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CONVEXITY OF AVERAGE OPERATORS FOR SUBSOLUTIONS TO SUBELLIPTIC EQUATIONS

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We study convexity properties of the average integral operators naturally associated with divergence-form second-order subelliptic operators $\mathscr{L}$ with nonnegative characteristic form. When $\mathscr{L}$ is the classical Laplace operator, these average operators are the usual average integrals over Euclidean spheres. In our subelliptic setting, the average operators are (weighted) integrals over the level sets

$$\partial\Omega_r(x) = \{y : \Gamma(x, y) = 1/r\}$$

of the fundamental solution $\Gamma(x, y)$ of $\mathscr{L}$. We shall obtain characterizations of the $\mathscr{L}$-subharmonic functions u (that is, the weak solutions to $-\mathscr{L}u \le 0$) in terms of the convexity (w.r.t. a power of r) of the average of u over $\partial\Omega_r(x)$, as a function of the radius r. Solid average operators will be considered as well. Our main tools are representation formulae of the (weak) derivatives of the average operators w.r.t. the radius. As applications, we shall obtain Poisson–Jensen and Bôcher type results for $\mathscr{L}$.

1. Introduction and main results

1A. ***Notation and definitions.*** Let u be a subharmonic function in an open set $\Omega \subseteq \mathbb{R}^N$, $N \ge 2$. Then, with fixed $x \in \Omega$, the map

$$\begin{gathered} m_r(u)(x) : (0, R(x)) \longrightarrow (-\infty, \infty), \\ r \mapsto m_r(u)(x) := \frac{1}{H^{N-1}(\partial B_r(x))} \int_{\partial B_r(x)} u(y)\, dH^{N-1}(y) \end{gathered} \tag{1-1}$$

is convex with respect to $\log r$ if $N = 2$, and $1/r^{N-2}$ if $N \ge 3$. In (1-1), $B_r(x)$ denotes the Euclidean ball of radius r and center x; $R(x)$ stands for $\sup\{r > 0 : B_r(x) \subset \Omega\}$; H^{N-1} is the Hausdorff $(N-1)$-dimensional measure in $\mathbb{R}^N$. This quite well-known classical result has many important consequences and applications; see [Armitage and Gardiner 2001, Section 3.5; Hayman and Kennedy 1976, Section 2.7; Hörmander 1994, Section 3.2]. Of these applications, we only mention the Hadamard three-circles theorem, the Liouville-type theorem for bounded above subharmonic functions in $\mathbb{R}^2$, the applications to the theory of Hardy spaces, and the Bôcher theorem for harmonic functions in punctured balls (see [Armitage and Gardiner 2001, Chapter 3], for example).

The aim of the present paper is to study analogous properties for some weighted average operators acting on subsolutions to

$$-\mathscr{L}u = 0 \quad \text{in } \Omega \subseteq \mathbb{R}^N,\ N \ge 3,$$

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where $\mathscr{L}$ is a linear second order PDO with nonnegative characteristic form. Precisely, the operators we are dealing with are of the form

$$\mathscr{L} := \sum_{i,j=1}^{N} \partial_{x_i}(a_{i,j}(x)\partial_{x_j}) = \operatorname{div}(A(x)\nabla),$$

where $\nabla = (\partial_{x_1}, \dots, \partial_{x_N})^T$ and $A(x) = (a_{i,j}(x))_{i,j}$ is a symmetric matrix with smooth entries that is nonnegative definite at any point $x \in \mathbb{R}^N$. In Section 2 we will precisely fix our hypotheses on $\mathscr{L}$. Here we only need to mention the crucial ones: $\mathscr{L}$ is not totally degenerate, hypoelliptic, and endowed with a fundamental solution

$$\Gamma : \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y\} \longrightarrow (0, \infty),$$

with pole at any point of the diagonal $\{x = y\}$ and vanishing at infinity. For example, besides the classical Laplace operator on $\mathbb{R}^N$ ($N \geq 3$), any sub-Laplacian operator on a stratified Lie group (with homogeneous dimension ≥ 3) enjoys all these hypotheses; see, for example, [Bonfiglioli et al. 2007].

The main objects of our investigation are the average operators on the level sets of Γ, that is, on the sets

$$\partial\Omega_r(x) = \{y \in \mathbb{R}^N : \Gamma(x, y) = 1/r\}, \quad x \in \mathbb{R}^N,\ r > 0,$$

together with their solid counterparts, the average operators on the sets

$$\Omega_r(x) = \{y \in \mathbb{R}^N : \Gamma(x, y) > 1/r\}, \quad x \in \mathbb{R}^N,\ r > 0.$$

We call $\partial\Omega_r(x)$ and $\Omega_r(x)$, respectively, the $\mathscr{L}$-sphere and the $\mathscr{L}$-ball with radius r and center x. Owing to Sard's theorem, since Γ is smooth (in view of the hypoellipticity of $\mathscr{L}$), any $\mathscr{L}$-sphere is an $(N-1)$-dimensional manifold of class C^∞, for almost every radius. (For simplicity, we assume this to be true for every positive radius.)

If $\Omega \subseteq \mathbb{R}^N$ is open, given an upper semicontinuous (u.s.c.) function $u : \Omega \to [-\infty, \infty)$, for any $\mathscr{L}$-ball $\Omega_r(x)$ with closure contained in $\Omega_r(x)$, we set[1]

$$m_r(u)(x) := \int_{\partial\Omega_r(x)} u(y) k(x, y)\, dH^{N-1}(y),$$
$$M_r^\alpha(u)(x) := \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x)} u(y) K_\alpha(x, y)\, dy$$

for any $\alpha > -1$. Set $\Gamma_x := \Gamma(x, \cdot)$. The weights k, K_α are defined on $\mathbb{R}^N \setminus \{x\}$ by

$$k(x, \cdot) := \frac{|\nabla_{\mathscr{L}}\Gamma_x|^2}{|\nabla\Gamma_x|}, \quad K_\alpha(x, \cdot) := \frac{|\nabla_{\mathscr{L}}\Gamma_x|^2}{\Gamma_x^{2+\alpha}}, \tag{1-2}$$

where $|\nabla_{\mathscr{L}}\Gamma_x(y)|^2 := \langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\rangle$. The average operators m_r and M_r^α can be used to characterize the solutions to $\mathscr{L}u = v$. Indeed, for every $u \in C^2(\Omega, \mathbb{R})$, the following representation formulae

[1] Obviously, in order to define $m_r(u)(x)$, we only need to require that Ω contains $\partial\Omega_r(x)$.

hold true [Bonfiglioli and Lanconelli 2013, Section 11]:

$$\begin{aligned} u(x) &= m_r(u)(x) - \int_{\Omega_r(x)} \left(\Gamma(x, y) - \frac{1}{r}\right) \mathscr{L}u(y)\, dy, \\ u(x) &= M_r^\alpha(u)(x) - \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{\Omega_\rho(x)} \left(\Gamma(x, y) - \frac{1}{\rho}\right) \mathscr{L}u(y)\, dy \right) d\rho \end{aligned} \tag{1-3}$$

for every $\mathscr{L}$-ball $\Omega_r(x)$ with closure contained in Ω. Thus, given $x \in \Omega$, the above formula is satisfied for any positive r such that $r < R(x)$, where

$$R(x) := \sup\{r > 0 : \Omega_r(x) \subset \Omega\}. \tag{1-4}$$

For $u \equiv 1$, these formulae give

$$1 = m_r(1)(x) = M_r^\alpha(1)(x) \quad \text{for every } x \in \mathbb{R}^N \text{ and } r > 0.$$

Therefore, since the kernels k and K_α are nonnegative (recall that $A(y) \geq 0$), $m_r(u)(x)$ and $M_r^\alpha(u)(x)$ are well-posed (possibly $-\infty$) for every u.s.c. function u. (Actually, as was recently proved in [Abbondanza and Bonfiglioli 2013], $k(x, \cdot)$ and $K_\alpha(x, \cdot)$ are positive on an open dense subset of $\mathbb{R}^N \setminus \{x\}$ for every $x \in \mathbb{R}^N$.)

It is also worth noticing that

$$M_r^\alpha(u)(x) = \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha m_\rho(u)(x)\, d\rho. \tag{1-5}$$

This can be proved by using Federer's co-area formula and suitable approximation arguments for u.s.c. functions.

In what follows, given a u.s.c. function u on an open set $\Omega \subseteq \mathbb{R}^N$, we say that

(1) u is m-continuous in Ω if $u(x) = \lim_{r \to 0+} m_r(u)(x)$ for every $x \in \Omega$;

(2) u is M^α-continuous in Ω if $u(x) = \lim_{r \to 0+} M_r^\alpha(u)(x)$ for every $x \in \Omega$.

A smooth function u will be called $\mathscr{L}$-harmonic in Ω if $\mathscr{L}u = 0$ in Ω. We call a u.s.c. function $u : \Omega \to [-\infty, \infty)$ $\mathscr{L}$-subharmonic in Ω if

(1) the set $\Omega(u) := \{x \in \Omega : u(x) > -\infty\}$ contains at least one point of every connected component of Ω;

(2) for every bounded open set $V \subset \overline{V} \subset \Omega$ and for every $\mathscr{L}$-harmonic function h in V, continuous up to ∂V, $u \leq h$ holds whenever $u \leq h$ on ∂V.

The family of the $\mathscr{L}$-subharmonic functions in Ω is a cone denoted by $\underline{\mathscr{S}}(\Omega)$.

In [Bonfiglioli and Lanconelli 2013, Section 8] it is proved that u is $\mathscr{L}$-subharmonic in Ω if and only if $u \in L^1_{\text{loc}}(\Omega)$, $\mathscr{L}u \geq 0$ in the weak sense of distributions, and u is M^α-continuous in Ω. For this reason the $\mathscr{L}$-subharmonic functions are also said to be the subsolutions of $-\mathscr{L}$. As a consequence of the cited characterization, by the classical Riesz representation theorem, it follows that, given $u \in \underline{\mathscr{S}}(\Omega)$, there exists a nonnegative Radon measure μ_u on the Borel subsets of Ω (called the $\mathscr{L}$-Riesz measure of u) such that $\mathscr{L}u = \mu_u$ in Ω, in the weak sense of distributions.

Several other characterizations of the $\mathscr{L}$-subharmonicity have been provided in [Bonfiglioli and Lanconelli 2013] in terms of the average operators m_r and M_r^α. For our aim it is convenient to recall [Bonfiglioli and Lanconelli 2013, Theorem 4.2]; see also the notation in (1-4). Let $u : \Omega \to [-\infty, \infty)$ be a u.s.c. function such that $\Omega(u)$ contains at least one point of every connected component of Ω. Then $u \in \underline{\mathscr{S}}(\Omega)$ if and only if one of the following conditions is satisfied:

(A.1) $u(x) \le m_r(u)(x)$ for every $x \in \Omega$ and every $r \in (0, R(x))$;

(A.2) u is m-continuous in Ω and, for every $x \in \Omega$, $r \mapsto m_r(u)(x)$ is monotone nondecreasing on $(0, R(x))$.

One obtains further equivalent conditions by replacing, in (A.1) and (A.2), the surface average m_r with the solid average M_r^α, with $\alpha > -1$.

The following result will be used frequently in what follows.

Remark 1.1. By [Bonfiglioli and Lanconelli 2013, Proposition 6.10], if $u \in \underline{\mathscr{S}}(\Omega)$, the map $r \mapsto m_r(u)(x)$ is finite-valued and continuous on $(0, R(x))$ for every $x \in \Omega$. This follows from [Bonfiglioli and Lanconelli 2013, Theorem 6.4] and

$$m_r(\Gamma(\cdot\,, z))(x) = \min\{\Gamma(x, z), 1/r\} \tag{1-6}$$

jointly with a Riesz representation argument decomposing u, locally, as an $\mathscr{L}$-harmonic function plus the convolution of Γ with the Riesz measure of u. As a consequence, whenever $\alpha > 0$, the map $r \mapsto M_r^\alpha(u)(x)$ is finite-valued and continuous on $(0, R(x))$ for every $x \in \Omega$. This follows at once from (1-5) and (1-6), since $\rho^{\alpha-1}$ is integrable on $(0, r)$ for any positive α. The solid average $M_r^\alpha(u)(x)$ is finite-valued and continuous also when $-1 < \alpha \le 0$, provided that $x \in \Omega(u)$. To obtain this fact, it suffices to keep in mind identity (1-5) and the inequalities $-\infty < u(x) \le m_r(u)(x)$, valid for $x \in \Omega(u)$ and $0 < r < R(x)$.

In order to list the main results of this paper, we need a few more definitions. Let $I \subseteq \mathbb{R}$ be an interval and suppose that $\varphi : I \to \mathbb{R}$ is a strictly monotone continuous function. Following [Armitage and Gardiner 2001, Section 3.5], we say that $f : I \to \mathbb{R}$ is φ-convex if

$$f(r) \le \frac{\varphi(r_2) - \varphi(r)}{\varphi(r_2) - \varphi(r_1)} f(r_1) + \frac{\varphi(r) - \varphi(r_1)}{\varphi(r_2) - \varphi(r_1)} f(r_2) \tag{1-7}$$

for every $r_1, r, r_2 \in I$ such that $r_1 < r < r_2$. When $\varphi(r) = r$, (1-7) gives back the standard definition of a convex function. Moreover, clearly f is φ-convex if and only if $f \circ \varphi^{-1}$ is convex on the interval $\varphi(I)$, in the usual sense.

Finally, given a function $f : I \to \mathbb{R}$, we say that

(1) f is locally absolutely continuous (locally a.c.) if f is absolutely continuous on every compact subinterval of I;

(2) f is essentially monotone if there exists a monotone function $f^* : I \to \mathbb{R}$ such that $f = f^*$ almost everywhere in I.

1B. *Main theorems.* Our crucial results concern the derivative with respect to r of the average operators $m_r(u)(x)$ and $M_r^\alpha(u)(x)$, when u is $\mathscr{L}$-subharmonic. These are given in the following theorem.

Theorem 1.2 (derivatives of $m_r(u)$ and $M_r^\alpha(u)$). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let u be an $\mathscr{L}$-subharmonic function in Ω with $\mathscr{L}$-Riesz measure μ_u.*

(i) *For every $x \in \Omega$, the map $r \mapsto m_r(u)(x)$ is locally a.c. on $(0, R(x))$, and*

$$\frac{d}{dr} m_r(u)(x) = \frac{\mu_u(\Omega_r(x))}{r^2} \quad \text{for almost every } r \text{ in } (0, R(x)). \tag{1-8}$$

(ii) *For every $x \in \Omega$ and $\alpha > 0$, the map $r \mapsto M_r^\alpha(u)(x)$ is of class C^1 on $(0, R(x))$, and*

$$\frac{d}{dr} M_r^\alpha(u)(x) = \frac{\alpha+1}{r^{\alpha+2}} \int_{\Omega_r(x)} \left(f_\alpha(r) - f_\alpha\left(\frac{1}{\Gamma(x,y)}\right)\right) d\mu_u(y) \tag{1-9}$$

for every r in $(0, R(x))$, where f_α denotes an antiderivative of $r^{\alpha-1}$:

$$f_\alpha(r) := \begin{cases} \ln r, & \text{if } \alpha = 0, \\ r^\alpha/\alpha, & \text{if } \alpha \neq 0. \end{cases} \tag{1-10}$$

This also holds for $-1 < \alpha \le 0$ if $x \in \Omega(u)$.

A straightforward consequence of this theorem is the following corollary.

Corollary 1.3 (Poisson–Jensen type formula). *Let $u \in \underline{\mathcal{S}}(\Omega)$ and let μ_u be its $\mathscr{L}$-Riesz measure. The maps $r \mapsto m_r(u)(x)$ and $r \mapsto M_r^\alpha(u)(x)$ (for $\alpha > -1$) can be prolonged with continuity up to $r = 0$ if and only if $x \in \Omega(u)$.*

Furthermore, for every $x \in \Omega$ and $r \in (0, R(x))$, one has the following representation formulae (of Poisson–Jensen type):

$$u(x) = m_r(u)(x) - \int_0^r \frac{\mu_u(\Omega_\rho(x))}{\rho^2}\, d\rho = m_r(u)(x) - \int_{\Omega_r(x)} \left(\Gamma(x,y) - \frac{1}{r}\right) d\mu_u(y), \tag{1-11}$$

and, for $\alpha > 0$,

$$\begin{aligned} u(x) &= M_r^\alpha(u)(x) - \int_0^r \frac{\alpha+1}{\rho^{\alpha+2}} \left(\int_{\Omega_\rho(x)} \left(f_\alpha(\rho) - f_\alpha\left(\frac{1}{\Gamma(x,y)}\right)\right) d\mu_u(y)\right) d\rho \\ &= M_r^\alpha(u)(x) - \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{\Omega_\rho(x)} \left(\Gamma(x,y) - \frac{1}{\rho}\right) d\mu_u(y)\right) d\rho. \end{aligned} \tag{1-12}$$

When $x \notin \Omega(u)$, all the sides of the previous formulae (1-11) and (1-12) are $-\infty$, and this happens if and only if $\mu_u(\{x\}) > 0$.

Formula (1-12) holds true also for $-1 < \alpha \le 0$, provided that $x \in \Omega(u)$.

Theorem 1.2, together with the following real analysis lemma, easily implies convexity properties of our average operators, and these will characterize the $\mathscr{L}$-subharmonic functions.

Lemma 1.4. *Let $I = (0, a)$ be an interval in $(0, \infty)$, and let $f : I \to \mathbb{R}$.*

(i) *If f is bounded from above and $r^{-\beta}$-convex for a real $\beta > 0$, then f is monotone nondecreasing.*

(ii) *Let f be locally a.c., and let $\beta \neq 0$. Then f is $r^{-\beta}$-convex if and only if $r \mapsto r^{\beta+1} f'(r)$ is essentially monotone nondecreasing.*

Here are our main results concerning convexity of the average operators.

Theorem 1.5 (subharmonicity and convexity of the average operators). *Suppose that $\Omega \subseteq \mathbb{R}^N$ is an open set, and let $u : \Omega \to [-\infty, \infty)$ be an u.s.c. function such that $\Omega(u)$ intersects every connected component of Ω.*

Then the following statements are equivalent.

(1) $u \in \underline{\mathcal{S}}(\Omega)$.

(2) *u is m-continuous and the map $r \mapsto m_r(u)(x)$ is $1/r$-convex on $(0, R(x))$ for every $x \in \Omega$.*

(3) *u is m-continuous and the map $r \mapsto m_r(u)(x)$ is $1/r$-convex on $(0, R(x))$ for every $x \in \Omega(u)$.*

(4) *u is M^α-continuous and for every $x \in \Omega$, the map $r \mapsto M_r^\alpha(u)(x)$ is $1/r^{\alpha+1}$-convex on $(0, R(x))$ for some (or for every) $\alpha > 0$.*

(5) *u is M^α-continuous and, for every $x \in \Omega(u)$, the map $r \mapsto M_r^\alpha(u)(x)$ is $1/r^{\alpha+1}$-convex on $(0, R(x))$ for some (or for every) $\alpha > -1$.*

We observe that, to the best of our knowledge, the implications (2), (3), (4), (5) $\Rightarrow$ (1) appear here for the first time, even when $\mathcal{L}$ is the classical Laplace operator.

Moreover, we shall prove that (in statements (2), (3), (4), (5) above) we can replace r^{-1}-convexity or $r^{-(\alpha+1)}$-convexity with $r^{-\gamma}$-convexity for infinitely many other values of $\gamma > 0$ (see Theorems 5.1 and 5.2 for the precise statements).

We observe that the convexity (w.r.t. suitable powers of r) of the maps $r \mapsto m_r(u)(x)$, $M_r^\alpha(u)(x)$ in Theorem 1.5 ensures that these functions have more regularity properties than those provided so far in Theorem 1.2: by Alexandrov's theorem, they are *twice differentiable* almost everywhere on $(0, R(x))$.

1C. *Ring-shaped domains, applications, and further developments.* Suitable versions of Theorems 1.2 and 1.5 hold true for $\mathcal{L}$-subharmonic functions in ring-shaped domains. Given a, b such that $0 \le a < b \le \infty$, and given $x_0 \in \mathbb{R}^N$, we define the Γ-annulus of center x_0 and radii a, b as follows:

$$A_{a,b}(x_0) := \left\{ x \in \mathbb{R}^N : a < \frac{1}{\Gamma(x_0, x)} < b \right\}. \tag{1-13}$$

The conventions $1/\infty = 0$ and $1/0 = \infty$ apply.

The following results (Corollary 1.7 and Theorems 1.8 and 1.9) improve [Bonfiglioli and Lanconelli 2007, Theorems 1.5, 1.8, and 1.9], proved in the case of sub-Laplacians $\mathcal{L}$ on stratified groups.

Theorem 1.6. *Let $u \in \underline{\mathcal{S}}(A_{a,b}(x_0))$ and let μ_u be its $\mathcal{L}$-Riesz measure. The map*

$$(a, b) \ni r \mapsto m_r(u)(x_0) \in \mathbb{R}$$

is locally a.c. and $1/r$-convex. Moreover, for every fixed α, β such that $a < \alpha < \beta < b$, there exists a constant $c \in \mathbb{R}$ (depending on $a, \alpha, \beta, b, u, x_0$) such that

$$r^2 \frac{d}{dr} m_r(u)(x_0) = \mu_u(A_{\alpha,r}(x_0)) + c \tag{1-14}$$

for almost every r in (α, β).

From this theorem we obtain the following result.

Corollary 1.7. *Suppose u is $\mathscr{L}$-harmonic in the Γ-annulus $A_{a,b}(x_0)$. Then*

$$m_r(u)(x_0) = \frac{c_1}{r} + c_2, \quad r \in (a, b),$$

for some real constants c_1, c_2.

As an application of the previous results on $\mathscr{L}$-subharmonic functions on ring-shaped domains, we will show a symmetry result, from which a Bôcher-type theorem for $\mathscr{L}$ will follow. The latter improves a result in [Bonfiglioli and Lanconelli 2007].

For our application we need (together with the structural assumptions (H1) and (H2) in Section 2) the following extra assumption on $\mathscr{L}$, a homogeneous Harnack inequality on Γ-spheres.

(HH) For every fixed $x_0 \in \mathbb{R}^N$ and every $0 < b < \infty$, there exist positive constants $C = C(x_0, b) > 1$ and $\theta = \theta(x_0, b) < 1$ such that

$$\sup_{\partial\Omega_r(x_0)} h \le C \inf_{\partial\Omega_r(x_0)} h$$

for every r such that $0 < r < \theta b$ and every $\mathscr{L}$-harmonic nonnegative function h in the Γ-annulus $A_{0,b}(x_0)$.

By standard arguments (see, for example, [Bony 1969]), this hypothesis is satisfied for the sum of squares of Hörmander vector fields $\mathscr{L} = \sum_{j=1}^m X_j^2$. Moreover, (HH) is fulfilled for $x_0 = 0$, when $\mathscr{L}$ is homogeneous of positive degree (in the sense recalled in Remark 7.1) w.r.t. a group of dilations; see [Bonfiglioli et al. 2007, Theorem 5.16.5, page 327].

Theorem 1.8. *Suppose $\mathscr{L}$ satisfies condition (HH) above.*

Let w be nonnegative and $\mathscr{L}$-harmonic in the Γ-annulus $A_{0,b}(x_0) = \Omega_b(x_0) \setminus \{x_0\}$ (where $b < \infty$) and suppose that w is also continuous up to $\partial\Omega_b(x_0)$ and $w \equiv 0$ on $\partial\Omega_b(x_0)$. Then w is affine w.r.t. Γ, that is,

$$w(x) = c(\Gamma(x_0, x) - 1/b), \quad x \in A_{0,b}(x_0),$$

for some positive constant c.

We prove this theorem as a consequence of Corollary 1.7, by following an idea exploited by Axler, Bourdon, and Ramey [Axler et al. 1992] in the classical case of the Laplace operator. From Theorem 1.8 one easily obtains the following Bôcher-type result.

Theorem 1.9 (Bôcher's theorem for $\mathscr{L}$). *Suppose $\mathscr{L}$ satisfies condition (HH).*

Let $\Omega \subseteq \mathbb{R}^N$ be open and $x_0 \in \Omega$. Let u be nonnegative and $\mathscr{L}$-harmonic in $\Omega \setminus \{x_0\}$. Then there exists an $\mathscr{L}$-harmonic function h on Ω and a constant $c \geq 0$ such that

$$u(x) = c\Gamma(x_0, x) + h(x) \quad \textit{for every } x \in \Omega \setminus \{x_0\}.$$

Further developments. We end the introduction by pointing out further applications of the results of this paper: they can be used for the investigations of *convex functions in Carnot groups* (as introduced in [Danielli et al. 2003; Lu et al. 2004]). Indeed (see [Juutinen et al. 2007] for the relevant results), since the so-called v-convex functions on Carnot groups are characterized in terms of their $\mathscr{L}$-subharmonicity w.r.t. the family of the sub-Laplacians $\{\mathscr{L}\}$ (a class of operators comprised in our present paper), by Theorem 1.5 it turns out that v-convexity can be characterized by the usual (Euclidean) convexity of the family of the real-variable functions $\{r \mapsto m_r(u)(x)\}$ (or of $\{r \mapsto M_r^{\alpha}(u)(x)\}$), as the average operators vary with $\{\mathscr{L}\}$. The characterization of v-convexity in [Bonfiglioli and Lanconelli 2012] can also be exploited to further simplify the investigation.

Finally, since our results apply to any Hörmander sum of squares of vector fields, we can use our characterization of v-convexity in order to obtain a new notion of convexity in more general frameworks than the Carnot setting (for instance, in the framework of Hörmander vector fields), as was done by Magnani and Scienza [2012]. We plan to develop this topic in a forthcoming study.

2. Main assumptions on $\mathscr{L}$ and recalls on $r^{-\beta}$-convexity

2A. ***Assumptions on $\mathscr{L}$.*** Throughout the paper, we let

$$\mathscr{L} := \sum_{i,j=1}^{N} \partial_{x_i}(a_{i,j}(x)\partial_{x_j}) \tag{2-1}$$

be a linear second order PDO in $\mathbb{R}^N$, in divergence form, with C^∞ coefficients, such that the matrix $A(x) := (a_{i,j}(x))_{i,j \leq N}$ is symmetric and nonnegative definite at every point $x \in \mathbb{R}^N$. The operator $\mathscr{L}$ is self-adjoint and it is (possibly) degenerate elliptic. However, we always assume without further comments that $\mathscr{L}$ is not totally degenerate, that is, there exists $i \in \{1, \dots, N\}$ such that $a_{i,i}(x) > 0$ for every $x \in \mathbb{R}^N$. As is well-known, this ensures that $\mathscr{L}$ satisfies the weak maximum principle on every bounded open subset of $\mathbb{R}^N$.

Our main assumptions on $\mathscr{L}$ are as follows.

(H1) *$\mathscr{L}$ is a C^∞-hypoelliptic differential operator*, that is, for every open set $\Omega \subseteq \mathbb{R}^N$, and for every $f \in C^\infty(\Omega, \mathbb{R})$, if $u \in \mathscr{D}'(\Omega)$ is a solution of $\mathscr{L}u = f$ in the weak sense of distributions, u can be identified with a C^∞ function on Ω.

(H2) We assume that $\mathscr{L}$ is equipped with a *global fundamental solution*

$$\Gamma : D = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y\} \longrightarrow (0, \infty)$$

with the following properties:

(a) $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}^N) \cap C^\infty(D, \mathbb{R})$;

(b) for every fixed $x \in \mathbb{R}^N$, we have $\lim_{y\to x} \Gamma(x, y) = \infty$ and $\lim_{y\to\infty} \Gamma(x, y) = 0$;

(c) for every $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$ and every $x \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} \Gamma(x, y)\mathscr{L}\varphi(y)\, dy = -\varphi(x). \tag{2-2}$$

If $\Omega \subseteq \mathbb{R}^N$ is open, we say that u is $\mathscr{L}$-harmonic on Ω if $u \in C^\infty(\Omega, \mathbb{R})$ and $\mathscr{L}u = 0$ in Ω. A bounded open set $V \subset \mathbb{R}^N$ is said to be $\mathscr{L}$-regular if the following property is satisfied: for every $f \in C(\partial V, \mathbb{R})$, there exists a (unique) $\mathscr{L}$-harmonic function in V, denoted by H^V_f, satisfying $\lim_{y\to x} H^V_f(y) = f(x)$ for every $x \in \partial V$.

As described in [Bonfiglioli and Lanconelli 2013, Remark 2.2], $\mathscr{L}$ endows $\mathbb{R}^N$ with the structure of a $\mathfrak{S}^*$-harmonic space, in the sense of [Bonfiglioli et al. 2007, Definition 6.10.1]: this is a consequence of hypothesis (H1). As a very particular byproduct, we can use Bouligand's theorem to derive that the Γ-balls $\Omega_r(x)$ are $\mathscr{L}$-regular open sets (we shall use this last fact in the proof of Bôcher's Theorem 1.9).

2B. *Background results on $r^{-\beta}$-convexity.* Next we prove some results on φ-convexity, as introduced in Section 1. We begin by remarking that, obviously, given intervals $I, J \subseteq \mathbb{R}$ and given a function $\psi : J \to I$ which is monotone and continuous, a function $u : I \to \mathbb{R}$ is φ-convex on I if and only if $u \circ \psi$ is $(\varphi \circ \psi)$-convex on $\psi^{-1}(I)$. Another very simple lemma is in order.

Lemma 2.1. *Suppose $\beta \neq 0$. Let $I \subseteq (0, \infty)$ be an interval and let $u : I \to \mathbb{R}$. The following assertions are equivalent*:

(1) *$u(r)$ is $r^{-\beta}$-convex on I;*

(2) *$u(r^{-1/\beta})$ is convex on $\varphi(I)$, where $\varphi(r) = r^{-\beta}$;*

(3) *$r^\beta u(r)$ is r^β-convex on I.*

Proof. The equivalence of (1) and (2) follows from the remark preceding the lemma. Taking $\varphi(r) = 1/r^\beta$, a simple computation shows that (1-7) is equivalent to

$$r^\beta u(r) \le \frac{r_2^\beta - r^\beta}{r_2^\beta - r_1^\beta} r_1^\beta u(r_1) + \frac{r^\beta - r_1^\beta}{r_2^\beta - r_1^\beta} r_2^\beta u(r_2),$$

which is equivalent to the r^β-convexity of $r^\beta u(r)$. □

The following result will be crucial later.

Lemma 2.2. *Let $a > 0$. Suppose $f : (0, a) \to \mathbb{R}$ is bounded from above and $r^{-\beta}$-convex on $(0, a)$ for some $\beta > 0$. Then f is monotone nondecreasing.*

This lemma proves Lemma 1.4(i).

Proof. Let f be as in the assertion; by Lemma 2.1(2), $g(r) := f(r^{-1/\beta})$ is convex on $I := (a^{-\beta}, \infty)$. Since f is bounded from above on $(0, a)$, g is bounded from above on I. From elementary properties of convex functions, since I is unbounded, we infer that g is monotone nonincreasing on I; since $\beta > 0$, this means that f is monotone nondecreasing on $(0, a)$. □

We prove a condition for $r^{-\beta}$-convexity under a weak-differentiability assumption.

Lemma 2.3. *Suppose $\beta \neq 0$ and let $I \subseteq (0, \infty)$ be an open interval. Suppose that $u : I \to \mathbb{R}$ is a locally absolutely continuous function. Then u is $r^{-\beta}$-convex on I if and only if $r^{\beta+1} u'(r)$ is essentially monotone nondecreasing on I.*

This lemma proves Lemma 1.4(ii).

Proof. By Lemma 2.1(2), u is $r^{-\beta}$-convex if and only if $F(r) := u(r^{-1/\beta})$ is convex in its domain in the usual sense. On the other hand, since F is continuous, standard results (which we may omit) imply that F is convex if and only if F' is essentially nondecreasing. Summing up,

$$u \text{ is } r^{-\beta}\text{-convex if and only if } F' \text{ is essentially nondecreasing.} \tag{2-3}$$

In turn, F' is essentially nondecreasing if and only if the map $\rho \mapsto -\beta F'(\rho^{-\beta})$ is essentially nondecreasing on its domain. (Indeed, notice that if $\beta > 0$, then $-\beta < 0$ and $\rho^{-\beta}$ is decreasing; if $\beta < 0$, then $-\beta > 0$ and $\rho^{-\beta}$ is increasing.) Since

$$F'(r) = -\beta^{-1} r^{-(\beta+1)/\beta} u'(r^{-1/\beta}),$$

we get $-\beta F'(\rho^{-\beta}) = \rho^{\beta+1} u'(\rho)$. As a consequence, F' is essentially nondecreasing if and only this is true of $r^{\beta+1} u'(r)$, and this ends the proof, in view of (2-3). □

Convexity of a monotone C^2 function with respect to a power of r brings along convexity with respect to many other functions, as the following result shows.

Lemma 2.4. *Let $I \subseteq (0, \infty)$ be an open interval and suppose that $u : I \to \mathbb{R}$ is monotone nondecreasing and locally a.c. If u is $r^{-\gamma}$-convex on I, it is $r^{-\beta}$-convex of I for every $\beta \geq \gamma$.*

Proof. Suppose u is monotone nondecreasing, locally a.c., and $r^{-\gamma}$-convex on I. From Lemma 2.3, we know that $r^{\gamma+1} u'(r)$ is essentially nondecreasing on I. Since $u'(r) \geq 0$ almost everywhere on I, if $\beta \geq \gamma$, then $r^{\beta+1} u'(r) = r^{\beta-\gamma}(r^{\gamma+1} u'(r))$ is essentially nondecreasing as well. Again by Lemma 2.3, we deduce that u is $r^{-\beta}$-convex on I. □

We now investigate convexity properties of an average integral function.

Corollary 2.5. *Let $a > 0$ and $f : (0, a] \to \mathbb{R}$. Assume furthermore that $\alpha > -1$ and $r^\alpha f(r)$ is integrable on $(0, a)$. Let us consider the function*

$$F(r) = \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha f(\rho)\, d\rho, \qquad r \in (0, a].$$

(a) *If $\beta \neq 0$ and f is r^β-convex on $(0, a]$, the same is true of $F(r)$.*

(b) *Suppose that f is also continuous. Then $F(r)$ is $r^{-(\alpha+1)}$-convex on $(0, a]$ if and only if $f(r)$ is monotone nondecreasing.*

As $\alpha > -1$, note that the integrability of $r^\alpha f(r)$ is ensured, for example, whenever f is bounded on $(0, a)$ (for example, when f extends continuously on $[0, a]$).

Proof. We prove (a). Fix $r \in (0, a]$. The change of variable $\rho = rs$ gives $F(r) = (\alpha+1)\int_0^1 s^\alpha f(rs)\,ds$. Setting $r = t^{1/\beta}$, we have

$$F(t^{1/\beta}) = (\alpha+1)\int_0^1 s^\alpha f(t^{1/\beta}s)\,ds.$$

For every fixed $s \in [0, 1]$, the function $t \mapsto f(t^{1/\beta}s)$ is convex, since $f(t^{1/\beta}s) = f((ts^\beta)^{1/\beta})$, and since $r \mapsto f(r^{1/\beta})$ is convex by the assumption of r^β-convexity of f. This immediately gives the convexity of $F(t^{1/\beta})$, that is, the r^β-convexity of $F(r)$.

We finally prove (b). By Lemma 2.1(3) (with $\beta=\alpha+1$), $F(r)$ is $r^{-(\alpha+1)}$-convex if and only if $r^{\alpha+1}F(r)$ is $r^{\alpha+1}$-convex. In turn, this last condition is equivalent to the fact that the function $G(r) := r^{-\alpha}(r^{\alpha+1}F(r))'$ is nondecreasing, this time by applying Lemma 2.3 to $u(r) = r^{\alpha+1}F(r)$ and $\beta = -\alpha - 1$. Now, the fundamental theorem of integral calculus ensures that $G(r) = (\alpha+1)f(r)$, and this function is monotone nondecreasing if and only if the same is true of $f(r)$. □

3. Derivatives of the average operators in the C^2 case

In order to prove Theorem 1.2, we first need the derivatives of $r \mapsto m_r(u)(x)$ and $M_r^\alpha(u)(x)$ for u of class C^2. An approximation argument will eventually yield the weak derivatives in the $\mathscr{L}$-subharmonic case (see Section 4).

Proposition 3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in C^2(\Omega, \mathbb{R})$. For every fixed $x \in \Omega$, the functions*

$$(0, R(x)) \ni r \mapsto m_r(u)(x),\ M_r^\alpha(u)(x)$$

are differentiable and their derivatives are given by

$$\frac{d}{dr}m_r(u)(x) = \frac{1}{r^2}\int_{\Omega_r(x)} \mathscr{L}u(y)\,dy, \tag{3-1}$$

$$\frac{d}{dr}M_r^\alpha(u)(x) = \frac{\alpha+1}{r^{\alpha+2}}\int_{\Omega_r(x)}\left(f_\alpha(r) - f_\alpha\left(\frac{1}{\Gamma(x,y)}\right)\right)\mathscr{L}u(y)\,dy, \tag{3-2}$$

where f_α is an antiderivative of $r^{\alpha-1}$ on $(0,\infty)$ (see (1-10)).

Proof. We fix the notation in the statement of the proposition. From the first mean-value formula for $\mathscr{L}$ in (1-3), we get

$$\begin{aligned}
\frac{d}{dr}m_r(u)(x) &= \frac{d}{dr}\left(u(x) + \int_{\Omega_r(x)}\left(\Gamma_x - \frac{1}{r}\right)\mathscr{L}u\right) \qquad \text{(by the co-area formula)}\\
&= \frac{d}{dr}\int_0^r\left(\int_{t=1/\Gamma_x}\left(\Gamma_x - \frac{1}{r}\right)\mathscr{L}u\frac{dH^{N-1}}{|\nabla(1/\Gamma_x)|}\right)dt\\
&= \int_{r=1/\Gamma_x}\left(\Gamma_x - \frac{1}{r}\right)\mathscr{L}u\frac{dH^{N-1}}{|\nabla(1/\Gamma_x)|} + \int_0^r\left(\int_{t=1/\Gamma_x}\frac{1}{r^2}\mathscr{L}u\frac{dH^{N-1}}{|\nabla(1/\Gamma_x)|}\right)dt\\
&= \frac{1}{r^2}\int_{\Omega_r(x)}\mathscr{L}u
\end{aligned}$$

(the first integral is 0; we use the co-area formula again in the second one). We next prove (3-2). From the second mean-value formula for $\mathscr{L}$ (1-3), we get

$$\frac{d}{dr}M_r^\alpha(u)(x) = -\frac{(\alpha+1)^2}{r^{\alpha+2}}\int_0^r \rho^\alpha\Big(\int_{\Omega_\rho(x)}\Big(\Gamma_x-\frac{1}{\rho}\Big)\mathscr{L}u\Big)\,d\rho+\frac{\alpha+1}{r}\int_{\Omega_r(x)}\Big(\Gamma_x-\frac{1}{r}\Big)\mathscr{L}u =: -\mathrm{I}+\mathrm{II}.$$

By applying Fubini's theorem to the summand I we get

$$\mathrm{I}=\frac{(\alpha+1)^2}{r^{\alpha+2}}\int_{\Omega_r(x)}\mathscr{L}u(y)\Big(\int_{1/\Gamma(x,y)}^r\Big(\rho^\alpha\Gamma(x,y)-\rho^{\alpha-1}\Big)\,d\rho\Big)\,dy.$$

By recalling (1-10), since the inner integral in ρ is equal to

$$f_\alpha\Big(\frac{1}{\Gamma(x,y)}\Big)-f_\alpha(r)+\frac{r^{\alpha+1}}{\alpha+1}\Big(\Gamma(x,y)-\frac{1}{r^{\alpha+1}\Gamma^\alpha(x,y)}\Big),$$

we derive for $-\mathrm{I}+\mathrm{II}$ the expression

$$\begin{aligned}\frac{(\alpha+1)^2}{r^{\alpha+2}}\int_{\Omega_r(x)}\mathscr{L}u\Big(f_\alpha(r)-f_\alpha\Big(\frac{1}{\Gamma_x}\Big)\Big)-\frac{\alpha+1}{r}\int_{\Omega_r(x)}\mathscr{L}u\Big(\Gamma_x-\frac{1}{r^{\alpha+1}\Gamma_x^\alpha}\Big)+\frac{\alpha+1}{r}\int_{\Omega_r(x)}\Big(\Gamma_x-\frac{1}{r}\Big)\mathscr{L}u\\ =\frac{(\alpha+1)^2}{r^{\alpha+2}}\int_{\Omega_r(x)}\mathscr{L}u\Big(f_\alpha(r)-f_\alpha\Big(\frac{1}{\Gamma_x}\Big)\Big)+\frac{\alpha+1}{r}\int_{\Omega_r(x)}\mathscr{L}u\Big(\frac{1}{r^{\alpha+1}\Gamma_x^\alpha}-\frac{1}{r}\Big)\\ =\frac{\alpha+1}{r^{\alpha+2}}\int_{\Omega_r(x)}\mathscr{L}u\Big((\alpha+1)f_\alpha(r)-(\alpha+1)f_\alpha\Big(\frac{1}{\Gamma_x}\Big)+\frac{1}{\Gamma_x^\alpha}-r^\alpha\Big).\end{aligned}$$

Now, the inner term in parentheses is equal to

$$\begin{cases} f_\alpha(r)-f_\alpha\Big(\dfrac{1}{\Gamma_x}\Big), & \text{if } \alpha=0,\\ (\alpha+1)f_\alpha(r)-(\alpha+1)f_\alpha\Big(\dfrac{1}{\Gamma_x}\Big)+\alpha f_\alpha\Big(\dfrac{1}{\Gamma_x}\Big)-\alpha f_\alpha(r), & \text{if } \alpha\neq 0,\end{cases}$$

and, in turn, this equals $f_\alpha(r)-f_\alpha(1/\Gamma_x)$ after a cancelation in the formula for $\alpha\neq 0$. Because $(d/dr)M_r^\alpha(u)(x)=-\mathrm{I}+\mathrm{II}$, the proof is complete. □

Proposition 3.1 allows us to prove the needed characterization of the $\mathscr{L}$-subharmonicity in the C^2 case.

Proposition 3.2. *Let $\Omega\subseteq\mathbb{R}^N$ be an open set, and let $u\in C^2(\Omega,\mathbb{R})$. Then the following conditions are equivalent (here $\alpha>-1$).*

(1) *u is $\mathscr{L}$-subharmonic on Ω.*

(2) *$\mathscr{L}u\geq 0$ on Ω.*

(3) *For every $x\in\Omega$, the function $r\mapsto m_r(u)(x)$ is $1/r$-convex on $(0, R(x))$.*

(4) *For every $x\in\Omega$, the function $r\mapsto M_r^\alpha(u)(x)$ is $1/r^{\alpha+1}$-convex on $(0, R(x))$.*

The interval $(0, R(x))$ in (3) and (4) above can be replaced with $(0,\varepsilon(x))$ (for some $\varepsilon(x)>0$), that is, two other characterizations hold true:

(5) *for every $x\in\Omega$, there exists $0<\varepsilon(x)\leq R(x)$ such that the function $r\mapsto m_r(u)(x)$ is $1/r$-convex on $(0,\varepsilon(x))$;*

(6) *for every* $x \in \Omega$, *there exists* $0 < \varepsilon(x) \le R(x)$ *such that the function* $r \mapsto M_r^\alpha(u)(x)$ *is* $1/r^{\alpha+1}$*-convex on* $(0, \varepsilon(x))$.

Proof. Owing to the submean characterizations of the $\mathcal{L}$-subharmonicity recalled in Section 1, $u \in \underline{\mathcal{S}}(\Omega)$ if and only if $u(x) \le m_r(u)(x)$ for every $x \in \Omega$ and every $r \in (0, R(x))$. When u is C^2, due to the representation formula (1-3), this is clearly equivalent to $\mathcal{L}u \ge 0$ on Ω (recall that $\Gamma(x, y) - 1/r$ is positive on $\Omega_r(x)$). This proves the equivalence of conditions (1) and (2) above.

We now prove the equivalence of conditions (2) and (3). Since $m_r(u)(x)$ is differentiable w.r.t. r (see Proposition 3.1), by Lemma 2.3 we obtain that condition (3) holds true if and only if the function

$$F(r) := r^2 \frac{d}{dr} m_r(u)(x)$$

is monotone nondecreasing on $(0, R(x))$. By (3-1), we have $F(r) = \int_{\Omega_r(x)} \mathcal{L}u$, and this function is nondecreasing if and only if $\mathcal{L}u \ge 0$ (indeed, recall that $\Omega_r(x)$ shrinks to $\{x\}$ as $r \to 0$). This shows the equivalence of (2) and (3).

The equivalence of (2) and (4) can be proved analogously, by showing that

$$F_\alpha(r) := r^{\alpha+2} \frac{d}{dr} M_r^\alpha(u)(x)$$

is monotone nondecreasing on $(0, R(x))$, this time by using (3-2) (and the fact that f_α is strictly increasing *for every* α; see (1-10)).

Obviously, condition (3) implies condition (5), and (4) implies (6).

Finally, we prove that conditions (5) and (6) imply condition (2). Suppose by contradiction that $\mathcal{L}u(x) < 0$ at some point $x \in \Omega$, and hence on some neighborhood $U \subset \Omega$ of x. Due to our hypothesis (H2)(b) on the fundamental solution Γ, we can choose $r_2 > 0$ so small that $r_2 < \varepsilon(x)$ and such that $\overline{\Omega_{r_2}(x)} \subset U$. If r_1 is any positive number less than r_2, we derive that $F(r_2) < F(r_1)$ and $F_\alpha(r_2) < F_\alpha(r_1)$, with the notations above for F and F_α. This shows that conditions (5) and (6) cannot be true, since they are equivalent to the nondecreasing monotonicity on $(0, \varepsilon(x))$ of F and F_α, respectively (by Lemma 2.3). This ends the proof. □

Remark 3.3. We observe that the equivalence "(2) ⇔ (4)" may also be proved as follows, without the aid of formula (3-2). By (3-1), condition (2) holds true if and only if $m_r(u)(x)$ is nondecreasing w.r.t. r on $(0, R(x))$; now we can apply Corollary 2.5(b), which ensures that this last condition is satisfied if and only if

$$r \mapsto \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha m_\rho(u)(x)\, d\rho \text{ is } \frac{1}{r^{\alpha+1}}\text{-convex on } (0, R(x)).$$

Owing to (1-5), this last assertion is nothing but condition (4).

4. Weak derivatives of the average operators of $u \in \underline{\mathcal{S}}(\Omega)$

Our next task is to prove analogues of (3-1) and (3-2) (in the sense of weak derivatives) *for arbitrary* $\mathcal{L}$*-subharmonic functions*. To this end, we need to recall that the $\mathcal{L}$-Riesz measure μ_u of u is characterized by the identity

$$\int_\Omega u(x)\mathcal{L}\varphi(x)\,dx = \int_\Omega \varphi(x)\,d\mu_u(x) \quad \text{for every } \varphi \in C_0^\infty(\Omega, \mathbb{R}). \tag{4-1}$$

We notice that, fixing a positive r, the average operators $m_r(u)(x)$ and $M_r^\alpha(u)(x)$ are well posed, as functions of the center x, for any $x \in \Omega^r$, where

$$\Omega^r := \{x \in \Omega : \overline{\Omega_r(x)} \subset \Omega\}, \tag{4-2}$$

if this set is nonempty. By our hypothesis (H2)(b) on the fundamental solution Γ, it is easy to see that $\Omega^\varepsilon \uparrow \Omega$ as $\varepsilon \downarrow 0$. Moreover, it is not difficult to prove that

$$\text{for every compact set } K \subset \Omega, \text{ there exists } \varepsilon > 0 \text{ such that } K \subset \Omega^\varepsilon. \tag{4-3}$$

We are ready to give the following keystone result, whose proof is quite delicate.

Theorem 4.1 (derivatives of $m_r(u)$ and $M_r^\alpha(u)$). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in \underline{\mathcal{S}}(\Omega)$ with $\mathcal{L}$-Riesz measure μ_u on Ω. Finally let $x \in \Omega$ be fixed.*

(i) *The function $r \mapsto m_r(u)(x)$ is locally absolutely continuous, hence it is almost everywhere differentiable and its weak derivative (coinciding with its derivative at the points where the latter exists) is given by*

$$\frac{d}{dr} m_r(u)(x) = \frac{\mu_u(\Omega_r(x))}{r^2}. \tag{4-4}$$

Moreover, $m_r(u)(x)$ can be prolonged with continuity at $r = 0$ if and only if $x \in \Omega(u)$, and in this case one has, for every $r \in [0, R(x))$,

$$\begin{aligned} m_r(u)(x) &= u(x) + \int_0^r \frac{\mu_u(\Omega_\rho(x))}{\rho^2}\,d\rho \\ &= u(x) + \int_{\Omega_r(x)} \left(\Gamma(x,y) - \frac{1}{r}\right) d\mu_u(y). \end{aligned} \tag{4-5}$$

(ii) *Let $\alpha > 0$. The function $r \mapsto M_r^\alpha(u)(x)$ is of class C^1 on $(0, R(x))$; its derivative is*

$$\frac{d}{dr} M_r^\alpha(u)(x) = \frac{\alpha+1}{r^{\alpha+2}} \int_{\Omega_r(x)} \left(f_\alpha(r) - f_\alpha\left(\frac{1}{\Gamma(x,y)}\right)\right) d\mu_u(y), \tag{4-6}$$

where f_α is as in (1-10). Moreover, $M_r^\alpha(u)(x)$ can be prolonged with continuity at $r = 0$ if and only if $x \in \Omega(u)$, and in this case one has, for $r \in [0, R(x))$,

$$\begin{aligned} M_r^\alpha(u)(x) &= u(x) + \int_0^r \frac{\alpha+1}{\rho^{\alpha+2}} \left(\int_{\Omega_\rho(x)} \left(f_\alpha(\rho) - f_\alpha\left(\frac{1}{\Gamma(x,y)}\right)\right) d\mu_u(y)\right) d\rho \\ &= u(x) + \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{\Omega_\rho(x)} \left(\Gamma(x,y) - \frac{1}{\rho}\right) d\mu_u(y)\right) d\rho. \end{aligned} \tag{4-7}$$

(iii) *The same result as in* (ii) *holds true also for $-1 < \alpha \le 0$, provided that $x \in \Omega(u)$ (in which case (4-7) is also satisfied).*

We observe that Theorem 4.1 proves Theorem 1.2.

Remark 4.2. We remark that, for $\alpha > 0$ and $x \in \Omega$ (and for $-1 < \alpha \le 0$, provided that $x \in \Omega(u)$), (4-5) and (4-7) produce the representation formulae

$$u(x) = m_r(u)(x) - \int_{\Omega_r(x)} \left(\Gamma(x, y) - \frac{1}{r}\right) d\mu_u(y),$$
$$u(x) = M_r^\alpha(u)(x) - \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{\Omega_\rho(x)} \left(\Gamma(x, y) - \frac{1}{\rho}\right) d\mu_u(y) \right) d\rho.$$

This demonstrates Corollary 1.3. The above formulae are the analogues, of *Poisson–Jensen* type, of the representation formulae (1-3).

Proof of Theorem 4.1. Let us fix $\varepsilon > 0$. Given $u \in \underline{\mathcal{S}}(\Omega)$, by the smoothing result in [Bonfiglioli and Lanconelli 2013, Theorem 7.1] (requiring the C^∞-hypoellipticity of $\mathcal{L}$), there exists a nonincreasing sequence u_n of smooth $\mathcal{L}$-subharmonic functions on the set Ω^ε (see (4-2)) converging point-wise to u on Ω^ε. Given $x \in \Omega^\varepsilon$, if we set

$$R^\varepsilon(x) := \sup\{r > 0 : \Omega_r(x) \subseteq \Omega^\varepsilon\},$$

then $\lim_{\varepsilon\to 0^+} R^\varepsilon(x) = R(x)$ holds. This is a direct consequence of (4-3).

Hence, the theorem is proved if we show that, for any given $x \in \Omega^\varepsilon$, the functions of $r \in [0, R^\varepsilon(x))$ given by $m_r(u)(x)$ and $M_r^\alpha(u)(x)$ are locally a.c. on $(0, R^\varepsilon(x))$, and that their weak derivatives are given by (4-4) and (4-6).

Since $u_n \in C^\infty(\Omega^\varepsilon, \mathbb{R})$, from (3-1) we have

$$\frac{d}{dr} m_r(u_n)(x) = \frac{1}{r^2} \int_{\Omega_r(x)} \mathcal{L} u_n$$

for every $x \in \Omega^\varepsilon$ and every $r \in (0, R^\varepsilon(x))$. Let $\psi(r)$ be a smooth function compactly supported in $(0, R^\varepsilon(x))$; we multiply both sides of the above equality by $\psi(r)$, we integrate with respect to $r \in (0, R^\varepsilon(x))$, and we use integration by parts in the left-hand side, thus getting

$$\int \psi'(r) m_r(u_n)(x)\, dr = \int \psi(r) \left(\frac{1}{r^2} \int_{\Omega_r(x)} \mathcal{L} u_n \right) dr. \tag{4-8}$$

We aim to let $n \to \infty$ in this identity. To begin with, we claim that

$$\lim_{n\to\infty} \int \psi'(r) m_r(u_n)(x)\, dr = \int \psi'(r) m_r(u)(x)\, dr. \tag{4-9}$$

To prove this claim, we observe that, by arguing as in the proof of (5-2), we have

$$\lim_{n\to\infty} m_r(u_n)(x) = m_r(u)(x) \quad \text{for all } x \in \Omega^\varepsilon, r \in (0, R^\varepsilon(x)). \tag{4-10}$$

As a consequence of (4-10), (4-9) holds true if we prove that, in the left-hand side of (4-9), it is possible to apply the dominated convergence theorem. This is indeed possible as a direct consequence of the bounds

$$u \le u_n \le u_1 \Longrightarrow -\infty < m_r(u)(x) \le m_r(u_n)(x) \le m_r(u_1)(x) < \infty.$$

We next investigate the right-hand side of (4-8). If we denote by $[a, b]$ the support of ψ (recall that $0 < a < b < R^\varepsilon(x)$), by Fubini's theorem we have

$$\begin{aligned}\int \psi(r)\Big(\frac{1}{r^2}\int_{\Omega_r(x)} \mathscr{L}u_n\Big)\,dr &= \int \psi(r)\Big(\frac{1}{r^2}\int_{\Gamma(x,y)>1/r} \mathscr{L}u_n(y)\,dy\Big)\,dr\\ &= \int_{\Omega_b(x)} \mathscr{L}u_n(y)\Big(\int_{\max\{1/\Gamma(x,y),a\}}^b \psi(r)\frac{dr}{r^2}\Big)\,dy =: \int_{\Omega_b(x)} \mathscr{L}u_n(y)\Psi(y)\,dy.\end{aligned}$$

Now the function Ψ is supported in $\Omega_b(x)$, it is identically equal to the constant function $\int_a^b \psi(r)\,dr/r^2$ on $\Omega_a(x)$, and it is smooth because $\Gamma(x,\,\cdot)$ is smooth outside x. Hence we can integrate by parts two times to derive

$$\int \psi(r)\Big(\frac{1}{r^2}\int_{\Omega_r(x)} \mathscr{L}u_n\Big)\,dr = \int u_n(y)\mathscr{L}\Psi(y)\,dy.$$

From $u \le u_n \le u_1$ we get $|u_n| \le \max\{|u|, |u_1|\}$; hence, by recalling that $\mathscr{L}$-subharmonic functions are locally integrable [Negrini and Scornazzani 1987], and by observing that $\mathscr{L}\Psi \in C_0^\infty(\Omega^\varepsilon)$, a dominated convergence argument finally proves that

$$\lim_{n\to\infty} \int \psi(r)\Big(\frac{1}{r^2}\int_{\Omega_r(x)} \mathscr{L}u_n\Big)\,dr = \int u(y)\mathscr{L}\Psi(y)\,dy \overset{(4\text{-}1)}{=} \int \Psi(y)\,d\mu_u(y).$$

On the other hand, again by Fubini's theorem, we infer that

$$\begin{aligned}\int \Psi(y)\,d\mu_u(y) &= \int_{\Omega_b(x)} \Big(\int_{\max\{1/\Gamma(x,y),a\}}^b \psi(r)\frac{dr}{r^2}\Big)\,d\mu_u(y)\\ &= \int \psi(r)\Big(\frac{1}{r^2}\int_{\Omega_r(x)} d\mu_u(y)\Big)\,dr = \int \psi(r)\frac{\mu_u(\Omega_r(x))}{r^2}\,dr.\end{aligned}$$

Summing up, we have proved that

$$\lim_{n\to\infty} \int \psi(r)\Big(\frac{1}{r^2}\int_{\Omega_r(x)} \mathscr{L}u_n\Big)\,dr = \int \psi(r)\frac{\mu_u(\Omega_r(x))}{r^2}\,dr. \tag{4-11}$$

Gathering together (4-9) and (4-11), from (4-8) we derive

$$\int \psi'(r)m_r(u)(x)\,dr = \int \psi(r)\frac{\mu_u(\Omega_r(x))}{r^2}\,dr.$$

From the arbitrariness of $\psi \in C_0^\infty((0, R^\varepsilon(x)), \mathbb{R})$, this shows that $m_r(u)(x)$ possesses a weak derivative on $(0, R^\varepsilon(x))$, and this is equal to $\mu_u(\Omega_r(x))/r^2$. From the arbitrariness of $\varepsilon > 0$, we infer that $m_r(u)(x)$ is weakly differentiable on (α, β) for every α, β such that $0 < \alpha < \beta < R(x)$, and its weak derivative is $\mu_u(\Omega_r(x))/r^2$. Note that this function is integrable on (α, β), since

$$\int_\alpha^\beta \frac{\mu_u(\Omega_r(x))}{r^2}\,dr \le \frac{\mu_u(\overline{\Omega_\beta(x)})}{\alpha^2}(\beta-\alpha) < \infty,$$

the last inequality following from the finiteness of μ_u on the compact subsets of Ω.

This proves that $m_r(u)(x)$ is equal almost everywhere to a continuous function on $(0, R(x))$, say $m(r)$, and $m(r)$ is locally a.c. on $(0, R(x))$, with weak derivative given by $\mu_u(\Omega_r(x))/r^2$; since absolutely continuous functions are almost everywhere differentiable, we also get $m'(r) = \mu_u(\Omega_r(x))/r^2$ for almost every $r \in (0, R(x))$. Moreover, since absolutely continuous functions satisfy the fundamental theorem of calculus, we also have

$$m(r) = m(r_1) + \int_r^{r_2} \frac{1}{\rho^2} \mu_u(\Omega_\rho(x))\, d\rho,$$

whenever $0 < r_1 < r < R(x)$.

As $m_r(u)(x)$ is monotone (see (A.2) in Section 1), it can be equal almost everywhere to $m(r)$ (which is a continuous function) only if $m_r(u)(x) = m(r)$ for every $r \in (0, R(x))$. Thus $m_r(u)(x)$ inherits all the above properties of $m(r)$. In particular, whenever $0 < r_1 < r < R(x)$, we get

$$m_r(u)(x) - m_{r_1}(u)(x) = \int_{r_1}^{r} \frac{1}{\rho^2} \mu_u(\Omega_\rho(x))\, d\rho.$$

Letting $r_1 \to 0^+$, by Beppo Levi's theorem and by exploiting the m-continuity of $\mathscr{L}$-subharmonic functions (see property (A.2)), we obtain

$$m_r(u)(x) - u(x) = \int_0^r \frac{1}{\rho^2} \mu_u(\Omega_\rho(x))\, d\rho,$$

where both sides are $+\infty$ if and only if $u(x) = -\infty$ (recall that $m_r(u)(x)$ is always finite). Otherwise, when $x \in \Omega(u)$ both sides are finite, and we get the first formula in (4-5). In this latter case, we derive that $\mu_u(\Omega_\rho(x))/\rho^2$ is integrable on every compact subinterval of $[0, R(x))$, so that the function $r \mapsto m_r(u)(x)$ (defined as $u(x)$ when $r = 0$) is locally a.c. on $[0, R(x))$. The second formula in (4-5) can be obtained by Tonelli's theorem, since

$$\begin{aligned}\int_0^r \frac{1}{\rho^2} \mu_u(\Omega_\rho(x))\, d\rho &= \int_0^r \frac{1}{\rho^2} \left(\int_{1/\Gamma(x,y) < \rho} d\mu_u(y) \right) d\rho \\ &= \int_{1/\Gamma(x,y) < r} \left(\int_{1/\Gamma(x,y)}^{r} \frac{1}{\rho^2}\, d\rho \right) d\mu_u(y) = \int_{\Omega_r(x)} \left(\Gamma(x, y) - \frac{1}{r} \right) d\mu_u(y).\end{aligned}$$

This completes the proof of the theorem where surface average operators are concerned. The case of solid average operators can be proved analogously, this time starting from (3-2), and by recalling that $M_r^\alpha(u)(x)$ is always finite if $\alpha > 0$, and it is finite for $-1 < \alpha \le 0$ if $x \in \Omega(u)$.

Note that the fact that $M_r^\alpha(u)(x)$ is of class C^1 is a consequence of identity (1-5), together with the continuity of $m_r(u)(x)$ up to $r = 0$ (when $x \in \Omega(u)$). The fact that the two formulae in (4-7) are equivalent to one another can be proved by direct computations, by taking into account that

$$\int_0^r \left(\int_{\Omega_\rho(x)} g(\rho, y)\, d\mu_u(y) \right) d\rho = \int_{\Omega_r(x)} \left(\int_{1/\Gamma(x,y)}^{r} g(\rho, y)\, d\rho \right) d\mu_u(y)$$

for every integrable function $g(\rho, y)$. □

5. Subharmonicity and convexity of the average operators

We are ready to give the proof of Theorem 1.5. We highlight the fact that, over the course of this section, we shall provide finer versions of Theorem 1.5, namely, Theorems 5.1 and 5.2 below.

Proof of Theorem 1.5. We split the proof into six short parts.

(1) ⇒ (2). If $u \in \underline{\mathcal{S}}(\Omega)$ and $x \in \Omega$, by Theorem 4.1, $r \mapsto m(r) := m_r(u)(x)$ is locally a.c. on $(0, R(x))$ and, due to identity (4-4), one has (for almost every $r \in (0, R(x))$)

$$r^2 m'(r) = \mu_u(\Omega_r(x)),$$

the latter being a nondecreasing function of r. This shows that $r^2 m'(r)$ is essentially monotone nondecreasing on $(0, R(x))$. By Lemma 2.3 (for $\beta = 1$) we see that $m(r)$ is r^{-1}-convex. Finally, the m-continuity of u is contained in (A.2). This proves statement (2) of the theorem.

(2) ⇒ (3). This is obvious.

(3) ⇒ (1). Let $x \in \Omega(u)$. By the assumption (3), the map $r \mapsto m(r) := m_r(u)(x)$ is r^{-1}-convex on $(0, R(x))$. On the other hand, for $0 < r \le a < R(x)$, one has

$$m(r) \le \sup\{u(y) : y \in \overline{\Omega_a(x)}\} < \infty,$$

due to $m_r(1)(x) = 1$, the upper semicontinuity of u, and the compactness of $\overline{\Omega_a(x)}$. Thus $m(r)$ is bounded from above on $(0, a)$ for every positive $a < R(x)$. An application of Lemma 2.2 (for $\beta = 1$) shows that $m(r)$ is monotone nondecreasing on $(0, R(x))$. Since u is m-continuous by assumption (3), this gives

$$u(x) = \lim_{r \to 0^+} m_r(u)(x) \le m_r(u)(x) \quad \text{for all } x \in \Omega(u), r \in (0, R(x)).$$

On the other hand, the inequality $u(x) \le m_r(u)(x)$ is trivially satisfied when $x \notin \Omega(u)$ (because this means that $u(x) = -\infty$). Therefore, one has $u(x) \le m_r(u)(x)$ for every $r \in (0, R(x))$ and every $x \in \Omega$. By the characterization (A.1) of the $\mathcal{L}$-subharmonicity, we deduce that $u \in \underline{\mathcal{S}}(\Omega)$.

(1) ⇒ (4). Let $\alpha > 0$. If $u \in \underline{\mathcal{S}}(\Omega)$ and $x \in \Omega$, by Theorem 4.1, the function $r \mapsto M(r) := M_r^\alpha(u)(x)$ is C^1 on $(0, R(x))$ and, due to identity (4-6), one has

$$r^{\alpha+2} M'(r) = (\alpha+1) \int_{\Omega_r(x)} \left(f_\alpha(r) - f_\alpha\left(\frac{1}{\Gamma(x, y)} \right) \right) d\mu_u(y),$$

where f_α is as in (1-10). Note that the function in the right-hand side is nondecreasing w.r.t. r, because this is true of f_α (and $r > 1/\Gamma(x, y)$ on $\Omega_r(x)$). This shows that $r^{\alpha+2} M'(r)$ is monotone nondecreasing on $(0, R(x))$. An application of Lemma 2.3 (for $\beta = \alpha + 1$) proves that $M(r)$ is $r^{-(\alpha+1)}$-convex. Finally, the M^α-continuity of u is contained in (A.2) (with m_r replaced with M_r^α). This proves statement (4) of the theorem.

(4) ⇒ (1). Suppose there exists $\alpha > 0$ such that $r \mapsto M_r^\alpha(u)(x)$ is $r^{-(\alpha+1)}$-convex on $(0, R(x))$ for every $x \in \Omega$. By arguing as in the above proof of "(3) ⇒ (1)", an application of Lemma 2.2 (for $\beta = \alpha + 1$) shows that $M_r^\alpha(u)(x)$ is monotone nondecreasing on $(0, R(x))$. Since u is M^α-continuous by assumption

(4), we get (see the above argument) $u(x) \le M_r^\alpha(u)(x)$ for every $x \in \Omega$ and $r \in (0, R(x))$. By the characterization (A.1) of the $\mathscr{L}$-subharmonicity (with m_r replaced with M_r^α), we deduce that $u \in \underline{\mathcal{S}}(\Omega)$.

(1) $\Leftrightarrow$ (5). This can be proved by using similar arguments as above (this time invoking identity (4-6) for $\alpha \in (-1, 0]$ and $x \in \Omega(u)$; note that f_α is increasing also for nonpositive values of α; see (1-10)). □

We next turn to proving a more refined versions of the implications (1) ⇒ (2), (3), (4), (5) of Theorem 1.5.

Theorem 5.1 (subharmonicity implies convexity of the average operators). *Suppose that $\Omega \subseteq \mathbb{R}^N$ is an open set, and let $u \in \underline{\mathcal{S}}(\Omega)$. Then we have the following.*

1. *For every $x \in \Omega$ the average operator $m_r(u)(x)$ is $1/r$-convex on $(0, R(x))$; furthermore, it is $1/r^\beta$-convex also for $\beta \ge 1$.*
2. *When $\alpha > 0$, for every $x \in \Omega$, the average operator $M_r^\alpha(u)(x)$ is $1/r^{\alpha+1}$-convex on $(0, R(x))$; furthermore, it is $1/r^\beta$-convex also for $\beta \ge 1$.*
3. *When $-1 < \alpha \le 0$, for every $x \in \Omega(u)$, the average operator $M_r^\alpha(u)(x)$ is $1/r^{\alpha+1}$-convex on $(0, R(x))$; furthermore, it is $1/r^\beta$-convex also for $\beta \ge \alpha + 1$.*

Proof. Let us fix $\varepsilon > 0$. Given $u \in \underline{\mathcal{S}}(\Omega)$, by the smoothing result in [Bonfiglioli and Lanconelli 2013, Theorem 7.1] (recall that we assumed $\mathscr{L}$ to be C^∞-hypoelliptic), there exists a nonincreasing sequence u_n of smooth $\mathscr{L}$-subharmonic functions on the set Ω^ε (see (4-2)) converging point-wise to u on Ω^ε. Given $x \in \Omega^\varepsilon$, if we set

$$R^\varepsilon(x) := \sup\{r > 0 : \Omega_r(x) \subseteq \Omega^\varepsilon\},$$

then $\lim_{\varepsilon \to 0^+} R^\varepsilon(x) = R(x)$ holds. (This is a direct consequence of (4-3).)

Hence, the theorem is proved if we show that, for any given $x \in \Omega^\varepsilon$, the functions of $r \in (0, R^\varepsilon(x))$ given by $m_r(u)(x)$ and $M_r^\alpha(u)(x)$ are $r^{-\beta}$-convex, respectively, for $\beta \ge 1$ and for $\beta \ge \min\{1, \alpha + 1\}$.

To this end, let us observe that, since $u_n \in \underline{\mathcal{S}}(\Omega^\varepsilon) \cap C^\infty(\Omega^\varepsilon, \mathbb{R})$, from Proposition 3.2 we know the following.

- $m_r(u_n)(x)$ is r^{-1}-convex on $(0, R^\varepsilon(x))$; since this function is smooth w.r.t. r and monotone nondecreasing (see (3-1) and recall that $\mathscr{L}u_n \ge 0$), by Lemma 2.4 we infer that it is also $r^{-\beta}$-convex for every $\beta \ge 1$.
- $M_r^\alpha(u_n)(x)$ is $r^{-(\alpha+1)}$-convex on $(0, R^\varepsilon(x))$; from the r^{-1}-convexity of the surface mean $m_r(u_n)(x)$ we derive that $M_r^\alpha(u_n)(x)$ is also r^{-1}-convex, owing to Corollary 2.5(a); since $M_r^\alpha(u_n)(x)$ is smooth w.r.t. r and monotone nondecreasing (see (3-2)), by Lemma 2.4 we infer that it is also $r^{-\beta}$-convex for every $\beta \ge \min\{1, \alpha + 1\}$.

We now show that the above properties are inherited by $m_r(u)(x)$ and $M_r^\alpha(u)(x)$, by passing to the limit as $n \to \infty$. We prove it for solid average operators, the argument for surface average operators being completely analogous. Let $\beta \ge \min\{1, \alpha + 1\}$. We know that (setting $\varphi(r) = r^{-\beta}$)

$$M_r^\alpha(u_n)(x) \le \frac{\varphi(r_2) - \varphi(r)}{\varphi(r_2) - \varphi(r_1)} M_{r_1}^\alpha(u_n)(x) + \frac{\varphi(r) - \varphi(r_1)}{\varphi(r_2) - \varphi(r_1)} M_{r_2}^\alpha(u_n)(x) \tag{5-1}$$

for every $r_1, r, r_2 \in (0, R^\varepsilon(x))$ such that $r_1 < r < r_2$. We claim that

$$\lim_{n\to\infty} M_r^\alpha(u_n)(x) = M_r^\alpha(u)(x) \quad \text{for all } x \in \Omega^\varepsilon, r \in (0, R^\varepsilon(x)). \tag{5-2}$$

Once this claim is proved, letting $n \to \infty$ in (5-1), we get

$$M_r^\alpha(u)(x) \le \frac{\varphi(r_2)-\varphi(r)}{\varphi(r_2)-\varphi(r_1)} M_{r_1}^\alpha(u)(x) + \frac{\varphi(r)-\varphi(r_1)}{\varphi(r_2)-\varphi(r_1)} M_{r_2}^\alpha(u)(x)$$

for every $r_1, r, r_2 \in (0, R^\varepsilon(x))$ such that $r_1 < r < r_2$. This is precisely what we aim to prove, that is, $M_r^\alpha(u)(x)$ is an $r^{-\beta}$-convex function of r on $(0, R^\varepsilon(x))$.

We finally turn to prove the claimed (5-2). We fix any $x \in \Omega^\varepsilon$ and any $r \in (0, R^\varepsilon(x))$. Let us consider the sequence v_n defined by

$$v_n(x) := u_1(x) - u_n(x), \qquad x \in \Omega^\varepsilon.$$

Since $\{u_n\}_n$ is monotone nonincreasing, we infer that $\{v_n\}_n$ is monotone nondecreasing and nonnegative. Moreover, by construction of u_n, we have $v_n \to u_1 - u$, as $n \to \infty$, point-wise on Ω^ε. As $K_\alpha \ge 0$ (see (1-2)), we are therefore entitled to apply the monotone convergence theorem to derive that

$$\lim_{n\to\infty} \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x)} v_n(y) K_\alpha(x,y)\,dy = \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x)} (u_1(y)-u(y)) K_\alpha(x,y)\,dy.$$

Recalling that $M_r^\alpha(u)(x)$ is finite valued (see Remark 1.1) for any $\alpha > 0$, and also for $-1 < \alpha \le 0$ provided that $x \in \Omega(u)$, we obtain the following identity from the above one (whenever $M_r^\alpha(u)(x) > -\infty$):

$$\lim_{n\to\infty} (M_r^\alpha(u_1)(x) - M_r^\alpha(u_n)(x)) = M_r^\alpha(u_1)(x) - M_r^\alpha(u)(x).$$

By canceling out $M_r^\alpha(u_1)(x)$ (when it is finite), we get (5-2) and the proof of statements (2) and (3) of the theorem is complete. The proof of (1) is analogous, taking into account that $m_r(u)(x)$ is always finite (see Remark 1.1). □

The next result provides the reverse implication of Theorem 5.1. Also, it proves refined versions of the implications (2), (3), (4), (5) ⇒ (1) of Theorem 1.5.

Theorem 5.2 (convexity of the average operators implies subharmonicity). *Suppose that $\Omega \subseteq \mathbb{R}^N$ is an open set and $\alpha > -1$. Let $u : \Omega \to [-\infty, \infty)$ be an u.s.c. function such that $\Omega(u)$ intersects every connected component of Ω.*

Then, any of the following conditions implies that u is $\mathscr{L}$-subharmonic in Ω:

(1) *u is m-continuous in Ω and, for every fixed $x \in \Omega(u)$, the average operator $m_r(u)(x)$ is $1/r^\gamma$-convex on $(0, R(x))$ for some $\gamma > 0$.*

(2) *u is M^α-continuous in Ω and, for every fixed $x \in \Omega(u)$, the average operator $M_r^\alpha(u)(x)$ is $1/r^\gamma$-convex on $(0, R(x))$ for some $\gamma > 0$.*

We explicitly point out that this result holds true for every sub-Laplacian $\mathscr{L}$ on any Carnot group of homogeneous dimension $Q > 2$, since $\mathscr{L}$ satisfies all the properties in Section 2 (see, for example, [Bonfiglioli et al. 2007]); hence, as a very special case, *Theorem 5.2 holds true for the classical Laplace operator* Δ *on* $\mathbb{R}^N$, *with* $N \geq 3$. This result seems to be new in the literature.

Proof. Since u is u.s.c., u is locally bounded from above. This ensures that, for every fixed $x \in \Omega$, $m_r(u)(x)$ and $M_r^\alpha(u)(x)$ are bounded from above on $(0, a]$ for every positive $a < R(x)$. If condition (1) of Theorem 5.2 holds true (respectively condition (2)), we can apply Lemma 2.2 to derive that, for every $x \in \Omega(u)$, the average operator $m_r(u)(x)$ (respectively $M_r^\alpha(u)(x)$) is monotone nondecreasing on $(0, a]$ for every $a < R(x)$. Hence it is nondecreasing on the whole of $(0, R(x))$. Since u is supposed to be m-continuous (respectively M^α-continuous), we infer that, for every $x \in \Omega(u)$, one has

$$u(x) = \lim_{r \to 0^+} m_r(u)(x) \leq m_r(u)(x) \quad \text{(respectively } u(x) = \lim_{r \to 0^+} M_r^\alpha(u)(x) \leq M_r^\alpha(u)(x))$$

for every $r \in (0, R(x))$. On the other hand, the inequality $u(x) \leq m_r(u)(x)$ (respectively $u(x) \leq M_r^\alpha(u)(x)$) is trivially satisfied when $x \notin \Omega(u)$ (since this means that $u(x) = -\infty$). Therefore, one has $u(x) \leq m_r(u)(x)$ (respectively $u(x) \leq M_r^\alpha(u)(x)$) for every $r \in (0, R(x))$ and every $x \in \Omega$. By the characterization (A.1) of the $\mathscr{L}$-subharmonicity (respectively the analogue of (A.1) with m_r replaced with M_r^α), we deduce that u is $\mathscr{L}$-subharmonic in Ω. □

6. The case of Γ-annuli

In this section we use the following notation: given a, b such that $0 \leq a < b \leq \infty$, and given $x_0 \in \mathbb{R}^N$, we set

$$A_{a,b}(x_0) := \left\{ x \in \mathbb{R}^N : \frac{1}{b} < \Gamma(x_0, x) < \frac{1}{a} \right\} \tag{6-1}$$

(with the convention that $1/\infty = 0$ and $1/0 = \infty$), and we say that $A_{a,b}(x_0)$ is the Γ-*annulus* of center x_0 and radii a, b. The notation $A_{a,b}$ will apply instead of $A_{a,b}(x_0)$ whenever x_0 is understood. Our main task is to prove the following result, from which applications will be derived in Section 7.

Theorem 6.1. *Let* $0 \leq a < b \leq \infty$ *and* $x_0 \in \mathbb{R}^N$ *be fixed. Suppose* u *is* $\mathscr{L}$*-subharmonic on* $A_{a,b}(x_0)$. *Then the function*

$$(a, b) \ni r \mapsto m_r(u)(x_0)$$

is r^{-1}*-convex and locally absolutely continuous on* (a, b). *For every* α, β *such that* $a < \alpha < \beta < b$, *there exists* c (*depending on* $a, \alpha, \beta, b, u, x_0$) *such that the* (*weak*) *derivative of* $m_r(u)(x_0)$ *on* (α, β) *is given by*

$$\frac{d}{dr} m_r(u)(x_0) = \frac{1}{r^2}(\mu_u(A_{\alpha,r}(x_0)) + c) \tag{6-2}$$

for almost every $r \in (\alpha, \beta)$. *As usual,* μ_u *is the* $\mathscr{L}$*-Riesz measure of* u *on* $A_{a,b}(x_0)$.

This proves Theorem 1.6.

Remark 6.2. We cannot expect that analogues of Theorems 5.1 and 5.2 will hold true in the case of Γ-annuli, since, in the case of a Γ-annulus

- $\mathscr{L}$-subharmonicity does not necessarily imply $r^{-\beta}$-convexity, when $\beta > 1$;

- solid α-means are not well-posed;
- $m_r(u)(x_0)$ is not necessarily monotone nondecreasing.

See Remark 6.5 at the end of the section for related results (and a converse of Theorem 6.1 for C^2 functions which are "radial" with respect to Γ).

In order to prove Theorem 6.1, we need a substitute for identity (3-1). This is given in the next result.

Lemma 6.3. *Let $x_0 \in \mathbb{R}^N$ and $0 \le R_1 < R_2 \le \infty$ be fixed, and suppose that $u \in C^2(A_{R_1,R_2}(x_0), \mathbb{R})$. Given any $R \in (R_1, R_2)$, one has*

$$\frac{d}{dr}\Big|_{r=R} m_r(u)(x_0) = \frac{1}{R^2}\left(\int_{\partial\Omega_\rho(x_0)} \langle A\nabla u, \nu\rangle \, dH^{N-1} + \int_{A_{\rho,R}(x_0)} \mathscr{L}u\right) \tag{6-3}$$

for every $\rho \in (R_1, R)$. In particular, we have

$$r_2^2 \frac{d}{dr}\Big|_{r=r_2} m_r(u)(x_0) - r_1^2 \frac{d}{dr}\Big|_{r=r_1} m_r(u)(x_0) = \int_{A_{r_1,r_2}(x_0)} \mathscr{L}u \tag{6-4}$$

for every r_1, r_2 such that $R_1 < r_1 < r_2 < R_2$.

Proof. Let $\Omega \subset \mathbb{R}^N$ be any bounded open set whose boundary is regular enough to support the divergence theorem. The divergence form (2-1) of $\mathscr{L} = \operatorname{div}(A\nabla)$ gives

$$\int_\Omega (u\mathscr{L}v - v\mathscr{L}u) = \int_{\partial\Omega} (u\langle A\nabla v, \nu_{\text{est}}\rangle - v\langle A\nabla u, \nu_{\text{est}}\rangle) \, dH^{N-1} \tag{6-5}$$

for every $u, v \in C^2(\overline{\Omega}, \mathbb{R})$. Here ν_{est} denotes the exterior normal unit vector on $\partial\Omega$. Let $u \in C^2(A_{R_1,R_2}(x_0))$ and let us take any ρ, r such that $R_1 < \rho < r < R_2$. Choosing $\Omega = A_{\rho,r}(x_0)$ and $v = -\Gamma_{x_0}$, and since

$$\nu_{\text{est}}(x) = \begin{cases} -\nu(x) := +\nabla\Gamma_{x_0}(x)/|\nabla\Gamma_{x_0}(x)|, & \text{if } x \in \partial\Omega_\rho(x_0), \\ +\nu(x) := -\nabla\Gamma_{x_0}(x)/|\nabla\Gamma_{x_0}(x)|, & \text{if } x \in \partial\Omega_r(x_0), \end{cases} \tag{6-6}$$

from (6-5) we derive (recalling that Γ_{x_0} is $\mathscr{L}$-harmonic on $\mathbb{R}^N \setminus \{x_0\}$)

$$\int_{A_{\rho,r}(x_0)} \Gamma_{x_0}\mathscr{L}u = m_r(u)(x_0) - m_\rho(u)(x_0) + \frac{1}{r} J_r(u)(x_0) - \frac{1}{\rho} J_\rho(u)(x_0), \tag{6-7}$$

where m_r is the usual surface average operator, while

$$J_R(u)(x_0) := \int_{\partial\Omega_R(x_0)} \langle A\nabla u, \nu\rangle \, dH^{N-1} \quad \text{for } R = r \text{ and } R = \rho, \tag{6-8}$$

and ν is as in (6-6) (note that ν is the normal unit vector on $\partial\Omega_R(x_0)$ which is exterior to the set $\Omega_R(x_0)$). If in (6-5) we take $v \equiv -1$ and $\Omega = A_{\rho,r}(x_0)$, we get

$$\int_{A_{\rho,r}(x_0)} \mathscr{L}u = J_r(u)(x_0) - J_\rho(u)(x_0). \tag{6-9}$$

We set $f(r) := m_r(u)(x_0)$ for brevity and we differentiate both sides of (6-7) w.r.t. r:

$$\frac{d}{dr}\int_{A_{\rho,r}(x_0)} \Gamma_{x_0}\mathscr{L}u = f'(r) - \frac{1}{r^2} J_r(u)(x_0) + \frac{1}{r}\frac{d}{dr} J_r(u)(x_0). \tag{6-10}$$

On the one hand, owing to the co-area formula, we have

$$\begin{aligned}
&\frac{d}{dr}\int_{A_{\rho,r}(x_0)}\Gamma_{x_0}\mathcal{L}u \\
&\quad=\frac{d}{dr}\int_{A_{\rho,r}(x_0)}\left(\frac{1}{r}+\Gamma_{x_0}-\frac{1}{r}\right)\mathcal{L}u \\
&\quad=-\frac{1}{r^2}\int_{A_{\rho,r}(x_0)}\mathcal{L}u+\frac{1}{r}\frac{d}{dr}\int_{A_{\rho,r}(x_0)}\mathcal{L}u+\frac{d}{dr}\int_{\rho}^{r}\left(\int_{1/\Gamma_{x_0}=t}\left(\Gamma_{x_0}-\frac{1}{r}\right)\mathcal{L}u\frac{dH^{N-1}}{|\nabla(1/\Gamma_{x_0})|}\right)dt \\
&\quad=-\frac{1}{r^2}\int_{A_{\rho,r}(x_0)}\mathcal{L}u+\frac{1}{r}\frac{d}{dr}\int_{A_{\rho,r}(x_0)}\mathcal{L}u \\
&\qquad\qquad+\int_{1/\Gamma_{x_0}=r}\left(\Gamma_{x_0}-\frac{1}{r}\right)\mathcal{L}u\frac{dH^{N-1}}{|\nabla(1/\Gamma_{x_0})|}+\int_{\rho}^{r}\left(\int_{1/\Gamma_{x_0}=t}\frac{1}{r^2}\mathcal{L}u\frac{dH^{N-1}}{|\nabla(1/\Gamma_{x_0})|}\right)dt.
\end{aligned}$$

The third summand is 0, while the fourth is the opposite of the first one. Thus

$$\frac{d}{dr}\int_{A_{\rho,r}(x_0)}\Gamma_{x_0}\mathcal{L}u=\frac{1}{r}\frac{d}{dr}\int_{A_{\rho,r}(x_0)}\mathcal{L}u\overset{(6\text{-}9)}{=}\frac{1}{r}\frac{d}{dr}J_r(u)(x_0).$$

This shows that the left-hand side of (6-10) and the last summand of its right-hand side are equal. Thus (6-10) is equivalent to

$$f'(r)=\frac{1}{r^2}J_r(u)(x_0).$$

Taking into consideration (6-9) again, we get

$$f'(r)=\frac{1}{r^2}\left(J_\rho(u)(x_0)+\int_{A_{\rho,r}(x_0)}\mathcal{L}u\right),\qquad R_1<\rho<r<R_2. \tag{6-11}$$

This proves (6-3). Equivalently, we also obtain that

$$r^2f'(r)=J_\rho(u)(x_0)+\int_{A_{\rho,r}(x_0)}\mathcal{L}u,\qquad R_1<\rho<r<R_2. \tag{6-12}$$

If r_1, r_2 are such that $R_1<r_1<r_2<R_2$, we can choose any ρ satisfying $R_1<\rho<r_1$. Taking $r=r_2$ in (6-12) and subtracting side by side what we get by taking $r=r_1$ in (6-12), we finally obtain

$$r_2^2f'(r_2)-r_1^2f'(r_1)=\int_{A_{\rho,r_2}(x_0)}\mathcal{L}u-\int_{A_{\rho,r_1}(x_0)}\mathcal{L}u=\int_{A_{r_1,r_2}(x_0)}\mathcal{L}u,$$

which is (6-4). □

We remark that, if $u\in C^2(\Omega_{R_2}(x_0),\mathbb{R})$, letting $\rho\to 0^+$ in (6-3), one gets back formula (3-1). Indeed,

$$\lim_{\rho\to 0^+}\int_{\partial\Omega_\rho(x_0)}\langle A\nabla u,\nu\rangle\,dH^{N-1}=0,$$

as it follows from the identity $\int_{\partial\Omega_\rho(x_0)}\langle A\nabla u,\nu\rangle\,dH^{N-1}=\int_{\partial\Omega_\rho(x_0)}\mathcal{L}u$ (a consequence of (6-5) taking $v\equiv -1$ and $\Omega=\Omega_\rho(x_0)$).

Proof of Theorem 6.1. First we observe that Theorem 6.1 holds true if, together with the other assumptions, u is of class C^2. Indeed, if u is C^2 and $\mathscr{L}$-subharmonic, we have $\mathscr{L}u \geq 0$ on $A_{a,b}$; thus (6-4) proves that $r^2(d/dr)(m_r(u)(x_0))$ is monotone nondecreasing on (a, b). Lemma 2.1(3) ensures that $m_r(u)(x_0)$ is r^{-1}-convex on (a, b) and that formula (6-4) holds true.

The general case of $u \in \underline{\mathscr{S}}(A_{a,b})$ can be proved by the very same approximation technique as in the proofs of Theorems 4.1 and 5.1. □

Remark 6.4. Another example of a convex function naturally associated to an $\mathscr{L}$-subharmonic function is

$$B(r) := \sup_{\partial\Omega_r(x_0)} u.$$

Indeed, let us prove that, *if $u \in \underline{\mathscr{S}}(A_{a,b}(x_0))$, then $B(r)$ is an r^{-1}-convex function of $r \in (a, b)$.* Fix any r_1, r_2 such that $a < r_1 < r_2 < b$. We need to prove that $B(r) \leq I(r)$ for every $r \in (r_1, r_2)$, where

$$I(r) = \frac{1/r_2 - 1/r}{1/r_2 - 1/r_1} B(r_1) + \frac{1/r - 1/r_1}{1/r_2 - 1/r_1} B(r_2).$$

We remark that $I(r_i) = B(r_i)$ for $i = 1, 2$ and

$$I(r) = \frac{1}{r} a + b, \quad \text{where } a = \frac{B(r_2) - B(r_1)}{1/r_2 - 1/r_1}, \; b = \frac{B(r_1)/r_2 - B(r_2)/r_1}{1/r_2 - 1/r_1}.$$

With these same notations, we set $v(x) := I(1/\Gamma(x_0, x)) = a\Gamma(x_0, x) + b$. Clearly v is $\mathscr{L}$-harmonic in $\mathbb{R}^N \setminus \{x_0\}$; moreover, for every $x \in \partial\Omega_{r_i}(x_0)$, one has

$$v(x) = I(r_i) = B(r_i) = \sup_{\partial\Omega_{r_i}(x_0)} u \geq u(x).$$

By the weak maximum principle for the $\mathscr{L}$-subharmonic function $u - v$ on the bounded open set $A_{r_1,r_2}(x_0)$, we infer that $u(x) \leq v(x)$ for every $x \in A_{r_1,r_2}(x_0)$. In particular, if we take $x \in \partial\Omega_r(x_0)$, we get $u(x) \leq v(x) = I(r)$; taking the supremum over $\partial\Omega_r(x_0)$, we get exactly the needed inequality $B(r) \leq I(r)$.

Remark 6.5. (a) *Surface average operators of $\mathscr{L}$-subharmonic functions on a Γ-annulus need not be monotone nondecreasing.* Indeed, if for example $\mathscr{L} = \Delta$ is the classical Laplace operator on $\mathbb{R}^3$, the function

$$u(x) = (\|x\| - 2)^2, \quad \text{where } \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

is subharmonic on the annulus $\{4/3 < \|x\| < 3\}$, but $m_r(u)(0) = (r-2)^2$ is not monotone on $(4/3, 3)$.

(b) *A converse of Theorem 6.1 holds true for C^2 functions which are "radial" with respect to Γ.* More precisely, suppose u has the form

$$u(x) = f(\Gamma(x_0, x)), \quad x \in A_{a,b}(x_0),$$

for some $f \in C^2((1/b, 1/a), \mathbb{R})$. A direct computation based on (2-1) and on the $\mathscr{L}$-harmonicity of $\Gamma(x_0, \cdot)$ on $\mathbb{R}^N \setminus \{x_0\}$, proves that

$$\mathscr{L}u = f'(\Gamma_{x_0})\mathscr{L}\Gamma_{x_0} + f''(\Gamma_{x_0}) \sum_{i,j} a_{i,j} \partial_i \Gamma_{x_0} \partial_j \Gamma_{x_0} = f''(\Gamma_{x_0}) \langle A\nabla\Gamma_{x_0}, \nabla\Gamma_{x_0}\rangle.$$

Thus u is $\mathscr{L}$-subharmonic on $A_{a,b}(x_0)$ (that is, $\mathscr{L}u \geq 0$) if and only if (recall that A is positive semidefinite) $f'' \geq 0$ on $(1/b, 1/a)$. On the other hand, if $r \in (a, b)$,

$$m_r(u)(x_0) = \int_{\Gamma(x_0,x)=1/r} f(\Gamma(x_0, x))k(x_0, x)\, dH^{N-1}(x) = f(1/r).$$

Thus, $m_r(u)(x_0)$ is r^{-1}-convex on (a, b) if and only if $f(r)$ is convex on $(1/b, 1/a)$. This proves that u is $\mathscr{L}$-subharmonic on $A_{a,b}(x_0)$ if and only if $m_r(u)(x_0)$ is an r^{-1}-convex function on (a, b).

(c) If u is as in part (b), then $m_r(u)(x_0)$ is $r^{-\beta}$-convex on (a, b) if and only if (see Lemma 2.1) $f(r^{1/\beta})$ is convex on $(b^{-\beta}, a^{-\beta})$; this last condition holds if and only if

$$f''(\rho) - (\beta - 1)\frac{f'(\rho)}{\rho} \geq 0 \quad \text{for all } \rho \in (b^{-1}, a^{-1}).$$

Now, when $\beta > 1$, it is very simple to produce a function f satisfying this last condition on some open interval (b^{-1}, a^{-1}), but violating $f'' \geq 0$ on the same interval (recall that this last condition is equivalent to u being $\mathscr{L}$-subharmonic on $A_{a,b}$): for instance, $f(\rho) = -\rho^\beta$ does the job. With this choice of f, the associated function $u(x) = -(\Gamma(x_0, x))^\beta$ is *not* $\mathscr{L}$-subharmonic on any annulus $A_{a,b}(x_0)$, but $m_r(u)(x_0)$ is $r^{-\beta}$-convex on every subinterval (b^{-1}, a^{-1}) of $(0, \infty)$.

7. Applications

We are ready to give the following proofs.

Proof of Corollary 1.7. From (6-4) we derive that $r^2(d/dr)m_r(u)(x_0)$ is constant on (a, b), that is, there exists $c_1 \in \mathbb{R}$ such that

$$\frac{d}{dr}(m_r(u)(x_0)) = -\frac{c_1}{r^2} = \frac{d}{dr}\left(\frac{c_1}{r}\right)$$

for every r in the interval (a, b). □

We now prove the Γ-symmetry result in Theorem 1.8. Hypothesis (HH) in Section 1 is assumed.

Remark 7.1. Thanks to our hypoellipticity assumption (H1), by the strong maximum principle for $\mathscr{L}$ (proved in [Abbondanza and Bonfiglioli 2013, Theorem 3.4]) we infer that the harmonic sheaf associated with $\mathscr{L}$ is elliptic (in the sense of [Constantinescu and Cornea 1972]). By standard techniques, hypothesis (HH) is then fulfilled, for example, in the following cases:

(1) if $\mathscr{L}$ can be put in the form $\mathscr{L} = \sum_{j=1}^m X_j^2$, where $X_1, \dots, X_m$ are smooth vector fields satisfying Hörmander's rank condition on $\mathbb{R}^N$;

(2) for $x_0 = 0$, if $\mathscr{L}$ is homogeneous w.r.t. some group of dilations on $\mathbb{R}^N$ (this is true, for example, if $\mathscr{L}$ is a sub-Laplacian on a Carnot group).

Here we agree to say that a family of maps $\{\delta_\lambda\}_{\lambda>0}$ is a group of dilations if

$$\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_\lambda(x_1, \dots, x_N) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_N} x_N),$$

where the exponents σ_j are strictly positive real numbers. Moreover, we say that $\mathscr{L}$ is δ_λ-homogeneous of positive degree if there exists $\sigma > 0$ such that $\mathscr{L}(u \circ \delta_\lambda) = \lambda^\sigma (\mathscr{L}u) \circ \delta_\lambda$ for every $u \in C^2(\mathbb{R}^N, \mathbb{R})$.

Proof of Theorem 1.8. We follow an idea [Axler et al. 1992] exploited in the classical case of the Laplace operator. The proof is split into three steps.

(I) We set $A := A_{0,b}(x_0)$. Given $u \in C(A, \mathbb{R})$, we introduce the operator

$$S(u)(x) := m_{1/\Gamma(x_0,x)}(u)(x_0), \quad x \in A.$$

Clearly, one has $S(1) \equiv 1$ and, moreover, Γ_{x_0} is another fixed function for S, since

$$\begin{aligned} S(\Gamma_{x_0})(x) = m_{1/\Gamma(x_0,x)}(\Gamma_{x_0})(x_0) &= \int_{\Gamma(x_0,y)=\Gamma(x_0,x)} \Gamma(x_0, y)k(x_0, y)\, dH^{N-1}(y) \\ &= \Gamma(x_0, x)S(1)(x) = \Gamma_{x_0}(x). \end{aligned}$$

We observe that if u is $\mathscr{L}$-harmonic in A, then (by Corollary 1.7)

$$S(u)(x) = c\Gamma(x_0, x) + c_2, \quad x \in A, \tag{7-1}$$

for some constants c, c_2. In particular, if u is $\mathscr{L}$-harmonic in A, then $S(u)$ is $\mathscr{L}$-harmonic in A (actually $S(u)$ extends to an $\mathscr{L}$-harmonic function in $\mathbb{R}^N \setminus \{x_0\}$). Furthermore, by the above results ensuring that 1 and Γ_{x_0} are fixed functions for S, we infer that

$$\textit{if } u \textit{ is } \mathscr{L}\textit{-harmonic in } A, \textit{ then } S(S(u)) = S(u). \tag{7-2}$$

Next we see how S behaves on a function w enjoying the hypotheses of the theorem. First notice that, since w vanishes on $\partial\Omega_b(x_0)$ with continuity, the same is true of $S(w)$. Moreover $S(w)$ is $\mathscr{L}$-harmonic in A (since this is true of w) and

$$S(w)(x) = c(\Gamma(x_0, x) - 1/b). \tag{7-3}$$

Here we used (7-1), observing that $c_2 = -c/b$ is the only choice for c_2 which ensures the vanishing of $S(w)$ on $\partial\Omega_b(x_0)$.

Comparing (7-3) with the thesis of the theorem, we recognize that *the theorem is proved if we are able to show that w is fixed by S*, that is, $S(w) = w$ on A.

(II) We let $\boldsymbol{c} := C^{-1}$, where C is the constant in hypothesis (HH). Note that $0 < \boldsymbol{c} < 1$. We claim that the following property holds true:

If h is $\mathscr{L}$-harmonic in A and continuous up to $\partial\Omega_b(x_0)$

$$\text{with } h \equiv 0 \text{ on } \partial\Omega_b(x_0) \text{ and } h \geq 0 \text{ on } A, \text{ then } h \geq \boldsymbol{c}S(h) \text{ on } A. \tag{7-4}$$

With this result in hand, the proof of Theorem 1.8 follows. Indeed, suppose w enjoys the hypothesis of the theorem; let us prove by induction that, setting

$$\boldsymbol{c}_n := 1 - (1 - \boldsymbol{c})^n, \quad n \in \mathbb{N} \cup \{0\}, \tag{7-5}$$

we have

$$w \geq \boldsymbol{c}_n S(w) \text{ on } A \quad \text{for any } n \in \mathbb{N} \cup \{0\}. \tag{7-6}$$

The case $n = 0$ follows from the nonnegativity of w on A and $c_0 = 0$.

Suppose (7-6) holds true, and let us prove it for $n+1$ replacing n. The function $h := w - \boldsymbol{c}_n S(w)$ satisfies the hypothesis of statement (7-4): indeed, from the last remarks of part I above, it follows that h is $\mathscr{L}$-harmonic in A, continuous up to $\partial\Omega_b(x_0)$ and vanishing there. Finally $h \geq 0$ on A is the inductive assumption.

Consequently, from the claimed result in (7-4), we have on A

$$\begin{aligned} 0 \leq h - \boldsymbol{c}S(h) &= w - \boldsymbol{c}_n S(w) - \boldsymbol{c}S(w - \boldsymbol{c}_n S(w)) \\ &= w - \boldsymbol{c}_n S(w) - \boldsymbol{c}S(w) + \boldsymbol{c}\boldsymbol{c}_n S(w) \\ &= w - \boldsymbol{c}_{n+1} S(w). \end{aligned}$$

Here we used (7-2) together with $\boldsymbol{c}_n + \boldsymbol{c} - \boldsymbol{c}\boldsymbol{c}_n = \boldsymbol{c}_{n+1}$ (see the very definition (7-5) of $\boldsymbol{c}_n$). Thus (7-6) is proved by induction.

Letting $n \to \infty$ in it, we infer $w \geq S(w)$ on A, since $\boldsymbol{c}_n \to 1$, as $0 < 1 - \boldsymbol{c} < 1$. Recalling what we proved in part I, we are done if we can also prove the reverse inequality $w \leq S(w)$. Suppose by contradiction that $w(\overline{x}) > S(w)(\overline{x})$ for some $\overline{x} \in A$; by (7-2), this gives $S(w)(\overline{x}) > S\big(S(w)\big)(\overline{x}) = S(w)(\overline{x})$, a contradiction. Note that the above inequality is a consequence of $S(1) = 1$ and of the fact that S is a nondecreasing operator (that is, if $u \leq v$ on A, then $S(u) \leq S(v)$ on A).

(III) We are thus left with the proof of the claimed (7-4). Notice that (HH) can be restated as follows:

$$\boldsymbol{c}h(z) \leq h(x), \quad \text{whenever } (\theta b)^{-1} < \Gamma(x_0, z) = \Gamma(x_0, x) < \infty \text{ and } h \geq 0 \text{ is } \mathscr{L}\text{-harmonic in } A. \tag{7-7}$$

Let h be as in (7-4). Arguing as in part I of the proof, we infer that $H := h - \boldsymbol{c}S(h)$ is $\mathscr{L}$-harmonic in A, continuous up to $\partial\Omega_b(x_0)$, and $H = 0$ on $\partial\Omega_b(x_0)$. Let us fix any arbitrary $r \in (0, \theta b)$. We take $x, z \in \partial\Omega_r(x_0)$; recall that this means

$$\Gamma(x_0, x) = \Gamma(x_0, z) = 1/r.$$

Let us consider the inequality in the left-hand side of (7-7), which is fulfilled since $(\theta b)^{-1} < 1/r < \infty$; by multiplication by $k(x_0, z)$ (see the notation in (1-2)), and by integration w.r.t. $z \in \partial\Omega_r(x_0)$, we get $\boldsymbol{c}m_r(h)(x_0) \leq h(x)$. Recalling that $r = 1/\Gamma(x_0, x)$, we infer

$$\boldsymbol{c}m_{1/\Gamma(x_0,x)}(h)(x_0) \leq h(x), \quad \text{that is,} \quad \boldsymbol{c}S(h)(x) \leq h(x).$$

The arbitrariness of $x \in \partial\Omega_r(x_0)$ implies that $H(x) \geq 0$ on $\partial\Omega_r(x_0)$. By the weak minimum principle applied to the $\mathscr{L}$-harmonic function H and to the bounded open set $A_{r,b}(x_0)$, we derive $H \geq 0$ on $A_{r,b}(x_0)$. Since $r \in (0, \theta b)$ is arbitrary, this yields $H \geq 0$ on $A_{0,b}(x_0) = A$, that is, $h \geq \boldsymbol{c}S(h)$ on A. This proves (7-4). □

We end the paper by giving the following proof.

Proof of Theorem 1.9. Let $\varepsilon > 0$ be so small that $\overline{\Omega_\varepsilon(x_0)} \subset \Omega$. Since $V := \Omega_\varepsilon(x_0)$ is an $\mathscr{L}$-regular open set, setting $f := u|_{\partial\Omega_\varepsilon(x_0)}$, we can consider H_f^V, the unique $\mathscr{L}$-harmonic function in V, continuous up to

∂V, coinciding with u on ∂V. Let

$$w(x) := u(x) - H_f^V(x) + \Gamma(x_0, x) - 1/\varepsilon, \quad x \in O := \Omega_\varepsilon(x_0) \setminus \{x_0\}.$$

The function w is $\mathscr{L}$-harmonic in O and continuous up to $\partial\Omega_\varepsilon(x_0)$, where it vanishes; moreover, $\liminf_{x\to x_0} w(x) \geq -H_f^V(x_0) - 1/\varepsilon + \lim_{x\to x_0} \Gamma(x_0, x) = \infty$, the inequality following from the hypothesis $u \geq 0$. The weak minimum principle for w and for the bounded open set O proves that $w \geq 0$ on O. Note that O is the Γ-annulus $A_{0,\varepsilon}(x_0)$. We are therefore entitled to apply Theorem 1.8 and derive that $w = c_1(\Gamma_{x_0} - 1/\varepsilon)$ on O, for some constant c_1. As a consequence, we get $u = c\Gamma_{x_0} + H$ on O, where $c = c_1 - 1$ and $H = H_f^V - c/\varepsilon$. From $u = c\Gamma_{x_0} + H$, the finiteness of $H(x_0)$ and the hypothesis $u \geq 0$, we get $c \geq 0$. This proves that the function h defined on $\Omega \setminus \{x_0\}$ by $h(x) := u(x) - c\Gamma(x_0, x)$ is not only $\mathscr{L}$-harmonic, but (as it coincides with H on O) it extends $\mathscr{L}$-harmonically through x_0. □

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