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Let F be a real quadratic field in which a fixed prime p is inert, and E_0 be an imaginary quadratic field in which p splits; put $E = E_0F$. Let X be the fiber over $\mathbb{F}_{p^2}$ of the Shimura variety for $G(U(1, n-1) \times U(n-1, 1))$ with hyperspecial level structure at p for some integer $n \geq 2$. We show that under some genericity conditions the middle-dimensional Tate classes of X are generated by the irreducible components of its supersingular locus. We also discuss a general conjecture regarding special cycles on the special fibers of unitary Shimura varieties, and on their relation to Newton stratification.

1. Introduction

The study of the geometry of Shimura varieties lies at the heart of the Langlands program. Arithmetic information of Shimura varieties builds a bridge relating the world of automorphic representations and the world of Galois representations.

One of the interesting topics in this area is to understand the supersingular locus of the special fibers of Shimura varieties, or more generally, any interesting stratifications (e.g., Newton or Ekedahl–Oort stratification) of the special fibers of Shimura varieties. The case of unitary Shimura varieties has been extensively

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studied. Vollaard and Wedhorn [2011] showed that the supersingular locus of the special fiber of the $GU(1, n-1)$-Shimura variety at an inert prime is a union of Deligne–Lusztig varieties. Further, Howard and Pappas [2014] studied the case of $GU(2, 2)$ at an inert prime, and Rapoport, Terstiege and Wilson proved similar results for $GU(n-1, 1)$ at a ramified prime. Finally, we remark that Görtz and He [2015] studied the basic loci in a slightly more general class of Shimura varieties.

In all the work mentioned above, the authors use the uniformization theorem of Rapoport–Zink to reduce the problem to the study of certain Rapoport–Zink spaces. In this paper, we take a different approach. Instead of using the uniformization theorem, we study the basic locus (or more generally other Newton strata) of certain unitary Shimura varieties by considering correspondences between unitary Shimura varieties of different signatures. This method was introduced by the first author in [Helm 2010; 2012], and applied successfully to quaternionic Shimura varieties by the second and the third authors [Tian and Xiao 2016].

Another new aspect of this work is that we study not only the global geometry of the supersingular locus, but also their relationship with the Tate conjecture for Shimura varieties over finite fields. We show that the basic locus contributes to all "generic" middle-dimensional Tate cycles of the special fiber of the Shimura variety. Similar results have been obtained by the second and the third authors for even-dimensional Hilbert modular varieties at an inert prime [Tian and Xiao 2014]. We believe that, this phenomenon is a general philosophy which holds for more general Shimura varieties. Our slogan is: *irreducible components of the basic locus of a Shimura variety should generate all Tate classes under some genericity condition on the automorphic representations.*

We explain in more detail the main results of this paper. Let F be a real quadratic field, E_0 be an imaginary quadratic field, and $E = E_0F$. Let p be a prime number inert in F, and split in E_0. Let $\mathfrak{p}$, $\bar{\mathfrak{p}}$ denote the two places of E above p so that $E_{\mathfrak{p}}$ and $E_{\bar{\mathfrak{p}}}$ are both isomorphic to $\mathbb{Q}_{p^2}$, the unique unramified quadratic extension of $\mathbb{Q}_p$. For an integer $n \geq 1$, let G be the similitude unitary group associated to a division algebra over E equipped with an involution of second kind. In the notation of Section 3.6, our G is denoted $G_{1,n-1}$. This is an algebraic group over $\mathbb{Q}$ such that $G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_{\mathfrak{p}})$ and $G(\mathbb{R})$ is the unitary similitude group with signature $(1, n-1)$ and $(n-1, 1)$ at the two archimedean places. (For a precise definition, see Section 2.2.)

Let $\mathbb{A}$ denote the ring of finite adeles of $\mathbb{Q}$, and $\mathbb{A}^\infty$ be its finite part. Fix a sufficiently small open compact subgroup $K \subseteq G(\mathbb{A}^\infty)$ with $K_p = \mathbb{Z}_p^\times \times \mathrm{GL}_n(\mathbb{Z}_{p^2}) \subseteq G(\mathbb{Q}_p)$, where $\mathbb{Z}_{p^2}$ is the ring of integers of $\mathbb{Q}_{p^2}$. Let $\mathcal{S}h(G)_K$ be the Shimura variety associated to G of level K.[1]

[1]Strictly speaking, the moduli space $\mathcal{S}h(G)_K$ is $\#\ker^1(\mathbb{Q}, G)$-copies of the classical Shimura variety whose $\mathbb{C}$-points are given by the double coset space $G(\mathbb{Q}) \setminus G(\mathbb{A})/K_\infty K$, where $K_\infty \subseteq G(\mathbb{R})$ is the maximal compact subgroup modulo center. See [Kottwitz 1992b, page 400] for details.

According to Kottwitz [1992b], when K^p is neat, $\mathcal{S}h(G)_K$ admits a proper and smooth integral model over $\mathbb{Z}_{p^2}$ which parametrizes certain polarized abelian schemes with K-level structure (See Section 2.3). Let $\mathrm{Sh}_{1,n-1}$ denote the special fiber of $\mathcal{S}h(G)_K$ over $\mathbb{F}_{p^2}$. This is a proper smooth variety over $\mathbb{F}_{p^2}$ of dimension $2(n-1)$. Let $\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}$ denote the supersingular locus of $\mathrm{Sh}_{1,n-1}$, i.e., the reduced closed subvariety of $\mathrm{Sh}_{1,n-1}$ that parametrizes supersingular abelian varieties. We will see in Proposition 4.14 that $\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}$ is equidimensional of dimension $n-1$.

Fix a prime $\ell \neq p$. There is a natural action by $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^2}) \times \overline{\mathbb{Q}}_\ell[K\backslash G(\mathbb{A}^\infty)/K]$ on the ℓ-adic étale cohomology group $H^{2(n-1)}_{\mathrm{et}}(\mathrm{Sh}_{1,n-1,\bar{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))$. We will take advantage of the Hecke action to consider a variant of the Tate conjecture for $\mathrm{Sh}_{1,n-1}$.

Fix an irreducible admissible representation π of $G(\mathbb{A}^\infty)$ (with coefficients in $\overline{\mathbb{Q}}_\ell$). The K-invariant subspace of π, denoted by π^K, is a finite-dimensional irreducible representation of the Hecke algebra $\overline{\mathbb{Q}}_\ell[K\backslash G(\mathbb{A}^\infty)/K]$. We denote the π^K-isotypic component of $H^{2(n-1)}_{\mathrm{et}}(\mathrm{Sh}_{1,n-1,\bar{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))$ by $H^{2(n-1)}_{\mathrm{et}}(\mathrm{Sh}_{1,n-1,\bar{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))_\pi$ and put

$$H^{2(n-1)}_{\mathrm{et}}(\mathrm{Sh}_{1,n-1,\bar{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))^{\mathrm{fin}}_\pi := \bigcup_{\mathbb{F}_q/\mathbb{F}_{p^2}} H^{2(n-1)}_{\mathrm{et}}(\mathrm{Sh}_{1,n-1,\bar{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))^{\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_q)}_\pi,$$

where $\mathbb{F}_q$ runs through all finite extensions of $\mathbb{F}_{p^2}$. By projecting to the π^K-isotypic component, we have an ℓ-adic cycle class map:

$$\mathrm{cl}^{n-1}_\pi : A^{n-1}(\mathrm{Sh}_{1,n-1,\bar{\mathbb{F}}_p}) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell \to H^{2(n-1)}_{\mathrm{et}}(\mathrm{Sh}_{1,n-1,\bar{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))^{\mathrm{fin}}_\pi, \qquad (1.0.1)$$

where $A^{n-1}(\mathrm{Sh}_{1,n-1,\bar{\mathbb{F}}_p})$ is the abelian group of codimension $n-1$ algebraic cycles on $\mathrm{Sh}_{1,n-1,\bar{\mathbb{F}}_p}$. Then the Tate conjecture for $\mathrm{Sh}_{1,n-1}$ predicts that the above map is surjective. Our main result confirms exactly this statement under some "genericity" assumptions on π.

From now on, we assume that π satisfies Hypothesis 2.5 to ensure the nontriviality of the π-isotypic component of the cohomology groups. In particular, π is the finite part of an automorphic cuspidal representation of $G(\mathbb{A})$, and $H^{2(n-1)}_{\mathrm{et}}(\mathrm{Sh}_{1,n-1,\bar{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))_\pi \neq 0$. Let π_p denote the p-component of π, which is an unramified principal series as K_p is hyperspecial. Since $G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_\mathfrak{p})$, we write $\pi_p = \pi_{p,0} \otimes \pi_\mathfrak{p}$, where $\pi_{p,0}$ is a character of $\mathbb{Q}_p^\times$ and $\pi_\mathfrak{p}$ is an irreducible admissible representation of $\mathrm{GL}_n(E_\mathfrak{p})$.

Our main theorem is the following.

Theorem 1.1. *Suppose π is the finite part of an automorphic representation of $G(\mathbb{A})$ that admits a cuspidal base change to $\mathrm{GL}_n(\mathbb{A}_E) \times \mathbb{A}^\times_{E_0}$, and the Satake parameters of $\pi_\mathfrak{p}$ are distinct modulo roots of unity. Then $H^{2(n-1)}_{\mathrm{et}}(\mathrm{Sh}_{1,n-1,\bar{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))^{\mathrm{fin}}_\pi$ is generated by the cohomological classes of the irreducible components of the supersingular locus $\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}$. In particular, the cycle class map (1.0.1) is surjective.*

This theorem will be restated in a more precise form in Theorem 4.18. Here, the assumption that the Satake parameters of $\pi_{\mathfrak{p}}$ are distinct modulo roots of unity is crucial for our method. It is closely tied to our geometric description of the irreducible components. This condition will be reformulated in Theorem 4.18 in terms of the Frobenius eigenvalues of certain Galois representation attached to $\pi_{\mathfrak{p}}$ via the unramified local Langlands correspondence. The other automorphic assumption on π is of technical nature. It is imposed here to ensure certain equalities on the automorphic multiplicity on π (See Remark 4.19). The method of our paper may be extended to more general representations π if we have more knowledge of the multiplicity of automorphic forms on unitary groups.

What we will prove is more precise than stated in Theorem 1.1. We need another unitary group $G' = G_{0,n}$ over $\mathbb{Q}$ for E/F as in Lemma 2.9, which is the unique inner form of G such that $G'(\mathbb{A}^\infty) \simeq G(\mathbb{A}^\infty)$ and the signatures of G' at the two archimedean places are $(0, n)$ and $(n, 0)$. Let $\mathrm{Sh}_{0,n}$ denote the (zero-dimensional) Shimura variety over $\mathbb{F}_{p^2}$ associated to G'. We will show in Proposition 4.14 that the supersingular locus $\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}$ is a union of n closed subvarieties Y_j with $1 \leq j \leq n$ such that each of Y_j admits a fibration over $\mathrm{Sh}_{0,n}$ of the same level $K \subseteq G(\mathbb{A}^\infty) \simeq G'(\mathbb{A}^\infty)$ with fibers isomorphic to a certain proper and smooth closed subvariety in a product of Grassmannians. In other words, each Y_j is an algebraic correspondence between $\mathrm{Sh}_{1,n-1}$ and $\mathrm{Sh}_{0,n}$:

$$\mathrm{Sh}_{0,n} \leftarrow Y_j \rightarrow \mathrm{Sh}_{1,n-1}\,.$$

This can be viewed as a geometric realization of the Jacquet–Langlands correspondence between G and G' in the sense of [Helm 2010]. Alternatively, we may view these Y_j as Hecke correspondences between special fibers of unitary Shimura varieties of different signatures. To prove Theorem 1.1, it suffices to show that, when the Satake parameters of $\pi_{\mathfrak{p}}$ are distinct modulo roots of unity, $H^{2(n-1)}_{\mathrm{et}}(\mathrm{Sh}_{1,n-1,\bar{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(n-1))^{\mathrm{fin}}_\pi$ is generated by the cohomology classes of the irreducible components of Y_j. The key point is to show that the π-projection of the intersection matrix of Y_j is nondegenerate under the assumption above on $\pi_{\mathfrak{p}}$.

We briefly describe the structure of this paper. In Section 2, we consider a more general setup of unitary Shimura varieties, and propose a general conjecture, which roughly predicts the existence of certain algebraic correspondences between the special fibers of Shimura varieties with hyperspecial level at p associated to unitary groups with different signatures at infinity (Conjecture 2.12). Theorem 1.1 is a special case of Conjecture 2.12. We believe that our conjecture will provide a new perspective to understand the special fibers of Shimura varieties. In Section 3, we review some Dieudonné theory and Grothendieck–Messing deformation theory that will be frequently used in later sections. Section 4 is devoted to the study of the supersingular locus $\mathrm{Sh}^{\mathrm{ss}}_{1,n-1}$, and constructing the subvarieties Y_j mentioned above. In Section 5, we compute certain intersection numbers on products of Grassmannian varieties. These numbers will play a fundamental role in our later computation of the intersection matrix of the Y_j. In Section 6, we will compute explicitly the intersection matrix of the Y_j (Theorem 6.7), and show that its π-isotypic projection of the intersection matrix is nondegenerate as long as the Satake parameters of $\pi_{\mathfrak{p}}$ are distinct (as opposed to being distinct modulo roots of unity). Then an easy cohomological computation allows us to conclude the proof of our main theorem. In Section 7, we will generalize the construction of the cycles Y_j to the Shimura variety associated to unitary group for E/F of signature $(r, s) \times (s, r)$ at infinity. In this case, we only obtain some partial results on these cycles predicted by Conjecture 2.12: the union of these cycles is exactly the supersingular locus of the unitary Shimura variety in question (Theorem 7.8).

2. The conjecture on special cycles

We will only discuss certain unitary Shimura varieties so that the description becomes explicit. We will discuss after Conjecture 2.12 on how to possibly extend this conjecture to more general Shimura varieties.

2.1. ***Notation.*** We fix a prime number p throughout this paper. We fix an isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$. Let $\mathbb{Q}_p^{\mathrm{ur}}$ be the maximal unramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$.

Let F be a totally real field of degree f in which p is inert. We label all real embeddings of F, or equivalently (via ι_p), all p-adic embeddings of F (into $\mathbb{Q}_p^{\mathrm{ur}}$) by $\tau_1, \dots, \tau_f$ so that post-composition by the Frobenius map takes τ_i to τ_{i+1}. Here the subindices are taken modulo f. Let E_0 be an imaginary quadratic extension of $\mathbb{Q}$ in which p splits. Put $E = E_0F$. Denote by v and $\bar{v}$ the two p-adic places of E_0. Then p splits into two primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ in E, where $\mathfrak{p}$ (resp. $\bar{\mathfrak{p}}$) is the p-adic place above v (resp. $\bar{v}$). Let q_i denote the embedding $E \to E_{\mathfrak{p}} \cong F_p \xrightarrow{\tau_i} \overline{\mathbb{Q}}_p$ and $\bar{q}_i$ the analogous embedding which factors through $E_{\bar{\mathfrak{p}}}$ instead. Composing with ι_p^{-1}, we regard q_i and $\bar{q}_i$ as complex embeddings of E, and we put $\Sigma_{\infty,E} = \{q_1, \dots, q_f, \bar{q}_1, \dots, \bar{q}_f\}$.

2.2. ***Shimura data.*** Let D be a division algebra of dimension n^2 over its center E, equipped with a positive involution $*$ which restricts to the complex conjugation c on E. In particular, $D^{\mathrm{opp}} \cong D \otimes_{E,c} E$. We assume that D splits at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$, and we fix an isomorphism

$$D \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathrm{M}_n(E_{\mathfrak{p}}) \times \mathrm{M}_n(E_{\bar{\mathfrak{p}}}) \cong \mathrm{M}_n(\mathbb{Q}_{p^f}) \times \mathrm{M}_n(\mathbb{Q}_{p^f}),$$

where $*$ switches the two direct factors. We use $\mathfrak{e}$ to denote the element of $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ corresponding to the $(1, 1)$-elementary matrix[2] in the first factor. Let $a_\bullet = (a_i)_{1 \le i \le f}$

[2] By a $(1, 1)$-elementary matrix, we mean an $n \times n$-matrix whose $(1, 1)$-entry is 1 and whose other entries are zero.

be a tuple of f numbers with $a_i \in \{0, \dots, n\}$. Assume that there exists an element $\beta_{a_\bullet} \in (D^\times)^{*=-1}$ such that the following condition is satisfied:[3]

Let $G_{a_\bullet}$ be the algebraic group over $\mathbb{Q}$ such that $G_{a_\bullet}(R)$ for a $\mathbb{Q}$-algebra R consists of elements $g \in (D^{\mathrm{opp}} \otimes_{\mathbb{Q}} R)^\times$ with $g\beta_{a_\bullet} g^* = c(g)\beta_{a_\bullet}$ for some $c(g) \in R^\times$. If $G^1_{a_\bullet}$ denotes the kernel of the similitude character $c : G_{a_\bullet} \to \mathbb{G}_{m,\mathbb{Q}}$, then there exists an isomorphism

$$G^1_{a_\bullet}(\mathbb{R}) \simeq \prod_{i=1}^{f} U(a_i, n - a_i),$$

where the i-th factor corresponds to the real embedding $\tau_i : F \hookrightarrow \mathbb{R}$.

Note that the assumption on D at p implies that

$$G_{a_\bullet}(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_{\mathfrak{p}}) \cong \mathbb{Q}_p^\times \times \mathrm{GL}_n(\mathbb{Q}_{p^f}).$$

We put $V_{a_\bullet} = D$ and view it as a left D-module. Let $\langle -, - \rangle_{a_\bullet} : V_{a_\bullet} \times V_{a_\bullet} \to \mathbb{Q}$ be the perfect alternating pairing given by

$$\langle x, y \rangle_{a_\bullet} = \mathrm{Tr}_{D/\mathbb{Q}}(x\beta_{a_\bullet} y^*) \quad \text{for } x, y \in V_{a_\bullet}.$$

Then $G_{a_\bullet}$ is identified with the similitude group associated to $(V_{a_\bullet}, \langle -, - \rangle_{a_\bullet})$, i.e., for all $\mathbb{Q}$-algebra R, we have

$$G_{a_\bullet}(R) = \{g \in \mathrm{End}_{D \otimes_{\mathbb{Q}} R}(V_{a_\bullet} \otimes_{\mathbb{Q}} R) \mid \langle gx, gy \rangle_{a_\bullet} = c(g)\langle x, y \rangle_{a_\bullet} \text{ for some } c(g) \in R^\times\}.$$

Consider the homomorphism of $\mathbb{R}$-algebraic groups $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to G_{a_\bullet,\mathbb{R}}$ given by

$$h(z) = \prod_{i=1}^{f} \mathrm{Diag}(\underbrace{z, \dots, z}_{a_i}, \underbrace{\bar{z}, \dots, \bar{z}}_{n-a_i}), \quad \text{for } z = x + \sqrt{-1}y. \tag{2.2.1}$$

Let $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to G_{a_\bullet,\mathbb{C}}$ be the composite of $h_{\mathbb{C}}$ with the map

$$\mathbb{G}_{m,\mathbb{C}} \to \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_{\mathbb{C}} \cong \mathbb{C}^\times \times \mathbb{C}^\times, \quad z \mapsto (z, 1).$$

Here, the first copy of $\mathbb{C}^\times$ in $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_{\mathbb{C}}$ is the one indexed by the identity element in $\mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$, and the other copy of $\mathbb{C}^\times$ is indexed by the complex conjugation.

Let E_h be the reflex field of μ_h, i.e., the minimal subfield of $\mathbb{C}$ where the conjugacy class of μ_h is defined. It has the following explicit description. The group $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{C})$ acts naturally on $\Sigma_{\infty,E}$, and hence on the functions on $\Sigma_{\infty,E}$. Then E_h is the subfield of $\mathbb{C}$ fixed by the stabilizer of the $\mathbb{Z}$-valued function a on $\Sigma_{\infty,E}$ defined by $a(q_i) = a_i$ and $a(\bar{q}_i) = n - a_i$. The isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$

[3] As explained in the proof of [Harris and Taylor 2001, Lemma I.7.1], when n is odd, such $\beta_{a_\bullet}$ always exists, and when n is even, existence of $\beta_{a_\bullet}$ depends on the parity of $a_1 + \cdots + a_f$. See also the proof of Lemma 2.9.

defines a p-adic place $\wp$ of E_h. By our hypothesis on E, the local field $E_{h,\wp}$ is an unramified extension of $\mathbb{Q}_p$ contained in $\mathbb{Q}_{p^f}$, the unique unramified extension over $\mathbb{Q}_p$ of degree f.

2.3. ***Unitary Shimura varieties of PEL-type.*** Let $\mathcal{O}_D$ be a $*$-stable order of D and $\Lambda_{a_\bullet}$ an $\mathcal{O}_D$-lattice of $V_{a_\bullet}$ such that $\langle \Lambda_{a_\bullet}, \Lambda_{a_\bullet} \rangle_{a_\bullet} \subseteq \mathbb{Z}$ and $\Lambda_{a_\bullet} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is self-dual under the alternating pairing induced by $\langle -, - \rangle_{a_\bullet}$. We put $K_p = \mathbb{Z}_p^\times \times \mathrm{GL}_n(\mathcal{O}_{E_\mathfrak{p}}) \subseteq G_{a_\bullet}(\mathbb{Q}_p)$, and fix an open compact subgroup $K^p \subseteq G_{a_\bullet}(\mathbb{A}^{\infty,p})$ such that $K = K^p K_p$ is *neat*, i.e., $G_{a_\bullet}(\mathbb{Q}) \cap gKg^{-1}$ is torsion free for any $g \in G_{a_\bullet}(\mathbb{A}^\infty)$.

Following [Kottwitz 1992b], we have a unitary Shimura variety $\mathcal{S}h_{a_\bullet}$ defined over $\mathbb{Z}_{p^f}$;[4] it represents the functor that takes a locally noetherian $\mathbb{Z}_{p^f}$-scheme S to the set of isomorphism classes of tuples (A, λ, η), where

(1) A is an fn^2-dimensional abelian variety over S equipped with an action of $\mathcal{O}_D$ such that the induced action on $\mathrm{Lie}(A/S)$ satisfies the *Kottwitz determinant condition*, that is, if we view the *reduced* relative de Rham homology $H_1^{\mathrm{dR}}(A/S)^\circ := \mathfrak{e} H_1^{\mathrm{dR}}(A/S)$ and its quotient $\mathrm{Lie}^\circ_{A/S} := \mathfrak{e} \cdot \mathrm{Lie}_{A/S}$ as a module over $F_p \otimes_{\mathbb{Z}_p} \mathcal{O}_S \cong \bigoplus_{i=1}^f \mathcal{O}_S$, they, respectively, decompose into the direct sums of locally free $\mathcal{O}_S$-modules $H_1^{\mathrm{dR}}(A/S)_i^\circ$ of rank n and, their quotients, locally free $\mathcal{O}_S$-modules $\mathrm{Lie}^\circ_{A/S,i}$ of rank $n - a_i$;

(2) $\lambda : A \to A^\vee$ is a prime-to-p $\mathcal{O}_D$-equivariant polarization such that the Rosati involution induces the involution $*$ on $\mathcal{O}_D$;

(3) η is a collection of, for each connected component S_j of S with a geometric point $\bar{s}_j$, a $\pi_1(S_j, \bar{s}_j)$-invariant K^p-orbit of isomorphisms $\eta_j : \Lambda_{a_\bullet} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)} \simeq T^{(p)}(A_{\bar{s}_j})$ such that the following diagram commutes for an isomorphism $\nu(\eta_j) \in \mathrm{Hom}(\widehat{\mathbb{Z}}^{(p)}, \widehat{\mathbb{Z}}^{(p)}(1))$:

$$\begin{tikzcd}
\Lambda_{a_\bullet} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)} \times \Lambda_{a_\bullet} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)} \arrow[r, "\langle -,- \rangle"] \arrow[d, "\eta_j \times \eta_j"] & \widehat{\mathbb{Z}}^{(p)} \arrow[d, "\nu(\eta_j)"] \\
T^{(p)} A_{\bar{s}_j} \times T^{(p)} A_{\bar{s}_j} \arrow[r, "\text{Weil pairing}"] & \widehat{\mathbb{Z}}^{(p)}(1),
\end{tikzcd}$$

where $\widehat{\mathbb{Z}}^{(p)} = \prod_{\ell \neq p} \mathbb{Z}_\ell$ and $T^{(p)}(A_{\bar{s}_j})$ denotes the product of the ℓ-adic Tate modules of $A_{\bar{s}_j}$ for all $\ell \neq p$.

The Shimura variety $\mathcal{S}h_{a_\bullet}$ is smooth and projective over $\mathbb{Z}_{p^f}$ of relative dimension $d(a_\bullet) := \sum_{i=1}^f a_i(n - a_i)$. Note that if $a_i \in \{0, n\}$ for all i, then $\mathcal{S}h_{a_\bullet}$ is of relative dimension zero; we call it a *discrete Shimura variety*.

[4] Although one can descend $\mathcal{S}h_{a_\bullet}$ to the subring $\mathcal{O}_{E_{h,\wp}}$ of $\mathbb{Z}_{p^f}$, we ignore this minor improvement.

We denote by $\mathcal{S}h_{a_\bullet}(\mathbb{C})$ the complex points of $\mathcal{S}h_{a_\bullet}$ via the embedding

$$\mathbb{Z}_{p^f} \hookrightarrow \overline{\mathbb{Q}}_p \xrightarrow{\iota_p^{-1}} \mathbb{C}.$$

Let $K_\infty \subseteq G_{a_\bullet}(\mathbb{R})$ be the stabilizer of h (2.2.1) under the conjugation action, and let X_∞ denote the $G_{a_\bullet}(\mathbb{R})$-conjugacy class of h. Then K_∞ is a maximal compact-modulo-center subgroup of $G_{a_\bullet}(\mathbb{R})$. According to [Kottwitz 1992b, page 400], the complex manifold $\mathcal{S}h_{a_\bullet}(\mathbb{C})$ is the disjoint union of $\#\ker^1(\mathbb{Q}, G_{a_\bullet})$ copies of

$$G_{a_\bullet}(\mathbb{Q}) \backslash (G_{a_\bullet}(\mathbb{A}^\infty) \times X_\infty)/K \cong G_{a_\bullet}(\mathbb{Q}) \backslash G_{a_\bullet}(\mathbb{A})/K \times K_\infty. \tag{2.3.1}$$

Here, if n is even, then $\ker^1(\mathbb{Q}, G_{a_\bullet}) = (0)$, while if n is odd then

$$\ker^1(\mathbb{Q}, G_{a_\bullet}) = \mathrm{Ker}(F^\times/\mathbb{Q}^\times N_{E/F}(E^\times) \to \mathbb{A}_F^\times/\mathbb{A}^\times N_{E/F}(\mathbb{A}_E^\times)).$$

In either case, $\ker^1(\mathbb{Q}, G_{a_\bullet})$ depends only on the CM extension E/F and the parity of n but not on the tuple $a_\bullet$.

Let $\mathrm{Sh}_{a_\bullet} := \mathcal{S}h_{a_\bullet} \otimes_{\mathbb{Z}_{p^f}} \mathbb{F}_{p^f}$ denote the special fiber of $\mathcal{S}h_{a_\bullet}$, and let $\overline{\mathrm{Sh}}_{a_\bullet} := \mathrm{Sh}_{a_\bullet} \otimes_{\mathbb{F}_{p^f}} \bar{\mathbb{F}}_p$ denote the geometric special fiber.

2.4. *ℓ-adic cohomology.* We fix a prime number $\ell \neq p$, and an isomorphism $\iota_\ell : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. Let ξ be an algebraic representation of $G_{a_\bullet}$ over $\overline{\mathbb{Q}}_\ell$, and $\xi_{\mathbb{C}}$ be the base change via ι_ℓ^{-1}. The theory of automorphic sheaves [Milne 1990, Section III] or just reading off from the rational ℓ-adic Tate modules of the universal abelian variety allows us to attach to ξ a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\xi$ over $\mathcal{S}h_{a_\bullet}$. For example, if ξ is the representation of $G_{a_\bullet}$ on the vector space $V_{a_\bullet}$ (Section 2.2), the corresponding ℓ-adic local system is given by the rational ℓ-adic Tate module (tensored with $\overline{\mathbb{Q}}_\ell$) of the universal abelian scheme over $\mathcal{S}h_{a_\bullet}$.

We assume that ξ is irreducible. Let $\mathscr{H}_K = \mathscr{H}(K, \overline{\mathbb{Q}}_\ell)$ be the Hecke algebra of compactly supported K-bi-invariant $\overline{\mathbb{Q}}_\ell$-valued functions on $G_{a_\bullet}(\mathbb{A}^\infty)$. The étale cohomology group $H^{d(a_\bullet)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{a_\bullet}, \mathcal{L}_\xi)$ is equipped with a natural action of $\mathscr{H}_K \times \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^f})$. Since $\mathrm{Sh}_{a_\bullet}$ is proper and smooth, there is no continuous spectrum and we have a canonical decomposition of $\mathscr{H}_K \times \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^f})$-modules (see, e.g., [Harris and Taylor 2001, Proposition III.2.1])

$$H^{d(a_\bullet)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{a_\bullet}, \mathcal{L}_\xi) = \bigoplus_{\pi \in \mathrm{Irr}(G_{a_\bullet}(\mathbb{A}^\infty))} \iota_\ell(\pi^K) \otimes R_{a_\bullet,\ell}(\pi), \tag{2.4.1}$$

where $\mathrm{Irr}(G_{a_\bullet}(\mathbb{A}^\infty))$ is the set of irreducible admissible representations of $G_{a_\bullet}(\mathbb{A}^\infty)$ with coefficients in $\mathbb{C}$, π^K is the K-invariant subspace of $\pi \in \mathrm{Irr}(G_{a_\bullet}(\mathbb{A}^\infty))$ and $R_{a_\bullet,\ell}(\pi)$ is a certain ℓ-adic representation of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^f})$ which we specify below.

We write $H^{d(a_\bullet)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{a_\bullet}, \mathcal{L}_\xi)_\pi$ for the π-*isotypic component* of the cohomology group, that is, the direct summand of (2.4.1) labeled by π. We make the following assumptions on π.

Hypothesis 2.5. (1) We have $\pi^K \neq 0$.

(2) There exists an admissible irreducible representation π_∞ of $G_{a_\bullet}(\mathbb{R})$ such that $\pi \otimes \pi_\infty$ is a cuspidal automorphic representation of $G_{a_\bullet}(\mathbb{A})$,

(2a) π_∞ is *cohomological in degree $d(a_\bullet)$ for ξ* in the sense that

$$H^{d(a_\bullet)}(\mathrm{Lie}(G_{a_\bullet}(\mathbb{R})), K_\infty, \pi_\infty \otimes \xi_{\mathbb{C}}) \neq 0, \text{[5]} \tag{2.5.1}$$

where K_∞ is a maximal compact subgroup of $G_{a_\bullet}(\mathbb{R})$,

(2b) and $\pi \otimes \pi_\infty$ admits a base change to a *cuspidal* automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E) \times \mathbb{A}_{E_0}^\times$.

Note that Hypothesis 2.5(1) implies that the p-component π_p is unramified. Hypothesis 2.5 (2a) ensures that $R_{a_\bullet,\ell}(\pi)$ is nontrivial. Moreover, by [Caraiani 2012, Theorem 1.2], this hypothesis implies that the base change of $\pi \otimes \pi_\infty$ to $\mathrm{GL}_{n,E}$ is tempered at all finite places, and hence π_p is tempered.

We recall now an explicit description, due to Kottwitz [1992a], of the Galois module $R_{a_\bullet,\ell}(\pi)$. As $G_{a_\bullet}(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_{\mathfrak{p}})$, we may write $\pi_p = \pi_{p,0} \otimes \pi_{\mathfrak{p}}$, where $\pi_{p,0}$ is a character of $\mathbb{Q}_p^\times$ trivial on $\mathbb{Z}_p^\times$, and $\pi_{\mathfrak{p}}$ is an irreducible admissible representation of $\mathrm{GL}_n(E_{\mathfrak{p}})$ such that $\pi_{\mathfrak{p}}^{\mathrm{GL}_n(\mathcal{O}_{E_{\mathfrak{p}}})} \neq 0$. Choose a square root $\sqrt{p}$ of p in $\overline{\mathbb{Q}}$. Depending on this choice of $\sqrt{p}$, one has an (unramified) local Langlands parameter attached to π_p:

$$\varphi_{\pi_p} = (\varphi_{\pi_{p,0}}, \varphi_{\pi_{\mathfrak{p}}}) : W_{\mathbb{Q}_p} \to {}^L(G_{a_\bullet,\mathbb{Q}_p}) \simeq \mathbb{C}^\times \times (\mathrm{GL}_n(\mathbb{C})^{\mathbb{Z}/f\mathbb{Z}} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)).$$

Here, $W_{\mathbb{Q}_p}$ is the Weil group of $\mathbb{Q}_p$, and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ permutes cyclically the f copies of $\mathrm{GL}_n(\mathbb{C})$ though the quotient $\mathrm{Gal}(\mathbb{Q}_{p^f}/\mathbb{Q}_p) \cong \mathbb{Z}/f\mathbb{Z}$. The image of $\varphi_{\pi_p}|_{W_{\mathbb{Q}_{p^f}}}$ lies in $({}^L G_{a_\bullet})^\circ \simeq \mathbb{C}^\times \times \mathrm{GL}_n(\mathbb{C})^{\mathbb{Z}/f\mathbb{Z}}$. The cocharacter $\mu_h : \mathbb{G}_{m,E_h} \to G_{a_\bullet,E_h}$ induces a character $\check{\mu}_h$ of $({}^L G_{a_\bullet})^\circ$ over E_h. Let r_{μ_h} denote the algebraic representation of $({}^L G_{a_\bullet})^\circ$ with extreme weight $\check{\mu}_h$. Denote by Frob_{p^f} a *geometric* Frobenius element in $W_{\mathbb{Q}_{p^f}}$. Let $\overline{\mathbb{Q}}_\ell(1/2)$ denote the unramified representation of $W_{\mathbb{Q}_{p^f}}$ which sends Frob_{p^f} to $(\sqrt{p})^{-f}$. Then $R_{a_\bullet,\ell}(\pi)$ can be described in terms of φ_{π_p} as follows.

Theorem 2.6 [Kottwitz 1992a, Theorem 1]. *Under the hypothesis and notation above, we have an equality in the Grothendieck group of $W_{\mathbb{Q}_{p^f}}$-modules*:

$$[R_{a_\bullet,\ell}(\pi)] = \#\ker^1(\mathbb{Q}, G_{a_\bullet}) m_{a_\bullet}(\pi) \big[\iota_\ell(r_{\mu_h} \circ \varphi_{\pi_p}) \otimes \overline{\mathbb{Q}}_\ell\big(-\tfrac{1}{2} d(a_\bullet)\big)\big],$$

where $m_{a_\bullet}(\pi)$ is a certain integer related to the automorphic multiplicities of automorphic representations of $G_{a_\bullet}$ with finite part π. [6]

[5] This automatically implies that π_∞ has the same central and infinitesimal characters as the *contragradient* of $\xi_{\mathbb{C}}$.

[6] Rigorously speaking, Kottwitz's theorem describes the direct sum of the π-component of all cohomological degrees. Since our π_p is tempered, so π appears only in the middle degree for purity reasons because $\mathrm{Sh}_{a_\bullet}$ is compact.

In our case, one can make Kottwitz's theorem more transparent. Define an ℓ-adic representation

$$\rho_{\pi_{\mathfrak{p}}} = \iota_\ell(\varphi^{(1),\vee}_{\pi_{\mathfrak{p}}}) \otimes \overline{\mathbb{Q}}_\ell\big(\tfrac{1}{2}(1-n)\big) : W_{\mathbb{Q}_{p^f}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell), \tag{2.6.1}$$

where $\varphi^{(1),\vee}_{\pi_{\mathfrak{p}}} : W_{\mathbb{Q}_{p^f}} \to \mathrm{GL}_n(\mathbb{C})$ denotes the *contragredient* of the projection to the first (or any) copy of $\mathrm{GL}_n(\mathbb{C})$. Both $\varphi_{\pi_{\mathfrak{p}}}$ and $\overline{\mathbb{Q}}_\ell\big(\frac{1}{2}\big)$ depend on the choice of $\sqrt{p}$, but $\rho_{\pi_{\mathfrak{p}}}$ does not. Explicitly, $\rho_{\pi_{\mathfrak{p}}}(\mathrm{Frob}_{p^f})$ is semisimple with the characteristic polynomial given by [Gross 1998, (6.7)]:

$$X^n + \sum_{i=1}^{n} (-1)^i (N\mathfrak{p})^{i(i-1)/2} a^{(i)}_{\mathfrak{p}} X^{n-i}, \tag{2.6.2}$$

where $a^{(i)}_{\mathfrak{p}}$ is the eigenvalue on $\pi_{\mathfrak{p}}^{\mathrm{GL}_n(\mathcal{O}_{E_{\mathfrak{p}}})}$ of the Hecke operator

$$T^{(i)}_{\mathfrak{p}} = \mathrm{GL}_n(\mathcal{O}_{E_{\mathfrak{p}}}) \cdot \mathrm{Diag}(\underbrace{p, \dots, p}_{i}, \underbrace{1, \dots, 1}_{n-i}) \cdot \mathrm{GL}_n(\mathcal{O}_{E_{\mathfrak{p}}}).$$

An easy computation shows that $r_{\mu_h} = \mathrm{Std}^{-1}_{\mathbb{Q}_p^\times} \otimes \bigotimes_{i=1}^{f} (\wedge^{a_i} \mathrm{Std}^\vee)$. Since the projection of $\varphi_{\pi_p}|_{W_{\mathbb{Q}_{p^f}}}$ to each copy of $\mathrm{GL}_n(\mathbb{C})$ is conjugate to all others, Theorem 2.6 is equivalent to

$$\begin{aligned}[]
&[R_{a_\bullet,\ell}(\pi)] \\
&\quad = \#\ker^1(\mathbb{Q}, G_{a_\bullet}) \cdot m_{a_\bullet}(\pi) \Big[\rho_{a_\bullet}(\pi_{\mathfrak{p}}) \otimes \chi^{-1}_{\pi_{p,0}} \otimes \overline{\mathbb{Q}}_\ell\Big(\sum_i \tfrac{1}{2} a_i(a_i-1)\Big)\Big],
\end{aligned} \tag{2.6.3}$$

where $\rho_{a_\bullet}(\pi_{\mathfrak{p}}) = r_{a_\bullet} \circ \rho_{\pi_{\mathfrak{p}}}$ with $r_{a_\bullet} = \bigotimes_{i=1}^{f} \wedge^{a_i} \mathrm{Std}$, and $\chi_{\pi_{p,0}}$ denotes the character of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^f})$ sending Frob_{p^f} to $\iota_\ell(\pi_{p,0}(p^f))$.

Remark 2.7. The reason why we normalize the Galois representation as above is the following: By Hypothesis 2.5, π is the finite part of an automorphic representation of $G_{a_\bullet}(\mathbb{A})$ which admits a base change to a cuspidal automorphic representation $\Pi \otimes \chi$ of $\mathrm{GL}_n(\mathbb{A}_E) \times \mathbb{A}^\times_{E_0}$. If ρ_Π denotes the Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ attached to Π, then $\rho_{\pi_{\mathfrak{p}}}$ is the semisimplification of the restriction of ρ_Π to $W_{E_{\mathfrak{p}}}$ (See [Caraiani 2012, Theorem 1.1]).

2.8. ***Tate conjecture.*** We recall first the Tate conjecture [1966] over finite fields. Let X be a projective smooth variety over a finite field $\mathbb{F}_q$ of characteristic p. Put $\overline{X} = X_{\bar{\mathbb{F}}_p}$. For each prime $\ell \neq p$ and integer $r \leq \dim(X)$, we have a cycle class map

$$\mathrm{cl}^r_X : A^r(X) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell \to H^{2r}_{\mathrm{et}}(\overline{X}, \overline{\mathbb{Q}}_\ell(r))^{\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_q)},$$

where $A^r(X)$ denotes the abelian group of codimension r algebraic cycles in X defined over $\mathbb{F}_q$. Then the Tate conjecture predicts that this map is surjective. One has a geometric variant of the Tate conjecture, which claims that the geometric cycle class map:

$$\mathrm{cl}^r_{\overline{X}} : A^r(\overline{X}) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell \to H^{2r}_{\mathrm{et}}(\overline{X}, \overline{\mathbb{Q}}_\ell(r))^{\mathrm{fin}} := \bigcup_{m \geq 1} H^{2r}_{\mathrm{et}}(\overline{X}, \overline{\mathbb{Q}}_\ell(r))^{\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{q^m})}$$

is surjective. Here, the superscript "fin" means the subspace on which $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_q)$ acts through a finite quotient. Note that the surjectivity of $\mathrm{cl}^r_{\overline{X}}$ implies that of cl^r_X by taking the $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_q)$-invariant subspace.

Consider the case $X = \mathrm{Sh}_{a_\bullet}$ with $d(a_\bullet)$ even. Let π be an irreducible admissible representation of $G_{a_\bullet}(\mathbb{A}^\infty)$ as in Theorem 2.6. By Theorem 2.6, the π-isotypic component of $H^{d(a_\bullet)}_{\mathrm{et}}\big(\overline{\mathrm{Sh}}_{a_\bullet}, \overline{\mathbb{Q}}_\ell\big(\tfrac{1}{2}d(a_\bullet)\big)\big)^{\mathrm{fin}}$ is, up to Frobenius semisimplification[7], isomorphic to $\dim(\pi^K) \cdot \#\ker^1(\mathbb{Q}, G_{a_\bullet}) \cdot m_{a_\bullet}(\pi)$ copies of

$$\left(\rho_{a_\bullet}(\pi_{\mathfrak{p}}) \otimes \chi^{-1}_{\pi_{p,0}} \otimes \overline{\mathbb{Q}}_\ell\left(\frac{(n-1)}{2}\sum_{i=1}^{f} a_i\right)\right)^{\mathrm{fin}}. \tag{2.8.1}$$

Note that $\chi_{\pi_{p,0}}(\mathrm{Frob}_{p^f}) = \pi_{p,0}(p^f)$ is a root of unity. Hence, the dimension of (2.8.1) is equal to the sum of the dimensions of the Frob_{p^f}-eigenspaces of $\rho_{a_\bullet}(\pi_{\mathfrak{p}})$ with eigenvalues $(p^f)^{(n-1)/2\sum_i a_i}\zeta$ for some root of unity ζ. In many examples, this space is known to be nonzero.

For instance, when $f = 2$, $a_1 = r$ and $a_2 = n - r$ for some $1 \leq r \leq n-1$, we have $d(a_\bullet) = 2r(n-r)$ and

$$\rho_{a_\bullet}(\pi_{\mathfrak{p}}) = \wedge^r \rho_{\pi_{\mathfrak{p}}} \otimes \wedge^{n-r} \rho_{\pi_{\mathfrak{p}}}.$$

Let $V_{\pi_{\mathfrak{p}}, a_\bullet}$ denote the space of representation $\rho_{a_\bullet}(\pi_{\mathfrak{p}})$. If $\rho_{\pi_{\mathfrak{p}}}(\mathrm{Frob}_{p^f})$ has distinct eigenvalues $\alpha_1, \dots, \alpha_n$, then the eigenvalues of Frob_{p^f} on $V_{\pi_{\mathfrak{p}}, a_\bullet}$ are given by $\alpha_{i_1} \cdots \alpha_{i_r} \cdot \alpha_{j_1} \cdots \alpha_{j_{n-r}}$, for distinct subscripts $i_1, \dots, i_r$ and distinct subscripts $j_1, \dots, j_{n-r}$. This product is exactly $(p^f)^{n(n-1)/2} a^{(n)}_{\mathfrak{p}}$ (note that $a^{(n)}_{\mathfrak{p}}$ is a root of unity) if the set $\{i_1, \dots, i_r\}$ and the set $\{j_1, \dots, j_{n-r}\}$ are the complement of each other as subsets of $\{1, \dots, n\}$. On the other hand, if the subsets $\{i_1, \dots, i_r\}$ and $\{j_1, \dots, j_{n-r}\}$ are not the complement of each other and if the α_i are "sufficiently generic"[8], the eigenvalue $\alpha_{i_1} \cdots \alpha_{i_r} \cdot \alpha_{j_1} \cdots \alpha_{j_{n-r}}$ is not a root of unity. In other words, the dimension of (2.8.1) is "generically" equal to $\binom{n}{r}$. As predicted by the Tate conjecture, these cohomology classes should come from algebraic cycles. Our main conjecture addresses exactly this, and it predicts that those desired "generic"

[7] Conjecturally, the Frobenius action on the étale ℓ-adic cohomology groups of a projective smooth variety over a finite field is always semisimple.

[8] For example, if $r = 1$ and $\alpha_1 = \alpha_2$, the eigenvalues $\alpha_1 \cdot \alpha_1\alpha_3\alpha_4 \cdots \alpha_n$ is equal to $\alpha_1 \cdots \alpha_n$ and hence is $p^{n(n-1)}$ times a root of unity. So to be in the generic case, we will need to require that α_i/α_j for $i \neq j$ is not a root of unity if $r = 1$. For another example, if $r = 2$, "generic" will mean that α_i/α_j for $i \neq j$ and $\alpha_i\alpha_{i'}/\alpha_j\alpha_{j'}$ for $\{i, i'\} \neq \{j, j'\}$ are not roots of unity.

algebraic cycles can be given by the irreducible components of the basic locus, and are birationally equivalent to certain fiber bundles over the special fiber of some other Shimura varieties associated to inner forms of $G_{a_\bullet}$. To make this precise, we need the following lemma.

Lemma 2.9. *Let $b_\bullet = (b_i)_{1\le i\le f}$ be a tuple with $b_i \in \{0, \dots, n\}$ such that $\sum_{i=1}^f b_i \equiv \sum_{i=1}^f a_i \pmod 2$ if n is even. Then there exists $\beta_{b_\bullet} \in (D^\times)^{*=-1}$ such that*

- *the alternating D-Hermitian space $(V_{b_\bullet}, \langle -, -\rangle_{b_\bullet})$ defined using $\beta_{b_\bullet}$ in place of $\beta_{a_\bullet}$ is isomorphic to $(V_{a_\bullet}, \langle -, -\rangle_{a_\bullet})$ when tensored with $\mathbb{A}^\infty$, and*
- *if $G_{b_\bullet}$ denotes the corresponding algebraic group over $\mathbb{Q}$ defined in the similar way with $\beta_{a_\bullet}$ replaced by $\beta_{b_\bullet}$, then*

$$G^1_{b_\bullet}(\mathbb{R}) \simeq \prod_{i=1}^f U(b_i, n-b_i).$$

Proof. We reduce the problem to the existence of a certain cohomology class. Note that $G^1_{a_\bullet} = \mathrm{Aut}(V_{a_\bullet}, \langle -, -\rangle_{a_\bullet})$ is the Weil restriction to $\mathbb{Q}$ of a unitary group $U_{a_\bullet}$ over F. The cohomology set $H^1(\mathbb{Q}, G^1_{a_\bullet}) \cong H^1(F, U_{a_\bullet})$ is in bijection with the isomorphism classes of one-dimensional skew-Hermitian D-modules V. As $U_{a_\bullet} \times_F E \simeq \mathrm{GL}_{n,E}$, Hilbert's Theorem 90 for GL_n implies that the inflation map induces an isomorphism

$$H^1(E/F, U_{a_\bullet}) \xrightarrow{\sim} H^1(F, U_{a_\bullet}).$$

Denote by $g \mapsto g^{\sharp_{\beta_{a_\bullet}}} = \beta_{a_\bullet} g^* \beta_{a_\bullet}^{-1}$ the involution on D induced by the alternating pairing $\langle -, -\rangle_{a_\bullet}$. Then a 1-cocycle of $\mathrm{Gal}(E/F)$ with values in $U_{a_\bullet}$ is given by an element $\alpha \in D^\times$ such that $\alpha = \alpha^{\sharp_{\beta_{a_\bullet}}}$, and $\alpha_1, \alpha_2 \in D^\times$ define the same cohomology class in $H^1(F, U_{a_\bullet})$ if and only if there exists $g \in D^\times$ such that $g\alpha_1 g^{\sharp_{\beta_{a_\bullet}}} = \alpha_2$. Explicitly, given such an α, the corresponding skew-Hermitian D-module is given by $V = D$ equipped with the alternating pairing

$$\langle -, -\rangle_\alpha : V \times V \to \mathbb{Q}, \qquad (x, y) \mapsto \mathrm{Tr}_{D/\mathbb{Q}}(x\alpha\beta_{a_\bullet} y^*).$$

For a place v of F, we denote by

$$\mathrm{loc}_v : H^1(F, U_{a_\bullet}) \to H^1(F_v, U_{a_\bullet})$$

the canonical localization map. By [Helm 2012, Proposition 8.1], if $\sum_{i=1}^f b_i \equiv \sum_{i=1}^f a_i \mod 2$, there exists a cohomology class $[\alpha] \in H^1(F, U_{a_\bullet})$ such that

- $\mathrm{loc}_v([\alpha])$ is trivial for every finite place v of F, and
- if $v = \tau_i$ with $1 \le i \le n$ is an archimedean place, one has an isomorphism of unitary groups over $\mathbb{R}$: $\mathrm{Aut}(V \otimes_{F,\tau_i} \mathbb{R}, \langle -, -\rangle_\alpha) \simeq U(b_i, n-b_i)$.

Then the element $\beta_{b_\bullet} = \alpha\beta_{a_\bullet}$ meets the requirements of Lemma 2.9. ☐

In the sequel, we always fix a choice of $\beta_{b_\bullet}$, and as well as an isomorphism $\gamma_{a_\bullet,b_\bullet} : V_{a_\bullet} \otimes_{\mathbb{Q}} \mathbb{A}^\infty \xrightarrow{\sim} V_{b_\bullet} \otimes_{\mathbb{Q}} \mathbb{A}^\infty$, which induces an isomorphism $G_{a_\bullet}(\mathbb{A}^\infty) \simeq G_{b_\bullet}(\mathbb{A}^\infty)$. Recall that we have chosen a lattice $\Lambda_{a_\bullet} \subseteq V_{a_\bullet}$ to define the moduli problem for $\mathcal{S}h_{a_\bullet}$. We put $\Lambda_{b_\bullet} := V_{b_\bullet} \cap \gamma_{a_\bullet,b_\bullet}(\Lambda_{a_\bullet} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$. Then applying the construction of Section 2.3 to the lattice $\Lambda_{b_\bullet} \subseteq V_{b_\bullet}$ and the open compact subgroup $K^p \subseteq G_{a_\bullet}(\mathbb{A}^{\infty,p}) \simeq G_{b_\bullet}(\mathbb{A}^{\infty,p})$, we get a Shimura variety $\mathcal{S}h_{b_\bullet}$ over $\mathbb{Z}_{p^f}$ of level K^p as well as its special fiber $\mathrm{Sh}_{b_\bullet}$. Moreover, an algebraic representation ξ of $G_{a_\bullet}$ over $\overline{\mathbb{Q}}_\ell$ corresponds, via the fixed isomorphism $G_{a_\bullet}(\mathbb{A}^\infty) \simeq G_{b_\bullet}(\mathbb{A}^\infty)$, to an algebraic representation of $G_{b_\bullet}$ over $\overline{\mathbb{Q}}_\ell$. We use the same notation $\mathcal{L}_\xi$ to denote the étale sheaf on $\mathcal{S}h_{a_\bullet}$ and $\mathcal{S}h_{b_\bullet}$ defined by ξ.

2.10. ***Gysin/trace maps.*** Before stating the main conjecture of this paper, we recall the general definition of Gysin maps. Let $f : Y \to X$ be a proper morphism of smooth varieties over an algebraically closed field k. Let d_X and d_Y be the dimensions of X and Y respectively. Recall that the derived direct image Rf_* on the derived category of constructible ℓ-adic étale sheaves has a left adjoint $f^!$. Since both X and Y are smooth, the ℓ-adic dualizing complex of X (resp. Y) is $\overline{\mathbb{Q}}_\ell(d_X)[2d_X]$ (resp. $\overline{\mathbb{Q}}_\ell(d_Y)[2d_Y]$). Therefore, one has

$$f^!(\overline{\mathbb{Q}}_\ell(d_X)[2d_X]) = \overline{\mathbb{Q}}_\ell(d_Y)[2d_Y].$$

The adjunction map $Rf_* f^! \overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell$ induces a canonical morphism

$$\mathrm{Tr}_f : Rf_* \overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell(d_X - d_Y)[2(d_X - d_Y)].$$

More generally, if $\mathcal{L}$ is a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X, it induces a Gysin/trace map

$$Rf_*(f^*\mathcal{L}) \cong \mathcal{L} \otimes Rf_*(\overline{\mathbb{Q}}_\ell) \xrightarrow{1\otimes \mathrm{Tr}_f} \mathcal{L}(d_X - d_Y)[2(d_X - d_Y)],$$

where the first isomorphism is the projection formula [SGA 4_2 1972, XVII 5.2.9]. When f is flat with equidimensional fibers of dimension $d_Y - d_X$, this is the trace map as defined in [SGA 4_2 1972, XVIII 2.9]. When f is a closed immersion of codimension $r = d_X - d_Y$, it is the usual Gysin map. For any integer q, the Gysin/trace map induces a morphism on cohomology groups:

$$f_! : H^q_{\mathrm{et}}(Y, f^*\mathcal{L}) \to H^{q+2(d_X-d_Y)}_{\mathrm{et}}(X, \mathcal{L}(d_X - d_Y)). \tag{2.10.1}$$

2.11. ***Representation theory of*** $\mathbf{GL}_n$. As suggested by the description of Galois representations appearing in the middle cohomology group of Shimura varieties in Theorem 2.6, as well as by the Tate conjecture, we need to understand the representation theory of GL_n embedded diagonally into the Langlands dual group

$$({}^L G_{a_\bullet})^\circ \simeq \mathbb{C}^\times \times \mathrm{GL}_n(\mathbb{C})^{\mathbb{Z}/f\mathbb{Z}}.$$

The Hodge cocharacter μ of $G_{a_\bullet}$ gives rise to the representation $r_{a_\bullet} = \bigotimes_{i=1}^{f} (\wedge^{a_i} \mathrm{Std})$ of the diagonal GL_n. If λ is a dominant weight of GL_n (with respect to the usual diagonal torus and upper triangular Borel subgroup) appearing in $r_{a_\bullet}$, we can write this weight λ as the sum of f dominant minuscule weights $\omega_{b_1} + \cdots + \omega_{b_f}$, where ω_i for $0 \le i \le n$ is the weight of GL_n that takes $\mathrm{Diag}(t_1, \dots, t_n)$ to $t_1 \cdots t_i$. The set $\{b_1, \dots, b_f\}$ (counted with multiplicity) is unique, which we denote by B_λ. Explicitly, if λ takes $\mathrm{Diag}(t_1, \dots, t_n)$ to $t_1^{\beta_1} \cdots t_n^{\beta_n}$ (necessarily $\beta_1 \le f$), then

$$B_\lambda = \{\underbrace{n, \dots, n}_{\beta_n}, \underbrace{n-1, \dots, n-1}_{\beta_{n-1}-\beta_n}, \dots, \underbrace{1, \dots, 1}_{\beta_1-\beta_0}, \underbrace{0, \dots, 0}_{f-\beta_1}\}.$$

Moreover, we always have $\sum a_i = \sum b_i$. In particular, this implies by Lemma 2.9 that the Shimura variety $\mathrm{Sh}_{b_\bullet}$ makes sense, and the étale sheaf $\mathcal{L}_\xi$ is well defined on $\mathrm{Sh}_{b_\bullet}$.

We write $m_\lambda(a_\bullet)$ for the multiplicity of the weight λ in $r_{a_\bullet}$.

Conjecture 2.12. *Let* $\mathrm{Sh}_{a_\bullet}$ *and* $\mathcal{L}_\xi$ *be as in Section 2.4. Let* λ *be a dominant weight that appears in the representation* $r_{a_\bullet}$ *as in Section 2.11. Define* B_λ *and* $m_\lambda(a_\bullet)$ *as in Section 2.11.*

Then there exist varieties $Y_1, \dots, Y_{m_\lambda(a_\bullet)}$ *of dimension* $\frac{1}{2}(d(a_\bullet) + d(b_\bullet))$ *over* $\mathbb{F}_{p^f}$*, equipped with natural action of prime-to-*p *Hecke correspondences, such that each* Y_j *fits into a diagram*

$$\begin{tikzcd} & Y_j \arrow[dl, "\mathrm{pr}_{a_\bullet}^{(j)}"'] \arrow[dr, "\mathrm{pr}_{b_\bullet^{(j)}}"] & \\ \mathrm{Sh}_{a_\bullet} & & \mathrm{Sh}_{b_\bullet^{(j)}} \end{tikzcd}$$

satisfying the following properties.

(1) *For each* j, $b_\bullet^{(j)} = (b_1^{(j)}, \dots, b_f^{(j)})$ *is a reordering of the elements of the set* B_λ*, and both* $\mathrm{pr}_{a_\bullet}^{(j)}$ *and* $\mathrm{pr}_{b_\bullet^{(j)}}$ *are equivariant for the prime-to-*p *Hecke correspondences.*

(2) *The morphism* $\mathrm{pr}_{a_\bullet}^{(j)}$ *is a proper morphism and is birational onto the image. The morphism* $\mathrm{pr}_{b_\bullet^{(j)}}$ *is proper and generically smooth of relative dimension* $\frac{1}{2}(d(a_\bullet) - d(b_\bullet))$ *(note that* $d(b_\bullet) \equiv d(a_\bullet) \pmod 2$ *since* $\sum_i a_i = \sum_i b_i$*).*

(3) *There exists a* p*-isogeny of abelian schemes over* Y_j

$$\phi_{b_\bullet^{(j)}, a_\bullet} : \mathrm{pr}_{b_\bullet^{(j)}}^*(\mathcal{A}_{b_\bullet^{(j)}}) \to \mathrm{pr}_{a_\bullet}^{(j),*}(\mathcal{A}_{a_\bullet}),$$

where $\mathcal{A}_{a_\bullet}$ *and* $\mathcal{A}_{b_\bullet^{(j)}}$ *denote respectively the universal abelian scheme on* $\mathrm{Sh}_{a_\bullet}$ *and* $\mathrm{Sh}_{b_\bullet^{(j)}}$*. Let*

$$\phi_{b_\bullet^{(j)}, a_\bullet, *} : \mathrm{pr}_{b_\bullet^{(j)}}^* \mathcal{L}_\xi \xrightarrow{\sim} \mathrm{pr}_{a_\bullet}^{(j),*} \mathcal{L}_\xi.$$

be the isomorphism of the ℓ-adic sheaves induced by $\phi_{b_\bullet^{(j)},a_\bullet}$ via the construction in Section 2.4.[9]

(4) *Let π be an irreducible admissible representation of $G_{a_\bullet}(\mathbb{A}^\infty) \simeq G_{b_\bullet^{(j)}}(\mathbb{A}^\infty)$ satisfying Hypothesis 2.5 for both $a_\bullet$ and $b_\bullet$, and assume that $m_{a_\bullet}(\pi) = m_{b_\bullet^{(j)}}(\pi)$ for all j* [10]*. Suppose that the n eigenvalues $\alpha_1, \ldots, \alpha_n$ of $\rho_{\pi_\mathfrak{p}}(\mathrm{Frob}_{p^f})$ are "sufficiently generic" in the sense that the generalized eigenspace decomposition of $\rho_{a_\bullet}(\mathrm{Frob}_{p^N})$ for any large N is the same as the weight space decomposition of the algebraic representation $r_{a_\bullet}$. Then the natural homomorphism of π-isotypic components*[11] *of the cohomology groups*

$$\bigoplus_{j=1}^{m_\lambda(a_\bullet)} H_{\mathrm{et}}^{d(b_\bullet)}\big(\overline{\mathrm{Sh}}_{b_\bullet^{(j)}}, \mathcal{L}_\xi\big(\tfrac{1}{2}d(b_\bullet)\big)\big)_\pi^{\mathrm{Frob}_{p^f}=\lambda}$$

$$\xrightarrow{\oplus \mathrm{pr}^*_{b_\bullet^{(j)}}} \bigoplus_{j=1}^{m_\lambda(a_\bullet)} H_{\mathrm{et}}^{d(b_\bullet)}\big(\bar{Y}_j, \mathrm{pr}^*_{b_\bullet^{(j)}} \mathcal{L}_\xi\big(\tfrac{1}{2}d(b_\bullet)\big)\big)_\pi^{\mathrm{Frob}_{p^f}=\lambda}$$

$$\xrightarrow{\oplus \phi_{b_\bullet^{(j)},a_\bullet,*}} \bigoplus_{j=1}^{m_\lambda(a_\bullet)} H_{\mathrm{et}}^{d(b_\bullet)}\big(\bar{Y}_j, \mathrm{pr}^*_{a_\bullet} \mathcal{L}_\xi\big(\tfrac{1}{2}d(b_\bullet)\big)\big)_\pi^{\mathrm{Frob}_{p^f}=\lambda}$$

$$\xrightarrow{\sum \mathrm{pr}^{(j)}_{a_\bullet,!}} H_{\mathrm{et}}^{d(a_\bullet)}\big(\overline{\mathrm{Sh}}_{a_\bullet}, \mathcal{L}_\xi\big(\tfrac{1}{2}d(a_\bullet)\big)\big)_\pi^{\mathrm{Frob}_{p^f}=\lambda}$$

is an isomorphism, where $\mathrm{pr}^{(j)}_{a_\bullet,!}$ is the Gysin map (2.10.1) and the superscript $\mathrm{Frob}_{p^f} = \lambda$ means taking the (direct sum of) generalized Frob_{p^f}-eigenspace with eigenvalues in the Weyl group orbit

$$\lambda \circ \rho_{\pi_\mathfrak{p}}(\mathrm{Frob}_{p^f}) \cdot \chi^{-1}_{\pi_{p,0}}(p^f)(\sqrt{p})^{-f(n-1)\sum_i b_i}.$$

Here, since the semisimple conjugacy classes of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is in natural bijection with the orbits of $T(\overline{\mathbb{Q}}_\ell)$ under the Weyl group of GL_n, it makes sense to evaluate a dominant weight of T on $\rho_{\pi_\mathfrak{p}}(\mathrm{Frob}_{p^f})$ to get an orbit under the action of the Weyl group of GL_n; hence the notation $\lambda \circ \rho_{\pi_\mathfrak{p}}(\mathrm{Frob}_{p^f})$.

In particular, when ξ is the trivial representation and the weight λ is a power of the determinant (so automatically, $\sum_i a_i$ is divisible by n, and $d(a_\bullet)$ is even), the cycles given by the images of $Y_1, \ldots, Y_{m_\lambda(a_\bullet)}$ parameterized by the ***discrete*** *Shimura varieties $\mathrm{Sh}_{b_\bullet^{(j)}}$, generate the Tate classes of $H_{\mathrm{et}}^{d(a_\bullet)}\big(\overline{\mathrm{Sh}}_{a_\bullet}, \overline{\mathbb{Q}}_\ell\big(\tfrac{1}{2}d(a_\bullet)\big)\big)_\pi$ when $\rho_{\pi_\mathfrak{p}}(\mathrm{Frob}_{p^f})$ is "sufficiently generic".*

[9] This isomorphism depends on the choice of the isomorphism $\gamma_{a_\bullet,b_\bullet}$ made earlier.

[10] This assumption is satisfied when π is the finite part of an automorphic cuspidal representation of $G_{a_\bullet}(\mathbb{A})$ which admits a base change to a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E) \times \mathbb{A}_E^\times$. Indeed, in this case, White [2012, Theorem E] proved that $m_{a_\bullet}(\pi) = m_{b_\bullet^{(j)}}(\pi) = 1$.

[11] The π-isotypic component is the same as the π^p-isotypic component according to Lemma 4.17.

Remark 2.13. (1) A key feature of this conjecture is that the codimension of the cycle map $\mathrm{pr}_{a_\bullet}: Y_j \to \mathrm{Sh}_{a_\bullet}$ is the same as the fiber dimension of $\mathrm{pr}_{b_\bullet^{(j)}}: Y_j \to \mathrm{Sh}_{b_\bullet^{(j)}}$.

(2) It seems that the fiber of $\mathrm{pr}_{b_\bullet^{(j)}}: Y_j \to \mathrm{Sh}_{b_\bullet^{(j)}}$ over a generic point $\eta \in \mathrm{Sh}_{b_\bullet^{(j)}}$ is likely to be isomorphic to a certain "iterated Deligne–Lusztig variety," that is, a tower of maps $Y_{j,\eta} = Z_\alpha \to \cdots \to Z_0 = \eta$ such that each $Z_i \to Z_{i-1}$ is a fiber bundle with certain Deligne–Lusztig varieties as fibers.

(3) Xinwen Zhu pointed out to us that since the universal abelian varieties $\mathcal{A}_{a_\bullet}$ and $\mathcal{A}_{b_\bullet}$ are isogenous over each Y_j, the union of the images of $Y_1, \dots, Y_{m_\lambda(a_\bullet)}$ on $\mathrm{Sh}_{a_\bullet}$ is contained in the closure of the Newton strata, where the slope is the same as the μ-ordinary slope of the universal abelian varieties on $\mathrm{Sh}_{b_\bullet^{(j)}}$ (for different j, they have the same μ-ordinary slopes). In fact, one should expect the union of images to be the same as the closure of this Newton stratum.

When λ is central (i.e., a power of the determinant), Conjecture 2.12 says: *irreducible components of the basic locus of the special fiber of a Shimura variety, generically, contribute to all Tate cycles in the cohomology*. Implicitly, this means that the dimension of the basic locus is half of the dimension of the Shimura variety if and only if the Galois representations of the Shimura variety has generically nontrivial Tate classes. Here two appearances of "generic" both mean that we only consider those π-isotypic components where the Satake parameter for π_p is sufficiently generic as in Conjecture 2.12(4). For example, the supersingular locus of Hilbert modular surface at a split prime or the supersingular locus of a Siegel modular variety (over $\mathbb{Q}$) is not half the dimension. This is related to the fact that the π-isotypic component of the cohomology of the Shimura varieties are not expected to have Tate classes, at least when the Satake parameter of π_p is sufficiently general.[12]

(4) These varieties Y_j may be viewed as Hecke correspondences at p *between the special fibers of two different Shimura varieties* $\mathrm{Sh}_{a_\bullet}$ and $\mathrm{Sh}_{b_\bullet^{(j)}}$. These correspondences certainly cannot be lifted to characteristic zero. We hope that the conjecture will bring interests into the study of such Hecke correspondences.

Remark 2.14. (1) The assumption on the decomposition of the place p in $E/\mathbb{Q}$ and working with unitary Shimura varieties is to simplify our presentation and to get to a situation where most terms can be defined. We certainly expect the validity of analogous conjectures for the special fibers of Shimura varieties of PEL-type or more generally of abelian type (using the integral model of M. Kisin [2010]). This would be a more precise version of the Tate conjecture in the context of special fibers of Shimura varieties: if Sh_G and $\mathrm{Sh}_{G'}$ are the

[12] The Siegel varieties are Shimura varieties associated to $\mathrm{GSp}_{2g}(\mathbb{Q})$. The Langlands dual group is isogenous to $\mathrm{Spin}(2g+1)$ and the associated representation r_μ is the spin representation, which is minuscule and hence does not contain trivial weight subspace.

special fibers of two unitary Shimura varieties associated to the groups G and G' such that $G(\mathbb{A}_f) \simeq G'(\mathbb{A}_f)$, then, generically, the cycles on the product $\mathrm{Sh}_G \times \mathrm{Sh}_{G'}$ predicted by the Tate conjectures are likely to be constructed by understanding the "isogenies" between the corresponding universal abelian varieties, and are closely related to the Newton stratifications of Sh_G and $\mathrm{Sh}_{G'}$. In the case of Shimura varieties of abelian type, we expect some technical difficulties in reinterpreting the meaning of isogenies of abelian varieties in terms of certain "G-crystals".

For example, consider a real quadratic field $F/\mathbb{Q}$ in which a prime p is inert. Let Sh_G denote the special fiber of the Hilbert–Siegel modular variety for $G := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GSp}_{2g}$, with hyperspecial level structure at p. Then by Langlands's prediction of the cohomology of Sh_G, we should look at the representation $r_{\mathrm{spin}}^{\otimes 2}$ of the "essential part" Spin_{2g+1} of the Langlands dual group, where r_{spin} is the 2^g-dimensional spin representation.[13] The central weight space of $r_{\mathrm{spin}}^{\otimes 2}$ has dimension 2^g. So we expect that the supersingular locus of Sh_G is the union of 2^g collection of varieties parameterized by the discrete Shimura variety $\mathrm{Sh}_{G'}$ where G' is the inner form of G which is split at all finite places and is compact modulo center at both archimedean places. Unfortunately, the moduli problem that describes G' uses a different division algebra from that describing G. We do not know how to interpret the meaning of isogenies of universal abelian varieties in this case, and the method of our paper does not apply directly to this case.

(2) Xinwen Zhu pointed out to us that even if p is ramified, we should expect Conjecture 2.12 continue to hold for (the special fiber of) the "splitting models" of Pappas and Rapoport [2005]. Some evidences of this have already appeared in the case of Hilbert modular varieties; see [Rapoport et al. 2014; Reduzzi and Xiao 2017].

(3) In our setup, we took advantage of many coincidences that ensures that for example the Shimura variety is compact and there is no endoscopy. It would be certainly an interesting future question to study the case involving Eisenstein series, as well as the case when the representations come from endoscopy transfers.

(4) As explained in Remark 2.13(3), the images of Y_j are expected to form the closure of a certain Newton polygon where the slopes are related to λ. Conjecture 2.12(1)–(3) may have a degenerate situation: when $\sum_i a_i$ is not divisible by n, the representation $V_{a_\bullet}$ does not contain a weight corresponding to a power of the determinant (which corresponds to the basic locus). So our

[13] As pointed out above, we have to work with the Hilbert–Siegel setup as opposed to the usual Siegel setup because r_{spin} is a minuscule representation.

conjecture does not describe the basic locus of $\mathrm{Sh}_{a_\bullet}$, and it is indeed not of half dimension of $\mathrm{Sh}_{a_\bullet}$.

Yet, this basic locus may still have a good description as the union of some fiber bundles over the special fibers of some other Shimura varieties for reductive groups which are *not* quasisplit at p. For example, the supersingular locus of modular curve is related to the Shimura variety associated to the definite quaternion algebra which is ramified at p, by a theorem of Serre and Deuring [Serre 1996]. More such examples are given in [Tian and Xiao 2016] and [Vollaard and Wedhorn 2011].

2.15. ***Known cases of Conjecture 2.12.*** Conjecture 2.12 is largely inspired by the work of Tian and Xiao [2014; 2016], where we proved the analogous conjecture for the special fibers of the Hilbert modular varieties assuming that p is inert in the totally real field.

Another strong evidence of Conjecture 2.12 is the work of Vollaard and Wedhorn [2011], where they considered certain stratification of the supersingular locus of the Shimura variety for $GU(1, n-1)$ with $s \in \mathbb{N}$ at an *inert* prime p. What concerns us is the case when $n-1$ is even. In this case, it is hidden in the writing of their Section 6 that one gets a correspondence (in the notation of *loc. cit.*)

$$
\begin{tikzcd}[column sep=small]
 & I(\mathbb{Q}) \backslash N_n \times^{C_p^{(n)}} J(\mathbb{Q}_p) \times \mathbb{G}(\mathbb{A}_f^{(p)})/C^p \arrow[dl] \arrow[dr] & \\
I(\mathbb{Q}) \backslash I(\mathbb{A}_f)/C^p C_p^{(n)} & & \mathcal{M}^{ss}_{C^p} \subset \mathcal{M}_{C^p}.
\end{tikzcd}
\tag{2.15.1}
$$

Note that $I(\mathbb{A}_f) \simeq \mathbb{G}(\mathbb{A}_f)$. Here N_n is a certain Deligne–Lusztig variety. In [Vollaard and Wedhorn 2011], the parameterizing space, namely the first term in (2.15.1), is interpreted very differently, in terms of Bruhat–Tits building. The method of this paper should be applicable to their situation to verify the analogous Conjecture 2.12. In fact, in their case, there will be only one collection of cycles as given by (2.15.1), but the computation of the intersection matrix (only essentially one entry in this case) of them requires some nontrivial Schubert calculus similar to Section 5.

When $n-1$ is odd, the result of [Vollaard and Wedhorn 2011] is related to the degenerate version of the Conjecture 2.12 in the sense of Remark 2.14(4).

The aim of the rest of the paper is to provide evidence for Conjecture 2.12 for some large rank groups. In particular, we will construct cycles in the case of the unitary group $G(U(r, s) \times U(s, r))$ with $s, r \in \mathbb{N}$ (Section 7). While we expect these cycles to verify Conjecture 2.12, we do not know how to compute the "intersection matrix" in general. Nonetheless, when $r = 1$, we are able to make the computation and prove Conjecture 2.12 (with trivial coefficients for the sake of a simple presentation) in this case; see Section 4–6. We point out that our method should be applicable to many other examples, and even in general reduce Conjecture 2.12 to a question of a combinatorial nature. This combinatorics problem is the heart of the question. In the Hilbert case [Tian and Xiao 2014], we model the combinatorics question by the so-called periodic semimeander (for GL_2). The generalization of the usual (as opposed to periodic) semimeander to other groups has been introduced; see [Fontaine et al. 2013] for the corresponding references. The straightforward generalization to the periodic case does seem to agree with some of our computations with small groups. Nonetheless, the corresponding Gram determinant formula seems to be extremely difficult. Even in the nonperiodic case, we only know it for a special case; see [Di Francesco 1997].

We also mention that in a very recent work [Xiao and Zhu 2017] of Zhu and the last author, we relate Conjecture 2.12 with the geometric Satake theory of Zhu [2017] in mixed characteristic, and we proved many new cases of Conjecture 2.12.

3. Preliminaries on Dieudonné modules and deformation theory

We first introduce the basic tools that we will use in this paper.

3.1. ***Notation.*** Recall that we have an isomorphism

$$\mathcal{O}_D\otimes_{\mathbb{Z}}\mathbb{Z}_{p^f}\cong\bigoplus_{i=1}^{f}(\mathcal{O}_D\otimes_{\mathcal{O}_E,q_i}\mathbb{Z}_{p^f}\oplus\mathcal{O}_D\otimes_{\mathcal{O}_E,\bar{q}_i}\mathbb{Z}_{p^f})\simeq\bigoplus_{i=1}^{f}(\mathrm{M}_n(\mathbb{Z}_{p^f})\oplus\mathrm{M}_n(\mathbb{Z}_{p^f})).$$

Let S be a locally noetherian $\mathbb{Z}_{p^f}$-scheme. An $\mathcal{O}_D\otimes_{\mathbb{Z}}\mathcal{O}_S$-module M admits a canonical decomposition

$$M=\bigoplus_{i=1}^{f}(M_{q_i}\oplus M_{\bar{q}_i}),$$

where M_{q_i} (resp. $M_{\bar{q}_i}$) is the direct summand of M on which $\mathcal{O}_E$ acts via q_i (resp. via $\bar{q}_i$). Then each M_{q_i} has a natural action by $\mathrm{M}_n(\mathcal{O}_S)$. Let $\mathfrak{e}$ denote the element of $\mathrm{M}_n(\mathcal{O}_S)$ whose $(1,1)$-entry is 1 and whose other entries are 0. We put $M_i^\circ:=\mathfrak{e}M_{q_i}$, and call it the *reduced part* of M_{q_i}.

Let A be an fn^2-dimensional abelian variety over an $\mathbb{F}_{p^f}$-scheme S, equipped with an $\mathcal{O}_D$-action. The de Rham homology $H_1^{\mathrm{dR}}(A/S)$ has a Hodge filtration

$$0\to\omega_{A^\vee/S}\to H_1^{\mathrm{dR}}(A/S)\to\mathrm{Lie}_{A/S}\to 0,$$

compatible with the natural action of $\mathcal{O}_D\otimes_{\mathbb{Z}}\mathcal{O}_S$ on $H_1^{\mathrm{dR}}(A/S)$. We call $H_1^{\mathrm{dR}}(A/S)_i^\circ$ (resp. $\omega^\circ_{A^\vee/S,i}$, $\mathrm{Lie}^\circ_{A/S,i}$) the *reduced de Rham homology* of A/S (resp. the *reduced invariant* 1*-forms* of $A^\vee/S$, the *reduced Lie algebra* of A/S) at q_i. In particular, the former is a locally free $\mathcal{O}_S$-module of rank n and the latter is a subbundle[14] of the former; when $A \to S$ satisfies the moduli problem in Section 2.3, $\omega^\circ_{A^\vee/S,i}$ is locally free of rank a_i.

The Frobenius morphism $A \to A^{(p)}$ induces a natural homomorphism

$$V : H_1^{\mathrm{dR}}(A/S)^\circ_i \to H_1^{\mathrm{dR}}(A/S)^{\circ,(p)}_{i-1},$$

where the index i is considered as an element of $\mathbb{Z}/f\mathbb{Z}$, and the superscript "(p)" means the pullback via the absolute Frobenius of S. The image of V is exactly $\omega^{\circ,(p)}_{A^\vee/S,i-1}$. Similarly, the Verschiebung morphism $A^{(p)} \to A$ induces a natural homomorphism[15]

$$F : H_1^{\mathrm{dR}}(A/S)^{\circ,(p)}_{i-1} \to H_1^{\mathrm{dR}}(A/S)^\circ_i.$$

We have $\mathrm{Ker}(F) = \mathrm{Im}(V)$ and $\mathrm{Ker}(V) = \mathrm{Im}(F)$.

When $S = \mathrm{Spec}(k)$ with k a perfect field containing $\mathbb{F}_{p^f}$, let $W(k)$ denote the ring of Witt vectors in k. Let $\tilde{\mathcal{D}}(A)$ denote the (covariant) Dieudonné module associated to the p-divisible group of A. This is a free $W(k)$-module of rank $2fn^2$ equipped with a Frob-linear action of F and a Frob^{-1}-linear action of V such that $FV = VF = p$. The $\mathcal{O}_D$-action on A induces a natural action of $\mathcal{O}_D$ on $\tilde{\mathcal{D}}(A)$ that commutes with F and V. Moreover, there is a canonical isomorphism $\tilde{\mathcal{D}}(A)/p\tilde{\mathcal{D}}(A) \cong H_1^{\mathrm{dR}}(A/k)$ compatible with all structures on both sides. For each $i \in \mathbb{Z}/f\mathbb{Z}$, we have the reduced part $\tilde{\mathcal{D}}(A)^\circ_i := \mathfrak{e}\tilde{\mathcal{D}}(A)_{q_i}$. The Verschiebung and the Frobenius induce natural maps

$$V : \tilde{\mathcal{D}}(A)^\circ_i \to \tilde{\mathcal{D}}(A)^\circ_{i-1}, \quad F : \tilde{\mathcal{D}}(A)^\circ_i \to \tilde{\mathcal{D}}(A)^\circ_{i+1}.$$

Note that $\tilde{\mathcal{D}}(A)_{q_i} = (\tilde{\mathcal{D}}(A)^\circ_i)^{\oplus n}$, and $\bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} \tilde{\mathcal{D}}(A)_{q_i}$ is the covariant Dieudonné module of the p-divisible group $A[\mathfrak{p}^\infty]$.

For any fn^2-dimensional abelian variety A' over k equipped with an $\mathcal{O}_D$-action, an $\mathcal{O}_D$-equivariant isogeny $A' \to A$ induces a morphism $\tilde{\mathcal{D}}(A')^\circ_i \to \tilde{\mathcal{D}}(A)^\circ_i$ compatible with the actions of F and V. Conversely, we have the following.

Proposition 3.2. *Let A be an abelian variety of dimension fn^2 over prefect field k which contains $\mathbb{F}_{p^f}$, equipped with an $\mathcal{O}_D$-action and an $\mathcal{O}_D$-compatible prime-to-p polarization λ. Suppose given an integer $m \geq 1$ and a $W(k)$-submodule $\tilde{\mathcal{E}}_i \subseteq \tilde{\mathcal{D}}(A)^\circ_i$ for each $i \in \mathbb{Z}/f\mathbb{Z}$ such that*

$$p^m\tilde{\mathcal{D}}(A)^\circ_i \subseteq \tilde{\mathcal{E}}_i, \quad F(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{i+1}, \quad \textit{and} \quad V(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{i-1}. \tag{3.2.1}$$

[14] Here and after, by a subbundle of a locally free coherent sheaf, we mean a locally free coherent sheaf that is Zariski locally a direct factor.

[15] The notation F for Frobenius was also used to denote the real quadratic field. But we think the chance for confusion is minimal.

Then there exists a unique abelian variety A' over k (depending on m) equipped with an $\mathcal{O}_D$-action, a prime-to-p polarization λ', and an $\mathcal{O}_D$-equivariant p-isogeny $\phi: A' \to A$ such that the natural inclusion $\tilde{\mathcal{E}}_i \subseteq \tilde{\mathcal{D}}(A)^\circ_i$ is naturally identified with the map $\phi_{*,i}: \tilde{\mathcal{D}}(A')^\circ_i \to \tilde{\mathcal{D}}(A)^\circ_i$ induced by ϕ and such that $\phi^\vee \circ \lambda \circ \phi = p^m\lambda'$. Moreover, we have

(1) *If* $\dim \omega^\circ_{A^\vee/k,i} = a_i$ *and* $\mathrm{length}_{W(k)}(\tilde{\mathcal{D}}(A)^\circ_i/\tilde{\mathcal{E}}_i) = \ell_i$ *for $i \in \mathbb{Z}/f\mathbb{Z}$, then*

$$\dim \omega^\circ_{A'^\vee/k,i} = a_i + \ell_i - \ell_{i+1}. \tag{3.2.2}$$

(2) *If A is equipped with a prime-to-p level structure η in the sense of Section 2.3(1), then there exists a unique prime-to-p level structure η' on A' such that $\eta = \phi \circ \eta'$.*

Proof. By Dieudonné theory, the Dieudonné submodule

$$\bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} (\tilde{\mathcal{E}}_i/p^m\tilde{\mathcal{D}}(A)^\circ_i)^{\oplus n} \subseteq \bigoplus_{i\in\mathbb{Z}/f\mathbb{Z}} (\tilde{\mathcal{D}}(A)^\circ_i/p^m\tilde{\mathcal{D}}(A)^\circ_i)^{\oplus n}$$

corresponds to a closed subgroup scheme $H_{\mathfrak{p}} \subseteq A[\mathfrak{p}^m]$. The prime-to-$p$ polarization λ induces a perfect pairing

$$\langle -, -\rangle_\lambda : A[\mathfrak{p}^m] \times A[\bar{\mathfrak{p}}^m] \to \mu_{p^m}.$$

Let $H_{\bar{\mathfrak{p}}} = H_{\mathfrak{p}}^\perp \subseteq A[\bar{\mathfrak{p}}^m]$ denote the orthogonal complement of $H_{\mathfrak{p}}$. Put $H_p = H_{\mathfrak{p}} \oplus H_{\bar{\mathfrak{p}}}$. Let $\psi: A \to A'$ be the canonical quotient with kernel H_p, and $\phi: A' \to A$ be the quotient with kernel $\psi(A[\mathfrak{p}^m])$ so that $\psi \circ \phi = p^m\,\mathrm{id}_{A'}$ and $\phi \circ \psi = p^m\,\mathrm{id}_A$. By construction, $H_p \subseteq A[p^m]$ is a maximal totally isotropic subgroup. By [Mumford 2008, §23, Theorem 2], there is a prime-to-p polarization λ' on A' such that $p^m\lambda = \psi^\vee \circ \lambda' \circ \psi$. It follows also that $p^m\lambda' = \phi^\vee \circ \lambda \circ \phi$. The fact that $\phi_{*,i}: \tilde{\mathcal{D}}(A')^\circ_i \to \tilde{\mathcal{D}}(A)^\circ_i$ is identified with the natural inclusion $\tilde{\mathcal{E}}_i \subseteq \tilde{\mathcal{D}}(A)^\circ_i$ follows from the construction. The existence and uniqueness of the tame level structure is clear. The dimension of the differential forms can be computed as follows:

$$\begin{aligned}\dim_k \omega^\circ_{A'^\vee/k,i} &= \dim_k \frac{V(\tilde{\mathcal{D}}(A')^\circ_{i+1})}{p\tilde{\mathcal{D}}(A')^\circ_i} = \dim_k \frac{V(\tilde{\mathcal{E}}_{i+1})}{p\tilde{\mathcal{E}}_i}\\ &= \dim_k \frac{V(\tilde{\mathcal{D}}(A)^\circ_{i+1})}{p\tilde{\mathcal{D}}(A)^\circ_i} - \mathrm{length}_{W(k)} \frac{V(\tilde{\mathcal{D}}(A)^\circ_{i+1})}{V(\tilde{\mathcal{E}}_{i+1})} + \mathrm{length}_{W(k)} \frac{p\tilde{\mathcal{D}}(A)^\circ_i}{p\tilde{\mathcal{E}}_i}\\ &= a_i - \ell_{i+1} + \ell_i.\end{aligned}$$

□

3.3. ***Deformation theory.*** We shall frequently use Grothendieck–Messing deformation theory to compare the tangent spaces of moduli spaces. We make this explicit in our setup.

Let $\hat{R}$ be a noetherian $\mathbb{F}_{p^f}$-algebra and $\hat{I} \subset \hat{R}$ an ideal such that $\hat{I}^2 = 0$. Put $R = \hat{R}/\hat{I}$. Let $\mathscr{C}_{\hat{R}}$ denote the category of tuples $(\hat{A}, \hat{\lambda}, \hat{\eta})$, where $\hat{A}$ is an fn^2-dimensional abelian variety over $\hat{R}$ equipped with an $\mathcal{O}_D$-action, $\hat{\lambda}$ is a polarization on $\hat{A}$ such that the Rosati involution induces the $*$-involution on $\mathcal{O}_D$, and $\hat{\eta}$ is a level structure as in Section 2.3(3). We define $\mathscr{C}_R$ in the same way. For an object (A, λ, η) in the category $\mathscr{C}_R$, let $H_1^{\mathrm{cris}}(A/\hat{R})$ be the evaluation of the first relative crystalline homology (i.e., dual crystal of the first crystalline cohomology) of A/R at the divided power thickening $\hat{R} \to R$, and $H_1^{\mathrm{cris}}(A/\hat{R})_i^\circ := \mathfrak{e}H_1^{\mathrm{cris}}(A/\hat{R})_{q_i}$ be the i-th reduced part. We denote by $\mathfrak{Def}(R, \hat{R})$ the category of tuples $(A, \lambda, \eta, (\hat{\omega}_i^\circ)_{i=1,\dots,f})$, where (A, λ, η) is an object in $\mathscr{C}_R$, and $\hat{\omega}_i^\circ \subseteq H_1^{\mathrm{cris}}(A/\hat{R})_i^\circ$ for each $i \in \mathbb{Z}/f\mathbb{Z}$ is a subbundle that lifts $\omega_{A^\vee/R,i}^\circ \subseteq H_1^{\mathrm{dR}}(A/R)_i^\circ$. The following is a combination of Serre–Tate and Grothendieck–Messing deformation theory.

Theorem 3.4 (Serre–Tate, Grothendieck–Messing). *The functor*

$$(\hat{A}, \hat{\lambda}, \hat{\eta}) \mapsto (\hat{A} \otimes_{\hat{R}} R, \lambda, \eta, \omega_{\hat{A}^\vee/\hat{R},i}^\circ),$$

where λ and η are the natural induced polarization and level structure on $\hat{A} \otimes_{\hat{R}} R$, is an equivalence of categories between $\mathscr{C}_{\hat{R}}$ and $\mathfrak{Def}(R, \hat{R})$.

Proof. The main theorem of the crystalline deformation theory (cf., [Grothendieck 1974, pp. 116–118], [Mazur and Messing 1974, Chapter II §1]) says that the category $\mathscr{C}_{\hat{R}}$ is equivalent to the category of objects (A, λ, η) in $\mathscr{C}_R$ together with a lift of $\omega_{A^\vee/R} \subseteq H_1^{\mathrm{cris}}(A/R)$ to a subbundle $\hat{\omega}$ of $H_1^{\mathrm{cris}}(A/\hat{R})$, such that $\hat{\omega}$ is stable under the induced $\mathcal{O}_D$-action and is isotropic for the pairing on $H_1^{\mathrm{cris}}(A/\hat{R})$ induced by the polarization λ. But the additional information $\hat{\omega}$ is clearly equivalent to the subbundles $\hat{\omega}_i^\circ \subseteq H_1^{\mathrm{cris}}(A/\hat{R})_i^\circ$ lifting $\omega_{A^\vee/R,i}^\circ$. □

Corollary 3.5. *If $\mathcal{A}_{a_\bullet}$ denotes the universal abelian variety over $\mathrm{Sh}_{a_\bullet}$, then the tangent space $T_{\mathrm{Sh}_{a_\bullet}}$ of $\mathrm{Sh}_{a_\bullet}$ is*

$$\bigoplus_{i=1}^{f} \mathrm{Lie}^\circ_{\mathcal{A}^\vee_{a_\bullet}/\mathrm{Sh}_{a_\bullet},i} \otimes \mathrm{Lie}^\circ_{\mathcal{A}_{a_\bullet}/\mathrm{Sh}_{a_\bullet},i}\,.$$

Proof. Even though this is a well-known statement often referred to as the Kodaira–Spencer isomorphism (e.g., [Lan 2013, Proposition 2.3.4.2]), we include a short proof, as the proof serves as a toy model of many arguments later. Let $\hat{R}$ be a noetherian $\mathbb{F}_{p^f}$-algebra and $\hat{I} \subset \hat{R}$ an ideal such that $\hat{I}^2 = 0$; put $R = \hat{R}/\hat{I}$. By Theorem 3.4, to lift an R-point (A, λ, η) of $\mathrm{Sh}_{a_\bullet}$ to an $\hat{R}$-point, it suffices to lift, for $i = 1, \dots, f$, the differentials $\omega_{A^\vee,i}^\circ \subseteq H_1^{\mathrm{cris}}(A/R)_i^\circ$ to a subbundle $\hat{\omega}_i \subseteq H_1^{\mathrm{cris}}(A/\hat{R})_i^\circ$. Such lifts form a torsor for the group

$$\mathrm{Hom}_R(\omega_{A^\vee/R,i}^\circ, \mathrm{Lie}_{A/R,i}^\circ) \otimes_R \hat{I}.$$

It follows from this

$$T_{\mathrm{Sh}_{a_\bullet}} \cong \bigoplus_{i=1}^{f} \mathcal{H}om(\omega^\circ_{\mathcal{A}^\vee_{a_\bullet}/\mathrm{Sh}_{a_\bullet},i}, \mathrm{Lie}^\circ_{\mathcal{A}_{a_\bullet}/\mathrm{Sh}_{a_\bullet},i}) \cong \bigoplus_{i=1}^{f} \mathrm{Lie}^\circ_{\mathcal{A}^\vee_{a_\bullet}/\mathrm{Sh}_{a_\bullet},i} \otimes \mathrm{Lie}^\circ_{\mathcal{A}_{a_\bullet}/\mathrm{Sh}_{a_\bullet},i}\,.$$

Note that this proof also shows that $\mathrm{Sh}_{a_\bullet}$ is smooth. □

3.6. ***Notation in the real quadratic case.*** For the rest of the paper, we assume $f = 2$ so that F is a real quadratic field in which p is inert. For nonnegative integers $r \le s$ such that $n = r + s$, we denote by $G_{r,s}$ the algebraic group previously denoted by $G_{a_\bullet}$ with $a_1 = r$ and $a_2 = s$; in particular, $G_{r,s}(\mathbb{R}) = G(U(r, s) \times U(s, r))$. If r', s' is another pair of nonnegative integers such that $n = r' + s'$ and $r' \le s'$, Lemma 2.9 gives an isomorphism $G_{r,s}(\mathbb{A}^\infty) \simeq G_{r',s'}(\mathbb{A}^\infty)$.

Let $\mathcal{S}h_{r,s}$ be the Shimura variety over $\mathbb{Z}_{p^2}$ attached to $G_{r,s}$ defined in Section 2.3 of some fixed sufficiently small prime-to-p level $K^p \subseteq G_{r,s}(\mathbb{A}^{\infty,p})$. Let $\mathrm{Sh}_{r,s}$ denote its special fiber over $\mathbb{F}_{p^2}$. Let $\mathcal{A} = \mathcal{A}_{r,s}$ denote the universal abelian variety over $\mathrm{Sh}_{r,s}$. It is a $2n^2$-dimensional abelian variety, equipped with an action of $\mathcal{O}_D$ and a prime-to-p polarization $\lambda_{\mathcal{A}}$. Moreover, $\omega^\circ_{\mathcal{A}^\vee/\mathrm{Sh}_{r,s},1}$ (resp. $\omega^\circ_{\mathcal{A}^\vee/\mathrm{Sh}_{r,s},2}$) is a locally free module over $\mathrm{Sh}_{r,s}$ of rank r (resp. rank s).

Remark 3.7. When $r = 0$ and $s = n$, the universal abelian variety $\mathcal{A} = \mathcal{A}_{0,n}$ over $\mathrm{Sh}_{0,n}$ is supersingular. Indeed, for each $\bar{\mathbb{F}}_p$-point z of $\mathrm{Sh}_{0,n}$, the Kottwitz condition implies that the Frobenius induces *isomorphisms*

$$\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_1 \xrightarrow{F} \tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_2 \xrightarrow{F} p\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_1.$$

In particular, $(1/p)F^2$ induces a σ^2-linear automorphism of $\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_1$. By Hilbert's Theorem 90, there exists a $\mathbb{Z}_{p^2}$-lattice $\mathbb{L}$ of $\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_1$ that is invariant under the action of $(1/p)F^2$; in other words, F^2 acts by multiplication by p for a basis chosen from this lattice. It follows that all slopes of the Frobenius on $\tilde{\mathcal{D}}(\mathcal{A}_z)$ are $\frac{1}{2}$, and hence $\mathcal{A}_z$ is supersingular.

4. The case of $G(U(1, n-1) \times U(n-1, 1))$

We will verify Conjecture 2.12 for $\mathrm{Sh}_{1,n-1}$, namely the existence of some cycles Y_j having morphisms to both $\mathrm{Sh}_{0,n}$ and $\mathrm{Sh}_{1,n-1}$ and generating Tate classes of $\mathrm{Sh}_{1,n-1}$ under a certain genericity hypothesis on the Satake parameters. We always fix an isomorphism $G_{1,n-1}(\mathbb{A}^\infty) \simeq G_{0,n}(\mathbb{A}^\infty)$, and write $G(\mathbb{A}^\infty)$ for either group.

Notation 4.1. For a smooth variety X over $\mathbb{F}_{p^2}$, we denote by T_X the tangent bundle of X, and for a locally free $\mathcal{O}_X$-module M, we put $M^* = \mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X)$.

4.2. ***Cycles on*** $\mathbf{Sh_{1,n-1}}$**.** For each integer j with $1 \le j \le n$, we first define the variety Y_j we briefly mentioned in the introduction. Let Y_j be the moduli space over $\mathbb{F}_{p^2}$ that associates to each locally noetherian $\mathbb{F}_{p^2}$-scheme S, the set of isomorphism classes of tuples $(A, \lambda, \eta, B, \lambda', \eta', \phi)$, where

- (A, λ, η) is an S-point of $\mathrm{Sh}_{1,n-1}$,
- (B, λ', η') is an S-point of $\mathrm{Sh}_{0,n}$ and
- $\phi : B \to A$ is an $\mathcal{O}_D$-equivariant isogeny whose kernel is contained in $B[p]$,

such that

- $p\lambda' = \phi^\vee \circ \lambda \circ \phi$,
- $\phi \circ \eta' = \eta$ and
- the cokernels of the maps

$$\phi_{*,1} : H_1^{\mathrm{dR}}(B/S)_1^\circ \to H_1^{\mathrm{dR}}(A/S)_1^\circ \quad \text{and} \quad \phi_{*,2} : H_1^{\mathrm{dR}}(B/S)_2^\circ \to H_1^{\mathrm{dR}}(A/S)_2^\circ$$

are locally free $\mathcal{O}_S$-modules of rank $j-1$ and j, respectively.

There is a unique isogeny $\psi : A \to B$ such that $\psi \circ \phi = p \cdot \mathrm{id}_B$ and $\phi \circ \psi = p \cdot \mathrm{id}_A$. We have

$$\mathrm{Ker}(\phi_{*,i}) = \mathrm{Im}(\psi_{*,i}) \quad \text{and} \quad \mathrm{Ker}(\phi_{*,i}) = \mathrm{Im}(\psi_{*,i}),$$

where $\psi_{*,i}$ for $i = 1, 2$ is the induced homomorphism on the reduced de Rham homology in the evident sense. This moduli space Y_j is represented by a scheme of finite type over $\mathbb{F}_{p^2}$. We have a natural diagram of morphisms:

$$\begin{tikzcd} & Y_j \arrow[dl, "\mathrm{pr}_j"'] \arrow[dr, "\mathrm{pr}'_j"] & \\ \mathrm{Sh}_{1,n-1} & & \mathrm{Sh}_{0,n}, \end{tikzcd} \tag{4.2.1}$$

where pr_j and pr'_j send a tuple $(A, \lambda, \eta, B, \lambda', \eta', \phi)$ to (A, λ, η) and to (B, λ', η'), respectively. Letting K^p vary, we see easily that both pr_j and pr'_j are equivariant under prime-to-p Hecke actions given by the double cosets $K^p \backslash G(\mathbb{A}^{\infty,p})/K^p$.

4.3. ***Some auxiliary moduli spaces.*** The moduli problem for Y_j is slightly complicated. We will introduce a more explicit moduli space Y'_j below and then show they are isomorphic.

Consider the functor $\underline{Y}'_j$ which associates to each locally noetherian $\mathbb{F}_{p^2}$-scheme S the set of isomorphism classes of tuples $(B, \lambda', \eta', H_1, H_2)$, where

- (B, λ', η') is an S-valued point of $\mathrm{Sh}_{0,n}$;
- $H_1 \subset H_1^{\mathrm{dR}}(B/S)_1^\circ$ and $H_2 \subset H_1^{\mathrm{dR}}(B/S)_2^\circ$ are $\mathcal{O}_S$-subbundles of rank j and $j-1$ respectively such that

$$V^{-1}(H_2^{(p)}) \subseteq H_1, \quad H_2 \subseteq F(H_1^{(p)}). \tag{4.3.1}$$

Here,

$$F : H_1^{\mathrm{dR}}(B/S)_1^{\circ,(p)} \xrightarrow{\sim} H_1^{\mathrm{dR}}(B/S)_2^\circ \quad \text{and} \quad V : H_1^{\mathrm{dR}}(B/S)_1^\circ \xrightarrow{\sim} H_1^{\mathrm{dR}}(B/S)_2^{\circ,(p)}$$

are respectively the Frobenius and Verschiebung homomorphisms, which are actually isomorphisms because of the signature condition on $\mathrm{Sh}_{0,n}$.

It follows from the moduli problem that the quotients $H_1/V^{-1}(H_2^{(p)})$, $F(H_1^{(p)})/H_2$ are both locally free $\mathcal{O}_S$-modules of rank one.

There is a natural projection $\pi'_j : \underline{Y}'_j \to \mathrm{Sh}_{0,n}$ given by $(B, \lambda', \eta', H_1, H_2) \mapsto (B, \lambda', \eta')$.

Proposition 4.4. *The functor $\underline{Y}'_j$ is representable by a scheme Y'_j smooth and projective over $\mathrm{Sh}_{0,n}$ of dimension $n-1$. Moreover, if $(\mathcal{B}, \lambda', \eta', \mathcal{H}_1, \mathcal{H}_2)$ denotes the universal object over Y'_j, then the tangent bundle of Y'_j is*

$$T_{Y'_j} \cong \big((\mathcal{H}_1/V^{-1}(\mathcal{H}_2^{(p)}))^* \otimes \big(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_1^\circ/\mathcal{H}_1\big) \oplus \big(\mathcal{H}_2^* \otimes F(\mathcal{H}_1^{(p)})/\mathcal{H}_2\big).$$

Proof. For each integer m with $0 \le m \le n$ and $i = 1, 2$, let $\boldsymbol{Gr}(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_i^\circ, m)$ be the Grassmannian scheme over $\mathrm{Sh}_{0,n}$ that parametrizes subbundles of the universal de Rham homology $H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_i^\circ$ of rank m. Then $\underline{Y}'_j$ is a closed subfunctor of the product of the Grassmannian schemes

$$\boldsymbol{Gr}(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_1^\circ, j) \times \boldsymbol{Gr}(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ, j-1).$$

The representability of $\underline{Y}'_j$ follows. Moreover, Y'_j is projective.

We show now that the structural map $\pi'_j : Y'_j \to \mathrm{Sh}_{0,n}$ is smooth of relative dimension $n-1$. Let $S_0 \hookrightarrow S$ be an immersion of locally noetherian $\mathbb{F}_{p^2}$-schemes with ideal sheaf I satisfying $I^2 = 0$. Suppose we are given a commutative diagram

$$\begin{tikzcd} S_0 \arrow[r, "g_0"] \arrow[d] & Y'_j \arrow[d, "\pi'_j"] \\ S \arrow[r, "h"] \arrow[ur, dashed, "g"] & \mathrm{Sh}_{0,n} \end{tikzcd}$$

with solid arrows. We have to show that, locally for the Zariski topology on S_0, there is a morphism $g : S \to Y'_j$ making the diagram commute. Let B be the abelian scheme over S given by h, and B_0 be the base change to S_0. The morphism g_0 gives rises to subbundles $\overline{H}_1 \subset H_1^{\mathrm{dR}}(B_0/S_0)_1^\circ$ and $\overline{H}_2 \subset H_1^{\mathrm{dR}}(B_0/S_0)_2^\circ$ with

$$F(\overline{H}_1^{(p)}) \supset \overline{H}_2, \quad V^{-1}(\overline{H}_2^{(p)}) \subset \overline{H}_1.$$

Finding g is equivalent to finding a subbundle $H_i \subset H_1^{\mathrm{dR}}(B/S)_i^\circ$ which lifts each $\overline{H}_i$ for $i = 1, 2$ and satisfies (4.3.1); this is certainly possible when passing to small enough affine open subsets of S_0. Thus $\pi'_j : Y'_j \to \mathrm{Sh}_{0,n}$ is formally smooth, and hence smooth. We note that $F_S^* : \mathcal{O}_S \to \mathcal{O}_S$ factors through $\mathcal{O}_{S_0}$. Hence $V^{-1}(H_2^{(p)})$ and $F(H_1^{(p)})$ actually depend only on $\overline{H}_1$, $\overline{H}_2$, but not on the lifts H_1 and H_2. Therefore, the possible lifts H_2 form a torsor under the group

$$\mathcal{H}om_{\mathcal{O}_{S_0}}(\overline{H}_2, F(\overline{H}_1^{(p)})/\overline{H}_2) \otimes_{\mathcal{O}_{S_0}} I,$$

and similarly the possible lifts H_1 form a torsor under the group

$$\mathcal{H}om_{\mathcal{O}_{S_0}}(\bar{H}_1/V^{-1}(\bar{H}_2^{(p)}),\, H_1^{\mathrm{dR}}(B_0/S_0)_1^\circ/\bar{H}_1) \otimes_{\mathcal{O}_{S_0}} I.$$

To compute the tangent bundle $T_{Y'_j}$, we take $S = \mathrm{Spec}(\mathcal{O}_{S_0}[\epsilon]/\epsilon^2)$ and $I = \epsilon\mathcal{O}_S$. The morphism $g_0 : S_0 \to Y'_j$ corresponds to an S_0-valued point of Y'_j, say y_0. Then the possible liftings g form the tangent space $T_{Y'_j}$ at y_0, denote by $T_{Y'_j,y_0}$. The discussion above shows that

$$T_{Y'_j,y_0} \cong \mathcal{H}om_{\mathcal{O}_{S_0}}(\bar{H}_2, F(\bar{H}_1^{(p)})/\bar{H}_2) \oplus \mathcal{H}om_{\mathcal{O}_{S_0}}(\bar{H}_1/V^{-1}(\bar{H}_2^{(p)}), H_1^{\mathrm{dR}}(B_0/S_0)_1^\circ/\bar{H}_1),$$

which is certainly a vector bundle over S_0 of rank $j-1+(n-j)=n-1$. Applying this to the universal case when $g_0 : S_0 \to Y'_j$ is the identity morphism, the formula of the tangent bundle follows. □

Remark 4.5. Let $(B, \lambda', \eta', H_1, H_2)$ be an S-point of Y'_j.

(a) If $j = n$, H_1 has to be $H_1^{\mathrm{dR}}(B/S)_1^\circ$, and H_2 is a hyperplane of $H_1^{\mathrm{dR}}(B/S)_2^\circ$. Condition (4.3.1) is trivial. In this case, Y'_n is the projective space over $\mathrm{Sh}_{0,n}$ associated to $H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ$, where $\mathcal{B}$ is the universal abelian scheme over $\mathrm{Sh}_{0,n}$. So it is geometrically a union of copies of $\mathbb{P}^{n-1}_{\bar{\mathbb{F}}_p}$.

(b) If $j = 1$, then H_1 is a line in $H_1^{\mathrm{dR}}(B/S)_1^\circ$ and $H_2 = 0$. So Y'_1 is the projective space over $\mathrm{Sh}_{0,n}$ associated to $(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_1^\circ)^*$.

(c) If $j = 2$, $H_2 \subseteq H_1^{\mathrm{dR}}(B/S)_2^\circ$ is a line, and $H_1 \subseteq H_1^{\mathrm{dR}}(B/S)_1^\circ$ is a subbundle of rank 2 such that $F(H_1^{(p)})$ contains both H_2 and $F(V^{-1}(H_2^{(p)})^{(p)})$. Therefore, if $H_2 \neq F(V^{-1}(H_2^{(p)}))^{(p)}$, H_1 is determined up to Frobenius pullback. If $H_2 = F(V^{-1}(H_2^{(p)})^{(p)})$, then H_1 could be any rank 2 subbundle containing $V^{-1}(H_2^{(p)})$.

We fix a geometric point $z = (B, \lambda', \eta') \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$. It is possible to find good bases for $H_1^{\mathrm{dR}}(B/\bar{\mathbb{F}}_p)_1^\circ$, $H_1^{\mathrm{dR}}(B/\bar{\mathbb{F}}_p)_2^\circ$ such that $F, V : H_1^{\mathrm{dR}}(B/\bar{\mathbb{F}}_p)_1^\circ \to H_1^{\mathrm{dR}}(B/\bar{\mathbb{F}}_p)_2^\circ$ are both given by the identity matrix. With these choices, we may identify the fiber $Y'_{2,z} = \pi_2'^{-1}(z)$ with a closed subvariety of

$$\boldsymbol{Gr}(\bar{\mathbb{F}}_p^n, 2) \times \boldsymbol{Gr}(\bar{\mathbb{F}}_p^n, 1).$$

Moreover, one may equip $\boldsymbol{Gr}(\bar{\mathbb{F}}_p^n, 1) \cong \mathbb{P}^{n-1}_{\bar{\mathbb{F}}_p}$ with an $\mathbb{F}_{p^2}$-rational structure such that $H_2 = F(V^{-1}(H_2^{(p)})^{(p)})$ if and only if $[H_2] \in \mathbb{P}^{n-1}_{\bar{\mathbb{F}}_p}$ is an $\mathbb{F}_{p^2}$-rational point. So $Y'_{2,z}$ is isomorphic to a "Frobenius twisted" blow-up of $\mathbb{P}^{n-1}_{\bar{\mathbb{F}}_p}$ at all of its $\mathbb{F}_{p^2}$-rational points. Here, "Frobenius twisted" means that each irreducible component of the exceptional divisor has multiplicity p. For instance, when $n = 3$, each $Y_{2,z}$ is isomorphic to the closed subscheme of $\mathbb{P}^2_{\bar{\mathbb{F}}_p} \times \mathbb{P}^2_{\bar{\mathbb{F}}_p}$ defined by

$$a_1 b_1^p + a_2 b_2^p + a_3 b_3^p = 0, \quad a_1^p b_1 + a_2^p b_2 + a_3^p b_3 = 0,$$

where $(a_1 : a_2 : a_3)$ and $(b_1 : b_2 : b_3)$ are the homogeneous coordinates on the two copies of $\mathbb{P}^2$.

Lemma 4.6. *Let $(A,\lambda,\eta,B,\lambda',\eta',\phi)$ be an S-point of Y_j. Then the image of $\phi_{*,1}$ contains both $\omega^\circ_{A^\vee/S,1}$ and $F(H_1^{\mathrm{dR}}(A/S)_2^{\circ,(p)})$, and the image of $\phi_{*,2}$ is contained in both $\omega^\circ_{A^\vee/S,2}$ and $F(H_1^{\mathrm{dR}}(A/S)_1^{\circ,(p)})$.*

Proof. By the functoriality, $\phi_{2,*}$ sends $\omega^\circ_{B^\vee/S,2}$ to $\omega^\circ_{A^\vee/S,2}$. Since $\omega^\circ_{B^\vee/S,2} = H_1^{\mathrm{dR}}(B/S)_2^\circ$ by the Kottwitz determinant condition, it follows that $\mathrm{Im}(\phi_{*,2})$ in contained in $\omega^\circ_{A^\vee/S,2}$. Similar arguments by considering $\psi_{*,1}$ shows that $\omega^\circ_{A^\vee/S,1} \subseteq \mathrm{Ker}(\psi_{*,1}) = \mathrm{Im}(\phi_{*,1})$. The fact that $\mathrm{Im}(\phi_{*,2})$ is contained in $F(H_1^{\mathrm{dR}}(A/S)_1^{\circ,(p)})$ follows from the commutative diagram

$$\begin{tikzcd} H_1^{\mathrm{dR}}(B/S)_1^{\circ,(p)} \arrow[r, "\phi_{*,1}^{(p)}"] \arrow[d, "F"', "\cong"] & H_1^{\mathrm{dR}}(A/S)_1^{\circ,(p)} \arrow[d, "F"] \\ H_1^{\mathrm{dR}}(B/S)_2^\circ \arrow[r, "\phi_{*,2}"] & H_1^{\mathrm{dR}}(A/S)_2^\circ \end{tikzcd} \tag{4.6.1}$$

and the fact that the left vertical arrow is an isomorphism. Similarly, the inclusion $F(H_1^{\mathrm{dR}}(A/S)_2^{\circ,(p)}) \subseteq \mathrm{Im}(\phi_{*,1}) = \mathrm{Ker}(\psi_{*,1})$ can be proved using the functoriality of Verschiebung homomorphisms. □

4.7. ***A morphism from Y_j to Y'_j.*** There is a natural morphism $\alpha : Y_j \to Y'_j$ for $1 \le j \le n$ defined as follows. For a locally noetherian $\mathbb{F}_{p^2}$-scheme S and an S-point $(A,\lambda,\eta,B,\lambda',\eta',\phi)$ of Y_j, we define

$$H_1 := \phi_{*,1}^{-1}(\omega^\circ_{A^\vee/S,1}) \subseteq H_1^{\mathrm{dR}}(B/S)_1^\circ, \quad H_2 := \psi_{*,2}(\omega^\circ_{A^\vee/S,2}) \subseteq H_1^{\mathrm{dR}}(B/S)_2^\circ. \tag{4.7.1}$$

In particular, H_1 and H_2 are $\mathcal{O}_S$-subbundles of rank j and $j-1$, respectively. Also, there is a canonical isomorphism $\omega^\circ_{A^\vee/S,2}/\mathrm{Im}(\phi_{*,2}) \xrightarrow{\sim} H_2$. From the commutative diagram (4.6.1), it is easy to see that $F(H_1^{(p)}) \subseteq \mathrm{Ker}(\phi_{*,2}) = \mathrm{Im}(\psi_{*,2})$, but comparing the rank forces this to be an equality. It follows that $H_2 \subseteq F(H_1^{(p)})$. Similarly, $V^{-1}(H_2^{(p)})$ is identified with $\mathrm{Im}(\psi_{*,1}) = \mathrm{Ker}(\phi_{*,1})$, hence $V^{-1}(H_2^{(p)}) \subseteq H_1$. From these, we deduce two canonical isomorphisms:

$$\begin{aligned} H_1/V^{-1}(H_2^{(p)}) &\xrightarrow{\sim} \omega^\circ_{A^\vee/S,1}, \\ F(H_1^{(p)})/H_2 &\xrightarrow{\sim} H_1^{\mathrm{dR}}(A/S)_2^\circ/\omega^\circ_{A^\vee/S,2} \cong \mathrm{Lie}^\circ_{A/S,2}. \end{aligned} \tag{4.7.2}$$

Therefore, we have a well-defined map $\alpha : Y_j \to Y'_j$ given by

$$\alpha : (A,\lambda,\eta,B,\lambda',\eta',\phi) \mapsto (B,\lambda',\eta',H_1,H_2).$$

Moreover, it is clear from the definition that $\pi'_j \circ \alpha = \mathrm{pr}'_j$.

Proposition 4.8. *The morphism α is an isomorphism.*

Proof. Let k be a perfect field containing $\mathbb{F}_{p^2}$. We first prove that α induces a bijection of points $\alpha : Y_j(k) \xrightarrow{\sim} Y'_j(k)$. It suffices to show that there exists a morphism of sets $\beta : Y'_j(k) \to Y_j(k)$ inverse to α. Let $y = (B, \lambda', \eta', H_1, H_2) \in Y'_j(k)$. We define $\beta(y) = (A, \lambda, \eta, B, \lambda', \eta', \phi)$ as follows. Let $\tilde{\mathcal{E}}_1 \subseteq \tilde{\mathcal{D}}(B)^\circ_1$ and $\tilde{\mathcal{E}}_2 \subseteq \tilde{\mathcal{D}}(B)^\circ_2$ be respectively the inverse images of $V^{-1}(H_2^{(p)}) \subseteq H_1^{\mathrm{dR}}(B/k)^\circ_1$ and $F(H_1^{(p)}) \subseteq H_1^{\mathrm{dR}}(B/k)^\circ_2$ under the natural reduction maps

$$\tilde{\mathcal{D}}(B)^\circ_i \to \tilde{\mathcal{D}}(B)^\circ_i / p\tilde{\mathcal{D}}(B)^\circ_i \cong H_1^{\mathrm{dR}}(B/k)^\circ_i \quad \text{for } i = 1, 2.$$

The condition (4.3.1) ensures that $F(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{3-i}$ and $V(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{3-i}$ for $i = 1, 2$. Applying Proposition 3.2 with $m = 1$, we get a triple (A, λ, η) and an $\mathcal{O}_D$-equivariant isogeny $\psi : A \to B$, where A is an abelian variety over k with an action of $\mathcal{O}_D$, λ is a prime-to-p polarization on A, and η is a prime-to-p level structure on A, such that $\psi^\vee \circ \lambda' \circ \psi = p\lambda$, $p\eta' = \psi \circ \eta$ and such that $\psi_{*,i} : \tilde{\mathcal{D}}(A)^\circ_i \to \tilde{\mathcal{D}}(B)^\circ_i$ is naturally identified with the inclusion $\tilde{\mathcal{E}}_i \hookrightarrow \tilde{\mathcal{D}}(B)^\circ_i$ for $i = 1, 2$. Moreover, the dimension formula (3.2.2) implies that $\omega^\circ_{A^\vee/k,1}$ has dimension 1, and $\omega^\circ_{A^\vee/k,2}$ has dimension $n-1$. Therefore, (A, λ, η) is a point of $\mathrm{Sh}_{1,n-1}$. Finally, we take $\phi : B \to A$ to be the unique isogeny such that $\phi \circ \psi = p \cdot \mathrm{id}_A$ and $\psi \circ \phi = p \cdot \mathrm{id}_B$. Thus we have $\phi \circ \eta' = \eta$. This finishes the construction of $\beta(y)$. It is direct to check that β is the set theoretic inverse to $\alpha : Y_j(k) \to Y'_j(k)$.

We show now that α induces an isomorphism on the tangent spaces at each closed point; as we have already shown that Y'_j is smooth, it will then follow that α is an isomorphism. Let $x = (A, \lambda, \eta, B, \lambda', \eta', \phi) \in Y_j(k)$ be a closed point. Consider the infinitesimal deformation over $k[\epsilon] = k[t]/t^2$. Note that (B, λ', η') has a unique deformation $(\hat{B}, \hat{\lambda}', \hat{\eta}')$ to $k[\epsilon]$, namely the trivial deformation. By the Serre–Tate and Grothendieck–Messing deformation theory (cf., Theorem 3.4), giving a deformation $(\hat{A}, \hat{\lambda}, \hat{\eta})$ of (A, λ, η) to $k[\epsilon]$ is equivalent to giving free $k[\epsilon]$-submodules $\hat{\omega}^\circ_{A^\vee,i} \subseteq H_1^{\mathrm{cris}}(A/k[\epsilon])^\circ_i$ for $i = 1, 2$ which lift $\omega^\circ_{A^\vee/k,i}$. The isogeny ϕ and the polarization λ deform to an isogeny $\hat{\phi} : \hat{B} \to \hat{A}$ and a polarization $\hat{\lambda} : \hat{A}^\vee \to \hat{A}$ (satisfying $p\hat{\lambda}' = \hat{\phi}^\vee \circ \hat{\lambda} \circ \hat{\phi}$), necessarily unique if they exist, if and only if

$$\hat{\omega}^\circ_{A^\vee,2} \supseteq \phi^{\mathrm{cris}}_{*,2}\big(H_1^{\mathrm{cris}}(B/k[\epsilon])^\circ_2\big) \quad \text{and} \quad \big(\phi^{\mathrm{cris}}_{*,1}(H_1^{\mathrm{cris}}(B/k[\epsilon])^\circ_1)\big)^\vee \subseteq (\hat{\omega}^\circ_{A^\vee,1})^\vee,$$

where the second inclusion comes from the consideration at the embedding $\bar{q}_2$ by taking duality using the polarization λ and is equivalent to $\hat{\omega}^\circ_{A^\vee,1} \subseteq \phi^{\mathrm{cris}}_{*,1}\big(H_1^{\mathrm{cris}}(B/k[\epsilon])^\circ_1\big)$.

As discussed before Proposition 4.8, we have $\mathrm{Ker}(\phi_{*,1}) = V^{-1}(H_2^{(p)})$ and $F(H_1^{(p)}) = \mathrm{Ker}(\phi_{*,2}) = \mathrm{Im}(\psi_{*,2})$. Then according to the relation between $\omega^\circ_{A^\vee/k,i}$ and H_1 in (4.7.1), giving such $\hat{\omega}^\circ_{A^\vee,i}$ for $i = 1, 2$ is equivalent to lifting each H_i to a free $k[\epsilon]$-submodule $\hat{H}_i \subseteq H_1^{\mathrm{dR}}(B/k)^\circ_i \otimes_k k[\epsilon] \cong H_1^{\mathrm{cris}}(B/k[\epsilon])^\circ_i$ for $i = 1, 2$ such that $\hat{H}_1 \supseteq V^{-1}(H_2^{(p)}) \otimes_k k[\epsilon]$ and $\hat{H}_2 \subseteq F(H_1^{(p)}) \otimes_k k[\epsilon]$. This is exactly the description of the tangent space of Y'_j at $\alpha(x)$. This concludes the proof. $\square$

In the sequel, we will always identify Y_j with Y'_j and pr'_j with π'_j. Before proceeding, we prove some results on the structure of $\mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$.

We turn to the study of the Shimura variety $\mathcal{S}h_{0,n}$. The following proposition was suggested by an anonymous referee of this article.

Proposition 4.9. (1) *The Shimura variety $\mathcal{S}h_{0,n}$ is finite and étale over $\mathbb{Z}_{p^2}$. In particular, the reduction map induces a bijection of geometric points*

$$\mathcal{S}h_{0,n}(\overline{\mathbb{Q}}_p) \xrightarrow{\sim} \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p).$$

(2) *Let $\tilde{x}_i = (\tilde{B}_i, \tilde{\lambda}_i, \tilde{\eta}_i) \in \mathcal{S}h_{0,n}(\overline{\mathbb{Q}}_p)$ for $i = 1, 2$ be two geometric points in characteristic* 0, *and $x_i = (B_i, \lambda_i, \eta_i) \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$ be their reductions. Then the reduction map on*

$$\mathrm{Hom}_{\mathcal{O}_D}(\tilde{B}_1, \tilde{B}_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_D}(B_1, B_2)$$

is an isomorphism.

Proof. (1) Let $\tilde{z} \in (\tilde{B}, \tilde{\lambda}, \tilde{\eta}) \in \mathcal{S}h_{0,n}(\mathbb{C})$. Put $H = H_1(\tilde{B}(\mathbb{C}), \mathbb{Q})$. It is a left D-module of rank 1 equipped with an alternating D-Hermitian pairing $\langle -, - \rangle_{\tilde{\lambda}}$ induced by the polarization $\tilde{\lambda}$. Let $(V_{0,n} = D, \langle -, - \rangle_{0,n})$ be the left D-module together with its alternating D-Hermitian pairing as in the definition of $\mathcal{S}h_{0,n}$. By results of Kottwitz [1992b, §8], for every place v of $\mathbb{Q}$, the skew-Hermitian $D_{\mathbb{Q}_v}$-modules $H_{\mathbb{Q}_v}$ and $V_{0,n,\mathbb{Q}_v}$ are isomorphic.[16] Then $\mathrm{End}_{\mathcal{O}_D}(\tilde{B}_{\mathbb{C}})_{\mathbb{Q}}$ consists of the elements of $D^{\mathrm{opp}} = \mathrm{End}_D(H)$ that preserves the complex structure on $H_{1,\mathbb{R}} \simeq V_{0,n,\mathbb{R}}$ induced the Deligne homomorphism by $h : \mathbb{C}^\times \to G_{0,n}(\mathbb{R})$. Since $h(i)$ is necessarily central (because $G^1_{0,n}$ is compact), it follows that $\mathrm{End}_{\mathcal{O}_D}(\tilde{B}_{\mathbb{C}})_{\mathbb{Q}} = D^{\mathrm{opp}}$, and

$$D \otimes_E D^{\mathrm{opp}} \simeq \mathrm{M}_{n^2}(E) \subseteq \mathrm{End}(\tilde{B})_{\mathbb{Q}}.$$

For dimension reasons, the inclusion above is an equality, and $\tilde{B}$ is isogenous to the product of n^2-copies of abelian varieties with complex multiplication by E. Therefore, $\tilde{B}$ is defined over a number field and has potentially good reduction everywhere. This implies that $\mathcal{S}h_{0,n}$ is proper over $\mathbb{Z}_{p^2}$.

To see that $\mathcal{S}h_{0,n}$ is finite and étale over $\mathbb{Z}_{p^2}$, it remains to show its étaleness over $\mathbb{Z}_{p^2}$. But this is clear from the description of its relative differential sheaf in Corollary 3.5, which is trivial as $\mathrm{Lie}_{\mathcal{A}^\vee/\mathcal{S}h_{0,n},1} = \mathrm{Lie}_{\mathcal{A}/\mathcal{S}h_{0,n},2} = 0$ by Kottwitz's determinant condition.

(2) In general, the reduction map

$$\mathrm{Hom}_{\mathcal{O}_D}(\tilde{B}_1, \tilde{B}_2) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_D}(B_1, B_2)$$

is injective. It remains to see that every element $f \in \mathrm{Hom}_{\mathcal{O}_D}(B_1, B_2)$ lifts to a homomorphism $\tilde{f} \in \mathrm{Hom}_{\mathcal{O}_D}(\tilde{B}_1, \tilde{B}_2)$. Note that points $\tilde{x}_1, \tilde{x}_2$ can be viewed over

[16] Note that the two skew-Hermitian forms $(H, \langle -, - \rangle)_{\tilde{\lambda}}$ and $(V_{0,n}, \langle -, - \rangle_{0,n})$ are not necessarily isomorphic over $\mathbb{Q}$. However, they differ at most only by a scalar in F, hence define the same similitude unitary group. See [Kottwitz 1992b, p. 400] for details.

$W(\bar{\mathbb{F}}_p)$. As recalled in Section 3, to show that f lifts to a map $\tilde{f}:\tilde{B}_1\to\tilde{B}_2$, it suffices to see that the induced map on crystalline homology

$$f^*: H_1^{\mathrm{cris}}(B_2/W(\bar{\mathbb{F}}_p))\to H_1^{\mathrm{cris}}(B_1/W(\bar{\mathbb{F}}_p))$$

preserves the Hodge filtrations

$$\omega_{\tilde{B}_i^\vee}\subseteq H_1^{\mathrm{dR}}(\tilde{B}_i/W(\bar{\mathbb{F}}_p))\cong H_1^{\mathrm{cris}}(B_i/W(\bar{\mathbb{F}}_p)).$$

It is clear that f^* preserves the decomposition

$$H_1^{\mathrm{dR}}(\tilde{B}_i/W(\bar{\mathbb{F}}_p))=H_1^{\mathrm{dR}}(\tilde{B}_i/W(\bar{\mathbb{F}}_p))_1\oplus H_1^{\mathrm{dR}}(\tilde{B}_i/W(\bar{\mathbb{F}}_p))_2$$

according to the two embeddings of $\mathcal{O}_E$ into $W(\bar{\mathbb{F}}_p)$. By the Kottwitz's determinant condition for $\mathcal{S}h_{0,n}$, the Hodge filtrations on $H_1^{\mathrm{dR}}(\tilde{B}_i/W(\bar{\mathbb{F}}_p))$ are trivial, namely,

$$\omega^\circ_{\tilde{B}_i^\vee/W(\bar{\mathbb{F}}_p),1}=0,\quad\text{and}\quad\omega^\circ_{\tilde{B}_i^\vee/W(\bar{\mathbb{F}}_p),2}=H_1^{\mathrm{dR}}(\tilde{B}_i/W(\bar{\mathbb{F}}_p))_2^\circ\quad\text{for } i=1,2.$$

It is clear now f^* preserves this trivial Hodge filtration, since it does so when tensoring with $\bar{\mathbb{F}}_p$. □

Fix a geometric point $z=(B,\lambda,\eta)\in\mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$. Put $C=\mathrm{End}_{\mathcal{O}_D}(B)_{\mathbb{Q}}$, and denote by † the Rosati involution on C induced by λ. Let I be the algebraic group over $\mathbb{Q}$ such that

$$I(R)=\{x\in C\otimes_{\mathbb{Q}}R\mid xx^\dagger\in R^\times\},\qquad\text{for all }\mathbb{Q}\text{-algebras } R.\tag{4.9.1}$$

Corollary 4.10. *We have an isomorphism of algebraic groups over* $\mathbb{Q}$: $I\simeq G_{0,n}$.

Proof. Let $\tilde{z}=(\tilde{B},\tilde{\lambda},\tilde{\eta})\in\mathcal{S}h_{0,n}(\overline{\mathbb{Q}}_p)$ denote the unique lift of z according to Proposition 4.9 (1). By 4.9 (2), we have a canonical isomorphism

$$\mathrm{End}_{\mathcal{O}_D}(\tilde{B})_{\mathbb{Q}}\xrightarrow{\sim}\mathrm{End}_{\mathcal{O}_D}(B)_{\mathbb{Q}}=C.$$

In the proof of 4.9, we have seen that $C=D^{\mathrm{opp}}$. Moreover, the Rosati involution on C corresponds to the involution $b\mapsto b^{\sharp_{\beta_{0,n}}}=\beta_{0,n}b^*\beta_{0,n}^{-1}$ on D^{opp}, where $\beta_{0,n}$ is the element in the definition of $\langle-,-\rangle_{0,n}$. It follows immediately that $I\simeq G_{0,n}$. □

Let $\mathrm{Isog}(z)\subseteq\mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$ denote the subset of points $z'=(B',\lambda',\eta')$ such that there exists an $\mathcal{O}_D$-equivariant quasi-isogeny $\phi:B'\to B$ such that $\phi^\vee\circ\lambda\circ\phi=c_0\lambda'$ for some $c_0\in\mathbb{Q}_{>0}$. We denote such a quasi-isogeny by $\phi:z'\to z$ for simplicity.

Corollary 4.11. *There exists a natural bijection of sets*

$$\Theta_z:\mathrm{Isog}(z)\xrightarrow{\sim}G_{0,n}(\mathbb{Q})\backslash G_{0,n}(\mathbb{A}^\infty)/K$$

Proof. First, we give the construction of Θ_z. Put $V^{(p)}(B)=T^{(p)}(B)\otimes_{\hat{\mathbb{Z}}^{(p)}}\mathbb{A}^{\infty,p}$. Then η determines an isomorphism

$$\tilde{\eta}:V_{0,n}^{(p)}\otimes_{\mathbb{Q}}\mathbb{A}^{\infty,p}\xrightarrow{\sim}V^{(p)}(B),$$

modulo right translation by K^p. For any $z' = (B', \lambda', \eta') \in \mathrm{Isog}(z)$ and a choice of $\phi : B' \to B$ as above. The quasi-isogeny ϕ induces an isomorphism $\phi_* : V^{(p)}(B') \xrightarrow{\sim} V^{(p)}(B)$. Then there exists a $g^p \in G_{0,n}(\mathbb{A}^{\infty,p})$, unique up to right multiplication by elements of K^p, such that the K^p-orbit of $\phi_*^{-1} \circ \tilde{\eta} \circ g$ gives η'.

We put

$$\mathbb{L}_z = \tilde{\mathcal{D}}(B)_1^{\circ, F^2=p} = \{v \in \tilde{\mathcal{D}}(B)_1^{\circ} : F^2(v) = pv\}. \tag{4.11.1}$$

Since B is supersingular (See Remark 3.7), this is a free $\mathbb{Z}_{p^2}$-module of rank n, and we have $\tilde{\mathcal{D}}(B)_1^{\circ} = \mathbb{L}_z \otimes_{\mathbb{Z}_{p^2}} W(\bar{\mathbb{F}}_p)$. Put $\mathbb{L}_z[1/p] = \mathbb{L}_z \otimes_{\mathbb{Z}_{p^2}} \mathbb{Q}_{p^2}$. Then ϕ induces an isomorphism $\phi_* : \mathbb{L}_{z'}[1/p] \xrightarrow{\sim} \mathbb{L}_z[1/p]$. Fix a $\mathbb{Z}_{p^2}$-basis for $\mathbb{L}_z$. Then there exists a $g_{\mathbb{L}} \in \mathrm{GL}_n(\mathbb{Q}_{p^2})$ such that $\phi_*(\mathbb{L}_{z'}) = g_{\mathbb{L}}(\mathbb{L}_z)$, and the right coset $g_{\mathbb{L}}\, \mathrm{GL}_n(\mathbb{Z}_{p^2})$ is independent of the choice of such a basis. We put $g_p = (c_0, g_{\mathbb{L}}) \in \mathbb{Q}_p^{\times} \times \mathrm{GL}_n(\mathbb{Q}_{p^2}) \simeq G_{0,n}(\mathbb{Q}_p)$, which is well defined up to right multiplication by elements of $K_p = \mathbb{Z}_{p^2}^{\times} \times \mathrm{GL}_n(\mathbb{Z}_{p^2})$.

Finally, note that the quasi-isogeny $\phi' : B' \to B$ is well determined by z' up to left composition with an element $\gamma \in I(\mathbb{Q}) = G_{0,n}(\mathbb{Q})$. If we replace ϕ by $\gamma \circ \phi$, then $g := (g^p, g_p) \in G_{0,n}(\mathbb{A}^{\infty})$ is replaced by $\gamma g = (\gamma g^p, \gamma g_p)$. Therefore, the map

$$\Theta_z : \mathrm{Isog}(z) \to G_{0,n}(\mathbb{Q}) \backslash G_{0,n}(\mathbb{A}^{\infty})/K, \quad z' \mapsto G_{0,n}(\mathbb{Q}) g K$$

is well defined. The fact that Θ_z is a bijection follows from the similar classical statement in characteristic 0. $\square$

Remark 4.12. It follows from Proposition 4.9 and the description of $\mathcal{S}h_{0,n}(\mathbb{C})$ in Section 2.3 that $\mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$ consists of $\#\ker^1(\mathbb{Q}, G_{0,n})$ isogeny classes of abelian varieties equipped with additional structures.

Lemma 4.13. *Let N be a fixed nonnegative integer. Up to replacing K^p by an open compact subgroup of itself, the following properties are satisfied: if (B, λ, η) is an $\bar{\mathbb{F}}_p$-point of $\mathrm{Sh}_{0,n}$ and $f : B \to B$ is an $\mathcal{O}_D$-quasi-isogeny such that $p^N f \in \mathrm{End}_{\mathcal{O}_D}(B)$, $f^{\vee} \circ \lambda \circ f = \lambda$ and $f \circ \eta = \eta$, then $f = \mathrm{id}$.*

Proof. It suffices to prove the lemma for (B, λ, η) in a fixed isogeny class $\mathrm{Isog}(z)$ of $\mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$. We write $G_{0,n}(\mathbb{A}^{\infty}) = \coprod_{i \in I} G_{0,n}(\mathbb{Q}) g_i K$ with $K = K^p K_p$, where $g_i = g_i^p g_{i,p}$, with $g_i^p \in G_{0,n}(\mathbb{A}^{\infty,p})$ and $g_{i,p} \in G_{0,n}(\mathbb{Q}_p)$, runs through a finite set of representatives of the double coset

$$G_{0,n}(\mathbb{Q}) \backslash G_{0,n}(\mathbb{A}^{\infty})/K.$$

Let (B, λ, η) be a point of $\mathrm{Sh}_{0,n}$ corresponding to $G_{0,n}(\mathbb{Q}) g_i K$ for some $i \in I$, and f be an $\mathcal{O}_D$-quasi-isogeny of B as in the statement. Then f is given by an element of $G^1_{0,n}(\mathbb{Q})$. The condition that $f \circ \eta = \eta$ is equivalent to saying that the image of f in $G_{0,n}(\mathbb{A}^{\infty,p})$ lies in $g_i^p K^p g_i^{p,-1}$. Moreover, $p^N f \in \mathrm{End}_{\mathcal{O}_D}(B)$ implies that the image of f in $G_{0,n}(\mathbb{Q}_p)$ belongs to $\coprod_\delta g_{i,p}(K_p\delta K_p)g_{i,p}^{-1}$, where δ runs through the set

$$\big\{(1,\mathrm{Diag}(p^{a_1},p^{a_2},\dots,p^{a_n}))\in G_{0,n}(\mathbb{Q}_p)\simeq\mathbb{Q}_p^\times\times\mathrm{GL}_n(\mathbb{Q}_{p^2})\;\big|\; 0\geq a_1\geq a_2\geq\cdots\geq a_n\geq -N\big\}.$$

Write $\coprod_\delta K_p\delta K_p=\coprod_{j\in J}h_jK_p$ for some finite set J. Hence, it suffices to show that there exists an open compact subgroup $K'^p\subseteq K^p$ such that for all g_i,

$$G^1_{0,n}(\mathbb{Q})\cap g_i(K'^p\cdot h_jK_p)g_i^{-1}=\{1\}$$

if $h_jK_p=K_p$, and empty otherwise. Since K is neat, we have

$$G^1_{0,n}(\mathbb{Q})\cap g_i(K'^pK_p)g_i^{-1}=\{1\}\quad\text{for any } g_i \text{ and any } K'^p\subseteq K^p.$$

Note that this implies that, for each $i\in I$, $G^1_{0,n}(\mathbb{Q})\cap g_i(K^p\cdot h_jK_p)g_i^{-1}$ contains at most one element (because if it contains both x and y, then $x^{-1}y$ is contained in $G^1_{0,n}(\mathbb{Q})\cap g_iKg_i^{-1}=\{1\}$). Let $S\subset I\times J$ be the subset consisting of (i,j) such that $h_jK_p\neq K_p$ and $G^1_{0,n}(\mathbb{Q})\cap g_i(K^p\cdot h_jK_p)g_i^{-1}$ indeed contains one element, say $x_{i,j}$. Then $x_{i,j}\neq 1$ for all $(i,j)\in S$. Hence, one can choose a normal open compact subgroup $K'^p\subseteq K^p$ so that $x_{i,j}\notin g_i^pK'^pg_i^{p,-1}$ for all i. We claim that this choice of K'^p will satisfy the desired property. Indeed, if $K^p=\coprod_l b_lK'^p$, then the double coset $G_{0,n}(\mathbb{Q})\setminus G_{0,n}(\mathbb{A}^\infty)/K'^pK_p$ has a set of representatives of the form g_ib_l. Here, by abuse of notation, we consider b_l as an element of K with p-component equal to 1. Then one has, for $h_jK_p\neq K_p$,

$$G^1_{0,n}(\mathbb{Q})\cap g_ib_l(K'^ph_jK_p)b_l^{-1}g_i^{-1}=G^1_{0,n}(\mathbb{Q})\cap g_i(K'^ph_jK_p)g_i^{-1}=\varnothing.$$

The first equality uses the fact that K'^p is normal in K^p. This finishes the proof. $\square$

We come back to the discussion on the cycles $Y_j\subseteq\mathrm{Sh}_{1,n-1}$ for $1\leq j\leq n$.

Proposition 4.14. *Let $(\mathcal{A},\lambda,\eta,\mathcal{B},\lambda',\eta',\phi^{\mathrm{univ}})$ denote the universal object on Y_j for $1\leq j\leq n$, and $\mathcal{H}_i\subset H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})$ for $i=1,2$ be the universal subbundles on $Y'_j\cong Y_j$.*

(1) *The induced map $T_{Y_j}\to\mathrm{pr}_j^*T_{\mathrm{Sh}_{1,n-1}}$ is universally injective, and we have canonical isomorphisms*

$$\begin{aligned}N_{Y_j}(\mathrm{Sh}_{1,n-1})&:=\mathrm{pr}_j^*T_{\mathrm{Sh}_{1,n-1}}/T_{Y_j}\\&\cong\big(\mathcal{H}_1/V^{-1}(\mathcal{H}_2^{(p)})\big)^*\otimes V^{-1}(\mathcal{H}_2^{(p)})\\&\qquad\oplus\big(F(\mathcal{H}_1^{(p)})/\mathcal{H}_2\big)\otimes\big(H_1^{\mathrm{dR}}(\mathcal{B}/\mathrm{Sh}_{0,n})_2^\circ/F(\mathcal{H}_1^{(p)})\big)^*\\&\cong\mathrm{Lie}^\circ_{\mathcal{A}^\vee,1}\otimes\mathrm{Coker}(\phi^{\mathrm{univ}}_{*,1})\oplus\mathrm{Lie}^\circ_{\mathcal{A},2}\otimes\mathrm{Im}(\phi^{\mathrm{univ}}_{*,2})^*.\end{aligned}$$

(2) *Assume that K^p is sufficiently small so that the consequences of Lemma 4.13 hold for $N = 1$. For each fixed closed point $z \in \mathrm{Sh}_{0,n}$, the map $\mathrm{pr}_{j,z} := \mathrm{pr}_j|_{Y_{j,z}} : Y_{j,z} \to \mathrm{Sh}_{1,n-1}$ is a closed immersion, or equivalently, the morphism $(\mathrm{pr}_j, \mathrm{pr}'_j) : Y_j \to \mathrm{Sh}_{1,n-1} \times_{\mathrm{Spec}(\mathbb{F}_{p^2})} \mathrm{Sh}_{0,n}$ is a closed immersion.*

(3) *The union of the images of pr_j for all $1 \le j \le n$ is the supersingular locus of $\mathrm{Sh}_{1,n-1}$, i.e., the reduced closed subscheme of $\mathrm{Sh}_{1,n-1}$ where all the slopes of the Newton polygon of the p-divisible group $\mathcal{A}[\mathfrak{p}^\infty]$ are $1/2$.*

Proof. **(1)** Let S be an affine noetherian $\mathbb{F}_{p^2}$-scheme and let $y = (A, \lambda, \eta, B, \lambda', \eta', \phi)$ be an S-point of Y_j. Put $\hat{S} = S \times_{\mathrm{Spec}(\mathbb{F}_{p^2})} \mathrm{Spec}(\mathbb{F}_{p^2}[t]/t^2)$. Then we have a natural bijection

$$\mathfrak{Def}(y, \hat{S}) \cong \Gamma(S, y^* T_{Y_j}),$$

where $\mathfrak{Def}(y, \hat{S})$ is the set of deformations of y to $\hat{S}$. Similarly, $\mathfrak{Def}(\mathrm{pr}_j \circ y, \hat{S}) \cong \Gamma(S, y^* \mathrm{pr}_j^* T_{\mathrm{Sh}_{1,n-1}})$. To prove the universal injectivity of $T_{Y_j} \to \mathrm{pr}_j^* T_{\mathrm{Sh}_{1,n-1}}$, it suffices to show that the natural map $\mathfrak{Def}(y, \hat{S}) \to \mathfrak{Def}(\mathrm{pr}_j \circ y, \hat{S})$ is injective. By crystalline deformation theory (Theorem 3.4), giving a point of $\mathfrak{Def}(y, \hat{S})$ is equivalent to giving $\mathcal{O}_{\hat{S}}$-subbundles $\hat{\omega}^\circ_{A^\vee,i} \subseteq H_1^{\mathrm{cris}}(A/\hat{S})^\circ_i$ over $\hat{S}$ for $i = 1, 2$ such that

- $\hat{\omega}^\circ_{A^\vee,i}$ lifts $\omega^\circ_{A^\vee/S,i}$;
- $\hat{\omega}^\circ_{A^\vee,1} \subseteq \mathrm{Im}(\phi_{*,1}) \otimes \mathbb{F}_{p^2}[t]/t^2$ and $\mathrm{Im}(\phi_{*,2}) \otimes \mathbb{F}_{p^2}[t]/t^2 \subseteq \hat{\omega}^\circ_{A^\vee,2}$ are locally direct factors.

Hence, one sees easily that

$$\begin{aligned}\mathfrak{Def}(y, \hat{S}) &\cong \mathrm{Hom}_{\mathcal{O}_S}(\omega^\circ_{A^\vee/S,1}, \mathrm{Im}(\phi_{*,1})/\omega^\circ_{A^\vee/S,1}) \\ &\qquad\qquad \oplus \mathrm{Hom}_{\mathcal{O}_S}(\omega^\circ_{A^\vee/S,2}/\mathrm{Im}(\phi_{*,2}), H_1^{\mathrm{dR}}(A/S)^\circ_2/\omega^\circ_{A^\vee/S,2}) \\ &\cong \mathrm{Lie}^\circ_{A^\vee/S,1} \otimes (\mathrm{Im}(\phi_{*,1})/\omega^\circ_{A^\vee/S,1}) \oplus (\omega^\circ_{A^\vee/S,2}/\mathrm{Im}(\phi_{*,2}))^* \otimes \mathrm{Lie}^\circ_{A/S,2}.\end{aligned}$$

Similarly, $\mathfrak{Def}(\mathrm{pr}_j \circ y, \hat{S})$ is given by the lifts of $\omega^\circ_{A^\vee/S,i}$ to $\hat{S}$ for $i = 1, 2$. These lifts are classified by $\mathrm{Hom}_{\mathcal{O}_S}(\omega^\circ_{A^\vee/S,i}, H_1^{\mathrm{dR}}(A/S)^\circ_i/\omega^\circ_{A^\vee/S,i}) \cong \mathrm{Lie}^\circ_{A^\vee/S,i} \otimes_k \mathrm{Lie}^\circ_{A/S,i}$. Hence, $\mathfrak{Def}(\mathrm{pr}_j \circ y, \hat{S})$ is canonically isomorphic to

$$\mathrm{Lie}^\circ_{A^\vee/S,1} \otimes_{\mathcal{O}_S} \mathrm{Lie}^\circ_{A/S,1} \oplus \mathrm{Lie}^\circ_{A^\vee/S,2} \otimes_{\mathcal{O}_S} \mathrm{Lie}^\circ_{A/S,2}.$$

The natural map $\mathfrak{Def}(y, \hat{S}) \to \mathfrak{Def}(\mathrm{pr}_j \circ y, \hat{S})$ is induced by the natural maps

$$\begin{aligned}\mathrm{Im}(\phi_{*,1})/\omega^\circ_{A^\vee/S,1} &\hookrightarrow H_1^{\mathrm{dR}}(A/S)^\circ_1/\omega^\circ_{A^\vee/S,1} \cong \mathrm{Lie}^\circ_{A/S,1}, \\ (\omega^\circ_{A^\vee/S,2}/\mathrm{Im}(\phi_{*,2}))^* &\hookrightarrow \qquad \omega^{\circ,*}_{A^\vee/S,2} \qquad \cong \mathrm{Lie}^\circ_{A^\vee/S,2}.\end{aligned}$$

It follows that $\mathfrak{Def}(y, \hat{S}) \to \mathfrak{Def}(\mathrm{pr}_j \circ y, \hat{S})$ is injective. To prove the formula for $N_{Y_j}(\mathrm{Sh}_{1,n-1})$, we apply the arguments above to affine open subsets of Y_j. We see easily that

$$\begin{aligned}N_{Y_j}(\mathrm{Sh}_{1,n-1}) &\cong \mathrm{Lie}^\circ_{\mathcal{A}^\vee,1}\otimes_{\mathcal{O}_{Y_j}}\mathrm{Coker}(\phi^{\mathrm{univ}}_{*,1})\oplus \mathrm{Lie}^\circ_{\mathcal{A},2}\otimes_{\mathcal{O}_{Y_j}}\mathrm{Im}(\phi^{\mathrm{univ}}_{*,2})^*\\ &\cong \big(\mathcal{H}_1/V^{-1}(\mathcal{H}_2^{(p)})\big)^*\otimes V^{-1}(\mathcal{H}_2^{(p)})\\ &\qquad\oplus\big(F(\mathcal{H}_1^{(p)})/\mathcal{H}_2\big)\otimes\big(H_1^{\mathrm{dR}}(\mathcal{B}/Y_j)^\circ_2/F(\mathcal{H}_1^{(p)})\big)^*.\end{aligned}$$

Here, the last step uses (4.7.2) and the isomorphism

$$\mathrm{Im}(\phi^{\mathrm{univ}}_{*,2})\cong H_1^{\mathrm{dR}}(\mathcal{B}/Y_j)^\circ_2/\operatorname{Ker}(\phi^{\mathrm{univ}}_{*,2})\cong H_1^{\mathrm{dR}}(\mathcal{B}/Y_j)^\circ_2/F(\mathcal{H}_1^{(p)}).$$

(2) By statement (1), $\mathrm{pr}_{j,z}$ induces an injection of tangent spaces at each closed points of $Y_{j,z}$. To complete the proof, it suffices to prove that $\pi_{j,z}$ induces injections on the closed points. Write $z=(B,\lambda',\eta')\in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$. Assume y_1 and y_2 are two closed points of $Y_{j,z}$ with $\pi_j(y_1)=\pi_j(y_2)=(A,\lambda,\eta)$. Let $\phi_1,\phi_2:B\to A$ be the isogenies given by y_1 and y_2. Then the quasi-isogeny $\phi_{1,2}=\phi_2^{-1}\phi_1\in\mathrm{End}_{\mathcal{O}_D}(B)_{\mathbb{Q}}$ satisfies the conditions of Lemma 4.13 for $N=1$. Hence, we get $\phi_{1,2}=\mathrm{id}_B$, which is equivalent to $y_1=y_2$. This proves that $\pi_{j,z}$ is injective on closed points.

(3) The proof resembles the work of Vollaard and Wedhorn [2011]. Since all the points of $\mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$ are supersingular by Remark 3.7, it is clear that the image of each pr_j lies in the supersingular locus of $\mathrm{Sh}_{1,n-1}$. Suppose now we are given a supersingular point $x=(A,\lambda,\eta)\in\mathrm{Sh}_{1,n-1}(\bar{\mathbb{F}}_p)$. We have to show that there exists $(B,\lambda',\eta')\in\mathrm{Sh}_{0,n}$ and an isogeny $\phi:B\to A$ such that $(A,\lambda,\eta,\lambda',\eta';\phi)$ lies in Y_j for some $1\le j\le n$.

Consider

$$\mathbb{L}_{\mathbb{Q}}=(\tilde{\mathcal{D}}(A)^\circ_1[1/p])^{F^2=p}=\{a\in\tilde{\mathcal{D}}(A)^\circ_1[1/p]\mid F^2(a)=pa\}.$$

Since x is supersingular, $\mathbb{L}_{\mathbb{Q}}$ is a $\mathbb{Q}_{p^2}$-vector space of dimension n by the Dieudonné–Manin classification, and $\tilde{\mathcal{D}}(A)^\circ_1[1/p]=\mathbb{L}_{\mathbb{Q}}\otimes_{\mathbb{Q}_{p^2}}W(\bar{\mathbb{F}}_p)[1/p]$. We put $\tilde{\mathcal{E}}^\circ_1=(\mathbb{L}_{\mathbb{Q}}\cap\tilde{\mathcal{D}}(A)^\circ_1)\otimes_{\mathbb{Z}_{p^2}}W(\bar{\mathbb{F}}_p)$, and $\tilde{\mathcal{E}}^\circ_2=F(\tilde{\mathcal{E}}^\circ_1)\subseteq\tilde{\mathcal{D}}(A)^\circ_2$. Thus $\tilde{\mathcal{E}}^\circ=\tilde{\mathcal{E}}^\circ_1\oplus\tilde{\mathcal{E}}^\circ_2$ is a Dieudonné submodule of $\tilde{\mathcal{D}}(A)^\circ$. We claim that $\tilde{\mathcal{E}}^\circ$ contains $p\tilde{\mathcal{D}}(A)^\circ$ as a submodule. Then applying Proposition 3.2 with $m=1$, we get an $\mathcal{O}_D$-abelian variety (B,λ',η') together with an $\mathcal{O}_D$-isogeny $\phi:B\to A$ with $\phi^\vee\circ\lambda\circ\phi=p\lambda$. It is easy to see in this case that $(A,\lambda,\eta,B,\lambda',\eta',\phi)$ defines a point in Y_j with $j=\dim_{\bar{\mathbb{F}}_p}(\tilde{\mathcal{D}}(A)^\circ_2/\tilde{\mathcal{E}}^\circ_2)$.

It then suffices to prove the claim that $p\tilde{\mathcal{D}}(A)^\circ\subseteq\tilde{\mathcal{E}}^\circ$. Suppose not, then $\tilde{\mathcal{D}}(A)^\circ\nsubseteq(1/p)\tilde{\mathcal{E}}^\circ$. Consider $M_i:=\tilde{\mathcal{D}}(A)^\circ_i/\tilde{\mathcal{E}}^\circ_i$ for $i=1,2$. For any integer $\alpha\ge 0$, its p^α-torsion submodule is

$$M_i[p^\alpha]=\Big(\tilde{\mathcal{D}}(A)^\circ_i\cap\frac{1}{p^\alpha}\tilde{\mathcal{E}}^\circ_i\Big)/\tilde{\mathcal{E}}^\circ_i.$$

It follows easily that

$$M_i[p^{\alpha+1}]/M_i[p^\alpha]\cong\Big(\frac{1}{p^{\alpha+1}}\tilde{\mathcal{E}}^\circ_i\cap\Big(\tilde{\mathcal{D}}(A)^\circ_i+\frac{1}{p^\alpha}\tilde{\mathcal{E}}^\circ_i\Big)\Big)\Big/\frac{1}{p^\alpha}\tilde{\mathcal{E}}^\circ_i.$$

On the other hand, the Kottwitz's signature condition implies that both F and $V:\tilde{\mathcal{D}}(A)^\circ_1 \to \tilde{\mathcal{D}}(A)^\circ_2$ have cokernel isomorphic to $\bar{\mathbb{F}}_p$, and both F and $V:\tilde{\mathcal{E}}^\circ_1 \to \tilde{\mathcal{E}}^\circ_2$ are isomorphism. Therefore, the two induced morphisms

$$F \text{ and } V: M_1 \to M_2$$

are injective and both have cokernel isomorphic to $\bar{\mathbb{F}}_p$. It follows that the induced maps on the graded pieces

$$\begin{aligned} F \text{ and } V: \Big(\frac{1}{p^{\alpha+1}}\tilde{\mathcal{E}}^\circ_1 \cap \Big(\tilde{\mathcal{D}}(A)^\circ_1 + \frac{1}{p^\alpha}\tilde{\mathcal{E}}^\circ_1\Big)\Big)\Big/\frac{1}{p^\alpha}\tilde{\mathcal{E}}^\circ_1 \\ \to \Big(\frac{1}{p^{\alpha+1}}\tilde{\mathcal{E}}^\circ_2 \cap \Big(\tilde{\mathcal{D}}(A)^\circ_2 + \frac{1}{p^\alpha}\tilde{\mathcal{E}}^\circ_2\Big)\Big)\Big/\frac{1}{p^\alpha}\tilde{\mathcal{E}}^\circ_2 \end{aligned} \tag{4.14.1}$$

are injective maps, and are isomorphisms for all $\alpha \geq 0$ except for exactly one α.[17] The assumption $\tilde{\mathcal{D}}(A)^\circ \not\subseteq (1/p)\tilde{\mathcal{E}}^\circ$ implies that there are at least two $\alpha \geq 0$ for which the right hand side of (4.14.1) is nonzero. So there exists $\alpha \geq 0$ such that (4.14.1) are isomorphisms of *nonzero* $\bar{\mathbb{F}}_p$*-vector spaces*. Multiplication by p^α gives isomorphisms:

$$F \text{ and } V: \Big(\frac{1}{p}\tilde{\mathcal{E}}^\circ_1 \cap \Big(p^\alpha\tilde{\mathcal{D}}(A)^\circ_1 + \tilde{\mathcal{E}}^\circ_1\Big)\Big) \to \Big(\frac{1}{p}\tilde{\mathcal{E}}^\circ_2 \cap \Big(p^\alpha\tilde{\mathcal{D}}(A)^\circ_2 + \tilde{\mathcal{E}}^\circ_2\Big)\Big). \tag{4.14.2}$$

Now, Hilbert 90 theorem implies that $\mathbb{L}' = \big((1/p)\tilde{\mathcal{E}}^\circ_1 \cap (p^\alpha\tilde{\mathcal{D}}(A)^\circ_1 + \tilde{\mathcal{E}}^\circ_1)\big)^{F^2=p}$ in fact generates the source of (4.14.2). On the other hand, it is obvious that $\mathbb{L}' \subset \mathbb{L}_{\mathbb{Q}}$ and $\mathbb{L}' \subseteq p^\alpha\tilde{\mathcal{D}}(A)^\circ_1 + \tilde{\mathcal{E}}^\circ_1 \subseteq \tilde{\mathcal{D}}(A)^\circ_1$. This means that $\mathbb{L}'$, and hence $\mathbb{L}_{\mathbb{Q}} \cap \tilde{\mathcal{D}}(A)^\circ_1$, generates the entire $(1/p)\tilde{\mathcal{E}}^\circ_1 \cap (p^\alpha\tilde{\mathcal{D}}(A)^\circ_1 + \tilde{\mathcal{E}}^\circ_1)$, i.e., one has $(1/p)\tilde{\mathcal{E}}^\circ_1 \cap (p^\alpha\tilde{\mathcal{D}}(A)^\circ_1 + \tilde{\mathcal{E}}^\circ_1) = \tilde{\mathcal{E}}^\circ_1$. But this contradicts with the nontriviality of the vector spaces in (4.14.1) by our choice of α. Now the proposition is proved. □

Corollary 4.15. *The morphism* pr_1 (*resp.* pr_n) *is a closed immersion, and it identifies* Y_1 (*resp.* Y_n) *with the closed subscheme of* $\mathrm{Sh}_{1,n-1}$ *defined by the vanishing of* $V:\omega^\circ_{\mathcal{A}^\vee,2} \to \omega^{\circ,(p)}_{\mathcal{A}^\vee,1}$ (*resp.* $V:\omega^\circ_{\mathcal{A}^\vee,1} \to \omega^{\circ,(p)}_{\mathcal{A}^\vee,2}$).

Proof. We just prove the statement for pr_1, and the case of pr_n is similar. Let Z_1 be the closed subscheme of $\mathrm{Sh}_{1,n-1}$ defined by the condition that $V:\omega^\circ_{\mathcal{A}^\vee,2} \to \omega^{\circ,(p)}_{\mathcal{A}^\vee,1}$ vanishes. We show first that $\mathrm{pr}_1: Y_1 \to \mathrm{Sh}_{1,n-1}$ factors through the natural inclusion $Z_1 \hookrightarrow \mathrm{Sh}_{1,n-1}$. Let $y = (A, \lambda, \eta, B, \lambda', \eta', \phi)$ be an S-valued point of Y_1. By Lemma 4.6, $\mathrm{Im}(\phi_{2,*})$ has rank $n-1$ and contains both $\omega^\circ_{A^\vee/S,2}$ and $F(H_1^{\mathrm{dR}}(A/S)_1^{\circ,(p)})$, which are both $\mathcal{O}_S$-subbundles of rank $n-1$. This forces $\omega^\circ_{A^\vee/S,2} = F(H_1^{\mathrm{dR}}(A/S)_1^{\circ,(p)})$, and therefore $V:\omega^\circ_{A^\vee/S,2} \to \omega^{\circ,(p)}_{A^\vee/S,1}$ vanishes. This shows that $\mathrm{pr}_1(y) \in Z_1$.

[17] We point out that, for (4.14.1), F is an isomorphism if and only if V is an isomorphism, because this is equivalent to requiring the source and the target to have the same dimension.

To prove that $\mathrm{pr}_1 : Y_1 \to Z_1$ is an isomorphism, as Y_1 is smooth, it suffices to show that it induces a bijection between closed points and tangent spaces of Y_1 and Z_1. For any perfect field k containing $\mathbb{F}_{p^2}$, one constructs a map $\theta : Z_1(k) \to Y_1(k)$ inverse to $\mathrm{pr}_1 : Y_1(k) \to Z_1(k)$ as follows. Given $x = (A, \lambda, \eta) \in Z_1(k)$. Let $\tilde{\mathcal{E}}_1^\circ = \tilde{\mathcal{D}}(A)_1^\circ$ and $\tilde{\mathcal{E}}_2^\circ \subseteq \tilde{\mathcal{D}}(A)_2^\circ$ be the inverse image of $\omega^\circ_{A^\vee/k,2} \subseteq \tilde{\mathcal{D}}(A)_2^\circ / p\tilde{\mathcal{D}}(A)_2^\circ$. Then the condition that $y \in Z_1$ implies that $\tilde{\mathcal{E}}_1^\circ \oplus \tilde{\mathcal{E}}_2^\circ$ is stable under F and V. Applying Proposition 3.2 with $m = 1$, we get a tuple $(B, \lambda', \eta', \phi)$ such that $y = (A, \lambda, \eta, B, \lambda', \eta', \phi) \in Y_1(k)$. It is immediate to check that $x \mapsto y$ and pr_1 are the set theoretic inverse of each other. It remains to show that pr_1 induces a bijection between $T_{Y_1,y}$ and $T_{Z_1,x}$. Proposition 4.14 already implies that we have an inclusion $T_{Y_1,y} \hookrightarrow T_{Z_1,x} \hookrightarrow T_{\mathrm{Sh}_{1,n-1},x}$. It suffices to check that $\dim T_{Z_1,x} = n-1$. The tangent space $T_{Z_1,x}$ is the space of deformations $(\hat{A}, \hat{\lambda}, \hat{\eta})$ over $\hat{k} = k[\epsilon]/(\epsilon^2)$ of (A, λ, η) such that $V : \omega^\circ_{\hat{A}^\vee/\hat{k},2} \to \omega^{\circ,(p)}_{\hat{A}^\vee/\hat{k},1} = \omega^{\circ,(p)}_{A^\vee/k,1} \otimes_k \hat{k}$ vanishes. This uniquely determines the lift $\hat{\omega}^\circ_{A^\vee,2} = \omega^\circ_{\hat{A}^\vee/\hat{k},2}$. So by deformation theory (Theorem 3.4), the tangent space $T_{Z_1,x}$ is determined by the liftings $\hat{\omega}^\circ_{A^\vee,1} = \omega^\circ_{\hat{A}^\vee/\hat{k},1}$ of $\omega^\circ_{A^\vee/k,1}$. So it is of dimension $n-1$. This concludes the proof of the corollary. □

4.16. ***Geometric Jacquet–Langlands morphism.*** Let $\ell \neq p$ be a prime number. For $1 \leq j \leq n$, the diagram (4.2.1) gives rise to a natural morphism

$$\mathcal{JL}_j : H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{pr}'^*_j} H^0_{\mathrm{et}}(\bar{Y}_j, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{pr}_{j,!}} H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_\ell(n-1)), \quad (4.16.1)$$

where $\mathrm{pr}_{j,!}$ is (2.10.1), whose restriction to each $H^0_{\mathrm{et}}(Y_{j,z}, \overline{\mathbb{Q}}_\ell)$ for $z \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$ is the Gysin map associated to the closed immersion $Y_{j,z} \hookrightarrow \overline{\mathrm{Sh}}_{1,n-1}$. It is clear that the image of $\mathcal{JL}_j$ is the subspace generated by the cycle classes of $[Y_{j,z}] \in A^{n-1}(\overline{\mathrm{Sh}}_{1,n-1})$ with $z \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$. According to [Helm 2010], $\mathcal{JL}_j$ should be considered as a certain geometric realization of the Jacquet–Langlands transfer from $G_{0,n}$ to $G_{1,n-1}$. Putting all the $\mathcal{JL}_j$ together, we get a morphism

$$\mathcal{JL} = \sum_j \mathcal{JL}_j : \bigoplus_{j=1}^n H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n}, \overline{\mathbb{Q}}_\ell) \to H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_\ell(n-1)). \quad (4.16.2)$$

Recall that we have fixed an isomorphism $G_{1,n-1}(\mathbb{A}^\infty) \simeq G_{0,n}(\mathbb{A}^\infty)$, which we write uniformly as $G(\mathbb{A}^\infty)$. Denote by $\mathcal{H}(K^p, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell[K^p \backslash G(\mathbb{A}^{\infty,p})/K^p]$ the prime-to-p Hecke algebra. Then the homomorphism (4.16.2) is a homomorphism of $\mathcal{H}(K^p, \overline{\mathbb{Q}}_\ell)$-modules.

For an irreducible admissible representation π of $G(\mathbb{A}_\infty)$, we write $\pi = \pi^p \otimes \pi_p$, where π^p (resp. π_p) is the prime-to-p part (resp. the p-component) of π.

Lemma 4.17. *Let π_1 and π_2 be two admissible irreducible representations of $G(\mathbb{A}^\infty)$, and (r_i, s_i) for $i = 1, 2$ be two pairs of integers with $0 \leq r_i, s_i \leq n$ and $r_1 + s_1 \equiv r_2 + s_2 \mod 2$. Assume that π_1 satisfies Hypothesis 2.5 with $a_\bullet = (r_1, s_1)$,* *and there exists an admissible irreducible representation* $\pi_{2,\infty}$ *of* $G_{(r_2,s_2)}(\mathbb{R})$ *such that* $\pi_2 \otimes \pi_{2,\infty}$ *is a cuspidal automorphic representation of* $G_{(r_2,s_2)}(\mathbb{A})$. *If* π_1^p *and* π_2^p *are isomorphic as representations of* $G(\mathbb{A}^{p,\infty})$, *then* $\pi_{1,p} \simeq \pi_{2,p}$, *and* $\pi_2 \otimes \pi_{2,\infty}$ *admits a base change to a cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_E) \times \mathbb{A}_{E_0}^\times$; *in particular*, π_2 *satisfies Hypothesis 2.5 for* $a_\bullet = (r_2, s_2)$.

Proof. By assumption on π_1, there exists an irreducible admissible representation $\pi_{1,\infty}$ of $G_{(r_1,s_1)}(\mathbb{R})$ such that $\pi_1 \otimes \pi_{1,\infty}$ is a cuspidal automorphic representation of $G_{r_1,s_1}(\mathbb{A})$, which base changes to a cuspidal automorphic representation (Π_1, χ_1) of $\mathrm{GL}_n(\mathbb{A}_E) \times \mathbb{A}_{E_0}^\times$. On the other hand, by [Shin 2014, Theorem 1.1], there exists always a base change of $\pi_2 \otimes \pi_{2,\infty}$ to an automorphic representation (Π_2, χ_2) of $\mathrm{GL}_n(\mathbb{A}_E) \times \mathbb{A}_{E_0}^\times$. The base changes (Π_i, χ_i) with $i = 1, 2$ satisfy the following properties:

- Π_i is conjugate self-dual,
- for every unramified rational prime x, the x-component of (Π_i, ψ_i) depends only on the x-component of π_i and
- if $\pi_{i,p} = \pi_{i,0} \otimes \pi_{i,\mathfrak{p}}$ as representation of $G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_\mathfrak{p})$, then $\Pi_{i,p} = (\pi_{i,\mathfrak{p}} \otimes \check{\pi}_{i,\mathfrak{p}}^c)$ as a representation of $\mathrm{GL}_n(E \otimes \mathbb{Q}_p) \cong \mathrm{GL}_n(E_\mathfrak{p}) \times \mathrm{GL}_n(E_{\bar{\mathfrak{p}}})$, and $\psi_{i,p} = \pi_{i,0} \otimes \pi_{i,0}^{-1}$ as a representation of $(E_0 \otimes \mathbb{Q}_p)^\times = \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$. Here, $\check{\pi}_{i,\mathfrak{p}}^c$ denotes the complex conjugate of the contragredient of $\pi_{i,\mathfrak{p}}$.

As $\pi_1^p \simeq \pi_2^p$, (Π_1, ψ_1) and (Π_2, ψ_2) are isomorphic at almost all finite places. By the strong multiplicity one theorem for GL_n [Jacquet and Shalika 1981], we have $(\Pi_1, \psi_1) \simeq (\Pi_2, \psi_2)$; in particular, (Π_2, ψ_2) is cuspidal. By the description of $(\Pi_{i,p}, \psi_{i,p})$, it follows immediately that $\pi_{1,p} \simeq \pi_{2,p}$. □

Let $\mathscr{A}_K$ be the set of isomorphism classes of irreducible admissible representations π of $G(\mathbb{A}^\infty)$ satisfying Hypothesis 2.5 with $a_\bullet = (0, n)$. In particular, each $\pi \in \mathscr{A}_K$ is the finite part of an automorphic cuspidal representation of $G_{0,n}(\mathbb{A})$.

We fix such a $\pi \in \mathscr{A}_K$. Let

$$\mathcal{JL}_\pi : \bigoplus_{i=1}^n H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n}, \overline{\mathbb{Q}}_\ell)_{\pi^p} \to H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_\ell(n-1))_{\pi^p}$$

denote the homomorphism on the $(\pi^p)^{K^p}$-isotypic components induced by $\mathcal{JL}$, where for an $\mathscr{H}(K^p, \overline{\mathbb{Q}}_\ell)$-module M we put

$$M_{\pi^p} := \mathrm{Hom}_{\mathscr{H}(K^p,\overline{\mathbb{Q}}_\ell)}((\pi^p)^{K^p}, M) \otimes (\pi^p)^{K^p}.$$

Then Lemma 4.17 implies that π is completely determined by its prime-to-p part. Hence, taking the π^p-isotypic components is the same as taking the π-isotypic components. We can thus write M_π instead of M_{π^p} for a $\mathscr{H}(K, \overline{\mathbb{Q}}_\ell)$-module M.

Recall that the image of $\mathcal{JL}_\pi$ is included in $H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_\ell(n-1))^{\mathrm{fin}}_\pi$, which is the maximal subspace of $H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_\ell(n-1))_\pi$ where the action of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^2})$ factors through a finite quotient. Note that, at this moment, it is not clear if the target of $\mathcal{JL}_\pi$ is nonzero. But this will follow from the proof of our main Theorem 4.18 below.

Our main result claims that this inclusion is actually an equality under certain genericity conditions on π_p. To make this precise, write $\pi_p = \pi_{p,0} \otimes \pi_{\mathfrak{p}}$ as a representation of $G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_{\mathfrak{p}})$. Let

$$\rho_{\pi_{\mathfrak{p}}} : W_{\mathbb{Q}_{p^2}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

be the unramified representation of the Weil group of $\mathbb{Q}_{p^2}$ defined in (2.6.1). It induces a continuous ℓ-adic representation of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^2})$, which we denote by the same notation. Then $\rho_{\pi_{\mathfrak{p}}}(\mathrm{Frob}_{p^2})$ is semisimple with characteristic polynomial (2.6.2). Using this, we get an explicit description of $H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_\ell(n-1))_\pi$ and $H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n}, \overline{\mathbb{Q}}_\ell)_\pi$ in terms of $\rho_{\pi_{\mathfrak{p}}}$ by (2.4.1) and (2.6.3).

We can now state our main theorem.

Theorem 4.18. *Fix a π in $\mathscr{A}_K$. Let $\alpha_{\pi_{\mathfrak{p}},1}, \dots, \alpha_{\pi_{\mathfrak{p}},n}$ be the eigenvalues of $\rho_{\pi_{\mathfrak{p}}}(\mathrm{Frob}_{p^2})$.*

(1) *If $\alpha_{\pi_{\mathfrak{p}},1}, \dots, \alpha_{\pi_{\mathfrak{p}},n}$ are distinct, then the map $\mathcal{JL}_\pi$ is injective;*

(2) *Let $m_{1,n-1}(\pi)$ (resp. $m_{0,n}(\pi)$) denote the multiplicity for π appearing in Theorem 2.6 for $a_\bullet = (1, n-1)$ (resp. for $a_\bullet = (0, n)$). Assume that $m_{1,n-1}(\pi) = m_{0,n}(\pi)$ and that $\alpha_{\pi_{\mathfrak{p}},i}/\alpha_{\pi_{\mathfrak{p}},j}$ is not a root of unity for all $1 \le i, j \le n$. Then the map*

$$\mathcal{JL}_\pi : \bigoplus_{j=1}^{n} H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n}, \overline{\mathbb{Q}}_\ell)_\pi \to H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_\ell(n-1))^{\mathrm{fin}}_\pi$$

is an isomorphism. In other words, $H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_\ell(n-1))^{\mathrm{fin}}_\pi$ is generated by the cycle classes of the irreducible components of Y_j for $1 \le j \le n$.

The proof of this theorem will be given at the end of Section 6.

Remark 4.19. The equality $m_{1,n-1}(\pi) = m_{0,n}(\pi)$ is conjectured to be true according to Arthur's formula on the automorphic multiplicities of unitary groups, and is known to hold when π is the finite part of an automorphic representation of $G_{1,n-1}(\mathbb{A})$ whose base change to $\mathrm{GL}_n(\mathbb{A}_E) \times \mathbb{A}_{E_0}^\times$ is cuspidal, and $G_{1,n-1}$ is quasisplit at all finite places. See for instance [White 2012, Theorem E].

On the other hand, Theorem 4.18(1) gives partial results towards the equality $m_{1,n-1}(\pi) = m_{0,n}(\pi)$. Indeed, when combining with Kottwitz's description 2.6 of the π-isotypic components of the cohomology groups, Theorem 4.18(1) implies (under the assumption that the Satake parameters of $\pi_{\mathfrak{p}}$ are distinct) that $m_{1,n-1}(\pi) \ge m_{0,n}(\pi)$ without using Arthur's trace formula. If we use only the fact that $\mathcal{JL}_\pi$

is nonzero (which is an easy consequence of our computation of the intersection matrix in Theorem 6.7), we get the implication $m_{0,n}(\pi) \neq 0 \Rightarrow m_{1,n-1}(\pi) \neq 0$.

5. Fundamental intersection numbers

In this section, we will compute some intersection numbers on certain Deligne–Lusztig varieties. These numbers will play a key role in the computation in the next section of the intersection matrix of the cycles Y_j on $\mathrm{Sh}_{1,n-1}$.

Notation 5.1. Let X be an algebraic variety of pure dimension N over $\bar{\mathbb{F}}_p$. For an integer $r \geq 0$, let $A^r(X)$ (resp. $A_r(X)$) denote the group of algebraic cycles on X of codimension r (resp. of dimension r) modulo rational equivalences. If $Y \subseteq X$ is a subscheme equidimensional of codimension r, we denote by $[Y] \in A^r(X)$ the class of Y. We put $A^\star(X) = \bigoplus_{r=0}^N A^r(X)$. For a zero-dimensional cycle $\eta \in A^N(X)$, we denote by

$$\deg(\eta) = \int_X \eta$$

the *degree* of η. Let $\mathcal{V}$ be a vector bundle over X. We denote by $c_r(\mathcal{V}) \in A^r(X)$ the r-th Chern class of $\mathcal{V}$ for $0 \leq r \leq N$, and put $c(\mathcal{V}) = \sum_{r=0}^N c_r(\mathcal{V}) t^r$ in the free variable t.

5.2. ***A special Deligne–Lusztig variety.*** We fix an integer $n \geq 1$. For an integer $0 \leq k \leq n$, we denote by $\boldsymbol{Gr}(n,k)$ the Grassmannian variety over $\mathbb{F}_p$ classifying k-dimensional subspaces of $\mathbb{F}_p^{\oplus n}$. Given an integer k with $1 \leq k \leq n$, let $Z_k^{\langle n \rangle}$ be the subscheme of $\boldsymbol{Gr}(n,k) \times \boldsymbol{Gr}(n,k-1)$ whose S-valued points are isomorphism classes of pairs (L_1, L_2), where L_1 and L_2 are respectively subbundles of $\mathcal{O}_S^{\oplus n}$ of rank k and $k-1$ satisfying $L_2 \subseteq L_1^{(p)}$ and $L_2^{(p)} \subseteq L_1$ (with locally free quotients). The same arguments as in Proposition 4.4 show that $Z_k^{\langle n \rangle}$ is a smooth variety over $\mathbb{F}_p$ of dimension $n-1$. We denote the natural closed immersion by

$$i_k : Z_k^{\langle n \rangle} \hookrightarrow \boldsymbol{Gr}(n,k) \times \boldsymbol{Gr}(n,k-1).$$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the universal subbundles on $\boldsymbol{Gr}(n,k) \times \boldsymbol{Gr}(n,k-1)$ coming from the two factors, and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ the universal quotients, respectively. When there is no confusion, we still use $\mathcal{L}_i$ and $\mathcal{Q}_i$ for $i = 1, 2$ to denote their restrictions to $Z_k^{\langle n \rangle}$. We put

$$\mathcal{E}_k = (\mathcal{L}_1/\mathcal{L}_2^{(p)})^* \otimes \mathcal{L}_2^{(p)} \oplus (\mathcal{L}_1^{(p)}/\mathcal{L}_2) \otimes \mathcal{Q}_1^{*,(p)}, \tag{5.2.1}$$

which is a vector bundle of rank $n-1$ on $Z_k^{\langle n \rangle}$. (This vector bundle is modeled on the description of the normal bundle $N_{Y_j}(\mathrm{Sh}_{1,n-1})$ in Proposition 4.14(1), which is how our computation will be used in the next section; see Proposition 6.4.) We have the top Chern class $c_{n-1}(\mathcal{E}_k) \in A^{n-1}(Z_k^{\langle n\rangle})$. We define the *fundamental intersection number* on $Z_k^{\langle n\rangle}$ as

$$N(n,k) := \int_{Z_k^{\langle n\rangle}} c_{n-1}(\mathcal{E}_k). \tag{5.2.2}$$

The main theorem we prove in this section is the following:

Theorem 5.3. *For integers n, r with $0 \le r \le n$, let*

$$\binom{n}{r}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-r+1}-1)}{(q-1)(q^2-1)\cdots(q^r-1)}$$

be the Gaussian binomial coefficients, and let $d(n,k) = (2k-1)n - 2k(k-1) - 1$ denote the dimension of $\boldsymbol{Gr}(n,k) \times \boldsymbol{Gr}(n,k-1)$. Then, for $1 \le k \le n$, we have

$$N(n,k) = (-1)^{n-1} \sum_{\delta=0}^{\min\{k-1,n-k\}} (n-2\delta)\, p^{d(n-2\delta,k-\delta)} \binom{n}{\delta}_{p^2}. \tag{5.3.1}$$

Remark 5.4. We point out that this theorem seems to be more than a technical result. It is at the heart of the understanding of these cycles we constructed.

Proof. We first claim that $N(n,k) = N(n,n+1-k)$ for $1 \le k \le n$. Let (L_1, L_2) be an S-valued point of $\boldsymbol{Gr}(n,k) \times \boldsymbol{Gr}(n,k-1)$, and $Q_i = \mathcal{O}_S^{\oplus n}/L_i$ for $i = 1, 2$ be the corresponding quotient bundles. Then $(L_1, L_2) \mapsto (Q_2^*, Q_1^*)$ defines a duality isomorphism

$$\theta : \boldsymbol{Gr}(n,k) \times \boldsymbol{Gr}(n,k-1) \xrightarrow{\sim} \boldsymbol{Gr}(n,n+1-k) \times \boldsymbol{Gr}(n,n-k).$$

Since $L_2^{(p)} \subseteq L_1$ (resp. $L_2 \subseteq L_1^{(p)}$) is equivalent to $Q_1^* \subseteq Q_2^{*,(p)}$ (resp. to $Q_1^{*,(p)} \subseteq Q_2^*$), θ induces an isomorphism between $Z_k^{(n)}$ and $Z_{n+1-k}^{(n)}$. It is also direct to check that $\mathcal{E}_k = \theta^*(\mathcal{E}_{n+1-k})$. This verifies the claim. Now since the right hand side of (5.3.1) is also invariant under replacing k by $n+1-k$, it suffices to prove the theorem when $k \le \frac{1}{2}(n+1)$.

We reduce the proof of the theorem to an analogous situation where the twists are given on one of the L_i. Let $\tilde{Z}_k^{\langle n\rangle}$ be the subscheme of $\boldsymbol{Gr}(n,k) \times \boldsymbol{Gr}(n,k-1)$ whose S-valued points are the isomorphism classes of pairs $(\tilde{L}_1, \tilde{L}_2)$, where $\tilde{L}_1$ and $\tilde{L}_2$ are respectively subbundles of $\mathcal{O}_S^{\oplus n}$ of rank k and $k-1$ satisfying $\tilde{L}_2 \subseteq \tilde{L}_1$ and $\tilde{L}_2^{(p^2)} \subseteq \tilde{L}_1$. The relative Frobenius morphisms on the two Grassmannian factors induce two morphisms

$$\begin{array}{ccccc}
Z_k^{\langle n\rangle} & \xrightarrow{\quad\varphi\quad} & \tilde{Z}_k^{\langle n\rangle} & \xrightarrow{\quad\hat{\varphi}\quad} & (Z_k^{\langle n\rangle})^{(p)} \\
(L_1, L_2) & \longmapsto & (L_1^{(p)}, L_2) & & \\
& & (\tilde{L}_1, \tilde{L}_2) & \longmapsto & (\tilde{L}_1, \tilde{L}_2^{(p)}),
\end{array}$$

such that the composition is the relative Frobenius on $\tilde{Z}_k^{\langle n\rangle}$. Using a simple deformation computation, we see that φ has degree p^{n-k} and $\hat{\varphi}$ has degree p^{k-1}. Let $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$ denote the universal subbundles on $\boldsymbol{Gr}(n,k)\times\boldsymbol{Gr}(n,k-1)$ when restricted to $\tilde{Z}_k^{\langle n\rangle}$; let $\tilde{\mathcal{Q}}_1$ and $\tilde{\mathcal{Q}}_2$ denote the universal quotients, respectively. We put

$$\tilde{\mathcal{E}}_k = (\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)})^* \otimes \tilde{\mathcal{L}}_2^{(p^2)} \oplus (\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes \tilde{\mathcal{Q}}_1^*, \tag{5.4.1}$$

which is a vector bundle of rank $n-1$ on $\tilde{Z}_k^{\langle n\rangle}$.

Note that

$$\varphi^*(\tilde{\mathcal{E}}_k) = (\mathcal{L}_1^{(p)}/\mathcal{L}_2^{(p^2)})^*\otimes\mathcal{L}_2^{(p^2)}\oplus(\mathcal{L}_1^{(p)}/\mathcal{L}_2)\otimes\mathcal{Q}_1^{*,(p)}.$$

Comparing with $\mathcal{E}_k$, we see that $c_{n-1}(\varphi^*(\tilde{\mathcal{E}}_k)) = p^{k-1}c_{n-1}(\mathcal{E}_k)$, where the factor p^{k-1} comes from the Frobenius twist on the first factor. Thus, we have

$$\begin{aligned}\int_{\tilde{Z}_k^{\langle n\rangle}} c_{n-1}(\tilde{\mathcal{E}}_k) &= (\deg\varphi)^{-1}\int_{Z_k^{\langle n\rangle}} c_{n-1}(\varphi^*(\tilde{\mathcal{E}}_k))\\ &= p^{k-n}\int_{Z_k^{\langle n\rangle}} p^{k-1}c_{n-1}(\mathcal{E}_k) = p^{2k-n-1}N(n,k).\end{aligned} \tag{5.4.2}$$

Since $d(n-2\delta,k-\delta)+2k-n-1 = 2(k-\delta-1)(n-k-\delta+1)$, the theorem is in fact equivalent to the following (for each fixed k). $\square$

Proposition 5.5. *For $1\le k\le (n+1)/2$, we have*

$$\int_{\tilde{Z}_k^{\langle n\rangle}} c_{n-1}(\tilde{\mathcal{E}}_k) = (-1)^{n-1}\sum_{\delta=0}^{k-1}(n-2\delta)p^{2(k-\delta-1)(n-k-\delta+1)}\binom{n}{\delta}_{p^2}. \tag{5.5.1}$$

Remark 5.6. Before giving the proof of this proposition, we point out a variant of the construction of $\tilde{Z}_k^{\langle n\rangle}$. Let $\tilde{Z}_k^{\prime\langle n\rangle}$ be the subscheme of $\boldsymbol{Gr}(n,k)\times\boldsymbol{Gr}(n,k-1)$ whose S-valued points are the isomorphism classes of pairs $(\tilde{L}'_1,\tilde{L}'_2)$, where $\tilde{L}'_1$ and $\tilde{L}'_2$ are respectively subbundles of $\mathcal{O}_S^{\oplus n}$ of rank k and $k-1$ satisfying $\tilde{L}'_2\subseteq\tilde{L}'_1$ and $\tilde{L}'_2\subseteq\tilde{L}_1^{\prime(p^2)}$ (Note that the twist is on L'_1 as opposed to be on L'_2). This is again a certain partial-Frobenius twist of $Z_k^{\langle n\rangle}$; it is smooth of dimension $n-1$. Define the universal subbundles and quotient bundles $\tilde{\mathcal{L}}'_1$, $\tilde{\mathcal{L}}'_2$, $\tilde{\mathcal{Q}}'_1$, and $\tilde{\mathcal{Q}}'_2$ similarly. We put

$$\tilde{\mathcal{E}}'_k = (\tilde{\mathcal{L}}'_1/\tilde{\mathcal{L}}'_2)^*\otimes\tilde{\mathcal{L}}'_2\oplus(\tilde{\mathcal{L}}_1^{\prime(p^2)}/\tilde{\mathcal{L}}'_2)\otimes(\tilde{\mathcal{Q}}_1^{\prime *})^{(p^2)}.$$

Using the same argument as above, we see that, for every fixed k,

$$\int_{\tilde{Z}_k^{\prime\langle n\rangle}} c_{n-1}(\tilde{\mathcal{E}}'_k) = p^{n+1-2k}N(n,k).$$

Note that the exponent is different from (5.4.2). So Proposition 5.5 for each fixed k is equivalent to

$$\int_{\tilde{Z}_k'^{\langle n\rangle}} c_{n-1}(\tilde{\mathcal{E}}_k') = (-1)^{n-1}\sum_{\delta=0}^{k-1}(n-2\delta)p^{2(k-\delta)(n-k-\delta)}\binom{n}{\delta}_{p^2},$$

as $2(k-\delta)(n-k-\delta) = d(n-2\delta, k-\delta)+n-2k+1$.

Proof of Proposition 5.5. We first prove it in the case of $k=1,2$ and then we explain an inductive process to deal with the general case.

When $k=1$, $\tilde{Z}_1^{\langle n\rangle}$ classifies a line subbundle $\tilde{L}_1$ of $\mathcal{O}_S^{\oplus n}$ with no additional condition (as $\tilde{L}_2$ is zero); so $\tilde{Z}_1^{\langle n\rangle}\cong\mathbb{P}^{n-1}$ and $\tilde{\mathcal{L}}_1=\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. The vector bundle $\tilde{\mathcal{E}}_1$ is equal to $\tilde{\mathcal{L}}_1\otimes\tilde{\mathcal{Q}}_1^*$. It is straightforward to check that

$$c(\tilde{\mathcal{E}}_1)=\big(1+c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(-1))\big)^n \quad\text{and hence}\quad \int_{\tilde{Z}_1^{\langle n\rangle}} c_{n-1}(\tilde{\mathcal{E}}_1)=(-1)^{n-1}n;$$

the proposition is proved in this case.

When $k=2$, we consider a forgetful morphism

$$\psi:\quad \tilde{Z}_2^{\langle n\rangle}\to\tilde{Z}_1^{\langle n\rangle},\quad (\tilde{L}_1,\tilde{L}_2)\mapsto\tilde{L}_2.$$

This morphism is an isomorphism over the closed points $x\in\tilde{Z}_1^{\langle n\rangle}(\bar{\mathbb{F}}_p)$ for which $\tilde{L}_{2,x}\neq\tilde{L}_{2,x}^{(p^2)}$, because in this case $\tilde{L}_{1,x}$ is forced to be $\tilde{L}_{2,x}+\tilde{L}_{2,x}^{(p^2)}$. On the other hand, for a closed point $x\in\tilde{Z}_1^{\langle n\rangle}(\bar{\mathbb{F}}_p)$ where $\tilde{L}_{2,x}=\tilde{L}_{2,x}^{(p^2)}$, i.e., for $x\in\tilde{Z}_1^{\langle n\rangle}(\mathbb{F}_{p^2})\cong\mathbb{P}^{n-1}(\mathbb{F}_{p^2})$, $\psi^{-1}(x)$ is the space classifying a line $\tilde{L}_1$ in $\bar{\mathbb{F}}_p^{\oplus n}/\tilde{L}_{2,x}$; so $\psi^{-1}(x)\simeq\mathbb{P}^{n-2}$. A simple tangent space computation shows that ψ is the blowup morphism of $\tilde{Z}_1^{\langle n\rangle}\cong\mathbb{P}^{n-1}$ at all of its $\mathbb{F}_{p^2}$-points. We use E to denote the exceptional divisors, which is a disjoint union of $\binom{n}{1}_{p^2}$ copies of $\mathbb{P}^{n-2}$.

Note that the vanishing of the morphism $\tilde{\mathcal{L}}_2\to\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)}$ defines the divisor E (as we can see using deformation); so

$$\mathcal{O}_{\tilde{Z}_2^{\langle n\rangle}}(E)\cong\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)}\otimes\tilde{\mathcal{L}}_2^{-1}.$$

Put $\eta=c_1(\tilde{\mathcal{L}}_2)=\psi^*c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(-1))$ and $\xi=c_1(E)$. Then

$$\begin{aligned}c(\tilde{\mathcal{E}}_2)&=c\big((\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)})^*\otimes\tilde{\mathcal{L}}_2^{(p^2)}\big)\cdot c\big((\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes\tilde{\mathcal{Q}}_1^*\big)\\&=(1-\xi+(p^2-1)\eta)\cdot(1+\xi+p^2\eta)^n/(1+\xi+(p^2-1)\eta),\end{aligned}\tag{5.6.1}$$

where the computation of the second term comes from the following two exact sequences

$$0\to(\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes\tilde{\mathcal{Q}}_1^*\to(\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)^{\oplus n}\to(\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes\tilde{\mathcal{L}}_1^*\to 0;$$

$$0\to\mathcal{O}_{\tilde{Z}_2^{\langle n\rangle}}\to(\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes\tilde{\mathcal{L}}_1^*\to(\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes\tilde{\mathcal{L}}_2^*\to 0.$$

Note that $\int_{\tilde{Z}_2^{\langle n\rangle}} \xi^i \eta^j = 0$ unless $(i, j) = (n-1, 0)$ or $(0, n-1)$, in which case

$$\int_{\tilde{Z}_2^{\langle n\rangle}} \eta^{n-1} = (-1)^{n-1} \quad \text{and} \quad \int_{\tilde{Z}_2^{\langle n\rangle}} \xi^{n-1} = (-1)^n \binom{n}{1}_{p^2}.$$

Here, to prove the last formula, we used the fact that the restriction of $\mathcal{O}_{\tilde{Z}_2^{\langle n\rangle}}(E)$ to each irreducible component $\mathbb{P}^{n-2}$ of E is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-2}}(-1)$. So it suffices to compute

- the ξ^{n-1}-coefficient of (5.6.1), which is the same as the ξ^{n-1}-coefficient of $(1-\xi)(1+\xi)^{n-1}$ and is equal to $2-n$; and
- the η^{n-1}-coefficient of (5.6.1), which is the same as the η^{n-1}-coefficient of $(1+(p^2-1)\eta)(1+p^2\eta)^n/(1+(p^2-1)\eta) = (1+p^2\eta)^n$ and is equal to $np^{2(n-1)}$.

To sum up, we see that

$$\int_{\tilde{Z}_2^{\langle n\rangle}} c_{n-1}(\tilde{\mathcal{E}}_2) = (-1)^{n-1} np^{2(n-1)} + (-1)^n (2-n) \binom{n}{1}_{p^2},$$

which is exactly (5.5.1) for $k = 2$.

In general, we make an induction on k. Assume that the proposition is proved for $k-1 \geq 1$ and we now prove the proposition for k (assuming that $k \leq \frac{1}{2}(n+1)$). By Remark 5.6, we get the similar intersection formula for $\tilde{\mathcal{E}}'_{k-1}$ on $\tilde{Z}'^{\langle n\rangle}_{k-1}$:

$$\int_{\tilde{Z}'^{\langle n\rangle}_{k-1}} c_{n-1}(\tilde{\mathcal{E}}'_{k-1}) = (-1)^{n-1} \sum_{\delta=0}^{k-2} (n-2\delta) p^{2(k-\delta-1)(n-k-\delta+1)} \binom{n}{\delta}_{p^2}. \tag{5.6.2}$$

We consider the moduli space W over $\mathbb{F}_{p^2}$ whose S-points are tuples $(\tilde{L}_1, \tilde{L}_2 = \tilde{L}'_2, \tilde{L}'_3)$, where $\tilde{L}_1$, $\tilde{L}_2$ and $\tilde{L}'_3$ are respectively subbundles of $\mathcal{O}_S^{\oplus n}$ of rank k, $k-1$ and $k-2$ satisfying $\tilde{L}'_3 \subset \tilde{L}_2 \subset \tilde{L}_1$ and $\tilde{L}'_3 \subset \tilde{L}_2^{(p^2)} \subset \tilde{L}_1$. It is easy to use deformation theory to check that W is a smooth variety of dimension $n-1$. There are two natural morphisms

$$\begin{tikzcd} & W \arrow[dl, "\psi_{12}"'] \arrow[dr, "\psi_{23}"] & \\ \tilde{Z}_k^{\langle n\rangle} & & \tilde{Z}'^{\langle n\rangle}_{k-1} \end{tikzcd} \qquad \begin{tikzcd} & (\tilde{L}_1, \tilde{L}_2 = \tilde{L}'_2, \tilde{L}'_3) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (\tilde{L}_1, \tilde{L}_2) & & (\tilde{L}'_2, \tilde{L}'_3). \end{tikzcd}$$

Let E denote the subspace of W whose closed points $x \in W(\bar{\mathbb{F}}_p)$ are those such that $\tilde{L}_{2,x} = \tilde{L}_{2,x}^{(p^2)}$, i.e., $\tilde{L}_{2,x}$ is an $\mathbb{F}_{p^2}$-rational subspace of $\mathbb{F}_{p^2}^{\oplus n}$ of dimension $k-1$. It is clear that E is a disjoint union of $\binom{n}{k-1}_{p^2}$ copies (corresponding to the choices of $\tilde{L}_2$) of $\mathbb{P}^{n-k} \times \mathbb{P}^{k-2}$ (corresponding to the choice of $\tilde{L}_1$ and $\tilde{L}'_3$ respectively). It gives rise to a smooth divisor on W.

For a point $x \in (W \setminus E)(\bar{\mathbb{F}}_p)$, we have $\tilde{L}_{2,x} \neq \tilde{L}_{2,x}^{(p^2)}$ and hence it uniquely determines both $\tilde{L}_{1,x}$ and $\tilde{L}'_{3,x}$; so ψ_{12} and ψ_{23} are isomorphisms restricted to $W \setminus E$. On the other hand, when restricted to E, ψ_{12} contracts each copy of $\mathbb{P}^{n-k}\times\mathbb{P}^{k-2}$ of E into the first factor $\mathbb{P}^{n-k}$; whereas ψ_{23} contracts each copy of $\mathbb{P}^{n-k}\times\mathbb{P}^{k-2}$ of E into the second factor $\mathbb{P}^{k-2}$. It is clear from this (with a little bit of help from a deformation argument) that ψ_{12} is the blowup of $\tilde{Z}_k^{\langle n\rangle}$ along $\psi_{12}(E)$ and ψ_{23} is the blowup of $\tilde{Z}'^{\langle n\rangle}_{k-1}$ along $\psi_{23}(E)$; the divisor E is the exceptional divisor for both blowups.

A simple deformation theory argument shows that the normal bundle of E in W when restricted to each component $\mathbb{P}^{n-k}\times\mathbb{P}^{k-2}$ is $\mathcal{O}_{\mathbb{P}^{n-k}}(-1)\otimes\mathcal{O}_{\mathbb{P}^{k-2}}(-1)$. Moreover, we can characterize E as the zero locus of either one of the following natural homomorphisms

$$\tilde{\mathcal{L}}_2^{(p^2)}/\tilde{\mathcal{L}}'_3\to\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2,\quad \tilde{\mathcal{L}}_2/\tilde{\mathcal{L}}'_3\to\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)}.$$

So as a line bundle over W, we have

$$\mathcal{O}_W(E)\cong(\tilde{\mathcal{L}}_2^{(p^2)}/\tilde{\mathcal{L}}'_3)^{-1}\otimes(\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\cong(\tilde{\mathcal{L}}_2/\tilde{\mathcal{L}}'_3)^{-1}\otimes(\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)}).$$

We want to compare

$$\begin{aligned}\int_{\tilde{Z}_k^{\langle n\rangle}}c_{n-1}(\tilde{\mathcal{E}}_k)&=\int_W c_{n-1}(\psi_{12}^*(\tilde{\mathcal{E}}_k))\quad\text{and}\\ \int_{\tilde{Z}'^{\langle n\rangle}_{k-1}}c_{n-1}(\tilde{\mathcal{E}}'_{k-1})&=\int_W c_{n-1}(\psi_{23}^*(\tilde{\mathcal{E}}'_{k-1})).\end{aligned}\tag{5.6.3}$$

We will show that they differ by $(2k-n-2)(-1)^n\binom{n}{k-1}_{p^2}$ and this will conclude the proof of the proposition by inductive hypothesis (5.6.2). Indeed, we have

$$c\big(\psi_{12}^*(\tilde{\mathcal{E}}_k)\big)=c\big((\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)})^*\otimes\tilde{\mathcal{L}}_2^{(p^2)}\big)\cdot c\big((\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes\tilde{\mathcal{Q}}_1^*\big),\tag{5.6.4}$$

$$c\big(\psi_{23}^*(\tilde{\mathcal{E}}'_{k-1})\big)=c\big((\tilde{\mathcal{L}}_2/\tilde{\mathcal{L}}'_3)^*\otimes\tilde{\mathcal{L}}'_3\big)\cdot c\big((\tilde{\mathcal{L}}_2^{(p^2)}/\tilde{\mathcal{L}}'_3)\otimes\tilde{\mathcal{Q}}_2^{*,(p^2)}\big),\tag{5.6.5}$$

where $\tilde{\mathcal{Q}}_1$ and $\tilde{\mathcal{Q}}_2$ are the universal quotient vector bundles. Consider the following two exact sequences where the two last terms are identified:

$$\begin{tikzcd}
 & \mathcal{O}_W(E)\otimes(\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)})^{-1}\otimes\tilde{\mathcal{L}}_2^{(p^2)} \arrow[d, leftrightarrow, "\cong"] & & \\
0 \arrow[r] & (\tilde{\mathcal{L}}_2/\tilde{\mathcal{L}}'_3)^{-1}\otimes\tilde{\mathcal{L}}'_3 \arrow[r] & (\tilde{\mathcal{L}}_2/\tilde{\mathcal{L}}'_3)^{-1}\otimes\tilde{\mathcal{L}}_2^{(p^2)} \arrow[r] & (\tilde{\mathcal{L}}_2/\tilde{\mathcal{L}}'_3)^{-1}\otimes(\tilde{\mathcal{L}}_2^{(p^2)}/\tilde{\mathcal{L}}'_3) \arrow[r] \arrow[d, leftrightarrow, "\cong"] & 0 \\
0 \arrow[r] & (\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes\tilde{\mathcal{Q}}_1^* \arrow[r] & (\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes\tilde{\mathcal{Q}}_2^{*,(p^2)} \arrow[r] \arrow[d, leftrightarrow, "\cong"] & (\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes(\tilde{\mathcal{Q}}_2^{*,(p^2)}/\tilde{\mathcal{Q}}_1^*) \arrow[r] & 0. \\
 & & \mathcal{O}_W(E)\otimes(\tilde{\mathcal{L}}_2^{(p^2)}/\tilde{\mathcal{L}}'_3)\otimes\tilde{\mathcal{Q}}_2^{*,(p^2)} & &
\end{tikzcd}$$

Here the right vertical isomorphism is given by

$$\begin{aligned}(\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes(\tilde{\mathcal{Q}}_2^{*,(p^2)}/\tilde{\mathcal{Q}}_1^*) &\cong (\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes(\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)})^{-1}\\ &\cong \big((\tilde{\mathcal{L}}_2^{(p^2)}/\tilde{\mathcal{L}}_3')\otimes\mathcal{O}_W(E)\big)\otimes\big((\tilde{\mathcal{L}}_2/\tilde{\mathcal{L}}_3')\otimes\mathcal{O}_W(E)\big)^{-1}\\ &\cong (\tilde{\mathcal{L}}_2/\tilde{\mathcal{L}}_3')^{-1}\otimes(\tilde{\mathcal{L}}_2^{(p^2)}/\tilde{\mathcal{L}}_3').\end{aligned}$$

From these two exact sequences we see that

$$\begin{aligned}&c\big((\tilde{\mathcal{L}}_2/\tilde{\mathcal{L}}_3')^{-1}\otimes\tilde{\mathcal{L}}_3'\big)\cdot c\big(\mathcal{O}_W(E)\otimes(\tilde{\mathcal{L}}_2^{(p^2)}/\tilde{\mathcal{L}}_3')\otimes\tilde{\mathcal{Q}}_2^{*,(p^2)}\big)\\ &\qquad= c\big((\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes\tilde{\mathcal{Q}}_1^*\big)\cdot c\big(\mathcal{O}_W(E)\otimes(\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)})^{-1}\otimes\tilde{\mathcal{L}}_2^{(p^2)}\big).\end{aligned}$$

Comparing this with (5.6.5) and (5.6.4), we get

$$\begin{aligned}&c_{n-1}(\psi_{12}^*(\tilde{\mathcal{E}}_k))-c_{n-1}(\psi_{23}^*(\tilde{\mathcal{E}}'_{k-1}))\\ &=\big(c_{k-1}((\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)})^{-1}\otimes\tilde{\mathcal{L}}_2^{(p^2)})-c_{k-1}(\mathcal{O}_W(E)\otimes(\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2^{(p^2)})^{-1}\otimes\tilde{\mathcal{L}}_2^{(p^2)})\big)\cdot c_{n-k}((\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes\tilde{\mathcal{Q}}_1^*)\\ &-c_{k-2}((\tilde{\mathcal{L}}_2/\tilde{\mathcal{L}}_3')^{-1}\otimes\tilde{\mathcal{L}}_3')\cdot\big(c_{n-k+1}((\tilde{\mathcal{L}}_2^{(p^2)}/\tilde{\mathcal{L}}_3')\otimes\tilde{\mathcal{Q}}_2^{*,(p^2)})-c_{n-k+1}(\mathcal{O}_W(E)\otimes(\tilde{\mathcal{L}}_2^{(p^2)}/\tilde{\mathcal{L}}_3')\otimes\tilde{\mathcal{Q}}_2^{*,(p^2)})\big).\end{aligned}$$

Recall that E is the exceptional divisor for the blow-up ψ_{12} centered at a disjoint union of $\mathbb{P}^{n-k}$; so $c_1(E)$ kills $\psi_{12}^*(A^i(\tilde{Z}_k^{\langle n\rangle}))$ for $i\geq n-k+1$. Similarly, $c_1(E)$ kills $\psi_{23}^*(A^i(\tilde{Z}_{k-1}^{\prime\langle n\rangle}))$ for $i\geq k-1$. As a result, we can rewrite the above complicated formula as

$$\begin{aligned}&c_{n-1}(\psi_{12}^*(\tilde{\mathcal{E}}_k))-c_{n-1}(\psi_{23}^*(\tilde{\mathcal{E}}'_{k-1}))\\ &\quad=-c_1(E)^{k-2}|_E\cdot c_{n-k}((\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes\tilde{\mathcal{Q}}_1^*)+c_{k-2}((\tilde{\mathcal{L}}_2/\tilde{\mathcal{L}}_3')^{-1}\otimes\tilde{\mathcal{L}}_3')\cdot c_1(E)^{n-k}|_E\\ &\quad=(-1)^{k-1}c_{n-k}((\tilde{\mathcal{L}}_1/\tilde{\mathcal{L}}_2)\otimes\tilde{\mathcal{Q}}_1^*)|_{\psi_{12}(E)}+(-1)^{n-k}c_{k-2}((\tilde{\mathcal{L}}_2/\tilde{\mathcal{L}}_3')^{-1}\otimes\tilde{\mathcal{L}}_3')|_{\psi_{23}(E)}.\end{aligned}$$

For the first term, over each $\mathbb{P}^{n-k}$ of $\psi_{12}(E)$, it is to take the top Chern class of the canonical subbundle of rank $n-k$ twisted by $\mathcal{O}_{\mathbb{P}^{n-k}}(-1)$; so the degree of the first term is $(-1)^{n-k}(n-k+1)$ on each $\mathbb{P}^{n-k}$. Similarly, for the second term, over each $\mathbb{P}^{k-2}$, it is the top Chern class of the canonical subbundle of rank $k-2$ twisted by $\mathcal{O}_{\mathbb{P}^{k-2}}(-1)$; so the degree of the second term is $(-1)^{k-2}(k-1)$ on each $\mathbb{P}^{k-2}$. To sum up, we have

$$\begin{aligned}&\int_W c_{n-1}(\psi_{12}^*(\tilde{\mathcal{E}}_k))-\int_W c_{n-1}(\psi_{23}^*(\tilde{\mathcal{E}}'_{k-1}))\\ &\qquad=(-1)^{k-1}(-1)^{n-k}(n-k+1)\binom{n}{k-1}_{p^2}+(-1)^{n-k}(-1)^{k-2}(k-1)\binom{n}{k-1}_{p^2}\\ &\qquad=(-1)^{n-1}(n-2k+2)\binom{n}{k-1}_{p^2}.\end{aligned}\tag{5.6.6}$$

So by the inductive hypothesis,

$$\begin{aligned}\int_{\tilde{Z}_k^{\langle n\rangle}} c_{n-1}(\tilde{\mathcal{E}}_k) &\overset{(5.6.3)}{=} \int_W c_{n-1}(\psi_{12}^*(\tilde{\mathcal{E}}_k)) \\ &\overset{(5.6.6)}{=} \int_W c_{n-1}(\psi_{23}^*(\tilde{\mathcal{E}}_k)) + (-1)^{n-1}(n-2k+2)\binom{n}{k-1}_{p^2}. \\ &\overset{(5.6.3)}{=} \int_{\tilde{Z}_{k-1}^{\prime\langle n\rangle}} c_{n-1}(\tilde{\mathcal{E}}'_{k-1}) + (-1)^{n-1}(n-2k+2)\binom{n}{k-1}_{p^2} \\ &= (-1)^{n-1}\sum_{\delta=0}^{k-2}(n-2\delta)p^{2(k-\delta-1)(n-k-\delta+1)}\binom{n}{\delta}_{p^2} \\ &\qquad\qquad + (-1)^{n-1}(n-2k+2)\binom{n}{k-1}_{p^2} \\ &= (-1)^{n-1}\sum_{\delta=0}^{k-1}(n-2\delta)p^{2(k-\delta-1)(n-k-\delta+1)}\binom{n}{\delta}_{p^2}.\end{aligned}$$

This shows the statement of the proposition for k and hence concludes the proof. □

6. Intersection matrix of supersingular cycles on $\mathrm{Sh}_{1,n-1}$

Throughout this section, we fix an integer $n \geq 2$ and keep the notation as in Section 4. We will study the intersection theory of cycles Y_j for $1 \leq j \leq n$ on $\mathrm{Sh}_{1,n-1}$ considered in Section 4. For this, we may assume the following:

Hypothesis 6.1. We assume that the tame level structure K^p is taken sufficiently small so that Lemma 4.13 holds with $N = 2$.

6.2. ***Hecke correspondences on*** $\mathbf{Sh_{0,n}}$**.** Recall that we have an isomorphism

$$G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_\mathfrak{p}) \cong \mathbb{Q}_p^\times \times \mathrm{GL}_n(\mathbb{Q}_{p^2}).$$

Put $K_\mathfrak{p} = \mathrm{GL}_n(\mathcal{O}_{E_\mathfrak{p}})$ and $K_p = \mathbb{Z}_p^\times \times K_\mathfrak{p}$. The Hecke algebra $\mathbb{Z}[K_\mathfrak{p} \backslash \mathrm{GL}_n(E_\mathfrak{p})/K_\mathfrak{p}]$ can be viewed as a subalgebra of $\mathbb{Z}[K_p \backslash G(\mathbb{Q}_p)/K_p]$ (with trivial factor at the $\mathbb{Q}_p^\times$-component).

For $\gamma \in \mathrm{GL}_n(E_\mathfrak{p})$, the double coset $T_\mathfrak{p}(\gamma) := K_\mathfrak{p}\gamma K_\mathfrak{p}$ defines a Hecke correspondence on $\mathrm{Sh}_{0,n}$. It induces a set theoretic Hecke correspondence

$$T_\mathfrak{p}(\gamma) : \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p) \to \mathcal{S}(\mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)),$$

where $\mathcal{S}(\mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p))$ denotes the set of subsets of $\mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$. By Remark 4.12, $\mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$ is a union of $\#\ker^1(\mathbb{Q}, G_{0,n})$-isogeny classes of abelian varieties. Fix a base point $z_0 \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$. Let

$$\Theta_{z_0} : \mathrm{Isog}(z_0) \xrightarrow{\sim} G_{0,n}(\mathbb{Q}) \backslash (G(\mathbb{A}^{\infty,p}) \times G(\mathbb{Q}_p))/K^p \times K_p.$$

be the bijection constructed as in Corollary 4.11. Write $K_{\mathfrak{p}}\gamma K_{\mathfrak{p}} = \coprod_{i\in I}\gamma_i K_{\mathfrak{p}}$. If $z \in \mathrm{Isog}(z_0)$ corresponds to the class of $(g^p, g_p) \in G(\mathbb{A}^{\infty,p}) \times G(\mathbb{Q}_p)$ with $g_p = (g_{p,0}, g_{\mathfrak{p}})$, then $T_{\mathfrak{p}}(\gamma)(z)$ consists of points in $\mathrm{Isog}(z_0)$ corresponding to the class of $(g^p, (g_{p,0}, g_{\mathfrak{p}}\gamma_i))$ for all $i \in I$.

Alternatively, $T_{\mathfrak{p}}(\gamma)$ has the following description. Write $z = (A, \lambda, \eta)$, and let $\mathbb{L}_z$ denote the $\mathbb{Z}_{p^2}$-free module $\tilde{\mathcal{D}}(A)_1^{\circ, F^2=p}$. Then a point $z' = (B, \lambda', \eta') \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$ belongs to $T_{\mathfrak{p}}(\gamma)(z)$ if and only if there exists an $\mathcal{O}_D$-equivariant p-quasi-isogeny $\phi : B' \to B$ (i.e., $p^m\phi$ is an isogeny of p-power order for some integer m) such that

(1) $\phi^\vee \circ \lambda \circ \phi = \lambda'$,

(2) $\phi \circ \eta' = \eta$,

(3) $\phi_*(\mathbb{L}_{z'})$ is a lattice of $\mathbb{L}_z[1/p] = \mathbb{L}_z \otimes_{\mathbb{Z}_{p^2}} \mathbb{Q}_{p^2}$ with the property: there exists a $\mathbb{Z}_{p^2}$-basis $(e_1, \dots, e_n)$ for $\mathbb{L}_z$ such that $(e_1, \dots, e_n)\gamma$ is a $\mathbb{Z}_{p^2}$-basis for $\phi_*(\mathbb{L}_{z'})$.

When $\gamma = \mathrm{Diag}(p^{a_1}, \dots, p^{a_n})$ with $a_i \in \{-1, 0, 1\}$, For given z and z', such a ϕ is necessarily unique if it exists, by Lemma 4.13 (with $N = 2$). Therefore, $T_{\mathfrak{p}}(\gamma)(z)$ is in natural bijection with the set of $\mathbb{Z}_{p^2}$-lattices $\mathbb{L}' \subseteq \mathbb{L}_z[1/p]$ satisfying property (3) above.

For each integer i with $0 \le i \le n$, we put

$$T_{\mathfrak{p}}^{(i)} = T_{\mathfrak{p}}(\mathrm{Diag}(\underbrace{p, \dots, p}_{i}, \underbrace{1, \dots, 1}_{n-i})).$$

By the discussion above, one has a natural bijection

$$T_{\mathfrak{p}}^{(i)}(z) \xrightarrow{\sim} \{\mathbb{L}_{z'} \subseteq \mathbb{L}_z[1/p] \mid p\mathbb{L}_z \subseteq \mathbb{L}_{z'} \subseteq \mathbb{L}_z,\ \dim_{\mathbb{F}_{p^2}}(\mathbb{L}_z/\mathbb{L}_{z'}) = i\}$$

for $z \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$. Note that $T_{\mathfrak{p}}^{(0)} = \mathrm{id}$, and we put $S_{\mathfrak{p}} := T_{\mathfrak{p}}^{(n)}$. Then the Satake isomorphism implies $\mathbb{Z}[K_{\mathfrak{p}} \backslash \mathrm{GL}_n(E_{\mathfrak{p}})/K_{\mathfrak{p}}] \cong \mathbb{Z}[T_{\mathfrak{p}}^{(1)}, \dots, T_{\mathfrak{p}}^{(n-1)}, S_{\mathfrak{p}}, S_{\mathfrak{p}}^{-1}]$. More generally, for $0 \le a \le b \le n$, we put

$$R_{\mathfrak{p}}^{(a,b)} = T_{\mathfrak{p}}(\mathrm{Diag}(\underbrace{p^2, \dots, p^2}_{a}, \underbrace{p, \dots, p}_{b-a}, \underbrace{1, \dots, 1}_{n-b})).$$

Note that $R_{\mathfrak{p}}^{(0,i)} = T_{\mathfrak{p}}^{(i)}$, and $R_{\mathfrak{p}}^{(a,b)}S_{\mathfrak{p}}^{-1}$ is the Hecke operator

$$T_{\mathfrak{p}}(\mathrm{Diag}(\underbrace{p, \dots, p}_{a}, \underbrace{1, \dots, 1}_{b-a}, \underbrace{p^{-1}, \dots, p^{-1}}_{n-b})).$$

For the explicit relations between $R_{\mathfrak{p}}^{(a,b)}$ and $T_{\mathfrak{p}}^{(i)}$, see Proposition A.1.

6.3. *Refined Gysin homomorphism.* For an algebraic variety X over $\bar{\mathbb{F}}_p$ of pure dimension N and any integer $r \ge 0$, we write $A_r(X) = A^{N-r}(X)$ to denote the group of dimension r (codimension $N - r$) cycles in X modulo rational equivalence. Recall that the restriction of $\mathrm{pr}_j : Y_j \to \mathrm{Sh}_{1,n-1}$ to each $Y_{j,z}$ for $z \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$ and

$1 \le j \le n$ is a regular closed immersion (into $\overline{\mathrm{Sh}}_{1,n-1}$). There is a well-defined Gysin homomorphism

$$\mathrm{pr}_j^! : A_{n-1}(\overline{\mathrm{Sh}}_{1,n-1}) \to A_0(\bar{Y}_j) = \bigoplus_{z \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)} A_0(Y_{j,z}), \tag{6.3.1}$$

whose composition with the natural projection $A_0(\bar{Y}_j) \to A_0(Y_{j,z})$ is the refined Gysin map $(\mathrm{pr}_j\,|_{Y_{j,z}})^!$ defined in [Fulton 1998, 6.2] for regular immersions. Let $X \subseteq \overline{\mathrm{Sh}}_{1,n-1}$ be a closed subvariety of dimension $n-1$. Consider the Cartesian diagram

$$\begin{tikzcd} \bar{Y}_j \times_{\overline{\mathrm{Sh}}_{1,n-1}} X \arrow[r, "g_X"] \arrow[d, "g_j"'] & X \arrow[d] \\ \bar{Y}_j \arrow[r, "\mathrm{pr}_j"] & \overline{\mathrm{Sh}}_{1,n-1}. \end{tikzcd}$$

Assume that the restriction of g_X to each $Y_{j,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} X$ with $z \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$ is a regular closed immersion as well. Then $\mathrm{pr}_j^!([X]) \in A_0(\bar{Y}_j)$ can be described as follows. Put $N_{Y_{j,z}}(\overline{\mathrm{Sh}}_{1,n-1}) := \mathrm{pr}_j^*(T_{\overline{\mathrm{Sh}}_{1,n-1}})/T_{Y_{j,z}}$, and we define $N_{Y_{j,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} X}(X)$ in a similar way. We define the *excess vector bundle* as

$$\mathcal{E}(Y_{j,z}, X) := g_j^* N_{Y_{j,z}}(\overline{\mathrm{Sh}}_{1,n-1}) / N_{Y_{j,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} X}(X).$$

This is a vector bundle on $Y_{j,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} X$. Let r be its rank function, which is equal to the dimension of $\bar{Y}_j \times_{\overline{\mathrm{Sh}}_{1,n-1}} X$ on each of its connected component. Then the excess intersection formula [Fulton 1998, 6.3] shows that

$$\mathrm{pr}_j^!([X]) = \sum_{z \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)} \int_{Y_{j,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} X} c_r(\mathcal{E}(Y_{j,z}, X)), \tag{6.3.2}$$

where $c_r(\mathcal{E}(Y_{j,z}, X))$ is the top Chern class of $\mathcal{E}(Y_{j,z}, X)$ over $Y_{j,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} X$. The integration should be understood as the sum over all connected components of $Y_{j,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} X$ of the degrees of $c_r(\mathcal{E}(Y_{j,z}, X))$.

Proposition 6.4. *Let i, j be integers with $1 \le i \le j \le n$ and $z, z' \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$.*

(1) *The subvarieties $Y_{i,z}$ and $Y_{j,z'}$ of $\overline{\mathrm{Sh}}_{1,n-1}$ have nonempty intersection if and only if there exists an integer δ with $0 \le \delta \le \min\{n-j, i-1\}$ such that $z' \in R_{\mathfrak{p}}^{(j-i+\delta, n-\delta)} S_{\mathfrak{p}}^{-1}(z)$, or equivalently $z \in R_{\mathfrak{p}}^{(\delta, n+i-j-\delta)} S_{\mathfrak{p}}^{-1}(z')$, where $R_{\mathfrak{p}}^{(a,b)}$ and $S_{\mathfrak{p}}$ are the Hecke operators defined in Section 6.2.*

(2) *If the condition in (1) is satisfied for some δ, then $Y_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{j,z'}$ is isomorphic to the variety $\bar{Z}_{i-\delta}^{\langle n+i-j-2\delta \rangle}$ defined in Section 5.2. Moreover, the excess vector bundles $\mathcal{E}(Y_{i,z}, Y_{j,z'})$ and $\mathcal{E}(Y_{j,z'}, Y_{i,z})$ are both isomorphic to the vector bundle (5.2.1) on $\bar{Z}_{i-\delta}^{\langle n+i-j-2\delta \rangle}$.*

Proof. Let $(\mathcal{B}_z, \lambda_z, \eta_z)$ and $(\mathcal{B}_{z'}, \lambda_{z'}, \eta_{z'})$ be the universal polarized abelian varieties on $\overline{\mathrm{Sh}}_{0,n}$ at z and z', respectively. Then $Y_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{j,z'}$ is the moduli space of tuples $(A, \lambda, \eta, \phi, \phi')$ where $\phi : \mathcal{B}_z \to A$ and $\phi' : \mathcal{B}_{z'} \to A$ are isogenies such that $(A, \lambda, \eta, \mathcal{B}_z, \lambda_z, \eta_z, \phi)$ and $(A, \lambda, \eta, \mathcal{B}_{z'}, \eta_{z'}, \phi')$ are points of $Y_{i,z}$ and $Y_{j,z'}$ respectively.

Assume first that $Y_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{j,z'}$ is nonempty, and let $(A, \lambda, \eta, \phi, \phi')$ be an $\bar{\mathbb{F}}_p$-valued point of it. Denote by $\tilde{\omega}^\circ_{A^\vee,k} \subseteq \tilde{\mathcal{D}}(A)^\circ_k$ for $k = 1, 2$ the inverse image of $\omega^\circ_{A^\vee/\bar{\mathbb{F}}_p,k} \subseteq H_1^{\mathrm{dR}}(A/\bar{\mathbb{F}}_p)^\circ_k \cong \tilde{\mathcal{D}}(A)^\circ_k / p\tilde{\mathcal{D}}(A)^\circ_k$. We identify $\tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_k$ and $\tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_k$ with their images in $\tilde{\mathcal{D}}(A)^\circ_k$ via $\phi_{z,*,k}$ and $\phi_{z',*,k}$. Then we have a diagram of inclusions of $W(\bar{\mathbb{F}}_p)$-modules:

$$\begin{tikzcd}[column sep=small]
 & & & \tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_1 \arrow[dr, hook, "\delta"] & & \\
p\tilde{\mathcal{D}}(A)^\circ_1 \arrow[r, hook, "1"] & \tilde{\omega}^\circ_{A^\vee,1} \arrow[r, hook, "n-j-\delta"] & \tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_1 \cap \tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_1 \arrow[ur, hook, "j-i+\delta"] \arrow[dr, hook, "\delta"'] & & \tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_1 + \tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_1 \arrow[r, hook, "i-\delta-1"] & \tilde{\mathcal{D}}(A)^\circ_1. \\
 & & & \tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_1 \arrow[ur, hook, "j-i+\delta"'] & &
\end{tikzcd} \tag{6.4.1}$$

Here the numbers on the arrows indicate the $\bar{\mathbb{F}}_p$-dimensions of the cokernel of the corresponding inclusions, which we shall compute below. By the definition of Y_i and Y_j, we have

$$\dim_{\bar{\mathbb{F}}_p}(\tilde{\mathcal{D}}(A)^\circ_1 / \tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_1) = \dim_{\bar{\mathbb{F}}_p} \mathrm{Coker}(\phi_{*,1}) = i - 1,$$

and similarly, $\dim_{\bar{\mathbb{F}}_p}(\tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_1 / \tilde{\omega}^\circ_{A^\vee,1}) = n - j$. Therefore, if we put

$$\delta = \dim_{\bar{\mathbb{F}}_p} \big(\tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_1 + \tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_1\big) / \tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_1 = \dim_{\bar{\mathbb{F}}_p} \tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_1 / \big(\tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_1 \cap \tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_1\big),$$

we have $0 \le \delta \le \min\{i-1, n-j\}$. Moreover, the quasi-isogeny $\phi_{z,z'} = \phi^{-1} \circ \phi' : \mathcal{B}_{z'} \to \mathcal{B}_z$ makes $\mathcal{B}_{z'}$ an element of $\mathrm{Isog}(z)$. We identify $\mathbb{L}_{z'}$ defined in (4.11.1) with a $\mathbb{Z}_{p^2}$-lattice of $\mathbb{L}_z[1/p]$ via $\phi_{z',z,*,1}$. Then

$$\dim_{\mathbb{F}_{p^2}} (\mathbb{L}_z \cap \mathbb{L}_{z'}) / p\mathbb{L}_z = \dim_{\bar{\mathbb{F}}_p} (\tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_1 \cap \tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_1) / p\tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_1 = n + i - j - \delta.$$

Take a $\mathbb{Z}_{p^2}$-basis $(e_1, \dots, e_n)$ of $\mathbb{L}_z$ such that the image of $(e_{j-i+\delta+1}, \dots, e_n)$ in $\mathbb{L}_z / p\mathbb{L}_z$ form a basis of $(\mathbb{L}_z \cap \mathbb{L}_{z'}) / p\mathbb{L}_z$ and such that $p^{-1}e_{n-\delta+1}, \dots, p^{-1}e_n$ form a basis of $(\mathbb{L}_z + \mathbb{L}_{z'}) / \mathbb{L}_z$. Then

$$(pe_1, \dots, pe_{j-i+\delta}, e_{j-i+\delta+1}, \dots, e_{n-\delta}, p^{-1}e_{n-\delta+1}, \dots, p^{-1}e_n) \tag{6.4.2}$$

is a basis of $\mathbb{L}_{z'}$, that is $z' \in R_{\mathfrak{p}}^{(j-i+\delta, n-\delta)} S_{\mathfrak{p}}^{-1}(z)$ according to the convention of Section 6.2.

Conversely, assume that there exists δ with $1 \le \delta \le \min\{i-1, n-j\}$ such that the point $z' \in R_{\mathfrak{p}}^{(j-i+\delta,n-\delta)} S_{\mathfrak{p}}^{-1}(z)$. We have to prove statement (2), then the nonemptiness of $Y_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{j,z'}$ will follow automatically. Let $\phi_{z',z} : \mathcal{B}_{z'} \to \mathcal{B}_z$ be the unique quasi-isogeny which identifies $\mathbb{L}_{z'}$ with a $\mathbb{Z}_{p^2}$-lattice of $\mathbb{L}_z[1/p]$. By the definition of $R_{\mathfrak{p}}^{(j-i+\delta,n-\delta)} S_{\mathfrak{p}}^{-1}$, there exists a basis $e_1, \dots, e_n$ of $\mathbb{L}_z$ such that (6.4.2) is a basis of $\mathbb{L}_{z'}$. One checks easily that $p(\mathbb{L}_z + \mathbb{L}_{z'}) \subseteq \mathbb{L}_z \cap \mathbb{L}_{z'}$. We put

$$M_k = \big(\tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_k \cap \tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_k\big) / p\big(\tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_k + \tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_k\big)$$

for $k = 1, 2$. Then one has

$$\dim_{\bar{\mathbb{F}}_p}(M_k) = \dim_{\mathbb{F}_{p^2}}(\mathbb{L}_z \cap \mathbb{L}_{z'})/p(\mathbb{L}_z + \mathbb{L}_{z'}) = n+i-j-2\delta.$$

The Frobenius and Verschiebung on $\tilde{\mathcal{D}}(\mathcal{B}_z)$ induce two bijective Frobenius semilinear maps $F : M_1 \to M_2$ and $V^{-1} : M_2 \to M_1$. We denote their linearizations by the same notation if no confusions arise. Let $Z_\delta(M_\bullet)$ be the moduli space which attaches to each locally noetherian $\bar{\mathbb{F}}_p$-scheme S the set of isomorphism classes of pairs (L_1, L_2), where $L_1 \subseteq M_1 \otimes_{\bar{\mathbb{F}}_p} \mathcal{O}_S$ and $L_2 \subseteq M_2 \otimes_{\bar{\mathbb{F}}_p} \mathcal{O}_S$ are subbundles of rank $i-\delta$ and $i-1-\delta$ respectively such that

$$L_2 \subseteq F(L_1^{(p)}), \quad V^{-1}(L_2^{(p)}) \subseteq L_1.$$

Note that there exists a basis $(\varepsilon_{k,1}, \dots, \varepsilon_{k,n+i-j-2\delta})$ of M_k for $k = 1, 2$ under which the matrices of F and V^{-1} are both identity. Indeed, by solving a system of equations of Artin–Schreier type, one can take a basis $(\varepsilon_{1,\ell})_{1 \le \ell \le n+i-j-2\delta}$ for M_1 such that

$$V^{-1}(F(\varepsilon_{1,\ell})) = \varepsilon_{1,\ell} \qquad \text{for all } 1 \le \ell \le n+i-j-2\delta.$$

We put $\varepsilon_{2,\ell} = F(\varepsilon_{1,\ell})$. Using these bases to identify both M_1 and M_2 with $\bar{\mathbb{F}}_p^{n+i-j-2\delta}$, it is clear that $Z_\delta(M_\bullet)$ is isomorphic to the variety $\bar{Z}_{i-\delta}^{\langle n+i-j-2\delta \rangle}$ considered in Section 5.2.

We have to establish an isomorphism between $Z_\delta(M_\bullet)$ and $Y_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{j,z'}$. Let (L_1, L_2) be an S-point of $Z_\delta(M_\bullet)$. Note that there is a natural surjection

$$\big((\tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_k \cap \tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_k)/p\tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_k\big) \otimes_{\bar{\mathbb{F}}_p} \mathcal{O}_S \to M_k \otimes_{\bar{\mathbb{F}}_p} \mathcal{O}_S.$$

We define $H_{z,k}$ for $k = 1, 2$ to be the inverse image of L_k under this surjection. Then $H_{z,k}$ can be naturally viewed as a subbundle of $\mathcal{D}(\mathcal{B}_z)^\circ_k \otimes_{\bar{\mathbb{F}}_p} \mathcal{O}_S$ of rank $i+1-k$, and we have $H_{z,2} \subseteq F(H_{z,1}^{(p)})$ and $V^{-1}(H_{z,2}^{(p)}) \subseteq H_{z,1}$ since the pair (L_1, L_2) verifies similar properties. Therefore, $(L_1, L_2) \mapsto (\mathcal{B}_{z,S}, \lambda_{z,S}, \eta_{z,S}, H_{z,1}, H_{z,2})$ gives rise to a well-defined map $\varphi'_{i,z} : Z_\delta(M_\bullet) \to Y'_{i,z}$, where $(\mathcal{B}_{z,S}, \lambda_{z,S}, \eta_{z,S})$ is the base change of $(\mathcal{B}_z, \lambda_z, \eta_z)$ to S. Similarly, we have a morphism $\varphi'_{j,z'} : Z_\delta(M_\bullet) \to Y'_{j,z'}$ defined by $(L_1, L_2) \mapsto (\mathcal{B}_{z',S}, \lambda_{z',S}, \eta_{z',S}, H_{z',1}, H_{z',2})$, where $H_{z',k}$ is the inverse image of

L_k under the natural surjection:

$$\big((\tilde{\mathcal{D}}(\mathcal{B}_z)_k^\circ \cap \tilde{\mathcal{D}}(\mathcal{B}_{z'})_k^\circ)/p\tilde{\mathcal{D}}(\mathcal{B}_{z'})_k^\circ\big) \otimes_{\bar{\mathbb{F}}_p} \mathcal{O}_S \to M_k \otimes_{\bar{\mathbb{F}}_p} \mathcal{O}_S.$$

By Proposition 4.8, we get two morphisms

$$\varphi_{i,z} : Z_\delta(M_\bullet) \to Y_{i,z}, \qquad \varphi_{j,z'} : Z_\delta(M_\bullet) \to Y_{j,z'}.$$

We claim that $\mathrm{pr}_i \circ \varphi_{i,z} = \mathrm{pr}_j \circ \varphi_{j,z}$, so that $(\varphi_{i,z}, \varphi_{j,z'})$ defines a map

$$\varphi : Z_\delta(M_\bullet) \to Y_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{j,z'}.$$

Since $Y_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{j,z'}$ is separated, the locus where $\mathrm{pr}_i \circ \varphi_{i,z}$ coincides with $\mathrm{pr}_j \circ \varphi_{j,z}$ is a closed subscheme of $Z_\delta(M_\bullet)$. As $Z_\delta(M_\bullet)$ is reduced, it is enough to show $\mathrm{pr}_i(\varphi_{i,z}(x)) = \mathrm{pr}_j(\varphi_{j,z}(x))$ for each closed geometric point $x = (L_1, L_2) \in Z_\delta(M_\bullet)(\bar{\mathbb{F}}_p)$. Let $(A, \lambda, \eta, \mathcal{B}_z, \lambda_z, \eta_z, \phi)$ and $(A', \lambda', \eta', \mathcal{B}_{z'}, \lambda_{z'}, \eta'_{z'}, \phi')$ be respectively the image of (L_1, L_2) under $\varphi_{i,z}$ and $\varphi_{j,z'}$. To prove the claim, we have to show that there is an isomorphism $(A, \lambda, \eta) \cong (A', \lambda', \eta')$ as objects of $\overline{\mathrm{Sh}}_{1,n-1}$. We identify $\tilde{\mathcal{D}}(\mathcal{B}_{z'})$, $\tilde{\mathcal{D}}(A)$, $\tilde{\mathcal{D}}(A')$ with $W(\bar{\mathbb{F}}_p)$-lattices of $\tilde{\mathcal{D}}(\mathcal{B}_z)[1/p]$ via the quasi-isogenies $\phi_{z',z} : \mathcal{B}_{z'} \to \mathcal{B}_z$, $\phi^{-1} : A \to \mathcal{B}_z$ and $\phi_{z',z}^{-1} \circ \phi' : A' \to \mathcal{B}_z$. Then by the construction of A (cf., the proof of Proposition 4.8), $\tilde{\mathcal{D}}(A)_1^\circ$ and $\tilde{\omega}_{A^\vee,1}^\circ$ fit into the diagram (6.4.1) such that there is a canonical isomorphism

$$\begin{aligned} L_1 &\cong \tilde{\omega}_{A^\vee,1}^\circ / p\big(\tilde{\mathcal{D}}(\mathcal{B}_z)_1^\circ + \tilde{\mathcal{D}}(\mathcal{B}_{z'})_1^\circ\big) \\ &\subseteq \big(\tilde{\mathcal{D}}(\mathcal{B}_z)_1^\circ \cap \tilde{\mathcal{D}}(\mathcal{B}_{z'})_1^\circ\big) / p\big(\tilde{\mathcal{D}}(\mathcal{B}_z)_1^\circ + \tilde{\mathcal{D}}(\mathcal{B}_{z'})_1^\circ\big) = M_1. \end{aligned} \tag{6.4.3}$$

Similarly, we have

$$\begin{aligned} L_2 &\cong p\tilde{\omega}_{A^\vee,2}^\circ / p\big(\tilde{\mathcal{D}}(\mathcal{B}_z)_2^\circ + \tilde{\mathcal{D}}(\mathcal{B}_{z'})_2^\circ\big) \\ &\subseteq \big(\tilde{\mathcal{D}}(\mathcal{B}_z)_2^\circ \cap \tilde{\mathcal{D}}(\mathcal{B}_{z'})_2^\circ\big) / p\big(\tilde{\mathcal{D}}(\mathcal{B}_z)_2^\circ + \tilde{\mathcal{D}}(\mathcal{B}_{z'})_2^\circ\big) = M_2. \end{aligned} \tag{6.4.4}$$

It is easy to see that such relations determine $\tilde{\mathcal{D}}(A)$ uniquely from (L_1, L_2). But the same argument shows that the same relations are satisfied with A replaced by A'. Hence, we see that the quasi-isogeny f induces an isomorphism between the Dieudonné modules of A and A'. As f is a p-quasi-isogeny, this implies immediately that f is an isomorphism of abelian varieties, proving the claim.

It remains to prove that $\varphi : Z_\delta(M_\bullet) \xrightarrow{\sim} Y_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{j,z'}$ is an isomorphism. It suffices to show that φ induces bijections on closed points and tangents spaces. The argument is similar to the proof of Proposition 4.8. Indeed, given a closed point $x = (A, \lambda, \eta, \phi, \phi')$ of $Y_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{j,z'}$, one can construct a unique point $y = (L_1, L_2)$ of $Z_\delta(M_\bullet)$ with $\varphi(y) = x$ by the relations (6.4.3) and (6.4.4). It follows immediately that φ induces a bijection on closed points. Let x and y be as above. By the same argument as in Proposition 4.4, the tangent space of $Z_\delta(M_\bullet)$ at y is given by

$$T_{Z_\delta(M_\bullet),y} \cong (L_1/V^{-1}(L_2^{(p)}))^* \otimes (M_1/L_1) \oplus L_2^* \otimes F(L_1^{(p)})/L_2.$$

On the other hand, using Grothendieck–Messing deformation theory, one sees easily that the tangent space of $Y_{i,z}\times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{j,z'}$ at x is given by

$$T_{Y_{i,z}\times_{\overline{\mathrm{Sh}}_{1,n-1}}Y_{j,z'},x}\cong \mathrm{Hom}_{\bar{\mathbb{F}}_p}\big(\omega^\circ_{A^\vee,1},\big(\tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_1\cap\tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_1\big)/\tilde{\omega}^\circ_{A^\vee,1}\big)\\ \oplus\mathrm{Hom}_{\bar{\mathbb{F}}_p}\big(\tilde{\omega}^\circ_{A^\vee,2}/\big(\tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_2+\tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_2\big),\tilde{\mathcal{D}}(A)^\circ_2/\tilde{\omega}^\circ_{A^\vee,2}\big).$$

From (6.4.3) and (6.4.4), we see easily that

$$\omega^\circ_{A^\vee,1}\cong L_1/V^{-1}(L_2^{(p)}),\qquad \tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_1)/\tilde{\omega}^\circ_{A^\vee,1}\cong M_1/L_1,$$
$$\tilde{\omega}^\circ_{A^\vee,2}/\big(\tilde{\mathcal{D}}(\mathcal{B}_z)^\circ_2+\tilde{\mathcal{D}}(\mathcal{B}_{z'})^\circ_2\big)\cong L_2,\qquad \tilde{\mathcal{D}}(A)^\circ_2/\tilde{\omega}^\circ_{A^\vee,2}\cong F(L_1^{(p)})/L_2.$$

It follows that φ induces a bijection between $T_{Z_\delta(M_\bullet),y}$ and $T_{Y_{i,z}\times_{\overline{\mathrm{Sh}}_{1,n-1}}Y_{j,z'},x}$. This finishes the proof of Proposition 6.4. $\square$

6.5. ***Applications to cohomology.*** Recall that we have a morphism $\mathcal{JL}_j$ (4.16.1) for each $j=1,\dots,n$. We consider another map in the opposite direction:

$$\nu_j: H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1},\overline{\mathbb{Q}}_\ell(n-1))\xrightarrow{\mathrm{pr}_j^*}H^{2(n-1)}_{\mathrm{et}}(\bar{Y}_j,\overline{\mathbb{Q}}_\ell)\xrightarrow{\sim}H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n},\overline{\mathbb{Q}}_\ell),$$

where the second isomorphism is induced by the trace map

$$\mathrm{Tr}_{\mathrm{pr}'_j}:R^{2(n-1)}\,\mathrm{pr}'_{j,*}\,\overline{\mathbb{Q}}_\ell(n-1)\xrightarrow{\sim}\overline{\mathbb{Q}}_\ell.$$

For $1\le i,\,j\le n$, we define

$$m_{i,j}=\nu_j\circ\mathcal{JL}_i:H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n},\overline{\mathbb{Q}}_\ell)\xrightarrow{\mathcal{JL}_i}H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1},\overline{\mathbb{Q}}_\ell(n-1))\xrightarrow{\nu_j}H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n},\overline{\mathbb{Q}}_\ell).$$

Putting all the morphisms $\mathcal{JL}_i$ and ν_j together, we get a sequence of morphisms:

$$\bigoplus_{i=1}^n H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n},\overline{\mathbb{Q}}_\ell)\xrightarrow{\mathcal{JL}}H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1},\overline{\mathbb{Q}}_\ell(n-1))\\ \xrightarrow{\nu=(\nu_1,\dots,\nu_n)}\bigoplus_{j=1}^n H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n},\overline{\mathbb{Q}}_\ell).\quad(6.5.1)$$

We see that the composed morphism above is given by the matrix $M=(m_{i,j})_{1\le i,j\le n}$, and we call it the *intersection matrix* of cycles Y_j on $\mathrm{Sh}_{1,n-1}$. All these morphisms are equivariant under the natural action of the Hecke algebra $\mathscr{H}(K^p,\overline{\mathbb{Q}}_\ell)$. We describe the intersection matrix in terms of the Hecke action of $\overline{\mathbb{Q}}_\ell[K_\mathfrak{p}\backslash\mathrm{GL}_n(E_\mathfrak{p})/K_\mathfrak{p}]$ on $H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n},\overline{\mathbb{Q}}_\ell)$.

The group $H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n},\overline{\mathbb{Q}}_\ell)$ is the space of functions on $\mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$ with values in $\overline{\mathbb{Q}}_\ell$. For $z\in\mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$, let e_z denote the characteristic function of z. Then the image of z under $K_\mathfrak{p}\gamma K_\mathfrak{p}$ for $\gamma\in\mathrm{GL}_n(E_\mathfrak{p})$ is

$$[K_\mathfrak{p}\gamma K_\mathfrak{p}]_*(e_z)=\sum_{z'\in T_\mathfrak{p}(\gamma)(z)}e_{z'},$$

where $T_{\mathfrak{p}}(\gamma)(z)$ means the set theoretic Hecke correspondence defined in Section 6.2. In the sequel, we will use the same notation $T_{\mathfrak{p}}(\gamma)$ to denote the action of $[K_{\mathfrak{p}}\gamma K_{\mathfrak{p}}]$ on $H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n}, \overline{\mathbb{Q}}_\ell)$. In particular, we have Hecke operators $T^{(i)}_{\mathfrak{p}}$, $S_{\mathfrak{p}}$, $R^{(a,b)}_{\mathfrak{p}}$, ...

Proposition 6.6. *For $1 \le i \le j \le n$, we have*

$$m_{i,j} = \sum_{\delta=0}^{\min\{i-1,n-j\}} N(n+i-j-2\delta, i-\delta) R_{\mathfrak{p}}^{(j-i+\delta,n-\delta)} S_{\mathfrak{p}}^{-1},$$

$$m_{j,i} = \sum_{\delta=0}^{\min\{i-1,n-j\}} N(n+i-j-2\delta, i-\delta) R_{\mathfrak{p}}^{(\delta,n+i-j-\delta)} S_{\mathfrak{p}}^{-1},$$

where $N(n+i-j-2\delta, i-\delta)$ are the fundamental intersection numbers defined by (5.2.2).

Proof. We have a commutative diagram:

$$\begin{tikzcd}
A_{n-1}(\bar{Y}_i) \arrow[r, "\mathrm{pr}_{i,*}"] \arrow[d, "\mathrm{cl}"] & A_{n-1}(\overline{\mathrm{Sh}}_{1,n-1}) \arrow[r, "\mathrm{pr}_j^!"] \arrow[d, "\mathrm{cl}"] & A_0(\bar{Y}_j) \arrow[d, "\mathrm{cl}"] \\
H^0_{\mathrm{et}}(\bar{Y}_i, \overline{\mathbb{Q}}_\ell) \arrow[r, "\mathrm{Gys}_{\mathrm{pr}_i}"] & H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_\ell(n-1)) \arrow[r, "\mathrm{pr}_j^*"] & H^{2(n-1)}_{\mathrm{et}}(\bar{Y}_j, \overline{\mathbb{Q}}_\ell).
\end{tikzcd} \tag{6.6.1}$$

Here, the vertical arrows are cycle class maps, and $\mathrm{pr}_j^!$ is the refined Gysin map defined in (6.3.1). For $z \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)$, the image of e_z under $m_{i,j}$ is given by

$$\begin{aligned}
m_{i,j}(e_z) &= \mathrm{Tr}_{\mathrm{pr}'_j}\, \mathrm{pr}_j^*\, \mathrm{Gys}_{\mathrm{pr}_i}\, \mathrm{cl}([Y_{i,z}]) = \mathrm{Tr}_{\mathrm{pr}'_j}\big(\mathrm{cl}(\mathrm{pr}_j^!\, \mathrm{pr}_{i,*}[Y_{i,z}])\big) \\
&= \mathrm{Tr}_{\mathrm{pr}'_j}\Bigg(\sum_{z' \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)} \mathrm{cl}\big(c_{r(z,z')}(\mathcal{E}(Y_{j,z'}, Y_{i,z}))\big) \cdot \mathrm{cl}(Y_{j,z'} \times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{i,z}) \Bigg) \\
&= \sum_{z' \in \mathrm{Sh}_{0,n}(\bar{\mathbb{F}}_p)} \bigg(\int_{Y_{j,z'} \times_{\mathrm{Sh}_{1,n-1}} Y_{i,z}} c_{r(z,z')}(\mathcal{E}(Y_{j,z'}, Y_{i,z})) \bigg) e_{z'},
\end{aligned}$$

where $r(z, z')$ is the rank of $\mathcal{E}(Y_{j,z'}, Y_{i,z})$, and we used (6.3.2) in the second step. Indeed, Proposition 6.4(1) says that the schematic intersection $Y_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{j,z'}$ is smooth, so the closed immersion $Y_{i,z} \times_{\overline{\mathrm{Sh}}_{1,n-1}} Y_{j,z'} \hookrightarrow Y_{j,z'}$ is a regular immersion and the assumptions for (6.3.2) are thus satisfied here.

By Proposition 6.4(1), $e_{z'}$ has a nonzero contribution to the summation above if and only if there exists an integer δ with $0 \le \delta \le \min\{i-1, n-j\}$ such that $z' \in R_{\mathfrak{p}}^{(j-i+\delta,n-\delta)} S_{\mathfrak{p}}^{-1}(z)$. In that case, Proposition 6.4(2) implies that the coefficient of $e_{z'}$ is nothing but the fundamental intersection number $N(n+i-j-2\delta, i-\delta)$ defined in (5.2.2). The formula for $m_{i,j}$ now follows immediately. The formula for $m_{j,i}$ is proved in the same manner. □

If we express $m_{i,j}$ in terms of the elementary Hecke operators $T_{\mathfrak{p}}^{(k)}$, we get the following.

Theorem 6.7. *Put $d(n,k) = (2k-1)n - 2k(k-1) - 1$ for integers $1 \le k \le n$. Then, for $1 \le i \le j \le n$, we have*

$$m_{i,j} = \sum_{\delta=0}^{\min\{i-1,n-j\}} (-1)^{n+1+i-j}(n+i-j-2\delta)\, p^{d(n+i-j-2\delta,i-\delta)} T_{\mathfrak{p}}^{(j-i+\delta)} T_{\mathfrak{p}}^{(n-\delta)} S_{\mathfrak{p}}^{-1},$$

$$m_{j,i} = \sum_{\delta=0}^{\min\{i-1,n-j\}} (-1)^{n+1+i-j}(n+i-j-2\delta)\, p^{d(n+i-j-2\delta,i-\delta)} T_{\mathfrak{p}}^{(\delta)} T_{\mathfrak{p}}^{(n+i-j-\delta)} S_{\mathfrak{p}}^{-1}.$$

Proof. We prove only the statement for $m_{i,j}$, and that for $m_{j,i}$ is similar. By Proposition A.1 in Appendix A, the right hand side of the first formula above is

$$\begin{aligned}\sum_{\delta=0}^{\min\{i-1,n-j\}} & (-1)^{n+1-i-j}(n+i-j-2\delta)\, p^{d(n+i-j-2\delta,i-\delta)} \\ & \cdot \Bigg(\sum_{k=0}^{\delta} \binom{n+i-j-2\delta+2k}{k}_{p^2} R_{\mathfrak{p}}^{(j-i+\delta-k,n-\delta+k)} S_{\mathfrak{p}}^{-1}\Bigg) \\ & = \sum_{r=0}^{\min\{i-1,n-j\}} (\star)\, R_{\mathfrak{p}}^{(j-i+r,n-r)} S_{\mathfrak{p}}^{-1}.\end{aligned}$$

Here, we have put $r = \delta - k$, and the expression $\star$ in the parentheses is

$$\begin{aligned}\star = & \sum_{k=0}^{\min\{i-1-r,n-j-r\}} (-1)^{n+1+i-j}(n+i-j-2r-2k) \\ & \cdot p^{d(n+i-j-2r-2k,i-r-k)} \binom{n+i-j-2r}{k}_{p^2} \\ = & \, N(n+i-j-2r, i-r).\end{aligned}$$

Here, the last equality is Theorem 5.3. The statement for $m_{i,j}$ now follows from Proposition 6.6. $\square$

Example 6.8. We write down explicitly the intersection matrices when n is small.

(1) Consider first the case $n = 2$. This case is essentially the same as the Hilbert quadratic case studied in [Tian and Xiao 2014], and the intersection matrix can be written:

$$M = \begin{pmatrix} -2p & T_{\mathfrak{p}}^{(1)} \\ T_{\mathfrak{p}}^{(1)} S_{\mathfrak{p}}^{-1} & -2p \end{pmatrix}.$$

(2) When $n=3$, Theorem 6.7 gives

$$M=\begin{pmatrix} 3p^2 & -2pT_{\mathfrak{p}}^{(1)} & T_{\mathfrak{p}}^{(2)} \\ -2pT_{\mathfrak{p}}^{(2)}S_{\mathfrak{p}}^{-1} & 3p^4+T_{\mathfrak{p}}^{(1)}T_{\mathfrak{p}}^{(2)}S_{\mathfrak{p}}^{-1} & -2pT_{\mathfrak{p}}^{(1)} \\ T_{\mathfrak{p}}^{(1)}S_{\mathfrak{p}}^{-1} & -2pT_{\mathfrak{p}}^{(2)}S_{\mathfrak{p}}^{-1} & 3p^2 \end{pmatrix}.$$

(3) The intersection matrix for $n=4$ can be written:

$$M=\begin{pmatrix} -4p^3 & 3p^2T_{\mathfrak{p}}^{(1)} & -2pT_{\mathfrak{p}}^{(2)} & T_{\mathfrak{p}}^{(3)} \\ 3p^2T_{\mathfrak{p}}^{(3)}S_{\mathfrak{p}}^{-1} & -4p^7-2pT_{\mathfrak{p}}^{(1)}T_{\mathfrak{p}}^{(3)}S_{\mathfrak{p}}^{-1} & 3p^4T_{\mathfrak{p}}^{(1)}+T_{\mathfrak{p}}^{(2)}T_{\mathfrak{p}}^{(3)}S_{\mathfrak{p}}^{-1} & -2pT_{\mathfrak{p}}^{(2)} \\ -2pT_{\mathfrak{p}}^{(2)}S_{\mathfrak{p}}^{-1} & 3p^4T_{\mathfrak{p}}^{(3)}S_{\mathfrak{p}}^{-1}+T_{\mathfrak{p}}^{(1)}T_{\mathfrak{p}}^{(2)}S_{\mathfrak{p}}^{-1} & -4p^7-2pT_{\mathfrak{p}}^{(1)}T_{\mathfrak{p}}^{(3)}S_{\mathfrak{p}}^{-1} & 3p^2T_{\mathfrak{p}}^{(1)} \\ T_{\mathfrak{p}}^{(1)}S_{\mathfrak{p}}^{-1} & -2pT_{\mathfrak{p}}^{(2)}S_{\mathfrak{p}}^{-1} & 3p^2T_{\mathfrak{p}}^{(3)}S_{\mathfrak{p}}^{-1} & -4p^3 \end{pmatrix}.$$

6.9. ***Proof of Theorem 4.18(1).*** Let $\pi\in\mathscr{A}_K$ as in the statement of Theorem 4.18(1). Consider the $(\pi^p)^{K^p}$-isotypic direct factor of the $\mathscr{H}(K^p,\overline{\mathbb{Q}}_\ell)$-equivariant sequence (6.5.1):

$$\bigoplus_{i=1}^n H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n},\overline{\mathbb{Q}}_\ell)_{\pi^p}\xrightarrow{\mathcal{JL}_\pi} H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1},\overline{\mathbb{Q}}_\ell(n-1))_{\pi^p}\xrightarrow{\nu_\pi}\bigoplus_{j=1}^n H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n},\overline{\mathbb{Q}}_\ell)_{\pi^p}. \tag{6.9.1}$$

In particular, when $i=j=1$, $\nu_1\circ\mathcal{JL}_1$ is given by multiplication by $-np^{n-1}$. So the π^p-isotypic component of (6.9.1) is nonzero. This implies that π^p appears in $H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1},\overline{\mathbb{Q}}_\ell(n-1))$, i.e., there exist admissible irreducible representations π'_p of $G_{1,n-1}(\mathbb{Q}_p)$ and π'_∞ of $G_{1,n-1}(\mathbb{R})$, which is cohomological in degree $n-1$, such that $\pi^p\otimes\pi'_p\otimes\pi'_\infty$ is a cuspidal automorphic representation $\pi'\otimes\pi'_\infty$ of $G_{1,n-1}(\mathbb{A}_{\mathbb{Q}})$. By Lemma 4.17, $\pi'\simeq\pi$ satisfies Hypothesis 2.5(2) for $a_\bullet=(1,n-1)$. Thus, taking the π^p-isotypic component of (6.9.1) is the same as taking its π-isotypic component. From now on, we use subscript π in places of subscript π^p.

If $a_{\mathfrak{p}}^{(i)}$ denotes the eigenvalues of $T_{\mathfrak{p}}^{(i)}$ on $\pi_{\mathfrak{p}}^{K_{\mathfrak{p}}}$ for each $1\le i\le n$, then $T_{\mathfrak{p}}^{(i)}$ acts as the scalar $a_{\mathfrak{p}}^{(i)}$ on all the terms in (6.9.1). Therefore, $\nu_\pi\circ\mathcal{JL}_\pi$ is given by the matrix M_π, which is obtained by replacing $T_{\mathfrak{p}}^{(i)}$ by $a_{\mathfrak{p}}^{(i)}$ in each entry of M. By definition, the $\alpha_{\pi_{\mathfrak{p}},i}$ are the roots of the Hecke polynomial (2.6.2):

$$X^n+\sum_{i=1}^n(-1)^i p^{i(i-1)}a_{\mathfrak{p}}^{(i)}X^{n-i}.$$

Then Theorem 4.18(1) follows easily from the following.

Lemma 6.10. *We have*

$$\det(M_\pi)=\pm p^{\frac{n(n^2-1)}{3}}\frac{\prod_{i<j}(\alpha_{\pi_{\mathfrak{p}},i}-\alpha_{\pi_{\mathfrak{p}},j})^2}{\left(\prod_{i=1}^n\alpha_{\pi_{\mathfrak{p}},i}\right)^{n-1}}.$$

Here, $\pm$ means that the formula holds up to sign. In particular, $\nu_\pi\circ\mathcal{JL}_\pi$ is an isomorphism if the $\alpha_{\pi_{\mathfrak{p}},i}$ are distinct.

Proof. Put $\beta_i = \alpha_{\pi_{\mathfrak{p}},i}/p^{n-1}$ for $1 \le i \le n$. For $i = 1, \dots, n$, let s_i be the i-th elementary symmetric polynomial in $\beta_1, \dots, \beta_n$. Then we have $a_{\mathfrak{p}}^{(i)} = p^{i(n-i)} s_i$. It follows from Theorem 6.7 that the (i, j)-entry of M_π with $1 \le i \le j \le n$ is given by

$$m_{i,j}(\pi) = s_n^{-1} \sum_{\delta=0}^{\min\{i-1,n-j\}} (-1)^{n+1+i-j}(n+i-j-2\delta) \cdot p^{d(n+i-j-2\delta,i-\delta)+(j-i+\delta)(n+i-j-\delta)+\delta(n-\delta)} s_{j-i+\delta} s_{n-\delta}.$$

A direct computation shows that the exponent index on p in each term above is independent of δ, and is equal to $e(i, j) := (n+1)(i+j-1) - (i^2+j^2)$. The same holds when $i > j$. In summary, we get $m_{i,j}(\pi) = s_n^{-1} p^{e(i,j)} m'_{i,j}(\pi)$ with

$$m'_{i,j}(\pi) = \begin{cases} \sum_{\delta=0}^{\min\{i-1,n-j\}} (-1)^{n+1+i-j}(n+i-j-2\delta) s_{j-i+\delta} s_{n-\delta}, & \text{if } i \le j, \\ \sum_{\delta=0}^{\min\{j-1,n-i\}} (-1)^{n+1+j-i}(n+j-i-2\delta) s_\delta s_{n+j-i-\delta}, & \text{if } i > j. \end{cases}$$

For any n-permutation σ, we have

$$\sum_{i=1}^{n} e(i, \sigma(i)) = \frac{n(n^2-1)}{3}.$$

Thus we get $\det(M_\pi) = p^{n(n^2-1)/3} s_n^{-n} \det(m'_{i,j}(\pi))$. The rest of the computation is purely combinatorial, which is the case $q = -1$ of Theorem B.1 in Appendix B. □

Remark 6.11. We point out that the determinant of the intersection matrix computed by Theorem B.1 holds with an auxiliary variable q. A similar phenomenon also appeared in the case of Hilbert modular varieties [Tian and Xiao 2014], where the computation was related to the combinatorial model of periodic semimeanders. These motivate us to ask, out of curiosity, whether there might be some quantum version of the construction of cycles, or even Conjecture 2.12, possibly for the geometric Langlands setup.

6.12. ***Proof of Theorem 4.18(2).*** Given Theorem 4.18(1), it suffices to prove that

$$n \dim H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n}, \overline{\mathbb{Q}}_\ell)_\pi \ge \dim H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_\ell(n-1))^{\mathrm{fin}}_\pi. \tag{6.12.1}$$

Actually, by (2.4.1) and (2.6.3), we have

$$H^0_{\mathrm{et}}(\overline{\mathrm{Sh}}_{0,n}, \overline{\mathbb{Q}}_\ell)_\pi = \pi^K \otimes R_{(0,n),\ell}(\pi), \quad H^{2(n-1)}_{\mathrm{et}}(\overline{\mathrm{Sh}}_{1,n-1}, \overline{\mathbb{Q}}_\ell)_\pi = \pi^K \otimes R_{(1,n-1),\ell}(\pi).$$

Write $\pi_p = \pi_{p,0} \otimes \pi_{\mathfrak{p}}$ as a representation of $G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \mathrm{GL}_n(E_{\mathfrak{p}})$. Let $\chi_{\pi_{p,0}}$: $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^2}) \to \overline{\mathbb{Q}}_\ell^\times$ denote the character sending Frob_{p^2} to $\pi_{p,0}(p^2)$, and let $\rho_{\pi_{\mathfrak{p}}}$

be as in (2.6.1). According to (2.6.3), up to semisimplification, we have

$$[R_{(0,n),\ell}(\pi)] = \#\ker^1(\mathbb{Q}, G_{0,n}) m_{0,n}(\pi) \Big[\wedge^n \rho_{\pi_{\mathfrak{p}}} \otimes \chi_{\pi_{p,0}}^{-1} \otimes \overline{\mathbb{Q}}_\ell\big(\tfrac{1}{2}n(n-1)\big)\Big], \tag{6.12.2}$$

$$[R_{(1,n-1),\ell}(\pi)] = \#\ker^1(\mathbb{Q}, G_{1,n-1}) m_{0,n}(\pi) \Big[\rho_{\pi_{\mathfrak{p}}} \otimes \wedge^{n-1} \rho_{\pi_{\mathfrak{p}}} \otimes \chi_{\pi_{p,0}}^{-1} \otimes \overline{\mathbb{Q}}_\ell\big(\tfrac{1}{2}(n-1)(n-2)\big)\Big]. \tag{6.12.3}$$

Note that

$$\dim\Big(\rho_{\pi_{\mathfrak{p}}} \otimes \wedge^{n-1} \rho_{\pi_{\mathfrak{p}}} \otimes \chi_{\pi_{p,0}}^{-1} \otimes \overline{\mathbb{Q}}_\ell\Big(\frac{(n-1)(n-2)}{2}\Big)\Big)^{\mathrm{fin}} = \sum_{\zeta} \dim(\rho_{\pi_{\mathfrak{p}}} \otimes \wedge^{n-1} \rho_{\pi_{\mathfrak{p}}})^{\mathrm{Frob}_{p^2} = p^{n(n-1)}\zeta},$$

where the superscript "fin" means taking the subspace on which $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^2})$ acts through a finite quotient, and ζ runs through all roots of unity. If $\alpha_{\pi_{\mathfrak{p}},i}/\alpha_{\pi_{\mathfrak{p}},j}$ is not a root of unity for any pair $i \neq j$, the right hand side above is equal to the sum of the multiplicities of $\prod_{i=1}^n \alpha_{\pi_{\mathfrak{p}},i} = p^{n(n-1)}\zeta$ as eigenvalues of $(\rho_{\pi_{\mathfrak{p}}} \otimes \wedge^{n-1} \rho_{\pi_{\mathfrak{p}}})(\mathrm{Frob}_{p^2})$, which is n. Therefore, under these conditions on the $\alpha_{\pi_{\mathfrak{p}},i}$, we have by (6.12.3)

$$\dim R_{(1,n-1),\ell}(\pi)^{\mathrm{fin}} \leq n \cdot \#\ker^1(\mathbb{Q}, G_{1,n-1}) \cdot m_{1,n-1}(\pi),$$

and the equality holds if Frob_{p^2} is semisimple on $R_{(1,n-1),\ell}(\pi)$. On the other hand, we have from (6.12.2)

$$\dim R_{(0,n),\ell}(\pi) = \#\ker^1(\mathbb{Q}, G_{0,n}) \cdot m_{0,n}(\pi).$$

By a result of White [2012, Theorem E], the multiplicity $m_{a_\bullet}(\pi)$ above is equal to 1 for $a_\bullet = (1, n-1)$ and $a_\bullet = (0, n)$. Now the inequality (6.12.1) follows immediately from this and the fact that $\#\ker^1(\mathbb{Q}, G_{1,n-1}) = \#\ker^1(\mathbb{Q}, G_{0,n})$. This finishes the proof of Theorem 4.18(2). □

7. Construction of cycles in the case of $G(U(r, s) \times U(s, r))$

We keep the notation of Section 3.6. In this section, we will give the construction of certain cycles on Shimura varieties for $G(U(r, s) \times U(s, r))$. We always assume that $s \geq r$.

7.1. ***Description of the cycles in terms of Dieudonné modules.*** Let δ be a nonnegative integer with $\delta \leq r$. We consider the case of Conjecture 2.12 when $n = r + s$, $a_1 = r$, $a_2 = s$, $b_1 = r - \delta$, and $b_2 = s + \delta$. The representation $r_{a_\bullet}$ of GL_n involved is

$$r_{a_\bullet} = \wedge^r \mathrm{Std} \otimes \wedge^s \mathrm{Std}.$$

The weight λ of Conjecture 2.12 is

$$\lambda = (\underbrace{2,\dots,2}_{r-\delta}, \underbrace{1,\dots,1}_{s-r+2\delta}, \underbrace{0,\dots,0}_{r-\delta}).$$

By elementary calculation of representations of GL_n, the multiplicity of λ in $r_{a_\bullet}$ is $m_\lambda(a_\bullet) = \binom{s-r+2\delta}{\delta}$. Then Conjecture 2.12 thus predicts the existence of $\binom{s-r+2\delta}{\delta}$ cycles Y_j on $\mathrm{Sh}_{r,s}$, each of dimension

$$\tfrac{1}{2}(\dim \mathrm{Sh}_{r,s} + \dim \mathrm{Sh}_{r-\delta,s+\delta}) = \tfrac{1}{2}(2rs + 2(r-\delta)(s+\delta)) = 2rs - (s-r)\delta - \delta^2,$$

and each admits a rational map to $\mathrm{Sh}_{r-\delta,s+\delta}$. The principal goal of this section is to construct these cycles, at least conjecturally. We start with the description in terms of the Dieudonné modules at closed points.

Consider the interval $[r-\delta, s+\delta]$; it contains $s-r+2\delta$ unit segments with integer endpoints. We will parametrize the cycles on the Shimura variety by the subsets of these $s-r+2\delta$ unit segments of cardinality δ. There are exactly $\binom{s-r+2\delta}{\delta}$ such subsets. Let $\boldsymbol{j}$ be one of them. Then we can write the union of all the segments in $\boldsymbol{j}$ as

$$[j_{1,1}, j_{1,2}] \cup [j_{2,1}, j_{2,2}] \cup \cdots \cup [j_{\epsilon,1}, j_{\epsilon,2}] \tag{7.1.1}$$

such that all $j_{\alpha,i}$ are integers,

$$r-\delta \le j_{1,1} < j_{1,2} < j_{2,1} < j_{2,2} < \cdots < j_{\epsilon,1} < j_{\epsilon,2} \le s+\delta,$$

and we have $\sum_{\alpha=1}^{\epsilon}(j_{\alpha,2} - j_{\alpha,1}) = \delta$. For notational convenience, we put $j_{0,1} = j_{0,2} = 0$.

We define $Z_{\boldsymbol{j}}$ to be the subset of $\bar{\mathbb{F}}_p$-points z of $\mathrm{Sh}_{r,s}$ such that the reduced Dieudonné modules $\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_1$ and $\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_2$ contain submodules $\tilde{\mathcal{E}}_1$ and $\tilde{\mathcal{E}}_2$ satisfying (3.2.1) for $m = \epsilon$, i.e.,

$$p^\epsilon \tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_i \subseteq \tilde{\mathcal{E}}_i, \quad F(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{3-i}, \quad \text{and} \quad V(\tilde{\mathcal{E}}_i) \subseteq \tilde{\mathcal{E}}_{3-i}, \quad \text{for } i = 1, 2,$$

and the following condition for $i = 1, 2$:

$$\begin{aligned}\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_i/\tilde{\mathcal{E}}_i \simeq (W(\bar{\mathbb{F}}_p)/p^\epsilon)^{\oplus j_{1,i}} \oplus (W(\bar{\mathbb{F}}_p)/p^{\epsilon-1})^{\oplus (j_{2,i}-j_{1,i})} \oplus \cdots \\ \cdots \oplus (W(\bar{\mathbb{F}}_p)/p)^{\oplus (j_{\epsilon,i}-j_{\epsilon-1,i})}.\end{aligned} \tag{7.1.2}$$

We refer to the toy model discussed in Example 7.3 for the motivation of this condition. For technical reasons, we will not prove the set $Z_{\boldsymbol{j}}$ is the set of $\bar{\mathbb{F}}_p$-points of a closed subscheme of $\mathrm{Sh}_{r,s}$; instead we prove that a closely related subset of $Z_{\boldsymbol{j}}$ is. See Remark 7.5.

Applying Proposition 3.2 with $m = \delta$, the submodules $\tilde{\mathcal{E}}_1$ and $\tilde{\mathcal{E}}_2$ give rise to a polarized abelian variety $(\mathcal{A}'_z, \lambda'_z)$ over z with an $\mathcal{O}_D$-action and an $\mathcal{O}_D$-equivariant isogeny $\mathcal{A}'_z \to \mathcal{A}_z$. Moreover, by (3.2.2), we have

$$\begin{aligned}\dim \omega^{\circ}_{\mathcal{A}'^{\vee}_z/\bar{\mathbb{F}}_p,1} &= \dim \omega^{\circ}_{\mathcal{A}^{\vee}_z/\bar{\mathbb{F}}_p,1} + \sum_{\alpha=0}^{\epsilon-1}\big((\epsilon-\alpha)(j_{\alpha+1,1}-j_{\alpha,1})-(\epsilon-\alpha)(j_{\alpha+1,2}-j_{\alpha,2})\big)\\ &= r-\delta\end{aligned}$$

and similarly $\dim \omega^{\circ}_{\mathcal{A}'^{\vee}_z/\bar{\mathbb{F}}_p,2} = s+\delta$. So $\mathcal{A}'_z$ satisfies the moduli problem for $\mathrm{Sh}_{r-\delta,s+\delta}$; this suggests a geometric relationship between $Z_{\boldsymbol{j}}$ and $\mathrm{Sh}_{r-\delta,s+\delta}$ that we make precise in Definition 7.4.

We make an immediate remark that when $\delta = r$, the abelian variety $\mathcal{A}_z$ coming from a point z of $Z_{\boldsymbol{j}}$ is isogenous to an abelian variety $\mathcal{A}'_z$ that is a moduli object for the Shimura variety $\mathrm{Sh}_{0,n}$. Thus both $\mathcal{A}'_z$ and $\mathcal{A}_z$ are supersingular. So every $Z_{\boldsymbol{j}}$ is contained in the supersingular locus of $\mathrm{Sh}_{r,s}$. In fact, we shall show in Theorem 7.8 that the supersingular locus of $\mathrm{Sh}_{r,s}$ is exactly the union of these $Z_{\boldsymbol{j}}$.

7.2. ***Towards a moduli interpretation.*** We need to reinterpret in a more geometric manner the Dieudonné-theoretic condition defining $Z_{\boldsymbol{j}}$. For $\alpha = 0, \dots, \epsilon$, we define submodules

$$\tilde{\mathcal{E}}_{\alpha,1} := \tilde{\mathcal{D}}(\mathcal{A}_z)^{\circ}_1 \cap \frac{1}{p^{\epsilon-\alpha}}\tilde{\mathcal{E}}_1 \quad \text{and} \quad \tilde{\mathcal{E}}_{\alpha,2} := \tilde{\mathcal{D}}(\mathcal{A}_z)^{\circ}_2 \cap \frac{1}{p^{\epsilon-\alpha}}\tilde{\mathcal{E}}_2$$

of $\tilde{\mathcal{D}}(\mathcal{A}_z)^{\circ}_1$ and $\tilde{\mathcal{D}}(\mathcal{A}_z)^{\circ}_2$. They are easily seen to satisfy condition (3.2.1) with $m = \alpha$. Thus, Proposition 3.2 generates a polarized abelian variety $(A_\alpha, \lambda_\alpha)$ with $\mathcal{O}_D$-action and an $\mathcal{O}_D$-equivariant isogeny $A_\alpha \to \mathcal{A}_z$, where

$$\begin{aligned} r_\alpha &:= \dim \omega^{\circ}_{A^{\vee}_\alpha/\bar{\mathbb{F}}_p,1} = r - \sum_{\alpha'=1}^{\alpha}(j_{\alpha',2}-j_{\alpha',1}) \quad \text{and}\\ s_\alpha &:= \dim \omega^{\circ}_{A^{\vee}_\alpha/\bar{\mathbb{F}}_p,2} = n - \dim \omega^{\circ}_{A^{\vee}_\alpha/\bar{\mathbb{F}}_p,1}\end{aligned} \tag{7.2.1}$$

by the formula (3.2.2). In particular $r_0 = r$, $s_0 = s$, $r_\epsilon = r-\delta$ and $s_\epsilon = s+\delta$.

In fact, applying Proposition 3.2 (with $m = 1$) to the sequence of inclusions

$$\tilde{\mathcal{E}}_i = \tilde{\mathcal{E}}_{\epsilon,i} \subset \tilde{\mathcal{E}}_{\epsilon-1,i} \subset \cdots \subset \tilde{\mathcal{E}}_{0,i} = \tilde{\mathcal{D}}(\mathcal{A}_z)^{\circ}_i,$$

we obtain a sequence of isogenies (each with p-torsion kernels):

$$\mathcal{A}'_z = A_\epsilon \xrightarrow{\phi_\epsilon} A_{\epsilon-1} \xrightarrow{\phi_{\epsilon-1}} \cdots \xrightarrow{\phi_1} A_0 = \mathcal{A}_z. \tag{7.2.2}$$

We have $\ker\phi_\alpha \subseteq A_\alpha[p]$, so that there exists a unique isogeny $\psi_\alpha : A_{\alpha-1} \to A_\alpha$ such that $\psi_\alpha\phi_\alpha = p\cdot \mathrm{id}_{A_\alpha}$ and $\phi_\alpha\psi_\alpha = p\cdot\mathrm{id}_{A_{\alpha-1}}$.

For each α, the cokernel of the induced map on cohomology

$$\begin{gathered}\phi_{\alpha,*,i} : H_1^{\mathrm{dR}}(A_\alpha/\bar{\mathbb{F}}_p)^{\circ}_i \to H_1^{\mathrm{dR}}(A_{\alpha-1}/\bar{\mathbb{F}}_p)^{\circ}_i\\ (\text{resp. } \psi_{\alpha,*,i} : H_1^{\mathrm{dR}}(A_{\alpha-1}/\bar{\mathbb{F}}_p)^{\circ}_i \to H_1^{\mathrm{dR}}(A_\alpha/\bar{\mathbb{F}}_p)^{\circ}_i)\end{gathered}$$

is canonically isomorphic to $\tilde{\mathcal{E}}_{\alpha-1,i}/\tilde{\mathcal{E}}_{\alpha,i}$ (resp. $\tilde{\mathcal{E}}_{\alpha,i}/p\tilde{\mathcal{E}}_{\alpha-1,i}$), which has dimension $j_{\alpha,i}$ (resp. $n-j_{\alpha,i}$) over $\bar{\mathbb{F}}_p$ by a straightforward computation using (7.1.2).

The upshot is that all these numeric information of the chain of isogenies (7.2.2) can be used to reconstruct $\tilde{\mathcal{E}}_i$ inside $\tilde{\mathcal{D}}(\mathcal{A}_z)_i^\circ$. This idea will be made precise after this important example.

Example 7.3. We give a good toy model for the isogenies of Dieudonné modules. This is the inspiration of the construction of this section. We start with the Dieudonné module $\tilde{\mathcal{D}}(A_\epsilon)_1^\circ = \bigoplus_{i=1}^n W(\bar{\mathbb{F}}_p)\boldsymbol{e}_j$ and $\tilde{\mathcal{D}}(A_\epsilon)_2^\circ = \bigoplus_{j=1}^n W(\bar{\mathbb{F}}_p)\boldsymbol{f}_j$. The maps $V_1 : \tilde{\mathcal{D}}(A_\epsilon)_1^\circ \to \tilde{\mathcal{D}}(A_\epsilon)_2^\circ$ and $V_2 : \tilde{\mathcal{D}}(A_\epsilon)_2^\circ \to \tilde{\mathcal{D}}(A_\epsilon)_1^\circ$, with respect to the given bases, are given by the diagonal matrices

$$\mathrm{Diag}(\underbrace{1,\dots,1}_{s+\delta},\underbrace{p,\dots,p}_{r-\delta}) \quad \text{and} \quad \mathrm{Diag}(\underbrace{1,\dots,1}_{r-\delta},\underbrace{p,\dots,p}_{s+\delta}),$$

respectively. Using the isogenies ϕ_α we may naturally identify $\tilde{\mathcal{D}}(A_\alpha)_i^\circ$ as lattices in $\tilde{\mathcal{D}}(A_\epsilon)_i^\circ[1/p]$ with induced Frobenius and Verschiebung morphisms. For our toy model, we choose

$$\begin{aligned}\tilde{\mathcal{D}}(A_\alpha)_1^\circ = \mathrm{Span}_{W(\bar{\mathbb{F}}_p)}\Big\{&\frac{1}{p^{\epsilon-\alpha}}\boldsymbol{e}_1,\dots,\frac{1}{p^{\epsilon-\alpha}}\boldsymbol{e}_{j_{\alpha+1,1}},\frac{1}{p^{\epsilon-\alpha-1}}\boldsymbol{e}_{j_{\alpha+1,1}+1},\dots,\frac{1}{p^{\epsilon-\alpha-1}}\boldsymbol{e}_{j_{\alpha+2,1}},\\ &\frac{1}{p^{\epsilon-\alpha-2}}\boldsymbol{e}_{j_{\alpha+2,1}+1},\dots,\frac{1}{p}\boldsymbol{e}_{j_{\epsilon,1}},\boldsymbol{e}_{j_{\epsilon,1}+1},\dots,\boldsymbol{e}_n\Big\};\\ \tilde{\mathcal{D}}(A_\alpha)_2^\circ = \mathrm{Span}_{W(\bar{\mathbb{F}}_p)}\Big\{&\frac{1}{p^{\epsilon-\alpha}}\boldsymbol{f}_1,\dots,\frac{1}{p^{\epsilon-\alpha}}\boldsymbol{f}_{j_{\alpha+1,2}},\frac{1}{p^{\epsilon-\alpha-1}}\boldsymbol{f}_{j_{\alpha+1,2}+1},\dots,\frac{1}{p^{\epsilon-\alpha-1}}\boldsymbol{f}_{j_{\alpha+2,2}},\\ &\frac{1}{p^{\epsilon-\alpha-2}}\boldsymbol{f}_{j_{\alpha+2,2}+1},\dots,\frac{1}{p}\boldsymbol{f}_{j_{\epsilon,2}},\boldsymbol{f}_{j_{\epsilon,2}+1},\dots,\boldsymbol{f}_n\Big\}.\end{aligned}$$

In particular, the Verschiebung $V_1 : \tilde{\mathcal{D}}(A_\alpha)_1^\circ \to \tilde{\mathcal{D}}(A_\alpha)_2^\circ$ with respect to the bases above is given by

$$\mathrm{Diag}(\underbrace{1,\dots,1}_{j_{\alpha+1,1}},***\cdots***,\underbrace{p,\dots,p}_{r-\delta}),$$

where the $***$ part is p if the place is in $[j_{\alpha',1}+1,\ j_{\alpha',2}]$ for some $\alpha' \geq \alpha$, and is 1 otherwise. Similarly, the Verschiebung $V_2 : \tilde{\mathcal{D}}(A_0)_2^\circ \to \tilde{\mathcal{D}}(A_0)_1^\circ$ with respect to the bases above is given by

$$\mathrm{Diag}(\underbrace{1,\dots,1}_{r-\delta},\underbrace{p,\dots,p}_{j_{\alpha+1,1}-r+\delta},***\cdots***,\underbrace{p,\dots,p}_{n-j_{\alpha_\epsilon,2}}),$$

where the $***$ part is 1 if the place is in $[j_{\alpha',1}+1,\ j_{\alpha',2}]$ for some $\alpha' \geq \alpha$, and is p otherwise.

So the sheaf of differentials is given by

$$\omega^\circ_{A^\vee_\alpha/\bar{\mathbb{F}}_p,1} = \mathrm{Span}_{\bar{\mathbb{F}}_p}\Big\{\frac{1}{p^{\epsilon-\alpha}}\boldsymbol{e}_1,\dots,\frac{1}{p^{\epsilon-\alpha}}\boldsymbol{e}_{r-\delta},\frac{1}{p^{\epsilon-\alpha}}\boldsymbol{e}_{j_{\alpha,1}+1},\dots,\frac{1}{p^{\epsilon-\alpha}}\boldsymbol{e}_{j_{\alpha,2}},\\ \frac{1}{p^{\epsilon-\alpha-1}}\boldsymbol{e}_{j_{\alpha+1,1}+1},\dots,\frac{1}{p}\boldsymbol{e}_{j_{\epsilon-1,2}},\boldsymbol{e}_{j_{\epsilon,1}+1},\dots,\boldsymbol{e}_{j_{\epsilon,2}}\Big\};$$

$$\omega^\circ_{A^\vee_\alpha/\bar{\mathbb{F}}_p,2} = \mathrm{Span}_{\bar{\mathbb{F}}_p}\Big\{\frac{1}{p^{\epsilon-\alpha}}\boldsymbol{f}_1,\dots,\frac{1}{p^{\epsilon-\alpha}}\boldsymbol{f}_{j_{\alpha+1,1}},\frac{1}{p^{\epsilon-\alpha-1}}\boldsymbol{f}_{j_{\alpha+1,2}+1},\dots,\frac{1}{p^{\epsilon-\alpha-1}}\boldsymbol{f}_{j_{\alpha+2,1}},\\ \frac{1}{p^{\epsilon-\alpha-2}}\boldsymbol{f}_{j_{\alpha+2,1}},\dots,\frac{1}{p}\boldsymbol{f}_{j_{\epsilon,1}},\boldsymbol{f}_{j_{\epsilon,2}+1},\dots,\boldsymbol{f}_{s+\delta-1}\Big\}.$$

Definition 7.4. Let $\boldsymbol{j}$ be as above. Define the numbers $j_{\alpha,i}$ as in (7.1.1) and the numbers r_α, s_α as in (7.2.1). Let $\underline{Y}_{\boldsymbol{j}}$ be the functor taking a locally noetherian $\mathbb{F}_{p^2}$-scheme S to the set of isomorphism classes of tuples

$$(A_0,\dots,A_\epsilon,\lambda_0,\dots,\lambda_\epsilon,\eta_0,\dots,\eta_\epsilon,\phi_1,\dots,\phi_\epsilon,\psi_1,\dots,\psi_\epsilon) \tag{7.4.1}$$

such that:

(1) for each α, $(A_\alpha,\lambda_\alpha,\eta_\alpha)$ is an S-point of $\mathrm{Sh}_{r_\alpha,s_\alpha}$;

(2) for each α, ϕ_α is an $\mathcal{O}_D$-isogeny $A_\alpha\to A_{\alpha-1}$, with kernel contained in $A_\alpha[p]$, which is compatible with the polarizations in the sense that $p\lambda_\alpha=\phi_\alpha^\vee\circ\lambda_{\alpha-1}\circ\phi_\alpha$ and with the tame level structures in the sense that $\phi_\alpha\circ\eta_\alpha=\eta_{\alpha-1}$;

(3) ψ_α is the isogeny $A_{\alpha-1}\to A_\alpha$ such that $\phi_\alpha\psi_\alpha=p\cdot\mathrm{id}_{A_\alpha}$ and $\psi_\alpha\phi_\alpha=p\cdot\mathrm{id}_{A_{\alpha-1}}$;

(4) the cokernel of the induced map $\phi^{\mathrm{dR}}_{\alpha,*,i}:H^{\mathrm{dR}}_1(A_\alpha/S)^\circ_i\to H^{\mathrm{dR}}_1(A_{\alpha-1}/S)^\circ_i$ is a locally free $\mathcal{O}_S$-module of rank $j_{\alpha,i}$ for each α and $i=1,2$;

(5) the cokernel of the induced map $\psi^{\mathrm{dR}}_{\alpha,*,i}:H^{\mathrm{dR}}_1(A_{\alpha-1}/S)^\circ_i\to H^{\mathrm{dR}}_1(A_\alpha/S)^\circ_i$ is a locally free $\mathcal{O}_S$-module of rank $n-j_{\alpha,i}$ for each α and $i=1,2$;[18]

(6) for each α, $\mathrm{Ker}(\phi^{\mathrm{dR}}_{\alpha,*,2})$ is contained in $\omega^\circ_{A^\vee_\alpha/S,2}$;

(7) for each α, the $(r_{\alpha-1}-r_\alpha+r_\epsilon+1)$-st Fitting ideal of the cokernel of $\phi^{\mathrm{dR}}_{\alpha,*,1}:\omega^\circ_{A^\vee_\alpha/S,1}\to\omega^\circ_{A^\vee_{\alpha-1}/S,1}$ is zero, or equivalently, Zariski locally on S, if we represent the map $\phi^{\mathrm{dR}}_{\alpha,*,1}:\omega^\circ_{A^\vee_\alpha/S,1}\to\omega^\circ_{A^\vee_{\alpha-1}/S,1}$ by an $r_{\alpha-1}\times r_\alpha$-matrix (after choosing local bases), then all $(r_\alpha-r_\epsilon+1)\times(r_\alpha-r_\epsilon+1)$-minors vanish.

(8) the $(r_\alpha-r_\epsilon+1)$-st Fitting ideal of the cokernel of $\psi^{\mathrm{dR}}_{\alpha,*,1}:\omega^\circ_{A^\vee_{\alpha-1}/S,1}\to\omega^\circ_{A^\vee_\alpha/S,1}$ is zero for each α.

Note that conditions (6)–(8) are all closed conditions. So the moduli problem $\underline{Y}_{\boldsymbol{j}}$ is represented by a proper scheme $Y_{\boldsymbol{j}}$ of finite type over $\mathbb{F}_{p^2}$. The moduli space $Y_{\boldsymbol{j}}$ admits natural maps to $\mathrm{Sh}_{r,s}$ and $\mathrm{Sh}_{r-\delta,s+\delta}$ by sending the tuple (7.4.1) to

[18] This is in fact a corollary of (2) and (4).

(A_0, λ_0, η_0) and $(A_\epsilon, \lambda_\epsilon, \eta_\epsilon)$, respectively.

$$\begin{tikzcd} & Y_{\boldsymbol{j}} \arrow[dl, "\mathrm{pr}_{\boldsymbol{j}}"'] \arrow[dr, "\mathrm{pr}'_{\boldsymbol{j}}"] & \\ \mathrm{Sh}_{r,s} & & \mathrm{Sh}_{r-\delta,s+\delta} \end{tikzcd}$$

We also point out that conditions (2) and (3) together imply that, for each α and $i = 1, 2$, we have $\mathrm{Im}(\psi^{\mathrm{dR}}_{\alpha,*,i}) = \mathrm{Ker}(\phi^{\mathrm{dR}}_{\alpha,*,i})$ and $\mathrm{Im}(\phi^{\mathrm{dR}}_{\alpha,*,i}) = \mathrm{Ker}(\psi^{\mathrm{dR}}_{\alpha,*,i})$. We shall freely use this property later.

Remark 7.5. Conditions (6)–(8) in Definition 7.4 are satisfied by the toy model in Example 7.3. They did not appear in moduli problem in Section 4.2 because they trivially hold in that case. The purpose of keeping these conditions in the moduli problem and carefully formulating them is so that the moduli space may hope to have the correct irreducible components. We think the picture is the following: $Z_{\boldsymbol{j}}$ is probably or at least heuristically the set of $\bar{\mathbb{F}}_p$-points of a closed subscheme of $\mathrm{Sh}_{r,s}$. But this scheme has many irreducible components, which may have overlaps with other $Z_{\boldsymbol{j}'}$. Conditions (6)–(8) will help select one irreducible component that is "special" for $\boldsymbol{j}$. When taking the union of all images of the $Y_{\boldsymbol{j}}$, we should still get the union of the $Z_{\boldsymbol{j}}$. This is verified in the case of supersingular locus (i.e., $r = \delta$) in Theorem 7.8.

Notation 7.6. Let $Y_{\boldsymbol{j}}$ as above. It will be convenient to introduce some dummy notation:

- ϕ_0 is the identity map on A_0;
- ψ_ϵ is the identity map on A_ϵ.

We use $Y^\circ_{\boldsymbol{j}}$ to denote the open subscheme of $Y_{\boldsymbol{j}}$ representing the functor that takes a locally noetherian $\mathbb{F}_{p^2}$-scheme S to the subset of isomorphism classes of tuples

$$(A_0, \dots, A_\epsilon, \lambda_0, \dots, \lambda_\epsilon, \eta_0, \dots, \eta_\epsilon, \phi_1, \dots, \phi_\epsilon, \psi_1, \dots, \psi_\epsilon)$$

of $Y_{\boldsymbol{j}}(S)$ such that

(i) for each $\alpha = 1, \dots, \epsilon$, the sum $\phi_{\alpha,*,2}(\omega^\circ_{A^\vee_\alpha/S,2}) + \mathrm{Ker}(\phi^{\mathrm{dR}}_{\alpha-1,*,2})$ is an $\mathcal{O}_S$-subbundle of $H^{\mathrm{dR}}_1(A_{\alpha-1}/S)^\circ_2$ of rank

$$\mathrm{rank}\,\omega_{A^\vee_\alpha/S,2} - \mathrm{rank}\,\mathrm{Ker}(\phi^{\mathrm{dR}}_{\alpha,*,2}) + \mathrm{rank}\,\mathrm{Ker}(\phi^{\mathrm{dR}}_{\alpha-1,*,2}) = s_\alpha - j_{\alpha,2} + j_{\alpha-1,2},$$

(ii) for each $\alpha = 1, \dots, \epsilon$, $\mathrm{Ker}(\phi^{\mathrm{dR}}_{\alpha,*,1}) + \mathrm{Ker}(\psi^{\mathrm{dR}}_{\alpha+1,*,1})$ is an $\mathcal{O}_S$-subbundle of rank

$$\mathrm{rank}\,\mathrm{Ker}(\phi^{\mathrm{dR}}_{\alpha,*,1}) + \mathrm{rank}\,\mathrm{Ker}(\psi^{\mathrm{dR}}_{\alpha+1,*,1}) = j_{\alpha,1} + (n - j_{\alpha+1,1}),$$

(iii) for each α, the cokernel of $\phi^{\mathrm{dR}}_{\alpha,*,1} : \omega^\circ_{A^\vee_\alpha/S,1} \to \omega^\circ_{A^\vee_{\alpha-1}/S,1}$ is a locally free $\mathcal{O}_S$-module of rank $r_{\alpha-1} - (r_\alpha - r_\epsilon)$,

(iv) for each α, the cokernel of $\psi^{\mathrm{dR}}_{\alpha,*,1} : \omega^\circ_{A^\vee_{\alpha-1}/S,1} \to \omega^\circ_{A^\vee_\alpha/S,1}$ is a locally free $\mathcal{O}_S$-module of rank $r_\alpha - r_\epsilon$.

We note that the ranks in conditions (i) and (ii) are maximal possible and the ranks in conditions (iii) and (iv) are minimal possible, under the conditions in Definition 7.4. So Y_j° is an open subscheme of Y_j.

We point out an additional benefit of having conditions (ii)–(iv). By (iii), $\omega^\circ_{A_\alpha^\vee/S,1} \cap \mathrm{Ker}(\phi^{\mathrm{dR}}_{\alpha,*,1})$ is an $\mathcal{O}_S$-subbundle of $\omega^\circ_{A_\alpha^\vee/S,1}$ of rank r_ϵ, for $\alpha = 1, \dots, \epsilon$; by (iv), $\omega^\circ_{A_\alpha^\vee/S,1} \cap \mathrm{Ker}(\psi^{\mathrm{dR}}_{\alpha+1,*,1})$ is an $\mathcal{O}_S$-subbundle of $\omega^\circ_{A_\alpha^\vee/S,1}$ of rank $r_\alpha - r_\epsilon$, for $\alpha = 0, \dots, \epsilon - 1$. Combining these two rank estimates and condition (ii) which implies that $\mathrm{Ker}(\phi^{\mathrm{dR}}_{\alpha,*,1})$ and $\mathrm{Ker}(\psi^{\mathrm{dR}}_{\alpha+1,*,1})$ are disjoint subbundles, we arrive at a *direct sum* decomposition

$$\omega^\circ_{A_\alpha^\vee/S,1} = (\omega^\circ_{A_\alpha^\vee/S,1} \cap \mathrm{Ker}(\phi^{\mathrm{dR}}_{\alpha,*,1})) \oplus (\omega^\circ_{A_\alpha^\vee/S,1} \cap \mathrm{Ker}(\psi^{\mathrm{dR}}_{\alpha+1,*,1})), \tag{7.6.1}$$

for $\alpha = 1, \dots, \epsilon - 1$; and we know that $\omega^\circ_{A_0^\vee/S,1} \cap \mathrm{Ker}(\psi^{\mathrm{dR}}_{1,*,1})$ has rank $r_0 - r_\epsilon = \delta$ and $\omega^\circ_{A_\epsilon^\vee/S,1} \subseteq \mathrm{Ker}(\phi^{\mathrm{dR}}_{\epsilon,*,1})$.

We shall show below in Theorem 7.7 that Y_j° is smooth. Unfortunately, we do not know how to prove the nonemptiness of Y_j°, nor do we know if some Y_j is completely contained in some other Y_j; but the fact that the Dieudonné modules in Example 7.3 satisfy conditions (i)–(iv) above is good evidence for this nonemptiness. Of course, if one can compute the intersection matrix in the sense of Theorem 6.7 and calculate the determinant, one can then probably show that these Y_j are essentially different. But the difficulties of this computation lie in understanding the singularities at $Y_j \setminus Y_j^\circ$, which seems to be very combinatorially involved.

Theorem 7.7. *Each Y_j° is smooth of dimension $rs + (r-\delta)(s+\delta)$ (if not empty).*

Proof. Let $\hat{R}$ be a noetherian $\mathbb{F}_{p^2}$-algebra and $\hat{I} \subset \hat{R}$ an ideal such that $\hat{I}^2 = 0$. Put $R = \hat{R}/\hat{I}$. Say we want to lift an R-point

$$(A_0, \dots, A_\epsilon, \lambda_0, \dots, \lambda_\epsilon, \eta_0, \dots, \eta_\epsilon, \phi_1, \dots, \phi_\epsilon, \psi_1, \dots, \psi_\epsilon)$$

of Y_j° an $\hat{R}$-point and we try to compute the corresponding tangent space. By Serre–Tate and Grothendieck–Messing deformation theory we recalled in Theorem 3.4, it is enough to lift, for $i = 1, 2$ and each $\alpha = 0, \dots, \epsilon$, the differentials $\omega^\circ_{A_\alpha^\vee/R,i} \subseteq H_1^{\mathrm{dR}}(A_\alpha/R)_i^\circ$ to a subbundle $\hat{\omega}_{\alpha,i} \subseteq H_1^{\mathrm{cris}}(A_\alpha/\hat{R})_i^\circ$ such that

(a) $\phi^{\mathrm{cris}}_{\alpha,*,i}(\hat{\omega}_{\alpha,i}) \subseteq \hat{\omega}_{\alpha-1,i}$ and $\psi^{\mathrm{cris}}_{\alpha,*,i}(\hat{\omega}_{\alpha-1,i}) \subseteq \hat{\omega}_{\alpha,i}$ (so that both ϕ_α and ψ_α are lifted, which would automatically imply $\mathrm{Ker}(\phi_\alpha) \in A_\alpha[p]$),

(b) $\hat{\omega}_{\alpha,2} \supseteq \mathrm{Ker}(\phi^{\mathrm{cris}}_{\alpha,*,2})$, and

(c) the $\hat{R}$-modules $\hat{\omega}_{\alpha-1,1}/\phi^{\mathrm{cris}}_{\alpha,*,1}(\hat{\omega}_{\alpha,1})$ and $\hat{\omega}_{\alpha,1}/\psi^{\mathrm{cris}}_{\alpha,*,1}(\hat{\omega}_{\alpha-1,1})$ are flat and of rank $r_{\alpha-1} - (r_\alpha - r_\epsilon)$ and $r_\alpha - r_\epsilon$, respectively.

We shall see that condition (i) of Notation 7.6 is automatic. Also, condition (ii) already holds: since $H_1^{\mathrm{cris}}(A_\alpha/\hat{R})_1^\circ/(\mathrm{Ker}(\phi^{\mathrm{cris}}_{\alpha,*,1}) + \mathrm{Ker}(\psi^{\mathrm{cris}}_{\alpha+1,*,1}))$ is locally generated by $j_{\alpha+1,1} - j_{\alpha,1}$ elements after modulo $\hat{I}$, it is so prior to modulo $\hat{I}$

by Nakayama's lemma. Note that rank of $\mathrm{Ker}(\phi^{\mathrm{cris}}_{\alpha,*,1})$ and $\mathrm{Ker}(\psi^{\mathrm{cris}}_{\alpha+1,*,1})$ and the number of the generators of the quotient above add up to exactly n; it follows that $\mathrm{Ker}(\phi^{\mathrm{cris}}_{\alpha,*,1})+\mathrm{Ker}(\psi^{\mathrm{cris}}_{\alpha+1,*,1})$ is a direct sum and the sum is a subbundle of $H^{\mathrm{cris}}_1(A_\alpha/\hat{R})^\circ_1$.

We separate the discussion of lifts at q_1 and q_2, and show that the tangent space $T_{Y^\circ_j}$ is isomorphic to $T_1\oplus T_2$ for the contributions T_1 and T_2 from the two places. We first look at q_2, as it is easier. Note that condition (b) $\hat{\omega}_{\alpha,2}\supseteq\mathrm{Ker}(\phi^{\mathrm{cris}}_{\alpha,*,2})=\mathrm{Im}(\psi^{\mathrm{cris}}_{\alpha,*,2})$ automatically implies that $\psi^{\mathrm{cris}}_{\alpha,*,2}(\hat{\omega}_{\alpha-1,2})\subseteq\hat{\omega}_{\alpha,2}$; so we can proceed as follows:

Step 0: First lift $\omega^\circ_{A^\vee_\epsilon/R,2}$ to a subbundle $\hat{\omega}_{\epsilon,2}$ of $H^{\mathrm{cris}}_1(A_\epsilon/\hat{R})^\circ_2$ so that it contains $\mathrm{Ker}(\phi^{\mathrm{cris}}_{\epsilon,*,2})$,

Step 1: then lift $\omega^\circ_{A^\vee_{\epsilon-1}/R,2}$ to a subbundle $\hat{\omega}_{\epsilon-1,2}$ of $H^{\mathrm{cris}}_1(A_{\epsilon-1}/\hat{R})^\circ_2$ so that it contains $\phi^{\mathrm{cris}}_{\epsilon,*,2}(\hat{\omega}_{\epsilon,2})+\mathrm{Ker}(\phi^{\mathrm{cris}}_{\epsilon-1,*,2})$,

Step(s) $\boldsymbol{\alpha}$: then lift $\omega^\circ_{A^\vee_{\epsilon-\alpha}/R,2}$ to a subbundle $\hat{\omega}_{\epsilon-\alpha,2}$ of $H^{\mathrm{cris}}_1(A_{\epsilon-\alpha}/\hat{R})^\circ_2$ so that it contains $\phi^{\mathrm{cris}}_{\epsilon-\alpha+1,*,2}(\hat{\omega}_{\epsilon-\alpha+1,2})+\mathrm{Ker}(\phi^{\mathrm{cris}}_{\epsilon-\alpha,*,2})$,

Step $\boldsymbol{\epsilon}$: finally lift $\omega^\circ_{A^\vee_0/R,2}$ to a subbundle $\hat{\omega}_{0,2}$ of $H^{\mathrm{cris}}_1(A_0/\hat{R})^\circ_2$ so that it contains $\phi^{\mathrm{cris}}_{1,*,2}(\hat{\omega}_{1,2})$.

At Step 0, the choices form a torsor for the group

$$\mathrm{Hom}_R(\omega^\circ_{A^\vee_\epsilon/R,2}/\mathrm{Ker}(\phi^{\mathrm{dR}}_{\epsilon,*,2}),\mathrm{Lie}^\circ_{A_\epsilon/R,2})\otimes_R\hat{I};$$

the Hom space is a locally free R-module of rank $(s_\epsilon-j_{\epsilon,2})r_\epsilon$.

At Step $\alpha=1,\dots,\epsilon$, we observe that condition (i) of the moduli problem $\underline{Y}^\circ_j$ implies $\phi_{\epsilon-\alpha+1,*,2}(\omega^\circ_{A^\vee_{\epsilon-\alpha+1}/R,2})+\mathrm{Ker}(\phi^{\mathrm{dR}}_{\epsilon-\alpha,*,2})$ is an R-subbundle of $H^{\mathrm{dR}}_1(A_{\epsilon-\alpha}/R)^\circ_2$ of rank

$$s_{\epsilon-\alpha+1}-j_{\epsilon-\alpha+1,2}+j_{\epsilon-\alpha,2}=s_{\epsilon-\alpha}+(j_{\epsilon-\alpha+1,1}-j_{\epsilon-\alpha,2})\quad\text{if }\alpha=1,\dots,\epsilon-1,\tag{7.7.1}$$

and of rank $s_1-j_{1,2}$ if $\alpha=\epsilon$. So $\phi^{\mathrm{cris}}_{\epsilon-\alpha+1,*,2}(\hat{\omega}_{\epsilon-\alpha+1,2})+\mathrm{Ker}(\phi^{\mathrm{cris}}_{\epsilon-\alpha,*,2})$ is an $\hat{R}$-subbundle of $H^{\mathrm{cris}}_1(A_{\epsilon-\alpha}/\hat{R})^\circ_2$ of the same rank. The choices of the lifts $\hat{\omega}_{\epsilon-\alpha,2}$ form a torsor for the group

$$\mathrm{Hom}_R\big(\omega^\circ_{A^\vee_{\epsilon-\alpha}/R,2}/(\phi_{\epsilon-\alpha+1,*,2}(\omega^\circ_{A^\vee_{\epsilon-\alpha+1}/R,2})+\mathrm{Ker}(\phi^{\mathrm{dR}}_{\epsilon-\alpha,*,2})),\mathrm{Lie}^\circ_{A_{\epsilon-\alpha}/R,2}\big)\otimes_R\hat{I}.$$

By (7.7.1), this Hom space is a locally free R-module of rank $(j_{\epsilon-\alpha+1,1}-j_{\epsilon-\alpha,2})r_{\epsilon-\alpha}$ if $\alpha=1,\dots,\epsilon-1$ and of rank $(s_0-(s_1-j_{1,2}))r_0$ if $\alpha=\epsilon$. This implies that the contribution T_2 to the tangent space $T_{Y^\circ_j}$ at q_2 admits a filtration such that the subquotients are

$$\mathcal{H}om\big(\omega^\circ_{\mathcal{A}^\vee_{\epsilon-\alpha},2}/(\phi_{\epsilon-\alpha+1,*,2}(\omega^\circ_{\mathcal{A}^\vee_{\epsilon-\alpha+1},2})+\mathrm{Ker}(\phi^{\mathrm{dR}}_{\epsilon-\alpha,*,2})),\mathrm{Lie}^\circ_{\mathcal{A}_{\epsilon-\alpha},2}\big)$$

where the $\mathcal{A}_{\epsilon-\alpha}$ are the universal abelian varieties and $\phi_{\epsilon+1,*,2}(\omega^\circ_{\mathcal{A}^\vee_{\epsilon+1},2})$ is interpreted as zero. In particular, T_2 is a locally free sheaf on Y_j° of rank

$$(s_\epsilon - j_{\epsilon,2})r_\epsilon + (s_0-(s_1-j_{1,2}))r_0 + \sum_{\alpha=1}^{\epsilon-1}(j_{\epsilon-\alpha+1,1}-j_{\epsilon-\alpha,2})r_{\epsilon-\alpha}$$
$$= (s_\epsilon - j_{\epsilon,2})r_\epsilon + j_{1,1}r_0 + \sum_{\alpha=1}^{\epsilon-1}(j_{\alpha+1,1}-j_{\alpha,2})r_\alpha. \quad (7.7.2)$$

We now look at the place q_1. By condition (ii), $\phi^{\mathrm{cris}}_{\alpha,*,1}$ when restricted to $\mathrm{Ker}(\psi^{\mathrm{cris}}_{\alpha+1,*,1})$ is a saturated injection of $\hat{R}$-bundles; and $\psi^{\mathrm{cris}}_{\alpha,*,1}$ when restricted to $\mathrm{Ker}(\phi^{\mathrm{cris}}_{\alpha-1,*,1})$ is also a saturated injection of $\hat{R}$-bundles. We first recall from the discussion in Notation 7.6 especially (7.6.1) that, when $\alpha = 1,\dots,\epsilon-1$, $\omega^\circ_{A^\vee_\alpha/R,1}$ is the direct sum of

$$\omega^{\circ,\mathrm{Ker}\,\phi}_{A^\vee_\alpha/R,1} := \omega^\circ_{A^\vee_\alpha/R,1}\cap \mathrm{Ker}(\phi^{\mathrm{dR}}_{\alpha,*,1}) \quad\text{and}\quad \omega^{\circ,\mathrm{Ker}\,\psi}_{A^\vee_\alpha/R,1} := \omega^\circ_{A^\vee_\alpha/R,1}\cap \mathrm{Ker}(\psi^{\mathrm{dR}}_{\alpha+1,*,1}),$$

which are locally free R-modules of rank r_ϵ and $r_\alpha - r_\epsilon$, respectively. Similarly, put

$$\omega^{\circ,\mathrm{Ker}\,\phi}_{A^\vee_\epsilon/R,1} := \omega^\circ_{A^\vee_\epsilon/R,1}, \quad \omega^{\circ,\mathrm{Ker}\,\psi}_{A^\vee_\epsilon/R,1} := 0, \quad\text{and}\quad \omega^{\circ,\mathrm{Ker}\,\psi}_{A^\vee_0/R,1} = \omega^\circ_{A^\vee_0/R,1}\cap \mathrm{Ker}(\psi^{\mathrm{dR}}_{1,*,1});$$

they have ranks r_ϵ, 0, and $r_0 - r_\epsilon$, respectively. We shall avoid talking about $\omega^{\circ,\mathrm{Ker}\,\phi}_{A^\vee_0/R,1}$ (as it does not make sense) but only psychologically understand it as the process that enlarges $\omega^{\circ,\mathrm{Ker}\,\phi}_{A^\vee_0/R,1}$ to $\omega^\circ_{A^\vee_0/R,1}$.

For $\alpha = 1,\dots,\epsilon$, the lift $\hat{\omega}_{\alpha,1}$ takes the form of $\hat{\omega}^{\mathrm{Ker}\,\phi}_{\alpha,1}\oplus\hat{\omega}^{\mathrm{Ker}\,\psi}_{\alpha,1}$, where the two direct summands are $\hat{R}$-subbundles of $\mathrm{Ker}(\phi^{\mathrm{cris}}_{\alpha,*,1})$ and of $\mathrm{Ker}(\psi^{\mathrm{cris}}_{\alpha+1,*,1})$, lifting $\omega^{\circ,\mathrm{Ker}\,\phi}_{A^\vee_\alpha/R,1}$ and $\omega^{\circ,\mathrm{Ker}\,\psi}_{A^\vee_\alpha/R,1}$, respectively. Whereas, the lift $\hat{\omega}_{0,1}$ contains the lift $\hat{\omega}^{\mathrm{Ker}\,\psi}_{0,1}$ of $\omega^{\circ,\mathrm{Ker}\,\psi}_{A^\vee_0/R,1}$ as an $\hat{R}$-subbundle of $\mathrm{Ker}(\psi^{\mathrm{cris}}_{1,*,1})$. Now the compatibility conditions $\phi^{\mathrm{cris}}_{\alpha,*,1}(\hat{\omega}_{\alpha,1})\subseteq\hat{\omega}_{\alpha-1,1}$ and $\psi^{\mathrm{cris}}_{\alpha,*,1}(\hat{\omega}_{\alpha-1,1})\subseteq\hat{\omega}_{\alpha,1}$ together with the condition (c) are equivalent to

$$\phi^{\mathrm{cris}}_{\alpha,*,1}(\hat{\omega}^{\mathrm{Ker}\,\psi}_{\alpha,1})\subseteq\hat{\omega}^{\mathrm{Ker}\,\psi}_{\alpha-1,1} \quad\text{and}\quad \psi^{\mathrm{cris}}_{\alpha,*,1}(\hat{\omega}^{\mathrm{Ker}\,\phi}_{\alpha-1,1})\subseteq\hat{\omega}^{\mathrm{Ker}\,\phi}_{\alpha,1}.$$

(The condition (c) on ranks of the quotients are also automatic.) In particular, the tangent space T_1 has three contributions, coming from the lifts $\hat{\omega}^{\mathrm{Ker}\,\phi}_{\alpha,1}$ (for $\alpha = 1,\dots,\epsilon$), from the lifts $\hat{\omega}^{\mathrm{Ker}\,\psi}_{\alpha,1}$ (for $\alpha = 0,\dots,\epsilon$), and from lifting $\omega^\circ_{A^\vee_0/R,1}$ to an $\hat{R}$-subbundle $\hat{\omega}_{0,1}$ of $H_1^{\mathrm{cris}}(A_0/\hat{R})^\circ_1$ containing $\hat{\omega}^{\mathrm{Ker}\,\psi}_{0,1}$. We shall use $T_1^{\mathrm{Ker}\,\phi}$, $T_1^{\mathrm{Ker}\,\psi}$, and $T_1^{\mathrm{Ker}\,\phi,0}$ to denote these three parts of the tangent space; and they will sit in an exact sequence

$$0\to T_1^{\mathrm{Ker}\,\phi,0}\to T_1\to T_1^{\mathrm{Ker}\,\phi}\oplus T_1^{\mathrm{Ker}\,\psi}\to 0. \quad (7.7.3)$$

We first determine the lifts $\hat{\omega}^{\mathrm{Ker}\,\phi}_{\alpha,1}$ for $\alpha = 1,\dots,\epsilon$. For $\hat{\omega}^{\mathrm{Ker}\,\phi}_{1,1}$, it lifts $\omega^{\circ,\mathrm{Ker}\,\phi}_{A^\vee_1/R,1}$ as an $\hat{R}$-subbundle of $H_1^{\mathrm{cris}}(A_1/\hat{R})^\circ_1$ of rank r_ϵ (with no further constraint). Then due to the rank constraint (and the injectivity of $\psi^{\mathrm{cris}}_{\alpha,*,1}$ when restricted to $\mathrm{Ker}(\phi^{\mathrm{cris}}_{\alpha+1,*,1})$), the lift $\hat{\omega}^{\mathrm{Ker}\,\phi}_{\alpha,1}$ for each $\alpha = 2, \dots, \epsilon$ is then forced to be equal to the image

$$\psi^{\mathrm{cris}}_{\alpha,*,1} \circ \cdots \circ \psi^{\mathrm{cris}}_{1,*,1}(\hat{\omega}^{\mathrm{Ker}\,\phi}_{1,1}).$$

So it suffices to consider the choices of the lift $\hat{\omega}^{\mathrm{Ker}\,\phi}_{1,1}$, which form a torsor for the group

$$\mathrm{Hom}_R(\omega^{\circ,\mathrm{Ker}\,\phi}_{A_1^\vee/R,1}, \mathrm{Ker}(\phi^{\mathrm{dR}}_{1,*,1})/\omega^{\circ,\mathrm{Ker}\,\phi}_{A_1^\vee/R,1}) \otimes_R \hat{I}.$$

This Hom space is a locally free R-module of rank

$$r_\epsilon(j_{1,1} - r_\epsilon). \tag{7.7.4}$$

It follows that the tangent space $T_1^{\mathrm{Ker}\,\phi}$ is simply just

$$\mathcal{H}om(\omega^{\circ,\mathrm{Ker}\,\phi}_{\mathcal{A}_0^\vee,1}, \mathrm{Ker}(\phi^{\mathrm{dR}}_{1,*,1})/\omega^{\circ,\mathrm{Ker}\,\phi}_{\mathcal{A}_0^\vee,1}).$$

We now determine the lifts $\hat{\omega}^{\mathrm{Ker}\,\psi}_{\alpha,1}$ for $\alpha = 0, \dots, \epsilon$ following the steps below:

Step 0: We start with putting $\hat{\omega}^{\mathrm{Ker}\,\psi}_{\epsilon,1} = 0$ because $\omega^{\circ,\mathrm{Ker}\,\psi}_{A_\epsilon^\vee/R,1}$ is,

Step(s) $\boldsymbol{\alpha}$: lift $\omega^{\circ,\mathrm{Ker}\,\psi}_{A^\vee_{\epsilon-\alpha}/R,1}$ to a subbundle $\hat{\omega}^{\mathrm{Ker}\,\psi}_{\epsilon-\alpha,1}$ of $\mathrm{Ker}(\psi^{\mathrm{cris}}_{\epsilon-\alpha+1,*,1})$ so that it contains $\phi^{\mathrm{cris}}_{\epsilon-\alpha+1}(\hat{\omega}^{\mathrm{Ker}\,\psi}_{\epsilon-\alpha+1,1})$,

Step $\boldsymbol{\epsilon}$: finally lift $\omega^{\circ,\mathrm{Ker}\,\psi}_{A_0^\vee/R,1}$ to a subbundle $\hat{\omega}^{\mathrm{Ker}\,\psi}_{0,1}$ of $\mathrm{Ker}(\psi^{\mathrm{cris}}_{1,*,1})$ so that it contains $\phi^{\mathrm{cris}}_{1,*,1}(\hat{\omega}^{\mathrm{Ker}\,\psi}_{1,1})$.

At Step $\alpha = 1, \dots, \epsilon$, the choices of the lifts $\hat{\omega}^{\mathrm{Ker}\,\psi}_{\epsilon-\alpha,1}$ form a torsor for the group

$$\mathrm{Hom}_R\big(\omega^{\circ,\mathrm{Ker}\,\psi}_{A^\vee_{\epsilon-\alpha}/R,1}\big/\phi_{\epsilon-\alpha+1,*,1}(\omega^{\circ,\mathrm{Ker}\,\psi}_{A^\vee_{\epsilon-\alpha+1}/R,1}), \mathrm{Ker}(\psi^{\mathrm{dR}}_{\epsilon-\alpha+1,*,1})\big/\omega^{\circ,\mathrm{Ker}\,\psi}_{A^\vee_{\epsilon-\alpha}/R,1}\big) \otimes_R \hat{I}.$$

This Hom space is a locally free R-module of rank

$$\big((r_{\epsilon-\alpha} - r_\epsilon) - (r_{\epsilon-\alpha+1} - r_\epsilon)\big)\big((n - j_{\epsilon-\alpha+1,1}) - (r_{\epsilon-\alpha} - r_\epsilon)\big).$$

This implies that the tangent space $T_1^{\mathrm{Ker}\,\psi}$ admits a filtration such that the subquotients are

$$\mathcal{H}om\big(\omega^{\circ,\mathrm{Ker}\,\psi}_{\mathcal{A}^\vee_{\epsilon-\alpha},1}\big/\phi_{\epsilon+1-\alpha,*,1}(\omega^{\circ,\mathrm{Ker}\,\psi}_{\mathcal{A}^\vee_{\epsilon+1-\alpha},1}), \mathrm{Ker}(\psi^{\mathrm{dR}}_{\epsilon+1-\alpha,*,1})/\omega^{\circ,\mathrm{Ker}\,\psi}_{\mathcal{A}^\vee_{\epsilon-\alpha},1}\big).$$

In particular, $T_1^{\mathrm{Ker}\,\psi}$ is a locally free sheaf on Y_j° of rank

$$\sum_{\alpha=1}^{\epsilon}\big((r_{\epsilon-\alpha} - r_\epsilon) - (r_{\epsilon-\alpha+1} - r_\epsilon)\big)\big((n - j_{\epsilon-\alpha+1,1}) - (r_{\epsilon-\alpha} - r_\epsilon)\big)$$
$$= \sum_{\alpha=0}^{\epsilon-1}(r_\alpha - r_{\alpha+1})(s_\alpha - j_{\alpha+1,1} + r_\epsilon). \tag{7.7.5}$$

Finally, we discuss the $\hat{R}$-module $\hat{\omega}_{0,1}$ that lifts $\omega^{\circ}_{A_0^\vee/R,1}$ and contains $\hat{\omega}_{0,1}^{\mathrm{Ker}\,\psi}$ we obtained earlier. The lift is subject to one condition: $\hat{\omega}_{0,1} \subseteq (\psi^{\mathrm{cris}}_{1,*,1})^{-1}(\hat{\omega}_{1,1}^{\mathrm{Ker}\,\phi})$. So the choices of the lift form a torsor for the group

$$\mathrm{Hom}_R\big(\omega^{\circ}_{A_0^\vee/R,1}/\omega^{\circ,\mathrm{Ker}\,\psi}_{A_0^\vee/R,1}, (\psi^{\mathrm{dR}}_{1,*,1})^{-1}(\omega^{\circ,\mathrm{Ker}\,\phi}_{A_1^\vee/R,1})/\omega^{\circ}_{A_0^\vee/R,1}\big) \otimes_R \hat{I}.$$

This implies that

$$T_1^{\mathrm{Ker}\,\phi,0} = \mathcal{H}om\big(\omega^{\circ}_{\mathcal{A}_0^\vee,1}/\omega^{\circ,\mathrm{Ker}\,\psi}_{\mathcal{A}_0^\vee,1}, (\psi^{\mathrm{dR}}_{1,*,1})^{-1}(\omega^{\circ,\mathrm{Ker}\,\phi}_{\mathcal{A}_1^\vee,1})/\omega^{\circ}_{\mathcal{A}_0^\vee,1}\big),$$

which is locally free of rank

$$\big(r_0 - (r_0 - r_\epsilon)\big)\big((r_\epsilon + n - j_{1,1}) - r_0\big) = r_\epsilon(s_0 + r_\epsilon - j_{1,1}). \tag{7.7.6}$$

To sum up, the tangent space $T_{Y_j^\circ}$, as the direct sum $T_1 \oplus T_2$ with T_1 sitting in the exact sequence (7.7.3), is a locally free sheaf of rank given by (7.7.6) + (7.7.4) + (7.7.5) + (7.7.2), that is,

$$\begin{aligned}
r_\epsilon(s_0 + r_\epsilon - j_{1,1}) + r_\epsilon(j_{1,1} - r_\epsilon) + \sum_{\alpha=0}^{\epsilon-1}(r_\alpha - r_{\alpha+1})(s_\alpha - j_{\alpha+1,1} + r_\epsilon)& \\
+ (s_\epsilon - j_{\epsilon,2})r_\epsilon + j_{1,1}r_0 + \sum_{\alpha=1}^{\epsilon-1}(j_{\alpha+1,1} - j_{\alpha,2})r_\alpha& \\
= r_\epsilon s_0 + \sum_{\alpha=0}^{\epsilon-1} r_\alpha(s_\alpha - j_{\alpha+1,1} + r_\epsilon) - \sum_{\alpha=1}^{\epsilon} r_\alpha(s_{\alpha-1} - j_{\alpha,1} + r_\epsilon)& \\
+ (s_\epsilon - j_{\epsilon,2})r_\epsilon + j_{1,1}r_0 + \sum_{\alpha=1}^{\epsilon-1}(j_{\alpha+1,1} - j_{\alpha,2})r_\alpha& \\
= r_\epsilon s_0 + r_0(s_0 - j_{1,1} + r_\epsilon) + r_\epsilon(s_{\epsilon-1} - j_{\epsilon,1} + r_\epsilon) + (s_\epsilon - j_{\epsilon,2})r_\epsilon + j_{1,1}r_0& \\
+ \sum_{\alpha=1}^{\epsilon-1} r_\alpha\big((s_\alpha - j_{\alpha+1,1} + r_\epsilon) - (s_{\alpha-1} - j_{\alpha,1} + r_\epsilon) + (j_{\alpha+1,1} - j_{\alpha,2})\big).&
\end{aligned}$$

One easily checks that the first line adds up to $r_\epsilon s_\epsilon + r_0 s_0$, and the second line cancels to zero. This concludes the proof. □

In the special case of $\delta = r$, each abelian variety A_α appearing in the moduli problem of Y_j is isogenous to A_ϵ, which is a certain abelian variety parameterized by the discrete Shimura variety $\mathrm{Sh}_{0,n}$ and is hence supersingular (by Remark 3.7). So in particular, the image $\mathrm{pr}_j(Y_j)$ in this case is contained in the supersingular locus of $\mathrm{Sh}_{r,s}$. In fact, the converse is also true.

Theorem 7.8. *Assume $\delta = r$. The supersingular locus of* $\mathrm{Sh}_{r,s}$ *is the union of all* $\mathrm{pr}_j(Y_j)$.

Proof. We say a finite torsion $W(\bar{\mathbb{F}}_p)$-module has *divisible sequence* $(a_1, a_2, \dots, a_\epsilon)$ with nonnegative integers $a_1 \le \cdots \le a_\epsilon$ if it is isomorphic to

$$(W(\bar{\mathbb{F}}_p)/p^\epsilon)^{\oplus a_1} \oplus (W(\bar{\mathbb{F}}_p)/p^{\epsilon-1})^{\oplus(a_2-a_1)} \oplus \cdots \oplus (W(\bar{\mathbb{F}}_p)/p)^{\oplus(a_\epsilon - a_{\epsilon-1})}.$$

The following is an elementary linear algebra fact, whose proof we omit.

Claim: If $M_1 \subseteq M_2$ are two torsion $W(\bar{\mathbb{F}}_p)$-modules with divisible sequences $(a_{1,i}, \dots, a_{\epsilon,i})$ for $i = 1, 2$ respectively, then $a_{\alpha,1} \le a_{\alpha,2}$ for all $\alpha = 1, \dots, \epsilon$.

The proof of the theorem is similar to the proof of Proposition 4.14(3), which is a special case of this theorem. It suffices to look at the closed points of $\mathrm{Sh}_{r,s}$. Let $z = (\mathcal{A}_z, \lambda, \eta) \in \mathrm{Sh}_{r,s}(\bar{\mathbb{F}}_p)$ be a supersingular point. Consider

$$\mathbb{L}_{\mathbb{Q}} = (\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_1[1/p])^{F^2=p} = \{a \in \tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_1[1/p] \mid F^2(a) = pa\}.$$

Since x is supersingular, $\mathbb{L}_{\mathbb{Q}}$ is a $\mathbb{Q}_{p^2}$-vector space of dimension n, and $\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_1[1/p]$ may be identified with the extension of scalars of $\mathbb{L}_{\mathbb{Q}}$ from $\mathbb{Q}_{p^2}$ to $W(\bar{\mathbb{F}}_p)[1/p]$. Put

$$\tilde{\mathcal{E}}^\circ_1 = (\mathbb{L}_{\mathbb{Q}} \cap \tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_1) \otimes_{\mathbb{Z}_{p^2}} W(\bar{\mathbb{F}}_p) \quad \text{and} \quad \tilde{\mathcal{E}}^\circ_2 = F(\tilde{\mathcal{E}}^\circ_1) = V(\tilde{\mathcal{E}}^\circ_1) \subseteq \tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_2.$$

Then we have

$$\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_i/\tilde{\mathcal{E}}_i \simeq (W(\bar{\mathbb{F}}_p)/p^\epsilon)^{\oplus j_{1,i}} \oplus (W(\bar{\mathbb{F}}_p)/p^{\epsilon-1})^{\oplus(j_{2,i}-j_{1,i})} \oplus \cdots \cdots \oplus (W(\bar{\mathbb{F}}_p)/p)^{\oplus(j_{\epsilon,i}-j_{\epsilon-1,i})}, \tag{7.8.1}$$

for nondecreasing sequences $0 \le j_{1,i} \le j_{2,i} \le \cdots \le j_{\epsilon,i} \le n$ with $i = 1, 2$; in other words, $\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_i/\tilde{\mathcal{E}}_i$ has divisible sequence $(j_{1,i}, \dots, j_{\epsilon,i})$. Without loss of generality, we assume that $j_{1,1}$ and $j_{1,2}$ are not both zero. The essential part of the proof consists of checking the sequence of inequalities

$$0 \le j_{1,1} < j_{1,2} < j_{2,1} < j_{2,2} < \cdots < j_{\epsilon,1} < j_{\epsilon,2} \le n. \tag{7.8.2}$$

We first prove (7.8.2) with all strict inequalities replaced by nonstrict ones. Indeed, the obvious inclusion $F(\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_i) \subseteq \tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_{3-i}$ implies that

$$\begin{aligned} F(\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_1/\tilde{\mathcal{E}}_1) &= F(\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_1)/\tilde{\mathcal{E}}_2 \subseteq \tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_2/\tilde{\mathcal{E}}_2, \quad \text{and} \\ F(\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_2/\tilde{\mathcal{E}}_2) &= F(\tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_2)/p\tilde{\mathcal{E}}_1 \subseteq \tilde{\mathcal{D}}(\mathcal{A}_z)^\circ_1/p\tilde{\mathcal{E}}_1. \end{aligned}$$

By (7.8.1), the first inclusion embeds a torsion $W(\bar{\mathbb{F}}_p)$-module with divisible sequence $(j_{1,1}, \dots, j_{\epsilon,1})$ into a torsion $W(\bar{\mathbb{F}}_p)$-module with divisible sequence $(j_{1,2}, \dots, j_{\epsilon,2})$. The Claim above implies that $j_{\alpha,1} \le j_{\alpha,2}$ for all $\alpha = 1, \dots, \epsilon$. Similarly, by (7.8.1), the second inclusion embeds a torsion $W(\bar{\mathbb{F}}_p)$-module with divisible sequence $(j_{1,2}, \dots, j_{\epsilon,2})$ into a torsion $W(\bar{\mathbb{F}}_p)$-module with divisible sequence $(j_{1,1}, \dots, j_{\epsilon,1}, n)$. The Claim above implies that $j_{\alpha,2} \le j_{\alpha+1,1}$ for all $\alpha = 1, \dots, \epsilon - 1$, and $j_{\epsilon,2} \le n$.

We now use the construction of $\mathbb{L}_{\mathbb{Q}}$ to show the strict inequalities in (7.8.2). Suppose first that $j_{\alpha,1} = j_{\alpha,2}$ for some $\alpha = 1, \dots, \epsilon$. Then it follows that the maps

$$F,\ V : \left(p^{\epsilon-\alpha}\tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ \cap \frac{1}{p}\tilde{\mathcal{E}}_1^\circ\right) + \tilde{\mathcal{E}}_1^\circ \to \left(p^{\epsilon-\alpha}\tilde{\mathcal{D}}(\mathcal{A}_z)_2^\circ \cap \frac{1}{p}\tilde{\mathcal{E}}_2^\circ\right) + \tilde{\mathcal{E}}_2^\circ \tag{7.8.3}$$

are both isomorphisms (due to an easy length computation as $\tilde{\mathcal{E}}_2^\circ = F(\tilde{\mathcal{E}}_1^\circ) = V(\tilde{\mathcal{E}}_1^\circ)$). By the definition of $\mathbb{L}_{\mathbb{Q}}$ and $\tilde{\mathcal{E}}_1^\circ$, we must have

$$\left(\left(p^{\epsilon-\alpha}\tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ \cap \frac{1}{p}\tilde{\mathcal{E}}_1^\circ\right) + \tilde{\mathcal{E}}_1^\circ\right)^{F=V} \subseteq \mathbb{L}_{\mathbb{Q}} \cap \tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ \subseteq \tilde{\mathcal{E}}_1^\circ.$$

But this is absurd because the isomorphisms (7.8.3) implies by Hilbert's Theorem 90 that the left hand side above generates the source of (7.8.3), which is clearly not contained in $\tilde{\mathcal{E}}_1^\circ$.

Similarly, suppose that $j_{\alpha,2} = j_{\alpha+1,1}$ for some $\alpha = 1, \dots, \epsilon - 1$. Then the following morphisms are isomorphisms

$$F, V : \left(p^{\epsilon-\alpha}\tilde{\mathcal{D}}(\mathcal{A}_z)_2^\circ \cap \frac{1}{p}\tilde{\mathcal{E}}_2^\circ\right) + \tilde{\mathcal{E}}_2^\circ \to (p^{\epsilon-\alpha}\tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ \cap \tilde{\mathcal{E}}_1^\circ) + p\tilde{\mathcal{E}}_1^\circ, \tag{7.8.4}$$

since $p\tilde{\mathcal{E}}_1^\circ = F(\tilde{\mathcal{E}}_2^\circ) = V(\tilde{\mathcal{E}}_2^\circ)$ and for length reasons. By the definition of $\mathbb{L}_{\mathbb{Q}}$ and $\tilde{\mathcal{E}}_1^\circ$,

$$\left((p^{\epsilon-\alpha}\tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ \cap \tilde{\mathcal{E}}_1^\circ) + p\tilde{\mathcal{E}}_1^\circ\right)^{F^{-1}=V^{-1}} \subseteq \mathbb{L}_{\mathbb{Q}} \cap p\tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ \subseteq p\tilde{\mathcal{E}}_1^\circ.$$

(Note that $\epsilon - \alpha \geq 1$ now.) But this is absurd because the isomorphisms (7.8.4) imply by Hilbert's Theorem 90 that the left hand side above generates the target of (7.8.4), which is clearly not contained in $p\tilde{\mathcal{E}}_1^\circ$.

Summing up, we have proved the strict inequalities (7.8.2). So the $j_{\alpha,i}$ define a $\boldsymbol{j}$ as in the beginning of Section 7.1. We now construct a point of $Y_{\boldsymbol{j}}$ which maps to the point $z \in \mathrm{Sh}_{r,s}$. Put

$$\tilde{\mathcal{E}}_{\alpha,1} := \tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ \cap \frac{1}{p^{\epsilon-\alpha}}\tilde{\mathcal{E}}_1 \quad \text{and} \quad \tilde{\mathcal{E}}_{\alpha,2} := \tilde{\mathcal{D}}(\mathcal{A}_z)_2^\circ \cap \frac{1}{p^{\epsilon-\alpha}}\tilde{\mathcal{E}}_2. \tag{7.8.5}$$

Using the exact construction in Section 7.2, we get the sequence of isogenies of abelian varieties

$$A_\epsilon \underset{\psi_\epsilon}{\overset{\phi_\epsilon}{\rightleftarrows}} A_{\epsilon-1} \underset{\psi_{\epsilon-1}}{\overset{\phi_{\epsilon-1}}{\rightleftarrows}} \cdots \underset{\psi_1}{\overset{\phi_1}{\rightleftarrows}} A_0 = \mathcal{A}_z,$$

such that A_α together with the induced polarization λ_α and the tame level structure η_α gives an $\bar{\mathbb{F}}_p$-point of $\mathrm{Sh}_{r_\alpha,s_\alpha}$, and $\tilde{\mathcal{D}}(A_\alpha)_i^\circ = \tilde{\mathcal{E}}_{\alpha,i}$ for all α and $i = 1, 2$.

Conditions (2)–(5) of Definition 7.4 easily follow from the description of the quotients $\tilde{\mathcal{D}}(\mathcal{A}_z)_i^\circ/\tilde{\mathcal{E}}_i$ in (7.8.1). Condition (6) of Definition 7.4 is equivalent to

$$p\tilde{\mathcal{D}}(A_{\alpha-1})_2^\circ \subseteq V(\tilde{\mathcal{D}}(A_\alpha)_1^\circ).$$

By the construction of these Dieudonné modules in (7.8.5), this is equivalent to

$$p\Big(\tilde{\mathcal{D}}(\mathcal{A}_z)_2^\circ \cap \frac{1}{p^{\epsilon-\alpha+1}}\tilde{\mathcal{E}}_2\Big) \subseteq V\Big(\tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ \cap \frac{1}{p^{\epsilon-\alpha}}\tilde{\mathcal{E}}_1\Big).$$

But this follows from $p\tilde{\mathcal{D}}(\mathcal{A}_z)_2^\circ \subseteq V\tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ$ and $\tilde{\mathcal{E}}_2 = V\tilde{\mathcal{E}}_1$. Condition (7) of Definition 7.4 is equivalent to $\omega^\circ_{A_\alpha^\vee/\bar{\mathbb{F}}_p,1} \cap \operatorname{Ker}(\phi^{\mathrm{dR}}_{\alpha,*,1})$ having dimension r_ϵ, which is zero in our case. Translating it into the language of Dieudonné modules, this is equivalent to

$$V\tilde{\mathcal{D}}(A_\alpha)_2^\circ \cap p\tilde{\mathcal{D}}(A_{\alpha-1})_1^\circ = p\tilde{\mathcal{D}}(A_\alpha)_1^\circ.$$

By the construction of these Dieudonné module in (7.8.5), this is equivalent to

$$\Big(V\tilde{\mathcal{D}}(\mathcal{A}_z)_2^\circ \cap \frac{1}{p^{\epsilon-\alpha}}V\tilde{\mathcal{E}}_2\Big) \cap \Big(p\tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ \cap \frac{1}{p^{\epsilon-\alpha}}\tilde{\mathcal{E}}_1\Big) = p\tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ \cap \frac{1}{p^{\epsilon-\alpha-1}}\tilde{\mathcal{E}}_1,$$

which follows from observing that $V\tilde{\mathcal{D}}(\mathcal{A}_z)_2^\circ \supseteq p\tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ$ and $V\tilde{\mathcal{E}}_2 = p\tilde{\mathcal{E}}_1$. Condition (8) of Definition 7.4 is equivalent to $\omega^\circ_{A_{\alpha-1}^\vee/\bar{\mathbb{F}}_p,1} \subseteq \operatorname{Ker}(\psi^{\mathrm{dR}}_{\alpha,*,1})$ (note that $r_\epsilon = 0$ in our case). Translating it into the language of Dieudonné modules and using (7.8.5), this is equivalent to

$$V\tilde{\mathcal{D}}(A_{\alpha-1})_2^\circ \subseteq \tilde{\mathcal{D}}(A_\alpha)_1^\circ, \text{ or equivalently,}$$
$$V\tilde{\mathcal{D}}(\mathcal{A}_z)_2^\circ \cap \frac{1}{p^{\epsilon-\alpha+1}}V\tilde{\mathcal{E}}_2 \subseteq \tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ \cap \frac{1}{p^{\epsilon-\alpha}}\tilde{\mathcal{E}}_1,$$

which follows from observing that $V\tilde{\mathcal{D}}(\mathcal{A}_z)_2^\circ \subseteq \tilde{\mathcal{D}}(\mathcal{A}_z)_1^\circ$ and $V\tilde{\mathcal{E}}_2 = p\tilde{\mathcal{E}}_1$. This concludes the proof. □

Conjecture 7.9. *The varieties Y_j together with the natural morphisms to* $\mathrm{Sh}_{r-\delta,s+\delta}$ *and* $\mathrm{Sh}_{r,s}$ *satisfy condition* (3) *of* Conjecture 2.12. *Moreover, the union of the images of Y_j in* $\mathrm{Sh}_{r,s}$ *is the closure of the locus where the Newton polygon of the universal abelian variety has slopes* 0 *and* 1 *each with multiplicity* $2(r-\delta)n$, *and slope* $\frac{1}{2}$ *with multiplicity* $2(n-2r+2\delta)n$.

This conjecture in the case of $r = \delta = 1$ was proved in Theorem 4.18.

Appendix A: An explicit formula in the local spherical Hecke algebra for GL_n

In this appendix, let F be a local field with ring of integers $\mathcal{O}$, $\varpi \in \mathcal{O}$ be a uniformizer, $\mathbb{F} = \mathcal{O}/\varpi\mathcal{O}$ and $q = \#\mathbb{F}$. Fix an integer $n \geq 1$. We consider the spherical Hecke algebra $\mathscr{H}_K = \mathbb{Z}[K \backslash \mathrm{GL}_n(F)/K]$ with $K = \mathrm{GL}_n(\mathcal{O})$. Here, the product of two double cosets $u = KxK$ and $v = KyK$ in $\mathcal{H}_K$ is defined as

$$u \cdot v = \sum_w m(u, v; w)w, \text{[19]} \tag{A.0.1}$$

where the sum runs through all the double cosets $w = KzK$ contained in $KxKyK$, and the coefficient $m(u, v; w) \in \mathbb{Z}$ is determined as follows: If $KxK = \coprod_{i\in I} x_i K$ and $KyK = \coprod_{j\in J} y_j K$, then

$$m(u, v; w) = \#\{(i, j) \in I \times J \mid x_i y_j K = zK \text{ for a fixed element } z \text{ in } w\}. \tag{A.0.2}$$

By the theory of elementary divisors, all double cosets KxK are of the form

$$T(a_1, \dots, a_n) := K\,\mathrm{Diag}(\varpi^{a_1}, \dots, \varpi^{a_n})K \quad \text{for } a_i \in \mathbb{Z} \text{ with } a_1 \geq a_2 \geq \cdots \geq a_n.$$

They form a $\mathbb{Z}$-basis of $\mathcal{H}_K$. We put

$$T^{(r)} = T(\underbrace{1, \dots, 1}_{r}, \underbrace{0, \dots, 0}_{n-r}) \qquad \text{for } 0 \leq r \leq n,$$

$$R^{(r,s)} = T(\underbrace{2, \dots, 2}_{r}, \underbrace{1, \dots, 1}_{s-r}, \underbrace{0, \dots, 0}_{n-s}) \quad \text{for } 0 \leq r \leq s \leq n.$$

In particular, $R^{(0,s)} = T^{(s)}$ and $T^{(0)} = [K]$.

Because of the lack of references, we include a proof of the following:

Proposition A.1. *For $1 \leq r \leq n$, let*

$$\binom{n}{r}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-r+1} - 1)}{(q - 1)(q^2 - 1) \cdots (q^r - 1)} \tag{A.1.1}$$

be the Gaussian binomial coefficients, and put $\binom{n}{0}_q = 1$. Then for $0 \leq r \leq s \leq n$,

$$T^{(r)}T^{(s)} = \sum_{i=0}^{\min\{r,n-s\}} \binom{s-r+2i}{i}_q R^{(r-i,s+i)}.$$

Proof. We fix a set of representatives $\tilde{\mathbb{F}} \subseteq \mathcal{O}$ of $\mathbb{F} = \mathcal{O}/\varpi\mathcal{O}$ which contains 0. Then we have $T^{(r)} = \coprod_{x\in\mathcal{S}(n,r)} xK$, where $\mathcal{S}(n, r)$ is the set of $n \times n$ matrices $x = (x_{i,j})_{1\leq i,j\leq n}$ such that

- r of the diagonal entries are ϖ and the remaining $n - r$ ones are 1;
- if $i \neq j$, then $x_{i,j} = 0$ unless $i > j$, $x_{i,i} = 1$ and $x_{j,j} = \varpi$, in which case $x_{i,j}$ can take any values in $\tilde{\mathbb{F}}$.

[19] We may also view elements of $\mathcal{H}_K$ as $\mathbb{Z}$-valued locally constant and compactly supported functions on $\mathrm{GL}_n(F)$ which are bi-invariant under K, and define the product of $f, g \in \mathcal{H}_K$ as $(f * g)(x) = \int_{\mathrm{GL}_n(F)} f(y)g(y^{-1}x)\,dy$, where dy means the unique bi-invariant Haar measure on $\mathrm{GL}_n(F)$ with $\int_K dy = 1$. For the equivalence between these two definitions, see [Gross 1998, p. 4].

For instance, the set $\mathcal{S}(3,2)$ consists of matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ x_{2,1} & \varpi & 0 \\ x_{3,1} & 0 & \varpi \end{pmatrix}, \quad \begin{pmatrix} \varpi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x_{3,2} & \varpi \end{pmatrix}, \quad \begin{pmatrix} \varpi & 0 & 0 \\ 0 & \varpi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $x_{2,1}, x_{3,1}, x_{3,2} \in \tilde{\mathbb{F}}$. We have a similar decomposition $T^{(s)} = \coprod_{y \in \mathcal{S}(n,s)} yK$. We write $T^{(r)}T^{(s)}$ as a linear combination of $T(a_1, \dots, a_n)$ with $a_i \in \mathbb{Z}$ and $a_1 \geq \dots \geq a_n$. By looking at the diagonal entries of xy, we see easily that only $R^{(r-i,s+i)}$ with $0 \leq i \leq \min\{r, n-s\}$ have nonzero coefficients, namely, we have

$$T^{(r)}T^{(s)} = \sum_{i=0}^{\min\{r,n-s\}} C^{(r,s)}(n,i) R^{(r-i,s+i)} \quad \text{for some } C^{(r,s)}(n,i) \in \mathbb{Z}.$$

By (A.0.1), $C^{(r,s)}(n,i)$ is the number of pairs $(x, y) \in \mathcal{S}(n,r) \times \mathcal{S}(n,s)$ such that

$$xyK = \mathrm{Diag}(\underbrace{\varpi^2, \dots, \varpi^2}_{r-i}, \underbrace{\varpi, \dots, \varpi}_{s-r+2i}, \underbrace{1, \dots, 1}_{n-s-i})K.$$

In this case, x and y must be of the form

$$x = \begin{pmatrix} \varpi I_{r-i} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_{n-s-i} \end{pmatrix}, \quad y = \begin{pmatrix} \varpi I_{r-i} & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I_{n-s-i} \end{pmatrix},$$

where I_k denotes the $k \times k$ identity matrix, and $A \in \mathcal{S}(s-r+2i, i)$ and $B \in \mathcal{S}(s-r+2i, s-r+i)$ satisfy $AB \cdot \mathrm{GL}_{s-r+2i}(\mathcal{O}) = \varpi I_{s-r+2i}\, \mathrm{GL}_{s-r+2i}(\mathcal{O})$. By (A.0.1), we see that $C^{(r,s)}(n,i) = C^{(i,s-r+i)}(s-r+2i, i)$. Therefore, one is reduced to proving the following lemma, which is a special case of our proposition. □

Lemma A.2. *Under the notation and hypothesis of Proposition A.1, assume moreover that $n = r + s$. Then the coefficient of $R^{(0,n)}$ in the product $T^{(r)}T^{(s)}$ is $\binom{n}{r}_q$.*

Proof. We induct on $n \geq 1$. The case $n = 1$ is trivial. We assume thus $n > 1$, and that the statement is true when n is replaced by $n-1$. The case of $r = 0$ being trivial, we may assume that $r \geq 1$. We say a pair $(x, y) \in \mathcal{S}(n,r) \times \mathcal{S}(n, n-r)$ is admissible if $xyK = \varpi I_n K$. We have to show that the number of admissible pairs is equal to $\binom{n}{r}_q$. Let (x, y) be an admissible pair. Denote by I (resp. by J) the set integers $1 \leq i \leq n$ such that $x_{i,i} = \varpi$ (resp. $y_{i,i} = \varpi$). Note that (x, y) being admissible implies that $J = \{1, \dots, n\} \setminus I$.

Assume first that $x_{1,1} = 1$. Then x and y must be of the form $x = \left(\begin{smallmatrix} 1 & 0 \\ * & A \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & B \end{smallmatrix}\right)$ where $(A, B) \in \mathcal{S}(n-1, r) \times \mathcal{S}(n-1, n-1-r)$ admissible. Note that $xyK = \varpi I_n K$ always hold. We have $x_{i,1} = 0$ for $i \notin I$, and $x_{i,1}$ can take any values in $\mathbb{F}$ for $i \in I$. Therefore, the number of admissible pairs (x, y) with $x_{1,1} = 1$ is equal to $q^{\#I} = q^r$ times that of the admissible (A, B). The latter is equal to $\binom{n-1}{r}_q$ by the induction hypothesis.

Consider now the case $x_{1,1} = \varpi$. One can write $x = \begin{pmatrix} \varpi & 0 \\ 0 & A \end{pmatrix}$, and $y = \begin{pmatrix} 1 & 0 \\ * & B \end{pmatrix}$ with $(A, B) \in \mathcal{S}(n-1, r-1) \times \mathcal{S}(n-1, n-r)$ admissible. Put $z = xy$. Then an easy computation shows that $z_{j,1} = y_{j,1}$ if $j \in J$, and $z_{j,1} = 0$ if $j \notin J$. Hence, $xyK = \varpi I_n K$ forces that $y_{j,1} = 0$ for all $j > 1$. Therefore, the number of admissible (x, y) in this case is equal to that of the admissible (A, B), which is $\binom{n-1}{r-1}_q$ by the induction hypothesis. The lemma now follows immediately from the equality

$$\binom{n}{r}_q = q^r \binom{n-1}{r}_q + \binom{n-1}{r-1}_q. \qquad \square$$

Appendix B: A determinant formula

In this appendix, we prove the following:

Theorem B.1. *Let $\alpha_1, \dots, \alpha_n$ be n indeterminates. For $i = 1, \dots, n$, let s_i denote the i-th elementary symmetric polynomial in the α, and $s_0 = 1$ by convention. Let q be another indeterminate. We put $q_r = q^{r-1} + q^{r-3} + \cdots + q^{1-r}$. Consider the matrix $M_n(q) = (m_{i,j})$ given as follows:*

$$m_{i,j} = \begin{cases} \sum_{\delta=0}^{\min\{i-1,n-j\}} q_{n+i-j-2\delta} s_{j-i+\delta} s_{n-\delta} & \text{if } i \le j; \\ \sum_{\delta=0}^{\min\{j-1,n-i\}} q_{n+j-i-2\delta} s_\delta s_{n+j-i+\delta} & \text{if } i > j. \end{cases}$$

Then we have

$$\det(M_n(q)) = \alpha_1 \cdots \alpha_n \prod_{i \neq j} \Big(q\alpha_i - \frac{1}{q}\alpha_j \Big).$$

Proof. Let $N_n(q)$ be the resultant matrix of the polynomials $f(x) = \prod_{i=1}^n (x + q^{-1}\alpha_i)$ and $g(x) = \prod_{i=1}^n (x + q\alpha_i)$, that is, $N_n(q)$ is the $2n \times 2n$ matrix given by

$$N_n(q) = \begin{pmatrix} s_0 & q^{-1}s_1 & q^{-2}s_2 & \cdots & q^{1-n}s_{n-1} & q^{-n}s_n & 0 & \cdots & 0 \\ 0 & s_0 & q^{-1}s_1 & \cdots & q^{2-n}s_{n-2} & q^{1-n}s_{n-1} & q^{-n}s_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s_0 & q^{-1}s_1 & q^{-2}s_2 & \cdots & q^{-n}s_n \\ s_0 & qs_1 & q^2s_2 & \cdots & q^{n-1}s_{n-1} & q^n s_n & 0 & \cdots & 0 \\ 0 & s_0 & qs_1 & \cdots & q^{n-2}s_{n-2} & q^{n-1}s_{n-1} & q^n s_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s_0 & qs_1 & q^2s_2 & \cdots & q^n s_n \end{pmatrix}.$$

It is well known that $\det(N_n(q)) = \prod_{i,j}(-q^{-1}\alpha_i + q\alpha_j)$. Thus it suffices to show that $\det(N_n(q)) = (q - q^{-1})^n \det(M_n(q))$.

We first make the following row operations on $N_n(q)$: subtract row i from row $n+i$ for all $i = 1, \dots, n$. We obtain a matrix whose first column is all 0 except the first entry being 1; moreover, one can take out a factor $(q-q^{-1})$ from row $n+1,\dots,2n$. Let $N'_n(q)$ be the right lower $(2n-1)\times(2n-1)$ submatrix of the remaining matrix. Then we have

$$N'_n(q)=\begin{pmatrix} s_0 & q^{-1}s_1 & q^{-2}s_2 & \cdots & q^{1-n}s_{n-1} & q^{-n}s_n & 0 & \cdots & 0\\ 0 & s_0 & q^{-1}s_1 & \cdots & q^{2-n}s_{n-2} & q^{1-n}s_{n-1} & q^{-n}s_n & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & s_0 & q^{-1}s_1 & q^{-2}s_2 & \cdots & q^{-n}s_n\\ q_1s_1 & q_2s_2 & q_3s_3 & \cdots & q_{n-1}s_{n-1} & q_ns_n & 0 & \cdots & 0\\ 0 & q_1s_1 & q_2s_2 & \cdots & q_{n-2}s_{n-2} & q_{n-1}s_{n-1} & q_ns_n & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & q_1s_1 & q_2s_2 & \cdots & q_ns_n \end{pmatrix}$$

with $\det(N_n(q))=(q-q^{-1})^n\det(N'_n(q))$. Thus we are reduced to proving that $\det(N'_n(q))=\det(M_n(q))$. Consider the $(2n-1)\times(2n-1)$ matrix $R=\left(\begin{smallmatrix} I_{n-1} & 0\\ C & D\end{smallmatrix}\right)$ with the lower $n\times(2n-1)$ submatrix given by

$$\begin{pmatrix} C & D\end{pmatrix}=\begin{pmatrix} -q_1s_1 & -q_2s_2 & \cdots & -q_{n-1}s_{n-1} & 1 & q^{-1}s_1 & q^{-2}s_2 & \cdots & q^{2-n}s_{n-2} & q^{1-n}s_{n-1}\\ 0 & -q_1s_1 & \cdots & -q_{n-2}s_{n-2} & 0 & 1 & q^{-1}s_1 & \cdots & q^{3-n}s_{n-3} & q^{2-n}s_{n-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & -q_1s_1 & 0 & 0 & 0 & \cdots & 1 & q^{-1}s_1\\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

By a careful computation, one verifies without difficulty that $RN'_n(q)=\left(\begin{smallmatrix} U & *\\ 0 & M_n(q)\end{smallmatrix}\right)$, where U is an $(n-1)\times(n-1)$-upper triangular matrix with all diagonal entries equal to 1. Note that $\det(R)=\det(D)=\det(U)=1$, and it follows immediately that $\det(N'_n(q))=\det(M_n(q))$. □

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