

# *Algebra & Number Theory*

Volume 13

2019

No. 3

**Algebraic independence for values of integral curves**

Tiago J. Fonseca



# Algebraic independence for values of integral curves

Tiago J. Fonseca

We prove a transcendence theorem concerning values of holomorphic maps from a disk to a quasiprojective variety over  $\overline{\mathbb{Q}}$  that are integral curves of some algebraic vector field (defined over  $\overline{\mathbb{Q}}$ ). These maps are required to satisfy some integrality property, besides a growth condition and a strong form of Zariski-density that are natural for integral curves of algebraic vector fields.

This result generalizes a theorem of Nesterenko concerning algebraic independence of values of the Eisenstein series  $E_2, E_4, E_6$ . The main technical improvement in our approach is the replacement of a rather restrictive hypothesis of polynomial growth on Taylor coefficients by a geometric notion of *moderate growth* formulated in terms of value distribution theory.

1. Introduction	643
2. ZL-dense formal curves in quasiprojective varieties	651
3. Moderate growth and jet estimates on complex disks	658
4. Analytic curves of moderate growth in quasiprojective varieties	668
5. Construction of auxiliary sections	673
6. Derivatives of sections of line bundles along vector fields	680
7. Proof of Theorem 1.2	684
Appendix A. Philippon's algebraic independence criterion for projective varieties	688
Appendix B. $D$ -property and ZL-density in quasiprojective varieties	690
Acknowledgments	692
References	692

## 1. Introduction

**1A. A theorem of Nesterenko.** This work was motivated by questions related to the following algebraic independence result.

Let  $E_2, E_4$ , and  $E_6$  be the classical Eisenstein series, seen as holomorphic functions on the complex unit disk  $D := \{q \in \mathbb{C} \mid |q| < 1\}$ , explicitly defined by

$$E_2(q) = 1 - 24 \sum_{j=1}^{\infty} \sigma_1(j)q^j, \quad E_4(q) = 1 + 240 \sum_{j=1}^{\infty} \sigma_3(j)q^j, \quad E_6(q) = 1 - 504 \sum_{j=1}^{\infty} \sigma_5(j)q^j$$

*MSC2010:* primary 11J81; secondary 14G40, 37F75, 32A22.

*Keywords:* algebraic independence, transcendence, integral curves, modular forms, Eisenstein series, differential equations, Nevanlinna theory, integrality, zero lemma.

for every  $q \in D$ , where  $\sigma_k(j) := \sum_{d|j} d^k \in \mathbb{Z}$ . Let us also consider the  $q$ -expansion of the  $j$ -invariant

$$J(q) = 1728 \frac{E_4(q)^3}{E_4(q)^3 - E_6(q)^2} = \frac{1}{q} + 744 + \sum_{j=1}^{\infty} c(j)q^j.$$

**Theorem 1.1** [Nesterenko 1996]. *For every  $z \in D \setminus \{0\}$ , we have*

$$\text{trdeg}_{\mathbb{Q}} \mathbb{Q}(z, E_2(z), E_4(z), E_6(z)) \geq 3.$$

This result is an improvement of Barré-Sirieix, Diaz, Gramain, and Philibert's [Barré-Sirieix et al. 1996] breakthrough concerning the solution of a conjecture of Mahler: for every algebraic  $z \in D \setminus \{0\}$ ,  $J(z)$  is transcendental.

In order to fully motivate our contributions, we next sketch the main steps of Nesterenko's original proof.

In view of an algebraic independence criterion due to Philippon [1986, Théorème 2.11], see also [Nesterenko 1996, Lemma 2.5], it suffices to construct a sequence of polynomials with integral coefficients  $Q_n \in \mathbb{Z}[X_0, X_1, X_2, X_3]$ , for  $n \gg 0$ , such that  $\deg Q_n = O(n \log n)$ ,  $\log \|Q_n\|_{\infty} = O(n \log^2 n)$  — here,  $\|Q_n\|_{\infty}$  denotes the maximum of the absolute values of all the coefficients of  $Q_n$  — and

$$-an^4 \leq \log |Q_n(z, E_2(z), E_4(z), E_6(z))| \leq -bn^4$$

for some real constants  $a > b > 0$ .

For this, Nesterenko implemented a *method* benefiting from the fact that  $E_2$ ,  $E_4$ , and  $E_6$  have integral Taylor coefficients in their  $q$ -expansion and satisfy the so-called *Ramanujan equations*

$$q \frac{dE_2}{dq} = \frac{E_2^2 - E_4}{12}, \quad q \frac{dE_4}{dq} = \frac{E_2E_4 - E_6}{3}, \quad q \frac{dE_6}{dq} = \frac{E_2E_6 - E_4^2}{2}.$$

It is also essential in his construction that

- (i) (*growth condition*) for each  $k \in \{1, 2, 3\}$ , the sequence of Taylor coefficients  $(E_{2k}^{(j)}(0)/j!)_{j \geq 0}$  grows polynomially in  $j$ , and
- (ii) (*zero lemma*)<sup>1</sup> there exists a constant  $C > 0$  such that

$$\text{ord}_{q=0} P(q, E_2(q), E_4(q), E_6(q)) \leq C(\deg P)^4$$

for every nonzero polynomial  $P \in \mathbb{C}[X_0, X_1, X_2, X_3] \setminus \{0\}$ .

The first condition can be easily deduced from the explicit description of the Taylor coefficients of  $E_{2k}$  given above. The second, which may be regarded as a strong form of algebraic independence between the functions  $q$ ,  $E_2(q)$ ,  $E_4(q)$ , and  $E_6(q)$ , is a nontrivial consequence of Nesterenko's *D-property* [1996,

<sup>1</sup>In Diophantine approximation and transcendental number theory, “zero lemma” is an umbrella term covering several auxiliary results involving estimates of the number of zeros in a certain region, or zeros' multiplicities, of polynomials composed with analytic functions.

Section 6] (see [Definition B.1](#)), an algebraic property concerning the global behavior of the foliation in  $\mathbb{C}^4$  induced by the vector field

$$v := x_0 \frac{\partial}{\partial x_0} + \frac{(x_1^2 - x_2)}{12} \frac{\partial}{\partial x_1} + \frac{(x_1 x_2 - x_3)}{3} \frac{\partial}{\partial x_2} + \frac{(x_1 x_3 - x_2^2)}{2} \frac{\partial}{\partial x_3}. \tag{1-1}$$

A considerable part of [\[Nesterenko 1996\]](#) is devoted to a proof of a stronger form of the estimate in (ii).

Nesterenko’s method goes as follows:

(1) Using that the Taylor coefficients of  $E_2$ ,  $E_4$ , and  $E_6$  are integers of polynomial growth (property (i) above), we may apply Siegel’s lemma [\[Lang 1966, I.1, Lemma 1\]](#) to obtain *auxiliary polynomials* with integral coefficients  $P_n \in \mathbb{Z}[X_0, X_1, X_2, X_3] \setminus \{0\}$  such that  $\deg P_n = n$ ,  $\log \|P_n\|_\infty = O(n \log n)$ , and

$$\text{ord}_{q=0} P_n(q, E_2(q), E_4(q), E_6(q)) \geq cn^4$$

for some constant  $c > 0$ .

(2) For a fixed  $z \in D \setminus \{0\}$ , the next step consists in proving the existence of a sequence  $j_n = O(n \log n)$  and of constants  $\alpha > \beta > 0$  such that the composed function  $f_n(q) := P_n(q, E_2(q), E_4(q), E_6(q))$  satisfies

$$-\alpha n^4 \leq \log |f_n^{(j_n)}(z)| \leq -\beta n^4$$

for  $n \gg 0$ . The main point for obtaining the above lower bound is that, if all the Taylor coefficients of  $f_n$  at  $q = z$  up to a sufficiently large order are too small, then its first nonzero Taylor coefficient at  $q = 0$  will have absolute value  $< 1$ , thereby contradicting its integrality. Here, we also make essential use of property (ii) above. This is the most delicate part of the argument.

(3) Finally, for  $n \gg 0$ , if we consider the differential operator

$$v^{[j_n]} := 12^{j_n} v \circ (v - 1) \circ \dots \circ (v - (j_n - 1)),$$

then the Ramanujan equations imply that  $Q_n := v^{[j_n]}(P_n) \in \mathbb{Z}[X_0, X_1, X_2, X_3]$  satisfies

$$(12q)^{j_n} f_n^{(j_n)}(q) = Q_n(q, E_2(q), E_4(q), E_6(q))$$

for every  $q \in D$ . The required properties for  $Q_n$  are now easily deducible from (1) and (2).

**1B. A puzzling remark.** One of the most striking features of the above method is its generality.

Indeed, a close inspection of the previous arguments suggests that, if  $f_1, \dots, f_m$  are holomorphic functions on the unit disk  $D$  with integral Taylor coefficients at  $q = 0$ , satisfying some algebraic differential equations with rational coefficients, and verifying conditions akin to (i) and (ii) above, then, mutatis mutandis, the above method applied to the system  $(f_1, \dots, f_m)$  in place of  $(E_2, E_4, E_6)$  would produce another transcendence result.

This was certainly known to specialists; see, for instance, [\[Nesterenko 2001, Section 3\]](#), where the pertinent properties satisfied by  $E_2$ ,  $E_4$ , and  $E_6$  were axiomatized as above — more generally, see the notion of *K-functions* introduced in [\[Philippon 1998\]](#). Clearly, one may produce examples of such  $f_i$  ad libitum

by algebraically manipulating Eisenstein series, but this procedure does not lead to *new* transcendence results. The problem on the existence of functions  $f_1, \dots, f_m$  satisfying the above properties, but not “related” to classical modular forms (in some imprecise sense), was explicitly stated in [Zudilin 2003].

Since the publication of [Candelas et al. 1991a; 1991b; Morrison 1993], it became apparent that the phenomenon of mirror symmetry provides a large class of functions with integral Taylor coefficients with respect to some *canonical coordinate* and which satisfy natural algebraic differential equations — see [Lian and Yau 1998; Zudilin 2002; Kontsevich et al. 2006; Krattenthaler and Rivoal 2010; Delaygue et al. 2017] for integrality issues. Zudilin [2000] studied some candidates within mirror symmetry for playing the role of  $(f_1, \dots, f_m)$ , but the few cases where he is able to prove all the required properties of Nesterenko’s method — those linked to elliptic curves and  $K3$  surfaces — are all of modular nature.

It becomes clear in Zudilin’s work that one of the main obstructions in applying this method for such functions is condition (i) (and, in particular, that the radius of convergence is equal to 1), which is not verified in general. In a more basic level, computing radii of convergence or getting global information on the domain of definition of such functions pertaining to mirror symmetry is a current research problem; see [Krattenthaler and Rivoal 2012] for results on certain families of mirror maps.

The following phenomenon provides further evidence that condition (i) is overly restrictive. Let  $f$  be the holomorphic function on  $D$  given by  $f(q) = qJ(q)$  and set  $\theta := q(d/dq)$ . Since  $\mathbb{Q}(J, \theta J, \theta^2 J) = \mathbb{Q}(E_2, E_4, E_6)$  (see, for instance, the explicit formulas in [Nesterenko 1996, Paragraph 1]), it follows from Theorem 1.1 that

$$\text{trdeg}_{\mathbb{Q}} \mathbb{Q}(z, f(z), \theta f(z), \theta^2 f(z)) \geq 3$$

for any  $z \in D \setminus \{0\}$ . However, Nesterenko’s method *cannot* be *directly* applied to the system  $(f, \theta f, \theta^2 f)$  since the sequence  $c(j)$  does not grow polynomially in  $j$ .<sup>2</sup> All the other good properties are nevertheless satisfied:  $f, \theta f$ , and  $\theta^2 f$  have integral Taylor coefficients, the Ramanujan equations imply  $f$  satisfies a third-order algebraic differential equation with rational coefficients, and a condition similar to (ii) also holds.

This paper grew from an observation of J.-B. Bost and H. Randriambololona that the growth condition (i) in Nesterenko’s method could be replaced by a geometric notion of *moderate growth* formulated in terms of characteristic functions a la Nevanlinna theory. Besides being weaker than the growth condition in (i), which in principle enlarges the domain of application of Nesterenko’s method, this geometric growth condition is preserved under some algebraic manipulations on the input functions, thereby eliminating the odd phenomenon explained in last paragraph.

We next explain our main results. Further directions and open problems are indicated below.

**1C. Our main results: a geometric approach.** Our main theorem is a general geometric formulation of Nesterenko’s method valid for arbitrary rings of algebraic integers and more general quasiprojective ambient spaces.

<sup>2</sup>Actually,  $c(j) \sim e^{4\pi\sqrt{j}}/(\sqrt{2}j^{3/4})$ ; see [Petersson 1932; Rademacher 1938].

Let us first informally introduce the geometric notions which will replace conditions (i) and (ii) above.

**1C1. Moderate growth.** Let  $X$  be a smooth projective variety over  $\mathbb{C}$ , and  $h$  be a  $C^\infty$  Hermitian metric on the complex manifold  $X(\mathbb{C})$ . Let  $\omega := -\operatorname{Im} h$  be the positive real (1,1)-form on  $X(\mathbb{C})$  associated to  $h$ . To fix ideas, the reader may consider the example  $X = \mathbb{P}^n_{\mathbb{C}}$  endowed with the Fubini–Study metric, for which  $\omega$  is given in homogeneous coordinates  $z = (z_0 : \cdots : z_n)$  by

$$\omega = \frac{i}{2\pi} \partial\bar{\partial} \log |z|^2 = \frac{i}{2\pi} \left( \frac{\sum_{j=0}^n dz_j \wedge d\bar{z}_j}{|z|^2} - \frac{\sum_{j,k=0}^n \bar{z}_j z_k dz_j \wedge d\bar{z}_k}{|z|^4} \right),$$

where  $|z|^2 = \sum_{j=0}^n |z_j|^2$ .

Let  $R > 0$  be a real number and denote by  $D_R := \{z \in \mathbb{C} \mid |z| < R\}$  the complex disk of radius  $R$  centered at the origin. One may measure the growth of an analytic map  $\varphi : D_R \rightarrow X(\mathbb{C})$  as follows. For each  $t \in (0, R)$ , the area of the “disk”  $\varphi(D_t)$  in  $X(\mathbb{C})$  with respect to the metric  $h$  is given by

$$A_\varphi(t) := \int_{D_t} \varphi^* \omega.$$

We may then form the *characteristic function*

$$T_\varphi : (0, R) \rightarrow \mathbb{R}_{\geq 0}, \quad r \mapsto T_\varphi(r) := \int_0^r A_\varphi(t) d \log t,$$

and we say that  $\varphi$  has *moderate growth* in  $X$  if

$$\limsup_{r \rightarrow R^-} \frac{T_\varphi(r)}{\log \frac{1}{1-r/R}} < +\infty.$$

By the compactness of  $X(\mathbb{C})$ , moderate growth does not depend on the choice of Hermitian metric.

When  $R = 1$ , any analytic map  $\varphi : D \rightarrow \mathbb{C}^n \subset \mathbb{P}^n(\mathbb{C})$  whose coordinates have Taylor coefficients of polynomial growth has moderate growth in  $\mathbb{P}^n_{\mathbb{C}}$  (see [Example 4.5](#) below). Therefore, moderate growth generalizes the growth condition (i) in Nesterenko’s method.

Moderate growth is nonetheless more flexible than polynomial growth on Taylor coefficients. For instance, as long as the image of  $\varphi : D_R \rightarrow X(\mathbb{C})$  is Zariski-dense in  $X$ , moderate growth is a birational invariant in the following sense: if  $f : X \rightarrow Y$  is a birational morphism between smooth projective varieties over  $\mathbb{C}$ , then  $\varphi$  has moderate growth in  $X$  if and only if  $f \circ \varphi$  has moderate growth in  $Y$  (see [Theorem 4.11](#) and [Corollary 4.12](#) below). In particular, this allows us to define, via compactifications, an unambiguous notion of moderate growth in smooth *quasiprojective* varieties.

**1C2. ZL-density.** Let  $k$  be a field,  $X$  a projective variety over  $k$  of dimension  $n$ , and  $L$  an ample line bundle on  $X$ . Consider a parametrized *formal curve*  $\hat{\varphi} : \operatorname{Spf} k[[q]] \rightarrow X$  in  $X$ ; i.e.,  $\hat{\varphi}$  is a morphism of formal  $k$ -schemes.

We say that  $\hat{\varphi}$  is *ZL-dense* in  $X$  if there exists a constant  $C > 0$  such that, for every integer  $d \geq 1$  and every nonzero global section  $s \in \Gamma(X, L^{\otimes d}) \setminus \{0\}$ , we have

$$\operatorname{ord}_{q=0} \hat{\varphi}^* s \leq Cd^n.$$

The exponent  $n = \dim X$  in the above polynomial bound is the smallest possible (see [Proposition 2.6](#) below). Moreover, since  $L$  is ample, a ZL-dense formal curve has Zariski-dense image. Thus, ZL-density may be regarded as a strong form of Zariski-density.

For a formal curve  $\hat{\varphi} : \text{Spf } k[[q]] \rightarrow \mathbb{A}_k^n \subset \mathbb{P}_k^n$  the above notion boils down to a classical zero-lemma property — here,  $L = \mathcal{O}_{\mathbb{P}_k^n}(1)$ . For instance, taking  $k = \mathbb{C}$ ,  $n = 4$ , and  $\hat{\varphi}$  defined by the system of formal series  $(q, E_2(q), E_4(q), E_6(q))$ , the zero lemma in condition (ii) above amounts to asserting that  $\hat{\varphi}$  is ZL-dense in  $\mathbb{P}_{\mathbb{C}}^4$ .

We shall prove that ZL-density does not depend on the choice of  $L$  (see [Proposition 2.9](#) below). Actually, if  $X$  is only quasiprojective and the closed point  $\hat{\varphi}(0)$  in the image of  $\hat{\varphi}$  is a regular point of  $X$ , then we shall prove that ZL-density does not depend on the choice of a projective compactification of  $X$  (see [Corollary 2.16](#) below).

**1C3.** *Statement of our main theorem and proof method.* Let  $K$  be a number field and  $\mathcal{O}_K$  be its ring of integers. By an *arithmetic scheme* over  $\mathcal{O}_K$  we mean an integral scheme  $\mathcal{X}$  endowed with a separated and flat morphism of finite type  $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$ .

The following theorem formalizes and generalizes Nesterenko’s method.

**Theorem 1.2.** *Let  $\mathcal{X}$  be a quasiprojective arithmetic scheme over  $\mathcal{O}_K$  of relative dimension  $n \geq 2$ , with smooth generic fiber  $\mathcal{X}_K$ , and let  $\hat{\varphi} : \text{Spf } \mathcal{O}_K[[q]] \rightarrow \mathcal{X}$  be a morphism of formal  $\mathcal{O}_K$ -schemes such that, for every field embedding  $\sigma : K \hookrightarrow \mathbb{C}$ , the formal curve  $\hat{\varphi}_\sigma : \text{Spf } \mathbb{C}[[q]] \rightarrow \mathcal{X}_\sigma$ , obtained from  $\hat{\varphi}$  by base change, lifts to an analytic curve  $\varphi_\sigma : D_{R_\sigma} \subset \mathbb{C} \rightarrow \mathcal{X}_\sigma^{\text{an}}$  defined on a disk of radius  $R_\sigma > 0$  centered at the origin.*

Assume that

$$\prod_{\sigma:K \hookrightarrow \mathbb{C}} R_\sigma = 1$$

and that there exists a vector field  $v \in \Gamma(\mathcal{X}_K, T_{\mathcal{X}_K/K}) \setminus \{0\}$  on the generic fiber of  $\mathcal{X}$  such that  $\hat{\varphi}_K : \text{Spf } K[[q]] \rightarrow \mathcal{X}_K$  satisfies the differential equation

$$q \frac{d\hat{\varphi}_K}{dq} = v \circ \hat{\varphi}_K.$$

If, moreover,

- (1) the formal curve  $\hat{\varphi}_K$  is ZL-dense in  $\mathcal{X}_K$ , and
  - (2) for each field embedding  $\sigma : K \hookrightarrow \mathbb{C}$ , the analytic curve  $\varphi_\sigma : D_{R_\sigma} \rightarrow \mathcal{X}_\sigma^{\text{an}}$  has moderate growth,
- then, for every  $\sigma : K \hookrightarrow \mathbb{C}$ , and every  $z \in D_{R_\sigma} \setminus \{0\}$ , the field of definition  $K(\varphi_\sigma(z))$  of the complex point  $\varphi_\sigma(z)$  in  $\mathcal{X}_K$  satisfies

$$\text{trdeg}_{\mathbb{Q}} K(\varphi_\sigma(z)) \geq n - 1.$$

Let us remark that the conditions of *ZL-density* and of *moderate growth*, corresponding to conditions (i) and (ii) in Nesterenko’s method, are actually very mild hypotheses.

For instance, ZL-density is automatic whenever  $\hat{\varphi}$  is a smooth integral curve of some vector field satisfying Nesterenko's  $D$ -property. When the ambient space is an affine space, this is also a theorem of Nesterenko [1996, Theorem 6], which was recently extended to a geometric framework by Binyamini [2014]. In Appendix B we explain how to slightly modify Binyamini's arguments to prove a similar statement for any smooth quasiprojective variety.

Moderate growth, in turn, is satisfied for curves having uniformly bounded derivative on the disk (endowed with the Poincaré metric; see Example 4.3 for a precise statement). In particular, a theorem of Brunella [Fornæss and Sibony 2008, Theorem 16] (see also Theorem 15 of that paper) implies that, for a generic one-dimensional holomorphic foliation (with singularities)  $\mathcal{F}$  on  $\mathbb{P}^n(\mathbb{C})$ , any integral curve to  $\mathcal{F}$  parametrized by a disk has moderate growth.

In the broader context of transcendental number theory, our result may be regarded as complementary to the Siegel–Shidlovsky and Schneider–Lang theories, which also deal with algebraic independence or transcendence of values of integral curves of algebraic vector fields (see [Gasbarri 2010; 2013; Herblot 2012] for general geometric formulations). Indeed, while the Siegel–Shidlovsky and Schneider–Lang criteria handle curves parametrized by *parabolic* Riemann surfaces, our theorem deals with the *hyperbolic* case.

Our proof of Theorem 1.2 bears the same general structure of Nesterenko's method. We also start by reducing it to a Diophantine approximation statement: Theorem 7.1 below. This is done via the same algebraic independence criterion of Philippon; we explain in Appendix A how to generalize it to arbitrary quasiprojective varieties. The first step in the method, concerning the construction of “auxiliary polynomials”, is replaced in our geometric framework by a construction of “auxiliary sections” given by Theorem 5.1 below, the proof of which makes essential use of Bost's *method of slopes* [1996] in Arakelov theory; see also [Bost 2001; 2006]. The second step also involves estimating some higher-order derivative — here, our main tool is a general result comparing, for a section of a Hermitian line bundle on a disk, norms of jets at two distinct points; see Proposition 3.13 below and its corollaries. The third and last step is essentially the same trick using the differential equation as explained above.

**1D. Further directions and open problems.** With Theorem 1.2 in hand, we may turn the puzzling remark explained above into a precise mathematical question: *is there any example of an application of Theorem 1.2 whose resulting transcendence statement is not contained in Theorem 1.1?*

As promising as the potential candidates from the theory of mirror symmetry may seem, one must face, given our current state of knowledge, the logical possibility of a negative answer. However, let us remark that a *proof* of this fact would be as *remarkable* as the discovery of a new example, since it would imply that modular functions and their derivatives are the only ones satisfying the (quite general) hypotheses of Theorem 1.2, which make no explicit reference to their geometric nature in terms of moduli of elliptic curves.

This paper, specially our geometric formulation of Theorem 1.2, should also be seen as a first step in a larger program aiming to adapt and apply Nesterenko's method to the study of algebraic independence of abelian periods of genus  $g > 1$ .

Since values of quasimodular forms can be expressed in terms of periods of elliptic curves, the theorem of Nesterenko recovers in particular a famous result of Chudnovsky [1980] establishing Grothendieck’s period conjecture for complex multiplication elliptic curves. A family of analogs of  $(E_2, E_4, E_6)$  related to the Siegel moduli spaces  $\mathcal{A}_g$  was constructed in [Fonseca 2018]; in this setting involving analytic functions of several variables, in addition to being easier to read the relevant properties required by Nesterenko’s method from geometry, the appearance of truly geometric phenomena — such as positive-dimensional “special subvarieties” of  $\mathcal{A}_g$  — strongly suggests a geometric approach.

**1E. Organization of this article.** A great effort has been done to isolate all the different techniques intervening in Nesterenko’s method and to place them in their natural generality. This distillation process is aimed not only at improving the readability of our paper, but also at making these techniques suitable for other applications in Diophantine approximation.

Section 2 contains the definition of ZL-density and some of its basic properties; here we use elementary intersection theory. Section 3 defines characteristic functions and moderate growth for 2-forms on a disk and contains basic versions of the jet estimates we shall need later; our main result here is Proposition 3.13 (see also Corollary 3.16). Both Sections 2 and 3 are self-contained and are of independent interest.

Section 4 treats the special case of moderate growth for analytic curves (as explained in this introduction) and it depends only on the beginning of Section 3. Its main objective is to prove that, under a nondegeneracy hypothesis, this concept is a birational invariant of the target space (Theorem 4.11); this is essentially classical material on Nevanlinna theory.

Section 5 is devoted to the construction of “auxiliary sections” in a geometric context. Here, we combine the concepts of moderate growth developed in Sections 3 and 4 with Bost’s *slope inequality* to obtain Theorem 5.1. This section contains a review of the prerequisites in Arakelov theory.

In Section 6 we explain how vector fields induce derivations on global sections of line bundles and we provide some  $L^\infty$  estimates. This section is also self-contained and of independent interest.

Section 7 contains a proof of Theorem 1.2. The reader will recognize, in Lemmas 7.2, 7.3, and 7.4, natural generalizations of the three steps of Nesterenko’s method explained above.

Finally, Appendices A and B concern geometric generalizations of results of Philippon and Binyamini originally stated only for affine (or projective) spaces, as explained above. In Appendix B we make use of the basic constructions of Section 6.

## 1F. Terminology and notation.

**1F1.** By an (algebraic) *variety* over a field  $k$  we mean a separated integral scheme of finite type over  $k$ .

**1F2.** Recall that a line bundle  $L$  on a scheme  $X$  is *semiample* if there exists an integer  $m \geq 1$  such that  $L^{\otimes m}$  is generated by its global sections. Observe that ample line bundles are semiample, and that semiamplicity is preserved under pullbacks.

**1F3.** A real  $(1,1)$ -form  $\omega$  on a complex manifold  $M$  can always be written, in local coordinates  $(z_1, \dots, z_n)$  on  $M$ , as

$$\omega = \frac{i}{2} \sum_{k,l=1}^n h_{kl} dz_k \wedge d\bar{z}_l,$$

where  $H := (h_{kl})_{1 \leq k,l \leq n}$  is a Hermitian matrix. We say that  $\omega$  is *positive* (resp. *semipositive*) if the matrix  $H$  is positive-definite (resp. positive-semidefinite). Note that semipositive  $(1,1)$ -forms are stable under pullbacks.

**1F4.** By a Hermitian line bundle  $\bar{L} = (L, \|\cdot\|)$  on a complex manifold  $M$ , we mean a holomorphic line bundle  $L$  on  $M$  endowed with a  $C^\infty$  Hermitian metric  $\|\cdot\|$ . If  $\Theta$  denotes the curvature of the Chern connection on  $L$  associated to  $\|\cdot\|$  (locally,  $\Theta = -\partial\bar{\partial} \log \|e\|^2$  where  $e$  is some trivialization of  $L$ ), then we define the *Chern curvature* of  $\bar{L}$  by

$$c_1(\bar{L}) = \frac{i}{2\pi} \Theta.$$

This is a closed real  $C^\infty$   $(1,1)$ -form on  $M$  whose class in  $H^2(M, \mathbb{R})$  coincides with the first Chern class  $c_1(L)$ . We say that  $\bar{L}$  is *positive* (resp. *semipositive*) if  $c_1(\bar{L})$  is positive (resp. semipositive).

**1F5.** We use the standard notation

$$d^c = \frac{i}{4\pi} (\bar{\partial} - \partial) = \frac{1}{4\pi} \left( r \frac{\partial}{\partial r} \otimes d\theta - \frac{1}{r} \frac{\partial}{\partial \theta} \otimes dr \right),$$

so that  $dd^c = \frac{i}{2\pi} \partial\bar{\partial}$ .

**1F6.** The continuous function  $\log^+ : \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$\log^+ x = \begin{cases} \log x & \text{if } x \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

## 2. ZL-dense formal curves in quasiprojective varieties

In this section we introduce the purely algebraic concept of *ZL-dense* formal curves, and we prove some of its basic properties. This notion refines the property of being Zariski-dense and isolates the content of the zero lemma necessary in Nesterenko’s method; that is, a formal curve satisfies the zero lemma if and only if it is ZL-dense.

**2A. Degree of a divisor with respect to a line bundle.** Let  $k$  be a field and  $X$  be a variety over  $k$ .

Recall from [Fulton 1984, 2.5] that (the isomorphism class of) a line bundle  $L$  on  $X$  defines an additive operator

$$\alpha \mapsto c_1(L) \cap \alpha$$

on the abelian group of algebraic cycles in  $X$  modulo rational equivalence; if  $\alpha$  is the class of a subvariety  $V$  of  $X$ , then  $c_1(L) \cap \alpha$  is by definition the class of the cycle in  $V$  associated to any Cartier divisor  $D$  of  $V$  for which  $L|_V \cong \mathcal{O}_V(D)$ . The  $r$ -fold composition of this operator with itself is denoted by  $\alpha \mapsto c_1(L)^r \cap \alpha$ .

We say that a cycle class  $\alpha$  in  $X$  is *semipositive* if there exists an integer  $m \geq 1$  such that  $m\alpha$  can be represented by a nonnegative cycle in  $X$  (i.e., a cycle of the form  $\sum_i m_i [V_i]$ , with each  $m_i \geq 0$ ). For instance, the cycle class of a Cartier divisor  $D$  is semipositive if and only if some positive multiple of  $D$  is linearly equivalent to an effective divisor.

**Lemma 2.1.** *Let  $L$  be a semiample line bundle on  $X$  (see Section 1F2 for a definition). Then  $c_1(L) \cap \alpha$  is semipositive for any semipositive cycle class  $\alpha$  in  $X$ .*

*Proof.* Let  $m \geq 1$  be an integer such that  $m\alpha$  is represented by the cycle  $\sum_i m_i [V_i]$ , with each  $m_i \geq 0$ .

As  $L$  is semiample, there exists an integer  $n \geq 1$  such that  $L^{\otimes n}$  is generated by global sections. In particular, for any subvariety  $V$  of  $X$ , the line bundle  $L^{\otimes n}|_V$  on  $V$  admits a nonzero global section  $s_V$ .

For every  $i$ ,  $c_1(L^{\otimes n}) \cap [V_i]$  is the cycle class induced by the effective Cartier divisor  $\text{div}(s_{V_i})$  on  $V_i$ , so that  $nm c_1(L) \cap \alpha = c_1(L^{\otimes n}) \cap m\alpha$  is represented by the nonnegative cycle  $\sum_i m_i [\text{div}(s_{V_i})]$ . □

Still following the terminology of [Fulton 1984], for any line bundle  $L$  on  $X$ , and any  $r$ -cycle class  $\alpha$  in  $X$ , the  $L$ -degree of  $\alpha$  is defined by

$$\text{deg}_L \alpha = \text{deg}(c_1(L)^r \cap \alpha),$$

where  $\text{deg}$  denotes the degree function on zero-cycle classes. If  $D$  is a Cartier divisor on  $X$ , then we denote by

$$\text{deg}_L D = \text{deg}(c_1(L)^{\dim X - 1} \cap [D])$$

the  $L$ -degree of the cycle class  $[D]$  induced by  $D$ .

Observe that the degree of a semipositive zero-cycle class is nonnegative. In the next result, we use the following easy consequence of Lemma 2.1: if  $L$  is semiample and  $\alpha$  is semipositive, then  $\text{deg}_L \alpha \geq 0$  [Fulton 1984, Lemma 12.1].

**Proposition 2.2.** *Let  $X$  be a projective variety over a field  $k$  and  $L$  (resp.  $M$ ) be an ample (resp. semiample) line bundle on  $X$ . Then, there exists an integer  $m \geq 1$  such that, for any semipositive  $r$ -cycle class  $\alpha$  in  $X$ , we have*

$$0 \leq \text{deg}_M \alpha \leq m^r \text{deg}_L \alpha.$$

*Proof.* Let  $m \geq 1$  be an integer such that  $N := L^{\otimes m} \otimes M^\vee$  is semiample. For any  $r$ -cycle class  $\alpha$  in  $X$ , we have

$$m^r \text{deg}_L \alpha = \text{deg}_{L^{\otimes m}} \alpha = \text{deg}_{N \otimes M} \alpha = \sum_{s=0}^r \binom{r}{s} \text{deg}_M (c_1(N)^s \cap \alpha).$$

Since  $N$  is semiample and  $\alpha$  is semipositive, it follows from Lemma 2.1 that each  $c_1(N)^s \cap \alpha$  is semipositive. As  $M$  is also semiample, we conclude that each term in the right-hand side of the above equation is nonnegative, so that  $m^r \text{deg}_L \alpha \geq \text{deg}_M \alpha \geq 0$ . □

**Remark 2.3.** By combining the above proposition with an induction argument in  $r$ , one can actually prove the following generalization. Let  $X$  be a projective variety over a field  $k$ ,  $L$  be an ample line bundle

on  $X$ , and  $M$  be any line bundle on  $X$ . Then there exists a constant  $C > 0$  such that  $|\deg_M \alpha| \leq C \deg_L \alpha$  for every semipositive cycle class  $\alpha$  in  $X$ .

**Corollary 2.4.** *Let  $X$  be a projective variety over a field  $k$ . If  $L$  and  $M$  are ample line bundles on  $X$ , then there exist constants  $C_1, C_2 > 0$  such that*

$$C_1 \deg_M D \leq \deg_L D \leq C_2 \deg_M D$$

for any effective Cartier divisor  $D$  in  $X$ . □

**2B. ZL-dense formal curves in projective varieties.** Let  $X$  be an algebraic variety over a field  $k$  and consider the ring of formal power series  $k[[q]]$ . By a *formal curve* in  $X$  we mean a morphism of  $k$ -schemes  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$ , or, equivalently, a morphism of formal  $k$ -schemes  $\hat{\varphi} : \text{Spf } k[[q]] \rightarrow X$ . The  $k$ -point of  $X$  obtained by composing the  $k$ -point of  $\text{Spec } k[[q]]$  given by the ideal  $(q) \subset k[[q]]$  with  $\hat{\varphi}$  is denoted by  $\hat{\varphi}(0)$ .

Let  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$  be a formal curve in  $X$ , and  $D$  be an effective Cartier divisor in  $X$ . We define the *intersection multiplicity* of  $D$  with  $\hat{\varphi}$  (at  $\hat{\varphi}(0)$ ) by

$$\text{mult}_{\hat{\varphi}} D := \text{ord}_0 \hat{\varphi}^* f,$$

where  $f \in \mathcal{O}_{X, \hat{\varphi}(0)}$  is any local equation for  $D$  around  $\hat{\varphi}(0)$ . This clearly does not depend on the choice of  $f$ . The multiplicity function  $\text{mult}_{\hat{\varphi}}$  is additive and takes values in  $\mathbb{N} \cup \{+\infty\}$ .

**Definition 2.5.** Let  $X$  be a projective variety of dimension  $n$  over a field  $k$  and let  $L$  be any ample line bundle on  $X$ . We say that a formal curve  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$  in  $X$  is *ZL-dense* if there exists a constant  $C > 0$  such that

$$\text{mult}_{\hat{\varphi}} D \leq C(\deg_L D)^n \tag{2-1}$$

for every effective Cartier divisor  $D$  in  $X$ .

Observe that the choice of  $L$  in the above definition is irrelevant by [Corollary 2.4](#).

Let us remark that the exponent  $n = \dim X$  intervening in the polynomial bound [\(2-1\)](#) is the smallest possible one:

**Proposition 2.6.** *Let  $X$  be a projective variety of dimension  $n$  over  $k$  endowed with an ample line bundle  $L$ , and  $\hat{\varphi}$  be a formal curve in  $X$ . Then there exists  $\varepsilon > 0$  and sequence of effective Cartier divisors  $(D_i)_{i \geq 1}$  on  $X$  satisfying*

$$\lim_{i \rightarrow +\infty} \deg_L D_i = +\infty \quad \text{and} \quad \text{mult}_{\hat{\varphi}} D_i \geq \varepsilon(\deg_L D_i)^n \quad \text{for every } i \geq 1.$$

In the above statement we allow the possibility that  $\text{mult}_{\hat{\varphi}} D_i = +\infty$  (i.e., the divisor  $D_i$  vanishes identically along the formal curve  $\hat{\varphi}$ ) by adopting the standard convention that  $+\infty \geq t$  for every  $t \in \mathbb{R}$ .

*Proof.* Assume first that  $X = \mathbb{P}_k^n$  and  $L = \mathcal{O}(1)$ . Consider the natural projection  $\mathbb{A}_k^n \setminus \{0\} \rightarrow \mathbb{P}_k^n$  and lift  $\hat{\varphi}$  to some  $\hat{\psi} : \text{Spec } k[[q]] \rightarrow \mathbb{A}_k^n \setminus \{0\}$ . Let  $i \geq 1$  be an integer. If  $P = \sum_{|I|=i} a_I X^I \in k[X_0, \dots, X_n]$  is a homogeneous polynomial of degree  $i$ , seen as a regular function on  $\mathbb{A}_k^n \setminus \{0\}$ , then we may write

$$\hat{\psi}^*P = \sum_{j=0}^{\infty} \left( \sum_{|I|=i} a_I b_{I,j} \right) q^j \in k[[q]]$$

for some  $b_{I,j} \in k$  depending on the coefficients of the  $n + 1$  formal series defining  $\hat{\psi}$ . Since

$$\text{card}\{I \in \mathbb{N}^{n+1} \mid |I| = i\} = \binom{i+n}{n} > \frac{1}{n!}i^n,$$

it follows from elementary linear algebra that there exists a nonzero homogeneous polynomial  $P_i$  of degree  $i$  such that  $\text{ord}_0 \hat{\psi}^*P_i \geq (1/n!)i^n$ . By considering the Cartier divisors  $D_i$  on  $\mathbb{P}_k^n$  induced by  $P_i$ , we see that we may take  $\varepsilon = 1/n!$  in this case.

The general case follows from the above one by considering a finite surjective morphism  $f : X \rightarrow \mathbb{P}_k^n$  satisfying  $f^*\mathcal{O}(1) \cong L^m$  for some  $m \geq 1$ . □

**Remark 2.7.** It follows from the above proof that the Cartier divisors  $D_i$  can actually be taken in the linear system  $|L^{\otimes mi}|$  for some fixed integer  $m \geq 1$ .

Any ZL-dense formal curve  $\hat{\varphi}$  in a projective variety  $X$  has a dense image in the Zariski topology. Indeed, since  $X$  is projective, any Zariski-closed subset of  $X$  is contained in the support of some effective Cartier divisor of  $X$ ; then, one simply remarks that ZL-density implies that  $\text{mult}_{\hat{\varphi}} D < +\infty$  for any effective Cartier divisor  $D$  in  $X$ , so that the image of  $\hat{\varphi}$  is not contained in the support of  $D$ .

The following example shows that the converse is not true in general.

**Example 2.8** (lacunary series). Let  $k$  be a field and  $(n_i)_{i \geq 0}$  be an increasing sequence of natural numbers satisfying  $\lim_{i \rightarrow +\infty} n_{i+1}/n_i^2 = +\infty$ . If  $h \in k[[q]]$  is any formal series of the form

$$h(q) = \sum_{i \geq 0} a_i q^{n_i}, \quad a_i \neq 0,$$

then the formal curve  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow \mathbb{P}_k^2$ , given in homogeneous coordinates by  $\hat{\varphi}(q) = (1 : q : h(q))$ , is not ZL-dense. Indeed, for any integer  $d \geq 0$ , we may consider the homogeneous polynomial of degree  $n_d$

$$P_d = X_0^{n_d-1} X_2 - \sum_{i=0}^d a_i X_0^{n_d-n_i} X_1^{n_i}$$

so that

$$\frac{\text{mult}_{\hat{\varphi}} \text{div}(P_d)}{n_d^2} = \frac{n_{d+1}}{n_d^2} \rightarrow +\infty$$

as  $d \rightarrow +\infty$ .

Observe that the image of  $\hat{\varphi}$  is indeed Zariski-dense. By contradiction, if  $C \subset \mathbb{P}_k^2$  is an irreducible curve containing the image of  $\hat{\varphi}$ , then, for any effective Cartier divisor  $D$  in  $\mathbb{P}_k^2$  whose support does not contain  $C$ , we have  $\text{mult}_{\hat{\varphi}} D = i(\hat{\varphi}(0), C \cdot D) \leq \text{deg } C \cdot \text{deg } D$ . By construction, this is absurd for  $D = \text{div}(P_d)$  and  $d$  sufficiently large.

For natural examples of ZL-dense formal curves we refer to [Appendix B](#).

**2C. Reformulation in terms of sections of an ample line bundle.** Let  $X$  be an algebraic variety over a field  $k$  and  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$  be a formal curve. If  $L$  is any line bundle on  $X$ , and  $s$  is a section of  $L$  on a neighborhood of  $\hat{\varphi}(0)$ , we may consider the vanishing order of  $\hat{\varphi}^*s \in \Gamma(\text{Spec } k[[q]], \hat{\varphi}^*L)$  at  $q = 0$ , which coincides with the intersection multiplicity of the effective Cartier divisor  $\text{div}(s)$  with  $\hat{\varphi}$ :

$$\text{ord}_0 \hat{\varphi}^*s = \text{mult}_{\hat{\varphi}} \text{div}(s).$$

The next proposition shows that ZL-density is a condition that has to be checked only for Cartier divisors arising from sections of powers of some fixed ample line bundle.

**Proposition 2.9.** *Let  $X$  be a projective variety of dimension  $n$  over a field  $k$  and  $L$  be an ample line bundle on  $X$ . A formal curve  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$  is ZL-dense if and only if there exists a constant  $C > 0$  such that*

$$\text{ord}_0 \hat{\varphi}^*s \leq Cd^n$$

for any integer  $d \geq 1$ , and any  $s \in \Gamma(X, L^{\otimes d}) \setminus \{0\}$ .

*Proof.* The necessity follows from the fact that  $\deg_L \text{div}(s) = (\deg_L X)d$  for any  $s \in \Gamma(X, L^{\otimes d}) \setminus \{0\}$ .

To prove the sufficiency, fix any finite surjective morphism  $f : X \rightarrow \mathbb{P}_k^n$  such that  $f^*\mathcal{O}(1)$  is isomorphic to  $L^{\otimes m}$  for some  $m \geq 1$ . If  $E$  is an effective Cartier divisor in  $\mathbb{P}_k^n$ , then there exists a section  $s \in \Gamma(\mathbb{P}_k^n, \mathcal{O}(\deg_{\mathcal{O}(1)} E))$  satisfying  $E = \text{div}(s)$ , so that

$$\text{mult}_{f \circ \hat{\varphi}} E = \text{ord}_0(f \circ \hat{\varphi})^*s = \text{ord}_0 \hat{\varphi}^*(f^*s) \leq C(m \deg_{\mathcal{O}(1)} E)^n = Cm^n (\deg_{\mathcal{O}(1)} E)^n. \tag{2-2}$$

Let  $D$  be an effective Cartier divisor in  $X$ . Since  $f$  is finite and  $\mathbb{P}_k^n$  is normal, we may define the pushforward  $f_*D$  by taking norms: there is an open affine covering  $(U_i)_i$  of  $\mathbb{P}_k^n$  such that  $D$  admits a local equation  $h_i$  on each  $f^{-1}(U_i)$ , and we define  $f_*D = [(\text{Norm}_f(h_i), U_i)_i]$ . As  $f^*f_*D - D$  is effective, we obtain

$$\text{mult}_{\hat{\varphi}} D \leq \text{mult}_{\hat{\varphi}} f^*f_*D = \text{mult}_{f \circ \hat{\varphi}} f_*D.$$

Note that the Weil divisor associated to  $f_*D$  coincides with the pushforward (of cycles) of the Weil divisor associated to  $D$  [Fulton 1984, Proposition 1.4]. In particular, the projection formula gives

$$\deg_{\mathcal{O}(1)} f_*D = \deg_{L^{\otimes m}} D = m^{n-1} \deg_L D$$

so that, by (2-2),

$$\text{mult}_{\hat{\varphi}} D \leq Cm^n (\deg_{\mathcal{O}(1)} f_*D)^n = Cm^{n^2} (\deg_L D)^n. \quad \square$$

An advantage of considering the above equivalent form of ZL-density stems from the vector space structure of the sets  $\Gamma(X, L^{\otimes d})$ ,  $d \geq 1$ . In general, a formal curve  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$  induces, for every integer  $d \geq 1$ , a decreasing filtration by linear subspaces  $(E_d^i)_{i \geq 0}$  on the  $k$ -vector space  $E_d := \Gamma(X, L^{\otimes d})$  defined by  $E_d^i := \{s \in E_d \mid \text{ord}_0 \hat{\varphi}^*s \geq i\}$ .

**Remark 2.10.** Since  $E_d$  is finite-dimensional, there exists  $i_d \geq 1$  such that

$$E_d^{i_d} = \bigcap_{i \geq 0} E_d^i = \{s \in E_d \mid \hat{\varphi}^*s = 0\}.$$

In other words, for every  $s \in E_d$  such that  $\hat{\varphi}^*s \neq 0$ , we have  $\text{ord}_0 \hat{\varphi}^*s < i_d$ . In particular, this shows that one may replace in [Proposition 2.9](#) the condition “for any integer  $d \geq 1$ ” by the weaker “for any sufficiently large integer  $d$ ”.

As a first application of [Proposition 2.9](#), we use the filtration  $(E_d^i)_{i \geq 0}$  to show that ZL-density is a geometric property.

**Proposition 2.11.** *Let  $X$  be a geometrically integral projective variety over a field  $k$  and  $\hat{\varphi} : \text{Spf } k[[q]] \rightarrow X$  be a formal curve. Then, for any field extension  $K$  of  $k$ , the formal curve  $\hat{\varphi}_K : \text{Spf } K[[q]] \rightarrow X_K$ , obtained from  $\hat{\varphi}$  by base change, is ZL-dense in  $X_K$  if and only if  $\hat{\varphi}$  is ZL-dense in  $X$ .*

*Proof.* Let  $d \geq 1$  and  $i \geq 0$  be integers. Note that  $E_d \otimes_k K$  may be canonically identified with  $\Gamma(X_K, L_K^{\otimes d})$ . Moreover, since  $E_d^i$  is the kernel of the  $k$ -linear map

$$E_d \rightarrow \Gamma(\text{Spec } k[[q]], \hat{\varphi}^*L^{\otimes d}) \otimes_k k[[q]]/(q^i), \quad s \mapsto \hat{\varphi}^*s \bmod q^i,$$

we conclude that  $E_d^i \otimes_k K = \{t \in E_d \otimes_k K \mid \text{ord}_0 \hat{\varphi}_K^*t \geq i\}$ .

In particular, for every integer  $d \geq 1$ , and any real number  $\kappa > 0$ ,  $\text{ord}_0 \hat{\varphi}^*s \leq \kappa$  for every  $s \in \Gamma(X, L^{\otimes d}) \setminus \{0\}$  if and only if  $\text{ord}_0 \hat{\varphi}_K^*t \leq \kappa$  for every  $t \in \Gamma(X_K, L_K^{\otimes d}) \setminus \{0\}$ . □

As another application of [Proposition 2.9](#), we prove the following result which will be used in our proof of [Proposition 2.15](#).

**Proposition 2.12.** *Let  $f : X \rightarrow Y$  be a surjective morphism between projective varieties of dimension  $n$  over a field  $k$  and let  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$  be a formal curve in  $X$ . If  $\hat{\varphi}$  is ZL-dense in  $X$ , then  $f \circ \hat{\varphi}$  is ZL-dense in  $Y$ .*

*Proof.* Let  $L$  be an ample line bundle on  $X$  admitting a global section  $s \in \Gamma(X, L) \setminus \{0\}$ , and  $M$  be any ample line bundle on  $Y$ . Since  $f^*M$  is semiample,  $N := L \otimes f^*M$  is ample [[EGA II 1961](#), Proposition 4.5.6(ii)].

Let  $d \geq 1$  be an integer, and  $t \in \Gamma(Y, M^{\otimes d}) \setminus \{0\}$ . Since  $f$  is surjective,  $f^*t \neq 0$ . Thus  $s^{\otimes d} \otimes f^*t$  is a nonzero global section of  $N^{\otimes d}$  and, since  $\hat{\varphi}$  is ZL-dense in  $X$ , there exists a constant  $C > 0$  independent of  $d$  such that

$$\text{ord}_0 \hat{\varphi}^*(s^{\otimes d} \otimes f^*t) \leq Cd^n.$$

To complete the proof, it is sufficient to remark that

$$\text{ord}_0(f \circ \hat{\varphi})^*(t) = \text{ord}_0 \hat{\varphi}^*(f^*t) \leq \text{ord}_0 \hat{\varphi}^*(s^{\otimes d} \otimes f^*t). \quad \square$$

**Remark 2.13.** The above proposition, combined with the arguments in the proof of [Proposition 2.9](#), actually shows that for any finite surjective morphism  $f : X \rightarrow Y$  between projective varieties over a field  $k$ , with  $Y$  normal, a formal curve  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$  is ZL-dense in  $X$  if and only if  $f \circ \hat{\varphi}$  is ZL-dense in  $Y$ .

**2D. ZL-density in quasiprojective varieties.** We defined a notion of ZL-density for formal curves in *projective* varieties. In this subsection, under a mild technical condition, we extend this notion, via compactification, to formal curves in *quasiprojective* varieties. To assure that we obtain a well-defined notion, we must show that this does not depend on the choice of compactification.

We start by reformulating ZL-density in terms of *Weil divisors* in place of Cartier divisors. Let  $X$  be an algebraic variety over a field  $k$ , and  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$  be a formal curve. If  $\hat{\varphi}(0)$  is a regular point of  $X$ , then we may define the intersection multiplicity of a Weil divisor with  $\hat{\varphi}$ : if  $U$  is a regular open neighborhood of  $\hat{\varphi}(0)$  and  $Z$  is a Weil divisor on  $X$ , then  $Z \cap U$  is induced by some Cartier divisor  $D$  on  $U$ , and we define

$$\text{mult}_{\hat{\varphi}} Z = \text{mult}_{\hat{\varphi}} D.$$

We may thus mimic the proof of [Proposition 2.9](#) to obtain the following result.

**Proposition 2.14.** *Let  $X$  be a projective variety of dimension  $n$  over a field  $k$ ,  $L$  be an ample line bundle on  $X$ , and  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$  be a formal curve such that  $\hat{\varphi}(0)$  is a regular point of  $X$ . Then,  $\hat{\varphi}$  is ZL-dense in  $X$  if and only if there exists a constant  $C > 0$  such that, for every effective Weil divisor  $Z$  on  $X$ ,*

$$\text{mult}_{\hat{\varphi}} Z \leq C(\text{deg}_L Z)^n.$$

We are now in position to prove that a modification away from  $\hat{\varphi}$  does not affect ZL-density.

**Proposition 2.15.** *Let  $f : X \rightarrow Y$  be a proper morphism between projective varieties over a field  $k$ , and  $U$  be an open subset of  $Y$  such that  $f$  induces an isomorphism  $f^{-1}(U) \xrightarrow{\sim} U$ . If  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow f^{-1}(U) \subset X$  is a formal curve such that  $\hat{\varphi}(0)$  is a regular point of  $X$ , then  $\hat{\varphi}$  is ZL-dense in  $X$  if and only if  $f \circ \hat{\varphi}$  is ZL-dense in  $Y$ .*

*Proof.* Since  $f : X \rightarrow Y$  is a proper birational morphism, and  $Y$  is irreducible,  $f$  is surjective. By [Proposition 2.12](#), if  $\hat{\varphi}$  is ZL-dense in  $X$ , then  $f \circ \hat{\varphi}$  is ZL-dense in  $Y$ .

Conversely, suppose that  $f \circ \hat{\varphi}$  is ZL-dense in  $Y$ . Fix ample line bundles  $L$  on  $X$  and  $M$  on  $Y$ , and let  $Z$  be an effective Weil divisor on  $X$ . Since  $f$  is an isomorphism over  $U$  and  $\hat{\varphi}$  factors through  $f^{-1}(U)$ , we have

$$\text{mult}_{\hat{\varphi}} Z = \text{mult}_{f \circ \hat{\varphi}} f_* Z.$$

As  $f \circ \hat{\varphi}$  is ZL-dense in  $Y$ , there is a constant  $C_1 > 0$  (not depending on  $Z$ ) such that

$$\text{mult}_{f \circ \hat{\varphi}} f_* Z \leq C_1(\text{deg}_M f_* Z)^n,$$

where  $n = \dim Y = \dim X$ . By the projection formula,  $\text{deg}_M f_* Z = \text{deg}_{f^*M} Z$ . Since  $f^*M$  is semiample, it follows from [Proposition 2.2](#) that there exists a constant  $C_2 > 0$  such that  $\text{deg}_{f^*M} Z \leq C_2 \text{deg}_L Z$ . We conclude that

$$\text{mult}_{\hat{\varphi}} Z \leq C_1 C_2^n (\text{deg}_L Z)^n. \quad \square$$

Let us remark that it is essential in our proof above to consider the pushforward of a Weil divisor on  $X$  under the proper morphism  $f$ . This explains why we reformulated ZL-density in terms of Weil divisors, as the pushforward of a Cartier divisor under a proper morphism is not well-defined in general.

**Corollary 2.16.** *Let  $X$  be a quasiprojective variety over a field  $k$ , and  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$  be a formal curve such that  $\hat{\varphi}(0)$  is a regular point of  $X$ . If  $j_i : X \hookrightarrow \bar{X}_i$ ,  $i = 1, 2$ , are two projective compactifications of  $X$ , then  $j_1 \circ \hat{\varphi}$  is ZL-dense in  $\bar{X}_1$  if and only if  $j_2 \circ \hat{\varphi}$  is ZL-dense in  $\bar{X}_2$ .*

*Proof.* Consider the scheme-theoretic image  $\bar{X}$  of  $(j_1, j_2) : X \rightarrow \bar{X}_1 \times_k \bar{X}_2$  and apply Proposition 2.15 to the natural projections  $\bar{X} \rightarrow \bar{X}_i$ ,  $i = 1, 2$ . □

This enables us to define a good notion of ZL-density in a quasiprojective variety.

**Definition 2.17.** Let  $X$  be a quasiprojective variety over a field  $k$ , and  $\hat{\varphi} : \text{Spec } k[[q]] \rightarrow X$  be a formal curve such that  $\hat{\varphi}(0)$  is a regular point of  $X$ . We say that  $\hat{\varphi}$  is ZL-dense in  $X$  if there exists a projective compactification  $j : X \hookrightarrow \bar{X}$  of  $X$  such that  $j \circ \hat{\varphi}$  is ZL-dense in  $\bar{X}$ .

### 3. Moderate growth and jet estimates on complex disks

In this section we introduce *characteristic functions* and *moderate growth* of certain 2-forms on a complex disk; these are purely analytic notions. We then proceed to establishing natural estimates on jets of sections of holomorphic line bundles on disks. In a sense, our exposition is more basic than the usual accounts on value distribution theory, since characteristic functions of analytic curves will be a special case of our construction.

The kind of jet estimates we consider here play a central role in Diophantine approximation and transcendence proofs. They notably appear in such proofs using the formalism of Arakelov geometry, to estimate the height of evaluation maps, when applying Bost’s method of slopes; see, for instance, [Bost 1996; 2001; Graftieaux 2005; Gasbarri 2010; 2013; Herblot 2012].

**3A. Characteristic functions.** Let  $r > 0$  be a real number, and  $p \in D_r := \{z \in \mathbb{C} \mid |z| < r\}$ . Recall that the *Green’s function* of  $D_r$  at  $p$  is defined by

$$g_{D_r,p} : \mathbb{C} \rightarrow (-\infty, +\infty], \quad z \mapsto \log^+ \left| \frac{r^2 - \bar{p}z}{r(z - p)} \right|.$$

This is a superharmonic (thus locally integrable) function on  $\mathbb{C}$ , real-valued and continuous on  $\mathbb{C} \setminus \{p\}$ , strictly positive and harmonic (thus  $C^\infty$ ) on  $D_r \setminus \{p\}$ , and vanishing identically on  $\mathbb{C} \setminus D_r$ .

For any locally bounded 2-form  $\alpha$  defined on an open neighborhood of  $D_r$ , we define

$$T_{\alpha,p}(r) := \int_{\mathbb{C}} g_{D_r,p} \alpha.$$

**Remark 3.1.** An integration by parts with  $u(t) = \int_{D_t} \alpha$  and  $v(t) = \log t$  shows that

$$T_{\alpha,0}(r) = \int_0^r \left( \int_{D_t} \alpha \right) \frac{dt}{t}.$$

Since  $g_{D_r,p}$  is the composition of  $g_{D_r,0}$  with the automorphism of  $D_r$  given by

$$\sigma_{r,p}(z) = \frac{r^2(z-p)}{r^2 - \bar{p}z},$$

we obtain

$$T_{\alpha,p}(r) = \int_0^r \left( \int_{D_t} (\sigma_{r,p}^{-1})^* \alpha \right) \frac{dt}{t}.$$

Let  $R > 0$  be a real number, and  $\alpha$  be a locally bounded semipositive  $(1, 1)$ -form on the disk  $D_R$ . The nondecreasing function

$$T_\alpha : (0, R) \rightarrow \mathbb{R}_{\geq 0}, \quad r \mapsto T_{\alpha,0}(r),$$

is the *characteristic function* of  $\alpha$  in  $D_R$ .

We shall be particularly interested in the following special case. Let  $\bar{L} = (L, \|\cdot\|)$  be a semipositive Hermitian line bundle on  $D_R$ . The *characteristic function* of  $\bar{L}$  in  $D_R$  is defined by

$$T_{\bar{L}} := T_{c_1(\bar{L})}.$$

We also define  $T_{\bar{L},p}(r) = T_{c_1(\bar{L}),p}(r)$  for  $p \in D_r \subset D_R$ .

**Remark 3.2.** Let  $d \geq 1$  be an integer. As  $c_1(\bar{L}^{\otimes d}) = d \cdot c_1(\bar{L})$ , we have  $T_{\bar{L}^{\otimes d},p}(r) = d \cdot T_{\bar{L},p}(r)$ .

**3B. Forms of moderate growth.** Let  $R > 0$  be a real number, and  $\alpha$  be a locally bounded semipositive  $(1,1)$ -form on  $D_R$ .

**Definition 3.3.** We say that  $\alpha$  has *moderate growth* if there exist constants  $a, b > 0$  such that

$$T_\alpha(r) \leq a + b \log \frac{1}{1-r/R}$$

for any  $r \in (0, R)$ . When  $\alpha = c_1(\bar{L})$  for some semipositive Hermitian line bundle  $\bar{L}$  on  $D_R$ , we rather say that  $\bar{L}$  has *moderate growth*.

The motivating example of a form of moderate growth is the following one.

**Example 3.4** (Poincaré form). Let

$$d\mu_R := \frac{i}{2} \left( \frac{R}{R^2 - |z|^2} \right)^2 dz \wedge d\bar{z}$$

be the 2-form associated to the surface element of the *Poincaré metric*  $(R/(R^2 - |z|^2))|dz|$  on  $D_R$ . A direct computation shows that, for any  $r \in (0, R)$ ,

$$T_{d\mu_R}(r) = \frac{\pi}{2} \log \frac{1}{1+r/R} + \frac{\pi}{2} \log \frac{1}{1-r/R}.$$

Thus, the 2-form  $d\mu_R$  on  $D_R$  has moderate growth.

We have defined moderate growth for a form  $\alpha$  as a growth condition on  $T_{\alpha,0}(r)$  with respect to  $r$ . Our next result shows that a similar growth condition for  $T_{\alpha,p}(r)$  holds uniformly for  $p$  varying in a fixed compact subset.

**Lemma 3.5.** *Let  $0 < r_0 < r_1 < R_1 < R$  be real numbers. Then there exists a constant  $C > 0$  such that, for every  $p \in D_{r_0}$ , and every  $r \in [R_1, R)$ , we have*

$$g_{D_r,p} \leq g_{D_{r_1},p} + Cg_{D_r,0}.$$

*Proof.* Let  $r \in [R_1, R)$  and  $p \in D_{r_0}$ . We set

$$C_{r,p} := \max_{z \in \partial D_{r_1}} \frac{g_{D_r,p}(z)}{g_{D_r,0}(z)}$$

and

$$f_{r,p} := g_{D_r,p} - g_{D_{r_1},p} - C_{r,p}g_{D_r,0}.$$

By the definition of  $C_{r,p}$ , we see that  $f_{r,p} \leq 0$  over  $\partial D_{r_1}$ . Moreover,  $f_{r,p}$  vanishes identically on  $\partial D_r$ . Since  $f_{r,p}$  is subharmonic over the domains  $D_{r_1}$  and  $D_r \setminus \bar{D}_{r_1}$ , by the maximum principle, we conclude that  $f_{r,p} \leq 0$  everywhere.

To finish the proof, it is sufficient to remark that  $C_{r,p}$  is uniformly bounded for  $r \in [R_1, R)$  and  $p \in D_{r_0}$ . Indeed, for  $z \in \partial D_{r_1}$ , we have

$$\frac{g_{D_r,p}(z)}{g_{D_r,0}(z)} = \frac{\log \left| \frac{r^2 - \bar{p}z}{r(z-p)} \right|}{\log |r/z|} \leq \frac{\log \frac{R^2 + r_0r_1}{R_1(r_1 - r_0)}}{\log(R_1/r_1)}. \quad \square$$

**Proposition 3.6.** *Let  $R > 0$  be a real number,  $K \subset D_R$  be a compact subset, and  $\alpha$  be a locally bounded semipositive (1,1)-form on  $D_R$ . Fix  $R_1 \in (0, R)$  such that  $K \subset D_{R_1}$ . If  $\alpha$  has moderate growth, then there exist real numbers  $a, b > 0$  such that*

$$\sup_{p \in K} T_{\alpha,p}(r) \leq a + b \log \frac{1}{1 - r/R}$$

for every  $r \in [R_1, R)$ .

*Proof.* Let  $0 < r_0 < r_1$  be real numbers such that  $K \subset D_{r_0}$  and  $r_1 < R_1$ . By Lemma 3.5, there exists a constant  $C > 0$  such that

$$g_{D_r,p} \leq g_{D_{r_1},p} + Cg_{D_r,0}$$

for every  $r \in [R_1, R)$  and every  $p \in K$ , so that

$$T_{\alpha,p}(r) \leq T_{\alpha,p}(r_1) + CT_{\alpha,0}(r).$$

Since  $\alpha$  has moderate growth, to conclude it is sufficient to remark that the function  $p \mapsto T_{\alpha,p}(r_1)$  is continuous, thus bounded on the compact  $K$ . □

**3C. Jets and characteristic functions.** Let  $r > 0$  be a real number and  $p \in D_r$ . We define a probability measure  $\pi_{r,p}$  supported on  $\partial D_r$  by

$$\int \psi \pi_{r,p} = \frac{1}{2\pi} \int_0^{2\pi} \psi(\sigma_{r,p}^{-1}(re^{i\theta})) d\theta,$$

where  $\sigma_{r,p}$  is the function defined in Remark 3.1. For the next proposition, we shall need the following classical result.

**Lemma 3.7.** *As an equality of distributions on  $\mathbb{C}$ , we have*

$$-2dd^c g_{D_r,p} = \delta_p - \pi_{r,p}.$$

*Proof.* Apply Remark 3.1 and Stokes' theorem (see also Section 1F5). □

Let  $U$  be an open subset of  $\mathbb{C}$  and  $\bar{L} = (L, \|\cdot\|)$  be a Hermitian line bundle over  $U$ . If  $s \in \Gamma(U, L)$  and  $z \in U$ , the  $m$ -th jet of  $s$  at  $z$  is denoted by  $j_z^m s$ . When  $s$  has vanishing order at least  $m$  at  $z$ ,  $j_z^m s$  is simply an element of the fiber of  $L \otimes (\Omega_U^1)^{\otimes m}$  at  $z$ . In this case, if  $r$  is a real number strictly greater than  $|z|$ , we denote by

$$\|j_z^m s\|_r$$

the norm of  $j_z^m s$  with respect to the metric  $\|\cdot\|$  on  $L$  and the norm on  $\Omega_{D_r,z}^1$  given by the dual of the Poincaré metric  $(r/(r^2 - |z|^2))|dz|$  on  $D_r$ .

The following result, relating jets of sections with characteristic functions, is a basic tool in Nevanlinna theory (see, for instance, [Noguchi and Winkelmann 2014, Section 2.3]); variants of it were used in the context of Diophantine approximation in [Bost 2001, Proposition 4.14; 2004, Section 3; Gasbarri 2010, Theorem 5.13].

**Proposition 3.8.** *Let  $R > 0$  be a real number,  $\bar{L} = (L, \|\cdot\|)$  be a semipositive Hermitian line bundle on  $D_R$ , and  $p \in D_R$ . For every real number  $r \in (|p|, R)$  and every global section  $s \in \Gamma(D_R, L) \setminus \{0\}$ , if  $m := \text{ord}_p s$  denotes the vanishing order of  $s$  at  $p$ , we have*

$$\log \|j_p^m s\|_r = T_{\bar{L},p}(r) + \int \log \|s\| \pi_{r,p} - \int g_{D_r,p} \delta_{\text{div}(s)-m[p]}. \tag{3-1}$$

We start with a lemma that follows immediately from the explicit formula for the Green's functions on disks (see Section 3A).

**Lemma 3.9.** *With the above notation, if  $E \subset D_R$  denotes the support of the divisor  $\text{div}(s)$ , then the distribution  $\log \|s\| + mg_{D_r,p}$  on  $D_R$  defines a  $C^\infty$  function over  $(D_r \setminus E) \cup \{p\}$  and a continuous function over  $\partial D_r \setminus E$ . Moreover,*

$$\lim_{z \rightarrow p} (\log \|s(z)\| + mg_{D_r,p}(z)) = \log \|j_p^m s\|_r. \tag{3-2} \quad \square$$

Observe now that, for a fixed  $s$ , both sides in formula (3-1) are continuous with respect to  $r$ . Since  $E$  is discrete, we may thus assume that  $\partial D_r \cap E = \emptyset$ .

*Proof of Proposition 3.8.* The Poincaré–Lelong formula yields the identity of currents

$$dd^c \log \|s\|^2 = \delta_{\text{div}(s)} - c_1(\bar{L}).$$

Thus, by Lemma 3.7,

$$c_1(\bar{L}) = -2dd^c(\log \|s\| + mg_{D_r,p}) + \delta_{\text{div}(s)-m[p]} + m\pi_{r,p}.$$

Since  $\pi_{r,p}$  is supported on  $\partial D_r$ , and  $g_{D_r,p}$  vanishes identically on  $\partial D_r$ , we obtain

$$T_{\bar{L},p}(r) = \int g_{D_r,p} \cdot (-2dd^c(\log \|s\| + mg_{D_r,p})) + \int g_{D_r,p} \delta_{\text{div}(s)-m[p]}.$$

Note that, by our choice of  $r$  and by Lemma 3.9, the above products of distributions are well-defined.

By another application of Lemma 3.7,

$$\begin{aligned} T_{\bar{L},p}(r) &= \int (\log \|s\| + mg_{D_r,p})(\delta_p - \pi_{r,p}) + \int g_{D_r,p} \delta_{\text{div}(s)-m[p]} \\ &= \int (\log \|s\| + mg_{D_r,p}) \delta_p - \int \log \|s\| \pi_{r,p} + \int g_{D_r,p} \delta_{\text{div}(s)-m[p]}, \end{aligned}$$

where in the second equality we used once again that  $g_{D_r,p}$  vanishes identically on  $\partial D_r$ . To conclude, we apply once more Lemma 3.9, which ensures that the function  $\log \|s\| + mg_{D_r,p}$  tends to  $\log \|j_p^m s\|_r$  at  $p$ . □

**Corollary 3.10.** *Let us keep the notation of Proposition 3.8. Then,*

$$\log \|j_p^m s\|_r \leq T_{\bar{L},p}(r) + \log \|s\|_{L^\infty(\partial D_r)}. \tag{3-2}$$

*If, moreover,  $p'$  is another point of  $D_r$ , and  $m'$  denotes the vanishing order of  $s$  at  $p'$ , then*

$$\log \|j_{p'}^{m'} s\|_r \leq T_{\bar{L},p}(r) + \log \|s\|_{L^\infty(\partial D_r)} - m' g_{D_r,p}(p'). \tag{3-3}$$

*Proof.* Since  $\pi_{r,p}$  is a probability measure over  $\partial D_r$ , we have  $\int \log \|s\| \pi_{r,p} \leq \log \|s\|_{L^\infty(\partial D_r)}$ . Thus, the estimate (3-2) (resp. (3-3)) follows immediately from the nonnegativity both of the function  $g_{D_r,p}$  and of the distribution  $\delta_{\text{div}(s)-m[p]}$  (resp.  $\delta_{\text{div}(s)-m[p]-m'[p']}$ ). □

**3D. A first application of moderate growth.** We shall need the following elementary inequality.

**Lemma 3.11.** *Let  $A, B$ , and  $R$  be positive real numbers. Set*

$$r := R \frac{B}{A+B}.$$

*If  $\log(B/A) \geq 2$ , then*

$$A \log\left(\frac{1}{1-r/R}\right) - B \log r \leq 2A \log\left(\frac{B}{A}\right) - B \log R.$$

*Proof.* By homogeneity, we may assume that  $A = R = 1$ , so that  $r = B/(1+B)$  and our statement is equivalent to

$$\log(1+B) - B \log(B/(1+B)) \leq 2 \log B$$

when  $\log B \geq 2$ . By subtracting  $\log B$  from both sides, we see that this is yet equivalent to

$$(1 + B) \log(1 + 1/B) \leq \log B,$$

when  $\log B \geq 2$ . Now, this last inequality follows trivially from the fact that  $\log(1 + 1/B) \leq 1/B$ .  $\square$

Note that  $r$  as above is the minimum of the real function  $t \mapsto A \log(1/(1 - t/R)) - B \log t$  defined on the open interval  $(0, R)$ .

**Proposition 3.12.** *Let  $R > 0$  be a real number and  $\bar{L} = (L, \|\cdot\|)$  be a semipositive Hermitian line bundle on  $D_R$ . If  $\bar{L}$  has moderate growth, then there exist constants  $\kappa_1, \kappa_2 > 0$  such that for every integer  $d \geq 1$  and every bounded global section  $s \in \Gamma(D_R, L^{\otimes d}) \setminus \{0\}$ , if we set  $m := \text{ord}_0 s$ , then*

$$\log \|j_0^m s\|_R \leq \kappa_1 d + \kappa_2 d \log^+ m + \log \|s\|_{L^\infty(D_R)}.$$

*Proof.* Since  $\bar{L}$  has moderate growth, there exist real numbers  $a, b > 0$  such that

$$T_{\bar{L}}(r) \leq a + b \log\left(\frac{1}{1 - r/R}\right)$$

for any  $r \in (0, R)$ . We may assume that  $b > 1$ . By the jet estimate (3-2), for any  $r \in (0, R)$ , we have

$$\log \|j_0^m s\|_r \leq d \cdot T_{\bar{L}}(r) + \log \|s\|_{L^\infty(\partial D_r)} \leq d \cdot T_{\bar{L}}(r) + \log \|s\|_{L^\infty(D_R)}.$$

As

$$\|j_0^m s\|_r = \|j_0^m s\|_R \left(\frac{r}{R}\right)^m,$$

we obtain

$$\log \|j_0^m s\|_R \leq ad + bd \log\left(\frac{1}{1 - r/R}\right) - m \log r + m \log R + \log \|s\|_{L^\infty(D_R)}. \tag{3-4}$$

The result being trivial for  $m = 0$ , we may assume that  $m > 0$ . We now consider two cases. If  $\log(m/(bd)) < 2$ , then we may take  $r := R/2$  in (3-4) to obtain

$$\log \|j_0^m s\|_R \leq (a + (1 + e^2)b \log 2)d + \log \|s\|_{L^\infty(D_R)}.$$

If  $\log(m/(bd)) \geq 2$ , we apply Lemma 3.11 for  $A = bd$  and  $B = m$ :

$$\log \|j_0^m s\|_R \leq ad + 2bd \log\left(\frac{m}{bd}\right) + \log \|s\|_{L^\infty(D_R)} \leq ad + 2bd \log m + \log \|s\|_{L^\infty(D_R)}. \quad \square$$

**3E. Bounding jets via Taylor coefficients at another point.** In this subsection, we compare Taylor coefficients at different points. We start with a general result, and next we explain how moderate growth improves the estimate.

**Proposition 3.13.** *Let  $R > 0$  be a real number,  $\bar{L} = (L, \|\cdot\|)$  be a semipositive Hermitian line bundle on  $D_R$ , and  $q \in D_R \setminus \{0\}$ . Fix a real number  $R_0$  satisfying  $|q| < R_0 < R$ , and a global holomorphic section  $s_0 \in \Gamma(D_R, L)$  such that  $s_0(q) \neq 0$ . Then there exists a real number  $\kappa > 1$  such that, for every*

integer  $J \geq 1$ , every integer  $d \geq 1$ , and every global section  $s \in \Gamma(D_R, L^{\otimes d}) \setminus \{0\}$ , if  $f$  denotes the germ of holomorphic function at  $q$  such that  $s = f s_0^{\otimes d}$  in a neighborhood of  $q$ , and if  $m := \text{ord}_0 s$ , then

$$\log \|j_0^m s\|_R \leq \log \left( \left( \frac{R_0}{|q|} \right)^{-J} \left( \frac{R}{R_0} \right)^m \|s\|_{L^\infty(\partial D_{R_0})} + \kappa^{d+m+J} \max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!} \right) + d \cdot T_{\bar{L}}(R_0).$$

Let us first remark that if such a constant  $\kappa > 0$  exists for  $s_0 \in \Gamma(D_R, L)$  trivializing  $L$  at  $q$ , then an analogous constant  $\bar{\kappa} > 0$  will exist for *any other* trivialization  $\bar{s}_0$  of  $L$  in a neighborhood of  $q$  — we *do not* require  $\bar{s}_0$  to be a global section. Indeed, if we write  $s = \bar{f} \bar{s}_0^{\otimes d}$  and  $\bar{s}_0 = u s_0$  in a neighborhood of  $q$ , then  $f = \bar{f} u^d$  and

$$\max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!} \leq \max_{0 \leq j < J} \sum_{k+l=j} \frac{|\bar{f}^{(k)}(q)|}{k!} \frac{|(u^d)^{(l)}(q)|}{l!} \leq \left( J \max_{0 \leq j < J} \frac{|(u^d)^{(j)}(q)|}{j!} \right) \max_{0 \leq j < J} \frac{|\bar{f}^{(j)}(q)|}{j!}.$$

We conclude by the Cauchy inequalities, which ensure that  $\max_{0 \leq j < J} |(u^d)^{(j)}(q)|/j!$  grows at most exponentially in  $d + J$ .

*Proof.* By the above remark, up to replacing  $s_0$  by  $z^{-\text{ord}_0(s_0)} s_0$ , we can assume that  $s_0$  trivializes  $L$  both at  $q$  and at 0. Let  $a_j \in \mathbb{C}$  be defined by the expansion

$$f(z) = z^m \sum_{j=0}^{\infty} a_j (z - q)^j$$

in a neighborhood of  $q$ , and set

$$g(z) := z^m \sum_{j=0}^{J-1} a_j (z - q)^j.$$

Note that  $g$  extends uniquely to a holomorphic function on  $D_R$ . Let  $s_1, s_2 \in \Gamma(D_R, L^{\otimes d})$  be given by  $s_1 := g s_0^{\otimes d}$  and  $s_2 := s - s_1$ . Observe that both  $s_1$  and  $s_2$  have vanishing order at least  $m$  at 0.

Next, we estimate  $\|j_0^m s_i\|_{R_0}$ ,  $i = 1, 2$ ; for this, we shall first assume that each  $j_0^m s_i \neq 0$ . By the jet estimate (3-2) for  $p = 0$ , we have

$$\log \|j_0^m s_1\|_{R_0} \leq d \cdot T_{\bar{L}}(R_0) + \log \|s_1\|_{L^\infty(\partial D_{R_0})}.$$

Since  $\text{ord}_q s_2 \geq J$ , by the jet estimate (3-3) for  $p = 0$  and  $p' = q$ , we have

$$\log \|j_0^m s_2\|_{R_0} \leq d \cdot T_{\bar{L}}(R_0) + \log \|s_2\|_{L^\infty(\partial D_{R_0})} - \log \left( \frac{R_0}{|q|} \right) J.$$

Thus

$$\begin{aligned} \|j_0^m s\|_R &= \|j_0^m s\|_{R_0} \left( \frac{R}{R_0} \right)^m \leq (\|j_0^m s_1\|_{R_0} + \|j_0^m s_2\|_{R_0}) \left( \frac{R}{R_0} \right)^m \\ &\leq \left( \left( \frac{R}{R_0} \right)^m \|s_1\|_{L^\infty(\partial D_{R_0})} + \left( \frac{R_0}{|q|} \right)^{-J} \left( \frac{R}{R_0} \right)^m \|s_2\|_{L^\infty(\partial D_{R_0})} \right) \exp(d \cdot T_{\bar{L}}(R_0)). \end{aligned}$$

Using that  $\|s_2\|_{L^\infty(\partial D_{R_0})} \leq \|s_1\|_{L^\infty(\partial D_{R_0})} + \|s\|_{L^\infty(\partial D_{R_0})}$ , we get

$$\|j_0^m s\|_R \leq \left( \left( \frac{R_0}{|q|} \right)^{-J} \left( \frac{R}{R_0} \right)^m \|s\|_{L^\infty(\partial D_{R_0})} + \left( 1 + \left( \frac{R_0}{|q|} \right)^{-J} \right) \left( \frac{R}{R_0} \right)^m \|s_1\|_{L^\infty(\partial D_{R_0})} \right) \exp(d \cdot T_{\bar{L}}(R_0)).$$

It should be clear at this point that the same estimate holds if  $j_0^m s_1 = 0$  or  $j_0^m s_2 = 0$ .

We now estimate  $\|s_1\|_{L^\infty(\partial D_{R_0})}$ . For any  $z \in \partial D_{R_0}$ , we have

$$|g(z)| = R_0^m \left| \sum_{j=0}^{J-1} a_j (z-q)^j \right| \leq R_0^m \left( \sum_{j=0}^{J-1} (2R_0)^j \right) \max_{0 \leq j < J} |a_j| \leq J R_0^m \max\{1, (2R_0)^J\} \max_{0 \leq j < J} |a_j|,$$

so that

$$\|s_1\|_{L^\infty(\partial D_{R_0})} = \sup_{z \in \partial D_{R_0}} |g(z)| \|s_0(z)\|^d \leq \|s_0\|_{L^\infty(\partial D_{R_0})}^d J R_0^m \max\{1, (2R_0)^J\} \max_{0 \leq j < J} |a_j|.$$

To finish, we must bound the coefficients  $a_j$ . By definition, for any  $j \in \mathbb{N}$ ,

$$a_j = \frac{1}{j!} \frac{d^j}{dz^j} \Big|_{z=q} \left( \frac{f(z)}{z^m} \right) = \sum_{k=0}^j \left( \frac{(-1)^k}{q^{m+k}} \binom{k+m-1}{k} \frac{f^{(j-k)}(q)}{(j-k)!} \right).$$

If  $j < J$ , then, for any  $0 \leq k \leq j$ , we have the (crude but sufficient) estimate

$$\binom{k+m-1}{k} < \binom{J+m-1}{J} < 2^{m+J-1} < 2^{m+J},$$

so that

$$|a_j| \leq \left( \sum_{k=0}^j \frac{1}{|q|^{m+k}} \right) 2^{m+J} \max_{0 \leq k \leq j} \frac{|f^{(k)}(q)|}{k!}.$$

Thus,

$$\max_{0 \leq j < J} |a_j| \leq J (2 \max\{1, |q|^{-1}\})^{m+J} \max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!}. \quad \square$$

**Proposition 3.14.** *Let  $R > 0$  be a real number,  $\bar{L} = (L, \|\cdot\|)$  be a semipositive Hermitian line bundle over  $D_R$ , and let  $K \subset D_R$  be a compact subset. If  $\bar{L}$  has moderate growth, then there exist real numbers  $\kappa_0, \kappa_1 > 1$  and an integer  $d_0 \geq 1$  such that, for any integer  $d \geq d_0$  and any bounded section  $s \in \Gamma(D_R, L^{\otimes d}) \setminus \{0\}$  for which  $m := \text{ord}_0 s$  satisfies  $m \geq \kappa_0 d$ , we have*

$$\|s(z)\| \leq m^{\kappa_1 d} \left( \frac{|z|}{R} \right)^m \|s\|_{L^\infty(D_R)}$$

for every  $z \in K \setminus \{0\}$ .

*Proof.* Fix any  $R_1 \in (0, R)$  such that  $K \subset D_{R_1}$ . By Proposition 3.6, there exist real numbers  $a, b > 0$  such that

$$T_{\bar{L},z}(r) \leq a + b \log \frac{1}{1-r/R}$$

for any  $z \in K$  and every  $r \in [R_1, R)$ .

Let  $s \in \Gamma(D_R, L^{\otimes d}) \setminus \{0\}$  be a bounded section, and  $z \in K \setminus \{0\}$ . We may assume that  $\text{ord}_z s = 0$ . By the jet estimate (3-3) for  $p = z$  and  $p' = 0$ , we have, for every  $r \in [R_1, R)$ ,

$$\begin{aligned} \log \|s(z)\| &\leq d \cdot T_{\bar{L},z}^-(r) + \log \|s\|_{L^\infty(\partial D_r)} - m \log \frac{r}{|z|} \\ &\leq ad + bd \log \frac{1}{1-r/R} - m \log r + m \log |z| + \log \|s\|_{L^\infty(D_R)}. \end{aligned}$$

Assume  $m \geq e^2bd$  (i.e.,  $\log(m/(bd)) \geq 2$ ). It follows from Lemma 3.11 for  $A = bd$  and  $B = m$  that, if

$$r := R \frac{m}{bd + m},$$

then

$$bd \log \frac{1}{1-r/R} - m \log r \leq 2bd \log \frac{m}{bd} - m \log R.$$

If we also require that  $m \geq b(R_1/(R - R_1))d$ , then  $r \geq R_1$ , so that

$$\log \|s(z)\| \leq 2bd \log m + (ad - 2bd \log bd) + m \log \frac{|z|}{R} + \log \|s\|_{L^\infty(D_R)}.$$

Now, for every integer  $d \geq e^{a/b}/b$ , we have  $ad - 2bd \log bd \leq 0$ , and we get

$$\log \|s(z)\| \leq 2bd \log m + m \log \frac{|z|}{R} + \log \|s\|_{L^\infty(D_R)}.$$

We may thus take  $\kappa_0 := b \max\{e^2, R_1/(R - R_1)\}$ ,  $\kappa_1 := 2b$ , and  $d_0 := \lceil e^{a/b}/b \rceil$ . □

The following result is a combination of Proposition 3.13 together with the existence of a nonzero global section of  $L$  (see the remark following the statement; actually,  $L$  is holomorphically trivial on  $D_R$ ) and Proposition 3.14.

**Corollary 3.15.** *Let  $R > 0$  be a real number,  $\bar{L} = (L, \|\cdot\|)$  be a semipositive Hermitian line bundle on  $D_R$ ,  $q \in D_R \setminus \{0\}$ , and  $s_0$  be a holomorphic trivialization of  $L$  in a neighborhood of  $q$ . Assume moreover that  $\bar{L}$  has moderate growth. Then there exist real numbers  $\kappa_i > 1$ ,  $i = 0, \dots, 4$ , and an integer  $d_0 \geq 1$ , such that, for any integer  $J \geq 1$ , any integer  $d \geq d_0$ , and every bounded section  $s \in \Gamma(D_R, L^{\otimes d}) \setminus \{0\}$  for which  $m := \text{ord}_0 s$  satisfies  $m \geq \kappa_0 d$ , if  $f$  denotes the germ of holomorphic function at  $q$  such that  $s = f s_0^{\otimes d}$  in a neighborhood of  $q$ , we have*

$$\log \|j_0^m s\|_R \leq \log \left( \frac{m^{\kappa_1 d}}{\kappa_2^J} \|s\|_{L^\infty(D_R)} + \kappa_3^{d+m+J} \max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!} \right) + \kappa_4 d.$$

*Proof.* Fix any real number  $R_0 \in (|q|, R)$ . We take  $\kappa_2 := R_0/|q|$ ,  $\kappa_3 := \kappa$  given by Proposition 3.13, and  $\kappa_4 := T_{\bar{L}}^-(R_0)$ . Since  $\bar{L}$  has moderate growth, we may apply Proposition 3.14 to the compact  $K = \partial D_{R_0}$  to obtain real numbers  $\kappa_0, \kappa_1 > 0$  and an integer  $d_0 \geq 1$  such that

$$\|s\|_{L^\infty(\partial D_{R_0})} \leq m^{\kappa_1 d} \left( \frac{R_0}{R} \right)^m \|s\|_{L^\infty(D_R)}$$

for any integer  $d \geq d_0$  and any bounded section  $s \in \Gamma(D_R, L^{\otimes d}) \setminus \{0\}$  such that  $m := \text{ord}_0 s \geq \kappa_0 d$ . We conclude by combining this bound with the estimate given by [Proposition 3.13](#).  $\square$

In practice, we shall be concerned with the following particular situation.

**Corollary 3.16.** *Let  $R > 0$  be a real number,  $\bar{L} = (L, \|\cdot\|)$  be a semipositive Hermitian line bundle of moderate growth on  $D_R$ , and  $q \in D_R \setminus \{0\}$ . Fix a holomorphic trivialization  $s_0$  of  $L$  in a neighborhood of  $q$ , real constants  $c_0, c_1, c_2 > 0$ , with  $c_0 < c_1$ , and an integer  $n \geq 2$ . For any real number  $C > 0$ , there exist real numbers  $\gamma_0, \gamma_1 > 0$  such that, for any sufficiently large integer  $d$ , and any bounded section  $s \in \Gamma(D_R, L^{\otimes d})$  satisfying*

$$c_0 d^n \leq m := \text{ord}_0 s \leq c_1 d^n, \quad \log \|s\|_{L^\infty(D_R)} \leq c_2 d \log d,$$

and

$$\max_{0 \leq j < \lceil \gamma_0 d \log d \rceil} \log \frac{|f^{(j)}(q)|}{j!} \leq -\gamma_1 d^n,$$

where  $s = f s_0^{\otimes d}$  on a neighborhood of  $q$ , we have

$$\log \|j_0^m s\|_R \leq -C d \log d.$$

*Proof.* Let  $\kappa_i > 1$ ,  $i = 0, \dots, 4$ , be the constants given by [Corollary 3.15](#). We claim that it suffices to take  $\gamma_0 > (\log \kappa_2)^{-1}(n\kappa_1 + c_2 + C)$  and  $\gamma_1 > c_1 \log \kappa_3$ .

Indeed, let  $s \in \Gamma(D_R, L^{\otimes d})$  be as in the statement. Since  $n \geq 2$  and  $m \geq c_0 d^n$ , if  $d$  is sufficiently large, we have  $m \geq \kappa_0 d$ , so that the conclusion of [Corollary 3.15](#) for  $J := \lceil \gamma_0 d \log d \rceil$  applies:

$$\log \|j_0^m s\|_R \leq \log \left( \frac{m^{\kappa_1 d}}{\kappa_2^J} \|s\|_{L^\infty(D_R)} + \kappa_3^{d+m+J} \max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!} \right) + \kappa_4 d. \tag{3-5}$$

Since  $m \leq c_1 d^n$ ,  $\log \|s\|_{L^\infty(D_R)} \leq c_2 d \log d$ , and  $J \geq \gamma_0 d \log d$ , we obtain

$$\log \left( \frac{m^{\kappa_1 d}}{\kappa_2^J} \|s\|_{L^\infty(D_R)} \right) \leq (n\kappa_1 + c_2 - (\log \kappa_2)\gamma_0)d \log d + \kappa_1 (\log c_1)d.$$

Thus, by our choice of  $\gamma_0$ , if  $d$  is sufficiently large, we get

$$\log \left( \frac{m^{\kappa_1 d}}{\kappa_2^J} \|s\|_{L^\infty(D_R)} \right) \leq -(C + \varepsilon_1)d \log d \tag{3-6}$$

for some  $\varepsilon_1 > 0$ .

Since  $m \leq c_1 d^n$ ,  $J \leq \gamma_0 d \log d + 1$ , and

$$\max_{0 \leq j < J} \log \frac{|f^{(j)}(q)|}{j!} \leq -\gamma_1 d^n,$$

we have

$$\log \left( \kappa_3^{d+m+J} \max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!} \right) \leq (c_1 \log \kappa_3 - \gamma_1)d^n + \gamma_0 (\log \kappa_3)d \log d + (\log \kappa_3)(d + 1).$$

Thus, as  $n \geq 2$ , and by our choice of  $\gamma_1$ , if  $d$  is sufficiently large, we obtain

$$\log \left( \kappa_3^{d+m+J} \max_{0 \leq j < J} \frac{|f^{(j)}(q)|}{j!} \right) \leq -\varepsilon_2 d^n \tag{3-7}$$

for some  $\varepsilon_2 > 0$ .

We conclude by applying (3-6) and (3-7) in (3-5), and by taking  $d$  to be sufficiently large. □

#### 4. Analytic curves of moderate growth in quasiprojective varieties

This section contains mostly well-known techniques and results in Nevanlinna theory. These are nevertheless written in the literature in a form not suitable for our purposes. Although our proofs may vary, much of the theory concerning growth of entire analytic maps (“parabolic case”) easily translates into our hyperbolic situation; we refer the reader to the recent monograph [Noguchi and Winkelmann 2014] for a thorough exposition of the general parabolic theory (in several variables).

**4A. Analytic curves of moderate growth in compact complex manifolds.** Let  $R > 0$  be a real number,  $M$  be a compact complex manifold, and  $\varphi : D_R \rightarrow M$  be an analytic map. Fix any Hermitian metric  $h$  on  $M$ , and let  $\omega := -\operatorname{Im} h$  be the positive  $(1, 1)$ -form associated to  $h$ ; in other words, if  $h = \sum_{k,l=1}^n h_{kl} dz_k \otimes d\bar{z}_l$  in a local chart  $(z_1, \dots, z_n)$  of  $M$ , then  $\omega = \frac{i}{2} \sum_{k,l=1}^n h_{kl} dz_k \wedge d\bar{z}_l$ .

**Definition 4.1.** We say that  $\varphi : D_R \rightarrow M$  has *moderate growth* if the semipositive  $(1,1)$ -form  $\varphi^*\omega$  on  $D_R$  has moderate growth (see Definition 3.3).

This notion does not depend on the choice of the Hermitian metric  $h$ . Indeed, since  $M$  is compact, any two Hermitian metrics on  $M$  are “comparable”: if  $h_0$  is another Hermitian metric on  $M$ , then there exist real numbers  $\alpha, \beta > 0$  such that  $\alpha \| \cdot \|_{h_0} \leq \| \cdot \|_h \leq \beta \| \cdot \|_{h_0}$ .

**Remark 4.2.** It follows from Remark 3.1 that  $T_{\varphi^*\omega}(r) = \int_0^r (\int_{D_t} \varphi^*\omega) d \log t$  can be thought of as a logarithmic integral of the areas of the disks  $\varphi(D_t)$  in  $M$  for  $0 < t < r$ .

We next consider a simple example of curves of moderate growth.

**Example 4.3** (bounded derivative). Let  $\varphi : D_R \rightarrow M$  be an analytic map, and  $h$  be a Hermitian metric on  $M$ . Then we can write

$$\varphi^*\omega = \|\varphi'(z)\|_{R,h}^2 d\mu_R,$$

where  $d\mu_R$  is the Poincaré form defined in Example 3.4, and  $\|\varphi'(z)\|_{R,h}$  denotes the norm of the tangent map  $D_z\varphi : T_z D_R \rightarrow T_{\varphi(z)} M$  with respect to the Poincaré metric on  $D_R$ , and the Hermitian metric  $h$  on  $M$ . Since  $d\mu_R$  has moderate growth, the analytic curve  $\varphi$  has moderate growth in  $M$  whenever the function  $z \mapsto \|\varphi'(z)\|_{R,h}$  is bounded on  $D_R$  (e.g.,  $\varphi$  extends continuously to  $\bar{D}_R \subset \mathbb{C}$ ).

**4B. Nevanlinna’s characteristic function.** Let  $M$  be a complex manifold,  $\bar{L} = (L, \| \cdot \|)$  be a semipositive Hermitian line bundle on  $M$ , and  $s_0 \in \Gamma(M, L) \setminus \{0\}$  be a nonzero global section.

Let  $R > 0$  be a real number and  $\varphi : D_R \rightarrow M$  be an analytic map whose image is not contained in the support of  $\text{div}(s_0)$ . We define, for every  $r \in (0, R)$ ,

$$m_{\varphi, \bar{L}, s_0}(r) := \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{\|s_0(\varphi(re^{i\theta}))\|} d\theta$$

and

$$N_{\varphi, \bar{L}, s_0}(r) := (\text{ord}_0 \varphi^* s_0) \log r + \sum_{0 < |z| < r} (\text{ord}_z \varphi^* s_0) \log \frac{r}{|z|}.$$

Then we can form the *Nevanlinna characteristic function* on the interval  $(0, R)$

$$T_{\varphi, \bar{L}, s_0} := m_{\varphi, \bar{L}, s_0} + N_{\varphi, \bar{L}, s_0}.$$

For the next proposition, we introduce a temporary notation. If  $s$  is a global section of  $\varphi^*L$  and  $m = \text{ord}_0 s$ , we denote by  $\ell(s)$  the unique element of the fiber of  $\varphi^*L$  at  $0 \in D_R$  such that  $j_0^m s = \ell(s) \otimes dz^{\otimes m}$  (the “leading coefficient” of  $s$ ).

The following classical identity, see [Noguchi and Winkelmann 2014, Theorem 2.3.31], is an immediate corollary of Proposition 3.8 applied to the section  $s = \varphi^*s_0$  and the point  $p = 0$ .

**Proposition 4.4** (Nevanlinna’s first fundamental theorem). *For every  $r \in (0, R)$ , we have*

$$T_{\varphi^* \bar{L}}(r) = T_{\varphi, \bar{L}, s_0}(r) + \log \|\ell(\varphi^* s_0)\|. \quad \square$$

As an application we show that, when  $R = 1$ , polynomial growth of Taylor coefficients implies moderate growth.

**Example 4.5.** Let  $\varphi = (\varphi_1, \dots, \varphi_n) : D \rightarrow \mathbb{C}^n$  be an analytic map with coordinates  $\varphi_i(z) = \sum_{j=0}^\infty a_{ij} z^j$ . Assume that there exist a real number  $C > 1$  and an integer  $d \geq 1$  such that

$$|a_{ij}| \leq Cj^d$$

for every  $1 \leq i \leq n$  and  $j \geq 0$ . Then, when identifying  $\mathbb{C}^n$  with the open affine subset

$$U_0 = \{(p_0 : \dots : p_n) \in \mathbb{P}^n(\mathbb{C}) \mid p_0 \neq 0\}$$

of  $\mathbb{P}^n(\mathbb{C})$  via  $(z_1, \dots, z_n) \mapsto (1 : z_1 : \dots : z_n)$ , the analytic curve  $\varphi : D \rightarrow \mathbb{P}^n(\mathbb{C})$  has moderate growth.

Indeed, let  $\overline{\mathcal{O}(1)}$  denote the line bundle  $\mathcal{O}(1)$  on  $\mathbb{P}^n(\mathbb{C})$  endowed with the Fubini–Study metric; that is,

$$\|X_i(p)\| = \frac{|p_i|}{\sqrt{|p_0|^2 + \dots + |p_n|^2}}$$

for every  $0 \leq i \leq n$ , and  $p = (p_0 : \dots : p_n) \in \mathbb{P}^n(\mathbb{C})$ . Since  $N_{\varphi, \overline{\mathcal{O}(1)}, X_0}$  vanishes identically, by Proposition 4.4, it is sufficient to prove that there exist  $a, b > 0$  such that

$$m_{\varphi, \overline{\mathcal{O}(1)}, X_0}(r) \leq a + b \log \frac{1}{1-r}$$

for every  $r \in (0, 1)$ .

For any real numbers  $t_1, \dots, t_m \geq 0$ , we have  $\log^+(\sum_{i=1}^m t_i) \leq \sum_{i=1}^m \log^+ t_i + \log m$ , so that

$$\log \frac{1}{\|X_0(\varphi(re^{i\theta}))\|} = \log \sqrt{1 + \sum_{i=1}^n |\varphi_i(re^{i\theta})|^2} \leq \sum_{i=1}^n \log^+ |\varphi_i(re^{i\theta})| + \log(\sqrt{1+n}).$$

Since

$$|\varphi_i(re^{i\theta})| \leq C \sum_{j=0}^{\infty} j^d r^j \leq Cd! \left(\frac{1}{1-r}\right)^{d+1},$$

we may take  $a = \log(\sqrt{1+n}) + n \log(Cd!)$  and  $b = n(d+1)$ .

**4C. The field of moderate functions on a disk.** In this subsection we study more closely the case  $M = \mathbb{P}^1(\mathbb{C})$ . We refer to [Tsuji 1959, Chapters V–VII] for a survey on the classical work on this subject.

Let  $R > 0$  be a real number and  $f$  be a meromorphic function on  $D_R$ , i.e., an analytic map  $f : D_R \rightarrow \mathbb{P}^1(\mathbb{C})$  which is not constantly equal to  $\infty = (0 : 1)$ .

**Definition 4.6.** We say that  $f$  is a *moderate function* on  $D_R$  if the analytic map  $f : D_R \rightarrow \mathbb{P}^1(\mathbb{C})$  has moderated growth.

If  $\overline{\mathcal{O}(1)}$  denotes the line bundle  $\mathcal{O}(1)$  on  $\mathbb{P}^1(\mathbb{C})$  endowed with the Fubini–Study metric (see Example 4.5), we define

$$T_f := T_{f^* \overline{\mathcal{O}(1)}}.$$

By Proposition 4.4, we have

$$T_f = m_{f, \overline{\mathcal{O}(1)}, X_0} + N_{f, \overline{\mathcal{O}(1)}, X_0} + O(1),$$

where  $O(1)$  denotes a constant. To lighten the notation, we shall write  $m_f = m_{f, \overline{\mathcal{O}(1)}, X_0}$  and  $N_f = N_{f, \overline{\mathcal{O}(1)}, X_0}$ .

Let  $\mathcal{K}_{D_R}$  denote the field of meromorphic functions on  $D_R$ . It is classical (and easy to prove) that characteristic functions are compatible with the algebraic structure of  $\mathcal{K}_{D_R}$  in the following sense: for  $f, g \in \mathcal{K}_{D_R} \setminus \{0\}$  and  $n \in \mathbb{Z} \setminus \{0\}$ , we have

$$T_{f+g} \leq T_f + T_g + O(1), \quad T_{fg} \leq T_f + T_g + O(1), \quad T_{f^n} = |n|T_f + O(1). \tag{4-1}$$

It follows from the above relations that the subset  $\mathcal{K}_{D_R}^m$  of  $\mathcal{K}_{D_R}$  consisting of moderate meromorphic functions is a field.

**Proposition 4.7** (cf. [Noguchi and Winkelmann 2014, Lemma 2.5.15]). *Let  $f, f_1, \dots, f_n$  be meromorphic functions on  $D_R$ . If  $f$  is algebraic over the field  $\mathbb{C}(f_1, \dots, f_n) \subset \mathcal{K}_{D_R}$ , then there exist real numbers  $a, b > 0$  such that*

$$T_f \leq a + b \sum_{i=1}^n T_{f_i}.$$

*Proof.* Let  $d$  be the degree of  $f$  over  $\mathbb{C}(f_1, \dots, f_n)$ . If  $d = 0$ , then the result follows immediately from formulas (4-1). Assume that  $d \geq 1$ , and let  $P = X^d - g_{d-1}X^{d-1} - \dots - g_0 \in \mathbb{C}(f_1, \dots, f_n)[X]$  be the minimal polynomial of  $f$ . Since each  $g_i \in \mathbb{C}(f_1, \dots, f_n)$ , it suffices to prove that  $T_f \leq \sum_{i=0}^{d-1} T_{g_i} + O(1)$ .

By formulas (4-1), we have

$$T_{f^d} = T_{(g_{d-1}f^{d-2} + \dots + g_1)f + g_0} \leq T_{g_{d-1}f^{d-2} + \dots + g_1} + T_f + T_{g_0} + O(1).$$

By descending induction, we get

$$T_{f^d} \leq (d - 1)T_f + \sum_{i=0}^{d-1} T_{g_i} + O(1).$$

As  $T_{f^d} = d \cdot T_f + O(1)$ , we obtain

$$T_f \leq \sum_{i=0}^{d-1} T_{g_i} + O(1). \quad \square$$

**Corollary 4.8.** *The field of moderate functions  $\mathcal{K}_{D_R}^m$  is algebraically closed in  $\mathcal{K}_{D_R}$ .* □

In particular, since the inclusion  $D_R \rightarrow \mathbb{P}^1(\mathbb{C})$  is easily seen to be an analytic map of moderate growth (see Example 4.3), the field  $\mathcal{K}_{D_R}^m$  contains the field of (univalued) algebraic meromorphic functions on  $D_R$ .

**4D. Birational invariance and moderate growth in quasiprojective varieties.** In this subsection, we establish the birational invariance of moderate growth under a nondegeneracy hypothesis. Our arguments follow closely those of [Noguchi and Winkelmann 2014, Section 2.5]; we claim no originality here.

In what follows, if  $f$  is a meromorphic function on  $D_R$ , we denote the divisor of zeros of  $f$  by  $\text{div}_0(f)$  and the divisor of poles of  $f$  by  $\text{div}_\infty(f)$ , so that  $\text{div}(f) = \text{div}_0(f) - \text{div}_\infty(f)$ .

**Lemma 4.9** (cf. [Noguchi and Winkelmann 2014, Theorem 2.5.7]). *Let  $M$  be a compact complex manifold endowed with a semipositive Hermitian line bundle  $\bar{L} = (L, \|\cdot\|)$ . Fix global sections  $s_0, s_1 \in \Gamma(M, L)$ , with  $s_0 \neq 0$ . Then, for any analytic map  $\varphi : D_R \rightarrow M$  whose image is not contained in the support of  $\text{div}(s_0)$ , if we denote by  $f$  the unique meromorphic function on  $D_R$  such that  $f\varphi^*s_0 = \varphi^*s_1$ , we have*

$$T_f \leq T_{\varphi, \bar{L}, s_0} + O(1).$$

*Proof.* Let  $H$  be the support of  $\text{div}(s_0)$ . For  $p \in M \setminus H$ , we have

$$\log \sqrt{1 + \frac{\|s_1(p)\|^2}{\|s_0(p)\|^2}} = \log \frac{1}{\|s_0(p)\|} + \log \sqrt{\|s_0(p)\|^2 + \|s_1(p)\|^2}.$$

Since  $M$  is compact, the functions  $\|s_i\|$  on  $M$  are bounded, so that

$$\log \sqrt{1 + \frac{\|s_1\|^2}{\|s_0\|^2}} \leq \log \frac{1}{\|s_0\|} + O(1)$$

over  $M \setminus H$ . In particular, we get

$$m_f \leq m_{\varphi, \bar{L}, s_0} + O(1).$$

Since  $\text{div}_\infty(f) \leq \text{div}(\varphi^*s_0)$ , the bound

$$N_f \leq N_{\varphi, \bar{L}, s_0}$$

is trivial. □

For the next lemma, we endow the line bundle  $\mathcal{O}(1)$  over  $\mathbb{P}^n(\mathbb{C})$  with the Fubini–Study metric as in [Example 4.5](#). Moreover, if  $E = \sum_{z \in D_R} n_z [z]$  is a divisor in  $D_R$ , we set  $i(z, E) := n_z$ .

**Lemma 4.10.** *Let  $\varphi : D_R \rightarrow \mathbb{P}^n(\mathbb{C})$  be an analytic map whose image is not contained in the support of  $\text{div}(X_0)$ . For  $1 \leq j \leq n$ , let us denote by  $f_j$  the unique meromorphic function on  $D_R$  such that  $f_j \varphi^* X_0 = \varphi^* X_j$ . Then*

$$T_{\varphi, \overline{\mathcal{O}(1)}, X_0} \leq \sum_{j=1}^n T_{f_j} + O(1).$$

*Proof.* We first prove that  $m_{\varphi, \overline{\mathcal{O}(1)}, X_0} \leq \sum_{j=1}^n m_{f_j}$ . For any real numbers  $t_1, \dots, t_n \geq 0$ , we have

$$\log\left(1 + \sum_{j=1}^n t_j\right) \leq \sum_{j=1}^n \log(1 + t_j).$$

Thus, for any  $r \in (0, R)$ ,

$$\begin{aligned} m_{\varphi, \overline{\mathcal{O}(1)}, X_0}(r) &= \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{1 + \sum_{j=1}^n |f_j(re^{i\theta})|^2} d\theta \\ &\leq \sum_{j=1}^n \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{1 + |f_j(re^{i\theta})|^2} d\theta = \sum_{j=1}^n m_{f_j}(r). \end{aligned}$$

Next, observe that to prove that  $N_{\varphi, \overline{\mathcal{O}(1)}, X_0} \leq \sum_{j=1}^n N_{f_j}$  it suffices to show that

$$\text{div}(\varphi^* X_0) \leq \sum_{j=1}^n \text{div}_\infty(f_j).$$

Since each  $\text{div}_\infty(f_j)$  is an effective divisor, it is sufficient to prove that, for every  $z \in D_R$ , there exists  $1 \leq j \leq n$  such that  $i(z, \text{div}(\varphi^* X_0)) \leq i(z, \text{div}_\infty(f_j))$ . Now, for any  $1 \leq j \leq n$ , since  $f_j \varphi^* X_0 = \varphi^* X_j$ , we may write

$$\text{div}_\infty(f_j) = \text{div}(\varphi^* X_0) + \text{div}_0(f_j) - \text{div}(\varphi^* X_j).$$

Finally, we simply remark that for any  $z \in D_R$  for which  $X_0(\varphi(z)) = 0$  (i.e.,  $\text{ord}_z \varphi^* X_0 > 0$  or, equivalently,  $i(z, \text{div}(\varphi^* X_0)) > 0$ ), there exists  $1 \leq j \leq n$  such that  $X_j(\varphi(z)) \neq 0$  (i.e.,  $\text{ord}_z \varphi^* X_j = 0$ ), so that  $i(z, \text{div}_\infty(f_j)) = i(z, \text{div}(\varphi^* X_0)) + i(z, \text{div}_0(f_j)) \geq i(z, \text{div}(\varphi^* X_0))$ . □

Let  $f, g : I \rightarrow \mathbb{R}$  be real functions defined on some interval  $I \subset \mathbb{R}$ . We say that  $f$  and  $g$  are *comparable* if there exist real numbers  $a, b, c, d > 0$  such that

$$af - b \leq g \leq cf + d$$

everywhere on  $I$ .

**Theorem 4.11** (cf. [Noguchi and Winkelmann 2014, Theorem 2.5.18]). *Let  $R > 0$  be a real number,  $X$  be a smooth projective variety of dimension  $n$  over  $\mathbb{C}$ , and  $\varphi : D_R \rightarrow X^{\text{an}}$  be an analytic map whose image is Zariski-dense in  $X$ . Then, for any positive  $(1, 1)$ -form  $\omega$  on  $X$ , and any transcendence basis  $(f_1, \dots, f_n)$  of the function field  $\mathbb{C}(X)$  of  $X$ , the real functions  $T_{\varphi^*\omega}$  and  $\sum_{j=1}^n T_{f_j \circ \varphi}$  on  $(0, R)$  are comparable. In particular,  $\varphi$  has moderate growth in  $X$  if and only if  $f_j \circ \varphi$  are moderate functions on  $D_R$  for every  $1 \leq j \leq n$ .*

Observe that the Zariski-density hypothesis above ensures that, for any rational function  $f$  on  $X$ , the image of  $\varphi$  is not contained in the indeterminacy locus of  $f$ , so that  $f \circ \varphi$  is a well-defined meromorphic function on  $D_R$ .

*Proof.* Let  $i : X \rightarrow \mathbb{P}_{\mathbb{C}}^N = \text{Proj } \mathbb{C}[X_0, \dots, X_N]$  be a closed immersion such that  $i \circ \varphi(D_R)$  is not contained in the support of  $\text{div}(X_0)$ , and consider the rational functions  $g_j \in \mathbb{C}(X)$ ,  $1 \leq j \leq N$ , given by restriction of  $X_j/X_0$  to  $X$ .

It follows from the compactness of  $X^{\text{an}}$  (see remark following Definition 4.1), and from Proposition 4.4, that the functions  $T_{\varphi^*\omega}$  and  $T_{i \circ \varphi, \overline{\mathcal{O}(1)}, X_0}$  are comparable. By Lemmas 4.9 and 4.10, the functions  $T_{i \circ \varphi, \overline{\mathcal{O}(1)}, X_0}$  and  $\sum_{j=1}^N T_{g_j \circ \varphi}$  are comparable. Furthermore, as  $\mathbb{C}(X) = \mathbb{C}(g_1, \dots, g_N)$  is an algebraic extension of  $\mathbb{C}(f_1, \dots, f_n)$ , we deduce from formulas (4-1) and from Proposition 4.7 that  $\sum_{j=1}^N T_{g_j \circ \varphi}$  and  $\sum_{j=1}^n T_{f_j \circ \varphi}$  are comparable. Our statement follows by transitivity of comparability.  $\square$

In particular, moderate growth in projective varieties is a birational invariant.

**Corollary 4.12.** *Let  $f : X \rightarrow Y$  be a birational morphism between smooth projective varieties over  $\mathbb{C}$ . If  $R > 0$  is a real number, then an analytic map  $\varphi : D_R \rightarrow X^{\text{an}}$  with Zariski-dense image has moderated growth if and only if  $f \circ \varphi : D_R \rightarrow Y^{\text{an}}$  has moderate growth.*  $\square$

Combining the standard argument in the proof of Corollary 2.16 with a resolution of singularities yields the following.

**Corollary 4.13.** *Let  $X$  be a smooth quasiprojective variety over  $\mathbb{C}$ , and let  $j_i : X \hookrightarrow \bar{X}_i$ ,  $i = 1, 2$ , be smooth projective compactifications of  $X$ . If  $R > 0$  is a real number and  $\varphi : D_R \rightarrow X^{\text{an}}$  is an analytic map with Zariski-dense image, then  $j_1 \circ \varphi$  has moderate growth if and only if  $j_2 \circ \varphi$  has moderate growth.*  $\square$

We may thus define an unambiguous notion of moderate growth for Zariski-dense analytic curves in smooth quasiprojective varieties.

**Definition 4.14.** Let  $X$  be a smooth quasiprojective variety,  $R > 0$  be a real number, and  $\varphi : D_R \rightarrow X^{\text{an}}$  be an analytic map with Zariski-dense image. We say that  $\varphi$  has moderate growth if there exists a smooth projective compactification  $j : X \hookrightarrow \bar{X}$  of  $X$  such that  $j \circ \varphi : D_R \rightarrow \bar{X}^{\text{an}}$  has moderate growth.

### 5. Construction of auxiliary sections

We prove in this section Theorem 5.1 below, generalizing the construction of auxiliary polynomials in Nesterenko’s method. Our approach, based on Bost’s method of slopes, differs from the classical

combinatorial one. However, the backbone of the argument remains the same: Minkowski’s theorem on minima of lattices (see [Proposition 5.7](#) below).

**5A. Notation and statement.** Let  $K$  be a number field and  $\mathcal{O}_K$  be its ring of integers. Recall that, if  $\mathcal{X}$  is an arithmetic scheme over  $\mathcal{O}_K$  (i.e., an integral scheme  $\mathcal{X}$  with a separated and flat morphism of finite type  $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$ ) with smooth generic fiber  $\mathcal{X}_K$ , a Hermitian line bundle  $\bar{L} = (L, (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$  over  $\mathcal{X}$  is the data of a line bundle  $L$  on  $\mathcal{X}$  and a family of  $C^\infty$  Hermitian metrics  $\|\cdot\|_\sigma$  on the holomorphic line bundles  $L_\sigma$  over  $\mathcal{X}_\sigma^{\text{an}}$  deduced from  $L$  by the field embeddings  $\sigma : K \hookrightarrow \mathbb{C}$  that is invariant under complex conjugation.

If  $d \geq 1$  is an integer, and  $s \in \Gamma(\mathcal{X}, L^{\otimes d})$  is a global section, we define

$$\|s\|_{\mathcal{X}} := \max_{\sigma} \|s\|_{\sigma, L^\infty(\mathcal{X}_\sigma^{\text{an}})}.$$

This section is devoted to the proof of the following theorem.

**Theorem 5.1.** *Let  $\mathcal{X}$  be a projective arithmetic scheme of relative dimension  $n \geq 1$  over  $\mathcal{O}_K$  with smooth generic fiber  $\mathcal{X}_K$ , and fix any Hermitian line bundle  $\bar{L} = (L, (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$  on  $\mathcal{X}$  such that  $L_K$  is ample on  $\mathcal{X}_K$ . Let  $\hat{\varphi} : \text{Spf } \mathcal{O}_K[[q]] \rightarrow \mathcal{X}$  be a morphism of formal  $\mathcal{O}_K$ -schemes such that, for every field embedding  $\sigma : K \hookrightarrow \mathbb{C}$ , the formal curve  $\hat{\varphi}_\sigma : \text{Spf } \mathbb{C}[[q]] \rightarrow \mathcal{X}_\sigma$  lifts to an analytic map  $\varphi_\sigma : D_{R_\sigma} \rightarrow \mathcal{X}_\sigma^{\text{an}}$  defined on some complex disk of radius  $R_\sigma > 0$ , and assume that*

$$\prod_{\sigma:K \hookrightarrow \mathbb{C}} R_\sigma = 1.$$

If, moreover,

- (1) the image of  $\hat{\varphi}_K : \text{Spf } K[[q]] \rightarrow \mathcal{X}_K$  is Zariski-dense, and
- (2) for every field embedding  $\sigma : K \hookrightarrow \mathbb{C}$ ,  $\varphi_\sigma$  has moderate growth in  $\mathcal{X}_\sigma^{\text{an}}$ ,

then, there are constants  $c_1, c_2, c_3 > 0$  such that, for every large enough positive integer  $d$ , there exists  $s \in \Gamma(\mathcal{X}, L^{\otimes d}) \setminus \{0\}$  such that

$$m := \text{ord}_0 \hat{\varphi}^* s > c_1 d^n$$

and

$$\log \|s\|_{\mathcal{X}} \leq c_2 d + c_3 d \log m.$$

**Remark 5.2.** By [Section 4D](#) one could also assume that  $\mathcal{X}$  is only quasiprojective over  $\text{Spec } \mathcal{O}_K$ , and then construct “auxiliary sections” on any projective compactification of  $\mathcal{X}$  having smooth generic fiber.

If we require the stronger condition of ZL-density of  $\hat{\varphi}_K$  instead of Zariski-density, we obtain the following.

**Corollary 5.3.** *With hypotheses and notation as in [Theorem 5.1](#), if moreover  $\hat{\varphi}_K : \text{Spf } K[[q]] \rightarrow \mathcal{X}_K$  is ZL-dense, then there exist constants  $c_1, c_2 > 0$  such that, for every large enough positive integer  $d$ , there exists  $s \in \Gamma(\mathcal{X}, L^{\otimes d}) \setminus \{0\}$  such that*

$$\text{ord}_0 \hat{\varphi}^* s > c_1 d^n$$

and

$$\log \|s\|_{\mathcal{X}} \leq c_2 d \log d.$$

**5B. Recollections of Arakelov theory; the slope inequality.** For the convenience of the reader, we recollect in this subsection some fundamental notions and results concerning Hermitian vector bundles over rings of algebraic integers. Proofs and further developments can be found in [Bost 1996, Appendix A; 2001, Sections 4.1–4.2; Bost and Künnemann 2010, Sections 3.1–3.3].

Let  $K$  be number field,  $\mathcal{O}_K$  be its ring of integers, and set  $S := \text{Spec } \mathcal{O}_K$ . Recall that a *Hermitian vector bundle* over  $S$  is a pair  $\bar{E} = (E, (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$ , where  $E$  is a projective  $\mathcal{O}_K$ -module of finite type, and  $(\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}}$  is a family of Hermitian norms over  $E_\sigma := E \otimes_{\sigma: \mathcal{O}_K \hookrightarrow \mathbb{C}} \mathbb{C}$ , invariant under complex conjugation. If  $\text{rk } E = 1$ , we say that  $\bar{E}$  is a *Hermitian line bundle* over  $S$ .

The multilinear constructions in the category of projective modules over  $\mathcal{O}_K$  (e.g., tensor products, quotients, Hom) make sense in the category of Hermitian vector bundles over  $S$ .

**Definition 5.4.** Let  $\bar{E} = (E, (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$  be a Hermitian vector bundle over  $S$ , and fix  $s \in \det E \setminus \{0\}$ , where  $\det E := \bigwedge^{\text{rk } E} E$ . We define the *Arakelov degree* of  $\bar{E}$  by

$$\widehat{\text{deg}}(\bar{E}) := \log |(\det E) / \mathcal{O}_K s| - \sum_{\sigma:K \hookrightarrow \mathbb{C}} \log \|s\|_\sigma \in \mathbb{R}.$$

This is easily seen not to depend on the choice of  $s$ . We define moreover the *normalized Arakelov degree* of  $\bar{E}$  by

$$\widehat{\text{deg}}_n(\bar{E}) := \frac{1}{[K : \mathbb{Q}]} \widehat{\text{deg}}(\bar{E})$$

and the *slope* of  $\bar{E}$  by

$$\hat{\mu}(\bar{E}) := \frac{1}{\text{rk } E} \widehat{\text{deg}}_n(\bar{E})$$

when  $\text{rk } E > 0$ , and  $\hat{\mu}(\bar{E}) := -\infty$  when  $\text{rk } E = 0$ .

**Proposition 5.5** [Bost 2001, Section 4.1.1]. *The following properties hold:*

(1) *If  $\bar{L}$  and  $\bar{M}$  are Hermitian line bundles over  $S$ , we have*

$$\widehat{\text{deg}}(\bar{L} \otimes \bar{M}) = \widehat{\text{deg}}(\bar{L}) + \widehat{\text{deg}}(\bar{M}).$$

(2) *Let  $\bar{E}$  be a Hermitian vector bundle over  $S$  and*

$$E = E^0 \supset E^1 \supset \dots \supset E^N \supset \{0\}$$

*be a filtration of  $E$  by saturated  $\mathcal{O}_K$ -submodules. Then*

$$\widehat{\text{deg}}(\bar{E}) = \widehat{\text{deg}}(\bar{E}^N) + \sum_{i=0}^{N-1} \widehat{\text{deg}}(\overline{E^i / E^{i+1}}),$$

*where  $\bar{E}^N$  and  $\overline{E^i / E^{i+1}}$  denote the Hermitian vector bundles with underlying modules  $E^N$  and  $E^i / E^{i+1}$ , respectively, and Hermitian structure induced by  $\bar{E}$ .*

(3) *For every Hermitian vector bundle  $\bar{E}$  over  $S$ , and every Hermitian line bundle  $\bar{L}$  over  $S$ , we have*

$$\hat{\mu}(\bar{E} \otimes \bar{L}) = \hat{\mu}(\bar{E}) + \widehat{\text{deg}}_n(\bar{L}).$$

Let  $\bar{E}$  and  $\bar{F}$  be Hermitian vector bundles over  $S$ . For every maximal ideal  $\mathfrak{p}$  of  $\mathcal{O}_K$ , we denote by  $K_{\mathfrak{p}}$  the completion of  $K$  at  $\mathfrak{p}$  and by  $\|\cdot\|_{\mathfrak{p}}$  the nonarchimedean norm over  $\text{Hom}_{K_{\mathfrak{p}}}(E_{K_{\mathfrak{p}}}, F_{K_{\mathfrak{p}}})$  associated to the  $\mathcal{O}_{K_{\mathfrak{p}}}$ -lattice  $\text{Hom}_{\mathcal{O}_{K_{\mathfrak{p}}}}(E_{\mathcal{O}_{K_{\mathfrak{p}}}}, F_{\mathcal{O}_{K_{\mathfrak{p}}}})$ ; explicitly, if  $\varphi \in \text{Hom}_{K_{\mathfrak{p}}}(E_{K_{\mathfrak{p}}}, F_{K_{\mathfrak{p}}}) \setminus \{0\}$ , then  $\|\varphi\|_{\mathfrak{p}} := |\mathcal{O}_K/\mathfrak{p}|^{-v_{\mathfrak{p}}(\varphi)}$ , where  $v_{\mathfrak{p}}(\varphi) = \max\{n \in \mathbb{Z} \mid \pi_{\mathfrak{p}}^{-n}\varphi \in \text{Hom}_{\mathcal{O}_{K_{\mathfrak{p}}}}(E_{\mathcal{O}_{K_{\mathfrak{p}}}}, F_{\mathcal{O}_{K_{\mathfrak{p}}}})\}$  and  $\pi_{\mathfrak{p}}$  denotes some uniformizer of  $\mathcal{O}_{K_{\mathfrak{p}}}$ . For a field embedding  $\sigma : K \hookrightarrow \mathbb{C}$ , we consider the operator norm on  $\text{Hom}_{\mathbb{C}}(E_{\sigma}, F_{\sigma})$ :

$$\|\varphi\|_{\sigma} = \max_{v \in E_{\sigma} \setminus \{0\}} \frac{\|\varphi(v)\|_{\sigma}}{\|v\|_{\sigma}}.$$

Then, the *height* of a nonzero  $K$ -linear map  $\varphi : E_K \rightarrow F_K$  is defined by

$$h_{\bar{E}, \bar{F}}(\varphi) = \frac{1}{[K : \mathbb{Q}]} \left( \sum_{\mathfrak{p}} \log \|\varphi\|_{\mathfrak{p}} + \sum_{\sigma} \log \|\varphi\|_{\sigma} \right).$$

If  $\varphi = 0$ , our convention is that  $h_{\bar{E}, \bar{F}}(\varphi) := -\infty$ .

**Proposition 5.6** (slope inequality; [Bost 2001, Proposition 4.5]). *With the above notation, if  $\varphi : E_K \rightarrow F_K$  is injective, then*

$$\hat{\mu}(\bar{E}) \leq \hat{\mu}_{\max}(\bar{F}) + h_{\bar{E}, \bar{F}}(\varphi),$$

where  $\hat{\mu}_{\max}(\bar{F}) := \sup\{\hat{\mu}(\bar{F}') \mid F' \text{ is an } \mathcal{O}_K\text{-submodule of } F\}$ .

Let us point out that  $\hat{\mu}_{\max}(\bar{F})$  is attained by a saturated submodule of  $F$ ; see [Bost 1996, Section A.3]. In particular, if  $\text{rk } F = 1$ , then  $\hat{\mu}_{\max}(\bar{F}) = \hat{\mu}(\bar{F})$ .

**5C. Short vectors in filtered Hermitian vector bundles.** Let  $K$  be a field,  $\mathcal{O}_K$  be its ring of integers, and  $S = \text{Spec } \mathcal{O}_K$ . Let  $\bar{E} = (E, (\|\cdot\|_{\sigma})_{\sigma:K \hookrightarrow \mathbb{C}})$  be a nonzero Hermitian vector bundle over  $S$ ; we denote its *first successive minimum* by

$$\lambda_1(\bar{E}) := \inf\left\{ \max_{\sigma} \|s\|_{\sigma} \mid s \in E \setminus \{0\} \right\}.$$

Since  $s \mapsto s \otimes 1$  identifies  $E$  with a lattice in the  $\mathbb{R}$ -vector space  $E \otimes_{\mathbb{Z}} \mathbb{R}$ , the first successive minimum is attained by some element  $s \in E \setminus \{0\}$ .

**Proposition 5.7** (Minkowski). *Let  $\bar{E}$  be a nonzero Hermitian vector bundle over  $S$ . Then*

$$\log \lambda_1(\bar{E}) \leq -\hat{\mu}(\bar{E}) + \frac{1}{2} \log(\text{rk } E) + \frac{\log |\Delta_K|}{2[K : \mathbb{Q}]} + \frac{1}{2} \log [K : \mathbb{Q}],$$

where  $\Delta_K$  denotes the discriminant of  $K$  over  $\mathbb{Q}$ .

This statement may be obtained from [Bost and Künnemann 2010, pp. 1027–1028] by considering the Hermitian vector bundle over  $\text{Spec } \mathbb{Z}$  given by the direct image of  $\bar{E}$  via  $S \rightarrow \text{Spec } \mathbb{Z}$ .

Let  $(\bar{E}_d)_{d \geq 1}$  be a family of Hermitian vector bundles over  $S$  such that

$$r_d := \text{rk } E_d \rightarrow +\infty$$

as  $d \rightarrow +\infty$ . Assume that, for each  $d \geq 1$ , we are given a *separated* filtration

$$E_d^0 = E_d \supset E_d^1 \supset E_d^2 \supset \dots$$

by saturated  $\mathcal{O}_K$ -submodules. We endow each  $E_d^m$  with the Hermitian vector bundle structure induced from  $\bar{E}_d$ .

**Proposition 5.8.** *With the above notation, assume that there exists an integer  $k \geq 1$  and a double sequence  $(a_{d,m})_{d \geq 1, m \geq 0}$  of positive real numbers, nondecreasing in  $m$  for every  $d \geq 1$ , such that*

$$\text{rk}(E_d^m/E_d^{m+1}) \leq k \tag{5-1}$$

and

$$\hat{\mu}(\overline{E_d^m/E_d^{m+1}}) \leq a_{d,m} \tag{5-2}$$

for every  $d \geq 1$  and  $m \geq 0$ . Then, for every  $d \geq 1$  such that  $r_d \neq 0$ , there exists  $m \geq \lfloor r_d/(2k) \rfloor$  and  $s \in E_d^m \setminus E_d^{m+1}$  satisfying

$$\max_{\sigma} \log \|s\|_{\sigma} \leq \max\{0, -2\hat{\mu}(\bar{E}_d)\} + a_{d,m} + \frac{1}{2} \log r_d + \frac{\log |\Delta_K|}{2[K : \mathbb{Q}]} + \frac{1}{2} \log [K : \mathbb{Q}].$$

*Proof.* Let  $d \geq 1$  such that  $r_d \neq 0$  and set

$$m' := \left\lfloor \frac{r_d}{2k} \right\rfloor,$$

so that, by (5-1),

$$r_d^{m'} := \text{rk } E_d^{m'} \geq r_d - km' \geq \frac{1}{2}r_d > 0.$$

By Proposition 5.5 (2), we have

$$\hat{\mu}(\bar{E}_d) = \frac{r_d^{m'}}{r_d} \hat{\mu}(\overline{E_d^{m'}}) + \frac{1}{r_d} \sum_{0 \leq i < m'} \text{rk}(E_d^i/E_d^{i+1}) \hat{\mu}(\overline{E_d^i/E_d^{i+1}}).$$

(When  $E_d^i/E_d^{i+1} = 0$ , we set  $\text{rk}(E_d^i/E_d^{i+1}) = 0$  and  $\hat{\mu}(\overline{E_d^i/E_d^{i+1}}) = -\infty$ , so that  $\text{rk}(E_d^i/E_d^{i+1}) \hat{\mu}(\overline{E_d^i/E_d^{i+1}}) = 0$  by convention.) Using hypotheses (5-1) and (5-2), and that  $(a_{d,m})$  is nondecreasing in  $m$  for every  $d$ , we obtain

$$\hat{\mu}(\bar{E}_d) \leq \frac{r_d^{m'}}{r_d} \hat{\mu}(\overline{E_d^{m'}}) + \frac{m'ka_{d,m'}}{r_d},$$

or, equivalently,

$$-\hat{\mu}(\overline{E_d^{m'}}) \leq -\frac{r_d}{r_d^{m'}} \hat{\mu}(\bar{E}_d) + \frac{m'k}{r_d^{m'}} a_{d,m'}.$$

Since  $r_d \leq 2r_d^{m'}$  and  $m' \leq r_d/(2k)$ , we conclude that

$$-\hat{\mu}(\overline{E_d^{m'}}) \leq \max\{0, -2\hat{\mu}(\bar{E}_d)\} + a_{d,m'}.$$

Let  $s \in E_d^{m'} \setminus \{0\}$  be such that  $\max_\sigma \|s\|_\sigma = \lambda_1(\overline{E_d^{m'}})$ . Then Proposition 5.7 yields

$$\begin{aligned} \max_\sigma \log \|s\|_\sigma &\leq -\hat{\mu}(\overline{E_d^{m'}}) + \frac{1}{2} \log r_d^{m'} + \frac{\log |\Delta_K|}{2[K : \mathbb{Q}]} + \frac{1}{2} \log [K : \mathbb{Q}] \\ &\leq \max\{0, -2\hat{\mu}(\overline{E_d})\} + a_{d,m'} + \frac{1}{2} \log r_d^{m'} + \frac{\log |\Delta_K|}{2[K : \mathbb{Q}]} + \frac{1}{2} \log [K : \mathbb{Q}]. \quad \square \end{aligned}$$

Thus  $m := \max\{i \in \mathbb{N} \mid s \in E_d^i\} \geq m'$  satisfies the conclusion of our statement.

**5D. Proof of Theorem 5.1.** Consider the notation and hypotheses of Theorem 5.1. Let us first observe that if Theorem 5.1 holds for some particular choice of Hermitian metric  $(\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}}$  on  $L$ , then a similar statement holds for any other choice of metric, up to modifying the constant  $c_2$ . We may thus assume that each  $(L_\sigma, \|\cdot\|_\sigma)$  is a positive Hermitian line bundle on  $\mathcal{X}_\sigma^{\text{an}}$ .

For every integer  $d \geq 1$ ,

$$E_d := \Gamma(\mathcal{X}, L^{\otimes d})$$

is a projective  $\mathcal{O}_K$ -module of finite type. For each field embedding  $\sigma : K \hookrightarrow \mathbb{C}$ , we may consider the uniform norm  $\|\cdot\|_{L^\infty(\mathcal{X}_\sigma^{\text{an}})}$  on  $E_{d,\sigma}$  induced by the Hermitian metric  $\|\cdot\|_\sigma$  on  $L_\sigma$ .

Note that the norm  $\|\cdot\|_{L^\infty(\mathcal{X}_\sigma^{\text{an}})}$  is not Hermitian in general. We denote by  $\|\cdot\|_{d,\sigma}$  the *John norm* on  $E_{d,\sigma}$  attached to  $\|\cdot\|_{L^\infty(\mathcal{X}_\sigma^{\text{an}})}$  (see [Bost 2015, Appendix F]); this is a Hermitian norm on  $E_{d,\sigma}$  satisfying

$$\|\cdot\|_{L^\infty(\mathcal{X}_\sigma^{\text{an}})} \leq \|\cdot\|_{d,\sigma} \leq (2 \operatorname{rk} E_d)^{1/2} \|\cdot\|_{L^\infty(\mathcal{X}_\sigma^{\text{an}})}. \tag{5-3}$$

We may thus consider the Hermitian vector bundle over  $\operatorname{Spec} \mathcal{O}_K$

$$\overline{E}_d := (E_d, (\|\cdot\|_{d,\sigma})_{\sigma:K \hookrightarrow \mathbb{C}}).$$

We define a decreasing filtration  $(E_d^m)_{m \geq 0}$  by saturated submodules on  $E_d$  via

$$E_d^m := \{s \in E_d \mid \operatorname{ord}_0 \hat{\varphi}^* s \geq m\}.$$

Since the image of  $\hat{\varphi}_K : \operatorname{Spf} K[[q]] \rightarrow \mathcal{X}_K$  is Zariski-dense,  $(E_d^m)_{m \geq 0}$  is a separated filtration. The subquotients  $E_d^m / E_d^{m+1}$  bear Hermitian vector bundle structures  $\overline{E}_d^m / \overline{E}_d^{m+1}$  induced by  $\overline{E}_d$ .

Let us denote by  $\Omega$  the fiber of coherent sheaf  $\Omega_{\operatorname{Spf} \mathcal{O}_K[[q]]/\mathcal{O}_K}^1$  at the point of  $\operatorname{Spf} \mathcal{O}_K[[q]]$  given by the ideal  $(q) \subset \mathcal{O}_K[[q]]$ . This is a trivial  $\mathcal{O}_K$ -module generated by  $dq$ . In what follows, we endow  $\Omega$  with a structure of Hermitian line bundle  $\overline{\Omega}$ , defined by

$$\|\alpha dq\|_\sigma = |\sigma(\alpha)|$$

for any  $\alpha \in \mathcal{O}_K$  and any embedding  $\sigma : K \hookrightarrow \mathbb{C}$ . Observe that  $\overline{\Omega}$  is isomorphic to the trivial Hermitian line bundle, and therefore  $\widehat{\deg} \overline{\Omega} = 0$ .

Let  $\hat{\varphi}(0) : \operatorname{Spec} \mathcal{O}_K \rightarrow \mathcal{X}$  denote the reduction of  $\hat{\varphi}$  modulo  $q$ , i.e., the composition of  $\hat{\varphi}$  with the closed immersion  $\operatorname{Spec} \mathcal{O}_K \rightarrow \operatorname{Spf} \mathcal{O}_K[[q]]$  associated to the ideal  $(q)$ . The Hermitian structure on  $L$  endows  $\hat{\varphi}(0)^*L$  with the structure of a Hermitian line bundle over  $\operatorname{Spec} \mathcal{O}_K$ .

For all integers  $d \geq 1$  and  $m \geq 0$ , we have an injective  $\mathcal{O}_K$ -linear map

$$\gamma_d^m : E_d^m / E_d^{m+1} \rightarrow \hat{\varphi}(0)^* L^{\otimes d} \otimes_{\mathcal{O}_K} \Omega^{\otimes m}$$

defined by mapping the class  $[s] \in E_d^m / E_d^{m+1}$  of  $s \in E_d^m$  to  $j_0^m \hat{\varphi}^* s$  (the jet of order  $m$  at  $q = 0$  of  $\hat{\varphi}^* s$ ).

**Lemma 5.9.** *There exist constants  $\kappa_1, \kappa_2 > 0$  such that for all integers  $d \geq 1$  and  $m \geq 0$  we have*

$$h(\gamma_{d,K}^m) \leq \kappa_1 d + \kappa_2 d \log^+ m,$$

where  $h$  denotes the height of  $\gamma_{d,K}^m$  with respect to the Hermitian vector bundles  $\overline{E_d^m / E_d^{m+1}}$  and  $\hat{\varphi}(0)^* \bar{L}^{\otimes d} \otimes_{\mathcal{O}_K} \bar{\Omega}^{\otimes m}$ .

*Proof.* Let  $\sigma : K \hookrightarrow \mathbb{C}$  be a field embedding. Since  $\varphi_\sigma : D_{R_\sigma} \rightarrow \mathcal{X}_\sigma^{\text{an}}$  has moderate growth, it follows from Proposition 3.12 that there exist constants  $\kappa_{1,\sigma}, \kappa_{2,\sigma} > 0$  such that, for any integers  $d \geq 1$  and  $m \geq 0$  for which  $E_d^m / E_d^{m+1} \neq 0$ , and any  $s \in E_{d,\sigma}^m \setminus E_{d,\sigma}^{m+1}$ , we have

$$\log \|j_0^m \varphi_\sigma^* s\|_{R_\sigma} - \log \|s\|_{L^\infty(\mathcal{X}_\sigma^{\text{an}})} \leq \kappa_{1,\sigma} d + \kappa_{2,\sigma} d \log^+ m. \tag{5-4}$$

Here, the norm  $\|\cdot\|_{R_\sigma}$  is the norm  $\|\cdot\|_r$ , introduced before Proposition 3.8, in the special case where  $r = R_\sigma$  and  $\bar{L}$  is  $\varphi_\sigma^* L_\sigma$  equipped with the pullback of  $\|\cdot\|_\sigma$ .

Note that

$$\|j_0^m \hat{\varphi}^* s\|_\sigma = R_\sigma^{-m} \|j_0^m \varphi_\sigma^* s\|_{R_\sigma},$$

where  $\|j_0^m \hat{\varphi}^* s\|_\sigma$  denotes the norm of  $j_0^m \hat{\varphi}^* s = \gamma_d^m([s])$  with respect to the Hermitian structure of  $\hat{\varphi}(0)^* \bar{L}^{\otimes d} \otimes_{\mathcal{O}_K} \bar{\Omega}^{\otimes m}$ . The estimate (5-4), together with (5-3), shows that

$$\log \|\gamma_d^m\|_\sigma + m \log R_\sigma \leq \kappa_{1,\sigma} d + \kappa_{2,\sigma} d \log^+ m.$$

Since  $\prod_{\sigma:K \hookrightarrow \mathbb{C}} R_\sigma = 1$ , we obtain

$$\sum_\sigma \log \|\gamma_d^m\|_\sigma \leq \left( \sum_\sigma \kappa_{1,\sigma} \right) d + \left( \sum_\sigma \kappa_{2,\sigma} \right) d \log^+ m.$$

Since  $\gamma_d^m$  is defined over  $\mathcal{O}_K$ , we have  $\|\gamma_{d,K}^m\|_{\mathfrak{p}} \leq 1$  for every maximal ideal  $\mathfrak{p}$  of  $\mathcal{O}_K$ , so that

$$h(\gamma_{d,K}^m) \leq \frac{1}{[K : \mathbb{Q}]} \sum_\sigma \log \|\gamma_d^m\|_\sigma \leq \frac{1}{[K : \mathbb{Q}]} \left( \sum_\sigma \kappa_{1,\sigma} \right) d + \frac{1}{[K : \mathbb{Q}]} \left( \sum_\sigma \kappa_{2,\sigma} \right) d \log^+ m. \quad \square$$

*End of proof of Theorem 5.1.* Let us first remark that, as  $L_K$  is ample, we have

$$r_d = \text{rk } E_d = \dim \Gamma(\mathcal{X}_K, L_K^{\otimes d}) \sim_{d \rightarrow +\infty} \frac{\deg_{L_K} \mathcal{X}_K}{n!} d^n.$$

In particular,  $\log r_d = O(d)$  as  $d \rightarrow +\infty$ . We shall apply Proposition 5.8 for  $(E_d^m)_{d \geq 1, m \geq 0}$  defined as above. This suffices by the estimates (5-3).

Note that condition (5-1) is trivially verified for  $k = 1$ . Moreover, by the same argument of [Bost 2001, Proposition 4.4] (see also Lemma 4.1 of that paper) and by the estimates (5-3), there exists a constant

$c > 0$  such that  $-\hat{\mu}(\bar{E}_d) \leq cd$  for every  $d \geq 1$ . Thus, to finish our proof, it is sufficient to find constants  $a, b > 0$  such that

$$\hat{\mu}(\overline{E_d^m/E_d^{m+1}}) \leq a_{d,m} := ad + bd \log^+ m$$

for every  $d \geq 1$  and  $m \geq 0$  (condition (5-2)).

By Lemma 5.9, there exist constants  $\kappa_1, \kappa_2 > 0$  such that, for every  $d \geq 1$  and  $m \geq 0$  such that  $E_d^m/E_d^{m+1} \neq 0$ , we have

$$h(\gamma_{d,K}^m) \leq \kappa_1 d + \kappa_2 d \log^+ m.$$

Thus, since  $\gamma_{d,K}^m$  is injective, we may apply the slope inequality (Proposition 5.6) to obtain

$$\begin{aligned} \hat{\mu}(\overline{E_d^m/E_d^{m+1}}) &\leq \hat{\mu}(\hat{\varphi}(0)^* \bar{L}^{\otimes d} \otimes_{\mathcal{O}_K} \bar{\Omega}^{\otimes m}) + h(\gamma_{d,K}^m) \\ &\leq (\kappa_1 + \hat{\mu}(\hat{\varphi}(0)^* \bar{L}))d + \kappa_2 d \log^+ m. \end{aligned} \quad \square$$

### 6. Derivatives of sections of line bundles along vector fields

A crucial step in Nesterenko’s method involves applying a certain differential operator (deduced from the Ramanujan equations) to auxiliary polynomials. It is also important to understand how this differential operator affects the degree and the norm  $\| \cdot \|_\infty$  of a polynomial.

Our generalization of Nesterenko’s proof replaces polynomials of degree  $d$  by global sections of the  $d$ -th tensor power of some ample line bundle. In this section we explain how to differentiate global sections of tensor powers of a line bundle  $L$  along a vector field  $v$ . Under a projectivity hypothesis, we also explain how  $L^\infty$  norms with respect to some Hermitian metric on  $L$  are affected by a differential operator deduced from  $v$ .

**6A. The basic definition.** Let  $M$  be a compact connected complex manifold, and  $L$  be a line bundle over  $M$  endowed with a global holomorphic section  $s_0 \in \Gamma(M, L) \setminus \{0\}$ . To  $L$  is associated the graded ring  $R = \bigoplus_{d \geq 0} R_d$ , where  $R_d := \Gamma(M, L^{\otimes d})$ .

Let  $v$  be a meromorphic vector field on  $M$ , and assume that  $v$  is holomorphic on the open subset  $M_{s_0} := \{p \in M \mid s_0(p) \neq 0\}$ . Then there is a smallest integer  $k \geq 0$ , the “order of pole of  $v$  at  $\text{div}(s_0)$ ”, such that  $v \otimes s_0^{\otimes k}$  defines a global holomorphic section of  $TM \otimes L^{\otimes k}$ .

The vector field  $v$  induces a  $\mathbb{C}$ -derivation of degree  $k + 1$  of the graded ring  $R$

$$\partial_v : R \rightarrow R$$

given as follows. By definition,  $\partial_v$  is the zero map on  $R_0 \cong \mathbb{C}$ . Let  $d \geq 1$  be an integer,  $s \in R_d$ , and  $f : M_{s_0} \rightarrow \mathbb{C}$  be the holomorphic function for which  $s = f s_0^{\otimes d}$  over  $M_{s_0}$ . Then  $\partial_v s \in R_{d+k+1}$  is defined as the unique global section of  $L^{\otimes d+k+1}$  such that  $\partial_v s = v(f) s_0^{\otimes d+k+1}$  over  $M_{s_0}$ . The next lemma guarantees that this is well-defined.

**Lemma 6.1.** *With the above notation,  $v(f) s_0^{\otimes d+k+1}$  extends to a global holomorphic section of  $L^{\otimes d+k+1}$ .*

*Proof.* The pair  $(L, s_0)$  corresponds canonically to an effective analytic Cartier divisor  $E$  on  $M$ ; let  $(g_i, U_i)_{i \in I}$  be a family of local equations of  $E$  for some open covering  $M = \bigcup_{i \in I} U_i$ . By hypothesis,  $g_i^k v$  extends to a holomorphic vector field on  $U_i$  for every  $i \in I$ .

An element  $s \in R_d$  might be identified with a meromorphic function  $f = s/s_0^{\otimes d}$  on  $M$  having pole of order at most  $d$  on  $E$ , i.e., such that  $g_i^d f$  defines a holomorphic function on  $U_i$  for every  $i \in I$ . Under this identification, our statement is equivalent to the assertion that  $g_i^{d+k+1} v(f)$  defines a holomorphic function on  $U_i$  for every  $i \in I$ .

Now, for  $i \in I$ , we have

$$g_i^{k+1} v(g_i^d f) = g_i^{k+1} (d \cdot g_i^{d-1} v(g_i) f + g_i^d v(f)) = d \cdot g_i^k v(g_i) g_i^d f + g_i^{d+k+1} v(f),$$

so that  $g_i^{d+k+1} v(f)$  defines a holomorphic function on  $U_i$ . □

Finally, it is easy to see that the  $\mathbb{C}$ -linear map  $\partial_v : R \rightarrow R$  satisfies Leibniz's rule: if  $s \in R_d$  and  $t \in R_e$ , then

$$\partial_v(s \otimes t) = \partial_v s \otimes t + s \otimes \partial_v t$$

in  $R_{d+e+k+1}$ .

**6B. Estimates of uniform norms.** Let us keep the notation of the last subsection and fix once and for all some  $t \in R_{k+1}$  (recall that  $k$  denotes the “order of pole of  $v$  at  $\text{div}(s_0)$ ”).

For any integer  $j \geq 1$  and  $d \geq 1$ , we define a differential operator of degree  $j(k+1)$

$$\partial_v^{[j]} : R \rightarrow R$$

as the composition

$$\partial_v^{[j]} = \partial_v \circ (\partial_v - t) \circ \cdots \circ (\partial_v - (j-1)t).$$

**Proposition 6.2.** *With the above notation, assume moreover that  $L = i^* \mathcal{O}(1)$  for some projectively normal embedding  $i : M \hookrightarrow \mathbb{P}^n(\mathbb{C})$ . Let  $\| \cdot \|$  be a Hermitian metric on  $L$ . Then there exists a constant  $C > 0$  such that, for any integer  $j \geq 1$ , any sufficiently large positive integer  $d$ , and any  $s \in R_d$ , we have*

$$\| \partial_v^{[j]} s \|_{L^\infty(M)} \leq C^{j+d} (j+d)^j \|s\|_{L^\infty(M)},$$

where  $\| \partial_v^{[j]}(s) \|_{L^\infty(M)}$  and  $\|s\|_{L^\infty(M)}$  denote the uniform norms on  $M$  with respect to the Hermitian metrics on  $L^{\otimes d+j(k+1)}$  and  $L^{\otimes d}$ , respectively, induced by  $\| \cdot \|$ .

Our proof is a reduction to the case  $M = \mathbb{P}^n(\mathbb{C})$ . Let  $\| \cdot \|$  denote the Fubini–Study metric on the line bundle  $\mathcal{O}(1)$  over  $\mathbb{P}^n_{\mathbb{C}} = \text{Proj } \mathbb{C}[X_0, \dots, X_n]$  (see [Example 4.5](#)), and let us identify  $\Gamma(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(d))$  with the  $\mathbb{C}$ -vector space  $\mathbb{C}[X_0, \dots, X_n]_d$  of homogeneous polynomials of degree  $d$ . If  $P = \sum_{|I|=d} a_I X^I$ , we consider the norms

$$\|P\|_{\infty} := \max_{|I|=d} |a_I|, \quad \|P\|_1 := \sum_{|I|=d} |a_I|.$$

The uniform norm of  $P$ , seen as an element of  $\Gamma(\mathbb{P}^n(\mathbb{C}), \mathcal{O}(d))$ , with  $\mathcal{O}(1)$  equipped with the Fubini–Study metric, is given by

$$\|P\|_{L^\infty(\mathbb{P}^n(\mathbb{C}))} = \sup_{z \in \mathbb{C}^{n+1} \setminus \{0\}} \frac{|P(z)|}{\left(\sum_{i=0}^n |z_i|^2\right)^{d/2}}.$$

**Lemma 6.3.** *For any  $P \in \Gamma(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(d))$ , we have*

$$(n + 1)^{-d/2} \|P\|_\infty \leq \|P\|_{L^\infty(\mathbb{P}^n(\mathbb{C}))} \leq \|P\|_1 \leq \binom{d+n}{n} \|P\|_\infty.$$

*Proof.* If we write  $P = \sum_{|I|=d} a_I X^I$ , then Cauchy’s integral formula gives, for any multi-index  $I$ ,

$$a_I = \frac{1}{(2\pi i)^{n+1}} \int_{(\partial D)^{n+1}} \frac{P(z)}{z^{I+1}} dz_0 \cdots dz_n,$$

where  $D$  denotes the unit disk in  $\mathbb{C}$  and  $\mathbf{1}$  the multi-index of order  $n + 1$  having 1 at each coordinate. Thus, if  $[z]$  denotes the image in  $\mathbb{P}^n(\mathbb{C})$  of a point  $z \in \mathbb{C}^{n+1} \setminus \{0\}$ ,

$$|a_I| \leq \sup_{z \in (\partial D)^{n+1}} |P(z)| = (n + 1)^{d/2} \sup_{z \in (\partial D)^{n+1}} \|P([z])\|.$$

This proves that  $(n + 1)^{-d/2} \|P\|_\infty \leq \|P\|_{L^\infty(\mathbb{P}^n(\mathbb{C}))}$ .

For any  $z \in \mathbb{C}^{n+1} \setminus \{0\}$ , we have

$$\|P([z])\| = \frac{|P(z)|}{\left(\sum_{i=0}^n |z_i|^2\right)^{d/2}} \leq \frac{\sum_{|I|=d} |a_I| |z^I|}{\left(\sum_{i=0}^n |z_i|^2\right)^{d/2}} \leq \frac{\max_{|I|=d} |z^I|}{\left(\sum_{i=0}^n |z_i|^2\right)^{d/2}} \|P\|_1.$$

Now, if  $I = (i_0, \dots, i_n)$  is a multi-index satisfying  $|I| = d$ , then it is clear that

$$|z^I|^2 = (|z_0|^2)^{i_0} \cdots (|z_n|^2)^{i_n} \leq \left(\sum_{i=0}^n |z_i|^2\right)^d.$$

We thus obtain  $\|P\|_{L^\infty(\mathbb{P}^n(\mathbb{C}))} \leq \|P\|_1$ .

The inequality  $\|P\|_1 \leq \binom{n+d}{n} \|P\|_\infty$  is an immediate consequence of  $\dim \Gamma(\mathbb{P}^n(\mathbb{C}), \mathcal{O}(d)) = \binom{n+d}{n}$ .  $\square$

*Proof of Proposition 6.2.* Since  $M$  is compact, if the conclusion of the statement holds for some Hermitian metric  $\|\cdot\|$ , then, up to replacing the constant  $C$ , it also holds for any other Hermitian metric on  $L$ . We may thus assume that  $\|\cdot\|$  is induced by the Fubini–Study metric on  $\mathcal{O}(1)$  via the embedding  $i$ .

Let  $(X_0, \dots, X_n)$  denote the projective coordinates of  $\mathbb{P}^n(\mathbb{C})$ , seen as global sections of  $\mathcal{O}(1)$ , and let  $t_j \in R_1$  be the restriction of  $X_j$  to  $M$  for every  $0 \leq j \leq n$ . Since  $i : M \hookrightarrow \mathbb{P}^n(\mathbb{C})$  is projectively normal, for any integer  $d \geq 1$ ,  $R_d$  is generated as a  $\mathbb{C}$ -vector space by the monomials of degree  $d$  in  $t_0, \dots, t_n$ .

We lift  $v$  to  $\mathbb{P}^n(\mathbb{C})$  as follows. For every  $0 \leq j \leq n$ , let

$$P_j \in \Gamma(\mathbb{P}^n(\mathbb{C}), \mathcal{O}(k + 2)) = \mathbb{C}[X_0, \dots, X_n]_{k+2}$$

be a lifting of  $\partial_v t_j \in R_{k+2}$ . Then there exists a unique  $\mathbb{C}$ -derivation  $\partial$  of  $\bigoplus_{d \geq 0} \Gamma(\mathbb{P}^n(\mathbb{C}), \mathcal{O}(d)) = \mathbb{C}[X_0, \dots, X_n]$ , of degree  $k + 1$ , such that  $\partial X_j = P_j$  for every  $0 \leq j \leq n$ . It is easy to see that, for every

integer  $d \geq 0$ , the diagram

$$\begin{array}{ccc} \mathbb{C}[X_0, \dots, X_n]_d & \xrightarrow{\partial} & \mathbb{C}[X_0, \dots, X_n]_{d+k+1} \\ i^* \downarrow & & \downarrow i^* \\ R_d & \xrightarrow{\partial_v} & R_{d+k+1} \end{array}$$

commutes. Moreover, if  $Q \in \mathbb{C}[X_0, \dots, X_n]_{k+1}$  is any lifting of  $t \in R_{k+1}$ , then it is clear that, for any  $j \geq 1$ ,

$$\partial^{[j]} := \partial \circ (\partial - Q) \circ \dots \circ (\partial - (j-1)Q)$$

makes the diagram

$$\begin{array}{ccc} \mathbb{C}[X_0, \dots, X_n]_d & \xrightarrow{\partial^{[j]}} & \mathbb{C}[X_0, \dots, X_n]_{d+j(k+1)} \\ i^* \downarrow & & \downarrow i^* \\ R_d & \xrightarrow{\partial_v^{[j]}} & R_{d+j(k+1)} \end{array}$$

commute.

For every multi-index  $I \in \mathbb{N}^{n+1}$ , we have

$$\|\partial X^I\|_\infty \leq |I| \max_{0 \leq i \leq n} \|P_i\|_\infty.$$

This implies that, for any  $d \geq 1$  and any homogeneous polynomial  $P \in \mathbb{C}[X_0, \dots, X_n]_d$ ,

$$\|\partial P\|_\infty \leq d \left( \max_{0 \leq i \leq n} \|P_i\|_\infty \right) \|P\|_\infty.$$

Thus, if

$$\kappa := \|Q\|_\infty + (k+1) \max_{0 \leq i \leq n} \|P_i\|_\infty$$

and  $S \in \mathbb{C}[X_0, \dots, X_n]$  is a homogeneous polynomial of degree  $d + m(k+1)$  for some  $0 \leq m \leq j-1$ , we have

$$\|(\partial - mQ)S\|_\infty \leq ((d + m(k+1)) \left( \max_{0 \leq i \leq n} \|P_i\|_\infty \right) + m\|Q\|_\infty) \|S\|_\infty \leq \kappa(d+j)\|S\|_\infty.$$

By induction, we conclude that, for any  $d \geq 1$  and any  $P \in \mathbb{C}[X_0, \dots, X_n]_d$ , we have

$$\|\partial^{[j]}P\|_\infty \leq \kappa^j (d+j)^j \|P\|_\infty. \tag{6-1}$$

To complete our proof, we apply a lifting argument. By [Bost 2004, Proposition 3.5], there exists a constant  $C_0 > 0$  such that, for every sufficiently large integer  $d$  and every  $s \in \Gamma(M, L^{\otimes d})$ , there exists a lifting  $P \in \Gamma(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(d))$  of  $s$  such that

$$\|P\|_{L^\infty(\mathbb{P}^n(\mathbb{C}))} \leq C_0^d \|s\|_{L^\infty(M)}. \tag{6-2}$$

Thus, for any  $j \geq 1$ ,

$$\begin{aligned}
 \|\partial_v^{[j]}s\|_{L^\infty(M)} &\leq \|\partial^{[j]}P\|_{L^\infty(\mathbb{P}^n(\mathbb{C}))} \leq \|\partial^{[j]}P\|_1 \leq \binom{d+j(k+1)}{j(k+1)} \|\partial^{[j]}P\|_\infty && \text{by Lemma 6.3} \\
 &\leq 2^{d+j(k+1)} \|\partial^{[j]}P\|_\infty \\
 &\leq 2^{d+j(k+1)} \kappa^j (d+j)^j \|P\|_\infty && \text{by (6-1)} \\
 &\leq 2^{d+j(k+1)} \kappa^j (d+j)^j (n+1)^{d/2} \|P\|_{L^\infty(\mathbb{P}^n(\mathbb{C}))} && \text{by Lemma 6.3} \\
 &\leq 2^{d+j(k+1)} \kappa^j (d+j)^j (n+1)^{d/2} C_0^d \|s\|_{L^\infty(M)} && \text{by (6-2).} \quad \square
 \end{aligned}$$

**6C. The arithmetic case.** We shall actually need an arithmetic variant of the above constructions.

Consider the notation and terminology of Section 5A. Let  $K$  be a number field,  $\mathcal{X}$  be a projective arithmetic scheme over  $S = \text{Spec } \mathcal{O}_K$  with smooth generic fiber, and  $L$  be a line bundle over  $\mathcal{X}$  endowed with a global section  $s_0 \in \Gamma(\mathcal{X}, L) \setminus \{0\}$ . Arguing as above, we see that a section  $w \in \Gamma(\mathcal{X}_{s_0}, \text{Der}_{\mathcal{O}_S}(\mathcal{O}_{\mathcal{X}}))$  induces an  $\mathcal{O}_K$ -derivation  $\partial_w$  of the ring  $\bigoplus_{d \geq 0} \Gamma(\mathcal{X}, L^{\otimes d})$ .

Let us fix  $t \in \Gamma(\mathcal{X}, L^{k+1})$ , where  $k \geq 0$  is the “order of pole of  $w$  at  $\text{div}(s_0)$ ”, and consider the differential operators  $\partial_w^{[j]} = \partial_w \circ (\partial_w - t) \circ \dots \circ (\partial_w - (j-1)t)$ , for  $j \geq 0$ , as above.

By applying Proposition 6.2 for each projective embedding  $\sigma : K \hookrightarrow \mathbb{C}$ , we obtain the following corollary.

**Corollary 6.4.** *With the above notation, assume moreover that  $L = i^* \mathcal{O}(1)$  for some closed immersion  $i : \mathcal{X} \hookrightarrow \mathbb{P}_{\mathcal{O}_K}^n$  over  $S$  such that  $i_K : \mathcal{X}_K \hookrightarrow \mathbb{P}_K^n$  is projectively normal. Let  $(\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}}$  be a Hermitian structure on  $L$ . Then, there exists a constant  $C > 0$  such that, for any integer  $j \geq 1$ , any sufficiently large positive integer  $d$ , and any  $s \in \Gamma(\mathcal{X}, L^{\otimes d})$ , we have*

$$\|\partial_w^{[j]}s\|_{\mathcal{X}} \leq C^{j+d} (j+d)^j \|s\|_{\mathcal{X}}.$$

### 7. Proof of Theorem 1.2

Recall the notation and hypotheses of Theorem 1.2:  $\mathcal{X}$  is a quasiprojective arithmetic scheme over  $\mathcal{O}_K$  of relative dimension  $n \geq 2$  with smooth generic fiber, and  $\hat{\varphi} : \text{Spf } \mathcal{O}_K[[q]] \rightarrow \mathcal{X}$  is a morphism of formal  $\mathcal{O}_K$ -schemes such that

- (i) the formal curve  $\hat{\varphi}_K : \text{Spf } K[[q]] \rightarrow \mathcal{X}_K$  is ZL-dense in  $\mathcal{X}_K$  and satisfies the differential equation

$$q \frac{d\hat{\varphi}_K}{dq} = v \circ \hat{\varphi}_K;$$

- (ii) for any field embedding  $\sigma : K \hookrightarrow \mathbb{C}$ , the formal curve  $\hat{\varphi}_\sigma : \text{Spf } \mathbb{C}[[q]] \rightarrow \mathcal{X}_\sigma$  lifts to an analytic curve  $\varphi_\sigma : D_{R_\sigma} \subset \mathbb{C} \rightarrow \mathcal{X}_\sigma^{\text{an}}$  of moderate growth. We also assume that  $\prod_{\sigma:K \hookrightarrow \mathbb{C}} R_\sigma = 1$ .

Let  $\bar{\mathcal{X}}$  be some projective compactification with smooth generic fiber of the arithmetic variety  $\mathcal{X}$  over  $\mathcal{O}_K$ . Fix a Hermitian line bundle  $\bar{L} = (L, (\|\cdot\|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$  over  $\bar{\mathcal{X}}$  such that  $L_K$  is ample and  $(L_\sigma, \|\cdot\|_\sigma)$  over  $\mathcal{X}_\sigma^{\text{an}}$  is positive for every  $\sigma : K \hookrightarrow \mathbb{C}$ .

In view of Philippon’s algebraic independence criterion (Theorem A.1), Theorem 1.2 will be a direct consequence of the following.

**Theorem 7.1.** *With the above notation, for any field embedding  $\sigma : K \hookrightarrow \mathbb{C}$ , and any  $z \in D_{R_\sigma} \setminus \{0\}$ , there exist real constants  $c_0, c_1, c_2, c_3 > 0$  such that, for every sufficiently large positive integer  $d$ , there exists a positive integer  $d' \leq c_0 d \log d$ , and  $t \in \Gamma(\bar{\mathcal{X}}, L^{\otimes d'})$  satisfying*

$$\log \|t\|_{\bar{\mathcal{X}}} \leq c_1 d \log^2 d$$

and

$$-c_2 d^n \leq \log \|t(\varphi_\sigma(z))\|_\sigma \leq -c_3 d^n.$$

We shall prove this theorem in three steps corresponding to the next three lemmas.

**Lemma 7.2** (auxiliary sections). *There exist constants  $a, b, c > 0$  such that, for every sufficiently large positive integer  $d$ , there is a global section  $s \in \Gamma(\bar{\mathcal{X}}, L^{\otimes d})$  such that*

$$ad^n < \text{ord}_0 \hat{\varphi}^* s \leq bd^n \tag{7-1}$$

and

$$\log \|s\|_{\bar{\mathcal{X}}} \leq cd \log d. \tag{7-2}$$

*Proof.* Since  $\hat{\varphi}_K$  is ZL-dense in  $\mathcal{X}_K$ , and  $\varphi_\tau$  has moderate growth in  $\mathcal{X}_\tau^{\text{an}}$  for every embedding  $\tau : K \hookrightarrow \mathbb{C}$ , our statement follows immediately from Corollary 5.3. □

Fix a field embedding  $\sigma : K \hookrightarrow \mathbb{C}$  and  $z \in D_{R_\sigma} \setminus \{0\}$ . By the projective prime avoidance lemma, there is an integer  $k \geq 1$ , and a global section  $s_0 \in \Gamma(\bar{\mathcal{X}}, L^{\otimes k})$  such that  $\bar{\mathcal{X}}_{s_0} := (s_0 \neq 0) \subset \mathcal{X}$  and  $\varphi_\sigma(z) \in \bar{\mathcal{X}}_{s_0, \sigma}^{\text{an}}$ . Up to replacing  $L$  by  $L^{\otimes k}$ , we may assume that  $k = 1$  (see Remark A.2).

**Lemma 7.3.** *There exist constants  $\gamma_0, \gamma_1, \gamma_2 > 0$  such that, for every sufficiently large positive integer  $d$ , and every  $s \in \Gamma(\bar{\mathcal{X}}, L^{\otimes d})$  as in Lemma 7.2, there exists  $j \leq \gamma_0 d \log d$  such that, if we write  $s = f s_0^{\otimes d}$  over  $\bar{\mathcal{X}}_{s_0}$ , then*

$$-\gamma_1 d^n \leq \log |(\varphi_\sigma^* f)^{(j)}(z)| \leq -\gamma_2 d^n.$$

*Proof.* Let  $d$  be a sufficiently large positive integer and  $s \in \Gamma(\bar{\mathcal{X}}, L^{\otimes d})$  be as in Lemma 7.2. Set  $m := \text{ord}_0 \hat{\varphi}^* s$ .

According to Proposition 3.12 and to the bounds (7-1) and (7-2), for every embedding  $\tau \neq \sigma$ , there is a constant  $\kappa_\tau$  (not depending on  $d$  or  $s$ ) such that

$$\log \|j_0^m \varphi_\tau^* s\|_{R_\tau} \leq \kappa_\tau d \log d. \tag{7-3}$$

We recall that the norm  $\| \cdot \|_{R_\tau}$  above is the norm  $\| \cdot \|_r$ , introduced before Proposition 3.8, in the special case where  $r = R_\tau$  and  $\bar{L}$  is  $\varphi_\tau^* L_\tau$  equipped with the pullback of  $\| \cdot \|_\tau$ .

Fix any constant  $C > \sum_{\tau \neq \sigma} \kappa_\tau$ . Then Corollary 3.16 shows that there exist real numbers  $\gamma_0, \gamma_1 > 0$  such that, for sufficiently large  $d$ , if

$$\max_{0 \leq j \leq \lfloor \gamma_0 d \log d \rfloor} \log \frac{|(\varphi_\sigma^* f)^{(j)}(z)|}{j!} < -\gamma_1 d^n \tag{7-4}$$

then

$$\log \|j_0^m \varphi_\sigma^* s\|_{R_\sigma} \leq -Cd \log d. \tag{7-5}$$

By contradiction, assume that (7-4) holds. Observe that  $j_0^m \hat{\varphi}^* s$  is an element of  $\hat{\varphi}(0)^* L^{\otimes d} \otimes \Omega^{\otimes m}$ , and the Hermitian structure on  $\hat{\varphi}(0)^* \bar{L}^{\otimes d} \otimes \bar{\Omega}^{\otimes m}$  allows us to consider its norms  $(\|j_0^m \hat{\varphi}^* s\|_\tau)_{\tau:K \hookrightarrow \mathbb{C}}$ . For every field embedding  $\tau : K \hookrightarrow \mathbb{C}$ , we have

$$\|j_0^m \hat{\varphi}^* s\|_\tau = R_\tau^{-m} \|j_0^m \varphi_\tau^* s\|_{R_\tau}.$$

Thus, since  $\prod_{\tau:K \hookrightarrow \mathbb{C}} R_\tau = 1$ , we obtain from (7-3) and (7-5)

$$\sum_{\tau:K \hookrightarrow \mathbb{C}} \log \|j_0^m \hat{\varphi}^* s\|_\tau \leq -\left(C - \sum_{\tau \neq \sigma} \kappa_\tau\right) d \log d.$$

On the other hand, by definition of the Arakelov degree, we have

$$\sum_{\tau:K \hookrightarrow \mathbb{C}} \log \|j_0^m \hat{\varphi}^* s\|_\tau \geq -\widehat{\text{deg}}(\hat{\varphi}(0)^* \bar{L}^{\otimes d} \otimes \bar{\Omega}^{\otimes m}) = -\widehat{\text{deg}}(\hat{\varphi}(0)^* \bar{L})d.$$

This contradicts our choice of  $C$  for  $d \gg 0$ . We conclude that, for sufficiently large  $d$ , (7-4) cannot hold, so that there exists an integer  $j \leq \gamma_0 d \log d$  for which

$$\log |(\varphi_\sigma^* f)^{(j)}(z)| \geq \log \frac{|(\varphi_\sigma^* f)^{(j)}(z)|}{j!} \geq -\gamma_1 d^n. \tag{7-6}$$

Next, we bound  $\log |(\varphi_\sigma^* f)^{(j)}(z)|$  from above. Let  $\Delta$  be a disk centered in  $z$ , of radius  $\varepsilon > 0$  small enough so that  $\bar{\Delta} \subset \varphi_\sigma^{-1}(\mathcal{X}_{s_0, \sigma}^{\text{an}})$ . It follows from Proposition 3.14, and bounds (7-1) and (7-2), that there is a constant  $c' > 0$  such that

$$\log \max_{\zeta \in \partial \Delta} |\varphi_\sigma^* f(\zeta)| \leq -c' d^n.$$

By the Cauchy inequalities, we have

$$\frac{|(\varphi_\sigma^* f)^{(j)}(z)|}{j!} \leq \frac{\max_{\zeta \in \partial \Delta} |\varphi_\sigma^* f(\zeta)|}{\varepsilon^j},$$

so that

$$\log |(\varphi_\sigma^* f)^{(j)}(z)| \leq -c' d^n + \log j! - j \log \varepsilon.$$

Since  $j = O(d \log d)$ , we have  $\log j! = O(d \log^2 d)$ , and we conclude that there is a constant  $\gamma_2 > 0$  (not depending on  $d$  or  $s$ ) such that

$$\log |(\varphi_\sigma^* f)^{(j)}(z)| \leq -\gamma_2 d^n \tag{7-7}$$

for every sufficiently large  $d$ . □

Again by Remark A.2, up to replacing  $L$  by a sufficiently large tensor power of itself, we may assume that there exists a closed immersion  $i : \bar{\mathcal{X}} \hookrightarrow \mathbb{P}_{\mathcal{O}_K}^n$  over  $\mathcal{O}_K$  such that  $i_K : \bar{\mathcal{X}}_K \hookrightarrow \mathbb{P}_K^n$  is projectively normal and  $L = i^* \mathcal{O}(1)$ .

Fix any  $\alpha \in \mathcal{O}_K \setminus \{0\}$  that “clears the denominators of  $v$ ”, i.e., such that  $w := \alpha v$  defines a nonzero global section of  $Der_{\mathcal{O}_K}(\mathcal{O}_{\bar{X}})$ . Let  $k \geq 0$  be the smallest integer for which  $w \otimes s_0^{\otimes k}$  defines a global section of  $Der_{\mathcal{O}_K}(\mathcal{O}_{\bar{X}}) \otimes L^{\otimes k}$ , and let  $\partial_w$  be the  $\mathcal{O}_K$ -derivation of degree  $k + 1$  of the ring  $\bigoplus_{d \geq 0} \Gamma(\bar{X}, L^{\otimes d})$  defined in Section 6. For any integer  $j \geq 1$ , set

$$\partial_w^{[j]} = \partial_w \circ (\partial_w - \alpha s_0^{\otimes k+1}) \circ \dots \circ (\partial_w - (j-1)\alpha s_0^{\otimes k+1}).$$

**Lemma 7.4.** *There exist constants  $c_1, c_2, c_3 > 0$  such that, for every sufficiently large positive integer  $d$ , and every  $s \in \Gamma(\bar{X}, L^{\otimes d})$  as in Lemma 7.2, if  $j$  denotes the integer constructed in Lemma 7.3, then the section  $t := \partial_w^{[j]}(s) \in \Gamma(\bar{X}, L^{\otimes d+j(k+1)})$  satisfies*

$$\log \|t\|_{\bar{X}} \leq c_1 d \log^2 d$$

and

$$-c_2 d^n \leq \log \|t(\varphi_\sigma(z))\|_\sigma \leq -c_3 d^n.$$

*Proof.* Since  $j$  grows on the order of  $d \log d$ , by Corollary 6.4 and bound (7-2), there exists  $c_1 > 0$  such that

$$\log \|t\|_{\bar{X}} \leq c_1 d \log^2 d.$$

In order to bound  $\|t(\varphi_\sigma(z))\|_\sigma$ , we first remark that the formal identity of differential operators

$$q^j \frac{d^j}{dq^j} = q \frac{d}{dq} \left( q \frac{d}{dq} - 1 \right) \dots \left( q \frac{d}{dq} - (j-1) \right)$$

and the differential equation

$$D\hat{\varphi}_K \left( q \frac{d}{dq} \right) = \hat{\varphi}_K^* v$$

yield

$$\begin{aligned} \hat{\varphi}^* t &= \hat{\varphi}^* (\partial_w (\partial_w - \alpha s_0^{\otimes k+1}) \dots (\partial_w - (j-1)\alpha s_0^{\otimes k+1})(s)) \\ &= \alpha^j \left[ q \frac{d}{dq} \left( q \frac{d}{dq} - 1 \right) \dots \left( q \frac{d}{dq} - (j-1) \right) \hat{\varphi}^*(f) \right] \hat{\varphi}^*(s_0)^{\otimes d+j(k+1)} = (\alpha q)^j \frac{d^j \hat{\varphi}^*(f)}{dq^j} \hat{\varphi}^*(s_0)^{\otimes d+j(k+1)}. \end{aligned}$$

A similar formula holds for  $\varphi_\sigma$ . Thus

$$\log \|t(\varphi_\sigma(z))\|_\sigma = \log |(\varphi_\sigma^* f)^{(j)}(z)| + j \log |\alpha z| + (d + j(k + 1)) \log \|\varphi_\sigma^* s_0(z)\|_\sigma.$$

Since  $j$  grows on the order of  $d \log d$ , we conclude from (7-6) and (7-7) that there exist real constants  $c_2 > c_3 > 0$  such that

$$-c_2 d^n \leq \log \|t(\varphi_\sigma(z))\|_\sigma \leq -c_3 d^n$$

for sufficiently large  $d$ . □

To finish the proof, we simply remark that, if  $c_0$  is any real number satisfying  $c_0 > (k + 1)\gamma_0$ , then the degree  $d' := d + j(k + 1)$  of  $t$  constructed above satisfies  $d' \leq d + (k + 1)\gamma_0 d \log d \leq c_0 d \log d$ , for  $d$  sufficiently large.

**Appendix A: Philippon’s algebraic independence criterion for projective varieties**

Let  $K$  be a number field,  $\mathcal{X}$  be a projective arithmetic scheme over  $\mathcal{O}_K$  of relative dimension  $n \geq 2$  (see the definition in Section 5A), and  $\bar{L} = (L, (\| \cdot \|_\sigma)_{\sigma:K \hookrightarrow \mathbb{C}})$  be a Hermitian line bundle over  $\mathcal{X}$  with  $L$  relatively ample over  $\text{Spec } \mathcal{O}_K$ . Recall that, if  $s \in \Gamma(\mathcal{X}, L^{\otimes d})$  for some integer  $d \geq 1$ , then we set  $\|s\|_{\mathcal{X}} = \max_{\sigma} \|s\|_{\sigma, L^{\infty}(\mathcal{X}_{\sigma}^{\text{an}})}$ , where  $\sigma$  runs through the set of field embeddings of  $K$  in  $\mathbb{C}$ .

The proof of the main theorem of this article relies on the following generalized version of an algebraic independence criterion of Philippon [1986, Théorème 2.11]; see also [Nesterenko 1996, Lemma 2.5].

**Theorem A.1.** *Let  $\sigma : K \hookrightarrow \mathbb{C}$  be a field embedding and  $p \in \mathcal{X}_{\sigma}(\mathbb{C})$ . Suppose that there exist an integer  $m \in [2, n]$ , a nondecreasing sequence of positive real numbers  $(\ell_d)_{d \geq 1}$  satisfying  $\ell_d^{m-1} = o(d)$  as  $d \rightarrow +\infty$ , and real constants  $a > b > 0$  such that, for every sufficiently large positive integer  $d$ , there exists an integer  $d' \leq d\ell_d$  and a section  $s \in \Gamma(\mathcal{X}, L^{\otimes d'})$  satisfying*

$$\log \|s\|_{\mathcal{X}} \leq d\ell_d$$

and

$$-ad^m \leq \log \|s_{\sigma}(p)\|_{\sigma} \leq -bd^m.$$

Then the field of definition  $K(p)$  of the complex point  $p$  in  $\mathcal{X}_K$  satisfies

$$\text{trdeg}_{\mathbb{Q}} K(p) \geq m - 1.$$

**Remark A.2.** For any integer  $k \geq 1$ , the conditions in the above statement are verified for the Hermitian line bundle  $\bar{L}$  if and only if similar conditions hold for the tensor power  $\bar{L}^{\otimes k}$  of  $\bar{L}$  (up to multiplying  $\ell_d$ ,  $a$ , and  $b$  by suitable constants).

Moreover, since  $\mathcal{X}$  is proper over  $\text{Spec } \mathcal{O}_K$ , it is easy to see that if the above statement is true for a particular choice of Hermitian structure on  $L$ , then it also holds for any other Hermitian structure on  $L$ .

In what follows, we explain how to deduce the above statement from Philippon’s original result concerning  $\mathcal{X} = \mathbb{P}_{\mathcal{O}_K}^n$ . The main technical tool is the following “integral lifting lemma”.

**Lemma A.3.** *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be projective arithmetic schemes over  $\mathcal{O}_K$ ,  $\bar{L}$  be a Hermitian line bundle over  $\mathcal{X}$ , with  $L$  relatively ample over  $\text{Spec } \mathcal{O}_K$ , and  $\mathcal{Y} \rightarrow \mathcal{X}$  be a closed immersion over  $\text{Spec } \mathcal{O}_K$ . Endow  $L|_{\mathcal{Y}}$  with the induced Hermitian structure. Then, there exists a real number  $C > 0$  such that, for every sufficiently large positive integer  $d$ , any section  $s \in \Gamma(\mathcal{Y}, L|_{\mathcal{Y}}^{\otimes d})$  can be lifted to a section  $\tilde{s} \in \Gamma(\mathcal{X}, L^{\otimes d})$  satisfying*

$$\|\tilde{s}\|_{\mathcal{X}} \leq C^d \|s\|_{\mathcal{Y}}.$$

This type of result is well known in Arakelov geometry and goes back to Zhang’s work [1992] on arithmetic ampleness. For lack of reference, we sketch a proof.

*Sketch of the proof.* If  $R$  is a ring, we set  $\Gamma(\mathcal{X}, L^{\otimes d})_R := \Gamma(\mathcal{X}, L^{\otimes d}) \otimes_{\mathbb{Z}} R$ .

Let  $d$  be large enough so that the restriction map  $\rho_d : \Gamma(\mathcal{X}, L^{\otimes d}) \rightarrow \Gamma(\mathcal{Y}, L|_{\mathcal{Y}}^{\otimes d})$  is surjective. Since  $\bigoplus_{\sigma} \Gamma(\mathcal{X}_{\sigma}, L_{\sigma}^{\otimes d}) \cong \Gamma(\mathcal{X}, L^{\otimes d})_{\mathbb{C}}$ , we may apply [Bost 2004, Proposition 3.5] to obtain a constant  $C_0 > 0$ ,

not depending on  $d$  or  $s$ , and an element  $t \in \Gamma(\mathcal{X}, L^{\otimes d})_{\mathbb{R}} \subset \Gamma(\mathcal{X}, L^{\otimes d})_{\mathbb{C}}$  lifting  $s$  (i.e.,  $(\rho_d \otimes 1)(t) = s \otimes 1$ ) and satisfying

$$\|t\|_{\mathcal{X}} \leq C_0^d \|s\|_{\mathcal{Y}}.$$

The idea now is to define  $\tilde{s}$  as the element of  $\rho_d^{-1}(s)$  minimizing the distance  $\|t - \tilde{s}\|_{\mathcal{X}}$  in  $\Gamma(\mathcal{X}, L^{\otimes d})_{\mathbb{R}}$ .

To finish the proof, we must show that the diameter of the fundamental domain of the lattice  $\Gamma(\mathcal{X}, \mathcal{I}_{\mathcal{Y}} \otimes L^{\otimes d}) = \ker \rho_d$  in  $\Gamma(\mathcal{X}, \mathcal{I}_{\mathcal{Y}} \otimes L^{\otimes d})_{\mathbb{R}}$  grows at most exponentially in  $d$ . We mimic the argument in the proof of [Charles 2017, Proposition 2.5]. Since  $L$  is ample, there exists an integer  $n \geq 1$  such that, for any sufficiently large integer  $r$ , and any positive integer  $q$ , the morphism

$$\Gamma(\mathcal{X}, \mathcal{I}_{\mathcal{Y}} \otimes L^{\otimes r}) \otimes_{\mathbb{Z}} \Gamma(\mathcal{X}, L^{\otimes n})^{\otimes q} \rightarrow \Gamma(\mathcal{X}, \mathcal{I}_{\mathcal{Y}} \otimes L^{\otimes nq+r})$$

is surjective. Choose sufficiently large integers  $r_1, \dots, r_n$  forming a complete residue system modulo  $n$ . Fixing bases of the finite free  $\mathbb{Z}$ -modules  $\Gamma(\mathcal{X}, L^{\otimes n}), \Gamma(\mathcal{X}, \mathcal{I}_{\mathcal{Y}} \otimes L^{\otimes r_1}), \dots, \Gamma(\mathcal{X}, \mathcal{I}_{\mathcal{Y}} \otimes L^{\otimes r_n})$ , we see that there exists a constant  $B > 1$  such that any  $\Gamma(\mathcal{X}, \mathcal{I}_{\mathcal{Y}} \otimes L^{\otimes nq+r_i})$  admits a full-rank submodule having a basis whose elements have norm bounded by  $B^q$ . By [Zhang 1992, Lemma 1.7], the  $\mathbb{Z}$ -module  $\Gamma(\mathcal{X}, \mathcal{I}_{\mathcal{Y}} \otimes L^{\otimes nq+r_i})$  admits a basis whose elements have norm bounded by  $r B^q$ , where  $r$  denotes the rank of  $\Gamma(\mathcal{X}, \mathcal{I}_{\mathcal{Y}} \otimes L^{\otimes nq+r_i})$ . Since  $r$  grows polynomially in  $q$ , and  $r_1, \dots, r_n$  form a complete residue system modulo  $n$ , we conclude that there exists a constant  $\kappa > 0$  such that, for any sufficiently large integer  $d$ , the  $\mathbb{Z}$ -module  $\Gamma(\mathcal{X}, \mathcal{I}_{\mathcal{Y}} \otimes L^{\otimes d})$  admits a basis consisting of elements with norm bounded by  $\kappa^d$ .  $\square$

To handle the case  $\mathcal{X} = \mathbb{P}_{\mathcal{O}_K}^n$ , we compare the height  $\bar{h}(P)$  of a homogeneous polynomial  $P \in \mathcal{O}_K[X_0, \dots, X_n]$  of degree  $d$  used in [Philippon 1986] with the Fubini–Study norm  $\|s\|_{\mathbb{P}_{\mathcal{O}_K}^n}$  of the corresponding section  $s \in \Gamma(\mathbb{P}_{\mathcal{O}_K}^n, \mathcal{O}(d))$ . By definition,

$$\bar{h}(P) = \frac{1}{[K : \mathbb{Q}]} \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log^+ M_{\sigma}(P),$$

where, for any field embedding  $\sigma : K \hookrightarrow \mathbb{C}$ , we set

$$M_{\sigma}(P) := \exp\left(\frac{1}{(2\pi)^{n+1}} \int_0^{2\pi} \dots \int_0^{2\pi} \log |P^{\sigma}(e^{i\theta_0}, \dots, e^{i\theta_n})| d\theta_0 \dots d\theta_n\right).$$

**Lemma A.4.** *Let  $(\|\cdot\|_{\sigma})_{\sigma: K \hookrightarrow \mathbb{C}}$  denote the Fubini–Study Hermitian structure on the line bundle  $\mathcal{O}(1)$  over the arithmetic scheme  $\mathbb{P}_{\mathcal{O}_K}^n$ . For any integer  $d \geq 1$ , and any section  $s \in \Gamma(\mathbb{P}_{\mathcal{O}_K}^n, \mathcal{O}(d))$ , if  $P \in \mathcal{O}_K[X_0, \dots, X_n]$  denotes the homogeneous polynomial of degree  $d$  corresponding to  $s$ , then*

$$\bar{h}(P) \leq \log^+ \|s\|_{\mathbb{P}_{\mathcal{O}_K}^n} + \frac{(n+1)}{2}d.$$

*Proof.* For any field embedding  $\sigma : K \hookrightarrow \mathbb{C}$ , and any  $(\theta_0, \dots, \theta_n) \in [0, 2\pi]^{n+1}$ , we have

$$\|s_{\sigma}(e^{i\theta_0} : \dots : e^{i\theta_n})\|_{\sigma} = \frac{|P^{\sigma}(e^{i\theta_0}, \dots, e^{i\theta_n})|}{(n+1)^{d/2}},$$

so that

$$\log M_\sigma(P) \leq \log \|s\|_{\sigma, L^\infty(\mathbb{P}^n(\mathbb{C}))} + \frac{(n+1)}{2}d \leq \log \|s\|_{\mathbb{P}^n_{\mathcal{O}_K}} + \frac{(n+1)}{2}d.$$

Clearly, a similar inequality holds with  $\log^+$  in place of  $\log$ . The result follows by taking the arithmetic mean over all  $\sigma : K \hookrightarrow \mathbb{C}$ . □

*Proof of Theorem A.1.* The case where  $\mathcal{X} = \mathbb{P}^n_{\mathcal{O}_K}$  and  $\bar{L}$  is given by  $\mathcal{O}(1)$  endowed with the Fubini–Study metric follows from Lemma A.4 and [Philippson 1986, Théorème 2.11]; see also [Nesterenko 1996, Lemma 2.5].

The general case follows from this one by considering a closed immersion  $i : \mathcal{X} \rightarrow \mathbb{P}^n_{\mathcal{O}_K}$  over  $\mathcal{O}_K$  satisfying  $i^*\mathcal{O}(1) = L^{\otimes k}$  for some  $k \geq 1$ , and by applying Lemma A.3 and Remark A.2. □

### Appendix B: *D*-property and ZL-density in quasiprojective varieties

Let  $k$  be a field,  $X$  be a smooth quasiprojective variety over  $k$ , and  $\mathcal{F}$  be an  $\mathcal{O}_X$ -submodule of rank 1 of the tangent bundle  $T_{X/k}$  such that the quotient  $T_{X/k}/\mathcal{F}$  is torsion-free, i.e., a one-dimensional (possibly singular) foliation on  $X$ .

Let  $p \in X(k)$  be a  $k$ -point of  $X$ . We say that a formal curve  $\hat{\varphi} : \text{Spf } k[[q]] \rightarrow X$  is an *integral curve* of  $\mathcal{F}$  at  $p$  if  $\hat{\varphi}(0) = p$  and if the image of the tangent map

$$D\hat{\varphi} : T_{\text{Spf } k[[q]]/k} \rightarrow \hat{\varphi}^*T_{X/k}$$

factors through the subbundle  $\hat{\varphi}^*\mathcal{F}$  of  $\hat{\varphi}^*T_{X/k}$ . Moreover, if  $\mathcal{F}(p) := \Gamma(\text{Spec } k, p^*\mathcal{F})$  denotes the fiber of  $\mathcal{F}$  at  $p$ , we say that  $\hat{\varphi}$  is *smooth* if  $\hat{\varphi}'(0) := D_0\hat{\varphi}(d/dq) \in \mathcal{F}(p)$  is nonzero.

From now on, we assume that  $k$  has *characteristic 0*. By a formal version of the Frobenius theorem, for every  $p \in X(k)$  such that  $\mathcal{F}(p) \neq 0$ , there exists a unique smooth integral curve  $\hat{\varphi}$  of  $\mathcal{F}$  at  $p$ , up to composition by an automorphism of  $\text{Spf } k[[q]]$ .

We say that a closed subscheme  $Y$  of  $X$  is  *$\mathcal{F}$ -invariant* if the ideal of  $Y$  in  $\mathcal{O}_X$  is stable under the derivations of  $\mathcal{F} \subset T_{X/k} = \text{Der}_k(\mathcal{O}_X)$ .

**Definition B.1.** Let  $X$  be a smooth quasiprojective variety over the field  $k$ , let  $\mathcal{F}$  be a one-dimensional foliation on  $X$ , and let  $\hat{\varphi} : \text{Spf } k[[q]] \rightarrow X$  be a formal integral curve of  $\mathcal{F}$ . We say that  $\hat{\varphi}$  satisfies the  *$D$ -property* for  $\mathcal{F}$  if there exists a constant  $C > 0$  such that, for every  $\mathcal{F}$ -invariant closed subvariety  $Y$  of  $X$ , there exists a Cartier divisor  $D$  whose support contains  $Y$  satisfying

$$\text{mult}_{\hat{\varphi}} D \leq C.$$

Observe that, if  $\hat{\varphi}$  satisfies the  *$D$ -property*, then its image is Zariski-dense in  $X$ . Indeed, the Zariski-closure of the image of an integral curve of  $\mathcal{F}$  is  $\mathcal{F}$ -invariant.

**Theorem B.2** (Nesterenko–Binyamini). *Let  $X$  be a smooth quasiprojective variety over an algebraically closed field  $k$  of characteristic 0,  $v \in \Gamma(X, T_{X/k}) \setminus \{0\}$  be a vector field on  $X$ , and  $\hat{\varphi} : \text{Spf } k[[q]] \rightarrow X$  be a*

smooth formal curve satisfying the differential equation

$$q \frac{d\hat{\varphi}}{dq} = v \circ \hat{\varphi}.$$

If  $\hat{\varphi}$  satisfies the  $D$ -property for the foliation generated by  $v$ , then  $\hat{\varphi}$  is ZL-dense in  $X$ .

Note that  $\hat{\varphi}(0)$  is a singular point of  $v$ . In the nonsingular case, i.e.,  $\hat{\varphi}$  satisfies the differential equation  $d\hat{\varphi}/dq = v \circ \hat{\varphi}$ , stronger statements are true (see [Binyamini 2014, Theorem 2]), but an analog of the above result may be also obtained by virtually the same proof.

Binyamini’s original result [2014, Corollary 3] builds on ideas of Nesterenko and concerns the case of an analytic integral curve of a polynomial vector field on some affine space over  $\mathbb{C}$ . In what follows, we merely indicate how a slight modification of the geometric methods of Binyamini may be used to prove Theorem B.2 above.

We start by recasting the  $D$ -property into a more workable form.

**Proposition B.3.** *Let  $\bar{X}$  be any projective compactification of  $X$  and  $L$  be an ample line bundle on  $\bar{X}$ . Then, a formal curve  $\hat{\varphi} : \text{Spf } k[[q]] \rightarrow X$  satisfies the  $D$ -property for a one-dimensional foliation  $\mathcal{F}$  on  $X$  if and only if there exists a constant  $C > 0$  such that, for every  $\mathcal{F}$ -invariant closed subvariety  $Y$  of  $X$ , there exists an integer  $d \geq 1$ , and a global section  $s \in \Gamma(\bar{X}, L^{\otimes d})$  vanishing identically on  $Y$  such that  $\text{ord}_0 \hat{\varphi}^* s \leq C$ .*

*Proof.* The sufficiency is clear: consider the divisors  $\text{div}(s)$ .

Conversely, suppose that  $\hat{\varphi}$  satisfies the  $D$ -property for  $\mathcal{F}$  with constant  $C > 0$  and let  $Y$  be an  $\mathcal{F}$ -invariant closed subvariety of  $X$ . Since  $L$  is ample, we may assume that  $Y$  contains  $\hat{\varphi}(0)$ ; otherwise, there exists an integer  $d \geq 1$  and a section  $s \in \Gamma(\bar{X}, L^{\otimes d})$  vanishing on  $Y$  such that  $s(\hat{\varphi}(0)) \neq 0$ , so that  $\text{ord}_0 \hat{\varphi}^* s = 0 \leq C$ .

Let  $D$  be a divisor whose support contains  $Y$  such that  $\text{mult}_{\hat{\varphi}} D \leq C$ , and let  $f$  be a local equation for  $D$  on some open neighborhood  $U$  of  $\hat{\varphi}(0)$ . Since  $L$  is ample, there exists an integer  $m \geq 1$  and a section  $s_0 \in \Gamma(\bar{X}, L^{\otimes m})$  such that  $\hat{\varphi}(0) \in \bar{X}_{s_0}$  and  $\bar{X}_{s_0} \subset U$ . Now, there exists an integer  $n \geq 1$ , and a global section  $s \in \Gamma(\bar{X}, L^{\otimes mn})$  such that  $s = f s_0^{\otimes mn}$  over  $\bar{X}_{s_0}$ . It is clear that  $s$  vanishes identically on  $Y$  and satisfies  $\text{ord}_0 \hat{\varphi}^* s = \text{ord}_0 \hat{\varphi}^* f = \text{mult}_{\hat{\varphi}} D \leq C$ . □

Consider the hypotheses and notation of Theorem B.2. Fix a projective compactification  $\bar{X}$  of  $X$ , and an ample line bundle  $L$  on  $\bar{X}$  endowed with a global section  $s_0 \in \Gamma(\bar{X}, L)$  satisfying  $\bar{X}_{s_0} \subset X$ . Recall from Section 6 that  $v$  defines a  $k$ -derivation  $\partial_v$  on the ring  $\bigoplus_{d \geq 0} \Gamma(\bar{X}, L^{\otimes d})$ .

Let  $p = \hat{\varphi}(0)$ . By a formal cycle of  $X$  at  $p$ , we mean a cycle in the scheme  $\text{Spec } \hat{\mathcal{O}}_{X,p}$ , where  $\hat{\mathcal{O}}_{X,p}$  denotes the completion of the local ring  $\mathcal{O}_{X,p}$  with respect to its maximal ideal. Note that every (global) cycle of  $X$  induces, by localization and formal completion, a formal cycle of  $X$  at  $p$ .

Let  $Y$  be a prime formal cycle of  $X$  at  $p$  corresponding to the prime ideal  $\mathfrak{p}$  of  $\hat{\mathcal{O}}_{X,p}$  and denote by  $I_{\hat{\varphi}}$  the ideal of  $\text{im } \hat{\varphi}$  in  $\hat{\mathcal{O}}_{X,p}$ . Assume that  $I_{\hat{\varphi}}$  does not contain  $\mathfrak{p}$  (i.e.,  $Y$  does not contain the image of  $\hat{\varphi}$ ). Since  $\hat{\varphi} : \text{Spf } k[[q]] \rightarrow X$  is smooth, the image of  $I_{\hat{\varphi}}$  in the local ring  $\hat{\mathcal{O}}_{X,p}/\mathfrak{p}$  contains some power of the

maximal ideal. We may thus consider the *Samuel multiplicity*

$$\text{mult}_{\hat{\varphi}} Y := e_{I_{\hat{\varphi}}/I_{\hat{\varphi}} \cap \mathfrak{p}}(\hat{\mathcal{O}}_{X,p}/\mathfrak{p}).$$

By additivity, we may extend this definition to every formal cycle of  $X$  at  $p$  whose components do not contain the image of  $\hat{\varphi}$ . By abuse of notation, if  $Z$  is a (global) cycle of  $X$ , we denote by  $\text{mult}_{\hat{\varphi}} Z$  the multiplicity  $\text{mult}_{\hat{\varphi}} \hat{Z}$  of its completion at  $p$ .

**Proposition B.4.** *The multiplicity function constructed above satisfies the following properties:*

- (1) If  $Z = \text{div}(f)$ , for some  $f \in \hat{\mathcal{O}}_{X,p}$ , then  $\text{mult}_{\hat{\varphi}}(Z) = \text{ord}_0 \hat{\varphi}^* f$ .
- (2) If  $Z = p$ , then  $\text{mult}_{\hat{\varphi}}(Z) = 1$ .
- (3) For any closed subvariety  $Y$  of  $X$ , any integer  $d \geq 1$ , and any  $s \in \Gamma(\bar{X}, L^{\otimes d}) \setminus \{0\}$  vanishing identically on  $Y$ , we have  $\text{mult}_{\hat{\varphi}}(Y) \leq \text{ord}_0 \hat{\varphi}^* s \cdot \text{mult}_p(Y)$ .
- (4) For any closed subvariety  $Y$  of  $X$ , any integer  $d \geq 1$ , and any  $s \in \Gamma(\bar{X}, L^{\otimes d}) \setminus \{0\}$  vanishing identically on  $Y$  for which  $\partial_v s$  does not vanish identically on  $Y$ , we have  $\text{mult}_{\hat{\varphi}}(Y) \leq \text{mult}_{\hat{\varphi}}(Y \cdot \text{div}(\partial_v s))$ .
- (5) There is an integer  $n_0 \geq 0$  such that, for every closed subvariety  $Y$  of  $X$  not contained in a  $v$ -invariant subvariety of  $X$ , if  $d \geq 1$  is the smallest integer for which there is  $s \in \Gamma(\bar{X}, L^{\otimes d}) \setminus \{0\}$  vanishing identically on  $Y$ , then  $\min\{n \mid \partial_v^n s \text{ does not vanish identically on } Y\} \leq n_0$ .

Properties (1) and (2) are easy. For properties (3) and (4), see [Binyamini 2014, Lemma 8 and Proposition 9]. Finally, property (5) follows by an adaptation of the arguments in [loc. cit., Section 3].

Once this is established, the proof Theorem B.2 becomes completely analogous to the proof of [loc. cit., Theorem 3].

### Acknowledgments

This work was supported by a public grant as part of the FMJH project, and is part of my Ph.D. thesis at Université Paris-Sud, Orsay, under the supervision of Jean-Benoît Bost. I was also partially supported by the project Foliage of Agence Nationale de la Recherche, under agreement ANR-16-CE40-0008-01.

I thank Jean-Benoît Bost and Hugues Randriam for allowing me to use their preliminary unpublished notes on moderate growth as a starting point for this paper. I am also grateful to Dinh Tuan Huynh for a fruitful discussion on Nevanlinna theory and to Daniel Bertrand for pointing out a missing reference. Finally, I thank the Max-Planck-Institut für Mathematik, Bonn, for the excellent working conditions in which this paper was finished.

### References

- [Barré-Sirieix et al. 1996] K. Barré-Sirieix, G. Diaz, F. Gramain, and G. Philibert, “Une preuve de la conjecture de Mahler–Manin”, *Invent. Math.* **124**:1-3 (1996), 1–9. [MR](#) [Zbl](#)
- [Binyamini 2014] G. Binyamini, “Multiplicity estimates, analytic cycles, and Newton polytopes”, preprint, 2014. [arXiv](#)
- [Bost 1996] J.-B. Bost, “Périodes et isogenies des variétés abéliennes sur les corps de nombres”, exposé 795, pp. 115–161 in *Séminaire Bourbaki*, 1994/95, Astérisque **237**, Soc. Mat. de France, Paris, 1996. [MR](#) [Zbl](#)

- [Bost 2001] J.-B. Bost, “Algebraic leaves of algebraic foliations over number fields”, *Publ. Math. Inst. Hautes Études Sci.* **93**:1 (2001), 161–221. [MR](#) [Zbl](#)
- [Bost 2004] J.-B. Bost, “Germs of analytic varieties in algebraic varieties: canonical metrics and arithmetic algebraization theorems”, pp. 371–418 in *Geometric aspects of Dwork theory, I*, edited by A. Adolphson et al., Walter de Gruyter, Berlin, 2004. [MR](#) [Zbl](#)
- [Bost 2006] J.-B. Bost, “Evaluation maps, slopes, and algebraicity criteria”, pp. 537–562 in *International Congress of Mathematicians, II* (Madrid, 2006), edited by M. Sanz-Solé et al., Eur. Math. Soc., Zürich, 2006. [MR](#) [Zbl](#)
- [Bost 2015] J.-B. Bost, “Theta invariants of euclidean lattices and infinite-dimensional hermitian vector bundles over arithmetic curves”, preprint, 2015. [arXiv](#)
- [Bost and Künnemann 2010] J.-B. Bost and K. Künnemann, “Hermitian vector bundles and extension groups on arithmetic schemes, I: Geometry of numbers”, *Adv. Math.* **223**:3 (2010), 987–1106. [MR](#) [Zbl](#)
- [Candelas et al. 1991a] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, “An exactly soluble superconformal theory from a mirror pair of Calabi–Yau manifolds”, *Phys. Lett. B* **258**:1-2 (1991), 118–126. [MR](#)
- [Candelas et al. 1991b] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, “A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory”, *Nuclear Phys. B* **359**:1 (1991), 21–74.
- [Charles 2017] F. Charles, “Arithmetic ampleness and an arithmetic Bertini theorem”, preprint, 2017, available at [https://www.math.u-psud.fr/~fcharles/arithmetic\\_bertini.pdf](https://www.math.u-psud.fr/~fcharles/arithmetic_bertini.pdf).
- [Chudnovsky 1980] G. V. Chudnovsky, “Algebraic independence of values of exponential and elliptic functions”, pp. 339–350 in *Proceedings of the International Congress of Mathematicians* (Helsinki, 1978), edited by O. Lehto, Acad. Sci. Fennica, Helsinki, 1980. [MR](#) [Zbl](#)
- [Delaygue et al. 2017] E. Delaygue, T. Rivoal, and J. Roques, *On Dwork’s  $p$ -adic formal congruences theorem and hypergeometric mirror maps*, Mem. Amer. Math. Soc. **1163**, Amer. Math. Soc., Providence, RI, 2017. [MR](#) [Zbl](#)
- [EGA II 1961] A. Grothendieck, *Eléments de géométrie algébrique, II: Étude globale élémentaire de quelques classes de morphismes*, vol. 8, 1961. [MR](#) [Zbl](#)
- [Fonseca 2018] T. J. Fonseca, “Higher Ramanujan equations and periods of abelian varieties”, preprint, 2018. [arXiv](#)
- [Fornæss and Sibony 2008] J. E. Fornæss and N. Sibony, “Riemann surface laminations with singularities”, *J. Geom. Anal.* **18**:2 (2008), 400–442. [MR](#) [Zbl](#)
- [Fulton 1984] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **2**, Springer, 1984. [MR](#) [Zbl](#)
- [Gasbarri 2010] C. Gasbarri, “Analytic subvarieties with many rational points”, *Math. Ann.* **346**:1 (2010), 199–243. [MR](#) [Zbl](#)
- [Gasbarri 2013] C. Gasbarri, “Horizontal sections of connections on curves and transcendence”, *Acta Arith.* **158**:2 (2013), 99–128. [MR](#) [Zbl](#)
- [Graftieaux 2005] P. Graftieaux, “Théorème stéphanois et méthode des pentes”, pp. 179–213 in *Formes modulaires et transcendance* (Marseille, 2003), edited by S. Fischler et al., Sémin. Congr. **12**, Soc. Math. France, Paris, 2005. [MR](#) [Zbl](#)
- [Herblot 2012] M. Herblot, “Algebraic points on meromorphic curves”, preprint, 2012. [arXiv](#)
- [Kontsevich et al. 2006] M. Kontsevich, A. Schwarz, and V. Vologodsky, “Integrality of instanton numbers and  $p$ -adic B-model”, *Phys. Lett. B* **637**:1-2 (2006), 97–101. [MR](#) [Zbl](#)
- [Krattenthaler and Rivoal 2010] C. Krattenthaler and T. Rivoal, “On the integrality of the Taylor coefficients of mirror maps”, *Duke Math. J.* **151**:2 (2010), 175–218. [MR](#) [Zbl](#)
- [Krattenthaler and Rivoal 2012] C. Krattenthaler and T. Rivoal, “Analytic properties of mirror maps”, *J. Aust. Math. Soc.* **92**:2 (2012), 195–235. [MR](#) [Zbl](#)
- [Lang 1966] S. Lang, *Introduction to transcendental numbers*, Addison-Wesley, Reading, MA, 1966. [MR](#) [Zbl](#)
- [Lian and Yau 1998] B. H. Lian and S.-T. Yau, “Integrality of certain exponential series”, pp. 215–227 in *Algebra and geometry* (Taipei, 1995), edited by M.-c. Kang, Lect. Algebra Geom. **2**, Int. Press, Cambridge, MA, 1998. [MR](#) [Zbl](#)
- [Morrison 1993] D. R. Morrison, “Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians”, *J. Amer. Math. Soc.* **6**:1 (1993), 223–247. [MR](#) [Zbl](#)

- [Nesterenko 1996] Y. V. Nesterenko, “Modular functions and transcendence questions”, *Mat. Sb.* **187**:9 (1996), 65–96. In Russian; translated in *Sb. Math.* **187**:9 (1996), 1319–1348. [MR](#) [Zbl](#)
- [Nesterenko 2001] Y. V. Nesterenko, “Algebraic independence for values of Ramanujan functions”, pp. 27–46 in *Introduction to algebraic independence theory*, edited by Y. V. Nesterenko and P. Philippon, Lecture Notes in Math. **1752**, Springer, 2001. [MR](#) [Zbl](#)
- [Noguchi and Winkelmann 2014] J. Noguchi and J. Winkelmann, *Nevanlinna theory in several complex variables and Diophantine approximation*, Grundlehren der Mathematischen Wissenschaften **350**, Springer, 2014. [MR](#) [Zbl](#)
- [Petersson 1932] H. Petersson, “Über die Entwicklungskoeffizienten der automorphen Formen”, *Acta Math.* **58**:1 (1932), 169–215. [MR](#) [Zbl](#)
- [Philippon 1986] P. Philippon, “Critères pour l’indépendance algébrique”, *Inst. Hautes Études Sci. Publ. Math.* **64** (1986), 5–52. [MR](#) [Zbl](#)
- [Philippon 1998] P. Philippon, “Indépendance algébrique et  $K$ -fonctions”, *J. Reine Angew. Math.* **497** (1998), 1–15. [MR](#) [Zbl](#)
- [Rademacher 1938] H. Rademacher, “The Fourier coefficients of the modular invariant  $J(\tau)$ ”, *Amer. J. Math.* **60**:2 (1938), 501–512. [MR](#) [Zbl](#)
- [Tsuji 1959] M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959. [MR](#) [Zbl](#)
- [Zhang 1992] S. Zhang, “Positive line bundles on arithmetic surfaces”, *Ann. of Math. (2)* **136**:3 (1992), 569–587. [MR](#) [Zbl](#)
- [Zudilin 2000] W. Zudilin, “Number theory casting a look at the mirror”, preprint, 2000. [arXiv](#)
- [Zudilin 2002] V. V. Zudilin, “On the integrality of power expansions related to hypergeometric series”, *Mat. Zametki* **71**:5 (2002), 662–676. In Russian; translated in *Math. Notes* **71**:5 (2002), 604–646. [MR](#) [Zbl](#)
- [Zudilin 2003] W. Zudilin, “The hypergeometric equation and Ramanujan functions”, *Ramanujan J.* **7**:4 (2003), 435–447. [MR](#) [Zbl](#)

Communicated by Jonathan Pila

Received 2018-05-17      Accepted 2018-12-25

[tiago.jardimdafonseca@maths.ox.ac.uk](mailto:tiago.jardimdafonseca@maths.ox.ac.uk)      *Mathematical Institute, University of Oxford, Oxford, United Kingdom*

# Algebra & Number Theory

[msp.org/ant](http://msp.org/ant)

## EDITORS

### MANAGING EDITOR

Bjorn Poonen  
Massachusetts Institute of Technology  
Cambridge, USA

### EDITORIAL BOARD CHAIR

David Eisenbud  
University of California  
Berkeley, USA

### BOARD OF EDITORS

Richard E. Borcherds	University of California, Berkeley, USA	Raman Parimala	Emory University, USA
Antoine Chambert-Loir	Université Paris-Diderot, France	Jonathan Pila	University of Oxford, UK
J-L. Colliot-Thélène	CNRS, Université Paris-Sud, France	Anand Pillay	University of Notre Dame, USA
Brian D. Conrad	Stanford University, USA	Michael Rapoport	Universität Bonn, Germany
Samit Dasgupta	University of California, Santa Cruz, USA	Victor Reiner	University of Minnesota, USA
Hélène Esnault	Freie Universität Berlin, Germany	Peter Sarnak	Princeton University, USA
Gavril Farkas	Humboldt Universität zu Berlin, Germany	Joseph H. Silverman	Brown University, USA
Hubert Flenner	Ruhr-Universität, Germany	Michael Singer	North Carolina State University, USA
Sergey Fomin	University of Michigan, USA	Christopher Skinner	Princeton University, USA
Edward Frenkel	University of California, Berkeley, USA	Vasudevan Srinivas	Tata Inst. of Fund. Research, India
Andrew Granville	Université de Montréal, Canada	J. Toby Stafford	University of Michigan, USA
Joseph Gubeladze	San Francisco State University, USA	Pham Huu Tiep	University of Arizona, USA
Roger Heath-Brown	Oxford University, UK	Ravi Vakil	Stanford University, USA
Craig Huneke	University of Virginia, USA	Michel van den Bergh	Hasselt University, Belgium
Kiran S. Kedlaya	Univ. of California, San Diego, USA	Akshay Venkatesh	Institute for Advanced Study, USA
János Kollár	Princeton University, USA	Marie-France Vignéras	Université Paris VII, France
Philippe Michel	École Polytechnique Fédérale de Lausanne	Kei-Ichi Watanabe	Nihon University, Japan
Susan Montgomery	University of Southern California, USA	Melanie Matchett Wood	University of Wisconsin, Madison, USA
Shigefumi Mori	RIMS, Kyoto University, Japan	Shou-Wu Zhang	Princeton University, USA
Martin Olsson	University of California, Berkeley, USA		

## PRODUCTION

[production@msp.org](mailto:production@msp.org)

Silvio Levy, Scientific Editor

---

See inside back cover or [msp.org/ant](http://msp.org/ant) for submission instructions.

---

The subscription price for 2019 is US \$385/year for the electronic version, and \$590/year (+\$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

---

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

---

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2019 Mathematical Sciences Publishers

# Algebra & Number Theory

Volume 13    No. 3    2019

---

<a href="#">Fundamental gerbes</a>	531
NIELS BORNE and ANGELO VISTOLI	
<a href="#">A new proof of the Waldspurger formula, I</a>	577
RAHUL KRISHNA	
<a href="#">Algebraic independence for values of integral curves</a>	643
TIAGO J. FONSECA	
<a href="#">Generically split octonion algebras and <math>\mathbb{A}^1</math>-homotopy theory</a>	695
ARAVIND ASOK, MARC HOYOIS and MATTHIAS WENDT	