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Effective Matsusaka's theorem for surfaces in characteristic p

Gabriele Di Cerbo and Andrea Fanelli

We obtain an effective version of Matsusaka's theorem for arbitrary smooth algebraic surfaces in positive characteristic, which provides an effective bound on the multiple that makes an ample line bundle D very ample. The proof for pathological surfaces is based on a Reider-type theorem. As a consequence, a Kawamata–Viehweg-type vanishing theorem is proved for arbitrary smooth algebraic surfaces in positive characteristic.

1. Introduction

A celebrated theorem of Matsusaka [1972] states that for a smooth n-dimensional complex projective variety X and an ample divisor D on it, there exists a positive integer M, depending only on the Hilbert polynomial $\chi(X, \mathbb{O}_X(kD))$, such that mD is very ample for all $m \geq M$. Kollár and Matsusaka [1983] improved the result, showing that the integer M only depends on the intersection numbers (D^n) and $(K_X \cdot D^{n-1})$.

The first effective versions of this result are due to Siu [2002a; 2002b] and Demailly [1996a; 1996b]; their methods are cohomological and rely on vanishing theorems. See also [Lazarsfeld 2004b] for a full account of this approach.

Although the minimal model program for surfaces in positive characteristic has recently been established, thanks to the work of Tanaka [2014; 2012], some

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interesting effectivity questions remain open in this setting, after the influential papers [Ekedahl 1988] and [Shepherd-Barron 1991a].

The purpose of this paper is to present a complete solution for the following problem:

Question 1.1. Let X be a smooth surface over an algebraically closed field of positive characteristic, and let D and B be an ample and a nef divisor on X respectively. Then there exists an integer M depending only on (D^2), $(K_X \cdot D)$ and $(D \cdot B)$ such that

$$mD - B$$

is very ample for all $m \geq M$.

The analogous question in characteristic zero with $B = 0$ was totally solved in [Fernández del Busto 1996], and a modified technique allows one to partially extend the result in positive characteristic [Ballico 1996].

The main result of this paper is the following:

Theorem 1.2. *Let D and B be respectively an ample divisor and a nef divisor on a smooth surface X over an algebraically closed field k, with* $\operatorname{char} k = p > 0$. *Then $mD - B$ is very ample for any*

$$m > \frac{2D \cdot (H + B)}{D^2}((K_X + 2D) \cdot D + 1),$$

where:

- $H := K_X + 4D$ *if X is neither quasielliptic with $\kappa(X) = 1$ nor of general type.*
- $H := K_X + 8D$ *if X is quasielliptic with $\kappa(X) = 1$ and $p = 3$.*
- $H := K_X + 19D$ *if X is quasielliptic with $\kappa(X) = 1$ and $p = 2$.*
- $H := 2K_X + 4D$ *if X is of general type and $p \geq 3$.*
- $H := 2K_X + 19D$ *if X is of general type and $p = 2$.*

The effective bound obtained with $H = K_X + 4D$ is expected to hold for all surfaces. Note that this bound is not far from being sharp even in characteristic zero [Fernández del Busto 1996].

The proof of Theorem 1.2 does not rely directly on vanishing theorems, but rather on Fujita's conjecture on basepoint-freeness and very-ampleness of adjoint divisors, which is known to hold for smooth surfaces in characteristic zero [Reider 1988] and for smooth surfaces in positive characteristic which are neither quasielliptic with $\kappa(X) = 1$ nor of general type [Shepherd-Barron 1991a; Terakawa 1999].

Conjecture 1.3 (Fujita). *Let X be a smooth n-dimensional projective variety and let D be an ample divisor on it. Then $K_X + kD$ is basepoint free for $k \geq n + 1$ and very ample for $k \geq n + 2$.*

If Fujita's conjecture on very-ampleness holds then the bound of Theorem 1.2 with $H = K_X + 4D$ would work for arbitrary smooth surfaces in positive characteristic.

For surfaces which are quasielliptic with $\kappa(X) = 1$ or of general type we can prove the following effective result in the spirit of Fujita's conjecture (see Section 4):

Theorem 1.4. *Let X a smooth surface over an algebraically closed field of characteristic $p > 0$, let D an ample Cartier divisor on X and let $L(a, b) := aK_X + bD$ for positive integers a and b. Then $L(a, b)$ is very ample for the following values of a and b:*

(1) *If X is quasielliptic with $\kappa(X) = 1$ and $p = 3$, $a = 1$ and $b \geq 8$.*

(2) *If X is quasielliptic with $\kappa(X) = 1$ and $p = 2$, $a = 1$ and $b \geq 19$.*

(3) *If X is of general type with $p \geq 3$, $a = 2$ and $b \geq 4$.*

(4) *If X is of general type with $p = 2$, $a = 2$ and $b \geq 19$.*

The key ingredient of Theorem 1.4 is a combination of a Reider-type result due to Shepherd-Barron and bend-and-break techniques.

For other results on the geography of pathological surfaces of Kodaira dimension smaller than two, see [Langer 2014].

In Section 5, a Kawamata–Viehweg-type vanishing theorem is proved for surfaces that are quasielliptic with $\kappa(X) = 1$ or of general type (see Theorem 5.7 and Corollary 5.9); this generalizes the vanishing result in [Terakawa 1999].

The core of our approach is a beautiful construction first introduced by Tango [1972] for the case of curves and Ekedahl [1988] and Shepherd-Barron [1991a] for surfaces. The same strategy was generalized by Kollár [1996] in order to investigate the geography of varieties where Kodaira-type vanishing theorems fail, via bend-and-break techniques.

2. Preliminary results

In this section we recall some techniques we will need later in this paper.

2A. ***Volume of divisors.*** Let D be a Cartier divisor on a normal variety X, not necessarily a surface. The volume of D measures the asymptotic growth of the space of global sections of multiples of D. We will recall here few properties of the volume, and we refer to [Lazarsfeld 2004b] for more details.

Definition 2.1. Let D be a Cartier divisor on X, with $\dim(X) = n$. The volume of D is defined by

$$\mathrm{vol}(D) := \limsup_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

The volume of X is defined as $\mathrm{vol}(X) := \mathrm{vol}(K_X)$.

It is easy to show that if D is big and nef then $\mathrm{vol}(D) = D^n$. In general it is a hard invariant to compute, but thanks to Fujita's approximation theorem some of its properties can be deduced from the case where D is ample. For a proof of the theorem in characteristic zero we refer to [Lazarsfeld 2004b]. More recently, Takagi [2007] gave a proof of the same theorem in positive characteristic. In particular, we can deduce the log-concavity of the volume function even in positive characteristic. The proof is exactly the same as Theorem 11.4.9 in [Lazarsfeld 2004b].

Theorem 2.2. *Let D and D' be big Cartier divisors on a normal variety X defined over an algebraically closed field. Then*

$$\mathrm{vol}(D + D')^{1/n} \geq \mathrm{vol}(D)^{1/n} + \mathrm{vol}(D')^{1/n}.$$

2B. ***Bogomolov's inequality and Sakai's theorems.*** We start with the notion of semistability for rank-two vector bundles on surfaces. Let X be a smooth surface defined over an algebraically closed field.

Definition 2.3. A rank-two vector bundle $\mathscr{E}$ on X is *unstable* if it fits in a short exact sequence

$$0 \longrightarrow \mathscr{O}_X(D_1) \longrightarrow \mathscr{E} \longrightarrow \mathscr{I}_Z \cdot \mathscr{O}_X(D_2) \longrightarrow 0,$$

where D_1 and D_2 are Cartier divisors such that $D' := D_1 - D_2$ is big with $(D'^2) > 0$ and Z is an effective 0-cycle on X.

The vector bundle $\mathscr{E}$ is *semistable* if it is not unstable.

In characteristic zero, the following celebrated result holds:

Theorem 2.4 [Bogomolov 1978]. *Let X be defined over a field of characteristic zero. Then every rank-two vector bundle $\mathscr{E}$ for which $c_1^2(\mathscr{E}) > 4c_2(\mathscr{E})$ is unstable.*

As a consequence, one can deduce the following theorem, due to Sakai [1990, Proposition 1], which turns out to be equivalent to Theorem 2.4. This equivalence was shown in [Di Cerbo 2013].

Theorem 2.5. *Let D be a nonzero big divisor with $D^2 > 0$ on a smooth projective surface X over a field of characteristic zero. If $H^1(X, \mathscr{O}_X(K_X + D)) \neq 0$, then there exists a nonzero effective divisor E such that*:

- *$D - 2E$ is big.*
- *$(D - E) \cdot E \leq 0$.*

The previous result easily implies a weaker version of Reider's theorem:

Theorem 2.6. *Let D be a nef divisor with $D^2 > 4$ on a smooth projective surface X over a field of characteristic zero. Then $K_X + D$ has no basepoint unless there exists a nonzero effective divisor E such that $D \cdot E = 0$ and $(E^2) = -1$ or $D \cdot E = 1$ and $(E^2) = 0$.*

The following result, conjectured by Fujita, can be deduced for smooth surfaces in characteristic zero:

Corollary 2.7 (Fujita conjectures for surfaces in char 0). *Let $D_1, \dots, D_k$ be ample divisors on a smooth surface X over a field of characteristic zero. Then $K_X + D_1 + \cdots + D_k$ is basepoint free if $k \geq 3$ and very ample if $k \geq 4$.*

We remark that Theorem 2.5 is not known in general for smooth surfaces in positive characteristic, although Fujita's conjectures are expected to hold.

2C. ***Ekedahl's construction and Shepherd-Barron's theorem.*** In this section we recall some classical results on the geography of smooth surfaces in positive characteristic (see [Ekedahl 1988; Shepherd-Barron 1991a; 1991b]).

For a good overview on the geography of surfaces in positive characteristic, see [Liedtke 2013].

We discuss here a construction which is due to Tango [1972] for the case of curves and Ekedahl [1988] for surfaces. There are many variations on the same theme, but we will focus on the one which is more related to the stability of vector bundles. We need this fundamental result:

Theorem 2.8 (Bogomolov). *Let $\mathscr{E}$ be a rank-two vector bundle on a smooth projective surface X over a field of positive characteristic such that Bogomolov's inequality does not hold (i.e., such that $c_1^2(\mathscr{E}) > 4c_2(\mathscr{E})$). Then there exists a reduced and irreducible surface Y contained in the ruled threefold $\mathbb{P}(\mathscr{E})$ such that*:

- *The restriction $\rho : Y \to X$ is p^e-purely inseparable for some $e > 0$.*
- *$(F^*)^e(\mathscr{E})$ is unstable.*

Proof. See [Shepherd-Barron 1991a, Theorem 1] □

The previous result also provides an explicit construction of the purely inseparable cover (see [Shepherd-Barron 1991a]).

Construction 2.9. Take a rank-two vector bundle $\mathscr{E}$ such that Bogomolov's inequality does not hold, and let e be an integer such that $F^{e*}\mathscr{E}$ is unstable. We have the commutative diagram

$$\begin{tikzcd} \mathbb{P}(F^{e*}\mathscr{E}) \arrow[r, "G"] \arrow[d, "p'"'] & \mathbb{P}(\mathscr{E}) \arrow[d, "p"] \\ X \arrow[r, "F^e"] & X \end{tikzcd} .$$

The fact that $F^{e*}\mathscr{E}$ is unstable gives an exact sequence

$$0 \longrightarrow \mathcal{O}_X(D_1) \longrightarrow F^{e*}\mathscr{E} \longrightarrow \mathcal{I}_Z \cdot \mathcal{O}_X(D_2) \longrightarrow 0$$

and a quasisection X_0 of $\mathbb{P}(F^{e*}\mathscr{E})$ (i.e., $p'_{|X_0} : X_0 \to X$ is birational). Let Y be the image of X_0 via G. One can show that the induced morphism

$$\rho : Y \to X$$

is p^e-purely inseparable. Let us define $D' := D_1 - D_2$. One can show (see [Shepherd-Barron 1991a, Corollary 5]) that

$$K_Y \equiv \rho^*\left(K_X - \frac{p^e - 1}{p^e} D'\right).$$

Remark 2.10. We will be particularly interested in the case when the rank-two vector bundle $\mathscr{E}$ comes as a nontrivial extension

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}_X(D) \longrightarrow 0$$

associated to a nonzero element $\gamma \in H^1(X, \mathscr{O}_X(-D))$, where D is a big Cartier divisor such that $(D^2) > 0$. Indeed, the instability of $F^{e*}\mathscr{E}$ guarantees the existence of a diagram (keeping the notation as in Definition 2.3)

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathscr{O}_X(D' + D_2) \arrow[d, "f_1"] & & \\
0 \arrow[r] & \mathscr{O}_X \arrow[r, "g_1"] \arrow[dr, "\sigma"'] & F^{e*}\mathscr{E} \arrow[r, "g_2"] \arrow[d, "f_2"] & \mathscr{O}_X(p^e D) \arrow[r] & 0 \\
 & & \mathscr{I}_Z \cdot \mathscr{O}_X(D_2) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}.$$

First, we claim that the composition map σ is nonzero. Assume for a contradiction that $\sigma \equiv 0$. This gives a nonzero section $\sigma' : \mathscr{O}_X \to \mathscr{O}_X(D' + D_2)$. This forces the composition $\tau := g_2 \circ f_1$ to be zero. But this implies that $D' + D_2 \le 0$. This is a contradiction (see the proof of [Sakai 1990, Proposition 1] and [Shepherd-Barron 1991a, Lemma 16]).

This implies that $D_2 \simeq E \ge 0$; one can then rewrite the vertical exact sequence as

$$0 \longrightarrow \mathscr{O}_X(p^e D - E) \longrightarrow F^{e*}\mathscr{E} \longrightarrow \mathscr{I}_Z \cdot \mathscr{O}_X(E) \longrightarrow 0.$$

Since [Shepherd-Barron 1991a, Corollary 8] guarantees that Corollary 2.7 holds true for smooth surfaces in positive characteristic which are neither quasielliptic with $\kappa(X) = 1$ nor of general type, we need to deduce effective basepoint-freeness and very-ampleness results only for these two classes of surfaces.

We recall here the following key result from [Shepherd-Barron 1991a]:

Theorem 2.11. *Let $\mathscr{E}$ be a rank-two vector bundle on a smooth projective surface X over an algebraically closed field of positive characteristic such that Bogomolov's inequality does not hold and $\mathscr{E}$ is semistable.*

- *If X is not of general type, then X is quasielliptic with $\kappa(X) = 1$.*
- *If X is of general type and*

$$c_1^2(\mathscr{E}) - 4c_2(\mathscr{E}) > \frac{\mathrm{vol}(X)}{(p-1)^2},$$

then X is purely inseparably uniruled. More precisely, in the notation of Theorem 2.8, Y is uniruled.

Proof. This is [Shepherd-Barron 1991a, Theorem 7], since the volume of a surface X with minimal model X' equals $(K_{X'}^2)$. □

Corollary 2.12 [Shepherd-Barron 1991a, Corollary 8]. *Corollary 2.7 holds in positive characteristic if X is neither of general type nor quasielliptic.*

2D. *Bend-and-break lemmas.* We recall here a well-known result in birational geometry, based on a celebrated method due to Mori (see [Kollár 1996] for an insight into these techniques).

First we need to recall some notation. Mori theory deals with effective 1-cycles in a variety X; more specifically, we will consider nonconstant morphisms $h : C \to X$, where C is a smooth curve. In particular, these techniques allow us to deform curves for which

$$(K_X \cdot C) := \deg_C h^* K_X < 0.$$

In what follows, we will denote by $\overset{e}{\approx}$ the *effective algebraic equivalence* defined on the space of effective 1-cycles $Z_1(X)$ (see [Kollár 1996, Definition II.4.1]).

Theorem 2.13 (bend-and-break). *Let X be a variety over an algebraically closed field, and let C be a smooth, projective and irreducible curve with a morphism $h : C \to X$ such that X has local complete intersection singularities along $h(C)$ and $h(C)$ intersects the smooth locus of X. Assume the numerical condition*

$$(K_X \cdot C) < 0$$

holds. Then, for every point $x \in C$, there exists a rational curve C_x in X passing through x such that

$$h_*[C] \overset{e}{\approx} k_0[C_x] + \sum_{i \neq 0} k_i[C_i] \tag{1}$$

(*as algebraic cycles*), *with* $k_i \geq 0$ *for all* i *and*

$$-(K_X \cdot C_x) \leq \dim X + 1.$$

Proof. See [Kollár 1996, Theorem II.5.14 and Remark II.5.15]. The relation (1) can be deduced by looking directly at the proofs of the bend-and-break lemmas [Kollár 1996, Corollary II.5.6 and Theorem II.5.7]; our notation is slightly different, since in (1) we have isolated a rational curve with the required intersection properties. □

In this paper we will need the following consequence of the previous theorem:

Corollary 2.14. *Let* X *be a surface which fibers over a curve* C *via* $f : X \to C$ *and let* F *be the general fiber of* f. *Assume that* X *has only local complete intersection singularities along* F *and that* F *is a* (*possibly singular*) *rational curve such that*

$$(K_X \cdot F) < 0.$$

Then

$$-(K_X \cdot F) \leq 3.$$

Proof. The hypotheses of Theorem 2.13 hold here, so we can take a point x in the smooth locus of X and deduce the existence of a rational curve C' passing through x such that

$$-(K_X \cdot C') \leq 3 \quad \text{and} \quad [F] \overset{e}{\approx} k_0[C'] + \sum_{i \neq 0} k_i [C_i].$$

By Exercise II.4.1.10 in [Kollár 1996], the curves appearing on the right hand side of the previous equation must be contained in the fibers of f. Since F is the general fiber, the second relation implies that $k_0 = 1$ and $k_i = 0$ for all $i \neq 0$, and so $C' = F$. □

3. An effective Matsusaka's theorem

In this section we prove Theorem 1.2 assuming the results on effective very-ampleness that we will prove in the next section. If not specified, X will denote a smooth surface over an algebraically closed field of arbitrary characteristic.

First we recall the following numerical criterion for bigness, whose characteristic-free proof is based on Riemann–Roch [Lazarsfeld 2004a, Theorem 2.2.15].

Theorem 3.1. *Let* D *and* E *be nef* $\mathbb{Q}$-*divisors on* X *and assume that*

$$D^2 > 2(D \cdot E).$$

Then $D - E$ *is big.*

Before proving Theorem 1.2, we need some lemmas.

Lemma 3.2. *Let D be an ample divisor on X. Then $K_X + 2D + C$ is nef for any irreducible curve $C \subset X$.*

Proof. If $X = \mathbb{P}^2$ then the lemma is trivial. By the cone theorem and the classification of surfaces with extremal rays of maximal length, we have that $K_X + 2D$ is always a nef divisor. This implies that $K_X + 2D + C$ may have negative intersection number only when intersected with C. On the other hand, by adjunction, $(K_X + C) \cdot C = 2g - 2 \geq -2$, where g is the arithmetic genus of C. Since D is ample, the result follows. □

We can now prove one of the main results of this section (see [Lazarsfeld 2004b, Theorem 10.2.4]).

Theorem 3.3. *Let D be an ample divisor and let B be a nef divisor on X. Then $mD - B$ is nef for any*

$$m \geq \frac{2D \cdot B}{D^2}((K_X + 2D) \cdot D + 1) + 1.$$

Proof. To simplify the notation in the proof let us define the following numbers:

$$\eta = \eta(D, B) := \inf\{t \in \mathbb{R}_{>0} \mid tD - B \text{ is nef}\},$$
$$\gamma = \gamma(D, B) := \inf\{t \in \mathbb{R}_{>0} \mid tD - B \text{ is pseudoeffective}\}.$$

The theorem will follow if we find an upper bound on η. Note that $\gamma \leq \eta$ since a nef divisor is also pseudoeffective.

By definition $\eta D - B$ is in the boundary of the nef cone and by Nakai's theorem we have two possible cases: either

- $(\eta D - B)^2 = 0$, or
- there exists an irreducible curve C such that $(\eta D - B) \cdot C = 0$.

If $(\eta D - B)^2 = 0$, then it is easy to see that

$$\eta \leq 2\frac{D \cdot B}{D^2}.$$

So we can assume that there exists an irreducible curve C such that $\eta D \cdot C = B \cdot C$. Let us define $G := \gamma D - B$. Then

$$G \cdot C = (\gamma - \eta) D \cdot C \leq (\gamma - \eta).$$

Let us define $A := K_X + 2D$. By Lemma 3.2 and the definition of G, we have that $(A + C) \cdot G \geq 0$. Combining with the previous inequality we get

$$(\eta - \gamma) \leq -G \cdot C \leq A \cdot G = \gamma A \cdot D - A \cdot B.$$

In particular,

$$\eta \leq \gamma(A \cdot D + 1) - A \cdot B \leq \gamma(A \cdot D + 1).$$

The statement of our result follows from Theorem 3.1, which guarantees that $\gamma < (2D \cdot B)/(D^2)$. □

Remark 3.4. The previous proof is characteristic-free, although the new result is for surfaces in positive characteristic.

We can now prove our main theorem, assuming the results in the next section.

Proof of Theorem 1.2. By Corollary 2.12, if X is neither of general type nor quasielliptic and $H = K_X + 4D$ then $H + N$ is very ample for any nef divisor N. By Theorem 3.3, $mD - (H + B)$ is nef for any m as in the statement. Then $K_X + 4D + (mD - K_X - 4D - B)$ is very ample. For surfaces in the other classes use Theorem 4.10 and Theorem 4.12 to obtain the desired very ample divisor H. □

4. Effective very-ampleness in positive characteristic

The aim of this section is to complete the proof of Theorem 1.2 for quasielliptic surfaces of Kodaira dimension one and for surfaces of general type. Our ultimate goal is to prove Theorem 1.4 via a case-by-case analysis.

First we need some notation (cf. Theorem 2.5).

Definition 4.1. A big divisor D on a smooth surface X with $(D^2) > 0$ is m-unstable for a positive integer m if either:

- $H^1(X, \mathcal{O}_X(-D)) = 0$.
- $H^1(X, \mathcal{O}_X(-D)) \neq 0$ and there exists a nonzero effective divisor E such that:
  - $mD - 2E$ is big.
  - $(mD - E) \cdot E \leq 0$.

Remark 4.2. Theorem 2.5 tells us that in characteristic zero every big divisor D on a smooth surface X with $(D^2) > 0$ is 1-unstable. The same holds in positive characteristic, if we assume that the surface is neither of general type nor quasielliptic of maximal Kodaira dimension; this is a consequence of Corollary 2.12. Our goal here is to clarify the picture in the remaining cases.

We start our analysis with quasielliptic surfaces of maximal Kodaira dimension.

Proposition 4.3. *Let X be a quasielliptic surface with $\kappa(X) = 1$ and let D be a big divisor on X with $(D^2) > 0$.*

(1) *if $p = 3$, then D is 3-unstable.*

(2) *if $p = 2$, then D is 4-unstable.*

Proof. Assume that $p = 3$ and $H^1(X, \mathcal{O}_X(-D)) \neq 0$. This nonzero element gives a nonsplit extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X(D) \to 0.$$

Theorem 2.8 implies that $(F^*)^e\mathscr{E}$ is unstable for e sufficiently large. To prove the proposition in this case we need to show that $e = 1$. Assume $e \geq 2$ and let F be the general element of the pencil which gives the fibration in cuspidal curves $f : X \to B$. Let $\rho : Y \to X$ be the p^e-purely inseparable morphism of Construction 2.9. Then $\{C_i := \rho^* F\}$ is a family of movable rational curves in Y. Let us define $g := f \circ \rho$ and consider its Stein factorization:

$$\begin{tikzcd} Y \arrow[d, "\rho"] \arrow[dr, "h"] \arrow[dd, bend right, "g"'] & \\ X \arrow[d, "f"] & B' \arrow[dl, "c"] \\ B & \end{tikzcd}.$$

Since the curves in the family $\{C_i\}$ are precisely the fibers of h, we can use Corollary 2.14 on $h : Y \to B'$ (since Y is defined via a quasisection in a $\mathbb{P}^1$-bundle over X, it has hypersurface singularities along the general element of $\{C_i\}$) and deduce that

$$0 < -(K_Y \cdot C_i) \leq 3.$$

This gives a contradiction, since

$$\begin{aligned} 3 \geq -(K_Y \cdot C_i) &= \left(\rho^*\left(\frac{p^e-1}{p^e}(p^e D - 2E) - K_X\right) \cdot C_i\right) \\ &= p^e\left(\left(\frac{p^e-1}{p^e}(p^e D - 2E) - K_X\right) \cdot F\right) \\ &= ((p^e-1)(p^e D - 2E) \cdot F) \\ &\geq p^e - 1 \geq 8, \end{aligned}$$

where E is the divisor appearing in Remark 2.10.

The same proof works for $p = 2$, although in this case we can only prove that $e \leq 2$. □

We can now focus on the general type case. We need the following theorem of Shepherd-Barron [1991a, Theorem 12].

Theorem 4.4. *Let D be a big Cartier divisor on a smooth surface X of general type which satisfies one of the following hypotheses*:

- $p \geq 3$ *and* $(D^2) > \mathrm{vol}(X)$.
- $p = 2$ *and* $(D^2) > \max\{\mathrm{vol}(X), \mathrm{vol}(X) - 3\chi(\mathcal{O}_X) + 2\}$.

Then D is 1*-unstable.*

Since the bound of the previous theorem depends on $\chi(\mathcal{O}_X)$ if $p=2$, we need an additional result for this case. First we recall a result by Shepherd-Barron [1991b, Theorem 8].

Theorem 4.5. *Let X be a surface in characteristic $p=2$ of general type with $\chi(\mathcal{O}_X)<0$. Then there is a fibration $f: X\to C$ over a smooth curve C whose generic fiber is a singular rational curve with arithmetic genus $2\le g\le 4$.*

We can prove now our result.

Proposition 4.6. *Let D be a big Cartier divisor on a surface in characteristic $p=2$ of general type with $\chi(\mathcal{O}_X)<0$ such that $(D^2)>\operatorname{vol}(X)$. Then D is 4-unstable.*

Proof. Assume that $H^1(X,\mathcal{O}_X(-D))\neq 0$. As in the proof of Proposition 4.3 we have a nonsplit extension

$$0\to\mathcal{O}_X\to\mathcal{E}\to\mathcal{O}_X(D)\to 0.$$

Using Theorem 2.8 we deduce the instability of $(F^*)^e\mathcal{E}$ for e sufficiently large. Let F be the general element of the pencil which gives the fibration in singular rational curves given by Theorem 4.5. Let $\rho: Y\to X$ be the p^e-purely inseparable morphism of Construction 2.9. As in the proof of Proposition 4.3, we use Corollary 2.14 on Y and deduce that $0<-(K_Y\cdot C_i)\le 3$. This gives

$$\begin{aligned}3\ge -(K_Y\cdot C_i)&=\left(\rho^*\left(\frac{2^e-1}{2^e}(2^eD-2E)-K_X\right)\cdot C_i\right)\\&=2^e\left(\left(\frac{2^e-1}{2^e}(2^eD-2E)-K_X\right)\cdot F\right)\\&=(((2^e-1)(2^eD-2E)-2^eK_X)\cdot F)\ge 1.\end{aligned}$$

This implies that

$$(((2^e-1)(2^{e-1}D-E)-2^{e-1}K_X)\cdot F)=1.$$

As a consequence, we apply Theorem 4.5 to bound the intersection $(K_X\cdot F)$:

$$(2^e-1)((2^{e-1}D-E)\cdot F)=2^e(g-1)+1,$$

where g is the arithmetic genus of F. By some basic arithmetic the only possibilities for the pair (g,e) are $(2,1)$, $(3,1)$, $(3,2)$ and $(4,1)$. □

We will use Theorem 4.4 to prove a variant of Reider's theorem in positive characteristic. We state a technical proposition that we will need later (see [Sakai 1990, Proposition 2]).

Proposition 4.7. *Let $\pi: Y\to X$ be a birational morphism between two normal surfaces. Let $\widetilde{D}$ be a Cartier divisor on Y such that $\widetilde{D}^2>0$. Assume there is a nonzero effective divisor $\widetilde{E}$ such that*

- $\widetilde{D} - 2\widetilde{E}$ *is big*,
- $(\widetilde{D} - \widetilde{E}) \cdot \widetilde{E} \leq 0$.

Set $D := \pi_* \widetilde{D}$, $E := \pi_* \widetilde{E}$ *and* $\alpha = D^2 - \widetilde{D}^2$. *If* D *is nef and* E *is a nonzero effective divisor*, *then*

- $0 \leq D \cdot E < \alpha/2$,
- $D \cdot E - \alpha/4 \leq E^2 \leq (D \cdot E)^2 / D^2$.

The corollary we need is the following.

Corollary 4.8. *Let* $\pi : Y \to X$ *be a birational morphism between two smooth surfaces and let* $\widetilde{D}$ *be a big Cartier divisor on* Y *such that* $(\widetilde{D}^2) > 0$. *Assume that*

- $H^1(X, \mathcal{O}_X(-\widetilde{D})) \neq 0$,
- $\widetilde{D}$ *is* m*-unstable for some* $m > 0$.

Set $D := \pi_* \widetilde{D}$ *and* $\alpha = D^2 - \widetilde{D}^2$. *Then if* D *is nef*, *there exists a nonzero effective divisor* E *on* X *such that*

- $0 \leq D \cdot E < m\alpha/2$,
- $mD \cdot E - m^2\alpha/4 \leq E^2 \leq (D \cdot E)^2 / D^2$.

We can now derive our effective basepoint-freeness results. We will start with quasielliptic surfaces, applying Proposition 4.3 and the previous corollary.

Proposition 4.9. *Let* X *be a quasielliptic surface with maximal Kodaira dimension. Let* D *be a big and nef divisor on* X. *Then the following hold.*

- *For* $p = 3$:
  - *If* $D^2 > 4$ *and* $|K_X + D|$ *has a basepoint at* $x \in X$, *there exists a curve* C *such that* $D \cdot C \leq 5$.
  - *If* $D^2 > 9$ *and* $|K_X + D|$ *does not separate any two points* $x, y \in X$, *there exists a curve* C *such that* $D \cdot C \leq 13$.
- *For* $p = 2$:
  - *If* $D^2 > 4$ *and* $|K_X + D|$ *has a basepoint at* $x \in X$, *there exists a curve* C *such that* $D \cdot C \leq 7$.
  - *If* $D^2 > 9$ *and* $|K_X + D|$ *does not separate any two points* $x, y \in X$, *there exists a curve* C *such that* $D \cdot C \leq 17$.

Proof. We start with the case $p = 3$. Assume that $|K_X + D|$ has a basepoint at $x \in X$. Let $\pi : Y \to X$ be the blowup at x. Since x is a basepoint we have that $H^1(Y, \mathcal{O}_Y(K_Y + \pi^* D - 2F)) \neq 0$, where F is the exceptional divisor of π. Let $\widetilde{D} := \pi^* D - 2F$. By assumption we have that $\widetilde{D}^2 > 0$. By Proposition 4.3 we can find an effective divisor $\widetilde{E}$ such that $p\widetilde{D} - 2\widetilde{E}$ is big and $(p\widetilde{D} - \widetilde{E}) \cdot \widetilde{E} \leq 0$. The previous inequality easily implies that $\widetilde{E}$ is not a positive multiple of the exceptional divisor and in particular $E := \pi_* \widetilde{E}$ is a nonzero effective divisor. Moreover, $D = \pi_* \widetilde{D}$ is nef by assumption, thus we can apply Corollary 4.8. Since $\alpha = (D^2 - \widetilde{D}^2) = 4$, the first inequality of the corollary implies that $D \cdot E \leq 5$.

The statement on separation of points follows in exactly the same way. Note that we allow the case $x = y$.

The bounds for the case $p = 2$ can be obtained the same way, remarking that $\widetilde{D}$ is p^2-unstable in this case. □

The previous results can be used to derive effective very-ampleness statements for quasielliptic surfaces when D is an ample divisor.

Theorem 4.10. *Let D be an ample Cartier divisor on a smooth quasielliptic surface X with $\kappa(X) = 1$.*

- *If $p = 3$, the divisor $K_X + kD$ is basepoint-free for any $k \geq 4$ and it is very ample for any $k \geq 8$.*
- *If $p = 2$, the divisor $K_X + kD$ is basepoint-free for any $k \geq 5$ and it is very ample for any $k \geq 19$.*

In particular, if N is any nef divisor, $K_X + kD + N$ is always very ample for any $k \geq 8$ (resp. $k \geq 19$) in characteristic 3 (resp. 2).

Proof. The proof consists of explicitly computing the minimal multiple of D which contradicts the second inequality of Corollary 4.8.

Let us start with basepoint-freeness for $p = 3$. Assume that $k \geq 5$, $K_X + kD$ has a basepoint and define $D' := kD$. Then, by Proposition 4.9, we know that there exists an effective divisor E such that $(D' \cdot E) \leq 5$. This implies

$$(D \cdot E) \leq 1.$$

Now use the second inequality of Corollary 4.8 on D' to deduce

$$15 - 9 \leq 3(D' \cdot E) - 9 \leq \frac{(D' \cdot E)^2}{(D'^2)} \leq 1.$$

This is a contradiction.

Similar computations give the other bounds. □

We now deal with surfaces of general type. The analogue of Proposition 4.9 is the following.

Proposition 4.11. *Let X be a surface of general type and let D be a big and nef divisor on X. Then the following hold.*

- *For $p \geq 3$:*
  - *If $D^2 > \mathrm{vol}(X) + 4$ and $|K_X + D|$ has a basepoint at $x \in X$, there exists a curve C such that $D \cdot C \leq 1$.*

  - *If* $D^2 > \mathrm{vol}(X) + 9$ *and* $|K_X + D|$ *does not separate any two points* $x, y \in X$*, there exists a curve* C *such that* $D \cdot C \leq 2$*.*

- *For* $p = 2$*:*
  - *If* $D^2 > \mathrm{vol}(X) + 6$ *and* $|K_X + D|$ *has a basepoint at* $x \in X$*, there exists a curve* C *such that* $D \cdot C \leq 7$*.*
  - *If* $D^2 > \mathrm{vol}(X) + 11$ *and* $|K_X + D|$ *does not separate any two points* $x, y \in X$*, there exists a curve* C *such that* $D \cdot C \leq 17$*.*

Proof. The proof is basically the same as Proposition 4.9. Let $p \geq 3$ and assume that $|K_X + D|$ has a basepoint at $x \in X$. Using the same notation as Proposition 4.9, we can blow up x and deduce the existence of an effective divisor $\widetilde{E}$ such that $\widetilde{D} - 2\widetilde{E}$ is big and $(\widetilde{D} - \widetilde{E}) \cdot \widetilde{E} \leq 0$ (in order to deduce 1-instability we use Theorem 4.4). Also here, the first inequality of Corollary 4.8 implies that $(D \cdot E) \leq 1$.

The statement on separation of points follows in the same way.

For the bounds in the case $p = 2$ we use the same strategy, using a combination of Theorem 4.5 and Proposition 4.6. □

The following effective very-ampleness statement can be deduced. Applying Proposition 4.11 directly would provide bounds that depend on the volume. It is possible to get a uniform bound if we work with linear systems of the type $|2K_X + mD|$. Note that we get sharp statements for those linear systems.

Theorem 4.12. *Let* D *be an ample Cartier divisor on a smooth surface* X *of general type.*

- *If* $p \geq 3$*, the divisor* $2K_X + kD$ *is basepoint free for any* $k \geq 3$ *and it is very ample for any* $k \geq 4$*.*
- *If* $p = 2$ *the divisor* $2K_X + kD$ *is basepoint free for any* $k \geq 5$ *and it is very ample for any* $k \geq 19$*.*

In particular, if N *is any nef divisor,* $2K_X + kD + N$ *is always very ample for any* $k \geq 4$ *(resp.* $k \geq 19$*) in characteristic* $p \geq 3$ *(resp.* $p = 2$*).*

Proof. Since negative extremal rays of general type surfaces have length 1, if $m \geq 3$, we know that $L := K_X + mD$ is an ample divisor and $L \cdot C \geq 2$ for any irreducible curve $C \subset X$. Moreover, by log-concavity of the volume function (see Theorem 2.2) we have that

$$L^2 = \mathrm{vol}(L) \geq \mathrm{vol}(K_X) + 9D^2 > \mathrm{vol}(X) + 4.$$

Proposition 4.11 implies that $K_X + L = 2K_X + kD$ is basepoint free for any $k \geq 4$. A similar computation allows us to derive very-ampleness.

The same strategy gives the result for $p = 2$. □

Proof of Theorem 1.4. This is simply given by Theorem 4.10 and Theorem 4.12. □

Remark 4.13. In [Terakawa 1999], similar results can be found. Nonetheless our approach allows us to deduce effective basepoint-freeness and very-ampleness also on quasielliptic surfaces and arbitrary surfaces of general type.

5. A Kawamata–Viehweg-type vanishing theorem in positive characteristic

In this section we give an extension of the results in [Terakawa 1999]. There, Terakawa used the results in [Shepherd-Barron 1991a] to deduce a Kawamata–Viehweg-type theorem for nonpathological surfaces. Using our methods we are able to discuss pathological surfaces and obtain an effective Kawamata–Viehweg-type theorem in positive characteristic.

Let us first recall the classical Kawamata–Viehweg vanishing theorem in its general version (see [Kollár and Mori 1998] for the general notation).

Theorem 5.1. *Let (X, B) be a klt pair over an algebraically closed field of characteristic zero and let D be a Cartier divisor on X such that $D-(K_X+B)$ is big and nef. Then*

$$H^i(X, \mathbb{O}_X(D)) = 0$$

for any $i > 0$.

In positive characteristic, even for nonpathological smooth surfaces, there are counterexamples to Theorem 5.1: Xie [2010] provided examples of relatively minimal irregular ruled surfaces in every characteristic where Theorem 5.1 fails.

Nonetheless, assuming $B = 0$, we have the following result (see [Mukai 2013]).

Theorem 5.2. *Let X be a smooth surface in positive characteristic. Assume that there exists a big and nef Cartier divisor D on X such that*

$$H^1(X, \mathbb{O}_X(K_X + D)) \neq 0.$$

Then:

- *X is either quasielliptic of Kodaira dimension one or of general type.*
- *Up to a sequence of blowups, X has the structure of a fibered surface over a smooth curve such that every fiber is connected and singular.*

Furthermore, Terakawa [1999] deduced the following vanishing result using the techniques in [Shepherd-Barron 1991a].

Theorem 5.3. *Let X be a smooth projective surface over an algebraically closed field of characteristic $p > 0$ and let D be a big and nef Cartier divisor on X. Assume that either*:

(1) *$\kappa(X) \neq 2$ and X is not quasielliptic with $\kappa(X) = 1$.*

(2) *X is of general type with*

- $p \geq 3$ *and* $(D^2) > \mathrm{vol}(X)$; *or*
- $p = 2$ *and* $(D^2) > \max\{\mathrm{vol}(X), \mathrm{vol}(X) - 3\chi(\mathcal{O}_X) + 2\}$.

Then

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0$$

for all $i > 0$.

Our aim is to improve this theorem for arbitrary surfaces, via bend-and-break techniques.

More generally, we want to deduce some results on the injectivity of cohomological maps

$$H^1(X, \mathcal{O}_X(-D)) \xrightarrow{F^*} H^1(X, \mathcal{O}_X(-pD))$$

where D is a big divisor on X.

The following result by Kollár is an application of bend-and-break lemmas (cf. Theorem 2.13), specialized to our two-dimensional setting.

Theorem 5.4. *Let X be a smooth projective variety over a field of positive characteristic and let D be a Cartier divisor on X such that*:

(1) $H^1(X, \mathcal{O}_X(-mD)) \xrightarrow{F^*} H^1(X, \mathcal{O}_X(-pmD))$ *is not injective for some integer* $m > 0$.

(2) *There exists a curve C on X such that*

$$(p-1)(D \cdot C) - (K_X \cdot C) > 0.$$

Then through every point x of C there is a rational curve C_x such that

$$[C] \overset{e}{\approx} k_0[C_x] + \sum_{i \neq 0} k_i[C_i] \tag{2}$$

(*as algebraic cycles*), *with $k_i \geq 0$ for all i and*

$$(p-1)(D \cdot C_x) - (K_X \cdot C_x) \leq \dim(X) + 1.$$

Proof. This is essentially a slight modification of [Kollár 1996, Theorem II.6.2]. For the reader's convenience, we sketch it ab initio. Assumption (1) allows us to construct a finite morphism

$$\pi : Y \to X,$$

where Y is defined as a Cartier divisor in the projectivization of a nonsplit rank-two bundle over X (see [Kollár 1996, Construction II.6.1.6], which is a slight modification of Construction 2.9). Furthermore, the following property holds:

$$K_Y = \pi^*(K_X + (k(1-p)D)),$$

where k is the largest integer for which $H^1(X, -kD) \neq 0$.

Now take the curve given in (2) and consider $C' := \operatorname{red} \pi^{-1}(C)$. The hypothesis on the intersection numbers and the formula for the canonical divisor of Y guarantee that $(K_Y \cdot C') < 0$. Let $y \in C'$ be a preimage of x in Y. So we can apply Theorem 2.13 and deduce the existence of a rational curve C'_y passing through y. Using the projection formula, we obtain a curve C_x on X for which

$$(p-1)(D \cdot C_x) - (K_X \cdot C_x) \leq \dim(X) + 1. \qquad \square$$

If we assume the dimension to be two and the divisor D to be big and nef, the asymptotic condition

$$H^1(X, \mathcal{O}_X(-mD)) = 0$$

for m sufficiently large is guaranteed by [Szpiro 1979].

This remark gives us the following corollary.

Corollary 5.5. *Let X be a smooth projective surface over a field of positive characteristic and let D be a big and nef Cartier divisor on X such that $H^1(X, \mathcal{O}_X(-D)) \neq 0$. Assume there exists a curve C on X such that*

$$(p-1)(D \cdot C) - (K_X \cdot C) > 0.$$

Then through every point x of C there is a rational curve C_x such that

$$(p-1)(D \cdot C_x) - (K_X \cdot C_x) \leq 3.$$

We will show later how Corollary 5.5 can be used to deduce an effective version of Kawamata–Viehweg-type vanishing for arbitrary smooth surfaces.

In what follows, we will also need the following lemma on fibered surfaces, which explicitly gives a bound on the genus of the fiber with respect to the volume of the surface.

Lemma 5.6. *Let $f : X \to C$ be a fibered surface of general type and let g be the arithmetic genus of the general fiber F. Then*

$$\operatorname{vol}(X) \geq g - 4.$$

Proof. We divide our analysis into cases according to the genus b of the base, after having assumed the fibration is relatively minimal (i.e., that $K_{X/C}$ is nef).

$b \geq 2$: In this case we can deduce a better estimate. Indeed,

$$\operatorname{vol}(X) \geq (K_X^2) = (K_{X/C}^2) + 8(g-1)(b-1) \geq 8g - 8.$$

$b = 1$: In this case we need a more careful analysis, since in positive characteristic we cannot assume the semipositivity of $f_* K_{X/C}$. Nonetheless the following general formula holds:

$$\deg(f_* K_{X/C}) = \chi(\mathcal{O}_X) - (g-1)(b-1), \tag{3}$$

which specializes to

$$\deg(f_* K_{X/C}) = \chi(\mathcal{O}_X) \geq 0.$$

Formula (3) can be obtained via Riemann–Roch, since we know that $R^1 f_* K_{X/C} = \mathcal{O}_C$ and that $R^1 f_* n K_{X/C} = 0$ for $n \geq 2$ by relative minimality. The last inequality can be assumed by [Shepherd-Barron 1991b, Theorem 8]. Furthermore, one can apply the following formula

$$\deg(f_*(nK_{X/C})) = \deg(f_* K_{X/C}) + \frac{n(n-1)}{2}(K_{X/C}^2).$$

Since $K_{X/C}$ is big, we deduce that

$$\deg(f_*(2K_{X/C})) \geq 1.$$

As a consequence, we can apply the results of [Atiyah 1957] and deduce a decomposition of $f_*(2K_{X/C})$ into indecomposable vector bundles

$$f_*(2K_{X/C}) = \bigoplus_i E_i,$$

where we can assume that $\deg(E_1) \geq 1$. This implies that all quotient bundles of E_1 have positive degree. We want to show now that there exists a degree-one divisor L_1 on C such that $h^0(C, f_*(2K_{X/C}) \otimes \mathcal{O}_C(-L_1)) \neq 0$.

But this is clear, since, for any degree-one divisor L on C, one has that all quotient bundles of $f_*(2K_{X/C}) \otimes \mathcal{O}_C(-L)$ have degree zero and, up to a twisting by a degree-zero divisor on C, one can assume there exists a quotient

$$f_*(2K_{X/C} \otimes \mathcal{O}_C(-L_1)) \to \mathcal{O}_C \to 0.$$

This implies that $h^0(X, \mathcal{O}_X(2K_{X/C} - F))(= h^0(C, f_*(2K_{X/C}) \otimes \mathcal{O}_C(-L_1))) \neq 0$, where F is the general fiber of f and, since $K_X = K_{X/C}$ is nef, that

$$(K_X \cdot (2K_X - F)) \geq 0.$$

This gives the bound

$$\mathrm{vol}(X) \geq (K_X^2) \geq g-1.$$

$b = 0$: Also in this case we can assume that $\chi(\mathcal{O}_X) \geq 0$ and, as a consequence, that

$$\deg(f_* K_{X/\mathbb{P}^1}) = \chi(\mathcal{O}_X) + g - 1 \geq g-1.$$

If $g \geq 6$,

$$\deg(f_* K_{X/\mathbb{P}^1}) \geq 5.$$

This implies that $\deg(f_* K_X \otimes \mathcal{O}_{\mathbb{P}^1}(3)) \geq 0$ and, as a consequence of Grothendieck's theorem on vector bundles on $\mathbb{P}^1$,

$$h^0(X, \mathcal{O}_X(K_X - f^*\mathcal{O}_C(-3))) \neq 0.$$

As before, we have assumed that $K_{X/\mathbb{P}^1} = K_X + f^*\mathcal{O}_C(2)$ is nef, so

$$((K_X + f^*\mathcal{O}_C(2)) \cdot (K_X - f^*\mathcal{O}_C(-3))) \geq 0.$$

So in this case

$$\operatorname{vol}(X) \geq (K_X^2) \geq 2g - 2.$$

If $g \leq 5$, we simply use the trivial inequality $\operatorname{vol}(X) \geq 1$ to deduce

$$\operatorname{vol}(X) \geq g - 4. \qquad \square$$

Our result in this setting is an effective bound, depending only on the birational geometry of X, that guarantees the injectivity of the induced Frobenius map on the H^1s.

Theorem 5.7. *Let X be a smooth surface in characteristic $p > 0$ and let D be a big Cartier divisor D on X. Then, for all integers*

$$m > m_0 = \frac{2\operatorname{vol}(X) + 9}{p - 1},$$

the induced Frobenius map

$$H^1(X, \mathcal{O}_X(-mD)) \xrightarrow{F^*} H^1(X, \mathcal{O}_X(-pmD))$$

is injective. (*If* $\kappa(X) \neq 2$, *the volume* $\operatorname{vol}(X) = 0$).

Remark 5.8. The previous result is trivial if $H^1(X, \mathcal{O}_X(-D)) = 0$. Furthermore, combined with Corollary 5.5 it gives an effective version of the Kawamata–Viehweg theorem (cf. Corollary 5.9) in the case of big and nef divisors. Our hope is to generalize this strategy in order to deduce effective vanishing theorems also in higher dimension.

Proof. Assume, for the sake of contradiction, that

$$H^1(X, \mathcal{O}_X(-\lceil m_0 \rceil D)) \xrightarrow{F^*} H^1(X, \mathcal{O}_X(-p\lceil m_0 \rceil D))$$

has a nontrivial kernel. Then, after a sequence of blowups $f : X' \to X$, we can assume the existence of a (relatively minimal) fibration (possibly with singular general fiber) of arithmetic genus g

$$\pi : X' \to C.$$

We remark that we can reduce to proving our result on X', since $D' := f^*D$ is a big divisor and we have the following commutative diagram

$$\begin{tikzcd} H^1(X', \mathcal{O}_{X'}(-\lceil m_0 \rceil D')) \arrow[r, "F^*"] \arrow[d, "\simeq"'] & H^1(X', \mathcal{O}_{X'}(-p\lceil m_0 \rceil D')\ , \arrow[d, "\simeq"] \\ H^1(X, \mathcal{O}_X(-\lceil m_0 \rceil D)) \arrow[r, "F^*"] & H^1(X, \mathcal{O}_X(-p\lceil m_0 \rceil D)) \end{tikzcd}$$

where the vertical isomorphisms hold because of $R^1 f_* \mathcal{O}_{X'} = 0$. We can now apply Theorem 5.4 to $\lceil m_0 \rceil D'$: we can choose C to be a general fiber F of π, which certainly intersects D' positively, and we can use Lemma 5.6 to obtain

$$(p-1)\lceil m_0 \rceil (D' \cdot F) - (K_{X'} \cdot F) \geq (p-1)\lceil m_0 \rceil - (2g-2) > 3. \tag{4}$$

So we can apply Theorem 5.4: fix a point $x \in F$ and find a rational curve C_x such that

$$(p-1)m_0(D \cdot C_x) - (K_X \cdot C_x) \leq 3.$$

Notice that, by construction, $F = C_x$, because of (2) in Theorem 5.4. But this is a contradiction, because of (4). □

We finally obtain our effective vanishing theorem.

Corollary 5.9. *Let X be a smooth surface in characteristic $p > 0$ and let D be a big and nef Cartier divisor D on X. Then*

$$H^1(X, \mathcal{O}_X(K_X + mD)) = 0$$

for all integers $m > m_0$, where:

- $m_0 = 3/(p-1)$ *if X is quasielliptic with $\kappa(X) = 1$.*
- $m_0 = (2\,\mathrm{vol}(X) + 9)/(p-1)$ *if X is of general type.*

Proof. For surfaces of general type, one simply applies the previous result. For quasielliptic surfaces, a better bound can be obtained, since in this case $(K_X \cdot F) = 0$, where F is the general fiber. □

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