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of algebraic tori**

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# Cohomological invariants of algebraic tori

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Let  $G$  be an algebraic group over a field  $F$ . As defined by Serre, a cohomological invariant of  $G$  of degree  $n$  with values in  $\mathbb{Q}/\mathbb{Z}(j)$  is a functorial-in- $K$  collection of maps of sets  $\text{Tors}_G(K) \rightarrow H^n(K, \mathbb{Q}/\mathbb{Z}(j))$  for all field extensions  $K/F$ , where  $\text{Tors}_G(K)$  is the set of isomorphism classes of  $G$ -torsors over  $\text{Spec } K$ . We study the group of degree 3 invariants of an algebraic torus with values in  $\mathbb{Q}/\mathbb{Z}(2)$ . In particular, we compute the group  $H_{\text{nr}}^3(F(S), \mathbb{Q}/\mathbb{Z}(2))$  of unramified cohomology of an algebraic torus  $S$ .

## 1. Introduction

Let  $G$  be a linear algebraic group over a field  $F$  (of arbitrary characteristic). The notion of an *invariant* of  $G$  was defined in [Garibaldi et al. 2003] as follows. Consider the category  $\text{Fields}_F$  of field extensions of  $F$  and the functor

$$\text{Tors}_G : \text{Fields}_F \rightarrow \text{Sets}$$

taking a field  $K$  to the set  $\text{Tors}_G(K)$  of isomorphism classes of (right)  $G$ -torsors over  $\text{Spec } K$ . Let

$$H : \text{Fields}_F \rightarrow \text{Abelian Groups}$$

be another functor. An  $H$ -invariant of  $G$  is then a morphism of functors

$$i : \text{Tors}_G \rightarrow H,$$

viewing  $H$  with values in  $\text{Sets}$ , that is, a functorial in  $K$  collection of maps of sets  $\text{Tors}_G(K) \rightarrow H(K)$  for all field extensions  $K/F$ . We denote the group of  $H$ -invariants of  $G$  by  $\text{Inv}(G, H)$ .

An invariant  $i \in \text{Inv}(G, H)$  is called *normalized* if  $i(I) = 0$  for the trivial  $G$ -torsor  $I$ . The normalized invariants form a subgroup  $\text{Inv}(G, H)_{\text{norm}}$  of  $\text{Inv}(G, H)$  and there is a natural isomorphism

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$$\text{Inv}(G, H) \simeq H(F) \oplus \text{Inv}(G, H)_{\text{norm}},$$

so it is sufficient to study normalized invariants.

Typically,  $H$  is a cohomological functor given by Galois cohomology groups with values in a fixed Galois module. Of particular interest to us is the functor  $H$  which takes a field  $K/F$  to the Galois cohomology group  $H^n(K, \mathbb{Q}/\mathbb{Z}(j))$ , where the coefficients  $\mathbb{Q}/\mathbb{Z}(j)$  are defined as follows. For a prime integer  $p$  different from the characteristic of  $F$ , the  $p$ -component  $\mathbb{Q}_p/\mathbb{Z}_p(j)$  of  $\mathbb{Q}/\mathbb{Z}(j)$  is the colimit over  $n$  of the étale sheaves  $\mu_{p^n}^{\otimes j}$ , where  $\mu_m$  is the sheaf of  $m$ -th roots of unity. In the case  $p = \text{char}(F) > 0$ ,  $\mathbb{Q}_p/\mathbb{Z}_p(j)$  is defined via logarithmic de Rham–Witt differentials; see [Section 3b](#).

We write  $\text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$  for the group of *cohomological invariants of  $G$  of degree  $n$  with values in  $\mathbb{Q}/\mathbb{Z}(j)$* .

The second cohomology group  $H^2(K, \mathbb{Q}/\mathbb{Z}(1))$  is canonically isomorphic to the Brauer group  $\text{Br}(K)$  of the field  $K$ . In [Section 2c](#) we prove ([Theorem 2.4](#)) that if  $G$  is a connected group (reductive if  $F$  is not perfect), then

$$\text{Inv}(G, \text{Br})_{\text{norm}} \simeq \text{Pic}(G).$$

The group  $\text{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}}$  for a semisimple simply connected group  $G$  has been studied by Rost; see [\[Garibaldi et al. 2003\]](#).

An essential object in the study of cohomological invariants is the notion of a *classifying torsor*: a  $G$ -torsor  $E \rightarrow X$  for a smooth variety  $X$  over  $F$  such that every  $G$ -torsor over an infinite field  $K/F$  is isomorphic to the pull-back of  $E \rightarrow X$  along a  $K$ -point of  $X$ . If  $V$  is a generically free linear representation of  $G$  with a nonempty open subset  $U \subset V$  such that there is a  $G$ -torsor  $\pi : U \rightarrow X$ , then  $\pi$  is classifying. Such representations exist (see [Section 2b](#)).

The generic fiber of  $\pi$  is the *generic torsor* over  $\text{Spec } F(X)$  attached to  $\pi$ . Evaluation at the generic torsor yields a homomorphism

$$\text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H^n(F(X), \mathbb{Q}/\mathbb{Z}(j)), \tag{1-1}$$

and in [Section 3](#) we show that the image of this map is contained in the subgroup  $H_{\text{Zar}}^0(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$  of  $H^n(F(X), \mathbb{Q}/\mathbb{Z}(j))$ , where  $\mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))$  is the Zariski sheaf associated to the presheaf  $W \mapsto H^n(W, \mathbb{Q}/\mathbb{Z}(j))$  of the étale cohomology groups. In fact, the image is contained in the subgroup  $H_{\text{Zar}}^0(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\text{bal}}$  of *balanced* elements, that is, elements that have the same images under the pull-back homomorphisms with respect to the two projections  $(U \times U)/G \rightarrow X$ . Moreover, the balanced elements precisely describe the image and we prove ([Theorem 3.4](#)):

**Theorem A.** *Let  $G$  be a smooth linear algebraic group over a field  $F$ . We assume that  $G$  is connected if  $F$  is a finite field. Let  $E \rightarrow X$  be a classifying  $G$ -torsor with*

*E* a *G*-rational variety such that  $E(F) \neq \emptyset$ . Then (1-1) yields an isomorphism  $\text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \simeq H_{\text{Zar}}^0(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\text{bal}}$ .

At this point it is convenient to make use of a construction due to Totaro [1999]: because the Chow groups are homotopy invariant, the groups  $\text{CH}^n(X)$  do not depend on the choice of the representation  $V$  and the open set  $U \subset V$  provided the codimension of  $V \setminus U$  in  $V$  is large enough. This leads to the notation  $\text{CH}^n(BG)$ , the Chow groups of the so-called *classifying space*  $BG$ , although  $BG$  itself is not defined in this paper.

Unfortunately, the étale cohomology groups with values in  $\mathbb{Q}_p/\mathbb{Z}_p(j)$ , where  $p = \text{char}(F) > 0$ , are not homotopy invariant. In particular, we cannot use the theory of cycle modules of Rost [1996].

The main result of this paper is the exact sequence in Theorem 4.3 describing degree 3 cohomological invariants of an algebraic torus  $T$ . Writing  $\widehat{T}_{\text{sep}}$  for the character lattice of  $T$  over a separable closure of  $F$  and  $T^\circ$  for the dual torus, we prove our main result:

**Theorem B.** *Let  $T$  be an algebraic torus over a field  $F$ . Then there is an exact sequence*

$$0 \rightarrow \text{CH}^2(BT)_{\text{tors}} \rightarrow H^1(F, T^0) \xrightarrow{\alpha} \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} \rightarrow H^0(F, S^2(\widehat{T}_{\text{sep}}))/\text{Dec} \rightarrow H^2(F, T^0).$$

The homomorphism  $\alpha$  is given by  $\alpha(a)(b) = a_K \cup b$  for every  $a \in H^1(F, T^0)$  and  $b \in H^1(K, T)$  and every field extension  $K/F$ , where the cup-product is defined in (4-5), and Dec is the subgroup of decomposable elements in the symmetric square  $S^2(\widehat{T}_{\text{sep}})$  defined in Section A-II.

In the proof of the theorem we compute the group of balanced elements in the motivic cohomology group  $H^4(BT, \mathbb{Z}(2))$  and relate it, using an exact sequence of B. Kahn and Theorem A, with the group of invariants  $\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}}$ .

We also prove that the torsion group  $\text{CH}^2(BT)_{\text{tors}}$  is finite of exponent 2 (Theorem 4.7) and the last homomorphism in the sequence is also of exponent 2 (see the discussion before Theorem 4.13).

Moreover, if  $p$  is an odd prime, the group  $\text{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{norm}}$ , which is the  $p$ -primary component of  $\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}}$ , splits canonically into the direct sum of *linear* invariants (those that induce group homomorphisms from  $\text{Tors}_T$  to  $H^3$ ) and *quadratic* invariants, that is, the invariants  $i$  such that the function  $h(a, b) := i(a + b) - i(a) - i(b)$  is bilinear and  $h(a, a) = 2i(a)$  for all  $a$  and  $b$ . Furthermore, the groups of linear and quadratic invariants with values in  $\mathbb{Q}_p/\mathbb{Z}_p(2)$  are canonically isomorphic to  $H^1(F, T^\circ)\{p\}$  and  $(H^0(F, S^2(\widehat{T}_{\text{sep}}))/\text{Dec})\{p\}$ , respectively.

We also prove (Theorem 4.10) that the degree 3 invariants have control over the structure of all invariants. Precisely, the group  $\text{Inv}^3(T_K, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}}$  is trivial for all  $K/F$  if and only if  $T$  is *special*, that is,  $T$  has no nontrivial torsors over any field  $K/F$ , which in particular means  $T$  has no nonconstant  $H$ -invariants for every functor  $H$ .

Our motivation for considering invariants of tori comes from their connection with unramified cohomology (defined in Section 5). Specifically, this work began as an investigation of a problem posed by Colliot-Thélène [1995, p. 39]: for  $n$  prime to  $\text{char}(F)$  and  $i \geq 0$ , determine the unramified cohomology group  $H_{\text{nr}}^i(F(S), \mu_n^{\otimes(i-1)})$ , where  $F(S)$  is the function field of a torus  $S$  over  $F$ . The connection is provided by Theorem 5.7 where we show that the unramified cohomology of a torus  $S$  is calculated by the invariants of an auxiliary torus:

**Theorem C.** *Let  $S$  be a torus over  $F$  and let  $1 \rightarrow T \rightarrow P \rightarrow S \rightarrow 1$  be a flasque resolution of  $S$ , that is,  $T$  is flasque and  $P$  is quasisplit. Then there is a natural isomorphism*

$$H_{\text{nr}}^n(F(S), \mathbb{Q}/\mathbb{Z}(j)) \simeq \text{Inv}^n(T, \mathbb{Q}/\mathbb{Z}(j)).$$

By Theorem B and Theorem C, we have an exact sequence

$$0 \rightarrow \text{CH}^2(BT)_{\text{tors}} \rightarrow H^1(F, T^0) \xrightarrow{\alpha} \overline{H}_{\text{nr}}^3(F(S), \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^0(F, S^2(\widehat{T}_{\text{sep}}))/\text{Dec} \rightarrow H^2(F, T^0)$$

describing the reduced third cohomology group

$$\overline{H}_{\text{nr}}^3(F(S), \mathbb{Q}/\mathbb{Z}(2)) := H_{\text{nr}}^3(F(S), \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)).$$

Moreover, for an odd prime  $p$ , we have a canonical direct sum decomposition of the  $p$ -primary components:

$$\overline{H}_{\text{nr}}^3(F(S), \mathbb{Q}_p/\mathbb{Z}_p(2)) = H^1(F, T^0)\{p\} \oplus (H^0(F, S^2(\widehat{T}_{\text{sep}}))/\text{Dec})\{p\}.$$

Note that the torus  $S$  determines  $T$  up to multiplication by a quasisplit torus. If  $X$  is a smooth compactification of  $S$ , one can take the torus  $T$  with  $\widehat{T}_{\text{sep}} = \text{Pic}(X_{\text{sep}})$ ; see [Colliot-Thélène and Sansuc 1977, §2].

In the present paper,  $F$  denotes a field of arbitrary characteristic,  $F_{\text{sep}}$  a separable closure of  $F$ , and  $\Gamma$  the absolute Galois group  $\text{Gal}(F_{\text{sep}}/F)$  of  $F$ .

The word “scheme” over a field  $F$  means a separated scheme over  $F$  and, following [Fulton 1984], a “variety” over  $F$  is an integral scheme of finite type over  $F$ . If  $X$  is a scheme over  $F$  and  $L/F$  is a field extension then we write  $X_L$  for  $X \times_F \text{Spec } L$ . When  $L = F_{\text{sep}}$  we write simply  $X_{\text{sep}}$ .

A “linear algebraic group over  $F$ ” is an affine group scheme of finite type over  $F$ , not necessarily smooth.

## 2. Invariants of algebraic groups

**2a. Definitions and basic properties.** Let  $G$  be a linear algebraic group over a field  $F$ . Consider the functor

$$\mathrm{Tors}_G : \mathrm{Fields}_F \rightarrow \mathrm{Sets}$$

from the category of field extensions of  $F$  to the category of sets taking a field  $K$  to the set  $\mathrm{Tors}_G(K)$  of isomorphism classes of (right)  $G$ -torsors over  $\mathrm{Spec} K$ . Note that if  $G$  is a smooth group, then there is a natural bijection

$$\mathrm{Tors}_G(K) \simeq H^1(K, G) := H^1(\mathrm{Gal}(K_{\mathrm{sep}}/K), G(K_{\mathrm{sep}})).$$

Let  $H : \mathrm{Fields}_F \rightarrow \mathrm{Abelian\ Groups}$  be a functor. We also view  $H$  as a functor with values in  $\mathrm{Sets}$ . Following [Garibaldi et al. 2003], we define an  $H$ -invariant of  $G$  as a morphism of functors  $\mathrm{Tors}_G \rightarrow H$  from the category  $\mathrm{Fields}_F$  to  $\mathrm{Sets}$ . All the  $H$ -invariants of  $G$  form the abelian group of invariants  $\mathrm{Inv}(G, H)$ .

An invariant  $i \in \mathrm{Inv}(G, H)$  is called *constant* if there is an element  $h \in H(F)$  such that  $i(I) = h_K$  for every  $G$ -torsor  $I \rightarrow \mathrm{Spec} K$ , where  $h_K$  is the image of  $h$  under natural map  $H(F) \rightarrow H(K)$ . The constant invariants form a subgroup  $\mathrm{Inv}(G, H)_{\mathrm{const}}$  of  $\mathrm{Inv}(G, H)$  isomorphic to  $H(F)$ . An invariant  $i \in \mathrm{Inv}(G, H)$  is called *normalized* if  $i(I) = 0$  for the trivial  $G$ -torsor  $I$ . The normalized invariants form a subgroup  $\mathrm{Inv}(G, H)_{\mathrm{norm}}$  of  $\mathrm{Inv}(G, H)$  and we have the decomposition

$$\mathrm{Inv}(G, H) = \mathrm{Inv}(G, H)_{\mathrm{const}} \oplus \mathrm{Inv}(G, H)_{\mathrm{norm}} \simeq H(F) \oplus \mathrm{Inv}(G, H)_{\mathrm{norm}},$$

so it suffices to determine the normalized invariants.

**2b. Classifying torsors.** Let  $G$  be a linear algebraic group over a field  $F$ . A  $G$ -torsor  $E \rightarrow X$  over a smooth variety  $X$  over  $F$  is called *classifying* if for every field extension  $K/F$ , with  $K$  infinite, and for every  $G$ -torsor  $I \rightarrow \mathrm{Spec} K$ , there is a point  $x : \mathrm{Spec} K \rightarrow X$  such that the torsor  $I$  is isomorphic to the fiber  $E(x)$  of  $E \rightarrow X$  over  $x$ , that is,  $I \simeq E(x) := x^*(E) = \mathrm{Spec}(K) \times_X E$ . The generic fiber  $E_{\mathrm{gen}} \rightarrow \mathrm{Spec} F(X)$  of a classifying torsor is called a *generic  $G$ -torsor*; see [ibid., Part 1, §5.3].

If  $V$  is a generically free linear representation of  $G$  with a nonempty open subset  $U \subset V$  such that there is a  $G$ -torsor  $\pi : U \rightarrow X$ , then  $\pi$  is classifying; see [ibid., Part 1, §5.4]. We will write  $U/G$  for  $X$  and call  $\pi$  a *standard classifying  $G$ -torsor*. Standard classifying  $G$ -torsors exist: we can embed  $G$  into  $U := \mathbf{GL}_{n,F}$  for some  $n$  as a closed subgroup. Then  $U$  is an open subset in the affine space  $M_n(F)$  on which  $G$  acts linearly and the canonical morphism  $U \rightarrow X := U/G$  is a  $G$ -torsor. Note that  $U(F) \neq \emptyset$ .

We say that a  $G$ -variety  $Y$  is  *$G$ -rational* if there is an affine space  $V$  with a linear  $G$ -action such that  $Y$  and  $V$  have  $G$ -isomorphic nonempty open  $G$ -invariant



subvarieties. Note that if  $U \rightarrow U/G$  is a standard classifying  $G$ -torsor, then  $U$  is a  $G$ -rational variety.

Let  $E \rightarrow X$  be a classifying  $G$ -torsor and let  $H : \text{Fields}_F \rightarrow \text{Abelian Groups}$  be a functor. Define the map

$$\theta_G : \text{Inv}(G, H) \rightarrow H(F(X)), \quad i \mapsto i(E_{\text{gen}}), \tag{2-1}$$

by sending an invariant to its value at the generic torsor  $E_{\text{gen}}$ .

Consider the following property of the functor  $H$ :

**Property 2.1.** The map  $H(K) \rightarrow H(K((t)))$  is injective for any field extension  $K/F$ .

The following theorem, due to M. Rost, was proved in [Garibaldi et al. 2003, Part II, Theorem 3.3]. For completeness, we give a slightly modified proof in Section A-I.

**Theorem 2.2.** Let  $G$  be a smooth linear algebraic group over  $F$ . If a functor  $H : \text{Fields}_F \rightarrow \text{Abelian Groups}$  has Property 2.1, then the map  $\theta_G$  is injective, that is, every  $H$ -invariant of  $G$  is determined by its value at the generic  $G$ -torsor.

Let  $G'$  be a (closed) subgroup of  $G$  over  $F$ . The map of sets

$$H^1(K, G') \rightarrow H^1(K, G)$$

for every field extension  $K/F$  yields the restriction map

$$\text{res} : \text{Inv}(G, H) \rightarrow \text{Inv}(G', H).$$

Choose standard torsors  $\pi : U \rightarrow U/G$  and  $\pi' : U \rightarrow U/G'$  (for example, with  $U = \mathbf{GL}_{n,F}$  as above). The pull-back of  $\pi$  with respect to the natural morphism  $\alpha : U/G' \rightarrow U/G$  is the push-forward of  $\pi'$  via the inclusion  $G' \hookrightarrow G$ . It follows that the diagram

$$\begin{array}{ccc} \text{Inv}(G, H) & \xrightarrow{\text{res}} & \text{Inv}(G', H) \\ \theta_G \downarrow & & \theta_{G'} \downarrow \\ H(F(U/G)) & \xrightarrow{\alpha^*} & H(F(U/G')) \end{array}$$

is commutative.

**2c. The Brauer group invariants.** Let  $G$  be a smooth connected linear algebraic group over  $F$ . Every cohomological invariant of  $G$  of degree 1 is constant by [Knus et al. 1998, Proposition 31.15]. In this section we study (degree 2) Br-invariants for the Brauer group functor  $K \mapsto \text{Br}(K)$ . We assume that  $G$  is reductive if  $\text{char}(F) > 0$ .

**Lemma 2.3.** For any field extension  $K/F$  such that  $F$  is algebraically closed in  $K$ , the natural map  $\text{Pic}(G) \rightarrow \text{Pic}(G_K)$  is an isomorphism.

*Proof.* We may assume that  $G$  is reductive by factoring out the unipotent radical in the case that  $F$  is perfect. There is an exact sequence (see [Colliot-Thélène 2004, Theorem 1.2])

$$1 \rightarrow C \rightarrow G' \rightarrow G \rightarrow 1$$

with  $C$  a torus and  $G'$  a reductive group with  $\text{Pic}(G'_L) = 0$  for any field extension  $L/F$ . Let  $T$  be the factor group of  $G'$  by the semisimple part. The result follows from the exact sequence [Sansuc 1981, Proposition 6.10] (note that  $G$  is reductive if  $L$  is not perfect)

$$\widehat{T}(L) \rightarrow \widehat{C}(L) \rightarrow \text{Pic}(G_L) \rightarrow \text{Pic}(G'_L) = 0$$

with  $L = F$  and  $K$  since the groups  $\widehat{T}(F)$  and  $\widehat{C}(F)$  don't change when  $F$  is replaced by  $K$ . □

Since for any  $G_K$ -torsor  $E \rightarrow \text{Spec}(K)$  over a field extension  $K/F$  one has [Sansuc 1981, Proposition 6.10] the exact sequence

$$\text{Pic}(E) \rightarrow \text{Pic}(G_K) \xrightarrow{\delta} \text{Br}(K) \xrightarrow{\varepsilon} \text{Br}(E), \tag{2-2}$$

we obtain the homomorphism

$$\nu : \text{Pic}(G) \rightarrow \text{Inv}(G, \text{Br}),$$

which takes an element  $\alpha \in \text{Pic}(G)$  to the invariant that sends a  $G$ -torsor  $E$  over a field extension  $K/F$  to  $\delta(\alpha_K)$ . If  $E$  is a trivial torsor, that is,  $E(K) \neq \emptyset$ , then  $\varepsilon$  is injective and hence  $\delta = 0$ . It follows that the invariant  $\nu(\alpha)$  is normalized.

**Theorem 2.4.** *Let  $G$  be a smooth connected linear algebraic group over  $F$ . Assume that  $G$  is reductive if  $\text{char}(F) > 0$ . Then the map  $\nu : \text{Pic}(G) \rightarrow \text{Inv}(G, \text{Br})_{\text{norm}}$  is an isomorphism.*

*Proof.* Choose a standard classifying  $G$ -torsor  $U \rightarrow U/G$ . Write  $K$  for the function field  $F(U/G)$  and let  $U_{\text{gen}}$  be the generic  $G$ -torsor over  $K$ . Consider the commutative diagram

$$\begin{array}{ccccccc} \text{Pic}(G) & \xrightarrow{\nu} & \text{Inv}(G, \text{Br})_{\text{norm}} & & & & \\ & & \downarrow j & & \downarrow \theta_G & & \\ \text{Pic}(U_{\text{gen}}) & \longrightarrow & \text{Pic}(G_K) & \xrightarrow{\delta} & \text{Br}(K) & \xrightarrow{i} & \text{Br}(K(U_{\text{gen}})), \end{array}$$

where the bottom sequence is (2-2) for the  $G$ -torsor  $U_{\text{gen}} \rightarrow \text{Spec}(K)$  followed by the injection  $\text{Br}(U_{\text{gen}}) \rightarrow \text{Br}(K(U_{\text{gen}}))$  (see [Milne 1980, Chapter IV, Corollary 2.6]), and the map  $\theta_G$  is evaluation at the generic torsor  $U_{\text{gen}}$  given in (2-1) and is injective by Theorem 2.2. Since the generic torsor is split over  $K(U_{\text{gen}})$ ,  $\text{Im}(\theta_G) \subset \text{Ker}(i) = \text{Im}(\delta)$ . By Lemma 2.3,  $j$  is an isomorphism, hence  $\nu$  is surjective.



Note that  $U_{\text{gen}}$  is a localization of  $U$ , hence  $\text{Pic}(U_{\text{gen}}) = 0$  as  $\text{Pic}(U) = 0$ . It follows that  $\nu$  is injective.  $\square$

An algebraic group  $G$  over a field  $F$  is called *special* if  $H^1(K, G) = \{1\}$  for every field extension  $K/F$ , that is, all  $G$ -torsors over any field extension of  $F$  are trivial.

**Corollary 2.5.** *If the group  $G$  is special, then  $\text{Pic}(G) = 0$ .*

### 3. Invariants with values in $\mathbb{Q}/\mathbb{Z}(j)$

In this section we find a description for the group of cohomological invariants with values in  $\mathbb{Q}/\mathbb{Z}(j)$  by identifying the image of the embedding  $\theta_G$  in (2-1).

Let  $G$  be a linear algebraic group over a field  $F$ , let  $H \subset G$  be a subgroup and let  $E \rightarrow X$  be a  $G$ -torsor. Suppose that  $G/H$  is affine. Consider a  $G$ -action on  $E \times (G/H)$  by  $(e, g'H)g = (eg, g^{-1}g'H)$ . By [Milne 1980, Theorem I.2.23], the affine  $G$ -equivariant projection  $E \times (G/H) \rightarrow E$  descends to an affine morphism  $Y \rightarrow X$ . The (trivial right)  $H$ -torsor  $E \times G \rightarrow E \times (G/H)$  descends to an  $H$ -torsor  $E \rightarrow Y$ . We will write  $E/H$  for  $Y$ .

**Example 3.1.** Let  $G$  be a linear algebraic group over a field  $F$  and let  $E \rightarrow X$  be a  $G$ -torsor. Then for every  $n > 0$ ,  $E^n := E \times_F \cdots \times_F E$  ( $n$  times) is a  $G^n$ -torsor over  $X^n$ . Viewing  $G$  as the diagonal subgroup of  $G^n$ , we have the  $G$ -torsor  $E^n \rightarrow E^n/G$ .

**3a. Balanced elements.** Let  $G$  be a linear algebraic group over a field  $F$ . We assume that  $G$  is connected if  $F$  is finite. Let  $E \rightarrow X$  be a  $G$ -torsor such that  $E(F) \neq \emptyset$ . We write  $p_1$  and  $p_2$  for the two projections  $E^2/G = (E \times_F E)/G \rightarrow X$  (see Example 3.1).

**Lemma 3.2.** *Let  $K/F$  be a field extension and  $x_1, x_2 \in X(K)$ . Then the  $G$ -torsors  $E(x_1)$  and  $E(x_2)$  over  $K$  are isomorphic if and only if there is a point  $y \in (E^2/G)(K)$  such that  $p_1(y) = x_1$  and  $p_2(y) = x_2$ .*

*Proof.* “ $\Rightarrow$ ”: By construction, we have  $G$ -equivariant morphisms  $f_i : E(x_i) \rightarrow E$  for  $i = 1, 2$ . Choose an isomorphism  $h : E(x_1) \xrightarrow{\sim} E(x_2)$  of  $G$ -torsors over  $K$ . The morphism  $(f_1, f_2h) : E(x_1) \rightarrow E^2$  yields the required point  $\text{Spec } K = E(x_1)/G \rightarrow E^2/G$ .

“ $\Leftarrow$ ”: The pull-back of  $E \rightarrow X$  with respect to any projection  $E^2/G \rightarrow X$  coincides with the  $G$ -torsor  $E^2 \rightarrow E^2/G$ , hence

$$E(x_1) = x_1^*(E) = y^* p_1^*(E) \simeq y^*(E^2) \simeq y^* p_2^*(E) = x_2^*(E) = E(x_2). \quad \square$$

Let  $H$  be a (contravariant) functor from the category of schemes over  $F$  to the category of abelian groups. We have the two maps  $p_i^* : H(X) \rightarrow H(E^2/G)$ ,  $i = 1, 2$ . An element  $h \in H(X)$  is called *balanced* if  $p_1^*(h) = p_2^*(h)$ . We write  $H(X)_{\text{bal}}$  for the subgroup of balanced elements in  $H(X)$ . In other words,  $H(X)_{\text{bal}} = h_0(H(E^\bullet/G))$  in the notation of Section A-IV.

We can view  $H$  as a (covariant) functor  $\text{Fields}_F \rightarrow \text{Sets}$  taking a field  $K$  to  $H(K) := H(\text{Spec } K)$ .

**Lemma 3.3.** *Let  $h \in H(X)_{\text{bal}}$  be a balanced element,  $K/F$  a field extension and  $I$  a  $G$ -torsor over  $\text{Spec}(K)$ . Let  $x \in X(K)$  be a point such that  $E(x) \simeq I$ . Then the element  $x^*(h)$  in  $H(K)$  does not depend on the choice of  $x$ .*

*Proof.* Let  $x_1, x_2 \in X(K)$  be two points such that  $E(x_1) \simeq E(x_2)$ . By Lemma 3.2, there is a point  $y \in (E^2/G)(K)$  such that  $p_1(y) = x_1$  and  $p_2(y) = x_2$ . Therefore

$$x_1^*(h) = y^*(p_1^*(h)) = y^*(p_2^*(h)) = x_2^*(h). \quad \square$$

It follows from Lemma 3.3 that if the torsor  $E \rightarrow X$  is classifying with  $E(F) \neq \emptyset$ , then every element  $h \in H(X)_{\text{bal}}$  determines an  $H$ -invariant  $i_h$  of  $G$  as follows. Let  $I$  be a  $G$ -torsor over a field extension  $K/F$ . We claim that there is a point  $x \in X(K)$  such that  $E(x) \simeq I$ . If  $K$  is infinite, this follows from the definition of the classifying  $G$ -torsor. If  $K$  is finite then all  $G$ -torsors over  $K$  are trivial by [Lang 1956], as  $G$  is connected. Since  $E(K) \neq \emptyset$ , we can take for  $x$  the image in  $X(K)$  of any point in  $E(K)$ . Defining  $i_h(E) = x^*(h) \in H(K)$ , we have a group homomorphism

$$H(X)_{\text{bal}} \rightarrow \text{Inv}(G, H), \quad h \mapsto i_h.$$

**3b. Cohomology with values in  $\mathbb{Q}/\mathbb{Z}(j)$ .** For every integer  $j \geq 0$ , the coefficients  $\mathbb{Q}/\mathbb{Z}(j)$  are defined as the direct sum over all prime integers  $p$  of the objects  $\mathbb{Q}_p/\mathbb{Z}_p(j)$  in the derived category of sheaves of abelian groups on the big étale site of  $\text{Spec } F$ , where

$$\mathbb{Q}_p/\mathbb{Z}_p(j) = \text{colim}_n \mu_{p^n}^{\otimes j}$$

if  $p \neq \text{char } F$ , with  $\mu_{p^n}$  the sheaf of  $(p^n)$ -th roots of unity, and

$$\mathbb{Q}_p/\mathbb{Z}_p(j) = \text{colim}_n W_n \Omega_{\log}^j[-j]$$

if  $p = \text{char } F > 0$ , with  $W_n \Omega_{\log}^j$  the sheaf of logarithmic de Rham–Witt differentials; see [Illusie 1979, I.5.7; Kahn 1996].

We write  $H^m(X, \mathbb{Q}/\mathbb{Z}(j))$  for the étale cohomology of a scheme  $X$  with values in  $\mathbb{Q}/\mathbb{Z}(j)$ . Then

$$H^m(X, \mathbb{Q}/\mathbb{Z}(j))\{p\} = \text{colim}_n H^m(X, \mu_{p^n}^{\otimes j})$$

if  $p \neq \text{char } F$  and

$$H^m(X, \mathbb{Q}/\mathbb{Z}(j))\{p\} = \text{colim}_n H^{m-j}(X, W_n \Omega_{\log}^j)$$

if  $p = \text{char } F > 0$ . In the latter case, the group  $W_n \Omega_{\log}^j(F)$  is canonically isomorphic to  $K_j^M(F)/p^n K_j^M(F)$ , where  $K_j^M(F)$  is Milnor’s  $K$ -group of  $F$  (see

[Bloch and Kato 1986, Corollary 2.8]), hence by [Izhboldin 1991; Garibaldi et al. 2003, Part II, Appendix A],  $H^s(F, W_n\Omega_{\log}^j)$  is isomorphic to

$$H^s(F, K_j^M(F_{\text{sep}})/p^n K_j^M(F_{\text{sep}})) = \begin{cases} K_j^M(F)/p^n K_j^M(F) & \text{if } s = 0, \\ H^2(F, K_j^M(F_{\text{sep}}))_{p^n} & \text{if } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that in the case  $p = \text{char } F > 0$ , we have

$$H^m(F, \mathbb{Q}/\mathbb{Z}(j))\{p\} = \begin{cases} K_j^M(F) \otimes (\mathbb{Q}_p/\mathbb{Z}_p) & \text{if } m = j, \\ H^2(F, K_j^M(F_{\text{sep}}))\{p\} & \text{if } m = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

The *motivic complexes*  $\mathbb{Z}(j)$ , for  $j = 0, 1, 2$ , of étale sheaves on a smooth scheme  $X$  were defined by S. Lichtenbaum [1987; 1990]. We write  $H^*(X, \mathbb{Z}(j))$  for the étale (hyper)cohomology groups of  $X$  with values in  $\mathbb{Z}(j)$ .

The complex  $\mathbb{Z}(0)$  is equal to the constant sheaf  $\mathbb{Z}$  and  $\mathbb{Z}(1) = \mathbb{G}_{m,X}[-1]$ , thus  $H^n(X, \mathbb{Z}(1)) = H^{n-1}(X, \mathbb{G}_{m,X})$ . In particular,  $H^3(X, \mathbb{Z}(1)) = \text{Br}(X)$ , the cohomological Brauer group of  $X$ . The complex  $\mathbb{Z}(2)$  is concentrated in degrees 1 and 2 and there is a product map  $\mathbb{Z}(1) \otimes^L \mathbb{Z}(1) \rightarrow \mathbb{Z}(2)$ ; see [Lichtenbaum 1987, Proposition 2.5].

The exact triangle in the derived category of étale sheaves

$$\mathbb{Z}(j) \rightarrow \mathbb{Q} \otimes \mathbb{Z}(j) \rightarrow \mathbb{Q}/\mathbb{Z}(j) \rightarrow \mathbb{Z}(j)[1]$$

yields the connecting homomorphism

$$H^i(X, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H^{i+1}(X, \mathbb{Z}(j)),$$

which is an isomorphism if  $X = \text{Spec}(F)$  for a field  $F$  and  $i > j$  [Kahn 1993, Lemme 1.1].

Write  $\mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))$  for the Zariski sheaf on a smooth scheme  $X$  associated to the presheaf  $U \mapsto H^n(U, \mathbb{Q}/\mathbb{Z}(j))$  of étale cohomology groups.

Let  $G$  be a linear algebraic group over  $F$ . We assume that  $G$  is connected if  $F$  is a finite field and write  $\text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$  for the group of *degree  $n$  invariants* of  $G$  for the functor  $K \mapsto H^n(K, \mathbb{Q}/\mathbb{Z}(j))$ . Note that Property 2.1 holds for this functor by [Garibaldi et al. 2003, Part 2, Proposition A.9].

Choose a classifying  $G$ -torsor  $E \rightarrow X$  with  $E$  a  $G$ -rational variety such that  $E(F) \neq \emptyset$ . Applying the construction given in Section 3a to the functor  $U \mapsto H_{\text{Zar}}^0(U, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$ , we get a homomorphism

$$\varphi : H_{\text{Zar}}^0(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\text{bal}} \rightarrow \text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)).$$

**Theorem 3.4.** *Let  $G$  be a smooth linear algebraic group over a field  $F$ . We assume that  $G$  is connected if  $F$  is a finite field. Let  $E \rightarrow X$  be a classifying  $G$ -torsor*

with  $E$  a  $G$ -rational variety such that  $E(F) \neq \emptyset$ . Then the homomorphism  $\varphi$  is an isomorphism.

*Proof.* Let  $E_{\text{gen}} \rightarrow F(X)$  be the generic fiber of the classifying  $G$ -torsor  $E \rightarrow X$ . Note that since the pull-back of  $E \rightarrow X$  with respect to any of the two projections  $E^2/G \rightarrow X$  coincides with the  $G$ -torsor  $E^2 \rightarrow E^2/G$ , the pull-backs of the generic  $G$ -torsor  $E_{\text{gen}} \rightarrow \text{Spec } F(X)$  with respect to the two morphisms  $\text{Spec } F(E^2/G) \rightarrow \text{Spec } F(X)$  induced by the projections are isomorphic. It follows that for every invariant  $i \in \text{Inv}(G, H^*(\mathbb{Q}/\mathbb{Z}(j)))$  we have

$$p_1^*(i(E_{\text{gen}})) = i(p_1^*(E_{\text{gen}})) = i(p_2^*(E_{\text{gen}})) = p_2^*(i(E_{\text{gen}}))$$

in  $H^*(F(E^2/G), \mathbb{Q}/\mathbb{Z}(j))$ , that is,  $i(E_{\text{gen}}) \in H^*(F(X), \mathbb{Q}/\mathbb{Z}(j))_{\text{bal}}$ . By [Proposition A.9](#),  $\partial_x(h) = 0$  for every point  $x \in X$  of codimension 1, hence

$$\theta_G(i) = i(E_{\text{gen}}) \in H_{\text{Zar}}^0(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\text{bal}}$$

by [Proposition A.10](#). By [Theorem 2.2](#),  $\theta_G$  is injective and by construction, the composition  $\theta_G \circ \varphi$  is the identity. It follows that  $\varphi$  is an isomorphism.  $\square$

Write  $\overline{H}_{\text{Zar}}^0(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$  for the factor group of  $H_{\text{Zar}}^0(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$  by the natural image of  $H^n(F, \mathbb{Q}/\mathbb{Z}(j))$ .

**Corollary 3.5.** *The isomorphism  $\varphi$  yields an isomorphism*

$$\overline{H}_{\text{Zar}}^0(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\text{bal}} \xrightarrow{\sim} \text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))_{\text{norm}}.$$

### 4. Degree 3 invariants of algebraic tori

In this section we prove the main theorem that describes degree 3 invariants of an algebraic torus with values in  $\mathbb{Q}/\mathbb{Z}(2)$ .

**4a. Algebraic tori.** Let  $F$  be a field and  $\Gamma = \text{Gal}(F_{\text{sep}}/F)$  the absolute Galois group of  $F$ . An *algebraic torus* of dimension  $n$  over  $F$  is an algebraic group  $T$  such that  $T_{\text{sep}}$  is isomorphic to the product of  $n$  copies of the multiplicative group  $\mathbb{G}_m$ ; see [\[Colliot-Thélène and Sansuc 1977; Voskresenskiĭ 1998\]](#). For an algebraic torus  $T$  over a field  $F$ , we write  $\widehat{T}_{\text{sep}}$  for the  $\Gamma$ -module of characters  $\text{Hom}(T_{\text{sep}}, \mathbb{G}_m)$ . The group  $\widehat{T}_{\text{sep}}$  is a  $\Gamma$ -lattice, that is, a free abelian group of finite rank with a continuous  $\Gamma$ -action. The contravariant functor  $T \mapsto \widehat{T}_{\text{sep}}$  is an antiequivalence between the category of algebraic tori and the category of  $\Gamma$ -lattices: the torus  $T$  and the group  $T(F)$  can be reconstructed from the lattice  $\widehat{T}_{\text{sep}}$  by the formulas

$$T = \text{Spec}(F_{\text{sep}}[\widehat{T}_{\text{sep}}]^\Gamma),$$

$$T(F) = \text{Hom}_\Gamma(\widehat{T}_{\text{sep}}, F_{\text{sep}}^\times) = (\widehat{T}_{\text{sep}}^\circ \otimes F_{\text{sep}}^\times)^\Gamma,$$

where  $\widehat{T}_{\text{sep}}^\circ = \text{Hom}(\widehat{T}_{\text{sep}}, \mathbb{Z})$ .

We write  $\widehat{T}$  for the character group  $\text{Hom}_F(T, \mathbb{G}_m) = (\widehat{T}_{\text{sep}})^\Gamma$  and  $T^\circ$  for the *dual torus* having character lattice  $\widehat{T}_{\text{sep}}^\circ$ .

A torus  $T$  is called *quasisplit* if  $T$  is isomorphic to the group of invertible elements of an étale  $F$ -algebra, or equivalently, the  $\Gamma$ -lattice  $\widehat{T}_{\text{sep}}$  is permutation, that is,  $\widehat{T}_{\text{sep}}$  has a  $\Gamma$ -invariant  $\mathbb{Z}$ -basis. An *invertible* torus is a direct factor of a quasisplit torus.

A torus  $T$  is called *flasque* or *coflasque* if  $H^1(L, \widehat{T}_{\text{sep}}^\circ) = 0$  or  $H^1(L, \widehat{T}_{\text{sep}}) = 0$ , respectively, for every finite field extension  $L/F$ . A *flasque resolution* of a torus  $S$  is an exact sequence of tori  $1 \rightarrow T \rightarrow P \rightarrow S \rightarrow 1$  with  $T$  flasque and  $P$  quasisplit. By [Colliot-Thélène and Sansuc 1977, §4; Voskresenskii 1998, §4.7], the torus  $T$  in the flasque resolution is invertible if and only if  $S$  is a direct factor of a rational torus.

**4b. Products.** Let  $T$  be a torus over  $F$  and let  $\widehat{T}(i)$  denote the complex  $\widehat{T}_{\text{sep}} \otimes \mathbb{Z}(i)$  of étale sheaves over  $F$  for  $i = 0, 1, 2$ . Thus,  $\widehat{T}(0) = \widehat{T}_{\text{sep}}$  and

$$\widehat{T}(1) = (\widehat{T}_{\text{sep}} \otimes F_{\text{sep}}^\times)[-1] = T^\circ(F_{\text{sep}})[-1].$$

Let  $S$  and  $T$  be algebraic tori over  $F$  and let  $i$  and  $j$  be nonnegative integers with  $i + j \leq 2$ . For any smooth variety  $X$  over  $F$ , we have the product map

$$(\widehat{S}_{\text{sep}} \otimes \widehat{T}_{\text{sep}})^\Gamma \otimes H^p(X, \widehat{S}^\circ(i)) \otimes H^q(X, \widehat{T}^\circ(j)) \rightarrow H^{p+q}(X, \mathbb{Z}(i+j)) \quad (4-1)$$

taking  $a \otimes b \otimes c$  to  $a \cup b \cup c$ , via the canonical pairings between  $\widehat{S}_{\text{sep}}$  and  $\widehat{S}_{\text{sep}}^\circ$ ,  $\widehat{T}_{\text{sep}}$  and  $\widehat{T}_{\text{sep}}^\circ$ , and the product map  $\mathbb{Z}(i) \otimes^L \mathbb{Z}(j) \rightarrow \mathbb{Z}(i+j)$ .

Recall that there is an isomorphism  $H^n(F, \mathbb{Z}(k)) \simeq H^{n-1}(F, \mathbb{Q}/\mathbb{Z}(k))$  for  $n > k$ . In particular, we have the cup-product map

$$(\widehat{S}_{\text{sep}} \otimes \widehat{T}_{\text{sep}})^\Gamma \otimes H^p(F, S) \otimes H^q(F, T) \rightarrow H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \quad (4-2)$$

if  $p + q = 2$ .

If  $S = T^\circ$  is the dual torus, then  $(\widehat{S}_{\text{sep}} \otimes \widehat{T}_{\text{sep}})^\Gamma = \text{End}_\Gamma(\widehat{T}_{\text{sep}})$  contains the canonical element  $1_T$ . We then have the product map

$$H^p(X, \widehat{T}(i)) \otimes H^q(X, \widehat{T}^\circ(j)) \rightarrow H^{p+q}(X, \mathbb{Z}(i+j)) \quad (4-3)$$

and in particular, the product maps

$$H^1(F, \widehat{T}_{\text{sep}}) \otimes H^1(F, T) \rightarrow H^2(F, \mathbb{Q}/\mathbb{Z}(1)) = \text{Br}(F), \quad (4-4)$$

$$H^1(F, T^\circ) \otimes H^1(F, T) \rightarrow H^3(F, \mathbb{Q}/\mathbb{Z}(2)), \quad (4-5)$$

$$H^2(F, T^\circ) \otimes H^0(F, T) \rightarrow H^3(F, \mathbb{Q}/\mathbb{Z}(2)), \quad (4-6)$$

taking  $a \otimes b$  to  $1_T \cup a \cup b$  and applying (4-2).

As  $T$  is a commutative group, the set  $H^1(K, T)$  is an abelian group. An invariant  $i \in \text{Inv}(T, H)$  for a functor  $H$  is called *linear* if  $i_K : H^1(K, T) \rightarrow H(K)$  is a group

homomorphism for every  $K/F$ . In the next section we will see that a normalized degree 3 invariant of a torus need not be linear.

**4c. Main theorem.** Let  $T$  be a torus over  $F$  and choose a standard classifying  $T$ -torsor  $U \rightarrow U/T$  such that the codimension of  $V \setminus U$  in  $V$  is at least 3. Such a torsor exists by [Edidin and Graham 1998, Lemma 9].

By [Sansuc 1981, Proposition 6.10], there is an exact sequence

$$F_{\text{sep}}[U]^\times / F_{\text{sep}}^\times \rightarrow \widehat{T}_{\text{sep}} \rightarrow \text{Pic}((U/T)_{\text{sep}}) \rightarrow \text{Pic}(U_{\text{sep}}).$$

The codimension assumption implies that the side terms are trivial, hence the map  $\widehat{T}_{\text{sep}} \rightarrow \text{Pic}((U/T)_{\text{sep}})$  is an isomorphism. It follows that the classifying  $T$ -torsor  $U \rightarrow U/T$  is universal in the sense of [Colliot-Thélène and Sansuc 1987a].

Write  $K_*(F)$  for the (Quillen)  $K$ -groups of  $F$  and  $\mathcal{K}_*$  for the Zariski sheaf associated to the presheaf  $U \mapsto K_*(U)$ . Then the groups  $H_{\text{Zar}}^n(U/T, \mathcal{K}_2)$  are independent of the choice of the classifying torsor; see [Edidin and Graham 1998]. So we write  $H_{\text{Zar}}^n(BT, \mathcal{K}_2)$  for this group (see Section A-IV). As  $T_{\text{sep}}$  is a split torus, by the Künneth formula (see Example A.5),

$$H_{\text{Zar}}^n(BT_{\text{sep}}, \mathcal{K}_2) = \begin{cases} K_2(F_{\text{sep}}) & \text{if } n = 0, \\ \text{Pic}((U/T)_{\text{sep}}) \otimes F_{\text{sep}}^\times = \widehat{T}_{\text{sep}} \otimes F_{\text{sep}}^\times = T^\circ(F_{\text{sep}}) & \text{if } n = 1, \\ \text{CH}^2((U/T)_{\text{sep}}) = S^2(\widehat{T}_{\text{sep}}) & \text{if } n = 2. \end{cases}$$

Applying the calculation of the  $\mathcal{K}$ -cohomology groups to the standard classifying  $T$ -torsor  $U^i \rightarrow U^i/T$  for every  $i > 0$  instead of  $U \rightarrow U/T$ , by Proposition B.3, we have the exact sequence

$$0 \rightarrow H^1(F, T^\circ) \xrightarrow{\alpha} \overline{H}^4(U^i/T, \mathbb{Z}(2)) \rightarrow \overline{H}^4((U^i/T)_{\text{sep}}, \mathbb{Z}(2))^\Gamma \rightarrow H^2(F, T^\circ), \quad (4-7)$$

where  $\overline{H}^4(U^i/T, \mathbb{Z}(2))$  is the factor group of  $H^4(U^i/T, \mathbb{Z}(2))$  by  $H^4(F, \mathbb{Z}(2))$ , the map  $\alpha$  is given by  $\alpha(a) = q^*(a) \cup [U^i]$  with  $q : U^i/T \rightarrow \text{Spec } F$  the structure morphism,  $[U^i]$  the class of the  $T$ -torsor  $U^i \rightarrow U^i/T$  in  $H^1(U^i/T, T)$ , and the cup-product is taken for the pairing (B-6).

Taking the sequences (4-7) for all  $i$  (see Section A-IV), we get the exact sequence of cosimplicial groups

$$0 \rightarrow H^1(F, T^\circ) \xrightarrow{\alpha} \overline{H}^4(U^\bullet/T, \mathbb{Z}(2)) \rightarrow \overline{H}^4((U^\bullet/T)_{\text{sep}}, \mathbb{Z}(2))^\Gamma \rightarrow H^2(F, T^\circ).$$

The first and the last cosimplicial groups in the sequence are constant, hence by Lemma A.2, the sequence

$$0 \rightarrow H^1(F, T^\circ) \xrightarrow{\alpha} \overline{H}^4(U/T, \mathbb{Z}(2))_{\text{bal}} \rightarrow \overline{H}^4((U/T)_{\text{sep}}, \mathbb{Z}(2))_{\text{bal}}^\Gamma \rightarrow H^2(F, T^\circ) \quad (4-8)$$

is exact as  $h_0(\overline{H}^4(U^\bullet/T, \mathbb{Z}(2))) = H^4(U/T, \mathbb{Z}(2))_{\text{bal}}$ .

The following theorem was proved by B. Kahn [1996, Theorem 1.1]:

**Theorem 4.1.** *Let  $X$  be a smooth variety over  $F$ . Then there is an exact sequence*

$$0 \rightarrow \text{CH}^2(X) \rightarrow H^4(X, \mathbb{Z}(2)) \rightarrow H_{\text{Zar}}^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \rightarrow 0.$$

By Theorem 4.1, there is an exact sequence of cosimplicial groups

$$0 \rightarrow \text{CH}^2(U^\bullet/T) \rightarrow \overline{H}^4(U^\bullet/T, \mathbb{Z}(2)) \rightarrow \overline{H}_{\text{Zar}}^0(U^\bullet/T, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \rightarrow 0.$$

As the functor  $\text{CH}^2$  is homotopy invariant, by Lemma A.4, the first group in the sequence is constant. In view of Lemma A.2, and following the notation for the  $\mathcal{H}$ -cohomology, the sequence

$$0 \rightarrow \text{CH}^2(BT) \rightarrow \overline{H}^4(U/T, \mathbb{Z}(2))_{\text{bal}} \rightarrow \overline{H}_{\text{Zar}}^0(U/T, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))_{\text{bal}} \rightarrow 0 \quad (4-9)$$

is exact. By Corollary 3.5, the last group in the sequence is canonically isomorphic to  $\text{Inv}(T, H^3(\mathbb{Q}/\mathbb{Z}(2)))_{\text{norm}}$ .

As the torus  $T_{\text{sep}}$  is split, all the invariants of  $T_{\text{sep}}$  are trivial hence the sequence (4-9) over  $F_{\text{sep}}$  yields an isomorphism

$$\overline{H}^4((U/T)_{\text{sep}}, \mathbb{Z}(2))_{\text{bal}} \simeq \text{CH}^2(BT_{\text{sep}}) \simeq S^2(\widehat{T}_{\text{sep}}). \quad (4-10)$$

Combining (4-8), (4-9) and (4-10), we get the following diagram with an exact row and column:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & H^1(F, T^0) & & & \\
 & & & \alpha \downarrow & \searrow & & \\
 0 & \longrightarrow & \text{CH}^2(BT) & \longrightarrow & \overline{H}^4(U/T, \mathbb{Z}(2))_{\text{bal}} & \longrightarrow & \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} \longrightarrow 0. \\
 & & \searrow & & \downarrow & & \\
 & & & & S^2(\widehat{T}_{\text{sep}})^\Gamma & & \\
 & & & & \downarrow & & \\
 & & & & H^2(F, T^0) & & 
 \end{array}$$

Write  $\text{Dec} = \text{Dec}(\widehat{T}_{\text{sep}})$  for the subgroup of *decomposable elements* in  $S^2(\widehat{T}_{\text{sep}})^\Gamma$  (see Section A-II).

**Lemma 4.2.** *The image of the homomorphism*

$$\text{CH}^2(BT) \rightarrow \text{CH}^2(BT_{\text{sep}})^\Gamma = S^2(\widehat{T}_{\text{sep}})^\Gamma$$

*in the diagram coincides with Dec.*



*Proof.* Consider the Grothendieck ring  $K_0(BT)$  of the category of  $T$ -equivariant vector bundles over  $\text{Spec}(F)$ , or equivalently, of the category of finite dimensional linear representations of  $T$ . If  $T$  is split, every linear representation of  $T$  is a direct sum of one-dimensional representations. Therefore, there is an isomorphism between the group ring  $\mathbb{Z}[\widehat{T}]$  of all formal finite sums  $\sum_{x \in \widehat{T}} a_x e^x$  and  $K_0(BT)$ , taking  $e^x$  with  $x \in \widehat{T}$  to the class of the 1-dimensional representation given by  $x$ . In general, for every torus  $T$ , we have  $K_0(BT_{\text{sep}}) = \mathbb{Z}[\widehat{T}_{\text{sep}}]$  and  $K_0(BT) = \mathbb{Z}[\widehat{T}_{\text{sep}}]^\Gamma = K_0(BT_{\text{sep}})^\Gamma$ ; see [Merkurjev and Panin 1997, page 136]. The group  $\mathbb{Z}[\widehat{T}_{\text{sep}}]^\Gamma$  is generated by the sums  $\sum_{i=1}^n e^{\gamma_i x}$ , where  $\gamma_1, \gamma_2, \dots, \gamma_n$  are representatives of the left cosets of an arbitrary open subgroup  $\Gamma'$  in  $\Gamma$  and  $x \in (\widehat{T}_{\text{sep}})^{\Gamma'}$ .

The equivariant Chern classes were defined in [Edidin and Graham 1998, §2.4]. The first Chern class  $c_1 : K_0(BT_{\text{sep}}) \rightarrow \text{CH}^1(BT_{\text{sep}}) = \widehat{T}_{\text{sep}}$  takes  $e^x$  to  $x$ . In the diagram

$$\begin{array}{ccccc}
 \mathbb{Z}[\widehat{T}_{\text{sep}}]^\Gamma & \xlongequal{\quad} & K_0(BT) & \xrightarrow{c_2} & \text{CH}^2(BT) \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbb{Z}[\widehat{T}_{\text{sep}}] & \xlongequal{\quad} & K_0(BT_{\text{sep}}) & \xrightarrow{c_2} & \text{CH}^2(BT_{\text{sep}}) \xlongequal{\quad} S^2(\widehat{T}_{\text{sep}})
 \end{array}$$

the second Chern class maps  $c_2$  are surjective by [Esnault et al. 1998, Lemma C.3]. It follows from the formula  $c_2(a+b) = c_2(a) + c_1(a)c_1(b) + c_2(b)$  that the composition

$$\mathbb{Z}[\widehat{T}_{\text{sep}}]^\Gamma = K_0(BT) \rightarrow K_0(BT_{\text{sep}}) \xrightarrow{c_2} \text{CH}^2(BT_{\text{sep}}) = S^2(\widehat{T}_{\text{sep}}) \rightarrow S^2(\widehat{T}_{\text{sep}})/(\widehat{T})^2$$

is a homomorphism and its image is generated by the elements (see Section A-II)

$$c_2\left(\sum_{i=1}^n e^{\gamma_i x}\right) = \sum_{i < j} (\gamma_i x)(\gamma_j x) = \text{Qtr}(x). \quad \square$$

By the restriction-corestriction argument, the kernel of the homomorphism

$$\text{CH}^2(BT) \rightarrow \text{CH}^2(BT_{\text{sep}})^\Gamma = S^2(\widehat{T}_{\text{sep}})^\Gamma$$

coincides with the torsion subgroup  $\text{CH}^2(BT)_{\text{tors}}$  in  $\text{CH}^2(BT)$ .

The following theorem describes degree 3 invariants of an algebraic torus with values in  $\mathbb{Q}/\mathbb{Z}(2)$ :

**Theorem 4.3.** *Let  $T$  be an algebraic torus a field  $F$ . Then there is an exact sequence*

$$\begin{aligned}
 0 \rightarrow \text{CH}^2(BT)_{\text{tors}} \rightarrow H^1(F, T^0) \xrightarrow{\alpha} \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} \\
 \rightarrow S^2(\widehat{T}_{\text{sep}})^\Gamma / \text{Dec} \rightarrow H^2(F, T^0).
 \end{aligned}$$

The homomorphism  $\alpha$  is given by  $\alpha(a)(b) = a_K \cup b$  for every  $a \in H^1(F, T^0)$  and  $b \in H^1(K, T)$  and every field extension  $K/F$ , where the cup-product is defined in (4-5).

*Proof.* The exactness of the sequence follows from the diagram before [Lemma 4.2](#). It remains to describe the map  $\alpha$ . Take an  $a \in H^1(F, T^0)$  and consider the invariant  $i$  defined by  $i(b) = a_K \cup b$ , where the cup-product is given by [\(4-5\)](#). We need to prove that  $i = \alpha(a)$ . Choose a standard classifying  $T$ -torsor  $U \rightarrow U/T$  and set  $K = F(U/T)$ . Let  $U_{\text{gen}}$  be the generic fiber of the classifying torsor. By [Theorem 2.2](#), it suffices to show that  $i(U_{\text{gen}}) = \alpha(a)(U_{\text{gen}})$  over  $K$ . This follows from the description of the map  $\alpha$  in the exact sequence [\(4-7\)](#).  $\square$

**Remark 4.4.** In a similar (and much simpler) fashion one can describe degree 2 invariants of an algebraic torus  $T$  with values in  $\mathbb{Q}/\mathbb{Z}(1)$ , that is, invariants with values in the Brauer group by computing the étale motivic cohomology group  $H^3(U/T, \mathbb{Z}(1)) = H^2(U/T, \mathbb{G}_m) = \text{Br}(U/T)$  for a standard classifying  $T$ -torsor  $U \rightarrow U/T$ . One establishes canonical isomorphisms

$$H^1(F, \widehat{T}_{\text{sep}}) \simeq \overline{H}^3(U/T, \mathbb{Z}(1))_{\text{bal}} \simeq \text{Inv}^2(T, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} = \text{Inv}(T, \text{Br})_{\text{norm}}.$$

The composition takes an element  $a \in H^1(F, \widehat{T}_{\text{sep}})$  to the invariant  $b \mapsto a_K \cup b$  for  $b \in H^1(K, T)$  and a field extension  $K/F$ . This description shows that every normalized Br-invariant of  $T$  is linear.

**4d. Torsion in  $\text{CH}^2(BT)$ .** We investigate the group  $\text{CH}^2(BT)_{\text{tors}}$ , the first term of the exact sequence in [Theorem 4.3](#).

Let  $S$  be an algebraic torus over  $F$ . Using the Gersten resolution, [[Quillen 1973](#), Proposition 5.8] we identify the group  $H^0(S_{\text{sep}}, \mathcal{K}_2)$  with a subgroup in  $K_2(F_{\text{sep}}(S))$ . Set  $\overline{H}^0(S_{\text{sep}}, \mathcal{K}_2) := H^0(S_{\text{sep}}, \mathcal{K}_2)/K_2(F_{\text{sep}})$ . By [[Garibaldi et al. 2003](#), Part 2, §5.7], we have an exact sequence

$$0 \rightarrow \widehat{S}_{\text{sep}} \otimes F_{\text{sep}}^\times \rightarrow \overline{H}^0(S_{\text{sep}}, \mathcal{K}_2) \xrightarrow{\lambda} \bigwedge^2 \widehat{S}_{\text{sep}} \rightarrow 0 \tag{4-11}$$

of  $\Gamma$ -modules, where  $\lambda(\{e^x, e^y\}) = x \wedge y$  for  $x, y \in \widehat{S}_{\text{sep}}$ .

Consider the  $\Gamma$ -homomorphism

$$\begin{aligned} \gamma : \bigwedge^2 \widehat{S}_{\text{sep}} &\rightarrow \overline{H}^0(S_{\text{sep}}, \mathcal{K}_2) \\ x \wedge y &\mapsto \{e^x, e^y\} - \{e^y, e^x\}. \end{aligned}$$

We have  $\lambda \circ \gamma = 2 \cdot \text{Id}$ , hence the connecting homomorphism

$$\partial : H^i(F, \bigwedge^2 \widehat{S}_{\text{sep}}) \rightarrow H^{i+1}(F, \widehat{S}_{\text{sep}} \otimes F_{\text{sep}}^\times) \tag{4-12}$$

satisfies  $2\partial = 0$ .

**Lemma 4.5.** *If  $S$  is an invertible torus, then the sequence of  $\Gamma$ -modules [\(4-11\)](#) is split.*

*Proof.* Suppose first that  $S$  is quasisplit. Let  $\{x_1, x_2, \dots, x_m\}$  be a permutation basis for  $\widehat{S}_{\text{sep}}$ . Then the elements  $x_i \wedge x_j$  for  $i < j$  form a  $\mathbb{Z}$ -basis for  $\wedge^2 \widehat{S}_{\text{sep}}$ . The map  $\wedge^2 \widehat{S}_{\text{sep}} \rightarrow \overline{H}^0(S_{\text{sep}}, \mathcal{H}_2)$ , taking  $x_i \wedge x_j$  to  $\{e^{x_i}, e^{x_j}\}$  is a splitting for  $\gamma$ .

In general, find a torus  $S'$  such that  $S \times S'$  is quasisplit. The desired splitting is the composition

$$\wedge^2 \widehat{S}_{\text{sep}} \rightarrow \wedge^2 \widehat{S_{\text{sep}} \times S'} \xrightarrow{\alpha} \overline{H}^0(S_{\text{sep}} \times S'_{\text{sep}}, \mathcal{H}_2) \xrightarrow{\beta} \overline{H}^0(S_{\text{sep}}, \mathcal{H}_2),$$

where  $\alpha$  is a splitting for the torus  $S \times S'$  and  $\beta$  is the pull-back map for the canonical inclusion  $S \hookrightarrow S \times S'$ . □

Let

$$1 \rightarrow T \rightarrow P \rightarrow Q \rightarrow 1$$

be a coflasque resolution of  $T$ , that is,  $P$  is a quasisplit torus and  $Q$  is a coflasque torus; see [Colliot-Thélène and Sansuc 1977]. The torus  $P$  is an open set in the affine space of a separable  $F$ -algebra on which  $T$  acts linearly. Hence  $P \rightarrow Q$  is a standard classifying  $T$ -torsor. By Theorem 2.2, the natural map

$$\theta_T : \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(F(Q), \mathbb{Q}/\mathbb{Z}(2))$$

is injective.

Consider the spectral sequence (B-10) for the variety  $X = Q$ . By [Garibaldi et al. 2003, Part 2, Corollary 5.6], we have  $H^1(Q_{\text{sep}}, \mathcal{H}_2) = 0$ . In view of Proposition B.4, we have an injective homomorphism

$$\beta : H^2(F, \overline{H}^0(Q_{\text{sep}}, \mathcal{H}_2)) \rightarrow \overline{H}^4(Q, \mathbb{Z}(2)) \tag{4-13}$$

such that the composition of  $\beta$  with the homomorphism

$$H^2(F, Q^\circ) \rightarrow H^2(F, \overline{H}^0(Q_{\text{sep}}, \mathcal{H}_2))$$

is given by the cup-product with the class of the identity in  $H^0(Q, Q)$ .

**Lemma 4.6.** *For a coflasque torus  $Q$ , the group  $\text{CH}^2(Q)$  is trivial.*

*Proof.* By [Merkurjev and Panin 1997, Theorem 9.1], for every torus  $Q$ , the Grothendieck group  $K_0(Q)$  is generated by the classes of the sheaves  $i_*(P)$ , where  $P$  is an invertible sheaf on  $Q_L$ ,  $L/F$  a finite separable field extension and  $i : Q_L \rightarrow Q$  is the natural morphism. By definition of a coflasque torus,

$$\text{Pic}(Q_L) = H^1(L, \widehat{Q}_{\text{sep}}) = 0.$$

It follows that every invertible sheaf on  $Q_L$  is trivial, hence  $K_0(Q) = \mathbb{Z} \cdot 1$ . Since the group  $\text{CH}^2(Q)$  is generated by the second Chern classes of vector bundles on  $Q$  [Esnault et al. 1998, Lemma C.3], we have  $\text{CH}^2(Q) = 0$ . □

It follows from [Proposition A.10](#), [Theorem 4.1](#), and [Lemma 4.6](#) that the homomorphism

$$\kappa : \bar{H}^4(Q, \mathbb{Z}(2)) \rightarrow \bar{H}^4(F(Q), \mathbb{Z}(2)) = \bar{H}^3(F(Q), \mathbb{Q}/\mathbb{Z}(2)) \tag{4-14}$$

is injective.

Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Q^\circ(F_{\text{sep}}) & \longrightarrow & \bar{H}^0(Q_{\text{sep}}, \mathcal{K}_2) & \longrightarrow & \wedge^2 \widehat{Q}_{\text{sep}} \longrightarrow 0 \\ & & \parallel & & \downarrow s & & \downarrow t \\ 0 & \longrightarrow & Q^\circ(F_{\text{sep}}) & \longrightarrow & P^\circ(F_{\text{sep}}) & \longrightarrow & T^\circ(F_{\text{sep}}) \longrightarrow 0 \end{array}$$

where  $s$  is the composition of the natural map  $\bar{H}^0(Q_{\text{sep}}, \mathcal{K}_2) \rightarrow \bar{H}^0(P_{\text{sep}}, \mathcal{K}_2)$  and a splitting of  $P^\circ(F_{\text{sep}}) \rightarrow \bar{H}^0(P_{\text{sep}}, \mathcal{K}_2)$  (see [Lemma 4.5](#)).

We have the following diagram

$$\begin{array}{ccccc} & & H^1(F, T^\circ) & \xrightarrow{\alpha} & \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} \\ & \nearrow t^* & \downarrow \partial_1 & & \downarrow \theta_T \\ H^1(F, \bar{H}^0(Q_{\text{sep}}, \mathcal{K}_2)) & \xrightarrow{\varphi} & H^1(F, \wedge^2 \widehat{Q}_{\text{sep}}) & \xrightarrow{\partial} & H^2(F, Q^\circ) \xrightarrow{\sigma} \bar{H}^3(F(Q), \mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

with the bottom sequence a complex, where  $\sigma$  is the composition of the maps in [\(4-13\)](#) and [\(4-14\)](#):

$$\begin{aligned} H^2(F, Q^\circ) &\xrightarrow{\psi} H^2(F, \bar{H}^0(Q_{\text{sep}}, \mathcal{K}_2)) \xrightarrow{\beta} \bar{H}^4(Q, \mathbb{Z}(2)) \\ &\xrightarrow{\kappa} \bar{H}^4(F(Q), \mathbb{Z}(2)) = \bar{H}^3(F(Q), \mathbb{Q}/\mathbb{Z}(2)), \end{aligned}$$

with  $\varphi$  and  $\psi$  given by Galois cohomology applied to the exact sequence [\(4-11\)](#) for the torus  $Q$ . Note that the connecting map  $\partial_1$  is injective as  $H^1(F, P^\circ) = 0$  since  $P^\circ$  is a quasplit torus. As  $2\partial = 0$  in [\(4-12\)](#), we have  $2t^* = 0$ .

The commutativity of the triangle follows from the definition of  $t^*$ . We claim that the square in the diagram is anticommutative. Note that  $\partial_2(\xi) = [P_{\text{gen}}]$ , where  $\partial_2 : H^0(F, Q) \rightarrow H^1(F, T)$  is the connecting homomorphism,  $P_{\text{gen}}$  is the generic fiber of the morphism  $P \rightarrow Q$ , and  $\xi \in H^0(K, Q)$  is the generic point of  $Q$  with  $K = F(Q)$ . It follows from the description of the maps  $\alpha$  and  $\beta$  in [\(4-7\)](#) and [\(4-13\)](#), respectively, and [Lemma A.1](#) that

$$\sigma(\partial_1(a)) = \partial_1(a)_K \cup \xi = (-a_K) \cup \partial_2(\xi) = (-a_K) \cup [P_{\text{gen}}] = -\theta_T(\alpha(a))$$

for every  $a \in H^1(F, T^\circ)$ .

The maps  $\beta$  and  $\kappa$  are injective, hence the bottom sequence in the diagram is

exact. Thus, we have an exact sequence

$$H^1(F, \bar{H}^0(Q_{\text{sep}}, \mathcal{K}_2)) \rightarrow H^1(F, \wedge^2 \widehat{Q}_{\text{sep}}) \rightarrow \text{Ker}(\alpha) \rightarrow 0$$

and  $2 \cdot \text{Ker}(\alpha) = 2 \cdot \text{Im}(t^*) = 0$ . Furthermore,  $\text{Ker}(\alpha) \simeq \text{CH}^2(BT)_{\text{tors}}$  by [Theorem 4.3](#) and the group  $H^1(F, \wedge^2 \widehat{Q}_{\text{sep}})$  is finite as  $\wedge^2 \widehat{Q}_{\text{sep}}$  is a lattice.

We have proved:

**Theorem 4.7.** *Let  $1 \rightarrow T \rightarrow P \rightarrow Q \rightarrow 1$  be a coflasque resolution of a torus  $T$ . Then there is an exact sequence*

$$H^1(F, \bar{H}^0(Q_{\text{sep}}, \mathcal{K}_2)) \rightarrow H^1(F, \wedge^2 \widehat{Q}_{\text{sep}}) \rightarrow \text{CH}^2(BT)_{\text{tors}} \rightarrow 0.$$

Moreover,  $\text{CH}^2(BT)_{\text{tors}}$  is a finite group satisfying  $2 \cdot \text{CH}^2(BT)_{\text{tors}} = 0$ .

**Corollary 4.8.** *If  $T^\circ$  is a birational direct factor of a rational torus, or if  $T$  is split over a cyclic field extension, then  $\text{CH}^2(BT)_{\text{tors}} = 0$ , that is, the map  $\alpha$  in [Theorem 4.3](#) is injective.*

*Proof.* The exact sequence  $1 \rightarrow Q^\circ \rightarrow P^\circ \rightarrow T^\circ \rightarrow 1$  is a flasque resolution of  $T^\circ$ . If  $T^\circ$  is a birational direct factor of a rational torus, or if  $T$  is split over a cyclic field extension, the torus  $Q^\circ$ , and hence  $Q$ , is invertible; see [Section 4a](#) and [[Voskresenskii 1998](#), §4, [Theorem 3](#)]. By [Lemma 4.5](#), the sequence (4-11) for the torus  $Q$  is split, hence the first map in [Theorem 4.7](#) is surjective. □

**Question 4.9.** Is  $\text{CH}^2(BT)_{\text{tors}}$  trivial for every torus  $T$ ?

**4e. Special tori.** Let  $T$  be an algebraic torus over a field  $F$ . The *tautological invariant* of the torus  $T^\circ \times T$  is the normalized invariant taking a pair

$$(a, b) \in H^1(K, T^\circ) \times H^1(K, T)$$

to the cup-product  $a \cup b \in H^3(K, \mathbb{Q}/\mathbb{Z}(2))$  defined in (4-5).

The following theorem shows that if a torus  $T$  has only trivial degree 3 normalized invariants with values in  $\mathbb{Q}/\mathbb{Z}(2)$  universally, that is, over all field extensions of  $F$ , then  $T$  has no nonconstant invariants at all by the simple reason: every  $T$ -torsor over a field is trivial. Note that it follows from [Theorem 2.4](#) that  $T$  has no degree 2 normalized invariants with values in  $\mathbb{Q}/\mathbb{Z}(1)$  universally if and only if  $T$  is coflasque.

**Theorem 4.10.** *Let  $T$  be an algebraic torus over a field  $F$ . Then the following are equivalent:*

- (1)  $\text{Inv}^3(T_K, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} = 0$  for every field extension  $K$  of  $F$ .
- (2) The tautological invariant of the torus  $T^\circ \times T$  is trivial.
- (3) The torus  $T$  is invertible.
- (4) The torus  $T$  is special.

*Proof.* (1)  $\Rightarrow$  (2): Let  $K/F$  be a field extension and  $a \in H^1(K, T^\circ)$ . By assumption, the degree 3 normalized invariant  $i = \alpha(a)$  with values in  $\mathbb{Q}/\mathbb{Z}(2)$ , defined by  $i(b) = a \cup b$  for every  $b \in H^1(K, T)$ , is trivial. In other words, the tautological invariant of the torus  $T^\circ \times T$  is trivial.

(2)  $\Rightarrow$  (3): The image of the tautological invariant in the group

$$S^2(\widehat{T}_{\text{sep}}^\circ \oplus \widehat{T}_{\text{sep}})^\Gamma / \text{Dec}$$

is represented by the identity  $1_{\widehat{T}}$  in the direct factor  $(\widehat{T}_{\text{sep}}^\circ \otimes \widehat{T}_{\text{sep}})^\Gamma = \text{End}_\Gamma(\widehat{T}_{\text{sep}})$  of  $S^2(\widehat{T}_{\text{sep}}^\circ \oplus \widehat{T}_{\text{sep}})^\Gamma$  (see Section A-II). The projection of Dec on the direct summand  $(\widehat{T}_{\text{sep}}^\circ \otimes \widehat{T}_{\text{sep}})^\Gamma$  is generated by the traces  $\text{Tr}(a \otimes b)$  for all open subgroups  $\Gamma' \subset \Gamma$  and all  $a \in (\widehat{T}_{\text{sep}}^\circ)^{\Gamma'}$  and  $b \in (\widehat{T}_{\text{sep}})^{\Gamma'}$ . Hence  $1_{\widehat{T}} = \sum_i \text{Tr}(a_i \otimes b_i)$  for some open subgroups  $\Gamma_i \subset \Gamma$ ,  $a_i \in (\widehat{T}^\circ)^{\Gamma_i}$  and  $b_i \in (\widehat{T})^{\Gamma_i}$ . If  $P_i = \mathbb{Z}[\Gamma/\Gamma_i]$ , then  $a_i$  can be viewed as a  $\Gamma$ -homomorphism  $\widehat{T} \rightarrow P_i$  and  $b_i$  can be viewed as a  $\Gamma$ -homomorphism  $P_i \rightarrow \widehat{T}$  such that the composition

$$\widehat{T} \xrightarrow{(b_i)} P \xrightarrow{(a_i)} \widehat{T},$$

where  $P = \coprod P_i$ , is the identity. It follows that  $\widehat{T}$  is a direct summand of a permutation  $\Gamma$ -module  $P$  and hence  $T$  is invertible.

(3)  $\Rightarrow$  (4): Obvious as every invertible torus is special.

(4)  $\Rightarrow$  (1): Obvious. □

**Remark 4.11.** The equivalence (3)  $\Leftrightarrow$  (4) was essentially proved in [Colliot-Thélène and Sansuc 1987b, Proposition 7.4].

**4f. Linear and quadratic invariants.** Let  $T$  be a torus over  $F$ . By Theorem 4.3, we have a natural homomorphism to the group of linear invariants:

$$\alpha : H^1(F, T^\circ) \rightarrow \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{lin}}.$$

Let  $S$  and  $T$  be algebraic tori over  $F$ . For every field extension  $K/F$ , the cup-product (4-2) yields a homomorphism

$$\varepsilon : (\widehat{T}_{\text{sep}}^{\otimes 2})^\Gamma \rightarrow \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))$$

defined by  $\varepsilon(a)(b) = a_K \cup b \cup b$  for  $a \in (\widehat{T}_{\text{sep}}^{\otimes 2})^\Gamma$  and  $b \in H^1(K, T)$ .

We say that an invariant  $i \in \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))$  is *quadratic* if the function

$$h(a, b) := i(a + b) - i(a) - i(b)$$

is bilinear and  $h(a, a) = 2i(a)$  for all  $a$  and  $b$ . For example, the tautological invariant of the torus  $T^\circ \times T$  in Section 4e is quadratic. We write  $\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{quad}}$  for the subgroup of all quadratic invariants in  $\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))$ . The image of  $\varepsilon$  is contained in  $\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{quad}}$ .

**Lemma 4.12.** *The composition of  $\varepsilon$  with the map*

$$\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow S^2(\widehat{T}_{\text{sep}})^\Gamma / \text{Dec}$$

*in Theorem 4.3 is induced by the natural homomorphism  $\widehat{T}_{\text{sep}}^{\otimes 2} \rightarrow S^2(\widehat{T}_{\text{sep}})$ .*

*Proof.* Let  $U \rightarrow U/T =: X$  be a standard classifying  $T$ -torsor as in Section 4c. Consider the commutative diagram

$$\begin{array}{ccccc} (\widehat{T}_{\text{sep}}^{\otimes 2})^\Gamma \otimes H^1(X, T)^{\otimes 2} & \xrightarrow{\text{prod}} & \overline{H}^4(X, \mathbb{Z}(2))_{\text{bal}} & \longrightarrow & \text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} \\ \downarrow & & \downarrow & & \\ \widehat{T}_{\text{sep}}^{\otimes 2} \otimes H^1(X_{\text{sep}}, T)^{\otimes 2} & \xrightarrow{\text{prod}} & \overline{H}^4(X_{\text{sep}}, \mathbb{Z}(2))_{\text{bal}} & & \\ \downarrow \wr & & \downarrow \wr & & \\ \widehat{T}_{\text{sep}}^{\otimes 2} \otimes (\widehat{T}_{\text{sep}}^\circ)^{\otimes 2} \otimes \widehat{T}_{\text{sep}}^{\otimes 2} & \xrightarrow{\kappa} & S^2(\widehat{T}_{\text{sep}})^\Gamma / \text{Dec} & & \end{array}$$

where the product maps are given by (4-1),  $\eta$  identifies

$$H^1(X_{\text{sep}}, T) = \widehat{T}_{\text{sep}}^\circ \otimes \text{Pic}(X_{\text{sep}})$$

with  $\widehat{T}_{\text{sep}}^\circ \otimes \widehat{T}_{\text{sep}}$  and  $\kappa$  is given by the pairing between the first and second factors. Write  $[U]$  for the class of the classifying torsor in  $H^1(X, T)$ . The image of  $[U]$  in

$$H^1(X_{\text{sep}}, T_{\text{sep}}) = \widehat{T}_{\text{sep}}^\circ \otimes \widehat{T}_{\text{sep}} = \text{End}(\widehat{T}_{\text{sep}})$$

is the identity  $1_{\widehat{T}_{\text{sep}}}$ . Hence for every  $a \in (\widehat{T}_{\text{sep}}^{\otimes 2})^\Gamma$ , the image of  $a \otimes [U] \otimes [U]$  under the diagonal map in the diagram coincides with the canonical image of  $a$  in  $S^2(\widehat{T}_{\text{sep}})^\Gamma / \text{Dec}$ .  $\square$

The composition of the map  $S^2(\widehat{T}_{\text{sep}})^\Gamma \rightarrow (\widehat{T}_{\text{sep}}^{\otimes 2})^\Gamma$  given by  $a \cdot b \mapsto a \otimes b + b \otimes a$  with the natural map  $(\widehat{T}_{\text{sep}}^{\otimes 2})^\Gamma \rightarrow S^2(\widehat{T}_{\text{sep}})^\Gamma$  is multiplication by 2. Then by Lemma 4.12,  $2 \cdot S^2(\widehat{T}_{\text{sep}})^\Gamma / \text{Dec}$  is contained in the image of the map

$$\text{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow S^2(\widehat{T}_{\text{sep}})^\Gamma / \text{Dec}.$$

Theorem 4.3 then yields:

**Theorem 4.13.** *Let  $T$  be an algebraic torus over  $F$ . Then 2 times the homomorphism  $S^2(\widehat{T}_{\text{sep}})^\Gamma / \text{Dec} \rightarrow H^2(F, T^0)$  from Theorem 4.3 is trivial. If  $p$  is an odd prime,*

$$\text{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{norm}} = \text{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{lin}} \oplus \text{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{quad}}$$

*and there are natural isomorphisms  $\text{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{lin}} \simeq H^1(F, T^\circ)\{p\}$  and*

$$\text{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{quad}} \simeq (S^2(\widehat{T}_{\text{sep}})^\Gamma / \text{Dec})\{p\}.$$



**Example 4.14.** Let  $X = \{x_1, x_2, \dots, x_n\}$  be a set of  $n$  elements with the natural action of the symmetric group  $S_n$ . A continuous surjective group homomorphism  $\Gamma \rightarrow S_n$  yields a separable field extension  $L/F$  of degree  $n$ . Consider the torus  $T = R_{L/F}(\mathbb{G}_{m,L})/\mathbb{G}_m$ , where  $R_{L/F}$  is the Weil restriction; see [Voskresenskiĭ 1998, Chapter 1, §3.12]. Note that the generic maximal torus of the group  $\mathbf{PGL}_n$  is of this form (see Section 5b). The character lattice  $\widehat{T}_{\text{sep}}$  is the kernel of the augmentation homomorphism  $\mathbb{Z}[X] \rightarrow \mathbb{Z}$ .

The dual torus  $T^\circ$  is the norm one torus  $R_{L/F}^{(1)}(\mathbb{G}_{m,L})$ . For every field extension  $K/F$ , we have:

$$H^1(K, T) = \text{Br}(KL/K), \quad H^1(K, T^\circ) = K^\times/N(KL)^\times,$$

where  $KL := K \otimes L$ ,  $N$  is the norm map for the extension  $KL/K$  and

$$\text{Br}(KL/K) = \text{Ker}(\text{Br}(K) \rightarrow \text{Br}(KL)).$$

The pairing

$$K^\times/N(KL)^\times \otimes \text{Br}(KL/K) \rightarrow H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

defines linear degree 3 invariants of both  $T$  and  $T^\circ$ .

We claim that  $S^2(\widehat{T}_{\text{sep}})^\Gamma/\text{Dec} = 0$  and  $S^2(\widehat{T}_{\text{sep}}^\circ)^\Gamma/\text{Dec} = 0$ , that is,  $T$  and  $T^\circ$  have no nontrivial quadratic degree 3 invariants. We have  $\widehat{T}_{\text{sep}}^\circ = \mathbb{Z}[X]/\mathbb{Z}N_X$ , where  $N_X = \sum x_i$ . The group  $S^2(\widehat{T}_{\text{sep}}^\circ)^\Gamma$  is generated by  $S := \sum_{i < j} x_i \cdot x_j$ . As  $S \in \text{Dec}$ , we have  $S^2(\widehat{T}_{\text{sep}}^\circ)^\Gamma/\text{Dec} = 0$ .

Let  $D = \sum x_i^2$  and  $E := \text{Qtr}(x_1 - x_2) = 2S - (n - 1)D$ , where the quadratic map  $\text{Qtr}$  is defined in Section A-II. The group  $S^2(\widehat{T}_{\text{sep}})^\Gamma$  is generated by  $E$  if  $n$  is even and by  $E/2$  if  $n$  is odd. A computation shows that  $nE/2 = \text{Qtr}(nx_1 - N_X)$ . It follows that the generator of  $S^2(\widehat{T}_{\text{sep}})^\Gamma$  belongs to  $\text{Dec}$ , hence  $S^2(\widehat{T}_{\text{sep}})^\Gamma/\text{Dec}$  is trivial.

Note that as the torus  $T$  is rational, it follows from Theorem 4.3 and Corollary 4.8 that  $\text{Inv}^3(T^\circ, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} \simeq \text{Br}(L/F)$ .

### 5. Unramified invariants

Let  $K/F$  be a field extension and  $v$  a discrete valuation of  $K$  over  $F$  with valuation ring  $O_v$ . We say that an element  $a \in H^n(K, \mathbb{Q}/\mathbb{Z}(j))$  is *unramified with respect to*  $v$  if  $a$  belongs to the image of the map  $H^n(O_v, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H^n(K, \mathbb{Q}/\mathbb{Z}(j))$ ; see [Colliot-Thélène and Ojanguren 1989]. We write  $H_{\text{nr}}^n(K, \mathbb{Q}/\mathbb{Z}(j))$  for the subgroup of the elements in  $H^n(K, \mathbb{Q}/\mathbb{Z}(j))$  that are unramified with respect to all discrete valuations of  $K$  over  $F$ . We have a natural homomorphism

$$H^n(F, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_{\text{nr}}^n(K, \mathbb{Q}/\mathbb{Z}(j)). \tag{5-1}$$

A dominant morphism of varieties  $Y \rightarrow X$  yields a homomorphism

$$H_{\text{nr}}^n(F(X), \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_{\text{nr}}^n(F(Y), \mathbb{Q}/\mathbb{Z}(j)). \tag{5-2}$$

**Proposition 5.1.** *Let  $K/F$  be a purely transcendental field extension. Then the homomorphism (5-1) is an isomorphism.*

*Proof.* The statement is well known for the  $p$ -components if  $p \neq \text{char } F$ ; see, for example, [Colliot-Thélène and Ojanguren 1989, Proposition 1.2]. It suffices to consider the case  $K = F(t)$  and prove the surjectivity of (5-1). The coniveau spectral sequence for the projective line  $\mathbb{P}^1$  (see (A-1) in the Appendix) yields an exact sequence

$$H^n(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H^n(K, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow \coprod_{x \in \mathbb{P}^1} H_x^{n+1}(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j))$$

and, therefore, a surjective homomorphism  $H^n(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_{\text{nr}}^n(K, \mathbb{Q}/\mathbb{Z}(j))$ . By the projective bundle theorem (classical if  $p \neq \text{char}(F)$  and [Gros 1985, Theorem 2.1.11] if  $p = \text{char}(F) > 0$ ), we have

$$H^n(\mathbb{P}^1, \mathbb{Q}/\mathbb{Z}(j)) = H^n(F, \mathbb{Q}/\mathbb{Z}(j)) \oplus H^{n-2}(F, \mathbb{Q}/\mathbb{Z}(j-1))t,$$

where  $t$  is a generator of  $H^2(\mathbb{P}^1, \mathbb{Z}(1)) = \text{Pic}(\mathbb{P}^1) = \mathbb{Z}$ . As  $t$  vanishes over the generic point of  $\mathbb{P}^1$ , the result follows. □

Let  $G$  be a linear algebraic group over  $F$ . Choose a standard classifying  $G$ -torsor  $U \rightarrow U/G$ . An invariant  $i \in \text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$  is called *unramified* if the image of  $i$  under  $\theta_G : \text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j))$  is unramified. This is independent of the choice of standard classifying torsor. Indeed, if  $U' \rightarrow U'/G$  is another standard classifying  $G$ -torsor, then  $(U \times V')/G \rightarrow U/G$  and  $(V \times U')/G \rightarrow U'/G$  are vector bundles. Hence the field  $F((U \times U')/G)$  is a purely transcendental extension of  $F(U/G)$  and  $F(U'/G)$  and by Proposition 5.1,

$$\begin{aligned} H_{\text{nr}}^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) &\simeq H_{\text{nr}}^n(F((U \times U')/G), \mathbb{Q}/\mathbb{Z}(j)) \\ &\simeq H_{\text{nr}}^n(F(U'/G), \mathbb{Q}/\mathbb{Z}(j)). \end{aligned}$$

We write  $H_{\text{nr}}^n(F(BG), \mathbb{Q}/\mathbb{Z}(j))$  for this common value and  $\text{Inv}_{\text{nr}}^n(G, \mathbb{Q}/\mathbb{Z}(j))$  for the subgroup of unramified invariants. Similarly, we write  $\text{Br}_{\text{nr}}(F(BG))$  for the *unramified Brauer group*  $H_{\text{nr}}^2(F(BG), \mathbb{Q}/\mathbb{Z}(1))$ .

**Proposition 5.2.** *If  $G'$  be a subgroup of  $G$  and  $i \in \text{Inv}_{\text{nr}}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ , then*

$$\text{res}(i) \in \text{Inv}_{\text{nr}}^n(G', \mathbb{Q}/\mathbb{Z}(j)).$$

*Proof.* It is shown in [Section 2b](#) that there is a surjective morphism  $X' \rightarrow X$  of the respective classifying varieties of  $G'$  and  $G$ , such that  $\theta_G(i)_{F(X')} = \theta_{G'}(\text{res}(i))$ . Applying the homomorphism (5-2) we see that  $\text{res}(i)$  is unramified.  $\square$

**Proposition 5.3.** *Let  $G$  be a smooth linear algebraic group over a field  $F$ . The map  $\text{Inv}_{\text{nr}}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_{\text{nr}}^n(F(BG), \mathbb{Q}/\mathbb{Z}(j))$  induced by  $\theta_G$  is an isomorphism.*

*Proof.* By [Theorem 3.4](#), it suffices to show that

$$H_{\text{nr}}^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) \subset H^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j))_{\text{bal}}.$$

We follow Totaro’s approach; see [\[Garibaldi et al. 2003, p. 99\]](#). Consider the open subscheme  $W$  of  $(U^2/G) \times \mathbb{A}^1$  of all triples  $(u, u', t)$  such that  $(2-t)u + (t-1)u' \in U$ . We have the projection  $q : W \rightarrow U^2/G$ , the morphisms  $f : W \rightarrow U/G$  defined by  $f(u, u', t) = (2-t)u + (t-1)u'$ , and  $h_i : U^2/G \rightarrow W$  defined by  $h_i(u, u') = (u, u', i)$  for  $i = 1$  and  $2$ . The composition  $f \circ h_i$  is the projection  $p_i : U^2/G \rightarrow U/G$  and  $q \circ h_i$  is the identity of  $U^2/G$ .

Let  $w_i$  be the generic point of the preimage of  $i$  with respect to the projection  $W \rightarrow \mathbb{A}^1$  and write  $O_i$  for the local ring of  $W$  at  $w_i$ . The morphisms  $q, f$ , and  $h_i$  yield  $F$ -algebra homomorphisms  $F(U^2/G) \rightarrow O_i, F(U/G) \rightarrow O_i$  and  $O_i \rightarrow F(U/G)$ . Note that by [Proposition A.11](#), we have

$$H_{\text{nr}}^n(F(W), \mathbb{Q}/\mathbb{Z}(j)) \subset H^n(O_i, \mathbb{Q}/\mathbb{Z}(j)).$$

In the commutative diagram

$$\begin{array}{ccccc}
 H_{\text{nr}}^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) & \xrightarrow{f^*} & H_{\text{nr}}^n(F(W), \mathbb{Q}/\mathbb{Z}(j)) & \xleftarrow{\sim q^*} & H_{\text{nr}}^n(F(U^2/G), \mathbb{Q}/\mathbb{Z}(j)) \\
 \downarrow & & \downarrow & & \downarrow \\
 H^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) & \xrightarrow{f^*} & H^n(O_i, \mathbb{Q}/\mathbb{Z}(j)) & \xleftarrow{q^*} & H^n(F(U^2/G), \mathbb{Q}/\mathbb{Z}(j)) \\
 & \searrow p_i^* & \downarrow h_i^* & \swarrow & \\
 & & H^n(F(U^2/G), \mathbb{Q}/\mathbb{Z}(j)) & & 
 \end{array}$$

the top right map  $q^*$  is an isomorphism by [Proposition 5.1](#) since the field extension  $F(W)/F(U^2/G)$  is purely transcendental. It follows that the restriction of  $p_i^*$  on  $H_{\text{nr}}^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j))$  coincides with  $(q^*)^{-1} \circ f^*$  and hence is independent of  $i$ .  $\square$

**5a. Unramified invariants of tori.**

**Proposition 5.4.** *If  $T$  is a flasque torus, then every invariant in  $\text{Inv}^n(T, \mathbb{Q}/\mathbb{Z}(j))$  is unramified.*

*Proof.* Consider an exact sequence of tori  $1 \rightarrow T \rightarrow P \rightarrow Q \rightarrow 1$  with  $P$  quasisplit. Choose a smooth projective compactification  $X$  of  $Q$ ; see [Colliot-Thélène et al. 2005]. As  $T$  is flasque, by [Colliot-Thélène and Sansuc 1977, Proposition 9], there is a  $T$ -torsor  $E \rightarrow X$  extending the  $T$ -torsor  $P \rightarrow Q$ . The torsor  $E$  is classifying and  $T$ -rational. Choose an invariant in  $\text{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$  and consider its image  $a$  in  $H^n(F(X), \mathbb{Q}/\mathbb{Z}(j))_{\text{bal}}$  (see Theorem 3.4). We show that  $a$  is unramified with respect to every discrete valuation  $v$  on  $F(X)$  over  $F$ ; see [Colliot-Thélène 1995, Proposition 2.1.8]. By Proposition A.9,  $a$  is unramified with respect to the discrete valuation associated to every point  $x \in X$  of codimension 1, that is,  $\partial_x(a) = 0$ .

As  $X$  is projective, the valuation ring  $O_v$  of the valuation  $v$  dominates a point  $x \in X$ . It follows from Proposition A.11 that  $a$  belongs to the image of

$$H^n(O_{X,x}, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H^n(F(X), \mathbb{Q}/\mathbb{Z}(j)).$$

As the local ring  $O_{X,x}$  is a subring of  $O_v$ ,  $a$  belongs to the image of

$$H^n(O_v, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H^n(F(X), \mathbb{Q}/\mathbb{Z}(j))$$

and hence  $a$  is unramified with respect to  $v$ . □

Let  $T$  be a torus over  $F$ . By [Colliot-Thélène and Sansuc 1987b, Lemma 0.6], there is an exact sequence of tori  $1 \rightarrow T \rightarrow T' \rightarrow P \rightarrow 1$  with  $T'$  flasque and  $P$  quasisplit. The following theorem computes the unramified invariants of  $T$  in terms of the invariants of  $T'$ .

**Theorem 5.5.** *There is a natural isomorphism*

$$\text{Inv}_{\text{nr}}^n(T, \mathbb{Q}/\mathbb{Z}(j)) \simeq \text{Inv}^n(T', \mathbb{Q}/\mathbb{Z}(j)).$$

*Proof.* Choose an exact sequence  $1 \rightarrow T' \rightarrow P' \rightarrow S \rightarrow 1$  with  $P'$  a quasisplit torus. Let  $S'$  be the cokernel of the composition  $T \rightarrow T' \rightarrow P'$ . We have an exact sequence  $1 \rightarrow P \rightarrow S' \rightarrow S \rightarrow 1$ . As  $P$  is quasisplit, the latter exact sequence splits at the generic point of  $S$  and therefore,  $F(S')$  is a purely transcendental field extension of  $F(S)$ . It follows from Propositions 5.1, 5.3, and 5.4 that

$$\begin{aligned} \text{Inv}_{\text{nr}}^n(T, \mathbb{Q}/\mathbb{Z}(j)) &\simeq H_{\text{nr}}^n(F(S'), \mathbb{Q}/\mathbb{Z}(j)) \simeq H_{\text{nr}}^n(F(S), \mathbb{Q}/\mathbb{Z}(j)) \\ &\simeq \text{Inv}_{\text{nr}}^n(T', \mathbb{Q}/\mathbb{Z}(j)) = \text{Inv}^n(T', \mathbb{Q}/\mathbb{Z}(j)). \end{aligned} \quad \square$$

The following corollary is essentially equivalent to [Colliot-Thélène and Sansuc 1987b, Proposition 9.5] in the case when  $F$  is of zero characteristic.

**Corollary 5.6.** *With notation as above, the isomorphism*

$$\text{Inv}(T, \text{Br}) \xrightarrow{\sim} \text{Pic}(T) = H^1(F, \widehat{T})$$

identifies  $\text{Inv}_{\text{nr}}(T, \text{Br})$  with the subgroup  $H^1(F, \widehat{T}')$  of  $H^1(F, \widehat{T})$  of all elements that are trivial when restricted to all cyclic subgroups of the decomposition group of  $T$ .

*Proof.* The description of  $H^1(F, \widehat{T}')$  as a subgroup of  $H^1(F, \widehat{T})$  is given in [Colliot-Thélène and Sansuc 1987b, Proposition 9.5], and this part of the proof is characteristic free.  $\square$

In view of Propositions 5.1 and 5.3 we can calculate the group of unramified cohomology for the function field of an arbitrary torus in terms of the invariants of a flasque torus:

**Theorem 5.7.** *Let  $S$  be a torus over  $F$  and let  $1 \rightarrow T \rightarrow P \rightarrow S \rightarrow 1$  be a flasque resolution of  $S$ , that is,  $T$  is flasque and  $P$  is quasplit. Then there is a natural isomorphism*

$$H_{\text{nr}}^n(F(S), \mathbb{Q}/\mathbb{Z}(j)) \simeq \text{Inv}^n(T, \mathbb{Q}/\mathbb{Z}(j)).$$

Note that the torus  $S$  determines  $T$  up to multiplication by a quasplit torus. If  $X$  is a smooth compactification of  $S$ , then one can take a torus  $T$  with  $\widehat{T}_{\text{sep}} \simeq \text{Pic}(X_{\text{sep}})$ ; see [Colliot-Thélène and Sansuc 1977, Proposition 6; Voskresenskii 1998, §4.6].

**Corollary 5.8.** *A torus  $S$  has no nonconstant unramified degree 3 cohomology with values in  $\mathbb{Q}/\mathbb{Z}(2)$  universally, that is,  $H_{\text{nr}}^3(K(S), \mathbb{Q}/\mathbb{Z}(2)) = H^3(K, \mathbb{Q}/\mathbb{Z}(2))$  for any field extension  $K/F$ , if and only if  $S$  is a direct factor of a rational torus.*

*Proof.* If  $S$  is a direct factor of a rational torus, then  $S$  has no nonconstant unramified cohomology by Proposition 5.1.

Conversely, let  $1 \rightarrow T \rightarrow P \rightarrow S \rightarrow 1$  be a flasque resolution of  $S$ . By Theorem 5.7,  $\text{Inv}^3(T_K, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} = 0$  for every  $K/F$ . It follows from Theorem 4.10 that  $T$  is invertible and hence  $S$  is a factor of a rational torus (see Section 4a).  $\square$

Theorems 4.3, 5.7 and [Colliot-Thélène and Sansuc 1977, §2] yield the following proposition.

**Proposition 5.9.** *Let  $S$  be a torus over  $F$  and let  $1 \rightarrow T \rightarrow P \rightarrow S \rightarrow 1$  be a flasque resolution of  $S$ . Then we have an exact sequence*

$$\begin{aligned} 0 \rightarrow \text{CH}^2(BT)_{\text{tors}} \rightarrow H^1(F, T^0) \rightarrow \overline{H}_{\text{nr}}^3(F(S), \mathbb{Q}/\mathbb{Z}(2)) \\ \rightarrow H^0(F, S^2(\widehat{T}_{\text{sep}}))/\text{Dec} \rightarrow H^2(F, T^0). \end{aligned}$$

For an odd prime  $p$ , there is a canonical direct sum decomposition

$$\overline{H}_{\text{nr}}^3(F(S), \mathbb{Q}_p/\mathbb{Z}_p(2)) = H^1(F, T^0)\{p\} \oplus (H^0(F, S^2(\widehat{T}_{\text{sep}}))/\text{Dec})\{p\}.$$

If  $X$  is a smooth compactification of  $S$ , one can take the torus  $T$  with  $\widehat{T}_{\text{sep}} = \text{Pic}(X_{\text{sep}})$ .

**5b. The Brauer invariant for semisimple groups.** The following theorem was proved by Bogomolov [1987, Lemma 5.7] in characteristic zero:

**Theorem 5.10.** *Let  $G$  be a (connected) semisimple group over a field  $F$ . Then  $\text{Inv}_{\text{nr}}(G, \text{Br}) = \text{Inv}(G, \text{Br})_{\text{const}} = \text{Br}(F)$  and  $\text{Br}_{\text{nr}}(F(BG)) = \text{Br}(F)$ .*

*Proof.* Let  $G' \rightarrow G$  be a simply connected cover of  $G$  and  $C$  the kernel of  $G' \rightarrow G$ . By Theorem 2.4 and [Sansuc 1981, Lemme 6.9(iii)], we have

$$\text{Inv}(G, \text{Br})_{\text{norm}} = \text{Pic}(G) = \widehat{C}(F).$$

As the map  $\widehat{C}(F) \rightarrow \widehat{C}(F_{\text{sep}})$  is injective, we can replace  $F$  by  $F_{\text{sep}}$  and assume that the group  $G$  is split.

Consider the variety  $\mathcal{T}$  of maximal tori in  $G$  and the closed subscheme  $\mathcal{X} \subset G \times \mathcal{T}$  of all pairs  $(g, T)$  with  $g \in T$ . The generic fiber of the projection  $\mathcal{X} \rightarrow \mathcal{T}$  is the *generic torus*  $T_{\text{gen}}$  of  $G$ . Then  $T_{\text{gen}}$  is a maximal torus of  $G_K$ , where  $K := F(\mathcal{T})$ .

Every maximal torus in  $G$  is the factor torus of a maximal torus in  $G'$  by  $C$ . It follows that the variety  $\mathcal{T}'$  of maximal tori in  $G'$  is naturally isomorphic to  $\mathcal{T}$ . Moreover, as the generic torus  $T'_{\text{gen}}$  of  $G'$  is a maximal torus of  $G'_K$ , we have  $T_{\text{gen}} \simeq T'_{\text{gen}}/C_K$  and, therefore, an exact sequence of character groups

$$0 \rightarrow \widehat{T}_{\text{gen}} \rightarrow \widehat{T}'_{\text{gen}} \rightarrow \widehat{C}_K \rightarrow 0.$$

By Theorem 2.4, the composition of the natural homomorphism

$$\text{Inv}(G, \text{Br})_{\text{norm}} \rightarrow \text{Inv}(G_K, \text{Br})_{\text{norm}}$$

with the restriction  $\text{Inv}(G_K, \text{Br})_{\text{norm}} \rightarrow \text{Inv}(T_{\text{gen}}, \text{Br})_{\text{norm}}$  can be identified with the natural composition  $\text{Pic}(G) \rightarrow \text{Pic}(G_K) \rightarrow \text{Pic}(T_{\text{gen}})$  and hence with the connecting homomorphism  $\widehat{C}(F) = \widehat{C}(K) \rightarrow H^1(K, \widehat{T}_{\text{gen}})$ . Note that as  $F = F_{\text{sep}}$ , the decomposition group of  $T_{\text{gen}}$  coincides with the Weyl group  $W$  of  $G$  by [Voskresenskii 1988, Theorem 1], hence  $H^1(K, \widehat{T}_{\text{gen}}) \simeq H^1(W, \widehat{T}_{\text{gen}})$ .

Let  $w$  be a Coxeter element in  $W$ .<sup>1</sup> It is the product of reflections with respect to all simple roots (in some order). By [Humphreys 1990, Lemma, p. 76], 1 is not an eigenvalue of  $w$  on the space of weights  $\widehat{T}'_{\text{gen}} \otimes \mathbb{R}$ . Let  $W_0$  be the cyclic subgroup in  $W$  generated by  $w$ . It follows that the first term in the exact sequence

$$(\widehat{T}'_{\text{gen}})^{W_0} \rightarrow \widehat{C}(K) \rightarrow H^1(W_0, \widehat{T}_{\text{gen}})$$

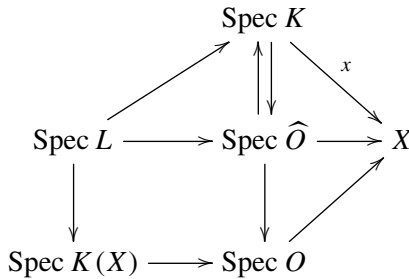
is trivial, that is, the second map is injective. Hence every nonzero character  $\chi$  in  $\widehat{C}(K)$  restricts to a nonzero element in  $H^1(W_0, \widehat{T}_{\text{gen}})$ . It follows that the image of  $\chi$  in  $H^1(W, \widehat{T}_{\text{gen}})$  is ramified by Corollary 5.6, hence so is  $\chi$  by Proposition 5.2.  $\square$

<sup>1</sup>We owe the idea to use the Coxeter element and the reference below to S. Garibaldi.

### Appendix A: Generalities

**A-I: Proof of Theorem 2.2.** Suppose that  $i(E_{\text{gen}}) = 0$  for an  $H$ -invariant  $i$  of  $G$ . Let  $K/F$  be a field extension and  $I \rightarrow \text{Spec } K$  a  $G$ -torsor. We need to show that  $i(I) = 0$  in  $H(K)$ .

Suppose first that  $K$  is infinite. Find a point  $x \in X(K)$  such that  $I$  is isomorphic to the pull-back of the classifying torsor with respect to  $x$ . Let  $x'$  be a rational point of  $X_K$  above  $x$  and write  $O$  for the local ring  $O_{X_K, x'}$ . The  $K$ -algebra  $O$  is a regular local ring with residue field  $K$ . Therefore, the completion  $\widehat{O}$  is isomorphic to  $K[[t_1, t_2, \dots, t_n]]$  over  $K$ . Let  $L$  be the quotient field of  $\widehat{O}$ , a field extension of  $K(X)$ . We have the following diagram of morphisms with a commutative square and three triangles:



The pull-back of the classifying torsor  $E \rightarrow X$  via  $\text{Spec } K(X) \rightarrow X$  is  $(E_{\text{gen}})_{K(X)}$ . The  $G$ -torsor  $I$  is the pull-back of  $E \rightarrow X$  with respect to  $x$ . Let  $\widehat{E}$  be the pull-back of  $E \rightarrow X$  via  $\text{Spec } \widehat{O} \rightarrow X$ . Therefore,  $I$  is the pull-back of  $\widehat{E}$ . Since  $G$  is smooth, by a theorem of Grothendieck [Demazure and Grothendieck 1970, XXIV, Proposition 8.1],  $\widehat{E}$  is the pull-back of  $I$  with respect to  $\text{Spec } \widehat{O} \rightarrow \text{Spec}(K)$ . It follows that  $I_L \simeq (E_{\text{gen}})_L$  as torsors over  $L$ . Hence the images of  $i(I)$  and  $i(E_{\text{gen}})$  in  $H(L)$  are equal and therefore,  $i(I)_L = 0$ . By Property 2.1, we have  $i(I) = 0$ .

If  $K$  is finite, we replace  $F$  by  $F((t))$  and  $K$  by  $K((t))$ . By the first part of the proof,  $i(I)$  belongs to the kernel of  $H(K) \rightarrow H(K((t)))$  and hence is trivial by Property 2.1 again.

**A-II: Decomposable elements.** Let  $\Gamma$  be a profinite group and  $A$  a  $\Gamma$ -lattice. Write  $A^\Gamma$  for the subgroup of  $\Gamma$ -invariant elements in  $A$ . Let  $\Gamma' \subset \Gamma$  be an open subgroup and choose representatives  $\gamma_1, \gamma_2, \dots, \gamma_n$  for the left cosets of  $\Gamma'$  in  $\Gamma$ . We have the trace map  $\text{Tr} : A^{\Gamma'} \rightarrow A^\Gamma$  defined by  $\text{Tr}(a) = \sum_{i=1}^n \gamma_i a$ .

Let  $S^2(A)$  be the symmetric square of  $A$ . Consider the quadratic trace map  $\text{Qtr} : A^{\Gamma'} \rightarrow S^2(A)^\Gamma$  defined by  $\text{Qtr}(a) = \sum_{i < j} (\gamma_i a)(\gamma_j a)$ . Write  $\text{Dec}(A)$  for the subgroup of decomposable elements in  $S^2(A)^\Gamma$  generated by the square  $(A^\Gamma)^2$  of  $A^\Gamma$  and the elements  $\text{Qtr}(a)$  for all open subgroups  $\Gamma' \subset \Gamma$  and all  $a \in A^{\Gamma'}$ .



Let  $B$  be another  $\Gamma$ -lattice. We write  $\text{Dec}(A, B)$  for the subgroup of  $(A \otimes B)^\Gamma$  generated by elements of the form  $\text{Tr}(a \otimes b)$  for all open subgroups  $\Gamma' \subset \Gamma$  and all  $a \in A^{\Gamma'}$ ,  $b \in B^{\Gamma'}$ .

There is a natural isomorphism  $S^2(A \oplus B) \simeq S^2(A) \oplus (A \otimes B) \oplus S^2(B)$ . Moreover, the equality  $\text{Qtr}(a + b) = \text{Qtr}(a) + (\text{Tr}(a) \otimes \text{Tr}(b) - \text{Tr}(a \otimes b)) + \text{Qtr}(b)$  yields the decomposition

$$\text{Dec}(A \oplus B) \simeq \text{Dec}(A) \oplus \text{Dec}(A, B) \oplus \text{Dec}(B).$$

**A-III: Cup-products.** Let  $1 \rightarrow T \rightarrow P \rightarrow Q \rightarrow 1$  be an exact sequence of tori. We consider the connecting maps

$$\partial_1 : H^p(F, \widehat{T}(i)) \rightarrow H^{p+1}(F, \widehat{Q}(i))$$

for the exact sequence  $0 \rightarrow \widehat{Q}_{\text{sep}} \rightarrow \widehat{P}_{\text{sep}} \rightarrow \widehat{T}_{\text{sep}} \rightarrow 0$  of character  $\Gamma$ -lattices and

$$\partial_2 : H^q(F, \widehat{Q}^\circ(j)) \rightarrow H^{q+1}(F, \widehat{T}^\circ(j))$$

for the dual sequence of lattices (see notation in Section 4b).

**Lemma A.1.** *Let  $a \in H^p(F, \widehat{T}(i))$  and  $b \in H^q(F, \widehat{Q}^\circ(j))$  with  $i + j \leq 2$ . Then  $\partial_1(a) \cup b = (-1)^{p+1} a \cup \partial_2(b)$  in  $H^{p+q+1}(F, \mathbb{Z}(i + j))$ , where the cup-product is defined in (4-3).*

*Proof.* By [Cartan and Eilenberg 1999, Chapter V, Proposition 4.1], the elements  $\partial_1(1_T)$  and  $\partial_2(1_Q)$  in

$$H^1(F, \widehat{T}_{\text{sep}}^\circ \otimes \widehat{Q}_{\text{sep}}) = \text{Ext}_\Gamma^1(\widehat{T}_{\text{sep}}, \widehat{Q}_{\text{sep}})$$

differ by a sign. Write  $\tau$  for the isomorphism induced by permutation of the factors. By the standard properties of the cup-product, we have

$$\begin{aligned} \partial_1(a) \cup b &= 1_T \cup \partial_1(a) \cup b = \partial_1(1_T) \cup a \cup b = (-1)^{pq} \tau(\partial_1(1_T) \cup b \cup a) \\ &= (-1)^{pq+1} \tau(\partial_2(1_Q) \cup b \cup a) = (-1)^{pq+1} \tau(1_Q \cup \partial_2(b) \cup a) \\ &= (-1)^{p+1} 1_Q \cup a \cup \partial_2(b) = (-1)^{p+1} a \cup \partial_2(b). \quad \square \end{aligned}$$

**A-IV: Cosimplicial abelian groups.** Let  $A^\bullet$  be a cosimplicial abelian group

$$A^0 \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \end{array} A^1 \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} A^2 \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \dots$$

and write  $h_*(A^\bullet)$  for the homology groups of the associated complex of abelian groups. In particular,

$$h_0(A^\bullet) = \text{Ker}[(d^0 - d^1) : A^0 \rightarrow A^1].$$

We say that the cosimplicial abelian group  $A^\bullet$  is *constant* if for every  $i$ , all the coface maps  $d_j : A^{i-1} \rightarrow A^i$ ,  $j = 0, 1, \dots, i$ , are isomorphisms. In this case all the  $d_j$  are equal as  $d_j = s_j^{-1} = d_{j+1}$ , where the  $s_j$  are the codegeneracy maps. For a constant cosimplicial abelian group  $A^\bullet$ , we have  $h_0(A^\bullet) = A^0$  and  $h_i(A^\bullet) = 0$  for all  $i > 0$ . We will need the following straightforward statement.

**Lemma A.2.** *Let  $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow D^\bullet$  be an exact sequence of cosimplicial abelian groups with  $A^\bullet$  a constant cosimplicial group. Then the sequence of groups  $0 \rightarrow A^0 \rightarrow h_0(B^\bullet) \rightarrow h_0(C^\bullet) \rightarrow h_0(D^\bullet)$  is exact.*

Let  $H$  be a contravariant functor from the category of schemes over  $F$  to the category of abelian groups. We say that  $H$  is *homotopy invariant* if for every vector bundle  $E \rightarrow X$  over  $F$ , the induced map  $H(X) \rightarrow H(E)$  is an isomorphism.

For an integer  $d > 0$  consider the following property of the functor  $H$ :

**Property A.3.** For every closed subscheme  $Z$  of a scheme  $X$  with  $\text{codim}_X(Z) \geq d$ , the natural homomorphism  $H(X) \rightarrow H(X \setminus Z)$  is an isomorphism.

Let  $G$  be a linear algebraic group over a field  $F$  and choose a standard classifying  $G$ -torsor  $U \rightarrow U/G$ . Let  $U^i$  denote the product of  $i$  copies of  $U$ . We have the  $G$ -torsors  $U^i \rightarrow U^i/G$ .

Consider the cosimplicial abelian group  $A^\bullet = H(U^\bullet/G)$  with  $A^i = H(U^{i+1}/G)$  and coface maps  $A^{i-1} \rightarrow A^i$  induced by the projections  $U^{i+1}/G \rightarrow U^i/G$ .

**Lemma A.4.** *Let  $H$  be a homotopy invariant functor satisfying [Property A.3](#) for some  $d$ . Let  $U \rightarrow U/G$  be a standard classifying  $G$ -torsor and  $U'$  an open subset of a  $G$ -representation  $V'$ .*

1. *If  $\text{codim}_{V'}(V' \setminus U') \geq d$ , then the natural homomorphism*

$$H(U/G) \rightarrow H((U \times U')/G)$$

*is an isomorphism.*

2. *If  $\text{codim}_V(V \setminus U) \geq d$ , then the cosimplicial group  $H(U^\bullet/G)$  is constant.*

*Proof.* 1. The scheme  $(U \times U')/G$  is an open subset of the vector bundle  $(U \times V')/G$  over  $U/G$  with complement of codimension at least  $d$ . The map in question is the composition  $H(U/G) \rightarrow H((U \times V')/G) \rightarrow H((U \times U')/G)$  and both maps in the composition are isomorphisms since  $H$  is homotopy invariant and satisfies [Property A.3](#).

2. By the first part of the lemma applied to the  $G$ -torsor  $U^i \rightarrow U^i/G$  and  $U' = U$ , the map  $H(U^i/G) \rightarrow H(U^{i+1}/G)$  induced by a projection  $U^{i+1}/G \rightarrow U^i/G$  is an isomorphism. □

By [Lemma A.4](#), if  $H$  is a homotopy invariant functor satisfying [Property A.3](#) for some  $d$ , then the group  $H(U/G)$  does not depend on the choice of the representation

$V$  and the open set  $U \subset V$  provided  $\text{codim}_V(V \setminus U) \geq d$ . Following [Totaro 1999], we denote this group by  $H(BG)$ .

**Example A.5.** The split torus  $T = (\mathbb{G}_m)^n$  over  $F$  acts freely on the product  $U$  of  $n$  copies of  $\mathbb{A}^{r+1} \setminus \{0\}$  with  $U/T \simeq (\mathbb{P}^r)^n$ , that is,  $BT$  is “approximated” by the varieties  $(\mathbb{P}^r)^n$  if “ $r \gg 0$ .” We then have  $\text{CH}^*(BT) = S^*(\widehat{T})$ , where  $S^*$  represents the symmetric algebra and  $\widehat{T}$  is the character group of  $T$ ; see [Edidin and Graham 1998, p. 607]. In particular,  $\text{Pic}(BT) = \text{CH}^1(BT) = \widehat{T}$ . More generally, by the Künneth formula [Esnault et al. 1998, Proposition 3.7],

$$H_{\text{Zar}}^*(BT, \mathcal{K}_*) \simeq \text{CH}^*(BT) \otimes K_*(F) \simeq S^*(\widehat{T}) \otimes K_*(F),$$

where  $K_n(F)$  is the Quillen  $K$ -group of  $F$  and  $\mathcal{K}_n$  is the Zariski sheaf associated to the presheaf  $U \mapsto K_n(U)$ .

**A-V: Étale cohomology.** For a scheme  $X$  and a closed subscheme  $Z \subset X$  we write  $H_Z^*(X, \mathbb{Q}/\mathbb{Z}(j))$  for the étale cohomology group of  $X$  with support in  $Z$  and values in  $\mathbb{Q}/\mathbb{Z}(j)$  [Milne 1980, Chapter III, §1]. Write  $X^{(i)}$  for the set of points in  $X$  of codimension  $i$ . For a point  $x \in X^{(1)}$  set

$$H_x^*(X, \mathbb{Q}/\mathbb{Z}(j)) = \text{colim}_{x \in U} H_{\{x\} \cap U}^*(U, \mathbb{Q}/\mathbb{Z}(j)),$$

where the colimit is taken over all open subsets  $U \subset X$  containing  $x$ . If  $X$  is a variety, write

$$\partial_x : H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_x^{*+1}(X, \mathbb{Q}/\mathbb{Z}(j))$$

for the residue homomorphisms arising from the *coniveau spectral sequence* [Colliot-Thélène et al. 1997, 1.2]

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} H_x^{p+q}(X, \mathbb{Q}/\mathbb{Z}(j)) \Rightarrow H^{p+q}(X, \mathbb{Q}/\mathbb{Z}(j)). \tag{A-1}$$

Let  $f : Y \rightarrow X$  be a dominant morphism of varieties over  $F$ ,  $y \in Y^{(1)}$ , and  $x = f(y)$ . If  $x \in X^{(1)}$ , there is a natural homomorphism

$$f_y^* : H_x^*(X, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_y^*(Y, \mathbb{Q}/\mathbb{Z}(j)).$$

The following lemma is straightforward.

**Lemma A.6.** *Let  $f : Y \rightarrow X$  be a dominant morphism of varieties over  $F$ ,  $y \in Y^{(1)}$  and  $x = f(y)$ .*

(1) *If  $x$  is the generic point of  $X$ , then the composition*

$$H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{f^*} H^*(F(Y), \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{\partial_y} H_y^{*+1}(Y, \mathbb{Q}/\mathbb{Z}(j))$$

*is trivial.*

(2) If  $x \in X^{(1)}$ , the diagram

$$\begin{array}{ccc}
 H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) & \xrightarrow{\partial_x} & H_x^{*+1}(X, \mathbb{Q}/\mathbb{Z}(j)) \\
 f^* \downarrow & & f_y^* \downarrow \\
 H^*(F(Y), \mathbb{Q}/\mathbb{Z}(j)) & \xrightarrow{\partial_y} & H_y^{*+1}(Y, \mathbb{Q}/\mathbb{Z}(j)).
 \end{array}$$

is commutative.

**Lemma A.7.** *Let  $X$  be a geometrically irreducible variety,  $Z \subset X$  a closed subvariety of codimension 1, and  $x$  the generic point of  $Z$ . Let  $P$  be a variety over  $F$  such that  $P(K)$  is dense in  $P$  for every field extension  $K/F$  with  $K$  infinite, and let  $y$  be the generic point of  $Z \times P$  in  $Y := X \times P$ . Then the homomorphism  $f_y^* : H_x^*(X, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_y^*(Y, \mathbb{Q}/\mathbb{Z}(j))$  induced by the projection  $f : Y \rightarrow X$  is injective.*

*Proof.* Assume first that the field  $F$  is infinite. An element  $\alpha \in H_x^*(X, \mathbb{Q}/\mathbb{Z}(j))$  is represented by an element  $h \in H_{Z \cap U}^*(U, \mathbb{Q}/\mathbb{Z}(j))$  for a nonempty open set  $U \subset X$  containing  $x$ . If  $\alpha$  belongs to the kernel of

$$f_y^* : H_x^*(X, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_y^*(Y, \mathbb{Q}/\mathbb{Z}(j)),$$

then there is an open subset  $W \subset U \times P$  containing  $y$  such that  $h$  belongs to the kernel of the composition

$$g : H_{Z \cap U}^*(U, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_{(Z \cap U) \times P}^*(U \times P, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_{(Z \times P) \cap W}^*(W, \mathbb{Q}/\mathbb{Z}(j)).$$

As  $F$  is infinite, by the assumption on  $P$ , there is a rational point  $t \in P$  in the image of the dominant composition  $(Z \times P) \cap W \hookrightarrow Z \times P \rightarrow P$ . We have  $(U \times t) \cap W = U' \times t$  for an open subset  $U' \subset U$  such that  $x \in U'$ . Composing  $g$  with the homomorphism  $H_{(Z \times P) \cap W}^*(W, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_{Z \cap U'}^*(U', \mathbb{Q}/\mathbb{Z}(j))$  induced by the morphism  $(U', Z \cap U') \rightarrow (W, (Z \times P) \cap W)$ ,  $u \mapsto (u, t)$ , we see that  $h$  belongs to the kernel of the restriction homomorphism  $H_{Z \cap U}^*(U, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_{Z \cap U'}^*(U', \mathbb{Q}/\mathbb{Z}(j))$ , hence the image of  $\alpha$  in  $H_x^*(X, \mathbb{Q}/\mathbb{Z}(j))$  is trivial.

Suppose now that  $F$  is a finite field. Choose a prime integer  $p$  and an infinite algebraic pro- $p$ -extension  $L/F$ . By the first part of the proof, the statement holds for the variety  $X_L$  over  $L$ . By the restriction-corestriction argument,  $\text{Ker}(f_y^*)$  is a  $p$ -primary torsion group. Since this holds for every prime  $p$ , we have  $\text{Ker}(f_y^*) = 0$ . □

**Corollary A.8.** *Let  $G$  be a linear algebraic group over  $F$ , let  $E \rightarrow X$  be a  $G$ -torsor over a geometrically irreducible variety  $X$  with  $E$  a  $G$ -rational variety and consider the first projection  $p : E^2/G \rightarrow X$ . Let  $x \in X$  and  $y \in E^2/G$  be points of*

*codimension 1 such that  $p(y) = x$ . Then the homomorphism*

$$p_y^* : H_x^*(X, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_y^*(E^2/G, \mathbb{Q}/\mathbb{Z}(j))$$

*is injective.*

*Proof.* Choose a linear  $G$ -space  $V$  and a nonempty  $G$ -variety  $U$  that is  $G$ -isomorphic to open subschemes of  $E$  and  $V$ . We can replace the variety  $E^2/G$  by  $(E \times U)/G$ , an open subscheme in the vector bundle  $(E \times V)/G$  over  $X$ . Shrinking  $X$  around  $x$ , we may assume that the vector bundle is trivial, that is,  $(E \times U)/G$  is isomorphic to an open subscheme in  $X \times V$ . The statement then follows from [Lemma A.7](#).  $\square$

**Proposition A.9.** *In the conditions of [Corollary A.8](#), let  $h \in H^*(F(X), \mathbb{Q}/\mathbb{Z}(j))_{\text{bal}}$ . Then  $\partial_x(h) = 0$  for every point  $x \in X$  of codimension 1.*

*Proof.* Let  $y \in E^2/G$  be the point of codimension 1 such that  $p_1(y) = x$ . As  $p_2(y)$  is the generic point of  $X$ , by [Lemma A.6\(1\)](#),  $\partial_y(h') = 0$ , where  $h' = p_1^*(h) = p_2^*(h)$  in  $H^*(F(E^2/G), \mathbb{Q}/\mathbb{Z}(j))$ . It follows from [Lemma A.6\(2\)](#) that  $\partial_x(h)$  is in the kernel of  $(p_1)_y^* : H_x^*(X, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_y^*(E^2/G, \mathbb{Q}/\mathbb{Z}(j))$  and hence is trivial by [Corollary A.8](#).  $\square$

The sheaf  $\mathcal{H}^*(\mathbb{Q}/\mathbb{Z}(j))$  defined in [Section 3](#) has a flasque resolution related to the Cousin complex by [[Colliot-Thélène et al. 1997](#), §2] (for the  $p$ -components with  $p \neq \text{char } F$ ) and [[Gros and Suwa 1988](#), Theorem 1.4] (for the  $p$ -component with  $p = \text{char } F > 0$ ):

$$0 \rightarrow \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)) \rightarrow \coprod_{x \in X^{(0)}} i_{x*} H_x^n(X, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow \coprod_{x \in X^{(1)}} i_{x*} H_x^{n+1}(X, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow \dots,$$

where  $i_x : \text{Spec } F(x) \rightarrow X$  are the canonical morphisms. In particular, we have:

**Proposition A.10.** *Let  $X$  be a smooth variety over  $F$ . The sequence*

$$0 \rightarrow H_{\text{Zar}}^0(X, \mathcal{H}^*(\mathbb{Q}/\mathbb{Z}(j))) \rightarrow H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{\partial} \coprod_{x \in X^{(1)}} H_x^{*+1}(X, \mathbb{Q}/\mathbb{Z}(j)),$$

where  $\partial = \coprod \partial_x$ , is exact.

**Proposition A.11.** *Let  $X$  be a smooth variety over  $F$  and  $x \in X$ . The sequence*

$$0 \rightarrow H^*(O_{X,x}, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H^*(F(X), \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{\partial} \coprod_{\substack{x' \in X^{(1)} \\ x' \in \{x\}}} H_{x'}^{*+1}(X, \mathbb{Q}/\mathbb{Z}(j))$$

*is exact.*

**Appendix B: Spectral sequences**

**B-I: Hochschild–Serre spectral sequence.** Let

$$\mathcal{A} \xrightarrow{W} \mathcal{B} \xrightarrow{V} \mathcal{C}$$

be additive left exact functors between abelian categories with enough injective objects. If  $W$  takes injective objects to  $V$ -acyclic ones, there is a spectral sequence

$$E_2^{p,q} = R^p V(R^q W(A)) \Rightarrow R^{p+q}(VW)(A)$$

for every complex  $A$  in  $\mathcal{A}$  bounded from below.

We have exact triangles in the derived category of  $\mathcal{B}$ :

$$\tau_{\leq n} RW(A) \rightarrow RW(A) \rightarrow \tau_{\geq n+1} RW(A) \rightarrow \tau_{\leq n} RW(A)[1], \tag{B-1}$$

$$\tau_{\leq n-1} RW(A) \rightarrow \tau_{\leq n} RW(A) \rightarrow R^n W(A)[-n] \rightarrow \tau_{\leq n-1} RW(A)[1]. \tag{B-2}$$

The filtration on  $R^n(VW)(A)$  is defined by

$$F^j R^n(VW)(A) = \text{Im}(R^n V(\tau_{\leq(n-j)} RW(A)) \rightarrow R^n V(RW(A)) = R^n(VW)(A)).$$

As  $\tau_{\geq n+1} RW(A)$  is acyclic in degrees less than or equal to  $n$ , the morphism

$$R^n V(\tau_{\leq n} RW(A)) \rightarrow R^n V(RW(A)) = R^n(VW)(A)$$

is an isomorphism, in particular,  $F^0 R^n(VW)(A) = R^n(VW)(A)$ .

The *edge* homomorphism is defined as the composition

$$R^n(VW)(A) \xrightarrow{\sim} R^n V(\tau_{\leq n} RW(A)) \rightarrow R^n V(R^n W(A)[-n]) = V(R^n W(A)).$$

Moreover, the kernel  $F^1 R^n(VW)(A)$  of the edge homomorphism is isomorphic to  $R^n V(\tau_{\leq n-1} RW(A))$ . We define the morphism  $d_n$  as the composition

$$d_n : F^1 R^n(VW)(A) \rightarrow R^n V(R^{n-1} W(A)[-n+1]) = R^1 V(R^{n-1} W(A)) = E_2^{1,n-1}.$$

**B-II: First spectral sequence.** Let  $X$  be a smooth variety over a field  $F$ . We have the functors

$$\text{Sheaves}_{\acute{e}t}(X) \xrightarrow{q_*} \text{Sheaves}_{\acute{e}t}(F) \xrightarrow{V} Ab,$$

where  $q_*$  is the push-forward map for the structure morphism  $q : X \rightarrow \text{Spec}(F)$  and  $V(M) = H^0(F, M)$ .

Consider the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(F, H^q(X_{\text{sep}}, \mathbb{Z}(2))) \Rightarrow H^{p+q}(X, \mathbb{Z}(2)). \tag{B-3}$$

Set  $\Delta(i) := Rq_*(\mathbb{Z}(i))$  for  $i = 1$  or  $2$ . Then  $\Delta(i)$  is the complex of étale sheaves on  $F$  concentrated in degrees  $\geq 1$ . The  $j$ -th term  $F^j H^n(X, \mathbb{Z}(i))$  of the filtration

on  $H^n(X, \mathbb{Z}(i))$  coincides with the image of the canonical homomorphism

$$H^n(F, \tau_{\leq(n-j)}\Delta(i)) \rightarrow H^n(F, \Delta(i)) = H^n(X, \mathbb{Z}(i)).$$

Let  $M$  be a  $\Gamma$ -lattice viewed as an étale sheaf over  $F$ . Note that there are canonical isomorphisms

$$H^*(F, M^\circ \otimes \Delta(i)) = \text{Ext}_F^*(M, \Delta(i)) = \text{Ext}_X^*(q^*M, \mathbb{Z}(i)), \tag{B-4}$$

where  $M^\circ := \text{Hom}(M, \mathbb{Z})$  is the dual lattice.

Consider also the following product map:

$$\mathbb{Z}(1) \otimes^L \Delta(1) \rightarrow Rq_*(q^*\mathbb{Z}(1) \otimes^L \mathbb{Z}(1)) \rightarrow Rq_*(\mathbb{Z}(1) \otimes^L \mathbb{Z}(1)) \rightarrow Rq_*(\mathbb{Z}(2)).$$

The complex  $\mathbb{Z}(1) \otimes^L \tau_{\leq 2}\Delta(1)$  is trivial in degrees greater than 3, hence we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}(1) \otimes^L \tau_{\leq 2}\Delta(1) & \xrightarrow{\text{prod}} & \tau_{\leq 3}Rq_*(\mathbb{Z}(2)) = \tau_{\leq 3}\Delta(2) \\ \downarrow & & \downarrow \\ \mathbb{Z}(1) \otimes^L \Delta(1) & \xrightarrow{\text{prod}} & Rq_*(\mathbb{Z}(2)) = \Delta(2). \end{array}$$

There are canonical morphisms from (B-2):

$$h_2 : \tau_{\leq 2}\Delta(1)[2] \rightarrow H^2(X_{\text{sep}}, \mathbb{Z}(1)) \quad \text{and} \quad h_3 : \tau_{\leq 3}\Delta(2)[3] \rightarrow H^3(X_{\text{sep}}, \mathbb{Z}(2)).$$

Consider an element

$$\delta \in H^1(F, M \otimes F_{\text{sep}}^\times) = \text{Ext}_F^1(M^\circ, \mathbb{G}_{m,F}) = \text{Ext}_F^2(M^\circ, \mathbb{Z}(1)),$$

and view  $\delta$  as a morphism  $\delta : M^\circ \rightarrow \mathbb{Z}(1)[2]$  in  $D^+(\text{Sheaves}_{\text{ét}}(F))$ .

The following diagram

$$\begin{array}{ccccc} M^\circ \otimes \Delta(1)[2] & \xrightarrow{\delta \otimes 1} & \mathbb{Z}(1) \otimes^L \Delta(1)[4] & \xrightarrow{\text{prod}} & \Delta(2)[4] \\ (1 \otimes i_2)[2] \uparrow & & (1 \otimes i_2)[4] \uparrow & & (i_3)[4] \uparrow \\ M^\circ \otimes \tau_{\leq 2}\Delta(1)[2] & \xrightarrow{\delta \otimes 1} & \mathbb{Z}(1) \otimes^L \tau_{\leq 2}\Delta(1)[4] & \xrightarrow{\text{prod}} & \tau_{\leq 3}\Delta(2)[4] \\ 1 \otimes h_2 \downarrow & & 1 \otimes h_2 \downarrow & & h_3 \downarrow \\ M^\circ \otimes H^2(X_{\text{sep}}, \mathbb{Z}(1)) & \xrightarrow{\delta \otimes 1} & \mathbb{Z}(1) \otimes^L H^2(X_{\text{sep}}, \mathbb{Z}(1))[2] & \xrightarrow{\text{prod}} & H^3(X_{\text{sep}}, \mathbb{Z}(2))[1] \end{array}$$

where  $i_2 : \tau_{\leq 2}\Delta(1) \rightarrow \Delta(1)$  and  $i_3 : \tau_{\leq 3}\Delta(2) \rightarrow \Delta(2)$  are natural morphisms, is commutative.

By (B-4), we have

$$H^0(F, M^\circ \otimes \Delta(1)[2]) = \text{Ext}_F^2(M, \Delta(1)) = \text{Ext}_X^2(q^*M, \mathbb{Z}(1)).$$



Furthermore, the diagram above yields a commutative square

$$\begin{array}{ccc}
 \mathrm{Ext}_X^2(q^*M, \mathbb{Z}(1)) & \xrightarrow{q^*(\delta) \cup -} & F^1 H^4(X, \mathbb{Z}(2)) \\
 d_2 \downarrow & & d_4 \downarrow \\
 \mathrm{Hom}_\Gamma(M, H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))) & \xrightarrow{j} & H^1(F, H^3(X_{\mathrm{sep}}, \mathbb{Z}(2))),
 \end{array}$$

where  $d_2$  is the edge map coming from the spectral sequence

$$\mathrm{Ext}_F^p(M, H^q(X_{\mathrm{sep}}, \mathbb{Z}(1))) \Rightarrow \mathrm{Ext}_X^{p+q}(q^*M, \mathbb{Z}(1)) \tag{B-5}$$

and  $j$  coincides with the composition

$$\begin{aligned}
 \mathrm{Hom}_\Gamma(M, H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))) &= H^0(F, M^\circ \otimes H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))) \\
 &\xrightarrow{\delta \cup -} H^1(F, F_{\mathrm{sep}}^\times \otimes H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))) \xrightarrow{\rho} H^1(F, H^3(X_{\mathrm{sep}}, \mathbb{Z}(2))),
 \end{aligned}$$

with  $\rho$  given by the product map.

Now suppose the group  $H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))$ , which is canonically isomorphic to  $\mathrm{Pic}(X_{\mathrm{sep}})$ , is a lattice. Let  $M = \mathrm{Pic}(X_{\mathrm{sep}})$  and consider the torus  $T$  over  $F$  with  $\widehat{T}_{\mathrm{sep}} = M$ . It follows that

$$\delta \in H^1(F, T^\circ) = H^1(F, \widehat{T}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^\times) = H^2(F, \widehat{T}_{\mathrm{sep}} \otimes \mathbb{Z}(1)),$$

where  $T^\circ$  is the dual torus. Note that  $\delta \cup 1_M = \delta$ , where

$$1_M \in H^0(F, M^\circ \otimes H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))) = \mathrm{End}_\Gamma(M)$$

is the identity.

The top map in the last diagram is given by the pairing

$$\begin{aligned}
 H^1(X, T^0) \otimes H^1(X, T) &\rightarrow F^1 H^4(X, \mathbb{Z}(2)), \\
 a \otimes b &\mapsto a \cup b,
 \end{aligned} \tag{B-6}$$

defined as the cup-product in (4-3),

$$H^2(X, \widehat{T}(1)) \otimes H^2(X, \widehat{T}^\circ(1)) \rightarrow F^1 H^4(X, \mathbb{Z}(2)),$$

if we identify  $\mathrm{Ext}_X^2(q^*M, \mathbb{Z}(1))$  with  $H^2(X, \widehat{T}^\circ(1)) = H^1(X, T)$ .

In this case, the homomorphism

$$\rho : H^1(F, T^\circ) \rightarrow H^1(F, H^3(X_{\mathrm{sep}}, \mathbb{Z}(2))) \tag{B-7}$$

is given by the product homomorphism

$$T^\circ(F_{\mathrm{sep}}) = F_{\mathrm{sep}}^\times \otimes \widehat{T}_{\mathrm{sep}} = F_{\mathrm{sep}}^\times \otimes \mathrm{Pic}(X_{\mathrm{sep}}) \rightarrow H^3(X_{\mathrm{sep}}, \mathbb{Z}(2)).$$

A  $T$ -torsor  $E \rightarrow X$  is called *universal* if the class of  $E$  in

$$H^1(X, T) = \text{Ext}_X^2(q^*M, \mathbb{Z}(1))$$

satisfies  $d_2([E]) = 1_M$ ; see [Colliot-Thélène and Sansuc 1987a].

Commutativity of the previous diagram gives:

**Proposition B.1.** *Let  $X$  be a smooth variety over  $F$  such that  $\text{Pic}(X_{\text{sep}})$  is a lattice. Let  $T$  be the torus over  $F$  satisfying  $\widehat{T}_{\text{sep}} = \text{Pic}(X_{\text{sep}})$  and let  $E$  be a universal  $T$ -torsor over  $X$  with the class  $[E] \in H^1(X, T)$ . Then for every  $\delta \in H^1(F, T^\circ)$ , we have*

$$d_4(q^*(\delta) \cup [E]) = \rho(\delta),$$

where  $d_4 : F^1 H^4(X, \mathbb{Z}(2)) \rightarrow H^1(F, H^3(X_{\text{sep}}, \mathbb{Z}(2)))$  is the map induced by the Hochschild–Serre spectral sequence (B-3) and the cup-product is taken for the pairing (B-6).

**B-III: Second spectral sequence.** We assume that  $H^3(X_{\text{sep}}, \mathbb{Z}(2)) = 0$ , hence in particular  $E_2^{0,3} = 0$  in the spectral sequence (B-3) and so  $E_\infty^{2,2} \subset E_2^{2,2}$ . Therefore, we have a canonical map

$$e_4 : F^2 H^4(X, \mathbb{Z}(2)) \rightarrow E_\infty^{2,2} \hookrightarrow E_2^{2,2} = H^2(F, H^2(X_{\text{sep}}, \mathbb{Z}(2))).$$

Let  $N$  be a  $\Gamma$ -lattice. Consider an element

$$\gamma \in H^2(F, N \otimes F_{\text{sep}}^\times) = \text{Ext}_F^2(N^\circ, \mathbb{G}_{m,F}) = \text{Ext}_F^3(N^\circ, \mathbb{Z}(1)),$$

and view  $\gamma$  as a morphism  $\gamma : N^\circ \rightarrow \mathbb{Z}(1)[3]$  in  $D^+(\text{Sheaves}_{\text{ét}}(F))$ .

As above, the commutative diagram

$$\begin{array}{ccccc} N^\circ \otimes \Delta(1)[1] & \xrightarrow{\gamma \otimes 1} & \mathbb{Z}(1) \otimes^L \Delta(1)[4] & \xrightarrow{\text{prod}} & \Delta(2)[4] \\ (1 \otimes i_1)[1] \uparrow & & (1 \otimes i_1)[4] \uparrow & & (i_2)[4] \uparrow \\ N^\circ \otimes \tau_{\leq 1} \Delta(1)[1] & \xrightarrow{\gamma \otimes 1} & \mathbb{Z}(1) \otimes^L \tau_{\leq 1} \Delta(1)[4] & \xrightarrow{\text{prod}} & \tau_{\leq 2} \Delta(2)[4] \\ 1 \otimes h_1 \downarrow & & 1 \otimes h_1 \downarrow & & h_2 \downarrow \\ N^\circ \otimes H^1(X_{\text{sep}}, \mathbb{Z}(1)) & \xrightarrow{\gamma \otimes 1} & \mathbb{Z}(1) \otimes^L H^1(X_{\text{sep}}, \mathbb{Z}(1))[3] & \xrightarrow{\text{prod}} & H^2(X_{\text{sep}}, \mathbb{Z}(2))[2] \end{array}$$

where  $i_1, i_2, h_1$  and  $h_2$  are defined in a similar fashion as in Section B-II, yields a commutative square

$$\begin{array}{ccc} \text{Ext}_X^1(q^*N, \mathbb{Z}(1)) & \xrightarrow{q^*(\gamma) \cup -} & F^2 H^4(X, \mathbb{Z}(2)) \\ d_1 \downarrow & & e_4 \downarrow \\ \text{Hom}_\Gamma(N, H^1(X_{\text{sep}}, \mathbb{Z}(1))) & \xrightarrow{k} & H^2(F, H^2(X_{\text{sep}}, \mathbb{Z}(2))) \end{array}$$

where  $d_1$  is the edge map coming from the spectral sequence

$$\mathrm{Ext}_F^p(N, H^q(X_{\mathrm{sep}}, \mathbb{Z}(1))) \Rightarrow \mathrm{Ext}_X^{p+q}(q^*N, \mathbb{Z}(1))$$

and  $k$  coincides with the composition

$$\begin{aligned} \mathrm{Hom}_\Gamma(N, H^1(X_{\mathrm{sep}}, \mathbb{Z}(1))) &= H^0(F, N^\circ \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1))) \\ &\xrightarrow{\gamma \cup -} H^2(F, F_{\mathrm{sep}}^\times \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1))) \rightarrow H^2(F, H^2(X_{\mathrm{sep}}, \mathbb{Z}(2))) \end{aligned}$$

with the last homomorphism given by the product map.

Suppose  $N$  is a  $\Gamma$ -lattice in  $F_{\mathrm{sep}}[X]^\times$  such that the composition

$$N \hookrightarrow F_{\mathrm{sep}}[X]^\times \rightarrow F_{\mathrm{sep}}[X]^\times / F_{\mathrm{sep}}^\times$$

is an isomorphism. Consider the torus  $Q$  with  $\widehat{Q}_{\mathrm{sep}} = N$ , so that  $\gamma \in H^2(F, Q^\circ)$ .

Note that  $\gamma \cup i_N = \gamma$ , where

$$i_N \in H^0(F, N^\circ \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1))) = \mathrm{Hom}_\Gamma(N, F_{\mathrm{sep}}[X]^\times)$$

is the embedding.

The top map in the previous diagram is given by the pairing

$$\begin{aligned} H^2(X, Q^0) \otimes H^0(X, Q) &\rightarrow F^2 H^4(X, \mathbb{Z}(2)), \\ a \otimes b &\mapsto a \cup b, \end{aligned} \tag{B-8}$$

defined as the cup-product in (4-3),

$$H^3(X, \widehat{Q}(1)) \otimes H^1(X, \widehat{Q}^\circ(1)) \rightarrow H^4(X, \mathbb{Z}(2)),$$

if we identify  $\mathrm{Ext}_X^1(q^*N, \mathbb{Z}(1))$  with  $H^1(X, \widehat{Q}^\circ(1)) = H^0(X, Q)$ .

The inclusion of  $\widehat{Q}_{\mathrm{sep}}$  into  $F_{\mathrm{sep}}[X]^\times$  yields a morphism  $\varepsilon : X \rightarrow Q$  that can be viewed as an element of  $H^0(X, Q)$ . Consider the map

$$\mu : H^2(F, Q^\circ) \rightarrow H^2(F, H^2(X_{\mathrm{sep}}, \mathbb{Z}(2))) \tag{B-9}$$

given by composition with the product homomorphism

$$Q^\circ(F_{\mathrm{sep}}) = F_{\mathrm{sep}}^\times \otimes \widehat{Q}_{\mathrm{sep}} \rightarrow F_{\mathrm{sep}}^\times \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1)) \rightarrow H^2(X_{\mathrm{sep}}, \mathbb{Z}(2)).$$

We have proved:

**Proposition B.2.** *Let  $X$  be a smooth variety over  $F$  such that  $H^3(X_{\mathrm{sep}}, \mathbb{Z}(2)) = 0$ . Let  $N$  be a  $\Gamma$ -lattice in  $F_{\mathrm{sep}}[X]^\times$  such that the composition*

$$N \hookrightarrow F_{\mathrm{sep}}[X]^\times \rightarrow F_{\mathrm{sep}}[X]^\times / F_{\mathrm{sep}}^\times$$

is an isomorphism. Let  $Q$  be the torus over  $F$  satisfying  $\widehat{Q}_{\text{sep}} = N$ . Then for every  $\gamma \in H^2(F, Q^\circ)$ , we have

$$e_4(q^*(\gamma) \cup \varepsilon) = \mu(\gamma),$$

where  $e_4 : F^2 H^4(X, \mathbb{Z}(2)) \rightarrow H^2(F, H^2(X_{\text{sep}}, \mathbb{Z}(2)))$  is the map induced by the Hochschild–Serre spectral sequence (B-3) and the cup-product is taken for the pairing (B-8).

**B-IV: Relative étale cohomology.** Let  $X$  be a smooth variety over  $F$ . Following B. Kahn [1996, §3], we define the relative étale cohomology groups as follows. Recall that  $\Delta(i) = Rq_*(\mathbb{Z}(i))$  for  $i = 1$  and  $2$ , where  $q : X \rightarrow \text{Spec}(F)$  is the structure morphism, and let  $\Delta'(i)$  be the cone of the natural morphism  $\mathbb{Z}(i) \rightarrow \Delta(i)$  in  $D_+(\text{Sheaves}_{\text{ét}}(F))$ . Define

$$H^*(X/F, \mathbb{Z}(2)) := H^*(F, \Delta'(2)).$$

(Note that our indexing is different from that in [Kahn 1996, §3].)

There is an infinite exact sequence

$$\dots \rightarrow H^i(F, \mathbb{Z}(2)) \rightarrow H^i(X, \mathbb{Z}(2)) \rightarrow H^i(X/F, \mathbb{Z}(2)) \rightarrow H^{i+1}(F, \mathbb{Z}(2)) \rightarrow \dots$$

If  $X$  has a rational point, we have

$$H^i(X/F, \mathbb{Z}(2)) = \overline{H}^i(X, \mathbb{Z}(2)) := H^i(X, \mathbb{Z}(2))/H^i(F, \mathbb{Z}(2)).$$

There is a Hochschild–Serre type spectral sequence [Kahn 1996, §3]

$$E_2^{p,q} = H^p(F, H^q(X_{\text{sep}}/F_{\text{sep}}, \mathbb{Z}(2))) \Rightarrow H^{p+q}(X/F, \mathbb{Z}(2)), \tag{B-10}$$

and we have by [Kahn 1996, Lemma 3.1] that

$$H^q(X_{\text{sep}}/F_{\text{sep}}, \mathbb{Z}(2)) = \begin{cases} 0 & \text{if } q \leq 0, \\ \text{uniquely divisible group} & \text{if } q = 1, \\ \overline{H}_{\text{Zar}}^0(X_{\text{sep}}, \mathcal{H}_2) & \text{if } q = 2, \\ H_{\text{Zar}}^1(X_{\text{sep}}, \mathcal{H}_2) & \text{if } q = 3. \end{cases}$$

It follows that  $E_2^{p,q} = 0$  if  $q \leq 1$  and  $p > 0$ . Comparing the spectral sequences (B-3) and (B-10), by Proposition B.1 we have:

**Proposition B.3.** *Let  $X$  be a smooth variety over  $F$  such that  $X(F) \neq \emptyset$ . If  $H_{\text{Zar}}^0(X_{\text{sep}}, \mathcal{H}_2) = K_2(F_{\text{sep}})$ , then the spectral sequence (B-10) yields an exact sequence*

$$\begin{aligned} 0 \rightarrow H^1(F, H_{\text{Zar}}^1(X_{\text{sep}}, \mathcal{H}_2)) &\xrightarrow{\alpha} \overline{H}^4(X, \mathbb{Z}(2)) \\ &\rightarrow \overline{H}^4(X_{\text{sep}}, \mathbb{Z}(2))^\Gamma \rightarrow H^2(F, H_{\text{Zar}}^1(X_{\text{sep}}, \mathcal{H}_2)). \end{aligned}$$

If, moreover, the group  $\text{Pic}(X_{\text{sep}})$  is a lattice and  $T$  is the torus over  $F$  such that  $\widehat{T}_{\text{sep}} = \text{Pic}(X_{\text{sep}})$ , then  $\alpha(\rho(\delta)) = q^*(\delta) \cup [E]$  for every  $\delta \in H^1(F, T^\circ)$ , where  $\rho$  is defined in (B-7) and  $E$  is a universal  $T$ -torsor over  $X$ .

Comparing the spectral sequences (B-3) and (B-10), by Proposition B.2 we have:

**Proposition B.4.** *Let  $X$  be a smooth variety over  $F$  such that  $X(F) \neq \emptyset$ . If  $H_{\text{Zar}}^1(X_{\text{sep}}, \mathcal{K}_2) = 0$ , then the spectral sequence (B-10) yields an exact sequence*

$$\begin{aligned} 0 \rightarrow H^2(F, \overline{H}_{\text{Zar}}^0(X_{\text{sep}}, \mathcal{K}_2)) &\xrightarrow{\beta} \overline{H}^4(X, \mathbb{Z}(2)) \\ &\rightarrow \overline{H}^4(X_{\text{sep}}, \mathbb{Z}(2))^\Gamma \rightarrow H^3(F, \overline{H}_{\text{Zar}}^0(X_{\text{sep}}, \mathcal{K}_2)). \end{aligned}$$

If  $N$  is a  $\Gamma$ -lattice in  $F_{\text{sep}}[X]^\times$  such that the composition

$$N \hookrightarrow F_{\text{sep}}[X]^\times \rightarrow F_{\text{sep}}[X]^\times / F_{\text{sep}}^\times$$

is an isomorphism and  $Q$  is the torus over  $F$  satisfying  $\widehat{Q}_{\text{sep}} = N$ , then  $\beta(\mu(\gamma)) = q^*(\gamma) \cup \varepsilon$  for every  $\gamma \in H^2(F, Q^\circ)$ , where  $\mu$  is defined in (B-9) and  $\varepsilon \in H^0(X, Q)$  is given by the inclusion of  $\widehat{Q}_{\text{sep}}$  into  $F_{\text{sep}}[X]^\times$ .

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