problem whether they are sufficient or necessary for equivalence of the Rademacher sequence (r_k) in $L^{p(\cdot)}$ to the unit vector basis in ℓ_2.

2. Preliminaries

If (Ω, Σ, μ) is a σ-finite measure space, then, as usual, $L^0 := L^0(\mu)$ denotes the space of all real-valued μ-measurable functions. We say that $(X, \|\cdot\|_X)$ is a *Banach function lattice* (in short, *Banach lattice*) on (Ω, Σ, μ) if X is an ideal in L^0 and $\|f\|_X \leq \|g\|_X$ whenever $f, g \in X$ and $|f| \leq |g|$. The Köthe dual space X' of X is a collection of all elements $g \in L^0$ such that

$$\|g\|_{X'} := \sup\left\{ \int_\Omega |fg|\, d\mu;\ \|f\|_X \leq 1 \right\} < \infty.$$

The space $(X', \|\cdot\|_{X'})$ is a Banach function lattice with the Fatou property. Recall that a Banach function lattice X is said *to have the Fatou property* if the conditions $\sup_{n\geq 1}\|x_n\|_X<\infty$ and $x_n\to x$ a.e. imply that $x\in X$ and $\|x\|_X\leq \liminf_{n\to\infty}||x_n||_X$. It is well known that X has the Fatou property if and only if the natural embedding of X into its second Köthe dual X'' is an isometric surjection.

Let $f\in L^0(I,m)$, where $I:=[0,1]$ is equipped with the Lebesgue measure m. The *distribution function* of f is defined by $d_f(\lambda)=\mu(\{t\in I\,;\,|f(t)|>\lambda\})$, $\lambda\geq 0$, and its *decreasing rearrangement* by $f^*(t)=\inf\{s>0; d_f(s)\leq t\}$, $t>0$. One says that functions f and g are *equimeasurable* if $f^*(t)=g^*(t),\ 0<t\leq 1$, or equivalently, $d_f(\lambda)=d_g(\lambda)$, $\lambda>0$.

Recall some definitions and auxiliary results from the theory of symmetric spaces (for more details see [Bennett and Sharpley 1988; Kreĭn et al. 1982]).

A Banach function lattice X on (I,m) is called a *symmetric space* if the conditions $f^*\leq g^*$ a.e. on I and $g\in X$ imply $f\in X$ and $\|f\|_X\leq\|g\|_X$. The fundamental function of a symmetric space X is given by $\varphi_X(t)=\|\chi_{[0,t)}\|_X$ for all $t\in I$. In what follows we will use the following obvious inequality for any symmetric space X on I,

$$f^*(t)\leq \frac{1}{\varphi_X(t)}\,\|f\|_X,\qquad f\in X,\ t\in(0,1]. \tag{1}$$

Important examples of symmetric spaces are Orlicz, Marcinkiewicz and Lorentz spaces. Recall that $\Phi:[0,\infty)\to[0,\infty)$ is called an Orlicz function if $\Phi(0)=0$ and Φ is positive, nondecreasing, convex and left-continuous on $(0,\infty)$. If Φ is such a function, the Orlicz space L_Φ consists of all $f\in L^0(m)$ for which there exists $\lambda>0$ such that

$$\int_I \Phi\big(|f|/\lambda\big)\,dm<\infty.$$

It is a symmetric space equipped with the norm

$$\|f\|_{L_\Phi}=\inf\left\{\lambda>0;\ \int_I\Phi\left(\frac{|f|}{\lambda}\right)dm\leq 1\right\}.$$

In what follows by L_N(resp., L_M) we will denote the Orlicz space on $[0,1]$ generated by the function $N(t)=\exp(t^2)-1$ (resp., $M(t)=\exp(t^2\log(t+1))-1$) for all $t\geq 0$.

Let $\varphi:I\to[0,\infty)$ be a quasiconcave function, that is $\varphi(0)=0$, $\varphi(t)>0$ for $t\in I$ and both φ and $t\mapsto\widetilde{\varphi}(t):=t/\varphi(t)$ are nondecreasing functions on $(0,1]$. The Marcinkiewicz space $M(\varphi)$ is defined to be the space of all $f\in L^0(m)$ equipped with the norm

$$\|f\|_{M(\varphi)}=\sup_{0<s\in I}\frac{1}{\varphi(s)}\int_0^s f^*(t)\,dt.$$

If $\varphi:I\to[0,\infty)$ is an increasing concave function, $\varphi(0)=0$, the Lorentz space $\Lambda(\varphi)$ consists of all $f\in L^0$ such that

$$\|f\|_{\Lambda(\varphi)}=\int_0^1 f^*(t)\,d\varphi(t)<\infty.$$

It is well known that L^1 and L^∞ are, respectively, the largest and the smallest symmetric spaces on I; moreover, if X is a symmetric space on I with the fundamental function φ, then φ is quasiconcave and the following continuous embeddings hold (see [Kreĭn et al. 1982, Theorems II.5.5 and II.5.7] or [Bennett and Sharpley 1988, Theorem II.5.13]):

$$\Lambda(\overline{\varphi}) \hookrightarrow X \hookrightarrow M(\widetilde{\varphi}),$$

where $\overline{\varphi}$ is the least concave majorant of φ. In what follows we will frequently use the well-known fact that the Orlicz space L_N generated by the function $N(t) = \exp(t^2) - 1$, $t \ge 0$, coincides up to equivalence of norms with the Marcinkiewicz space $M(\varphi)$ generated by the function $\varphi(t) = t \log^{1/2}(e/t),\ 0 < t \le 1$ (see [Lorentz 1951]).

Let (Ω, Σ, μ) be a σ-finite measure space. Given a measurable function $p : \Omega \to [1, \infty)$, we define the Nakano space $L^{p(\cdot)}(\mu)$ to be the space of all $f \in L^0(\mu)$ such that for some $\lambda > 0$

$$\rho_\lambda(f) = \int_\Omega \left(\frac{|f(t)|}{\lambda} \right)^{p(t)} d\mu < \infty.$$

$L^{p(\cdot)}(\mu)$ becomes a Banach function lattice with the Fatou property when equipped with the norm

$$\|f\|_{L^{p(\cdot)}} = \|f\|_{p(\cdot)} := \inf\{\lambda > 0; \rho_\lambda(f/\lambda) \le 1\}.$$

Throughout the paper a Nakano space defined on $[0, 1]$ equipped with the Lebesgue measure m is denoted for short $L^{p(\cdot)}$. Notice that $L^{p(\cdot)}$ is not a symmetric space unless the exponent p is a constant function, and in this case we write $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p}$.

Further, we shall frequently use the following lemma which is an immediate consequence of Theorem 3 from [Fiorenza and Rakotoson 2007].

Lemma 2.1. *Let $f : [0, 1] \to [0, \infty)$ and $p : [0, 1] \to [1, \infty)$ be two Lebesgue measurable functions. Then*

$$\|f\|_{L^{p(\cdot)}} \le 4\|f^*\|_{L^{p^*(\cdot)}}.$$

3. Main results

In this section we shall prove the main results of the paper. We recall that L_N and L_M are the Orlicz spaces on $[0, 1]$ generated by the functions $N(t) = \exp(t^2) - 1$ and $M(t) = \exp(t^2 \log(t + 1)) - 1)$.

Theorem 3.1. *Let $p : (0, 1] \to [1, \infty)$ be a Lebesgue measurable function and let $L^{p(\cdot)}$ be the Nakano space generated by p. Each of the following conditions implies the next*:

(i) $L_N \subset L^{p(\cdot)}$.

(ii) *The Rademacher system (r_n) is equivalent in the space $L^{p(\cdot)}$ to the unit vector basis in ℓ_2.*

(iii) *There is a constant $C > 0$ such that*

$$m(\{t \in [0, 1];\ p(t) > \lambda\}) \le C^\lambda \lambda^{-\lambda/2}, \quad \lambda \ge 1.$$

(iv) $L_M \subset L^{p(\cdot)}$.

We start with the following distribution estimate, which will be useful for us in the sequel:

Proposition 3.1. *Suppose that for each $k \in \mathbb{N}$ and $m \in \mathbb{N}$ there exists $\ell > m$ such that*

$$\Big\| \sum_{i=\ell+1}^{\ell+k} r_i \Big\|_{L^{p(\cdot)}} \le B\sqrt{k},$$

where $B > 0$ is independent of k and m. Then

$$m(\{t \in [0,1];\ p(t) > \lambda\}) \le 2(4B)^{\lambda} \lambda^{-\lambda/2}, \quad \lambda \ge 1.$$

Proof. Let $\lambda \ge 1$ be fixed. We put

$$E_\lambda := \{t \in [0,1]; p(t) > \lambda\}.$$

Without loss of generality, we can assume that $m(E_\lambda) > 0$. By the Sagher–Zhou local version of Khintchine inequality for L^1 (see [Sagher and Zhou 1990, Theorem 1]), it follows that there exists $n(\lambda)$ such that for all $n \ge n(\lambda)$, every Rademacher sum $R_n = \sum_{k=n}^{\infty} a_k r_k$ and arbitrary $(a_k) \in \ell_2$ with $\|(a_k)\|_{\ell_2} = 1$, we have

$$\int_{E_\lambda} |R_n(t)|\, dt \ge \alpha\, m(E_\lambda),$$

where $\alpha > 0$ is a universal constant. Since $\lambda \ge 1$,

$$\left(\frac{1}{m(E_\lambda)} \int_{E_\lambda} |R_n(t)|^\lambda\, dt\right)^{1/\lambda} \ge \frac{1}{m(E_\lambda)} \int_{E_\lambda} |R_n(t)|\, dt,$$

and so

$$\left(\int_{E_\lambda} |R_n(t)|^\lambda\, dt\right)^{1/\lambda} \ge \alpha (m(E_\lambda))^{1/\lambda}. \tag{2}$$

On the other hand, it is well known (in particular, it is a consequence of the above-cited Rodin–Semenov theorem) that there exists a constant $\beta > 0$ such that

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{L_N} \le \beta\, \|(a_k)\|_{\ell_2}, \quad (a_k) \in \ell_2, \tag{3}$$

where, as above, L_N is the Orlicz space generated by the function $N(t) = \exp(t^2) - 1$, $t \ge 0$. Since the fundamental function of L_N is given by $\varphi(t) = 1/N^{-1}(1/t) = \log^{-1/2}(1 + 1/t)$ for all $t \in (0,1]$, it follows by (1) and (3) that

$$\left(\sum_{n=1}^{\infty} a_k r_k\right)^*(t) \le \beta \log^{1/2}\left(1 + \frac{1}{t}\right) \le \beta \log^{1/2}\left(\frac{e}{t}\right), \quad t \in (0,1],$$

for all $(a_k) \in \ell_2$ with $\|(a_k)\|_{\ell_2} \le 1$. Hence, for every $\delta > 0$ and $E \subset [0,1]$ with $m(E) < \delta$, we obtain

$$\left(\int_E |R_n(t)|^\lambda\, dt\right)^{1/\lambda} \le \left(\int_0^\delta R_n^*(t)^\lambda\, dt\right)^{1/\lambda} \le \beta \left(\int_0^\delta \log^{\lambda/2}\left(\frac{e}{t}\right) dt\right)^{1/\lambda}.$$

Choose $\delta = \delta(\lambda) > 0$ so that

$$\int_0^\delta \log^{\lambda/2}\left(\frac{e}{t}\right) dt \le \beta^{-\lambda}\alpha^\lambda m(E_\lambda).$$

Then, from the preceding inequality and (2), it follows that

$$\left(\int_E |R_n(t)|^\lambda \, dt\right)^{1/\lambda} \le \alpha (m(E_\lambda))^{1/\lambda} \le \left(\int_{E_\lambda} |R_n(t)|^\lambda \, dt\right)^{1/\lambda}. \tag{4}$$

provided $m(E) < \delta$ and $\|(a_k)\|_{\ell_2} = 1$.

We denote by I_k^ν the dyadic interval $[(k-1)2^{-\nu}, k2^{-\nu}]$ for each $\nu \in \mathbb{Z}_+$ and each $1 \le k \le 2^\nu$. Then we can find a finite union of pairwise disjoint intervals $F = \bigcup_{j=1}^m I_{k_j}^{\nu_j}$, $1 \le k_j \le 2^{\nu_j}$, $1 \le j \le m$ such that

$$m(E_\lambda \triangle F) \le \max\left\{\delta, \tfrac{1}{2} m(E_\lambda)\right\}$$

(here, $A \triangle B := (A \setminus B) \cup (B \setminus A)$). Hence, $m(F) \ge m(E_\lambda) - m(E_\lambda \triangle F) \ge \frac{1}{2} m(E_\lambda)$, and for each sum $R_n = \sum_{k=n}^\infty a_k r_k$ with $\|(a_k)\|_{\ell_2} = 1$, by (4), we obtain

$$\left(\int_F |R_n(t)|^\lambda \, dt\right)^{1/\lambda} \le \left(\int_{E_\lambda} |R_n(t)|^\lambda \, dt\right)^{1/\lambda} + \left(\int_{E_\lambda \triangle F} |R_n(t)|^\lambda \, dt\right)^{1/\lambda} \le 2\left(\int_{E_\lambda} |R_n(t)|^\lambda \, dt\right)^{1/\lambda}.$$

This implies that

$$\left(\frac{1}{m(E_\lambda)} \int_{E_\lambda} |R_n(t)|^\lambda \, dt\right)^{1/\lambda} \ge \frac{1}{2}\left(\frac{1}{2m(F)} \int_F |R_n(t)|^\lambda \, dt\right)^{1/\lambda} \ge \frac{1}{4}\left(\frac{1}{m(F)} \int_F |R_n(t)|^\lambda \, dt\right)^{1/\lambda}.$$

Now, let a positive integer $m \ge n(\lambda)$ be such that all Rademacher functions r_k with $k \ge m$ change their sign at least once on each dyadic component of the set F. Then for any $(a_k) \in \ell_2$,

$$\left(\frac{1}{m(F)} \int_F \left|\sum_{k=m}^\infty a_k r_k(t)\right|^\lambda dt\right)^{1/\lambda} = \left\|\sum_{k=m}^\infty a_k r_k\right\|_\lambda.$$

Combining this equality with the above estimate, we obtain

$$\left(\frac{1}{m(E_\lambda)} \int_{E_\lambda} |R_m(t)|^\lambda \, dt\right)^{1/\lambda} \ge \tfrac{1}{4} \|R_m\|_\lambda \tag{5}$$

for every sum $R_m = \sum_{k=m}^\infty a_k r_k$, $\|(a_k)\|_{\ell_2} = 1$ (m depends on λ). Our hypothesis implies that for each $\lambda \ge 1$ we can find $\ell > m$ such that

$$\left\|\sum_{i=\ell+1}^{\ell+[\lambda]} r_i\right\|_{L^{p(\cdot)}} \le B\sqrt{[\lambda]}, \tag{6}$$

where, as usual, $[x]$ is the integer part of x. In the opposite direction, we will use the following well-known inequality (see, e.g., [Blei 2001, Lemma VII.30, p. 167]):

$$2\Big\|\sum_{j=1}^{k} r_j\Big\|_k \ge k, \quad k\in\mathbb{N}.$$

If $R_{\lambda,\ell} := \sum_{i=\ell+1}^{\ell+[\lambda]} r_i$, this inequality yields

$$2\,\|R_{\lambda,\ell}\|_\lambda \ge 2\,\|R_{\lambda,\ell}\|_{[\lambda]} = 2\Big\|\sum_{j=1}^{[\lambda]} r_j\Big\|_{[\lambda]} \ge 2\,[\lambda].$$

Let $\bar{R}_{\lambda,\ell} := R_{\lambda,\ell}/\sqrt{[\lambda]}$. Then, from the latter inequality it follows that

$$\|\bar{R}_{\lambda,\ell}\|_{[\lambda]} \ge 2\sqrt{[\lambda]} \ge \sqrt{\lambda}.$$

Moreover, it is easy to see that $\bar{R}_{\lambda,\ell} = \sum_{k=m}^{\infty} a_k' r_k$, with $\|(a_k')\|_{\ell_2} = 1$. Combining the preceding estimate with inequality (5), we obtain

$$\left(\frac{1}{m(E_\lambda)}\int_{E_\lambda} \left|\bar{R}_{\lambda,\ell}(t)\right|^\lambda dt\right)^{1/\lambda} \ge \tfrac{1}{4}\sqrt{\lambda},$$

or equivalently,

$$\|\bar{R}_{\lambda,\ell}\,\chi_{E_\lambda}\|_\lambda \ge \tfrac{1}{4}\sqrt{\lambda}\, m(E_\lambda)^{1/\lambda}. \tag{7}$$

where χ_{E_λ} is the characteristic function of the set E_λ. On the other hand, in view of (6) we have $\|\bar{R}_{\lambda,\ell}\|_{L^{p(\cdot)}} \le B$ and so, setting $\bar{E}_\lambda = \{t\in E_\lambda;\ |\bar{R}_{\lambda,\ell}(t)| \ge B\}$, by the definition of the norm in the Nakano space $L^{p(\cdot)}$, we deduce

$$\int_{E_\lambda}\left|\frac{\bar{R}_{\lambda,\ell}(t)}{B}\right|^\lambda dt \le \int_{\bar{E}_\lambda}\left|\frac{\bar{R}_{\lambda,\ell}(t)}{B}\right|^\lambda dt + \int_{E_\lambda\setminus\bar{E}_\lambda}\left|\frac{\bar{R}_{\lambda,\ell}(t)}{B}\right|^\lambda dt \le \int_0^1\left|\frac{\bar{R}_{\lambda,\ell}(t)}{B}\right|^{p(t)} dt + 1 \le 2. \tag{8}$$

Therefore, from (7) it follows that

$$2 \ge \int_{E_\lambda}\left|\frac{\bar{R}_{\lambda,\ell}(t)}{B}\right|^\lambda dt \ge \frac{\lambda^{\lambda/2}}{(4B)^\lambda} m(E_\lambda),$$

whence $m(E_\lambda) \le 2(4B)^\lambda \lambda^{-\lambda/2}$. This completes the proof. □

Proof of Theorem 3.1. (i) ⇒ (ii). First, by [Diening et al. 2011, Theorem 3.3.1], for any exponent $p(\cdot)$ we have

$$\|f\|_{L^1} \le 2\,\|f\|_{L^{p(\cdot)}}, \quad f\in L^{p(\cdot)}.$$

Combining this with the Khintchine inequality in L^1 (see [Szarek 1976]), we obtain

$$\Big\|\sum_{k=1}^{\infty} a_k\, r_k\Big\|_{L^{p(\cdot)}} \ge \frac{1}{2\sqrt{2}}\,\|(a_k)\|_{\ell_2}, \quad (a_k)\in\ell_2.$$

Thus our hypothesis and (3) imply that there exists a constant $C > 0$ such that

$$\Big\| \sum_{k=1}^{\infty} a_k\, r_k \Big\|_{L^{p(\cdot)}} \le C \, \|(a_k)\|_{\ell_2}, \quad (a_k) \in \ell_2.$$

The implication (ii) $\Rightarrow$ (iii) follows from Proposition 3.1.

(iii) $\Rightarrow$ (iv). Note that the Orlicz space L_M, where $M(t) = \exp(t^2 \log(1+t)) - 1$ for all $t \ge 0$, coincides with the Marcinkiewicz space with the fundamental function $\varphi := \varphi_{L_M}$ given by

$$\varphi(t) := \left(\frac{\log(e/t)}{\log\log(e^2/t)} \right)^{-1/2}, \quad 0 < t \le 1$$

(see, e.g., [Lorentz 1951] or [Astashkin 2009, Lemma 3.2]). Hence, L_M can be characterised as the set of all measurable functions x on $[0,1]$ for which there exists a constant $C > 0$ such that

$$x^*(t) \le \frac{C}{\varphi(t)}, \quad 0 < t \le 1.$$

Thus, since $L^{p(\cdot)}$ is a Banach lattice, the embedding $L^{p(\cdot)} \supset L_M$ will be proved if we show that the space $L^{p(\cdot)}$ contains all functions equimeasurable with the function

$$f_0(t) = \frac{1}{\varphi(t)}, \quad 0 < t \le 1.$$

By hypothesis and Lemma 2.1, it follows that we need only to check that for some $\lambda > 0$

$$\int_0^1 \left(\frac{f_0(t)}{\lambda} \right)^{g(t)} dt < \infty, \tag{9}$$

where g is a decreasing positive function on $(0,1]$ such that $g(t) \ge 1$ and

$$m(\{t \in (0,1];\, g(t) > x\}) = g^{-1}(x) = C^x x^{-x/2}, \quad x \ge 1,$$

for some $C \ge 1$.

For $x_0 \ge 1$, which can be chosen later, we have

$$\begin{aligned} \int_0^{g^{-1}(x_0)} \left(\frac{f_0(t)}{\lambda} \right)^{g(t)} dt &= - \int_{x_0}^{\infty} \left(\frac{f_0(C^x x^{-x/2})}{\lambda} \right)^{x} d(C^x x^{-x/2}) \\ &= \int_{x_0}^{\infty} \left(\frac{f_0(C^x x^{-x/2})}{\lambda} \right)^{x} C^x x^{-x/2} \log(C^{-1} e^{1/2} x^{1/2})\, dx. \end{aligned}$$

If x_0 is sufficiently large, then for all $x \ge x_0$ we infer

$$f_0(C^x x^{-x/2}) = \left(\frac{\log(e C^{-x} x^{x/2})}{\log\log(e^2 C^{-x} x^{x/2})} \right)^{1/2} = \frac{1}{\sqrt{2}} \left(\frac{x \log(C^{-x} e^{2/x} x)}{\log x + \log\big(\frac{1}{2}\log(C^{-2} e^{4/x} x)\big)} \right)^{1/2} \le x^{1/2}.$$

Therefore, the preceding inequality implies

$$\int_0^{g^{-1}(x_0)} \left(\frac{f_0(t)}{\lambda}\right)^{g(t)} dt \le \int_{x_0}^{\infty} \left(\frac{C}{\lambda}\right)^x \log(C^{-1}e^{1/2}x^{1/2})\, dx < \infty,$$

provided that $\lambda > C$. Clearly, we obtain (9).

Finally, implication (iv) $\Rightarrow$ (i) is an immediate consequence of the obvious embedding $L_N \subset L_M$, and the proof is complete. □

We do not know whether the distribution condition from (iii) implies the embedding $L_N \subset L^{p(\cdot)}$ or the equivalence of Rademacher system in $L^{p(\cdot)}$ to the unit vector basis in ℓ_2. However, the next result can be treated as an approach to the solution of these problems. In its first part we prove that some stronger condition on the distribution function of an exponent $p(\cdot)$ insures the embedding $L_N \subset L^{p(\cdot)}$ and in the second one we show that this result is in a sense sharp.

Theorem 3.2. *Let $p : (0, 1] \to [1, \infty)$ be a Lebesgue measurable function.*

(a) *If there exists a constant $C > 0$ such that*

$$m(\{t \in (0, 1]; p(t) > x\}) \le C^x (x \log x)^{-x/2}, \quad x \ge 1,$$

then $L_N \subset L^{p(\cdot)}$.

(b) *If there exists an increasing differentiable function θ such that $\lim_{x\to\infty} \theta(x) = \infty$, the function $x \mapsto \theta(x)x^{-1/2}\log^{-1/2} x$ is decreasing for large enough x, and $\liminf_{x\to\infty} m(\{t \in (0, 1];\ p(t) > x\}\, \theta(x)^{-x} x^{x/2} \log^{x/2} x > 0$,*

then $L_N \not\subset L^{p(\cdot)}$.

Proof. (a) It can be easily checked that the function $x \mapsto C^x(x \log x)^{-x/2}$ decreases if $x \ge x_0$, where $x_0 > 1$ is sufficiently large. Denote by q the function inverse to it on the interval $[0, t_0]$, where $q(t_0) = x_0$. Then, from our hypothesis on p, it follows that $p^*(t) \le q(t)$ for all $0 < t \le t_0$. Recall that the space L_N coincides with the Marcinkiewicz space whose fundamental function is given by $t \mapsto \log^{-1/2}(e/t)$, $t \in (0, 1)$. Therefore, thanks to Lemma 2.1, we need only to check that for some $\lambda > 0$

$$I_\lambda := \int_0^{t_0} \left(\frac{\log^{1/2}(e/t)}{\lambda}\right)^{q(t)} dt < \infty.$$

In fact,

$$\begin{aligned} I_\lambda &= -\int_{x_0}^{\infty} (\lambda^{-1} \log^{1/2}(eC^{-x}(x\log x)^{x/2}))^x \, d(C^x(x\log x)^{-x/2}) \\ &= \frac{1}{2}\int_{x_0}^{\infty} \lambda^{-x}\left(\frac{x}{2}\right)^{x/2} \log^{x/2}(e^{2/x}C^{-2}x\log x)\cdot C^x(x\log x)^{-x/2}\left(\log(C^{-2}x\log x) + \frac{\log x + 1}{\log x}\right) dx \\ &\le C_1 \int_{x_0}^{\infty} \left(\frac{C}{\lambda}\right)^x \left(\log(x\log x) + \frac{\log x + 1}{\log x}\right) dx < \infty, \end{aligned}$$

provided $\lambda > C$, and this completes the proof.

(b) It is sufficient to show that for every $\lambda > 0$ there exists a measure-preserving transformation ω of $(0,1]$ such that

$$\int_0^1 \big(\lambda^{-1}\log^{1/2}(e/\omega(t))\big)^{p(t)}\,dt = \infty. \tag{10}$$

In fact, from (10) it follows that $\log^{1/2}(e/\omega) \notin L^{p(\cdot)}$. On the other hand, since ω preserves measure, we have

$$\left(\log^{1/2}\left(\frac{e}{\omega(\cdot)}\right)\right)^*(t) = \log^{1/2}\left(\frac{e}{t}\right), \quad t \in (0,1].$$

Combining this with the fact that $L_N = M(\varphi)$, where $\varphi(t) = t\log^{1/2}(e/t)$, $0 < t \le 1$, we infer $\log^{1/2}(e/\omega) \in L_N$ and the desired result follows.

Let us prove (10). Without loss of generality, we can assume that

$$\theta(x) \le \log^{1/2} x, \text{ for large enough } x \tag{11}$$

(otherwise, instead of $\theta(x)$ we can take the function $\min\{\theta(x), \log^{1/2} x\}$). Moreover, our hypotheses on θ imply

$$\left(\frac{\theta(x)^2}{x\log x}\right)' = x^{-2}\log^{-2} x\,(2\theta'(x)\theta(x)x\log x - \theta^2(x)(1+\log x)) \le 0,$$

and so

$$\frac{2x\theta'(x)}{\theta(x)} \le \frac{1+\log x}{x}, \quad x \ge x_0, \tag{12}$$

if $x_0 \ge 1$ is sufficiently large.

By assumption, there exists $\alpha \in (0,1)$ such that for all $x \ge x_0$ we have

$$m\{t \in (0,1];\ p(t) > x\} \ge \alpha\,\psi(x)^x.$$

Hence, if g is the inverse function to the mapping $x \mapsto \alpha\psi(x)^x$, $x \ge x_0$, we obtain

$$p^*(t) \ge g(t), \quad 0 < t \le t_0, \tag{13}$$

for some $t_0 \in (0,1]$. If it is necessary, diminishing t_0 we can assume also, for a given $\lambda > 0$, the inequality $\log^{1/2}(e/t) \ge \lambda$ to be valid for all $t \in (0, t_0]$.

Let ω be a measure-preserving transformation of $(0,1]$ such that $p(t) = p^*(\omega(t))$ (see [Bennett and Sharpley 1988, Theorem 2.7.5]). From inequality (13) it follows that

$$p(t) \ge g(\omega(t)), \quad t \in E,$$

where $E = \omega^{-1}([0,t_0])$. As a consequence,

$$\begin{aligned} I_\lambda &:= \int_E \big(\lambda^{-1}\log^{1/2}(e/\omega(t))\big)^{p(t)}\,dt \ge \int_E \big(\lambda^{-1}\log^{1/2}(e/\omega(t))\big)^{g(\omega(t))}\,dt \\ &= \int_0^{t_0} (\lambda^{-1}\log^{1/2}(e/t))^{g(t)}\,dt, \end{aligned}$$

and by letting $x = g(t)$, we obtain

$$I_\lambda \geq -\alpha \int_{g(t_0)}^{\infty} \lambda^{-x} \log^{x/2}\Big(\frac{e}{\alpha\psi(x)^x}\Big)\, d(\psi(x)^x).$$

Together with the elementary calculations

$$\begin{aligned}(\psi(x)^x)' &= \Big(\exp\Big(-\frac{x}{2}\log(\theta(x)^{-2}x\log x)\Big)\Big)' \\ &= \psi(x)^x\Big(-\frac{1}{2}\log(\theta(x)^{-2}x\log x) - \frac{x}{2}\frac{\theta(x)^2}{x\log x}\theta^{-4}(x)\big((1+\log x)\theta^2(x) - 2\theta(x)\theta'(x)x\log x\big)\Big) \\ &= -\frac{1}{2}\psi(x)^x\Big(\log\frac{x\log x}{\theta^2(x)} + \frac{1+\log x}{\log x} - \frac{2x\theta'(x)}{\theta(x)}\Big),\end{aligned}$$

inequality (12) shows that

$$(\psi(x)^x)' \leq -\frac{1}{2}\psi(x)^x \log\frac{x\log x}{\theta^2(x)}, \quad x \geq x_0.$$

Combining this with the preceding inequality and (11), we obtain

$$\begin{aligned} I_\lambda &\geq \frac{\alpha}{2}\int_{g(t_0)}^{\infty} \lambda^{-x}\Big(\frac{x}{2}\Big)^{x/2} \log^{x/2}(\alpha^{-2/x}e^{2/x}\theta(x)^{-2}x\log x)\,\theta(x)^x x^{-x/2}\log^{-x/2} x\, \log\frac{x\log x}{\theta^2(x)}\, dx \\ &\geq \frac{\alpha}{2}\int_{g(t_0)}^{\infty} (\lambda\sqrt{2})^{-x}\theta(x)^x \log x\, dx. \end{aligned}$$

Since $\lim_{x\to\infty}\theta(x) = \infty$, from the last estimate it follows that $I_\lambda = \infty$, which implies (10).

The proof is complete. □

We conclude the paper with the result which can be treated as a complement to Theorem 3.1 showing that equivalence of the Rademacher system in $L^{q(\cdot)}$ with arbitrary exponent q, which is equimeasurable with a given p, to the unit vector basis in ℓ_2 implies the embedding $L_N \subset L^{p(\cdot)}$.

Given a Lebesgue measurable function $p : [0,1] \to [1,\infty)$ we let $\Omega(p)$ to be the set of all functions $q \in L^0(m)$ which are equimeasurable with p.

Theorem 3.3. *Suppose that for every $q \in \Omega(p)$ the Rademacher system is equivalent in the space $L^{q(\cdot)}$ to the standard basis in ℓ_2. Then $L_N \subset L^{q(\cdot)}$ for every $q \in \Omega(p)$.*

Proof. Our hypothesis yields that for any $q \in \Omega(p)$ there exits a constant $C_q > 0$ such that for every $a = (a_k) \in \ell_2$

$$\Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_{L^{q(\cdot)}} \leq C_q\,\|a\|_{\ell_2}. \tag{14}$$

We claim that there is a constant $C_0 > 0$ such that for every measure-preserving mapping $\omega : [0,1] \to [0,1]$

and all $a = (a_k) \in \ell_2$ we have

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{L^{p^*(\omega(\cdot))}} \le C_0 \, \|a\|_{\ell_2}. \tag{15}$$

To see this we define the linear operator $T_\omega : \ell_2 \to L^{p^*(\cdot)}$:

$$T_\omega(a_k) := \sum_{k=1}^{\infty} a_k \, r_k(\omega^{-1}), \qquad (a_k) \in \ell_2,$$

generated by an arbitrary measure-preserving mapping $\omega : [0, 1] \to [0, 1]$. Since for any $\lambda > 0$

$$\int_0^1 \Big| \frac{1}{\lambda} \, T_\omega a(t) \Big|^{p^*(t)} dt = \int_0^1 \Big(\frac{1}{\lambda} \Big| \sum_{k=1}^{\infty} a_k r_k(t) \Big| \Big)^{p^*(\omega(t))} dt \tag{16}$$

and the function $q := p^*(\omega) \in \Omega(p)$, from (14) it follows that the operator T_ω is bounded from ℓ_2 into $L^{p^*(\cdot)}$.

For a given sequence $b = (b_k) \in \ell_2$ we let $f = \big|\sum_{k=1}^{\infty} b_k \, r_k\big|$. Applying Theorem 2.7.5 from [Bennett and Sharpley 1988] once more, we can find a measure-preserving mapping $v : [0, 1] \to [0, 1]$ such that $f = f^*(v)$. Since $p^*(v) \in \Omega(p)$, by (14), we have

$$\|f\|_{L^{p^*(v)}} \le K := C_{p^*(v)} \, \|b\|_{\ell_2}.$$

Therefore,

$$\int_0^1 \Big(\frac{f^*(t)}{K} \Big)^{p^*(t)} dt = \int_0^1 \Big(\frac{f(v^{-1}(t))}{K} \Big)^{p^*(t)} dt = \int_0^1 \Big(\frac{f(t)}{K} \Big)^{p^*(v(t))} dt \le 1,$$

whence, by Lemma 2.1,

$$\int_0^1 \Big(\frac{f(t)}{K} \Big)^{p^*(\omega(t))} dt = \int_0^1 \Big(\frac{f(\omega^{-1}(t))}{K} \Big)^{p^*(t)} dt \le 3.$$

Combining the last inequality and equality (16), with $a = b$, we get

$$\|T_\omega b\|_{L^{p^*(\cdot)}} \le 3K = 3 \, C_{p^*(v)} \|b\|_{\ell_2},$$

where the constant $C_{p^*(v)}$ does not depend on ω. Thus, the family of operators $\{T_\omega\}_{\omega \in \Omega(p)}$ is pointwise bounded, and thanks to the uniform boundedness principle, we obtain

$$\|T_\omega a\|_{L^{p^*(\cdot)}} \le C_0 \|a\|_{\ell_2}$$

for some constant C_0 independent of ω. Clearly, inequality (15) is an immediate consequence of the latter inequality and (16).

Let us continue the proof of Theorem 3.3. As above, G is the closure L^∞ in the Orlicz space L_N. By [Astashkin and Semënov 2013, Theorem 4], for arbitrary $x \in G$ there exists a Rademacher sum

$f_1 = \sum_{k=1}^{\infty} a_k r_k$ such that

$$\|a\|_{\ell_2} \le C_1 \|x\|_{L_N} \quad \text{and} \quad x^*(t) \le C_2(\|a\|_{\ell_2} + f_1^*(t)), \qquad t \in (0,1]. \tag{17}$$

Take a measure-preserving mapping $\omega : [0,1] \to [0,1]$, for which $|f_1| = f_1^*(\omega)$. Then, from (17) and (15) it follows

$$\begin{aligned} \|x^*\|_{L^{p^*(\cdot)}} &\le C_2(\|a\|_{\ell_2} + \|f_1(\omega^{-1})\|_{L^{p^*(\cdot)}}) = C_2(\|a\|_{\ell_2} + \|f_1\|_{L^{p^*(\omega)}}) \\ &\le C_2(1+C_0)\,\|a\|_{\ell_2} \le C_1 C_2(1+C_0)\|x\|_{L_N}. \end{aligned}$$

Furthermore, letting $x_n(t) = \min\{n, \log^{1/2}(e/t)\}$, $t \in (0,1]$, we have $x_n = x_n^* \in G$ and $\|x_n\|_{L_N} \le \alpha := \|\log^{1/2}(e/t)\|_{L_N}$ for each $n \in \mathbb{N}$. Hence, from the previous inequality it follows that

$$\|x_n\|_{L^{p^*(\cdot)}} \le C_1 C_2(1+C_0)\alpha, \quad n \in \mathbb{N}.$$

Since the space $L^{p^*(\cdot)}$ has the Fatou property and $\lim_{t\to\infty} x_n(t) = \log^{1/2}(e/t)$, we infer that the function $t \mapsto \log^{1/2}(e/t)$ lies in $L^{p^*(\cdot)}$. Recall that L_N consists of all $x \in L^0(m)$ such that $x^*(t) \le C \log^{1/2}(e/t)$ for all $t \in (0,1]$ and some constant $C > 0$. Therefore, by Lemma 2.1, we obtain $L_N \subset L^{p^*(\cdot)}$. Combining this with the fact that L_N is a symmetric space, we deduce $L_N \subset L^{q(\cdot)}$ for arbitrary exponent $q \in \Omega(p)$, which completes the proof. □

Let us observe that, if a function p satisfies the conditions of Theorem 3.2(b), the Rademacher system (r_n) in $L^{q(\cdot)}$ is not equivalent to the unit vector basis in ℓ_2 for every $q \in \Omega(p)$ (otherwise we would arrive to contradiction by Theorem 3.3); therefore, we obtain

Corollary 3.1. *Suppose that a function p satisfies the conditions of Theorem 3.2(b). Then there exists a function $q \in \Omega(p)$ such that the Rademacher system is not equivalent in $L^{q(\cdot)}$ to the unit vector basis in ℓ_2.*

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Received 27 Mar 2015. Revised 24 Jul 2015. Accepted 7 Sep 2015.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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ANALYSIS & PDE

Volume 9 No. 1 2016