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THE NONLINEAR SCHRÖDINGER EQUATION GROUND STATES ON PRODUCT SPACES

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We study the nature of the nonlinear Schrödinger equation ground states on the product spaces $\mathbb{R}^n \times M^k$, where M^k is a compact Riemannian manifold. We prove that for small L^2 masses the ground states coincide with the corresponding $\mathbb{R}^n$ ground states. We also prove that above a critical mass the ground states have nontrivial M^k dependence. Finally, we address the Cauchy problem issue, which transforms the variational analysis into dynamical stability results.

1. Introduction

Our goal here is to study the nature of the nonlinear Schrödinger equation ground states when the problem is posed on the product spaces $\mathbb{R}^n \times M^k$, where M^k is a compact Riemannian manifold. We thus consider the Cauchy problems

$$\begin{cases} i\partial_t u - \Delta_{x,y} u - u|u|^\alpha = 0, & (t,x,y) \in \mathbb{R} \times \mathbb{R}^n_x \times M^k_y, \\ u(0,x,y) = \varphi(x,y), \end{cases} \tag{1-1}$$

where

$$\Delta_{x,y} = \sum_{j=1}^n \partial^2_{x_j} + \Delta_y$$

and Δ_y is the Laplace–Beltrami operator on M^k_y. Recall that the Laplace–Beltrami operator is defined in local coordinates by

$$\frac{1}{\sqrt{\det(g_{i,j}(y))}} \partial_{y_i} \sqrt{\det(g_{i,j}(y))}\, g^{i,j}(y) \partial_{y_j},$$

where $g^{i,j}(y) = (g_{i,j}(y))^{-1}$ and $g_{i,j}(y)$ is the metric tensor.

We assume that $0 < \alpha < 4/(n+k)$, which corresponds to L^2 subcritical nonlinearity. In this paper, we shall study the following two questions:

- the existence and stability of solitary waves for (1-1);
- the global well-posedness of the Cauchy problem associated to (1-1).

Equation (1-1) has two (at least formal) conservation laws: the energy

$$\mathscr{E}_{n,M^k,\alpha}(u) = \int_{M^k_y} \int_{\mathbb{R}^d_x} \left(\frac{1}{2} |\nabla_{x,y} u|^2 - \frac{1}{2+\alpha} |u|^{2+\alpha} \right) dx\, d\mathrm{vol}_{M^k_y}, \tag{1-2}$$

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and the L^2 mass,

$$\|u\|^2_{L^2(\mathbb{R}^n\times M^k)} = \int_{M^k_y}\int_{\mathbb{R}^n_x} |u|^2\,dx\,d\mathrm{vol}_{M^k_y}. \tag{1-3}$$

Here we denote by $d\mathrm{vol}_{M^k_y}$ the volume form on M^k. Recall that in local coordinates it can be written as $\sqrt{\det(g_{i,j}(y))}\,dy$. Moreover, the i-th component (in local coordinates) of the gradient $(\nabla_y u(y))$ is

$$g^{i,j}(y)\partial_{y_j} u.$$

One has the classical Gagliardo–Nirenberg inequality

$$\|u\|^{2+\alpha}_{L^{2+\alpha}(\mathbb{R}^n\times M^k)} \le C\|u\|^{\theta(\alpha)}_{H^1(\mathbb{R}^n\times M^k)}\|u\|^{2+\alpha-\theta(\alpha)}_{L^2(\mathbb{R}^n\times M^k)}, \tag{1-4}$$

where $\theta(\alpha) = (n+k)\alpha/2$. Thus $\theta(\alpha) < 2$ under our assumption $0 < \alpha < 4/(n+k)$. This implies that the conservation laws (1-2) and (1-3) imply a control on the H^1 norm which excludes an L^2 self-focusing blow-up, and thus one expects that (1-1) has well defined global dynamics. This problem seems quite delicate for a general M^k. However, if we replace M^k with $\mathbb{R}^k$, it is well known (see [Tsutsumi 1987; Cazenave 2003] and the references therein) that (1-1) has a global strong solution for every $L^2(\mathbb{R}^{n+k})$ initial data.

Our argument to construct stable solutions to (1-1) follows the one proposed in [Cazenave and Lions 1982]. Hence we shall look at the following minimization problems:

$$K^\rho_{n,M^k,\alpha} = \inf_{\substack{u\in H^1(\mathbb{R}^n\times M^k)\\ \|u\|_{L^2(\mathbb{R}^n\times M^k)}=\rho}} \mathscr{E}_{n,M^k,\alpha}(u) \tag{1-5}$$

and $\mathscr{E}_{n,M^k,\alpha}(u)$ is defined in (1-2). In the following we shall use the notation

$$\mathcal{M}^\rho_{n,M^k,\alpha} = \{v \in H^1(\mathbb{R}^n\times M^k) : \|v\|_{L^2(\mathbb{R}^n\times M^k)} = \rho \text{ and } \mathscr{E}_{n,M^k,\alpha}(v) = K^\rho_{n,M^k,\alpha}\}. \tag{1-6}$$

The first result we state concerns the compactness of minimizing sequences to (1-5).

Theorem 1.1. *Let M^k be a compact manifold and $0 < \alpha < 4/(n+k)$. Then*

$$K^\rho_{n,M^k,\alpha} > -\infty \quad \text{and} \quad \mathcal{M}^\rho_{n,M^k,\alpha} \ne \varnothing \quad \text{for all } \rho > 0. \tag{1-7}$$

Also, for any sequence $u_j \in H^1(\mathbb{R}^n\times M^k)$ such that $\|u_j\|_{L^2(\mathbb{R}^n\times M^k)} = \rho$ and $\lim_{j\to\infty}\mathscr{E}_{n,M^k,\alpha}(u_j) = K^\rho_{n,M^k,\alpha}$, there exists a subsequence u_{j_l} and $\tau_l \in \mathbb{R}^n_x$ such that

$$u_{j_l}(x+\tau_l, y) \text{ converges in } H^1(\mathbb{R}^n\times M^k). \tag{1-8}$$

The proof of Theorem 1.1 is based on the concentration compactness principle which will be given in the Appendix. Also, the following stability theorem follows from a standard argument, hence its classical proof will be recalled in the Appendix.

Theorem 1.2. *Let $\rho > 0$ be fixed and n, M^k, α as in Theorem 1.1. Assume moreover that*

$$\text{the Cauchy problem (1-1) is globally well posed for any data } \varphi \in \mathcal{U}, \tag{1-9}$$

where $\mathcal{U}$ is an $H^1(\mathbb{R}^n \times M^k)$-neighborhood of $\mathcal{M}^\rho_{n,M^k,\alpha}$. Then the set $\mathcal{M}^\rho_{n,M^k,\alpha}$ is orbitally stable; that is, for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that, for any $\varphi \in \mathcal{U}$ with $\inf_{v\in\mathcal{M}^\rho_{n,M^k,\alpha}} \|\varphi - v\|_{H^1(\mathbb{R}^n\times M^k)} < \delta(\epsilon)$, we have

$$\sup_{t\in\mathbb{R}} \inf_{v\in\mathcal{M}^\rho_{n,M^k,\alpha}} \|u_\varphi(t) - v\|_{H^1(\mathbb{R}^n\times M^k)} < \epsilon,$$

where $u_\varphi(t, x, y)$ is the unique global solution to (1-1).

Let us emphasize that the stability result stated in Theorem 1.2 has two major defaults: the first one is that we don't have an explicit description of the minimizers $\mathcal{M}^\rho_{n,M^k,\alpha}$; the second one is that it is subordinated to (1-9), that is, the global well-posedness of the Cauchy problem (1-1). The main contributions of this paper concern a partial understanding of the aforementioned questions.

Notice that [Cazenave 2003] a special family of solutions to (1-1) is given by

$$u(t, x, y) = e^{-i\omega t} u_{n,\omega,\alpha}(x),$$

where $\omega > 0$ and $u_{n,\omega,\alpha}(x)$ is defined as the unique radial solution to

$$-\Delta_x u_{n,\omega,\alpha} + \omega u_{n,\omega,\alpha} = u_{n,\omega,\alpha}|u_{n,\omega,\alpha}|^\alpha, \quad u_{n,\omega,\alpha} \in H^1(\mathbb{R}^n_x),\ u_{n,\omega,\alpha}(x) > 0,\ x \in \mathbb{R}^n_x. \tag{1-10}$$

Next, we set

$$\mathcal{N}_{n,\omega,\alpha} = \{e^{i\theta} u_{n,\omega,\alpha}(x+\tau) : \tau \in \mathbb{R}^n, \theta \in \mathbb{R}\}. \tag{1-11}$$

Notice that there is a natural embedding $H^1(\mathbb{R}^n_x) \subset H^1(\mathbb{R}^n_x \times M^k_y)$. In fact, every function in $H^1(\mathbb{R}^n_x)$ can be extended in a trivial way with respect to the y variable on $\mathbb{R}^n_x \times M^k_y$, and this extension will belong to $H^1(\mathbb{R}^n \times M^k)$. In particular, since now the set $\mathcal{N}_{n,\omega,\alpha}$ defined in (1-11) will be considered without any further comment both as a subset of $H^1(\mathbb{R}^n_x)$ and as a subset of $H^1(\mathbb{R}^n_x \times M^k_y)$, by a rescaling argument, one can prove that the function

$$(0, \infty) \ni \omega \to \|u_{n,\omega,\alpha}\|^2_{L^2(\mathbb{R}^n_x)} \in (0, \infty)$$

is strictly increasing for any $0 < \alpha < 4/n$ and

$$\lim_{\omega\to\infty} \|u_{n,\omega,\alpha}\|_{L^2(\mathbb{R}^n_x)} = \infty \quad \text{and} \quad \lim_{\omega\to 0} \|u_{n,\omega,\alpha}\|_{L^2(\mathbb{R}^n_x)} = 0.$$

As a consequence, for any fixed $0 < \alpha < 4/n$, we have

$$\text{for all } \rho > 0 \text{ there exists a unique } \omega(\rho) > 0 \text{ such that } \|u_{n,\omega(\rho),\alpha}\|_{L^2(\mathbb{R}^n_x)} = \rho. \tag{1-12}$$

In the next theorem, the set $\mathcal{N}_{n,\omega,\alpha}$ is the one defined in (1-11) and $\mathcal{M}^\rho_{n,M^k,\alpha}$ is defined in (1-6).

Theorem 1.3. *Let n, M^k, α be as in Theorem 1.2. There exists $\rho^* \in (0, \infty)$ such that*

$$\mathcal{M}^\rho_{n,M^k,\alpha} = \mathcal{N}_{n,\omega(\rho/\sqrt{\mathrm{vol}(M^k)}),\alpha} \text{ for all } \rho < \rho^* \tag{1-13}$$

and

$$\mathcal{M}^\rho_{n,M^k,\alpha} \cap \mathcal{N}_{n,\omega(\rho/\sqrt{\mathrm{vol}(M^k)}),\alpha} = \varnothing \text{ for all } \rho > \rho^*, \tag{1-14}$$

where $\omega\big(\rho/\sqrt{\mathrm{vol}(M^k)}\big)$ *is uniquely defined in (1-12). In particular for* $\rho > \rho^*$ *the elements of* $\mathcal{M}^{\rho}_{n,M^k,\alpha}$ *depend in a nontrivial way on the* M^k *variable.*

By the approach of Weinstein [1986] one may expect that $\mathcal{N}_{n,\omega,\alpha}$ is stable under (1-1) for $\alpha < 4/n$ and ω small enough; see [Rousset and Tzvetkov 2012] for a recent related work. It should however be pointed out that in such a stability result one would not get the variational description of $\mathcal{N}_{n,\omega,\alpha}$ as is the case in Theorem 1.3 ($\alpha < 4/(n+k)$). We underline that, by combining Theorem 1.2 and Theorem 1.3, we get a stable set for large values of the mass ρ, and in general it is independent of the solitary waves associated to the nonlinear Schrödinger equation in $\mathbb{R}^n$.

Next we shall focus on the question of the global well-posedness of the Cauchy problem associated to (1-1) in the particular case $n \geq 1$, $k = 1$. For every $n > 1$ we fix the numbers

$$p := p(n,\alpha) = \frac{4(2+\alpha)}{n\alpha} \quad \text{and} \quad q := q(n,\alpha) = 2+\alpha,$$

and for every $T > 0$ we define the localized norms

$$\|u(t,x,y)\|_{X_T} \equiv \|u(t,x,y)\|_{L^p((-T,T);L^q(\mathbb{R}^n_x;H^1(M^1_y))} \tag{1-15}$$

and

$$\|u(t,x,y)\|_{Y_T} \equiv \|\nabla_x u\|_{L^p((-T,T);L^q(\mathbb{R}^n_x;L^2(M^1_y))}. \tag{1-16}$$

Theorem 1.4. *Let* $n \geq 1$ *be fixed and* $\alpha < 4/(n+1)$. *Then, for every initial data* $\varphi \in H^1(\mathbb{R}^n \times M^1)$, *the Cauchy problem (1-1) has a unique global solution* $u(t,x,y)$ *satisfying*

$$u(t,x,y) \in \mathscr{C}((-T,T);H^1(\mathbb{R}^n \times M^1)) \cap X_T \cap Y_T \quad \textit{for all } T > 0.$$

Remark 1.5. The main difficulty in the analysis of the Cauchy problem (1-1) (compared with the Cauchy problem in the euclidean space) is related to the fact that the propagator $e^{-it\Delta_{x,y}}$ on $\mathbb{R}^n \times M^1_y$ does not satisfy the Strichartz estimates which are available for the propagator $e^{-it\Delta_{\mathbb{R}^{n+k}}}$ on the euclidean space $\mathbb{R}^{n+k}$.

Let us now describe some other known cases when (1-1) is well posed in $H^1(\mathbb{R}^n \times M^k)$ under the assumption $\alpha < 4/(n+k)$. Using the analysis of [Burq et al. 2004; Burq et al. 2003], one may prove such a well-posedness result in the case $\mathbb{R} \times M^2$, that is, $n = 1$ and $k = 2$. Moreover, using the analysis of [Herr et al. 2010; Ionescu and Pausader 2012], one may also prove such a well-posedness result in the cases $\mathbb{R}^2 \times \mathbb{T}^2$ and $\mathbb{R} \times \mathbb{T}^3$, respectively.

Notation. Next we fix some notations. We denote by L^p_x and H^s_x the spaces $L^p(\mathbb{R}^n_x)$ and $H^s(\mathbb{R}^n_x)$, respectively. We also use the notation $L^p_{x,y} = L^p(\mathbb{R}^n_x \times M^k_y)$ and $L^p_x L^q_y = L^p(\mathbb{R}^n_x; L^q(M^k_y))$. If $v(t)$ is a time dependent function defined on $\mathbb{R}_t$ and valued in a Banach space X, we define

$$\|v\|^p_{L^p_t(X)} = \int_{\mathbb{R}} \|v(t)\|^p_X \, dt.$$

For every $p \in [1,\infty]$ we denote by $p' \in [1,\infty]$ its conjugate Hölder exponent. We denote by $e^{-it\Delta_{x,y}}$ the free propagator associated to the Schrödinger equation on $\mathbb{R}^n_x \times M^k_y$.

2. Some useful results on the euclidean space $\mathbb{R}^n_x$ with $n \geq 1$

In this section we recall some well-known facts (see [Cazenave 2003]) related to the following minimization problem on $\mathbb{R}^n_x$:

$$I^{\rho}_{n,\alpha} = \inf_{\substack{u \in H^1_x \\ \|u\|_{L^2_x} = \rho}} \mathscr{E}_{n,\alpha}(u), \tag{2-1}$$

where, for $\alpha < 4/n$,

$$\mathscr{E}_{n,\alpha}(u) = \frac{1}{2}\int_{\mathbb{R}^n_x} |\nabla_x u|^2\,dx - \frac{1}{2+\alpha}\int_{\mathbb{R}^n_x} |u|^{2+\alpha}\,dx. \tag{2-2}$$

By an elementary rescaling argument we have

$$I^{\rho}_{n,\alpha} = \rho^{(8+4\alpha-2\alpha n)/(4-\alpha n)} I^1_{n,\alpha}. \tag{2-3}$$

It is well known that

$$-\infty < I^{\rho}_{n,\alpha} < 0, \quad \text{for all } \rho > 0, \tag{2-4}$$

and

$$\mathscr{M}^{\rho}_{n,\alpha} = \mathscr{N}_{n,\omega(\rho),\alpha}, \tag{2-5}$$

where $\mathscr{N}_{n,\omega,\alpha}$ is defined in (1-11),

$$\mathscr{M}^{\rho}_{n,\alpha} = \{u \in H^1_x \,|\, \|u\|_{L^2_x} = \rho \text{ and } \mathscr{E}_{n,\alpha}(u) = I^{\rho}_{n,\alpha}\} \tag{2-6}$$

and $\omega(\rho)$ is defined uniquely (see (1-12)) by the relation

$$\|u_{n,\omega(\rho),\alpha}\|_{L^2_x} = \rho.$$

We also recall that the functions $u_{n,\omega,\alpha}$ (defined as the unique radially symmetric and positive solution to (1-10)) satisfy the following Pohozaev type identity (for a proof of (2-7) see the proof of (3-21) in the next section):

$$\int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2\,dx = \frac{\alpha n}{2(\alpha+2)}\int_{\mathbb{R}^n_x} |u_{n,\omega,\alpha}|^{2+\alpha}\,dx. \tag{2-7}$$

On the other hand, if we multiply (1-10) by $u_{n,\omega,\alpha}$ and integrate by parts, we get

$$\int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2\,dx + \omega\|u_{n,\omega,\alpha}\|^2_{L^2_x} = \int_{\mathbb{R}^n_x} |u_{n,\omega,\alpha}|^{2+\alpha}\,dx,$$

which, in conjunction with (2-7), gives

$$\begin{aligned}\omega\|u_{n,\omega,\alpha}\|^2_2 = \frac{2\alpha+4-\alpha n}{\alpha n}&\int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2\,dx \\ &= \frac{4\alpha+8-2\alpha n}{\alpha n-4}\left(\frac{1}{2}\int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2\,dx - \frac{1}{2+\alpha}\int_{\mathbb{R}^n_x} |u_{n,\omega,\alpha}|^{2+\alpha}\,dx\right) \\ &= \frac{4\alpha+8-2\alpha n}{\alpha n-4} I^{\|u_{n,\omega,\alpha}\|_{L^2_x}}_{n,\alpha}\end{aligned} \tag{2-8}$$

(in the last step we have used the fact that due to (2-5) we have that $u_{n,\omega,\alpha}$ is a minimizer for $\mathscr{E}_{n,\alpha}$ on its associated constrained).

Finally notice that by (2-7) we deduce

$$I_{n,\alpha}^{\|u_{n,\omega,\alpha}\|_{L^2_x}} = \mathscr{E}_{n,\alpha}(u_{n,\omega,\alpha}) = \frac{\alpha n-4}{2\alpha n}\int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2\,dx. \tag{2-9}$$

3. An auxiliary problem

In this section we study the minimizers of the minimization problems

$$J_{n,M^k,\alpha,\lambda} = \inf_{\substack{u\in H^1(\mathbb{R}^n\times M^k)\\ \|u\|_{L^2_{x,y}}=1}} \mathscr{E}_{n,M^k,\alpha,\lambda}(u), \tag{3-1}$$

where

$$\mathscr{E}_{n,M^k,\alpha,\lambda}(u) = \int_{M^k_y}\int_{\mathbb{R}^n_x}\Big(\frac{\lambda}{2}|\nabla_y u|^2 + \tfrac{1}{2}|\nabla_x u|^2 - \frac{1}{2+\alpha}|u|^{2+\alpha}\Big)\,dx\,d\mathrm{vol}_{M^k_y}.$$

We also introduce the sets

$$\mathcal{M}_{n,M^k,\alpha,\lambda} = \{w \in H^1(\mathbb{R}^n\times M^k) : \|w\|_{L^2_{x,y}} = 1 \text{ and } \mathscr{E}_{n,M^k,\alpha,\lambda}(w) = J_{n,M^k,\alpha,\lambda}\}.$$

Theorem 3.1. *Let n, M^k, and $0<\alpha<4/(n+k)$ be given. There exists $\lambda^*\in(0,\infty)$ such that*

$$\mathcal{M}_{n,M^k,\alpha,\lambda} = \mathcal{N}_{n,\bar\omega,\alpha} \quad \textit{for all } \lambda>\lambda^* \tag{3-2}$$

and

$$\mathcal{M}_{n,M^k,\alpha,\lambda}\cap\mathcal{N}_{n,\bar\omega,\alpha} = \varnothing \quad \textit{for all } \lambda<\lambda^*, \tag{3-3}$$

where $\bar\omega$ is defined by the condition

$$\mathrm{vol}(M^k)\|u_{n,\bar\omega,\alpha}\|^2_{L^2_x} = 1.$$

We fix a sequence $\lambda_j\to\infty$ and a corresponding sequence of functions $u_{\lambda_j}\in\mathcal{M}_{n,M^k,\alpha,\lambda_j}$. In the sequel we shall assume that

$$u_{\lambda_j}(x,y)\ge 0 \quad \text{for all } (x,y)\in\mathbb{R}^n_x\times M^k_y. \tag{3-4}$$

Indeed, it is well known that if u_{λ_j} is a minimizer, $|u_{\lambda_j}|$ is also a minimizer. In particular there exists at least one minimizer which satisfies (3-4).

Notice that the functions u_{λ_j} depend in principle on the full set of variables (x,y). Our aim is to prove that, for j large and up to subsequence, the functions u_{λ_j} will not depend explicitly on the variable y.

First we prove some a priori bounds satisfied by $u_{\lambda_j}(x,y)$. Recall that the quantities $I^\rho_{n,\alpha}$ are defined in (2-1).

Lemma 3.2. *Make the same assumptions as in Theorem 3.1. Then we have*

$$\lim_{j\to\infty} J_{n,M^k,\alpha,\lambda_j} = \mathrm{vol}(M^k)\, I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}} \tag{3-5}$$

and

$$\lim_{j\to\infty} \lambda_j \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 \, dx \, d\mathrm{vol}_{M_y^k} = 0. \tag{3-6}$$

Proof. First notice that

$$J_{n,M^k,\alpha,\lambda_j} \le \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}. \tag{3-7}$$

In fact, let $w(x) \in H_x^1$ be such that $\|w\|_{L_x^2} = 1/\sqrt{\mathrm{vol}(M^k)}$ and $\mathscr{E}_{n,\alpha}(w) = I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}$. Then we easily get

$$\begin{aligned} J_{n,M^k,\alpha,\lambda_j} \le \mathscr{E}_{n,M^k,\alpha,\lambda_j}(w(x)) &= \mathrm{vol}(M^k)\biggl(\frac{1}{2}\int_{\mathbb{R}_x^n} |\nabla_x w|^2 \, dx - \frac{1}{2+\alpha}\int_{\mathbb{R}_x^n} |w|^{2+\alpha} \, dx\biggr) \\ &= \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}, \end{aligned}$$

which concludes the proof of (3-7).

Next we claim that

$$\lim_{j\to\infty} \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 \, dx \, d\mathrm{vol}_{M_y^k} = 0. \tag{3-8}$$

Assume for a contradiction that this is false. Then there exists a subsequence of λ_j (that we still denote by λ_j) such that

$$\lim_{j\to\infty} \lambda_j = \infty \quad \text{and} \quad \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 \, dx \, d\mathrm{vol}_{M_y^k} \ge \epsilon_0 > 0,$$

and, in particular,

$$\lim_{j\to\infty} (\lambda_j - 1) \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 \, dx \, d\mathrm{vol}_{M_y^k} = \infty. \tag{3-9}$$

On the other hand, by the classical Gagliardo–Nirenberg inequality (see (1-4)) we deduce the existence of $0 < \mu < 2$ such that

$$\begin{aligned} &\frac{1}{2}\int_{M_y^k} \int_{\mathbb{R}_x^n} (|\nabla_y v|^2 + |\nabla_x v|^2 + |v|^2) \, dx \, d\mathrm{vol}_{M_y^k} - \frac{1}{2+\alpha}\int_{M_y^k} \int_{\mathbb{R}_x^n} |v|^{2+\alpha} \, dx \, d\mathrm{vol}_{M_y^k} \\ &\quad \ge \frac{1}{2}\int_{M_y^k} \int_{\mathbb{R}_x^n} (|\nabla_y v|^2 + |\nabla_x v|^2 + |v|^2) \, dx \, d\mathrm{vol}_{M_y^k} - C\biggl[\int_{M_y^k} \int_{\mathbb{R}_x^n} (|\nabla_y v|^2 + |\nabla_x v|^2 + |v|^2) \, dx \, d\mathrm{vol}_{M_y^k}\biggr]^{\mu} \\ &\quad \ge \inf_{t>0}(1/2t^2 - Ct^{\mu}) = C(\mu) > -\infty \end{aligned}$$

for all $v \in H^1(\mathbb{R}^n \times M^k)$ such that $\|v\|_{L_{x,y}^2} = 1$. By the previous inequality we get

$$\mathscr{E}_{n,M^k,\alpha,\lambda_j}(v) - \tfrac{1}{2}(\lambda_j - 1)\int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y v|^2 \ge -\tfrac{1}{2} + C(\mu)$$

for all $v \in H^1(\mathbb{R}^n \times M^k)$ such that $\|v\|_{L_{x,y}^2} = 1$. In particular, if we choose $v = u_{\lambda_j}$, we get

$$J_{n,M^k,\alpha,\lambda_j} = \mathscr{E}_{n,M^k,\alpha,\lambda_j}(u_{\lambda_j}) \ge \tfrac{1}{2}(\lambda_j - 1)\int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 \, dx \, d\mathrm{vol}_{M_y^k} - \tfrac{1}{2} + C(\mu).$$

By (3-9) this implies $\lim_{n\to\infty} J_{n,M^k,\alpha,\lambda_j} = \infty$, which is in contradiction with (3-7). Hence (3-8) is proved.

Next we introduce the functions

$$w_j(y) = \|u_{\lambda_j}(x,y)\|^2_{L^2_x}.$$

Notice that

$$\|w_j(y)\|_{L^1_y} = 1 \tag{3-10}$$

and, moreover,

$$\begin{aligned}\int_{M^k_y} |\nabla_y w_j(y)|\, d\mathrm{vol}_{M^k_y} &\le C \int_{M^k_y}\int_{\mathbb{R}^n_x} |u_{\lambda_j}(x,y)|\,|\nabla_y u_{\lambda_j}(x,y)|\, dx\, d\mathrm{vol}_{M^k_y} \\ &\le C\|u_{\lambda_j}\|_{L^2_{x,y}} \|\nabla_y u_{\lambda_j}\|_{L^2_{x,y}}.\end{aligned}$$

Hence, due to (3-8), we get

$$\lim_{j\to\infty} \|\nabla_y w_j\|_{L^1_y} = 0. \tag{3-11}$$

By combining (3-10) and (3-11) with the Rellich compactness theorem and with the Sobolev embedding $W^{1,1}(M^1) \subset L^\infty(M^1)$ and $W^{1,1}(M^2) \subset L^2(M^2)$, we deduce in the cases $k = 1$ and $k = 2$ that (up to a subsequence)

$$\lim_{j\to\infty} \|w_j(y) - 1/\operatorname{vol}(M^1)\|_{L^r_y} = 0 \quad \text{for all } 1 \le r < \infty \tag{3-12}$$

and

$$\lim_{j\to\infty} \|w_j(y) - 1/\operatorname{vol}(M^2)\|_{L^r_y} = 0 \quad \text{for all } 1 \le r < 2, \tag{3-13}$$

respectively. For $k > 2$ we use the Sobolev embedding $H^1(M^k) \subset L^{2k/(k-2)}(M^k)$ and we get

$$\sup_j \|u_{\lambda_j}\|_{L^2_x L^{2k/(k-2)}_y} \le C \sup_j \|u_{\lambda_j}\|_{L^2_x H^1(M^k_y)} < \infty$$

(where in the last step we have used the fact that $\sup_j(\|u_{\lambda_j}\|_{L^2_{x,y}} + \|\nabla_y u_{\lambda_j}\|_{L^2_{x,y}}) < \infty$). By the Minkowski inequality the bound above implies $\sup_j \|u_{\lambda_j}\|_{L^{2k/(k-2)}_y L^2_x}$, which is equivalent to the condition

$$\sup_j \|w_j(y)\|_{L^{k/(k-2)}_y} < \infty \quad \text{for } k > 2. \tag{3-14}$$

By combining (3-10) and (3-11) with the Rellich compactness theorem, we deduce that up to a subsequence

$$\|w_j(y) - 1/\operatorname{vol}(M^k)\|_{L^1_y} = 0 \quad \text{for } k > 2,$$

and hence, by interpolation with (3-14), we get

$$\|w_j(y) - 1/\operatorname{vol}(M^k)\|_{L^r_y} = 0 \quad \text{for } k > 2,\ 1 \le r < k/(k-2). \tag{3-15}$$

By the definition of $I^\rho_{n,\alpha}$ (see (2-1)) and (2-3) we get

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^n_x} |\nabla_x u_{\lambda_j}(x,y)|^2\, dx - \frac{1}{2+\alpha}\int_{\mathbb{R}^n_x} |u_{\lambda_j}(x,y)|^{2+\alpha}\, dx \\ \ge I^{\|u_{\lambda_j}(\cdot,y)\|_{L^2_x}}_{n,\alpha} = I^1_{n,\alpha} \|u_{\lambda_j}(\cdot,y)\|^{(8+4\alpha-2\alpha n)/(4-\alpha n)}_{L^2_x} = I^1_{n,\alpha} w_j(y)^{(4+2\alpha-\alpha n)/(4-\alpha n)}\end{aligned} \tag{3-16}$$

for all $y \in M^k$ and all $j \in \mathbb{N}$. Next notice that, by definition,

$$\begin{aligned} J_{n,M^k,\alpha,\lambda_j} &= \mathscr{E}_{n,M^k,\alpha,\lambda_j}(u_{\lambda_j}) \\ &= \frac{1}{2}\int_{M^k_y}\int_{\mathbb{R}^n_x} (\lambda_j|\nabla_y u_{\lambda_j}|^2 + |\nabla_x u_{\lambda_j}|^2)\,dx\,dy - \frac{1}{2+\alpha}\int_{M^k_y}\int_{\mathbb{R}^n_x} |u|^{2+\alpha}\,dx\,d\mathrm{vol}_{M^k_y}, \end{aligned} \tag{3-17}$$

and we can continue

$$\begin{aligned} \cdots &\geq \int_{M^k_y}\left(\frac{1}{2}\int_{\mathbb{R}^n_x} |\nabla_x u_{\lambda_j}(x,y)|^2\,dx - \frac{1}{2+\alpha}\int_{\mathbb{R}^n_x} |u_{\lambda_j}(x,y)|^{2+\alpha}\,dx\right) d\mathrm{vol}_{M^k_y} \\ &\geq I^1_{n,\alpha}\int_{M^k_y} w_j(y)^{(4+2\alpha-\alpha n)/(4-\alpha n)}\,d\mathrm{vol}_{M^k_y} \\ &= I^1_{n,\alpha}\,\mathrm{vol}(M^k)\,\mathrm{vol}(M^k)^{-(4+2\alpha-\alpha n)/(4-\alpha n)} + o(1), \end{aligned} \tag{3-18}$$

where $o(1) \to 0$ as $j \to \infty$ and in the last step we have combined (3-12), (3-13), and (3-15) for $k = 1$, $k = 2$, and $k > 2$, respectively, and we used our assumption on α. By combining this fact with (2-3), we have

$$\liminf_{j\to\infty} J_{n,M^k,\alpha,\lambda_j} \geq \mathrm{vol}(M^k) I^{1/\sqrt{\mathrm{vol}(M^k)}}_{n,\alpha}. \tag{3-19}$$

Hence (3-5) follows by combining (3-7) with (3-19).

Next we prove (3-6). For that purpose, it suffices to keep the term $\lambda_j|\nabla_y u_{\lambda_j}|^2$ in the previous analysis. Namely, by combining (3-5) with (3-17) and (3-18), we get

$$\mathrm{vol}(M^k) I^{1/\sqrt{\mathrm{vol}(M^k)}}_{n,\alpha} + g(j) \geq \tfrac{1}{2}\lambda_j \int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_y u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M^k_y} + h(j), \tag{3-20}$$

where

$$\lim_{j\to\infty} g(j) = 0 \quad \text{and} \quad \liminf_{j\to\infty} h(j) \geq \mathrm{vol}(M^k) I^{1/\sqrt{\mathrm{vol}(M^k)}}_{n,\alpha}.$$

Hence (3-6) follows by (3-20). □

Lemma 3.3. *We have the identity*

$$\int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M^k_y} = \frac{\alpha n}{2(2+\alpha)}\int_{M^k_y}\int_{\mathbb{R}^n_x} |u_{\lambda_j}|^{2+\alpha}\,dx\,d\mathrm{vol}_{M^k_y}. \tag{3-21}$$

Moreover, there exist $J \in \mathbb{N}$ such that for all $j > J$ there exists $\omega(\lambda_j) > 0$ such that

$$-\lambda_j \Delta_y u_{\lambda_j} - \Delta_x u_{\lambda_j} + \omega(\lambda_j) u_{\lambda_j} = u_{\lambda_j}|u_{\lambda_j}|^\alpha, \tag{3-22}$$

and the following limit exists:

$$\lim_{j\to\infty} \omega(\lambda_j) = \bar{\omega} \in (0,\infty). \tag{3-23}$$

Proof. Since u_{λ_j} is a constrained minimizer for $\mathscr{E}_{n,M^k,\alpha,\lambda_j}$ on the ball of size 1 in $L^2(\mathbb{R}^n \times M^k)$, we get

$$\frac{d}{d\epsilon}[\mathscr{E}_{n,M^k,\alpha,\lambda_j}(\epsilon^{n/2} u_{\lambda_j}(\epsilon x, y))]_{\epsilon=1} = 0,$$

which is equivalent to

$$\frac{d}{d\epsilon}\Big[\tfrac{1}{2}\lambda_j\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_y u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}+\tfrac{1}{2}\epsilon^2\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}-\frac{1}{2+\alpha}\epsilon^{\alpha n/2}\|u_{\lambda_j}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big]_{\epsilon=1}=0.$$

By computing explicitly the derivative (in ϵ), we deduce (3-21).

Next notice that by using the Lagrange multiplier technique we get (3-22) for a suitable $\omega(\lambda_j)\in\mathbb{R}$. On the other hand, by (3-22), we get

$$\int_{M_y^k}\int_{\mathbb{R}_x^n}(\lambda_j|\nabla_y u_{\lambda_j}|^2+|\nabla_x u_{\lambda_j}|^2)\,dx\,d\mathrm{vol}_{M_y^k}+\omega(\lambda_j)\|u_{\lambda_j}\|^2_{L^2_{x,y}}=\int_{M_y^k}\int_{\mathbb{R}_x^n}|u_{\lambda_j}|^{2+\alpha}\,dx\,d\mathrm{vol}_{M_y^k},$$

which, by (3-21), gives

$$\omega(\lambda_j)=\frac{-\alpha n+4+2\alpha}{\alpha n}\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}-\lambda_j\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_y u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k},$$

and hence, by (3-6), we get

$$\omega(\lambda_j)=\frac{-\alpha n+4+2\alpha}{\alpha n}\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}+o(1),\tag{3-24}$$

where $\lim_{j\to\infty}o(1)=0$.

On the other hand, notice that, by (3-21), we get

$$J_{n,M^k,\alpha,\lambda_j}=\mathscr{E}_{n,M^k,\alpha,\lambda_j}(u_{\lambda_j})=\frac{\alpha n-4}{2\alpha n}\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}+\frac{1}{2}\int_{M_y^k}\int_{\mathbb{R}_x^n}\lambda_j|\nabla_y u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k},$$

and by (3-6)

$$\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}=\frac{2\alpha n}{\alpha n-4}J_{n,M^k,\alpha,\lambda_j}+o(1).\tag{3-25}$$

By (3-5) this implies

$$\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}=\frac{2\alpha n}{\alpha n-4}\,\mathrm{vol}(M^k)I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}+o(1),\tag{3-26}$$

which, in conjunction with (3-24) and (2-4), implies $\omega(\lambda_j)>0$ for j large enough. Moreover, (3-23) follows by (3-24) and (3-26). □

Next recall that the sets $\mathcal{M}^\rho_{n,\alpha}$ are the ones defined in (2-6).

Lemma 3.4. *Let $\bar\omega$ be as in (3-23) and let $v(x)\in\mathcal{M}_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}$ be such that $v(x)>0$. Then*

$$-\Delta_x v+\bar\omega v=v|v|^\alpha.$$

Proof. It is well known that

$$-\Delta_x v+\omega_1 v=v|v|^\alpha$$

for a suitable $\omega_1 > 0$. More precisely, we can assume that up to translation $v = u_{n,\omega_1,\alpha}$. Our aim is to prove that $\omega_1 = \bar{\omega}$. Notice that, by (2-8),

$$\omega_1 \frac{1}{\operatorname{vol}(M^k)} = \frac{4\alpha + 8 - 2\alpha n}{\alpha n - 4} I_{n,\alpha}^{\|v\|_{L_x^2}} = \frac{4\alpha + 8 - 2\alpha n}{\alpha n - 4} I_{n,\alpha}^{1/\sqrt{\operatorname{vol}(M^k)}}. \tag{3-27}$$

On the other hand, by (3-24) and (3-26), we get

$$\omega(\lambda_j) = \frac{-2\alpha n + 8 + 4\alpha}{\alpha n - 4} \operatorname{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\operatorname{vol}(M^k)}} + o(1),$$

and hence, passing to the limit in j, we get

$$\bar{\omega} = \frac{-2\alpha n + 8 + 4\alpha}{\alpha n - 4} \operatorname{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\operatorname{vol}(M^k)}}. \tag{3-28}$$

By combining (3-27) and (3-28), we get $\bar{\omega} = \omega_1$. $\square$

Lemma 3.5. *There exist a subsequence of λ_j (that we shall denote still by λ_j) and a sequence $\tau_j \in \mathbb{R}_x^n$ such that*

$$\lim_{j\to\infty} \|u_{\lambda_j}(x + \tau_j, y) - u_{\bar{\omega}}\|_{H^1(\mathbb{R}^n \times M^k)} = 0,$$

where $u_{\bar{\omega}} \in \mathcal{N}_{n,\bar{\omega},\alpha}$, $u_{\bar{\omega}} > 0$ and $\bar{\omega}$ is defined in (3-23).

Proof. By combining (3-6) and (3-26), and since $\|u_{\lambda_j}\|_{L_{x,y}^2} = 1$, we deduce that u_{λ_j} is bounded in $H^1(\mathbb{R}^n \times M^k)$. Moreover, by combining (3-5) with the fact that $I_{n,\alpha}^{1/\sqrt{\operatorname{vol}(M^k)}} < 0$ (see (2-4)), we get

$$\inf_j \|u_{\lambda_j}\|_{L_{x,y}^{2+\alpha}} > 0.$$

By using the localized version of the Gagliardo–Nirenberg inequality (A-5) (in the same spirit as in the Appendix), we get the existence (up to subsequence) of $\tau_j \in \mathbb{R}_x^n$ such that

$$u_{\lambda_j}(x + \tau_j, y) \rightharpoonup w \neq 0 \quad \text{in } H^1(\mathbb{R}^n \times M^k).$$

Moreover, due to (3-4), we can assume that

$$w(x, y) \geq 0 \quad \text{a.e. in } (x, y) \in \mathbb{R}_x^n \times M_y^k,$$

and by (3-6) we get $\nabla_y w = 0$. In particular w is y-independent.

By combining (3-6) and (3-23), we pass to the limit in (3-22) in the distribution sense, and we get

$$-\Delta_x w + \bar{\omega} w = w|w|^\alpha \text{ in } \mathbb{R}_x^n, \quad w(x) \geq 0,\ w \neq 0. \tag{3-29}$$

We claim that

$$\|w\|_{L_x^2} = \frac{1}{\sqrt{\operatorname{vol}(M^k)}}. \tag{3-30}$$

If not, we can assume $\|w\|_{L_x^2} = \beta < 1/\sqrt{\operatorname{vol}(M^k)}$, and since w solves (3-29) by (2-5), we get

$$w \in \mathcal{M}_{n,\alpha}^{\beta}. \tag{3-31}$$

On the other hand, by Lemma 3.4, (3-29) is satisfied by any $v \in \mathcal{M}_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}$. Hence, again by (2-5) and by the injectivity of the map $\rho \to \omega(\rho)$ (see (1-12)), we deduce that, necessarily, $\beta = 1/\sqrt{\mathrm{vol}(M^k)}$.

In particular, by (3-30), we deduce

$$\lim_{j\to\infty} \|u_{\lambda_j}(x+\tau_j, y) - w\|_{L^2_{x,y}} = 0.$$

Next notice that, by (3-6) and since we have already proved that $\nabla_y w = 0$, we can deduce that

$$\lim_{j\to\infty} \|\nabla_y u_{\lambda_j}(x+\tau_j, y)\|_{L^2_{x,y}} = 0 = \|\nabla_y w\|_{L^2_{x,y}}.$$

Hence, in order to conclude that $u_{\lambda_j}(x+\tau_j, y)$ converges strongly to w in $H^1(\mathbb{R}^n \times M^k)$, it is sufficient to prove that

$$\lim_{j\to\infty} \|\nabla_x u_{\lambda_j}(x+\tau_j, y)\|_{L^2_{x,y}} = \sqrt{\mathrm{vol}(M^k)}\|\nabla_x w\|_{L^2_x} = \|\nabla_x w\|_{L^2_{x,y}}.$$

This last fact follows by combining (2-9) (where we use the fact that $w \in \mathcal{N}_{n,\bar\omega,\alpha}$ by (3-29) and $\|w\|_{L^2_x} = 1/\sqrt{\mathrm{vol}(M^k)}$ by (3-30)) and (3-26). □

Lemma 3.6. *There exists* $j_0 > 0$ *such that*

$$\nabla_y u_{\lambda_j} = 0 \quad \textit{for all } j > j_0.$$

Proof. By Lemma 3.5 we can assume that

$$u_{\lambda_j} \to u_{\bar\omega} \quad \text{in } H^1(\mathbb{R}^n \times M^k). \tag{3-32}$$

We introduce $w_j = \sqrt{-\Delta_y} u_{\lambda_j}$. Notice that due to (3-22) the functions w_j satisfy

$$-\lambda_j \Delta_y w_j - \Delta_x w_j + \omega(\lambda_j) w_j = \sqrt{-\Delta_y}(u_{\lambda_j}|u_{\lambda_j}|^\alpha), \tag{3-33}$$

which, after multiplication by w_j, implies

$$\int_{M^k_y}\int_{\mathbb{R}^n_x} [\lambda_j |\nabla_y w_j|^2 + |\nabla_x w_j|^2 + \omega(\lambda_j)|w_j|^2 - \sqrt{-\Delta_y}(u_{\lambda_j}|u_{\lambda_j}|^\alpha) w_j]\, dx\, d\mathrm{vol}_{M^k_y} = 0. \tag{3-34}$$

In turn this gives

$$\begin{aligned} 0 = &\int_{M^k_y}\int_{\mathbb{R}^n_x} (\lambda_j - 1)|\nabla_y w_j|^2 - (\alpha+1)\sqrt{-\Delta_y}(u_{\lambda_j}|u_{\bar\omega}|^\alpha) w_j\, dx\, d\mathrm{vol}_{M^k_y} \\ &+ \int_{M^k_y}\int_{\mathbb{R}^n_x} (|\nabla_y w_j|^2 + |\nabla_x w_j|^2 + \bar\omega |w_j|^2 + \sqrt{-\Delta_y}(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha)) w_j\, dx\, d\mathrm{vol}_{M^k_y} \\ &+ \int_{M^k_y}\int_{\mathbb{R}^n_x} (\omega(\lambda_j) - \bar\omega)|w_j|^2\, dx\, dy \equiv I_j + II_j + III_j. \end{aligned} \tag{3-35}$$

Next we fix an orthonormal basis of eigenfunctions for $-\Delta_y$, that is, $-\Delta_y \varphi_k = \mu_k \varphi_k$ and $\varphi_0 = \mathrm{const}$.

We can write the following development:

$$w_j(x,y)=\sum_{k\in\mathbb{N}\setminus\{0\}} a_{j,k}(x)\varphi_k(y) \tag{3-36}$$

(where the eigenfunction φ_0 does not enter in the development). By using the representation in (3-36), we get

$$I_j \geq \sum_{k\neq 0}(\lambda_j-1)|\mu_k|^2\int_{\mathbb{R}^n_x}|a_{j,k}(x)|^2\,dx-(\alpha+1)\sum_{k\neq 0}\int_{\mathbb{R}^n_x}|u_{\bar\omega}(x)|^\alpha|a_{j,k}(x)|^2\,dx, \tag{3-37}$$

and by (3-23) we get

$$III_j=o(1)\|w_j\|^2_{L^2_{x,y}}. \tag{3-38}$$

By combining (3-37) with (3-38), we get

$$I_j+III_j\geq 0 \tag{3-39}$$

for j large enough. In order to estimate II_j, notice that, by the Cauchy–Schwartz inequality, we get

$$\begin{aligned}\left|\int_{M^k_y}\int_{\mathbb{R}^n_x}\sqrt{-\Delta_y}(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha-|u_{\lambda_j}|^\alpha))w_j\,dx\,d\mathrm{vol}_{M^k_y}\right|&\\ \leq \|\sqrt{-\Delta_y}(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha-|u_{\lambda_j}|^\alpha))\|_{L_x^{2(n+k)/(n+k+2)}L_y^{2(n+k)/(n+k+2)}}\|w_j\|_{L_{x,y}^{2(n+k)/(n+k-2)}}&\\ \leq C\|\nabla_y(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha-|u_{\lambda_j}|^\alpha))\|_{L_x^{2(n+k)/(n+k+2)}L_y^{2(n+k)/(n+k+2)}}\|w_j\|_{L_{x,y}^{2(n+k)/(n+k-2)}},&\end{aligned} \tag{3-40}$$

where in the last step we have used the following estimate: for all $p\in(1,\infty)$ there exist $c(p),C(p)>0$ such that

$$c(p)\|\sqrt{-\Delta_y}f\|_{L^p_y}\leq\|\nabla_y f\|_{L^p_y}\leq C(p)\|\sqrt{-\Delta_y}f\|_{L^p_y}. \tag{3-41}$$

Indeed, using [Sogge 1993, Theorem 3.3.1], we have that $\sqrt{-\Delta_y}$ is a first-order classical pseudodifferential operator on M with a principal symbol $(g^{i,j}(y)\xi_i\,\xi_j)^{1/2}$. Observe that

$$C_1\sum_{i,j}g^{i,j}(y)\xi_i\,\xi_j\leq\sum_i\left|\sum_j g^{i,j}(y)\xi_j\right|^2\leq C_2|\xi|^2\leq C_3\sum_{i,j}g^{i,j}(y)\xi_i\,\xi_j.$$

Moreover, one can assume that in (3-41) f has no zero frequency. Then one can deduce (3-41) by working in local coordinates, introducing a classical angular partition of unity according to the index $l\in[1,\dots,k]$ such that

$$\sum_{i,j}g^{i,j}(y)\xi_i\,\xi_j\leq c\left|\sum_j g^{l,j}(y)\xi_j\right|^2,$$

and, most importantly, using the L^p boundedness of zero-order pseudodifferential operators on $\mathbb{R}^k$ (for the proof of this fact we refer to [Sogge 1993, Theorem 3.1.6]).

Next, by the chain rule, we get

$$\nabla_y\big(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha-|u_{\lambda_j}|^\alpha)\big)=(\alpha+1)\nabla_y u_{\lambda_j}(|u_{\bar\omega}|^\alpha-|u_{\lambda_j}|^\alpha),$$

and by the Hölder inequality we can continue the estimate (3-40):

$$\cdots \le C \big\| \|\nabla_y u_{\lambda_j}\|_{L^q_y} \| |u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha \|_{L^r_y} \big\|_{L_x^{2(n+k)/(n+k+2)}} \|w_j\|_{L_{x,y}^{2(n+k)/(n+k-2)}},$$

where

$$\frac{1}{q}+\frac{1}{r}=\frac{n+k+2}{2(n+k)},$$

and, again by the Hölder inequality in the x-variable, we can continue

$$\cdots \le C \|\nabla_y u_{\lambda_j}\|_{L^q_{x,y}} \| |u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha \|_{L^r_{x,y}} \|w_j\|_{L_{x,y}^{2(n+k)/(n+k-2)}}.$$

Notice that if we fix

$$q=\frac{2(n+k)}{n+k-2} \quad \text{and} \quad r=\frac{n+k}{2},$$

then, by combining the Sobolev embedding

$$H^1_{x,y} \subset L^{2(n+k)/(n+k-2)}_{x,y} \tag{3-42}$$

with (3-32) and (3-41), we can continue the estimate:

$$\cdots \le o(1) \|\sqrt{-\Delta_y} u_{\lambda_j}\|_{L^q_{x,y}} \|w_j\|_{H^1_{x,y}} = o(1)\|w_j\|^2_{H^1_{x,y}},$$

where $\lim_{j\to\infty} o(1)=0$. By combining this information in conjunction with the structure of II_j, we get

$$II_j \ge \|w_j\|^2_{H^1_{x,y}}(1-o(1)) \ge 0 \quad \text{for } j > j_0. \tag{3-43}$$

By combining (3-35), (3-39), and (3-43), we deduce $w_j=0$ for j large enough. □

Proof of Theorem 3.1. By using the diamagnetic inequality, we deduce that (up to a remodulation factor $e^{i\theta}$) we can assume that $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$ is real valued. Moreover, if $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$, then also $|v| \in \mathcal{M}_{n,M^k,\alpha,\lambda}$. By a standard application of the strong maximum principle, we finally deduce that it is not restrictive to assume that $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$ and $v(x,y)>0$ for all $(x,y)\in \mathbb{R}^n_x \times M^k_y$.

First step: *there exists $\tilde\lambda>0$ such that for all $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$, $v(x,y)>0$ we have $\nabla_y v=0$ for all $\lambda>\tilde\lambda$.* Assume that the conclusion is false. Then there exists $\lambda_j\to\infty$ such that $u_{\lambda_j}(x,y)\in M_{n,M^k,\alpha,\lambda_j}$, $u_{\lambda_j}(x,y)>0$ and $\nabla_y u_{\lambda_j}\neq 0$. This is absurd due to Lemma 3.6.

Second step: *conclusion.* We define

$$\lambda^* = \inf_\lambda\{\lambda>0 : \nabla_y v = 0 \text{ for all } v\in \mathcal{M}_{n,M^k,\alpha,\lambda}\}.$$

By the first step, $\lambda^*<\infty$. Moreover, it is easy to deduce that if $\lambda>\lambda^*$, the minimizers of the problem $J_{n,M^k,\alpha,\lambda}$ are precisely the same minimizers as those of the problem $I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}$, which in turn are characterized in Section 2 (hence we get (3-2)).

Next we prove that $\lambda^*>0$. It is sufficient to show that

$$\lim_{\lambda\to 0} J_{n,M^k,\alpha,\lambda} < \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}} \tag{3-44}$$

(see (2-1) and (3-1) for a definition of the quantities involved in the inequality above). Let us fix $\rho(y) \in C^\infty(M^k)$ such that

$$\int_{M^k} |\rho|^2 \, d\mathrm{vol}_{M^k_y} = 1$$

and $\rho^2(y_0) \neq 1/\mathrm{vol}(M^k)$ for some $y_0 \in M^k$ (that is, $\rho(y)$ is not identically constant). Then we introduce the functions

$$\psi(x, y) = \rho(y)^{4/(4-\alpha n)} Q(\rho(y)^{\frac{2\alpha}{4-\alpha n}} x),$$

where $Q(x)$ is the unique radially symmetric minimizer for $I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}$. Then we get

$$\|\psi(x, y)\|^2_{L^2_x} = (\rho(y))^2 \quad \text{and} \quad \mathscr{E}_{n,\alpha}(\psi(x, y)) = I^1_{n,\alpha}(\rho(y))^{\frac{8+4\alpha-2\alpha n}{(4-\alpha n)}},$$

and, as a consequence, we deduce

$$\begin{aligned}\int_{M^k_y}\int_{\mathbb{R}^n_x}\left(\frac{1}{2}|\nabla_x\psi(x, y)|^2 - \frac{1}{2+\alpha}|\psi(x, y)|^{2+\alpha}\right) dx\, d\mathrm{vol}_{M^k_y} \\ = I^1_{n,\alpha}\int_{M^k_y}(\rho(y))^{\frac{8+4\alpha-2\alpha n}{4-\alpha n}}\, d\,\mathrm{vol}_{M^k_y} \\ < I^1_{n,\alpha}\left(\int_{M^k}(\rho(y))^2\, d\mathrm{vol}_{M^k_y}\right)^{\frac{4-\alpha n+2\alpha}{4-\alpha n}} \mathrm{vol}(M^k)^{-\frac{2\alpha}{4-\alpha n}} = I^1_{n,\alpha}\,\mathrm{vol}(M^k)^{-\frac{2\alpha}{4-\alpha n}},\end{aligned}$$

where in the last inequality we have used the fact that $I^1_{n,\alpha} < 0$ in conjunction with the Hölder inequality (moreover, we get the inequality $<$ since by hypothesis $\rho(y)$ is not identically constant). As a byproduct we get

$$\lim_{\lambda\to 0} \mathscr{E}_{n,M^k,\alpha,\lambda}(\psi(x, y)) < I^1_{n,\alpha}\,\mathrm{vol}(M^k)^{-2\alpha/(4-\alpha n)} = \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}$$

(where we have used (2-3)), which in turn implies (3-44).

Let us finally prove (3-3). It is sufficient to show that if $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$ for $\lambda < \lambda^*$, then $\nabla_y v \neq 0$. Assume for a contradiction that this is false. Then we get $\lambda_1 < \lambda^*$ and $v_1 \in \mathcal{M}_{n,M^k,\alpha,\lambda_1}$ such that $\nabla_y v_1 = 0$. Arguing as above implies that

$$J_{n,M^k,\alpha,\lambda_1} = \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}. \tag{3-45}$$

On the other hand, by the definition of λ^*, there exists $\lambda_2 \in (\lambda_1, \lambda^*]$ and $v_2 \in \mathcal{M}_{n,M^k,\alpha,\lambda_2}$ such that $\nabla_y v_2 \neq 0$. As a consequence, we deduce that

$$J_{n,M^k,\alpha,\lambda_1} < \mathscr{E}_{n,M^k,\alpha,\lambda_2}(v_2) = J_{n,M^k,\alpha,\lambda_2} \le \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}},$$

where in the last step we have used (3-7). Hence we get a contradiction with (3-45). □

4. Proof of Theorem 1.3

The homogeneity of the euclidean space $\mathbb{R}^n$ will play a key role in the sequel. Due to this property we shall be able to reduce the proof of Theorem 1.3 to the problem studied in the previous section.

In view of Section 2 it is sufficient to prove that there exists $\rho^* > 0$ such that

$$v \in \mathcal{M}^{\rho}_{n,M^k,\alpha} \quad \text{implies} \quad \nabla_y v = 0 \text{ for } \rho < \rho^* \tag{4-1}$$

and

$$v \in \mathcal{M}^{\rho}_{n,M^k,\alpha} \quad \text{implies} \quad \nabla_y v \neq 0 \text{ for } \rho > \rho^*. \tag{4-2}$$

By an elementary computation, we have that the map

$$S_1 \ni u \to \rho^{4/(4-\alpha n)} u(\rho^{2\alpha/(4-\alpha n)} x, y) \in S_\rho,$$

where

$$S_\lambda = \{v \in H^1(\mathbb{R}^n \times M^k) : \|v\|_{L^2_{x,y}} = \lambda\}$$

is a bijection. Moreover, we have

$$\begin{aligned}
&\mathscr{E}_{n,M_k,\alpha}\big(\rho^{4/(4-\alpha n)} u(\rho^{2\alpha/(4-\alpha n)} x, y)\big) \\
&\quad = \rho^{(8-2\alpha n)/(4-\alpha n)} \int_{M^k_y} \int_{\mathbb{R}^n_x} |\nabla_y u|^2 \, dx \, d\mathrm{vol}_{M^k_y} + \rho^{(8-2\alpha n+4\alpha)/(4-\alpha n)} \int_{M^k_y} \int_{\mathbb{R}^n_x} |\nabla_x u|^2 \, dx \, d\mathrm{vol}_{M^k_y} \\
&\qquad\qquad - \rho^{(8-2\alpha n+4\alpha)/(4-\alpha n)} \frac{1}{2+\alpha} \int_{M^k_y} \int_{\mathbb{R}^n_x} |u|^{2+\alpha} \, dx \, d\mathrm{vol}_{M^k_y} \\
&\quad = \rho^{(8-2\alpha n+4\alpha)/(4-\alpha n)} \bigg(\frac{1}{2} \rho^{-4\alpha/(4-\alpha n)} \int_{M^k_y} \int_{\mathbb{R}^n_x} |\nabla_y u|^2 \, dx \, d\mathrm{vol}_{M^k_y} \\
&\qquad\qquad + \frac{1}{2} \int_{M^k_y} \int_{\mathbb{R}^n_x} |\nabla_x u|^2 - \frac{1}{2+\alpha} |u|^{2+4/d} \, dx \, d\mathrm{vol}_{M^k_y} \bigg).
\end{aligned}$$

In particular, (4-1) and (4-2) are satisfied provided that there exists $\rho^* > 0$ such that

$$v \in \mathcal{M}_{n,M^k,\alpha,\rho^{-4\alpha/(4-\alpha n)}} \quad \text{implies} \quad \nabla_y v = 0 \text{ for } \rho < \rho^* \tag{4-3}$$

and

$$v \in \mathcal{M}_{n,M^k,\alpha,\rho^{-4\alpha/(4-\alpha n)}} \quad \text{implies} \quad \nabla_y v \neq 0 \text{ for } \rho > \rho^*, \tag{4-4}$$

which in turn follow by Theorem 3.1.

5. Proof of Theorem 1.4

The main tool we use is the following Strichartz type estimate (whose proof follows by [Tzvetkov and Visciglia 2012]).

Proposition 5.1. *For every manifold M^k_y, $n \geq 1$ and $p, q \in [2, \infty]$ such that*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad (p, n) \neq (2, 2),$$

there exists $C>0$ such that

$$\|e^{-it\Delta_{x,y}}f\|_{L^p_tL^q_xH^1_y}+\left\|\int_0^t e^{-i(t-s)\Delta_{x,y}}F(s)\,ds\right\|_{L^p_tL^q_xH^1_y}\le C(\|f\|_{L^2_xH^1_y}+\|F\|_{L^{p'}_tL^{q'}_xH^1_y}),\tag{5-1}$$

$$\|\nabla_x e^{-it\Delta_{x,y}}f\|_{L^p_tL^q_xL^2_y}+\left\|\nabla_x\int_0^t e^{-i(t-s)\Delta_{x,y}}F(s)\,ds\right\|_{L^p_tL^q_xL^2_y}\le C(\|\nabla_x f\|_{L^2_xL^2_y}+\|\nabla_x F\|_{L^{p'}_tL^{q'}_xL^2_y}),\tag{5-2}$$

and

$$\left\|\int_0^t e^{-i(t-s)\Delta_{x,y}}F(s)\,ds\right\|_{L^p_tL^q_xL^2_y}\le C\|F\|_{L^{p'}_tL^{q'}_xL^2_y}.\tag{5-3}$$

Moreover,

$$\|e^{-it\Delta_{x,y}}f\|_{L^\infty_tL^2_xH^1_y}+\left\|\int_0^t e^{-i(t-s)\Delta_{x,y}}F(s)\,ds\right\|_{L^\infty_tL^2_xH^1_y}\le C(\|f\|_{L^2_xH^1_y}+\|F\|_{L^{p'}_tL^{q'}_xH^1_y})\tag{5-4}$$

and

$$\|\nabla_x e^{-it\Delta_{x,y}}f\|_{L^\infty_tL^2_xL^2_y}+\left\|\nabla_x\int_0^t e^{-i(t-s)\Delta_{x,y}}F(s)\,ds\right\|_{L^\infty_tL^2_xL^2_y}\le C(\|\nabla_x f\|_{L^2_xL^2_y}+\|\nabla_x F\|_{L^{p'}_tL^{q'}_xL^2_y}).\tag{5-5}$$

Next we shall use the norms $\|\cdot\|_{X_T}$ and $\|\cdot\|_{Y_T}$ introduced in (1-15) and (1-16) for time dependent functions. We also introduce the space Z_T whose norm is defined by

$$\|v\|_{Z_T}\equiv\|v\|_{X_T}+\|v\|_{Y_T}$$

and the nonlinear operator associated to the Cauchy problem (1-1):

$$\mathcal{T}_\varphi(u)\equiv e^{-it\Delta_{x,y}}\varphi+\int_0^t e^{-i(t-s)\Delta_{x,y}}u(s)|u(s)|^\alpha\,ds.$$

We split the proof of Theorem 1.4 in several steps.

5A. *Local well-posedness.* We devote this subsection to proving the following: for all $\varphi\in H^1(\mathbb{R}^n\times M^1)$ there exists a $T=T(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)})>0$ and there exists a unique $v(t,x)\in Z_T\cap\mathscr{C}((-T,T);H^1(\mathbb{R}^n\times M^1))$ such that $\mathcal{T}_\varphi v(t)=v(t)$ for all $t\in(-T,T)$

First step: *for all $\varphi\in H^1(\mathbb{R}^n\times M^1)$ there exist $T=T(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)})>0$, $R=R(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)})>0$ such that $\mathcal{T}_\varphi(B_{Z_{\widetilde{T}}}(0,R))\subset B_{Z_{\widetilde{T}}}(0,R)$ for all $\widetilde{T}<T$.* First we estimate the nonlinear term:

$$\|u|u|^\alpha\|_{L^{p'}_tL^{q'}_xH^1_y}\le\big\|\|u^\alpha(t,x,\cdot)\|_{L^\infty_y}\|u(t,x,\cdot)\|_{H^1_y}\big\|_{L^{p'}_tL^{q'}_x}$$

(where (p,q) is the couple in (1-15) and (1-16)). After applying the Hölder inequality in (t,x), we get

$$\cdots\le\|u\|_{L^p_tL^q_xH^1_y}\|u\|^\alpha_{L^{\alpha\tilde{p}}_tL^{\alpha\tilde{q}}_xL^\infty_y}\le C\|u\|_{L^p_tL^q_xH^1_y}\|u\|^\alpha_{L^{\alpha\tilde{p}}_tL^{\alpha\tilde{q}}_xH^1_y},$$

where we have used the embedding $H^1_y\subset L^\infty_y$ and we have chosen

$$\frac{1}{\tilde{p}}+\frac{1}{p}=1-\frac{1}{p}\quad\text{and}\quad\frac{1}{\tilde{q}}+\frac{1}{q}=1-\frac{1}{q}.$$

By direct computation we have

$$\alpha\tilde{q} = q \quad \text{and} \quad \alpha\tilde{p} < p. \tag{5-6}$$

By combining the nonlinear estimate above with (5-1), (5-6), and the Hölder inequality (in the time variable), we get

$$\|\mathcal{T}_\varphi u\|_{X_T} \le C(\|\varphi\|_{L^2_x H^1_y} + T^{a(d)}\|u\|^{1+\alpha}_{X_T}) \tag{5-7}$$

with $a(d) > 0$.

Arguing as above, we get

$$\|\nabla_x(u|u|^\alpha)\|_{L^{p'}_t L^{q'}_x L^2_y} \le C\|\nabla_x u\|_{L^p_t L^q_x L^2_y}\|u^\alpha\|_{L^{\tilde{p}}_t L^{\tilde{q}}_x L^\infty_y} \le C\|u\|_{Y_T}\|u\|^\alpha_{L^{\alpha\tilde{p}}_t L^{\alpha\tilde{q}}_x H^1_y},$$

where $\tilde{p}$ and $\tilde{q}$ are as above and we have used the embedding $H^1_y \subset L^\infty_y$. As a consequence of this estimate and (5-2), we get

$$\|\mathcal{T}_\varphi u\|_{Y_T} \le C(\|\nabla_x\varphi\|_{L^2_{x,y}} + T^{a(d)}\|u\|_{Y_T}\|u\|^\alpha_{X_T}) \tag{5-8}$$

with $a(d) > 0$.

By combining (5-7) with (5-8), we get

$$\|\mathcal{T}_\varphi u\|_{Z_T} \le C(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)} + T^{a(d)}\|u\|_{Z_T}\|u\|^\alpha_{Z_T}).$$

The proof follows by a standard continuity argument.

Next we introduce the norm

$$\|w(t,x,y)\|_{\widetilde{Z}_T} \equiv \|w(t,x,y)\|_{L^p((-T,T);L^q_x L^2_y)}.$$

and we shall prove the following.

Second step: *Let $T, R > 0$ as in the previous step. Then there exists $T' = T'(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)}) < T$ such that $\mathcal{T}_\varphi$ is a contraction on $B_{Z_{T'}}(0,R)$ endowed with the norm $\|\cdot\|_{\widetilde{Z}_{T'}}$.* It is sufficient to prove

$$\|\mathcal{T}_\varphi v_1 - \mathcal{T}_\varphi v_2\|_{\widetilde{Z}_T} \le CT^{a(d)}\|v_1 - v_2\|_{\widetilde{Z}_T}\sup_{i=1,2}\{\|v_i\|_{Z_T}\}^\alpha \tag{5-9}$$

with $a(d) > 0$. Notice that we have

$$\begin{aligned}\|v_1|v_1|^\alpha - v_2|v_2|^\alpha\|_{L^{p'}((-T,T);L^{q'}_x L^2_y)} &\le C\Big\|\|v_1 - v_2\|_{L^2_y}(\|v_1\|_{L^\infty_y} + \|v_2\|_{L^\infty_y})^\alpha\Big\|_{L^{p'}((-T,T);L^{q'}_x)}\\ &\le CT^{a(d)}\|v_1 - v_2\|_{\widetilde{Z}_T}\sup_{i=1,2}\{\|v_i\|_{Z_T}\}^\alpha,\end{aligned}$$

where we have used the Sobolev embedding $H^1_y \subset L^\infty_y$ and the Hölder inequality in the same spirit as in the proof of (5-7) and (5-8). We conclude by combining the estimate above with the Strichartz estimate (5-3).

Third step: *existence and uniqueness of the solution in $Z_{T'}$, where T' is as in the previous step.* We apply the contraction principle to the map $\mathcal{T}_\varphi$ defined on the complete space $B_{Z_{T'}}(0,R)$ endowed with the topology induced by $\|\cdot\|_{\widetilde{Z}_{T'}}$. It is well known that this space is complete.

Fourth step: *regularity of the solution*. By combining the previous steps with the fixed point argument, we get the existence of a solution $v \in Z_{T'}$. In order to get the regularity $v \in \mathscr{C}((-T', T'); H^1(\mathbb{R}^n \times M^1))$, it is sufficient to argue as in the first step (to estimate the nonlinearity) in conjugation with the Strichartz estimates (5-4) and (5-5).

5B. *Global well-posedness.* Next we prove that the local solution (whose existence has been proved above) cannot blow up in finite time. The argument is standard and follows from the conservation laws

$$\|u(t)\|_{L^2_{x,y}} \equiv \|\varphi\|_{L^2_{x,y}}, \tag{5-10}$$

$$\mathscr{E}_{n,M^1,\alpha}(u(t)) + \tfrac{1}{2}\|u(t)\|^2_{L^2_{x,y}} \equiv \mathscr{E}_{n,M^1,\alpha}(\varphi) + \tfrac{1}{2}\|\varphi\|^2_{L^2_{x,y}}, \tag{5-11}$$

where $\mathscr{E}_{n,M^1,\alpha}$ is defined in (1-2). By the Gagliardo–Nirenberg inequality we deduce

$$\mathscr{E}_{n,M^1,\alpha}(u(t)) + \tfrac{1}{2}\|u(t)\|^2_{L^2_{x,y}} \geq \tfrac{1}{2}\|u(t)\|^2_{H^1(\mathbb{R}^n \times M^1)} - C\|u(t)\|^{2+\alpha-\mu}_{L^2_{x,y}}\|u(t)\|^{\mu}_{H^1(\mathbb{R}^n \times M^1)}$$

for a suitable $\mu \in (0, 2)$. By combining the estimate above with (5-10) and (5-11), we get

$$\tfrac{1}{2}\|u(t)\|^2_{H^1(\mathbb{R}^n \times M^1)} - C\|\varphi\|^{2+\alpha-\mu}_{L^2_{x,y}}\|u(t)\|^{\mu}_{H^1(\mathbb{R}^n \times M^1)} \leq \mathscr{E}_{n,M^1,\alpha}(\varphi) + \tfrac{1}{2}\|\varphi\|^2_{L^2_{x,y}}.$$

Since $\mu \in (0, 2)$, it implies that $\|u(t)\|_{H^1(\mathbb{R}^n \times M^1)}$ cannot blow up in finite time.

Appendix

For the sake of completeness we prove in this appendix Theorems 1.1 and 1.2. Our argument is heavily inspired by [Cazenave and Lions 1982] even if, in our opinion, the following presentation of Theorem 1.1 is simpler compared with the original one.

Proof of Theorem 1.1. For any given $\rho > 0$ we shall denote by $u_{j,\rho} \in H^1(\mathbb{R}^n \times M^k)$ any constrained minimizing sequence, that is,

$$\|u_{j,\rho}\|_{L^2_{x,y}} = \rho \quad \text{and} \quad \lim_{j\to\infty} \mathscr{E}_{n,M^k,\alpha}(u_{j,\rho}) = K^{\rho}_{n,M^k,\alpha}. \tag{A-1}$$

Next we split the proof into many steps.

First step: $K^{\rho}_{n,M^k,\alpha} > -\infty$ *and* $\sup_j \|u_{j,\rho}\|_{H^1_{x,y}} < \infty$ *for all* $\rho > 0$. By the classical Gagliardo–Nirenberg inequality (see (1-4)) we get the existence of $\mu \in (0, 2)$ such that

$$\begin{aligned}\mathscr{E}_{n,M^k,\alpha}(u_{j,\rho}) + \tfrac{1}{2}\rho^2 &\geq \frac{1}{2}\int_{M^k_y}\int_{\mathbb{R}^n} (|\nabla_{x,y}u_{j,\rho}|^2 + |u_{j,\rho}|^2)\,dx\,d\mathrm{vol}M^k_y - C(\rho)\|u_{j,\rho}\|^{\mu}_{H^1(\mathbb{R}^m \times M^k)} \\ &\geq \inf_{t>0}(1/2t^2 - C(\rho)t^{\mu}) > -\infty.\end{aligned}$$

The conclusion follows by a standard argument.

Second step: *the map* $(0,\infty)\ni\rho\to K^{\rho}_{n,M^k,\alpha}$ *is continuous.* Fix $\rho\in(0,\infty)$ and let $\rho_j\to\rho$. Then we have

$$\begin{aligned}
K^{\rho_j}_{n,M^k,\rho} &\le \mathscr{E}_{n,M^k,\alpha}\Big(\frac{\rho_j}{\rho}u_{j,\rho}\Big)\\
&=\Big(\frac{\rho_j}{\rho}\Big)^2\Big(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho}\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\Big(\frac{\rho_j}{\rho}\Big)^{\alpha}\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)\\
&=\Big(\frac{\rho_j}{\rho}\Big)^2\Big(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho}\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)+\frac{1}{2+\alpha}\Big(\frac{\rho_j}{\rho}\Big)^2\Big(1-\Big(\frac{\rho_j}{\rho}\Big)^{\alpha}\Big)\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\\
&=\Big(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho}\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)+\Big(\Big(\frac{\rho_j}{\rho}\Big)^2-1\Big)\Big(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho}\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)\\
&\qquad+\frac{1}{2+\alpha}\Big(\frac{\rho_j}{\rho}\Big)^2\Big(1-\Big(\frac{\rho_j}{\rho}\Big)^{\alpha}\Big)\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}.
\end{aligned}$$

Since we are assuming that $\rho_j\to\rho$ and $\sup_n\|u_{j,\rho}\|_{H^1(\mathbb{R}^n\times M^k)}<\infty$ (see the first step), we get

$$\limsup_{j\to\infty}K^{\rho_j}_{n,M^k,\alpha}\le K^{\rho}_{n,M^k,\alpha}.$$

To prove the opposite inequality, let us fix $u_j\in H^1(\mathbb{R}^n\times M^k)$ such that

$$\|u_j\|_{L^2_{x,y}}=\rho_j\quad\text{and}\quad \mathscr{E}_{n,M^k,\alpha}(u_j)<K^{\rho_j}_{n,M^k,\alpha}+\frac{1}{j}. \tag{A-2}$$

By looking at the proof of the first step, we also deduce that u_j can be chosen in such a way that

$$\sup_j\|u_j\|_{H^1(\mathbb{R}^n\times M^k)}<\infty. \tag{A-3}$$

Then we can argue as above and we get

$$\begin{aligned}
K^{\rho}_{n,M^k,\alpha}&\le\mathscr{E}_{n,M^k,\alpha}\Big(\frac{\rho}{\rho_j}u_j\Big)=\Big(\frac{1}{2}\|\nabla_{x,y}u_j\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_j\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)\\
&\quad+\Big(\Big(\frac{\rho}{\rho_j}\Big)^2-1\Big)\Big(\frac{1}{2}\|\nabla_{x,y}u_j\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_j\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)+\frac{1}{2+\alpha}\Big(\frac{\rho}{\rho_j}\Big)^2\Big(1-\Big(\frac{\rho}{\rho_j}\Big)^{\alpha}\Big)\|u_j\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}.
\end{aligned}$$

By using (A-2), (A-3), and the assumption $\rho_j\to\rho$, we get

$$K^{\rho}_{n,M^k,\alpha}\le\liminf_{j\to\infty}K^{\rho_j}_{n,M^k,\alpha}.$$

Third step: *for every* $\rho>0$ *we have (up to subsequence)* $\inf_j\|u_{j,\rho}\|_{L^{2+\alpha}_{x,y}}>0$. It is sufficient to prove that $K^{\rho}_{n,M^k,\alpha}<0$. In fact, we have

$$K^{\rho}_{n,M^k,\alpha}\le\operatorname{vol}(M^k)\mathscr{E}_{n,\alpha}(u_{n,\omega,\alpha})=\operatorname{vol}(M^k)I^{\rho/\sqrt{\operatorname{vol}(M^k)}}_{n,\alpha}<0, \tag{A-4}$$

where $\mathscr{E}_{n,\alpha}$ is the energy defined in (2-2) and ω is chosen in such a way that $\|u_{n,\omega,\alpha}\|_{L^2_x}=\rho/\sqrt{\operatorname{vol}(M^k)}$. Notice that in (A-4) we have used (2-4) and (2-5).

Fourth step: *for any minimizing sequence $u_{j,\rho}$, there exists $\tau_j \in \mathbb{R}^n$ such that (up to subsequence) $u_{j,\rho}(x+\tau_j, y)$ has a weak limit $\bar{u} \neq 0$.* We have the localized Gagliardo–Nirenberg inequality:

$$\|v\|_{L^{2+4/(n+k)}_{x,y}} \leq C \sup_{x \in \mathbb{R}^n} (\|v\|_{L^2_{Q^n_x \times M^k}})^{2/(n+k+2)} \|v\|^{(n+k)/(n+k+2)}_{H^1(\mathbb{R}^n \times M^k)}, \tag{A-5}$$

where

$$Q^n_x = x + [0,1]^n \quad \text{for all } x \in \mathbb{R}^n.$$

The estimate above can be proved as follows (see [Lions 1984] for a similar argument on the flat space $\mathbb{R}^{d+k}$). We fix $x_h \in \mathbb{R}^n$ in such a way that $\bigcup_h Q^n_{x_h} = \mathbb{R}^n$ and $\text{meas}_n(Q^n_{x_i} \cap Q^n_{x_j}) = 0$ for $i \neq j$, where meas_n denotes the Lebesgue measure in $\mathbb{R}^n$. By the classical Gagliardo–Nirenberg inequality we get

$$\|v\|^{2+4/(n+k)}_{L^{2+4/(n+k)}_{Q^n_{x_h} \times M^k}} \leq C \|v\|^{4/(n+k)}_{L^2_{Q^n_{x_h} \times M^k}} \|v\|^2_{H^1(Q^n_{x_h} \times M^k)}.$$

The proof of (A-5) follows by taking the sum of the previous estimates on $h \in \mathbb{N}$.

Due to the boundedness of $u_{j,\rho}$ in $H^1(\mathbb{R}^m \times M^k)$ (see the first step), we deduce by (A-5) that

$$0 < \epsilon_0 = \inf_j \|u_{j,\rho}\|_{L^{2+4/(n+k)}_{x,y}} \leq C \sup_{x \in \mathbb{R}^n} \|u_{j,\rho}\|^{2/(n+k+2)}_{L^2_{Q^n_x \times M^k}} \tag{A-6}$$

(the left side above follows by combining the Hölder inequality with the third step). The proof can be concluded by the Rellich compactness theorem once we choose a sequence $\tau_j \in \mathbb{R}^n_x$ in such a way that

$$\inf_j \|u_{j,\rho}\|_{L^2_{Q^n_{\tau_j} \times M^k}} > 0$$

(the existence of such a sequence τ_j follows by (A-6)).

Fifth step: *the map $(0, \bar{\rho}) \ni \rho \to \rho^{-2} K^{\rho}_{n,M^k,\alpha}$ is strictly decreasing.* Let us fix $\rho_1 < \rho_2$ and u_{j,ρ_1} a minimizing sequence for $K^{\rho_1}_{n,M^k,\alpha}$. Then we have

$$\begin{aligned}
K^{\rho_2}_{n,M^k,\alpha} &\leq \mathscr{E}_{n,M^k,\alpha}\left(\frac{\rho_2}{\rho_1} u_{j,\rho_1}\right) \\
&= \left(\frac{\rho_2}{\rho_1}\right)^2 \left(\frac{1}{2}\|\nabla_{x,y} u_{j,\rho_1}\|^2_{L^2_{x,y}} - \frac{1}{2+\alpha}\left(\frac{\rho_2}{\rho_1}\right)^{\alpha} \|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\right) \\
&= \left(\frac{\rho_2}{\rho_1}\right)^2 \left(\frac{1}{2}\|\nabla_{x,y} u_{j,\rho_1}\|^2_{L^2_{x,y}} - \frac{1}{2+\alpha}\|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\right) + \frac{1}{2+\alpha}\left(\frac{\rho_2}{\rho_1}\right)^2 \left(1 - \left(\frac{\rho_2}{\rho_1}\right)^{\alpha}\right) \|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} \\
&\leq \left(\frac{\rho_2}{\rho_1}\right)^2 \left(\frac{1}{2}\|\nabla_{x,y} u_{j,\rho_1}\|^2_{L^2_{x,y}} - \frac{1}{2+\alpha}\|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\right) + \frac{1}{2+\alpha}\left(\frac{\rho_2}{\rho_1}\right)^2 \left(1 - \left(\frac{\rho_2}{\rho_1}\right)^{\alpha}\right) \inf_j \|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}.
\end{aligned}$$

By recalling (see the third step) that $\inf_j \|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} > 0$, we get

$$K^{\rho_2}_{n,M^k,\alpha} < \left(\frac{\rho_2}{\rho_1}\right)^2 K^{\rho_1}_{n,M^k,\alpha}.$$

Sixth step: *Let $\bar u$ be as in the fourth step. Then $\|\bar u\|_{L^2_{x,y}} = \rho$.* Up to a subsequence we get

$$u_{j,\rho}(x+\tau_j, y) \to \bar u(x, y) \neq 0 \quad \text{a.e. in } (x, y) \in \mathbb{R}^n_x \times M^k_y,$$

and hence, by the Brezis–Lieb lemma [1983], we get

$$\|u_{j,\rho}(x+\tau_j, y) - \bar u(x, y)\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} = \|u_{j,\rho}(x+\tau_j, y)\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} - \|\bar u(x, y)\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} + o(1). \tag{A-7}$$

Assume that $\|\bar u\|_{L^2_{x,y}} = \theta$. Our aim is to prove $\theta = \rho$. Since $\bar u \neq 0$, necessarily $\theta > 0$. Notice that since $L^2_{x,y}$ is a Hilbert space, we have

$$\rho^2 = \|u_{j,\rho}(x+\tau_j, y)\|^2_{L^2_{x,y}} = \|u_{j,\rho}(x+\tau_j, y) - \bar u(x, y)\|^2_{L^2_{x,y}} + \|\bar u(x, y)\|^2_{L^2_{x,y}} + o(1), \tag{A-8}$$

and hence

$$\|u_{j,\rho}(x+\tau_j, y) - \bar u(x, y)\|^2_{L^2_{x,y}} = \rho^2 - \theta^2 + o(1). \tag{A-9}$$

By a similar argument,

$$\begin{aligned}
&\int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_x(u_{j,\rho}(x+\tau_j, y)) - \nabla_x \bar u(x, y)|^2\, dx\, dy \\
&+\int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_y(u_{j,\rho}(x+\tau_j, y)) - \nabla_y \bar u(x, y)|^2\, dx\, d\mathrm{vol}_{M^k_y} + \int_{M^k_y}\int_{\mathbb{R}^n_x} (|\nabla_x \bar u(x, y)|^2 + |\nabla_y \bar u(x, y)|^2)\, dx\, d\mathrm{vol}_{M^k_y} \\
&\qquad = \int_{M^k_y}\int_{\mathbb{R}^n_x} (|\nabla_x(u_{j,\rho}(x+\tau_j, y)|^2 + |\nabla_y u_{j,\rho}(x+\tau_j, y)|^2)\, dx\, d\mathrm{vol}_{M^k_y} + o(1).
\end{aligned} \tag{A-10}$$

By combining (A-10) with (A-7), we get

$$K^\rho_{n,M^k,\alpha} = \lim_{j\to\infty} \mathscr{E}_{n,M^k,\alpha}(u_{j,\rho}(x+\tau_j, y)) = \lim_{j\to\infty} \mathscr{E}_{n,M^k,\alpha}(u_{j,\rho}(x+\tau_j, y) - \bar u(x, y)) + \mathscr{E}_{n,M^k,\alpha}(\bar u), \tag{A-11}$$

and we can continue the estimate as follows:

$$\cdots \geq K^{\sqrt{\rho^2-\theta^2}+o(1)}_{n,M^k,\alpha} + K^\theta_{n,M^k,\alpha},$$

where we have used (A-9). Hence, by using the second step, we get

$$K^\rho_{n,M^k,\alpha} \geq K^{\sqrt{\rho^2-\theta^2}}_{n,M^k,\alpha} + K^\theta_{n,M^k,\alpha}.$$

Assume that $\theta < \rho$. Then, by using the monotonicity proved in the fifth step, we get

$$K^\rho_{n,M^k,\alpha} > \frac{\rho^2-\theta^2}{\rho^2} K^\rho_{n,M^k,\alpha} + \frac{\theta^2}{\rho^2} K^\rho_{n,M^k,\alpha} = K^\rho_{n,M^k,\alpha},$$

and we have an absurdity. □

Proof of Theorem 1.2. Assume for a contradiction that the conclusion is false. Then there exists ρ and two sequences $\varphi_j \in H^1(\mathbb{R}^n \times M^k)$ and $t_j \in \mathbb{R}$ such that

$$\lim_{j\to\infty} \mathrm{dist}_{H^1(\mathbb{R}^n\times M^k)}(\varphi_j, \mathcal{M}^\rho_{n,M^k,\alpha}) = 0 \tag{A-12}$$

and

$$\liminf_{j\to\infty} \operatorname{dist}_{H^1(\mathbb{R}^n\times M^k)}(u_{\varphi_j}(t_j), \mathcal{M}^{\rho}_{n,M^k,\alpha}) > 0, \tag{A-13}$$

where u_{φ_j} is the solution to (1-1) with Cauchy data φ_j. By (A-12) we deduce the following information:

$$\lim_{j\to\infty} \|\varphi_j\|_{L^2_{x,y}} = \rho \quad \text{and} \quad \lim_{j\to\infty} \mathscr{E}_{n,M^k,\alpha}(\varphi_j) = K^{\rho}_{n,M^k,\alpha},$$

and hence, due to the conservation laws satisfied by solutions to (1-1), we get

$$\lim_{j\to\infty} \|u_{\varphi_j}(t_j)\|_{L^2_{x,y}} = \rho \quad \text{and} \quad \lim_{j\to\infty} \mathscr{E}_{n,M^k,\alpha}(u_{\varphi_j}(t_j)) = K^{\rho}_{n,M^k,\alpha}.$$

In turn, by an elementary computation, we get

$$\|\tilde{u}_j\|_{L^2_{x,y}} = \rho \quad \text{and} \quad \lim_{j\to\infty} \mathscr{E}_{n,M^k,\alpha}(\tilde{u}_j) = K^{\rho}_{n,M^k,\alpha}$$

(more precisely $\tilde{u}_j$ is a constrained minimizing sequence for $K^{\rho}_{n,M^k,\alpha}$), where

$$\tilde{u}_j = \rho \frac{u_{\varphi_j}(t_j)}{\|u_{\varphi_j}(t_j)\|_{L^2_{x,y}}}.$$

Moreover, by (A-13), it is easy to deduce

$$\liminf_{j\to\infty} \operatorname{dist}_{H^1(\mathbb{R}^n\times M^k)}(\tilde{u}_j, \mathcal{M}^{\rho}_{n,M^k,\alpha}) > 0,$$

which is in contradiction with the compactness of minimizing sequences for $K^{\rho}_{n,M^k,\alpha}$ from Theorem 1.1. □

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