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We study the combinatorics of a subcomplex of the Bloch–Kriz cycle complex that was used to construct the category of mixed Tate motives. The algebraic cycles we consider properly contain the subalgebra of cycles that correspond to multiple logarithms (as defined by Gangl, Goncharov and Levin). We associate an algebra of graphs to our subalgebra of algebraic cycles. We give a purely combinatorial criterion for admissibility. We show that sums of bivalent graphs correspond to coboundary elements of the algebraic cycle complex. Finally, we compute the Hodge realization for an infinite family of algebraic cycles represented by sums of graphs that are not describable in the combinatorial language of Gangl et al.

1. Introduction

Let $\mathcal{M}_T$ denote the category of mixed Tate motives and denote its associated Galois group by G_T. This Galois group has been defined in the literature in at least two distinct contexts, first by [Bloch 1991; Bloch and Kriz 1994] but also by [Levine 1993] in what turned out to be Voevodsky's formalism (see [Deligne and Goncharov 2005], for example). Note that Spitzweck [2001; n.d.] and Levine [2005] have shown that the two definitions are equivalent.

We will take the Bloch–Kriz construction as our definition of M_T and G_T.

Although a significant amount of work has gone into understanding G_T, there is still much that is unknown about Tate motives, even over a number field k. In particular, the connection between G_T and the unipotent completions $\pi^1(\mathbb{P}^1_k - n \text{ points})^{\text{unip}}$ of $\pi^1(\mathbb{P}^1_k - n \text{ points})$ is still of current interest.

For $N \geq 1$, let k_N be the cyclotomic field over $\mathbb{Q}$ generated by an N-th root of unity, and $\mathcal{O}_{k_N}$ its ring of integers. Let $M_{T,N}$ denote the full Tannakian subcategory of M_T generated by the motivic fundamental group of $\mathbb{P}^1_{k_N} - \{0, \infty, \mu_N\}$, with associated motivic Galois group $G_{T,N}$ and algebra of periods $\mathcal{P}_{T,N}$. Here μ_N denotes the set of the N-th roots of unity, though geometrically it could just be a set of N distinct points of $\mathbb{C}^*$. A question, probably going back to Grothendieck, is

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how much of the motivic fundamental group G_T is measured by $G_{T,N}$, in particular $G_{T,1}$. This subcategory, and its integral analogues, were studied by Deligne and Goncharov [2005]. They showed that, over a number field, $\mathcal{P}_{T,N}(\mathbb{O}_{k_N})$ is generated as a $\mathbb{Q}$ vector space by values of multiple polylogarithms. There is a natural categorical inclusion $M_{T,N} \hookrightarrow M_T$, which induces surjections $\phi_N : G_T(\mathbb{O}_{k_N}) \twoheadrightarrow G_{T,N}(\mathbb{O}_{k_N})$ (equivalently an injection $\mathcal{P}_{T,N} \hookrightarrow \mathcal{P}_T$). Brown [2012], in the case $N = 1$, and Deligne [2010], in the cases $N \in \{2, 3, 4, 6, 8\}$, showed that ϕ was an isomorphism. Conversely, and more interestingly, Goncharov [2001a] showed that for most N, ϕ has a nontrivial kernel. Little is known about this kernel. Even less is known about this kernel if the ground field is a cyclotomic extension of a general number field (as opposed to a cyclotomic extension of $\mathbb{Q}$). In particular, all known constructions of elements of M_T lie in the subcategory $M_{T,N}$.

What is sorely needed is an approach to construct more general elements of M_T, especially ones that do not come from the motivic fundamental groups of $\mathbb{G}_m - \mu_N$. This paper is motivated in part by the desire to find a suitable framework to study this kernel. We do not claim to have found such a framework here, but are hopeful that we have taken a first step in the right direction.

The Bloch–Kriz definition of M_T relies heavily on the theory of algebraic cycles. While general enough to capture all mixed Tate motives, traditional methods of representing algebraic cycles (such as in terms of formal linear combinations of systems of polynomial equations) are notoriously difficult to work with, so progress in capitalizing on this description of the category to illuminate outstanding conjectures in the field has been slow. Gangl, Goncharov and Levin [Gangl et al. 2009] suggest a simpler way to understand a subcategory of $\mathcal{M}_T$ by relating specific algebraic cycles to rooted, decorated, binary trees. This approach necessarily restricts focus to motives generated by the motivic fundamental groups of $\mathbb{G}_m - \mu_N$. Any attempt to study the kernel of ϕ defined above requires a more general framework.

Soudères [2016a; 2016b] extends the family of algebraic cycles studied by Gangl et al. to include those over a more general base scheme, in particular giving a rigorous construction of unital values of the multiple polylogarithms, i.e., multiple zeta values, as periods (and not just nonunital values of the multiple logarithms). The combinatorial properties of these algebraic cycles, however, are as yet unexplored.

Let $\mathcal{A}$ be the differential graded algebra (DGA) of cycles introduced by Bloch and Kriz [1994]. In this paper we generalize the Gangl–Goncharov–Levin construction as follows: We define a subalgebra of cycles, $\mathcal{A}^{\times}_{1L} \subset \mathcal{A}$, that properly contains the subalgebra associated to multiple logarithms studied in [Gangl et al. 2009], and reinterprets $\mathcal{A}^{\times}_{1L}$ in terms of graphs. By considering graphs, as opposed to trees, and by loosening the valence restriction on the vertices, we enrich the tools available to study algebraic cycles. Therefore, we are able to consider a larger subcomplex of cycles. We hope this will lead to a better understanding of the complexity and richness underlying the Bloch–Kriz cycle complex, even in the restricted subclass we consider. In particular, in Section 4, we describe several examples of classes of algebraic cycles that define motives. Most of these cycles cannot be described by the trees presented in [Gangl et al. 2009]. In Section 5, we compute the Hodge realization of an infinite family of such classes. Furthermore, in Section 3, we present a purely graphical interpretation of admissibility for the family of algebraic cycles we consider. We also give valency requirements for which classes of algebraic cycles will always be coboundaries in $H^0(B(\mathcal{G}_{1L}))$. There is a lot of interesting combinatorial structure in the types of underlying graphs — and their linear combinations — that give rise to allowable classes of motives. We have barely begun to explore this structure and feel strongly that it deserves further study.

The plan for the paper is as follows. In Section 2, we review mixed Tate motives à la [Bloch and Kriz 1994] and introduce the subalgebra, $\mathscr{A}^{\times}_{1L}$, of $\mathbb{P}^1_k$-linear parametrizable cycles of the algebra $\mathscr{A}$ of admissible cycles. This subalgebra is the focus of our attention. We then define a subcomplex $B(\mathscr{A}^{\times}_{1L})$ of the bar construction on admissible cycles, $B(\mathscr{A})$. The category of comodules over $H^0(B(\mathscr{A}^{\times}_{1L}))$ is the (sub)category of motives we wish to study.

Section 3 introduces an algebra of graphs, $\mathcal{G}_{1L}$, that corresponds to the algebra $\mathscr{A}^{\times}_{1L}$. Theorem 3.63 shows that the two algebras are isomorphic as DGAs. Since $\mathscr{A}^{\times}_{1L}$ is a subalgebra of $\mathscr{A}$, this implies that there is an injection from the algebra of graphs developed in this paper to the full Bloch Kriz cycle complex. In the process, we show, in Theorem 3.59, that the conditions for an arbitrary irreducible $\mathbb{P}^1_k$-linear cycle to be admissible, that is, a generator of $\mathscr{A}^{\times}_{1L}$, can be defined and computed completely from the graphical properties of the corresponding graph in $\mathcal{G}_{1L}$.

In Section 4, we give examples of classes in and results about $H^0(B(\mathcal{G}_{1L}))$. In addition we show, in Corollary 4.14, that in any completely decomposable (sum of) graphs either each summand has a valence-two vertex, or none do. We further show, in Theorem 4.16, that if a completely decomposable (sum of) graphs has valence-two vertices, it is a coboundary in $B(\mathcal{G}_{1L})$.

In Section 5, following the algorithm as outlined in [Bloch and Kriz 1994; Gangl et al. 2009] and especially [Kimura 2013], we compute the Hodge realization of a projective system of classes whose defining cycles are not describable by trees. (All previously known explicit computations of the Bloch–Kriz Hodge realization have been of cycles that can be described by trees.)

2. A subcomplex of algebraic cycles

In this section, we define a particular subcomplex of the Bloch–Kriz cycle complex that we develop in this paper. We begin with a review of the general mixed Tate motive construction via algebraic cycles. Then we proceed to describe parametrized cycles, and finally define the subcomplex of $\mathbb{P}_{1L}$-cycles that we use in the remainder of this paper.

2A. ***A review of mixed Tate motives.*** We work with the category of mixed Tate motives over a field k, $\mathcal{M}(T)$, as constructed by [Bloch 1991; Bloch and Kriz 1994]. When k is a number field, this construction does not depend on any conjectures. In [Bloch and Kriz 1994], two conjectures are stated: that $\mathrm{gr}_r K_n(F) \otimes \mathbb{Q} \cong \mathrm{CH}^r(\mathrm{Spec}(F), n) \otimes \mathbb{Q}$, and that a certain algebra is quasiisomorphic to its Sullivan 1-model. The first conjecture was subsequently proved more generally for all varieties X independently by Bloch [1994; 1986], Levine [1994] and Spivakowsky (unpublished). The second conjecture, which is a strengthening of the Beilinson–Soulé conjecture for fields, is known for number fields by the work of Borel and Yang [1994] on the rank conjecture. (The Beilinson–Soulé conjecture was already known to be true for number fields by the work of Borel [1974]).

In the rest of this section we review some details of their construction, following [Bloch and Kriz 1994] closely.

We assume the reader is familiar with the concepts of algebraic cycles, higher Chow groups, minimal models, 1-minimal models and the bar construction for a commutative differential graded algebra (DGA) A. For the reader who wishes to refresh her memory: The concept of a generalized minimal model is due originally to Quillen (see [Quillen 1970], for example). In the form used here (extensions by free one-dimensional models) it is due originally to Sullivan [1977, discussion starting p. 316]. A good reference for the applications of minimal models we have in mind is the treatment in [Kriz and May 1995, Part IV]. The bar construction is due originally to Eilenberg and Mac Lane. Good references for the use of the bar construction in this paper are [Chen 1976; Bloch and Kriz 1994, Section 2].

In order to define the category of mixed Tate motives, $\mathcal{M}(T)$, it suffices to define its motivic Galois group G_T [Bloch and Kriz 1994]. Equivalently, one may work with its dual Hopf algebra, $\mathcal{H}_T$. This is defined from the DGA, $\mathcal{A}$, of admissible algebraic cycles.

Below, following [loc. cit.], we define how to derive a Hopf algebra from a commutative graded DGA, A, which is cohomologically connected. That is, $H^0(A) = \mathbb{Q}$ and $H^{-n}(A) = 0$ for $n > 0$. Our DGA, A, is not a Hopf algebra in general, as the differential does not decompose. The strategy, therefore, is to "linearize" A, i.e., form the minimal model $\mathcal{H}(A)$ of A, which, by construction, is a Hopf algebra which is quasiisomorphic to A. The minimal model can be constructed quite explicitly via the bar construction. We start with a few definitions.

Definition 2.1. (1) Consider the commutative DGA, $A = \bigoplus_i A_i$. Here, we refer to the grading on A by degree: $\deg(a) = i \Leftrightarrow a \in A_i$. The tensor algebra, $T(A) = \bigoplus_n A^{\otimes n}$, is a commutative algebra under the shuffle product, Ш.

(2) Let $D(A)$ be the ideal in $T(A)$ of degenerate tensor products, defined by

$$\{a_1 \otimes \cdots \otimes a_n \mid a_i \in A,\ a_j \in k \text{ for some } j\}.$$

(3) The bar construction on A is defined as

$$B(A) = T(A)/D(A).$$

It is a bigraded algebra, with grading given by tensor degree and algebraic degree. The total degree of a monomial $a_1 \otimes \cdots \otimes a_n \in B(A)$ is defined by a shift in the degree of the tensor components in A. That is,

$$\operatorname{tot}\deg(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n} (\deg(a_i) - 1).$$

Hence, the total degree of an element of the bar construction is the difference between the algebraic degree and the tensor degree. Write the bar construction as $B(A) = \bigoplus_{i,j} B(A)^i_j$, where

$$B(A)^i_j = \bigoplus_{\sum_1^i (j_k - 1) = j} A_{j_1} \otimes \cdots \otimes A_{j_i}$$

has total degree j.

Since A is a DGA, it is endowed with a differential structure $\partial : A \to A$ and a product structure $\mu : A \otimes A \to A$. These both extend to define differential structures on the bar construction $B(A)$, called the algebraic and multiplicative differentials, respectively. Thus $(B(A), \partial + \mu)$ is the following bicomplex:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r, "\mu"] & B(A)^3_0 \arrow[u, "d"] \arrow[r, "\mu"] & B(A)^2_1 \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^1_2 \arrow[u, "\partial"] \arrow[r, "\varepsilon"] & 0 \\
\cdots \arrow[r, "\mu"] & B(A)^3_{-1} \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^2_0 \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^1_1 \arrow[u, "\partial"] \arrow[r, "\varepsilon"] & 0 \\
\cdots \arrow[r, "\mu"] & B(A)^3_{-2} \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^2_{-1} \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^1_0 \arrow[u, "\partial"] \arrow[r, "\varepsilon"] & \mathbb{Q} \\
\cdots \arrow[r, "\mu"] & B(A)^3_{-3} \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^2_{-2} \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^1_{-1} \arrow[u, "\partial"] \arrow[r, "\varepsilon"] & 0 \\
 & \vdots \arrow[u, "\partial"] & \vdots \arrow[u, "\partial"] & \vdots \arrow[u, "\partial"] &
\end{tikzcd} \tag{2.2}$$

Further details and calculations involving the bar complex can be found in Section 4.

When A is connected, cohomologically connected and generated in degree one (a $K(\pi, 1)$ in the sense of Sullivan), then its minimal model is isomorphic to $\mathcal{H}(A) := H^0(B(A))$, where the cohomology is taken under the total derivative $\partial + \mu$. Note that $B(A)$ is a Hopf algebra, with a product structure given by the signed shuffle product and a coproduct structure given by deconcatenation, which satisfy all the axioms for a Hopf algebra. Note that while the product introduces a degree-dependent sign fact, the coproduct has no such sign. This induces a well-defined product, coproduct, and Hopf algebra structure on $\mathcal{H}(A)$.

Bloch and Kriz study a bar construction of a DGA of admissible cycles, $\mathcal{A} = \bigoplus_i \mathcal{A}_i$, defined below. The Hopf algebra H_T dual to the motivic Galois group G_T is exactly the Hopf algebra defined above for the algebra of admissible cycles.

Definition 2.3. (1) Denote $\mathbb{P}^1_k \setminus \{1\}$ by $\square$. Then we may write $\square^n = (\mathbb{P}^1_k \setminus \{1\})^n$. The boundary of this space is defined when one of the coordinates is set to 0 or ∞.

(2) For $I, J \subset \{1, \dots, n\}$ two disjoint subsets, write $F_{I,J}$ to indicate the codimension-$|I \cup J|$ face of $\square^n$ with the coordinates in I set to 0 and the coordinates in J set to ∞. Write $F_{\varnothing,\varnothing} = \square^n$ to indicate the entire space.

(3) As usual, let $\mathfrak{Z}^p(\square^n)$ be the free abelian group generated by algebraic cycles of codimension p in $\square^n$. These are the elements of weight p.

(4) Write $\mathfrak{Z}^p(\operatorname{Spec} k, n) \subset \mathfrak{Z}^p(\square^n)$ for the free abelian subgroup generated by *admissible* algebraic cycles. A cycle $\mathfrak{Z} \in \mathfrak{Z}^p(\operatorname{Spec} k, n)$ is one that intersects each face $F_{I,J}$ of $\square^n$ in codimension p, or not at all.

(5) Let Alt be the alternating projection with respect to the action of the group $\mathfrak{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ on $\mathfrak{Z}^p(\operatorname{Spec} k, n)$. Here the symmetric group $\mathfrak{S}_n$ acts by permutation of coordinates, and the i-th copy of $(\mathbb{Z}/2\mathbb{Z})^n$ acts by taking a coordinate to its multiplicative inverse.

(6) Write
$$\mathcal{A}^n_i = \operatorname{Alt}(\mathfrak{Z}^n(\operatorname{Spec} k, 2n - i) \otimes \mathbb{Q}),$$
where i is the degree of the algebraic cycle and n the codimension. This is a bigraded algebra, by weight and degree. The weight-graded pieces, $\mathcal{A}^n := \bigoplus_i \mathcal{A}^n_i = \bigoplus_i \operatorname{Alt}(\mathfrak{Z}^n(\operatorname{Spec} k, 2n-i) \otimes \mathbb{Q})$, define a complex by the differential operator defined in (2.6). Each degree-graded piece is $\mathcal{A}_i := \bigoplus_n \mathcal{A}^n_i = \bigoplus_n \operatorname{Alt}(\mathfrak{Z}^n(\operatorname{Spec} k, 2n-i) \otimes \mathbb{Q})$.

Remark 2.4. The main result of Section 3C is to identify which cycles are elements in $\mathcal{A}$. In order to determine *which* algebraic cycles are admissible, we must consider the space of *all* algebraic cycles, including those that are not admissible. Therefore, when we write $\mathfrak{Z}^p(\square^n)$, we mean the entire space of algebraic cycles. We denote admissible cycles by the notation $\mathfrak{Z}^p(\operatorname{Spec} k, n)$.

We now define the DGA structure of $\mathscr{A}$. Consider two admissible cycles,

$$Z_i \in \mathscr{Z}^n(\operatorname{Spec} k, 2n-i) \quad \text{and} \quad Z_j \in \mathscr{Z}^m(\operatorname{Spec} k, 2m-j).$$

Write $(Z_i, Z_j) \in \mathscr{Z}^{n+m}(\operatorname{Spec} k, 2(n+m)-(i+j))$ to indicate the admissible cycle defined by Z_i on the first $2n-i$ coordinates and Z_j on the last $2m-j$ coordinates. The product on the associated elements in $\mathscr{A}$ is given by

$$\mu(\operatorname{Alt}\mathscr{Z}_i \otimes \operatorname{Alt}\mathscr{Z}_j) = \operatorname{Alt}(\mathscr{Z}_i, \mathscr{Z}_j) = (-1)^{ij} \operatorname{Alt}(\mathscr{Z}_j, \mathscr{Z}_i),$$

where we drop the $\otimes\mathbb{Q}$ notation for simplicity. The last inequality comes from the properties of Alt, and defines a graded commutative structure on $\mathscr{A}$.

Definition 2.5. An element $Z \in \mathscr{A}$ is decomposable if it can be expressed as the product of two nontrivial cycles.

Next, we define the differential structure on $\mathscr{A}$. Consider $Z \in \mathscr{A}$. Let $\partial_{j,0}\mathscr{Z}$ indicate the intersection of $\mathscr{Z}$ with the face $F_{j,\varnothing}$. Similarly, let $\partial_{j,\infty}\mathscr{Z}$ indicate the intersection of $\mathscr{Z}$ with the face $F_{\varnothing,j}$. These two operators define the differential ∂ on $\mathscr{A}$:

$$\partial\mathscr{Z} = \sum_{j=1}^{2n-i} (-1)^{j-1}(\partial_{j,0} - \partial_{j,\infty})\mathscr{Z}. \tag{2.6}$$

Remark 2.7. It is difficult to identify elements of $\mathscr{A}$, that is, to classify the cycles that satisfy the condition of admissibility. One of the achievements of this paper is to give a clear, simple condition for identifying admissible cycles for a large subclass of cycles, called $\mathbb{P}^1_k$-linear cycles. In particular, see Theorem 3.59.

For an element $\varepsilon \in \bigoplus_n B(\mathscr{A})^n_i$ to define a class in $H^i(B(\mathscr{A}))$, each graded component must have decomposable algebraic boundary. This comes from the fact that $(\partial + \mu)(\varepsilon) = 0$. In order to define what it means for a cycle to have decomposable boundary, let π_m be the projection of ε onto the m-th tensor component. That is, $\pi_m(\varepsilon) \in B(\mathscr{A})^m_i$. Then, for each m, $\partial(\pi_m\varepsilon)$ is a decomposable element.

Definition 2.8. Consider an element $\varepsilon \in B(\mathscr{A})$.

(1) The projection, $\pi_i(\varepsilon) \in B(\mathscr{A})^n_i$, is decomposable if it has a decomposable algebraic boundary. That is, if there exists an $\varepsilon' \in B(\mathscr{A})^{n+1}_{i+1}$ such that $\partial(\pi_i(\varepsilon)) = -\mu(\varepsilon')$. That is, the coboundary of the projection $\pi_i(\varepsilon)$ is in the image of the product map μ.

(2) An element $\varepsilon \in B(\mathscr{A})$ is completely decomposable if $\pi_i(\varepsilon)$ is decomposable for all i, with

$$\partial\pi_i(\varepsilon) = -\mu\pi_{i+1}(\varepsilon).$$

Definition 2.9. We say that the element $\varepsilon \in \bigoplus_n B(\mathscr{A})_i^n$ is minimally decomposable if it is completely decomposable, and cannot be written as a sum of two nontrivial completely decomposable elements. That is, one cannot write $\varepsilon = \varepsilon_1 + \varepsilon_2$, where each $\varepsilon_i \neq 0$ and is completely decomposable.

Remark 2.10. Notice that if ε is minimally decomposable, it is determined (up to shuffle products) by $\pi_{n_0}(\epsilon)$, where n_0 is the smallest integer for which $\pi_n(\varepsilon) \neq 0$. Therefore, by abuse of notation, we say that $\pi_{n_0}(\varepsilon)$ defines a class in $H^i(B(\mathscr{A}))$. In all examples in this paper, $n_0 = 1$.

Next we give an example of an admissible cycle that defines a class in $H^0(B(\mathscr{A}))$.

Example 2.11. Consider the cycle $Z_T(a) = \mathrm{Alt}(t, 1-t, 1-a/t) \in \mathscr{A}_1^2$. This is a parametric representation of the algebraic cycle determined by the system of equations $\{x + y = 1,\ xz = x + a\}$. This is the Torato cycle [1992] with codimension 2 in $\square^3$. It is a degree-one element in $\mathscr{A}$, $Z_T(a) \in \mathscr{A}_1^2$,

We check that $Z_T(a)$ has a completely decomposable boundary. Therefore, it defines a class in $H^0(B(\mathscr{A}))$. To see this, compute $\partial Z_T(a)$. The intersections $\partial_{\infty,i} Z_T(a)$ give the empty cycles for $i \in \{1, 2, 3\}$. This is because setting one of the coordinates of $Z_T(a)$ to ∞ sets a different coordinate to 1. The same holds for $\partial_{0,1} Z_T(a)$ and $\partial_{0,2} Z_T(a)$. Therefore,

$$\partial Z_T(a) = \partial_{0,3} Z_T(a) = \mathrm{Alt}(a, 1-a) = \mu[\mathrm{Alt}(a) \mid \mathrm{Alt}(1-a)].$$

The last equality comes from the product structure on $\mathscr{A}$. Since (a) and $(1-a)$ are constant cycles, $\partial[\mathrm{Alt}(a) \mid \mathrm{Alt}(1-a)] = 0$ by the Leibnitz rule. Therefore, $Z_T(a) \oplus -[\mathrm{Alt}(a) \mid \mathrm{Alt}(1-a)] \in \ker(\partial + \mu)$. Since $Z_T(a)$ has total degree 0 in $B(\mathscr{A})$, it defines a class in $H^0(B(\mathscr{A}))$.

The Hodge realization functor associates the period $\mathrm{Li}_2(a)$ to the cycle $Z_T(a)$ [Bloch and Kriz 1994]. To do this, consider the $\mathscr{A}$ module, $\mathscr{T}$, defined by maps from n-simplices, Δ_n, to $\square^n$. There is an element $\zeta(a)$ in the circular bar construction $B(\mathscr{T}, \mathscr{A})$ such that $\zeta(a) + 1 \otimes Z_T(a)$ defines a class in $H^0(B(\mathscr{T}, \mathscr{A}))$. The summands of $\zeta(a)$ that are supported completely on Δ_2 define the integrand of the associated period.

This example hints at another shortcoming of the current state of technology surrounding algebraic cycles. We are interested in defining elements of $B(\mathscr{A})$ that define classes of $H^0(B(\mathscr{A}))$. In particular, we are interested in cycles with boundaries that can be written as products of other cycles, as is the case for the Torato cycle in Example 2.11. In Section 4B1, we provide several examples of such sums of cycles in weight 4. However, we have not yet addressed this issue of how to find such sums in general. We hope that the graphical point of view presented here will shed light on the problem of identifying cycles with completely decomposable boundaries. We leave this for future work.

2B. ***A subalgebra of $\mathscr{A}$.*** Unfortunately, the standard parametric notation for cycles as represented in [Bloch and Kriz 1994; Gangl et al. 2007; 2009; Totaro 1992] is rather misleading. For example, consider the usual form for the Totaro cycle, $Z_T(a) = \mathrm{Alt}(t, 1-t, 1-a/t) \in \mathscr{A}_1^2$, defined in Example 2.11, and in the literature [Totaro 1992; Gangl et al. 2009]. It is technically defined on $\square_k^3 = (\mathbb{P}_k^1 - \{1\})^3$, but is written as if it is defined on $\mathbb{A}_k^3 = (\mathbb{P}_k^1 - \{\infty\})^3$. In actuality, the Totaro cycle (for $a \in k^*$) is an algebraic cycle defined by the system of equations

$$\left\{x + y = 1,\ xz = x - a : (x, y, z) \in (\mathbb{P}_k^1 - \{1\})^3\right\}$$

together with a parametrization map $\mathbb{P}_k^1 \to (\mathbb{P}_k^1 - \{1\})^3$. However, when manipulated in practice, the cycle is understood

- to come equipped with a parametrization map, and
- to be defined at the hyperplanes with one coordinate equal to ∞, and not defined at the hyperplanes with one coordinate equal to 1.

This is unnecessarily obtuse. It can be described as the intersection of the image of

$$\mathbb{P}_k^1 \to (\mathbb{P}_k^1)^3, \quad (T : U) \mapsto \left(\frac{T}{U}, \frac{U-T}{U}, \frac{T-aU}{T}\right),$$

with the complement of the hyperplanes of $(\mathbb{P}_k^1)^3$ defined by setting some coordinate equal to 1.

In light of this example, we work with parametrized cycles.

Definition 2.12. A parametrized cycle is a pair, (Z, ϕ), consisting of an algebraic cycle $Z \in \mathscr{Z}^p(\square^n)$ and a parametrization $\phi : \mathbb{P}_k^{n-p} \to (\mathbb{P}_k^1)^n$ satisfying the following: ϕ induces a map on the group of algebraic cycles,

$$\phi_* : \mathscr{Z}^0(\mathbb{P}_k^{n-p}) \to \mathscr{Z}^p((\mathbb{P}_k^1)^n).$$

Then, given the inclusion $i : \square^n \hookrightarrow (\mathbb{P}_k^1)^n$, we have

$$Z = i^* \phi_*(\mathbb{P}_k^{n-p}),$$

where $\mathbb{P}_k^{n-p}$ is the generator of $\mathscr{Z}^0(\mathbb{P}_k^{n-p})$.

For $Z \in \mathscr{Z}^p(\square^n)$, write the parametrizing map $\phi = (\phi_1, \ldots, \phi_n)$, where each ϕ_i corresponds to the image in a coordinate of $\square^n$. There are, of course, multiple possible parametrizations of any cycle $Z \in \mathscr{Z}^p(\square^n)$. Here we are interested in the algebraic cycles themselves, not the particular parametrizations. If the same cycle Z can be represented by two different parametrizations, (Z, ϕ) and (Z, ϕ'), we say that ϕ and ϕ' are equivalent parametrizations. We are interested in cycles that *can* be endowed with a $\mathbb{P}_k^1$-linear parametrization.

Definition 2.13. A cycle $Z \in \mathcal{Z}^p(\square^n)$ is $\mathbb{P}^1_k$-linear if it can be parametrized by a ϕ such that each component can be written as

$$\phi_j \in \left\{ \left(1 - \frac{t_1}{a_j t_2}\right)^\zeta, \left(1 - \frac{t_2}{a_j t_1}\right)^\zeta, \left(\frac{t_1}{a_j t_2}\right)^\zeta \right\},$$

with $a_j \in k^\times$ and $\zeta \in \{\pm 1\}$. In particular, writing the j-th $\mathbb{P}^1_k$ in the image of ϕ as $[U_j : V_j]$, we define $\phi_j = U_j/V_j$, using the standard affine representation. Such a ϕ is called a $\mathbb{P}^1_k$-linear parametrization, and can be written as a map on $\mathbb{P}^1_k$ via the following commutative diagram:

$$\begin{tikzcd} \mathbb{P}^1_k \arrow[r] \arrow[d, "\phi_j"] & \mathbb{P}^{n-p}_k \arrow[d, "\phi"] \\ \mathbb{P}^1_k \arrow[r, hook, "i_j"'] & (\mathbb{P}^1_k)^n \end{tikzcd}$$

The top arrow is given by a map

$$(t_1 : t_2) \mapsto (0 : \cdots : 0 : t_1 : 0 : \cdots : 0 : t_2 : 0 : \cdots : 0),$$

and the bottom arrow is given by inclusion into the j-th coordinate.

Definition 2.14. Denote the free abelian groups of $\mathbb{P}^1_k$-linear cycles by $\mathcal{Z}^n_{1L}(\square^m)$. Write $\mathcal{Z}_{1L}{}^n(\operatorname{Spec} k, m)$ for the free abelian group of $\mathbb{P}^1_k$-linear *admissible* cycles.

The goal of this section is to define a sub-DGA of $\mathcal{A}$, the algebra of admissible cycles, that is generated by $\mathcal{Z}_{1L}{}^n(\operatorname{Spec} k, 2n-i)$. Call it

$$\mathcal{A}_{1L} = \bigoplus_i \mathcal{A}_{1L,i} = \bigoplus_{n,i} \operatorname{Alt} \mathcal{Z}_{1L}{}^n(\operatorname{Spec} k, 2n-i) \otimes \mathbb{Q}.$$

The graded commutative structure on $\mathcal{A}_{1L}$ comes from the product structure on $\mathcal{A}$, along with the fact that the product of two parametrizable cycles is still parametrizable. It remains to check that the differential structure on $\mathcal{A}$ is well-defined on $\mathcal{A}_{1L}$. The differential on $\mathcal{A}$ comes from intersecting each coordinate of an element $\operatorname{Alt} Z \in \mathcal{A}^n_i$ with the appropriate 0 and ∞ face of $\square^{2n-i}_k$. Consider $Z \in \mathcal{Z}_{1L}{}^n(\operatorname{Spec} k, 2n-i)$. Let ϕ be a parametrization on Z. Then the intersection of Z with a particular face corresponds to the pullback of ϕ by the appropriate face map. Therefore, the differential of Z is also a $\mathbb{P}^1_k$-linear parametrizable cycle.

If $\operatorname{Alt} Z \in \mathcal{A}_{1L}$ is a decomposable cycle of codimension i, write

$$\operatorname{Alt} Z = \operatorname{Alt}(Z_1, \ldots, Z_r)$$

as above. The Leibniz rule and properties of Alt show that $\partial \operatorname{Alt}(Z_1, \ldots, Z_r)$ is also parametrizable.

The algebra $\mathcal{A}_{1L}$ contains all the Totaro cycles. Moreover, it contain a large class of cycles which correspond to the multiple logarithms [Gangl et al. 2009]. Therefore, conjecturally, it contains all the cycles necessary to define the full category of mixed Tate motives. There has been some effort to understand subalgebras of $\mathcal{A}_{1L}$ in terms of polylogarithms and multiple logarithms [Gangl et al. 2009; Soudères 2016a]. Here we study a subalgebra $\mathcal{A}^{\times}_{1L} \subset \mathcal{A}_{1L}$ that specifically excludes the Totaro cycles, but still contains the multiple logarithms.

Definition 2.15. Let $\mathcal{A}^{\times}_{1L}$ be the algebra of $\mathbb{P}^1_k$-linear cycles, where

$$\phi_i \in \left\{ \left(1 - \frac{t_1}{a_i t_2}\right)^{\zeta}, \left(1 - \frac{t_2}{a_i t_1}\right)^{\zeta} \right\},$$

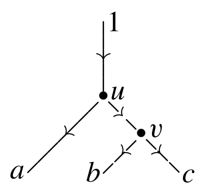
with $a_i \in k^{\times}$ and $\zeta \in \{\pm 1\}$.

The combinatorics of the cycles in $\mathcal{A}^{\times}_{1L}$ are studied in Section 3. The graphs introduced in Section 3 correspond to the subalgebra $\mathcal{A}^{\times}_{1L}$, which excludes cycles with coordinates of the form $a_i t_i / t_j$.

3. Motivic graphs

The first graphical description of some of the algebraic cycles that arise in the category $\mathcal{M}(T)$ of mixed Tate motives was given by [Gangl et al. 2007; 2009] in their description of R-deco trees. These provide a description of a particular proper sub-DGA of $\mathcal{A}^{\times}_{1L}$.

In particular, they represent a subalgebra of cycles by labeled oriented trees. For example,

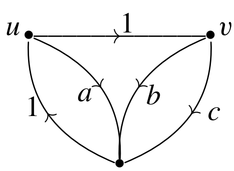


$$\mapsto \left[1 - \frac{1}{u}, 1 - \frac{u}{a}, 1 - \frac{u}{v}, 1 - \frac{v}{b}, 1 - \frac{v}{c}\right].$$

Note that this assignment depends on several choices, such as a choice of root vertex as well as a choice of affine patch.

In this section we give a more general graphical depiction that encapsulates all $\mathcal{A}^{\times}_{1L}$ cycles using decorated, oriented, non-simply connected graphs.

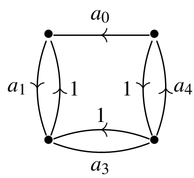
For example, the tree and cycle above come from the labeled oriented graph



$$\mapsto \left[1 - \frac{z}{u}, 1 - \frac{u}{az}, 1 - \frac{u}{v}, 1 - \frac{v}{bz}, 1 - \frac{v}{cz}\right],$$

by taking the affine patch at $z=1$. Graphically this amounts to removing the vertex labeled z and changing the labels from the edges of the graph to the leaves and root of the tree.

Our approach produces far more algebraic cycles that are not seen via the approach given in [Gangl et al. 2007; 2009]. In particular, we can study cycles represented by graphs that cannot be represented by a tree in any affine patch. For example, the graph



in this paper corresponds to the algebraic cycle

$$\mathrm{Alt}\Big[1-\frac{z}{x},1-\frac{x}{a_1 z},1-\frac{w}{z},1-\frac{z}{a_2 w},1-\frac{y}{w},1-\frac{w}{a_3 y},1-\frac{y}{a_0 x}\Big].$$

Yet there is no affine patch one can take (i.e., a vertex one can remove) that will result in a tree of the form studied in [Gangl et al. 2009].

The aim of Section 3 is to construct an algebra of graphs, $\mathcal{G}_{1L}=\bigoplus_{\bullet,\star}\mathcal{G}_{1L}{}^{\bullet}_{\star}$ that is isomorphic to the algebra of admissible cycles $\mathcal{A}^{\times}_{1L}$ as DGAs. The definition of this algebra is given at the end of Section 3D. Most of Sections 3A–3D are devoted to building up $\mathcal{G}_{1L}$ step by step. We begin with a general set of oriented graphs with labeled and ordered edges, $\mathcal{G}(k^\times)$. This corresponds to the set of generators of the free abelian group $\mathcal{Z}^{\bullet}_{1L}(\square^{2\bullet-\star})$. We define a monoid structure on the set so that $\mathcal{G}(k^\times)$ generates an algebra, $\mathbb{Q}[\mathcal{G}]$. Then we consider the alternating representation on the graphs, by imposing an equivalence relation on them by the ordering of their edges. This gives an algebra homomorphism from $\mathbb{Q}[\mathcal{G}]^{\bullet}_{\star}/{\sim_{\mathrm{ord}}}$ to the algebra of cycles $\mathrm{Alt}\,\mathcal{Z}^{\star}_{1L}(\square^{2\bullet-\star})$.

However, we wish for a DGA homomorphism to the algebra of admissible, $\mathbb{P}^1_k$-linear cycles, $\mathcal{A}^{\times}_{1L}\subset\mathcal{Z}^{\star}(\mathrm{Spec}\,k,2\bullet-\star)$. To do this, we define a subset of $\mathcal{G}_{\mathrm{ad}}(k^\times)\subset\mathcal{G}(k^\times)$, which we show corresponds to admissible graphs in Theorem 3.59. We write $\mathbb{Q}[\mathcal{G}_{\mathrm{ad}}]$ to indicate the algebra generated by $\mathcal{G}_{\mathrm{ad}}(k^\times)$. In order to establish a DGA isomorphism between $\mathcal{A}^{\times}_{1L}$ and $\mathbb{Q}[\mathcal{G}_{\mathrm{ad}}]$, we must define a differential operator on graphs. To do this, we need two further equivalence relations among graphs, which we call $\sim_{\mathrm{v}}$ and $\sim_{\mathrm{ori}}$. In Section 3C, we show that $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}},\sim_{\mathrm{v}})$ is a DGA of graphs.

In Section 3D, we show one of the main findings of this paper, that *admissibility of $\mathbb{P}^1_k$-linear cycles can be encoded purely by labeled oriented graphs.* In particular, there is no further algebraic input necessary. Imposing the third equivalence relation gives the desired isomorphism

$$\mathcal{G}_{1L}=\mathbb{Q}[\mathcal{G}_{\mathrm{ad}}]/(\sim_{\mathrm{ord}},\sim_{\mathrm{v}},\sim_{\mathrm{ori}})\simeq\mathcal{A}^{\times}_{1L}.$$

3A. ***An interesting algebra of graphs.*** In this section, we introduce a general set of biconnected graphs with oriented, labeled, and ordered edges. We impose a product structure on it. This defines an algebra of graphs that corresponds to the algebra of general (not necessarily admissible) algebraic cycles.

We work over a number field k.

Definition 3.1. Let $\mathcal{G}(k^\times)$ be the set of graphs with biconnected connected components, with oriented and ordered edges, each labeled by an element of $k^\times \times \mathbb{Z}/2\mathbb{Z}$.

In practice, we say that the edges of G are labeled by a nonzero number and a sign.

For a graph $G \in \mathcal{G}(k^\times)$, let $V(G)$ be the set of vertices of G, and $E(G)$ be the unordered set of edges of the graph. However, we are working with graphs with ordered edges. Therefore we must consider the ordered set of edges.

Definition 3.2. Let $\omega(G)$ be the ordered set of edges of G, where $\omega(e)$ expresses the ordinality of the edge $e \in E(G)$ in $\omega(G)$. Write $\mathrm{sgn}_{\omega(e)}$ to indicate the sign associated the edge e.

The loop number, or first Betti number, of a graph $G \in \mathcal{G}(k^\times)$ is

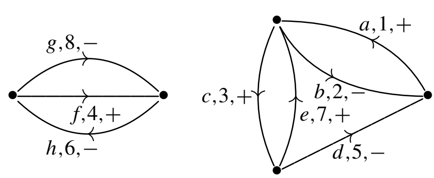
$$h^1(G) = |E(G)| - |V(G)| + h^0(G), \tag{3.3}$$

where $h^0(G)$ counts the number of connected components of the graph. The vector space $H^1(G)$ is spanned by graphical cycles of the unoriented graph underlying G.

Remark 3.4. There are multiple conventions regarding the definition of cycles in graphs in the literature. We take $L \subset E(G)$, together with an orientation (possibly different from the orientation on the individual edges in $E(L)$) is a graphical cycle of the graph G if it defines a path in G that starts and ends at the same vertex. Specifically, the path in G defined by the edges of L does not need to respect the orientation of the edges in L. A graphical *loop* is a graphical cycle that does not intersect itself until the final vertex.

We will concern ourselves only with graphical loops of $G \in \mathcal{G}(k^\times)$.

Example 3.5. Consider the disconnected graph G given by

$G =$ 

These are in $\mathcal{G}(k^\times)$, assuming $a, \ldots, g$ are all in $k^\times$. The second labels indicate the ordering of the edges; the final label give the signs.

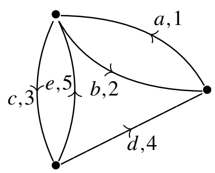
We impose a product structure on the set $\mathcal{G}(k^\times)$. For $G, G' \in \mathcal{G}(k^\times)$, let $G \sqcup G'$ be the disjoint union of the graphs, without an overall ordering imposed on the union of the edges. The product of two graphs $G \cdot G'$ is the graph $G \sqcup G'$, with the edges of G appearing before the edges of G'. In particular, this is a noncommutative product,

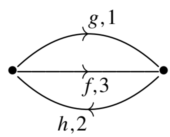
$$G \cdot G' \neq G' \cdot G,$$

as the ordering of the edge set, $E(G \sqcup G')$, in the two cases is not the same.

Example 3.6. In this example, we concern ourselves primarily with the ordering of the edges in the product. Therefore, we write label the edges with elements of $k^\times$ and the ordering, and neglect to indicate the sign. One may assume, without loss of generality, that the signs are all positive in the graphs below.

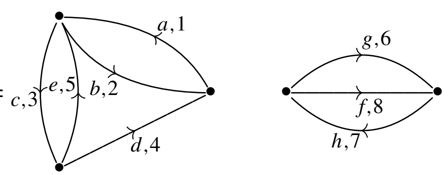
Consider the graphs

$G_1 =$ 

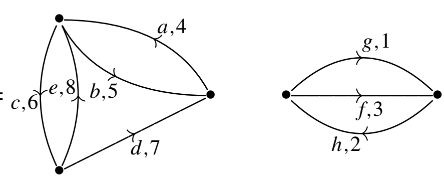
and $G_2 =$ 

First, notice that the graph in Example 3.5 cannot be written as the product of G_1 and G_2, since the edges of one connected component do not precede the edges of the other, as written.

The product in one order is

$G_1 \cdot G_2 =$ 

while the product in the other order is

$G_2 \cdot G_1 =$ 

It is the ordering on the two graphs that distinguishes the two products. Everything else about the labeled oriented graphs $G \cdot G'$ and $G' \cdot G$ is the same.

This noncommutative product gives $(\mathcal{G}(k^\times), \cdot)$ a free monoidal structure. The unit in the monoid is given by the empty graph, which has no loops and no edges, and therefore no labels.

Definition 3.7. Let $\mathbb{Q}[\mathcal{G}]$ be the free algebra generated by the monoid $(\mathcal{G}(k^\times), \cdot)$.

Just as with the cycles, we are not interested in the order of the coordinates, but their image under Alt. Therefore, we are also only interested in an alternating projection on the edges of the graphs. There is a $\mathfrak{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ action on the edges of a graph $G \in \mathcal{G}(k^\times)$. This action permutes the order of the edges in the graph, and changes the assigned signs. An element $g \in \mathfrak{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ is of the form $g = (\sigma, \vec{\operatorname{sgn}})$, where $\sigma \in \mathfrak{S}_{|E(G)|}$ and $\vec{\operatorname{sgn}} \in (\mathbb{Z}/2\mathbb{Z})^n$ is an ordered set of signs. Write $\vec{\operatorname{sgn}}_j$ for the j-th entry of the ordered set. Furthermore, write

$$\operatorname{sgn}(g) = \operatorname{sgn}(\sigma) \prod_j \vec{\operatorname{sgn}}_j,$$

where $\operatorname{sgn}(\sigma)$ indicates the sign of the permutation $\sigma \in \mathfrak{S}_{|E(G)|}$.

The action of $\mathfrak{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ on the algebra of graphs is as follows: $gG = 0$ if $|E(G)| \neq n$, and otherwise gG is given by

$$\begin{cases} \omega(gG) := \sigma(\omega(G)), \\ \vec{\operatorname{sgn}}_i(gG) = \vec{\operatorname{sgn}}_i \vec{\operatorname{sgn}}_i(G). \end{cases}$$

That is, if $|E(G)| = n$, the ordering and signs of the edges in gG for $g = (\sigma, \vec{\operatorname{sgn}})$ are determined by σ and $\vec{\operatorname{sgn}}$, respectively.

The action of $\mathfrak{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ defines an equivalence relation on $\mathbb{Q}[\mathcal{G}]$.

Lemma 3.8. *Letting n vary, any two monomials G and $G' \in \mathbb{Q}[\mathcal{G}]$ are equivalent if and only if there is an element $g \in \mathfrak{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ relating the two:*

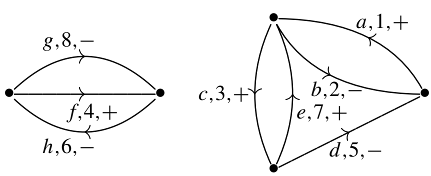
$$G \sim_{\text{ord}} \operatorname{sgn}(g) gG.$$

The proof comes from the identity, inverse and composition laws of the group $\mathfrak{S}_{|E(G)|} \rtimes (\mathbb{Z}/2\mathbb{Z})^n$, and we omit it.

In Lemma 3.14, we show that $\mathbb{Q}[\mathcal{G}]/{\sim_{\text{ord}}}$ is generated as an algebra by connected graphs. In other words, under the equivalence $\sim_{\text{ord}}$, any disconnected element of $\mathbb{Q}[\mathcal{G}]$ is no longer primitive.

First we give an example.

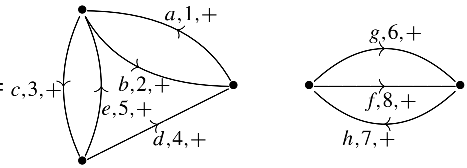
Example 3.9. To illustrate the equivalence relations from Lemma 3.8, consider the graph G in Example 3.5 as a monomial in $\mathbb{Q}[\mathcal{G}]$:

$G =$ 

with the edges ordered as indicated by the subscripts, as usual. This graph is a primitive element of $\mathbb{Q}[\mathcal{G}]$.

However, in the ring quotiented by the equivalence relation, $\mathbb{Q}[\mathcal{G}]/\sim_{\text{ord}}$, we see that $G \sim_{\text{ord}} G_1 \cdot G_2$, where G_1 and G_2 are the graphs defined in Example 3.6:

$$G \sim_{\text{ord}} G_1 \cdot G_2 =$$



which is not primitive. Notice that both signs and orderings have been changed in this example.

As an algebra, $\mathbb{Q}[\mathcal{G}]$ is bigraded by first Betti number, or weight, and degree of the graphs. That is, if $G \in \mathbb{Q}[\mathcal{G}]_\bullet^\star$, then $h_1(G) = \bullet$, while $\star = h_1(G) - V(G) + h_0(G)$. From the formula for the first Betti number of a graphs in (3.3), if $G \in \mathbb{Q}[\mathcal{G}]_\bullet^\star$,

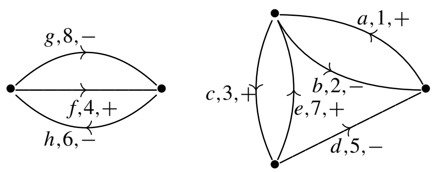
$$|E(G)| = 2\bullet - \star. \tag{3.10}$$

As the equivalence relation $\sim_{\text{ord}}$ does not affect the underlying topology of the graph, $\mathbb{Q}[\mathcal{G}]/\sim_{\text{ord}}$ is also bigraded by weight and degree of the graphs.

Remark 3.11. The unit of this algebra is in $\mathbb{Q}[\mathcal{G}]_0^0$. It is represented by the empty graph.

Example 3.12. For instance, consider the graph in Examples 3.5 and 3.9:

$$G =$$



This graph has five loops, five vertices and two connected components. Therefore, it is in $\mathbb{Q}[\mathcal{G}]_5^2/\sim_{\text{ord}}$.

Definition 3.13. Let $\mathcal{G}_0(k^\times) \subset \mathcal{G}(k^\times)$ be the subset of biconnected graphs with ordered, labeled, oriented edges. That is, there are no disconnected graphs in $\mathcal{G}_0(k^\times)$.

Lemma 3.14. *The algebra $\mathbb{Q}[\mathcal{G}]/\sim_{\text{ord}}$ is generated by the set $\mathcal{G}_0(k^\times)/\sim_{\text{ord}}$ as a skew symmetric bigraded algebra.*

Proof. For any disconnected graph $G \in \mathbb{Q}[\mathcal{G}]_n^i$, there is an element $g = (\sigma, \text{id})$ in $\mathfrak{S}_{2n-i} \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ that rearranges the order of the edges of each connected component consecutively. Since $\text{sgn}(g) = \text{sgn}(\sigma)$, by Lemma 3.8 we obtain

$$G \sim_{\text{ord}} \text{sgn}(g)(gG) = \text{sgn}(g) G_1 \cdot G_2 \cdots G_m,$$

with each $G_i \in \mathcal{G}_0(k^\times)$.

The product preserves the bigrading, as the zeroth and first Betti numbers as well as the sizes of the edge and vertex sets are additive under disjoint union. For $G \in \mathbb{Q}[\mathcal{G}]^i_n/\sim_{\mathrm{ord}}$ and $G' \in \mathbb{Q}[\mathcal{G}]^{i'}_{n'}/\sim_{\mathrm{ord}}$, we have

$$G \cdot G' \in \mathbb{Q}[\mathcal{G}]^{i+i'}_{n+n'}/\sim_{\mathrm{ord}}.$$

To see that this is skew symmetric, as above, write

$$G \cdot G' \sim_{\mathrm{ord}} (-1)^{|E(G)||E(G')|} G' \cdot G = (-1)^{ii'} G' \cdot G.$$

The last equality comes from the fact that $|E(G)| = 2n - i$ and $|E(G')| = 2n' - i'$. $\square$

Since

$$\mathbb{Q}[\mathcal{G}]/\sim_{\mathrm{ord}} = \mathbb{Q}[\mathcal{G}_0]/\sim_{\mathrm{ord}},$$

for the rest of this paper we consider only elements of $\mathcal{G}_0(k^\times)$.

3B. ***A brief interlude on algebraic cycles.*** In this section we introduce the relationship between the graphs defined above and algebraic cycles generating $\mathcal{Z}^p_{1L}(\square^n)$. As of yet, we make no claims on admissibility of cycles.

Definition 3.15. Define $\mathbb{Q}\mathcal{Z}_{1L}$ to be the group ring generated by the free abelian group of $\mathbb{P}^1_k$-linear cycles

$$\mathbb{Q}\mathcal{Z}_{1L} = \bigoplus_{p,i} \mathrm{Alt}(\mathcal{Z}^p_{1L}(\square^{2p-i}) \otimes \mathbb{Q}).$$

This is a skew symmetric algebra. Write $\mathbb{Q}\mathcal{Z}_{1L}{}^i_p = \mathrm{Alt}(\mathcal{Z}^n_{1L}(\square^{2p-i}) \otimes \mathbb{Q})$.

There is a homomorphism, Z, from $\mathbb{Q}[\mathcal{G}]^\star_\bullet/\sim_{\mathrm{ord}}$ to $\mathbb{Q}\mathcal{Z}_{1L}$. Note that $\mathcal{A}^\times_{1L} \subset \mathbb{Q}\mathcal{Z}_{1L}$. In Section 3E, we show that Z is a DGA homomorphism onto $\mathcal{A}^\times_{1L}$ that becomes an isomorphism of DGAs when $\mathbb{Q}[\mathcal{G}]^\star_\bullet$ is subjected to more equivalence relations. That is the isomorphism we seek. In this section, we only show that elements of $\mathbb{Q}[\mathcal{G}]$ correspond to parametrizations of $\mathbb{P}^1_k$-linear algebraic cycles on $\square_k^{|E(G)|}$.

Definition 3.16. Each connected graph $G \in \mathcal{G}(k^\times)$, with loop number p and n edges, defines a parametrization, $\phi : \mathbb{P}_k^{|V(G)|-1} \to (\mathbb{P}^1_k)^n$, of an algebraic cycle $Z(G) \in \mathcal{Z}^p_{1L}(\square^n)$. The $\omega(e)$-th coordinate of the cycle $Z(G)$ is

$$\phi_{\omega(e)} = \left(1 - \frac{x_{s(e)}}{a_e x_{t(e)}}\right)^{\mathrm{sgn}_{\omega(e)}},$$

where $x_{s(e)}$ and $x_{t(e)}$ are variables assigned to the vertices at the source and target of the edge $e \in E(G)$, and a_e is the label of edge e.

Recall from Definition 2.13, each $\phi_{\omega(e)}$ is the ratio of the projective coordinates defining the $\omega(e)$-th copy of $\mathbb{P}^1_k$ in the image.

Thus we have, for $\bullet = h_1(G)$ and $\star = h_1(G) - |V(G)| + h_0(G)$, a set map

$$Z : \mathcal{G}_0(k^\times) \to \bigoplus_{\bullet,\star} \mathcal{Z}^{\bullet}_{1L}(\square^{2\star-\bullet}), \quad G \mapsto [\phi_1, \dots, \phi_{|E(G)|}], \tag{3.17}$$

from graphs to parametrized $\mathbb{P}_k$-linear cycles.

To make this map concrete, we explicitly derive the system of polynomials defined by a graph G. First we introduce a function that relates edges of a graph to the loops of G.

Definition 3.18. For $e \in E(G)$, and L a loop of G, define

$$\epsilon(e, L) = \begin{cases} 1 & \text{if } e \in E(L), \text{ oriented as } L \text{ is,} \\ 0 & \text{if } e \notin E(L), \\ -1 & \text{if } e \in E(L), \text{ oriented opposite to } L. \end{cases}$$

Given this notation, we are ready to define the system of polynomials defined by a graph $G \in \mathcal{G}_0(k^\times)$.

Theorem 3.19. *For a graph $G \in \mathcal{G}_0(k^\times)/\sim_{\text{ord}}$, indicate the label of the edge $e \in E(G)$ as $a_e \in k^\times$. Suppose $h_1(G) = p$ and $|E(G)| = n$. Let $\beta = \{L_1, \dots, L_p\}$ be a loop basis of $H_1(G)$. The algebraic cycle $Z(G)$ is defined by the system of p polynomial equations, each associated to an element of the loop basis, and induced from the rational relations*

$$1 = \prod_{e \in E(G)} (a_e(1 - \phi_{\omega(e)}))^{\epsilon(e, L_i)}. \tag{3.20}$$

Proof. Given a loop basis β for $H_1(G)$, begin with a loop, call it L_1. Subsequent elements of the system of equations are similarly defined.

Consider an edge $e \in E(L_1)$. The $\omega(e)$-th coordinate of the cycle $Z(G)$ is defined by the function $\phi_{\omega(e)}(x, y)$, where x and y are the variables associated to the vertices at the endpoints of $e \in E(G)$. Suppose that, in the orientation inherent in L_1 as an element of a loop basis, L_1 flows from the vertex associated to x directly to the vertex associated to y. This is not necessarily the orientation of the edge connecting the vertices associated to x and y, but the second orientation on the edges induced by the orientation of L_1. Since we are working over $\mathbb{Q}[\mathcal{G}]^\star_\bullet/\sim_{\text{ord}}$, we may choose G such that all the signs on the edges of G are all positive. Then one can associate to the edge $e \in E(G)$ the equation

$$x = y(a_e(1 - \phi_{\omega(e)}))^{\epsilon(e, L_1)}. \tag{3.21}$$

There is a unique edge $e' \neq e$ in L_1 with an endpoint at the vertex associated to the variable y. As above, associate to the edge e' the equation

$$y = z(a_{e'}(1 - \phi_{\omega(e')}))^{\epsilon(e', L_1)}.$$

Substituting this into (3.21) gives

$$x = z(a_e(1-\phi_{\omega(e)}))^{\epsilon(e,L_1)}(a_{e'}(1-\phi_{\omega(e')}))^{\epsilon(e',L_1)}.$$

Continuing along the entire loop in this manner gives

$$x = x \prod_{e\in E(G)} (a_e(1-\phi_{\omega(e)}))^{\epsilon(e,L_1)},$$

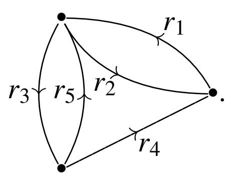
which simplifies to an expression of the form in (3.20):

$$1 = \prod_{e\in E(G)} (a_e(1-\phi_{\omega(e)}))^{\epsilon(e,L_1)}.$$

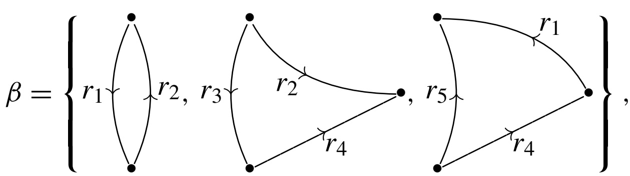
Since β is a loop basis, the function $\phi_{\omega(e)}$, associated to each edge of G is used in the system of equations defined in (3.20), and the functions thus derived are independent of each other. $\square$

Notice that the specific form of this system of equations depends on the loop basis for $H_1(G)$. However, a different loop basis will give an equivalent system of polynomials.

Example 3.22. Recall the graph in Example 3.5:



Define a basis



where all the loops in β are oriented counterclockwise.

A system of equations for this graph is given by the polynomials

$$1 = r_1 r_2(1-\phi_1)(1-\phi_2),$$
$$1 = \frac{r_3 r_4}{r_2}\frac{(1-\phi_3)(1-\phi_4)}{(1-\phi_2)},$$
$$1 = \frac{r_1 r_4}{r_5}\frac{(1-\phi_1)(1-\phi_4)}{(1-\phi_5)}.$$

This brings us to an important invariant of the graphs in $\mathbb{Q}[\mathcal{G}]^{\star}_{\bullet}/\sim_{\mathrm{ord}}$, the loop coefficient:

Definition 3.23. Given a loop L of G, the loop coefficient of L is defined by

$$\chi_G(L) = \prod_{E(G)} r_e^{\epsilon(e,L)}. \tag{3.24}$$

In this notation, we can restate the image of the map Z. For $G \in \mathbb{Q}[\mathcal{G}]_n^p/\sim_{\text{ord}}$ with $\beta = \{L_1, \ldots, L_p\}$ a basis of $H^1(G)$, the cycle $Z(G)$ is defined by the system of polynomial equations

$$\Big\{1 = \chi_G(L_i) \prod_{e \in E(L_i)} (1 - \phi_{\omega(e)})^{\epsilon(e,L_i)}\Big\}_{L_i \in \beta}. \tag{3.25}$$

We can extend the set map Z thus defined to the algebra $\mathbb{Q}[\mathcal{G}]^\star_\bullet/\sim_{\text{ord}}$, where $Z(G)$ maps a graph to an algebraic cycle under the alternating projection.

Theorem 3.26. *The set map Z in (3.17) induces a grading-preserving algebra homomorphism*

$$Z : \mathbb{Q}[\mathcal{G}]/\sim_{\text{ord}} \to \mathbb{Q}\mathcal{Z}_{1L}, \quad G \mapsto \operatorname{Alt}[\phi_1, \ldots, \phi_{|E(G)|}].$$

Proof. The equivalence relation $\sim_{\text{ord}}$ equates different orderings of edges of graphs as Alt combines different orderings of coordinates into a single generator of $\mathbb{Q}\mathcal{Z}_{1L}$. Therefore, Z maps generators of $\mathbb{Q}[\mathcal{G}]/\sim_{\text{ord}}$ to generators of $\mathbb{Q}\mathcal{Z}_{1L}$. Lemma 3.14 shows that the algebra structure of $\mathbb{Q}[\mathcal{G}]/\sim_{\text{ord}}$ matches the algebra structure of $\mathbb{Q}\mathcal{Z}_{1L}$.

It remains to check that if $G \in \mathbb{Q}[\mathcal{G}]^i_p/\sim_{\text{ord}}$ then $Z(G) \in \mathbb{Q}\mathcal{Z}_{1L}{}^i_p$. First notice that by the parametrization given in Definition 3.16, Writing $G = G_1 \cdots G_m$ in terms of its connected components, the cycle $Z(G)$ is parametrized by the map

$$\phi : \prod_{i=1}^m \mathbb{P}_k^{|V(G_i)|-1} \to \square_k^{|E(G)|}.$$

Therefore, the cycle $Z(G)$ has codimension

$$E(G) - V(G) + h_0(G) = h_1(G) = p$$

in $\square_k^{|E(G)|}$. By (3.10) this implies that $Z(G) \in \mathcal{Z}^p_{1L}(\square^{2p-i})$. □

Finally, in conjunction with Theorem 3.19, this allows for a statement about irreducible cycles.

Corollary 3.27. *If G is a generator of $\mathbb{Q}[\mathcal{G}]/\sim_{\text{ord}}$, i.e., a disconnected graph, then $Z(G)$ is a reducible cycle.*

Proof. Recall that a reducible cycle is one that arises from a reducible variety. □

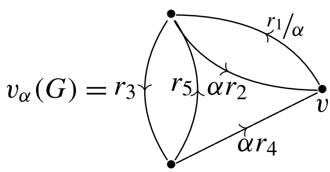
3C. *The DGA structure on graphs.* In this section, we define a differential structure on the algebra of graphs. In order to do this, we need to define an additional equivalence relation on $\mathcal{G}_0(k^\times)$.

In particular, we consider graphs that differ only by a rescaling of the labels of the edges attached to a particular vertex.

Definition 3.28. Consider $\alpha \in k^\times$ and $v \in V(G)$. The vertex rescaled graph $v_\alpha(G)$ is the labeled oriented graph G with labels changed as follows: for each edge e of G, if an edge terminates (starts) at v, multiply (divide) its label by α to get the label of the edge in $v_\alpha(G)$; otherwise, keep the same label for e. The signs associated to and the ordering of the edges of G by ω do not change.

Vertex rescaling a graph corresponds to rescaling all instances of a variable in the parametrized $\mathbb{P}_k$-linear cycle $Z(G)$ by a constant multiple. This does not affect the cycle at all. In other words, G and $v_\alpha(G)$ correspond to two different parametrizations of $Z(G)$. We call this procedure label rescaling with respect to a vertex, or label rescaling at v.

Example 3.29. For the graph G in Example 3.5, one can rescale the rightmost vertex by α to obtain the graph



where the ordering of the edges is given by the subscripts.

Remark 3.30. Vertex rescaling is an equivalence relation on the set $\mathcal{G}_0(k^\times)$. We write it as $\sim_{\mathrm{v}}$.

In the sequel, we consider the algebra of graphs up to this equivalence set. We are interested in graphs only as a tool to understand their corresponding algebraic cycles. We work with graphs up to this rescaling since two graphs that differ by a vertex rescaling correspond, under the homomorphism Z defined in Section 3B, to different parametrizations of the same cycle.

To see this, notice that vertex rescaling does not change the loop coefficient of the graph.

Lemma 3.31. *Loop coefficients are invariant under rescaling at vertices.*

Proof. Let L be a loop in G, with $G \in \mathcal{G}_0(k^\times)$. For $v \in V(L)$, a vertex in L, v is attached to exactly two edges e_1 and e_2 of L. We compare $\chi_G(L)$ and $\chi_{v_\alpha(G)}(L)$.

There are three cases to consider. If v is the terminal vertex of e_1 and the source vertex of e_2, then the respective coefficients are r_1 and r_2 in G, and $r_1\alpha$ and r_2/α in $v_\alpha(G)$. Both numbers either appear in the numerator or the denominator of the coefficient of L. Thus the contributions of α cancel in $\chi_{v_\alpha(G)}(L)$.

The other two cases are as follows. The vertex v is either the source or target vertex of both e_1 and e_2. Then the coefficients are r_1/α and r_2/α (or $r_1\alpha$ and $r_2\alpha$). One label appears in the numerator of the loop, the other in the denominator, so the contribution of α cancels $\chi_{v_\alpha(G)}(L)$.

Therefore,

$$\chi_G(L) = \chi_{v_\alpha(G)}(L),$$

as desired. □

Therefore, given the form of the system of polynomials defined by each of these graphs in (3.25), $Z(G) = Z(v_\alpha(G))$.

Theorem 3.32. *The parametrized cycles $Z(G)$ and $Z(v_\alpha(G))$ correspond to the same cycle, under different parametrizations*

$$Z(G) = Z(v_\alpha(G)) \in \mathscr{Z}^{h_1(G)}_{1L}(\square^{|E(G)|}).$$

Proof. Since, by Lemma 3.31, loop coefficients are invariant under vertex rescaling, from the system of equations defined in (3.25), we see that the cycles defined are the same. □

Therefore, the algebra homomorphism, Z, defined in Section 3B passes to an algebra homomorphism under the quotient $\sim_{\mathrm{v}}$

$$Z : \mathbb{Q}[\mathscr{G}]^{\star}_{\bullet}/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}) \to \mathrm{Alt}(\mathscr{Z}^{\bullet}_{1L}(\square^{2\bullet-\star}) \otimes \mathbb{Q}).$$

As we mentioned before, the algebra $\mathrm{Alt}\,\mathscr{Z}^{\bullet}_{1L}(\square^{2\bullet-\star})$ does not have a DGA structure. However, the algebra $\mathbb{Q}[\mathscr{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$ does. On individual graphs, this is defined by a modified contraction of the edges. We devote the rest of this subsection to developing this differential.

Definition 3.33. Consider $G \in \mathscr{G}_0(k^\times)$. For $e \in E(G)$, define the graph G/e to be that formed by contracting the edge e and identifying the vertices s_e and t_e as a new vertex v. If the edge $e \in E(G)$ has the same source and target vertex, then $G/e = 0$. If contracting the edge e leads to a one connected graph, split the graph into its biconnected components at the articulation vertex.

The above definition is not the standard definition of an edge contraction in graphs. The standard definition has been modified to fit the algebraic properties of the graphs we need, namely the splitting of graphs at the articulation vertex. Furthermore, the ordering of $G/e \in \mathbb{Q}[\mathscr{G}]$ is induced from the ordering of G.

Definition 3.34. Let $\omega(G)$ be the ordering of the edges of the graph G. Then $\hat{\omega}_e(G/e)$ is the ordering of the graph G/e which is the same as $\omega(G)$ with the $\omega(e)$-th element removed.

We are now ready to define a differential operator on $G \in \mathbb{Q}[\mathscr{G}]^{\star}_{\bullet}/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$.

Theorem 3.35. *Consider a monomial $G \in \mathbb{Q}[\mathscr{G}]^{\star}_{\bullet}/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$. For $e \in E(G)$ an edge, let r_e denote the label of this edge and let s_e denote the source vertex. There* *is a degree*-1 *differential operator*

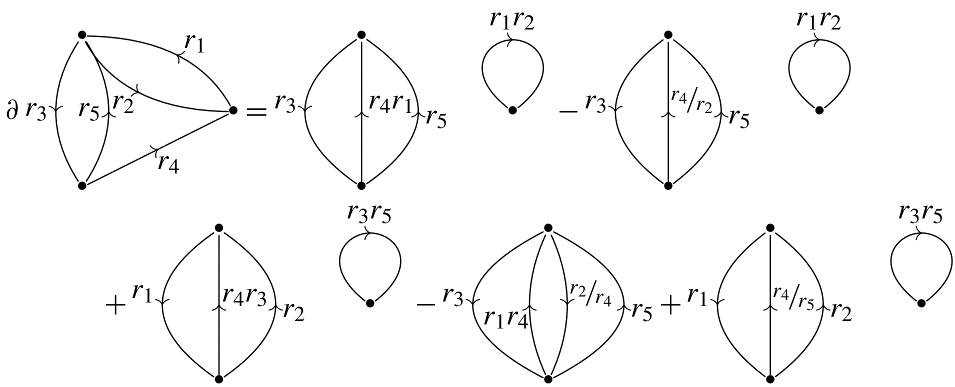
$$\partial : \mathbb{Q}[\mathcal{G}]^{\star}_{\bullet}/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}) \to \mathbb{Q}[\mathcal{G}]^{\star+1}_{\bullet}/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}),$$
$$(\omega, G) \mapsto \sum_{e \in E(G)} (-1)^{\omega(e)-1} (\hat{\omega}_e, ((s_e)_{r_e}(G))/e).$$

By direct calculation, one sees that this operator satisfies the Leibnitz rule

$$\partial(G \cdot G') = \partial(G) \cdot G' + (-1)^{\star} G \cdot \partial(G'). \tag{3.36}$$

We prove this theorem in steps. Before starting the proof, we give an example of the action of ∂. Recall that the notation $(s_e)_{r_e}$ in Theorem 3.35 is the vertex rescaling from Definition 3.28.

Example 3.37. For example, for the graph in Example 3.5, with ω ordered according to the numbering of the labels,



First, we define a contraction operator on graphs with labeled edges.

Definition 3.38. For $e \in E(G)$, we write the contraction of an edge as $\partial_e(G) = (s_e)_{r_e}(G)/e$.

In this notation, the operator defined in Theorem 3.35 can be rewritten as

$$\partial(G) = \sum_{e \in E(G)} (-1)^{\omega(e)-1} \partial_e(G).$$

Notice that if $r_e = 1$ then $\partial_e(G) = (G/e)$. This further implies that the loop coefficient is invariant under contraction.

Lemma 3.39. *Consider* $G \in \mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$. *Let* L *be a loop in* G *with more than one edge*, *and* $e \in E(L)$. *Then*

$$\chi_G(L) = \chi_{\partial_e G}(L/e).$$

Proof. It is sufficient to consider G connected. If s is the source vertex of e, and r the label, the equivalent graph $s_r(G)$ is such that the label of e equals 1.

In Lemma 3.43, we show that contraction is well-defined on $\mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}})$. Therefore, $\partial_e(G) \sim_{\text{v}} \partial_e(s_r(G))$. Since the label of e is 1, the contraction $\partial_e(s_r(G))$ equals $s_r(G)/e$, and

$$\chi_G(L) = \chi_{s_r(G)}(L) = \chi_{\partial_e s_r(G)}(L \setminus e) = \chi_{\partial_e G}(L \setminus e).$$

The first equality comes from Lemma 3.31. The second equality comes from the form of $\partial_e(s_r(G))$. Finally, the third equality comes from the equivalence of the two contractions (Lemma 3.43). □

Working under the equivalence relations $\sim_{\text{v}}$ gives an important representation of graphs that simplifies the calculation of the derivatives.

Lemma 3.40. *For any given $G \in \mathcal{G}_0(k^\times)$, and any subtree $T \subset G$, there is a graph G_T such that the labels of the edges in T are* 1 *and $G \sim_{\text{v}} G_T$. In particular, any monomial $G \in \mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}})$ can be rescaled such that any spanning forest of G is labeled by* 1.

Proof. Without loss of generality, assume that the graph $G \in \mathcal{G}_0(k^\times)$ is a connected graph. Otherwise, the following arguments apply to each connected component of G.

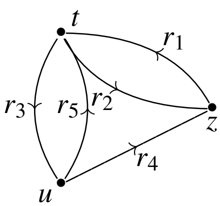
Let T be a spanning tree of G. Label the vertices $\{v_1, \ldots, v_{|V(G)|}\} \in V(G)$ such that v_1 has valence 1 in T. Let $\{r_2, \ldots, r_{|V(T)|}\}$ be the labels of the edges in $E(T)$, where r_i labels the edge connected to v_i.

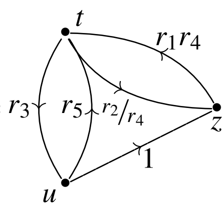
Rescale the graph G at the vertex v_2 by r_2 (resp. $1/r_2$) if v_2 is a source (resp. target) vertex of the edge labeled by r_2. In the rescaled graph $(v_2)_{r_2}(G)$ (resp. $(v_2)_{1/r_2}(G)$) the edge connecting v_1 and v_2 is labeled by 1. By similar logic, there is a series of rescaling coefficients, $\{\alpha_1, \ldots, \alpha_{|V(G)|-1}\}$, where each α_i is a rational function of the r_j such that edges of the spanning tree T in

$$(v_{|V(G)|-1})_{\alpha_{|V(G)|-1}}(\cdots((v_1)_{\alpha_1}(G))\cdots)$$

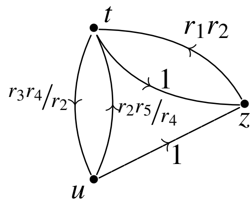
are all labeled by 1. □

Example 3.41. Consider again the graph in Example 3.5. The loop coefficient of the loop defined by the inner triangle of legs, oriented clockwise, is $r_2 r_5/r_4$. The same graph can be relabeled to have a spanning tree labeled with ones as follows:

$G =$ 

; $z_{1/r_4}(G) =$ 

;

$t_{r_2/r_4}(z_{1/r_4}(G)) =$ 

Contrary to appearance, we have made no choice in our definition of the derivative ∂_e. We could just as easily have written

$$\partial_e(G) = (-1)^{\omega(e)-1}(t_e)_{1/r_e}(G)/e,$$

where t_e is the target vertex of the edge e. This is because the two graphs are equivalent under vertex rescaling.

Lemma 3.42. *For $G \in \mathbb{Q}[\mathcal{G}]$, let t and s be the target and source vertices, respectively, of the edge $e \in E(G)$. Then*

$$t_{1/a}(G)/e \sim_{\mathrm{v}} s_a(G)/e.$$

Proof. We show that there is a vertex rescaling such that

$$t_{1/a}(G)/e \sim s_a(G)/e.$$

By construction, $e \notin E(G/e)$, and the vertices $t, s \in V(G)$ are replaced by a single vertex $v \in V(G/e)$.

In the graph $t_{1/a}(G)$, the label of e is multiplied by $1/a$, as are all the edges terminating on t. All edges starting at t are multiplied by a. The edges attached to s and not t are unaffected. Similarly, in the graph $s_a(G)$, the label of e is multiplied by $1/a$, as are all the edges starting at s. All edges terminating at s are multiplied by a. The edges attached to t and not s are unaffected.

Therefore, contracting e and identifying s with t at the new vertex in the contracted graph, we get a unique vertex $v = V(G/e) \setminus V(G)$,

$$v_{1/a}(s_a(G)/e) = t_{1/a}(G)/e.$$

Similarly, one may also write

$$s_a(G)/e = v_a(t_{1/a}(G)/e). \qquad \square$$

Choosing $a = r_e$, the label of the edge e, shows that, in $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$, it does not matter if ∂_e is defined according to the source vertex of e or the target vertex.

Next we show that the operator ∂ is well-defined under vertex rescaling.

Lemma 3.43. *The operator ∂ defined above is well-defined on $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$.*

Proof. Since $\partial = \sum_{e \in E(G)} (-1)^{\omega(e)-1} \partial_e$, for any $g \in \mathfrak{S}|E(G)| \rtimes \mathbb{Z}/2\mathbb{Z}^{|E(G)|}$

$$\partial G = \partial g G$$

in the quotient space $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$ for all $G \in \mathcal{G}_0(k^\times)$.

It remains to check that, for $G \in \mathcal{G}_0(k^\times)$,

$$\partial(G) \sim \partial(v_\alpha(G)) \tag{3.44}$$

for any $v \in V(G)$. Before proceeding, we note that vertex rescaling is multiplicative. That is, for $v \in V(G)$,

$$v_\alpha(v_\beta(G)) = v_{\alpha\beta}(G). \tag{3.45}$$

Fix $v \in V(G)$. For any edge e not incident upon v,

$$\partial_e(v_\alpha G) = v_\alpha \partial_e(G) \sim \partial_e(G).$$

Therefore, consider only the edges $e \in E(G)$ that are incident upon v. They are labeled by r_e. By Definition 3.38 and Lemma 3.42,

$$\partial_e(G) \sim_{\mathrm{v}} \begin{cases} v_{r_e}(G)/e, & v \text{ a source of } e, \\ v_{1/r_e}(G)/e, & v \text{ a target of } e. \end{cases}$$

Recall, by the definition of ∂_e, that if v is the source of e, the above equivalence is an exact equality.

Similarly,

$$\partial_e(v_\alpha(G)) \sim_{\mathrm{v}} \begin{cases} v_{r_e/\alpha}(v_\alpha(G))/e, & v \text{ a source of } e, \\ v_{1/r_e\alpha}(v_\alpha(G))/e, & v \text{ a target of } e. \end{cases}$$

By the multiplicativity of vertex rescaling (3.45), we rewrite this

$$\begin{aligned} \partial_e(v_\alpha(G)) \sim_{\mathrm{v}} & \begin{cases} v_{r_e}(G)/e, & v \text{ a source of } e, \\ v_{1/r_e}(G)/e, & v \text{ a target of } e, \end{cases} \\ & \sim_{\mathrm{v}} \partial_e(G). \end{aligned}$$

Therefore, $\partial(G) \sim_{\mathrm{v}} \partial(v_\alpha(G))$ for any $G \in \mathcal{G}_0(k^\times)$ and $v \in V(G)$. □

Thus far, we have shown that the operator ∂ is well-defined on $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$. Next we show that the operators ∂_e commute.

Lemma 3.46. *Contractions along different edges commute in* $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$, *that is*, $\partial_e \circ \partial_{e'} = \partial_{e'} \circ \partial_e$.

Proof. There are two cases to consider: when the edges e and e' form a cycle in G, and when they do not.

If $e \cup e'$ is a union of loops in G, then, by Definition 3.33, $\partial_e G = \partial_{e'} G = 0$. If $e \cup e'$ is a loop in G, then e' defines a loop in $\partial_e G$, and e a loop in $\partial_{e'} G$. Therefore, $\partial_e \circ \partial_{e'} G = \partial_{e'} \circ \partial_e G = 0$.

If $e \cup e'$ is not a cycle in G, there is a spanning tree T such that $e, e' \subset E(T)$. By Lemma 3.40, write G such that the edges of T are labeled by 1. In this case, $\partial_e \circ \partial_{e'} G = (G/e')/e = G/\{e' \cup e\} = \partial_{e'} \circ \partial_e G$. □

We are now ready to prove Theorem 3.35.

Proof of Theorem 3.35. Lemma 3.43 shows that the operator

$$\partial : \mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}) \to \mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$$

is well-defined.

To see that $\partial \circ \partial = 0$, write

$$\partial \circ \partial = \sum_{e \in E(G/e)} (-1)^{\omega(e)-1} \partial_e \biggl(\sum_{e' \in E(G)} (-1)^{\omega(e')-1} \partial'_e(G) \biggr).$$

Assume without loss of generality that $\omega(e) < \omega(e')$. Then the term $\partial_e \circ \partial_{e'}$ appears in $\partial \circ \partial$ with sign $(-1)^{\omega(e)}(-1)^{\omega(e')}$, while $\partial_{e'} \circ \partial_e$ appears with sign $(-1)^{\omega(e)-1}(-1)^{\omega(e')}$. By Lemma 3.46, $\partial_e \circ \partial_{e'} = \partial_{e'} \circ \partial_e$. Thus the two contributions cancel.

To see that ∂ is a degree-one operator, note that if G/e is not 0, then

$$h_1(G) = h_1(G/e).$$

However,

$$|V(G/e)| = |V(G)| - 1 + (h_0(G/e) - h_0(G)).$$

Recall from (3.10) that if $G \in \mathcal{G}_{1L_i^\bullet}$, the degree is given by

$$i = h_1(G) - |V(G)| + h_0(G).$$

Similarly, the degree of $(\hat{\omega}_e, G/e)$ is given by

$$\begin{aligned} h_1(G/e) - |V(G/e)| + h_0(G/e) \\ &= h_1(G) - \bigl(|V(G)| - 1 + (h_0(G/e) - h_0(G))\bigr) + h_0(G/e) \\ &= h_1(G) - |V(G)| + h_0(G) + 1 = i + 1. \end{aligned}$$

□

So far, we have shown that $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$ is a bigraded DGA and that Z is a homomorphism of algebras from $\mathbb{Q}[\mathcal{G}]^\star_\bullet/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$ to $\mathrm{Alt}\,\mathcal{Z}^\bullet_{1L}(\square^{2\bullet-\star})$. However, we are ultimately interested in graphs $\mathcal{G}_{1L}$ that correspond to $\mathcal{A}^\times_{1L}$ under the algebra homomorphism Z defined in Section 3B. In Section 3D, we define the algebra of admissible graphs, and show that $\mathcal{G}_{1L}$ is a DGA under the differential defined in this section. In Section 3E, we show that $\mathcal{G}_{1L}$ is isomorphic to $\mathcal{A}^\times_{1L}$ as a DGA.

3D. ***Admissible graphs.*** So far, we have said nothing about admissible cycles. By the arguments presented in Sections 3B and 3C, there is an algebra homomorphism

$$Z : \mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}) \to \mathrm{Alt}\,\mathcal{Z}^\bullet_{1L}(\square^{2\bullet-\star}).$$

Theorem 3.32 shows that generators of $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$ map to generic $\mathbb{P}^1_k$-linear cycles under Z, not necessarily to admissible ones. In this section, we define a subalgebra of admissible graphs, which, in Section 3E, we show corresponds to admissible cycles.

There is a compact way of reading off loop coefficients for graphs if the graph is parametrized as in Lemma 3.40, by setting each label of a spanning tree to 1.

Lemma 3.47. *Consider a connected graph $G \in \mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$. Each spanning tree T of G defines a loop basis of $H^1(G)$, the loop coefficients of which are the labels of the edges $E(G) \setminus E(T)$.*

Proof. Each spanning tree of a connected graph defines a set of loops in G as follows: For a spanning tree T, each oriented edge $e \in E(G) \setminus E(T)$ defines a graphical loop, L_e, in conjunction with a subset of $E(T)$. The orientation of the graphical loop is determined by the orientation of e. The rank of the loop space of G is $\operatorname{rk} H^1(G) = |E(G)| - |V(G)| + 1$. Since $|E(T)| = |V(G)| - 1$, we see that $\operatorname{rk} H^1(G) = |E(G) \setminus E(T)|$. Furthermore, $\bigcup_{e \in E(G) \setminus E(T)} E(L_e) = E(G)$. Therefore, the set $\{L_e\}_{e \in E(G) \setminus E(T)}$ defines a basis of $H^1(G)$.

By choosing a parametrization where T is labeled by ones, the graphical loop coefficient L_e is exactly the label of e. $\square$

We are now ready to define a class of graphs called admissible graphs. We show in Section 3E that these correspond to admissible cycles under the homomorphism Z defined in Section 3B.

Definition 3.48. A graph $G \in \mathcal{G}_0(k^\times)$ is admissible if:

(1) The connected components of G are strongly connected.

(2) There is no graphical loop in G that has loop coefficient 1.

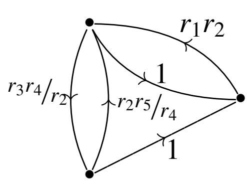
We recall the definition of a strongly connected graph in the first condition.

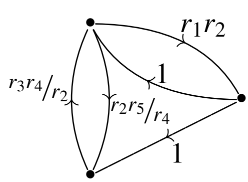
Definition 3.49. An oriented graph is strongly connected if, for any two vertices $v, w \in V(G)$, there is a path from v to w and one from w to v which respect the orientation of the edges of G.

By Lemma 3.47, Definition 3.48 implies that, if a graph $G \in \mathcal{G}_0(k^\times)$ can be parametrized such that there exists a loop with all edged labeled by ones, then G is not admissible.

Finally we add one more equivalence relation among graphs that is useful in Section 3E.

Definition 3.50. For $G \in \mathcal{G}$, let $\overline{G} \in \mathcal{G}$ be the graph with the same underlying labled unoriented graph structure, but with the orientations of every edge switched. Define an equivalence relation $\sim_{\mathrm{ori}}$ that relates graphs with all orientations switched: $G \sim_{\mathrm{ori}} \overline{G}$.

Example 3.51. $G =$ 

$\implies$ $\overline{G} =$ 

Switching the orientation of all edges of a graph corresponds to a reparametrization of $Z(G)$. If the $\omega(e)$-th edge of G corresponds to the parametrization $\phi_{\omega(e)} = 1 - t_i/(a_{\omega(e)}t_j)$, then the $\omega(e)$-th edge of $\overline{G}$ corresponds to the parametrization $\bar{\phi}_{\omega(e)} = 1 - t_j/(a_{\omega(e)}t_i)$, which differs from $\phi_{\omega(e)}$ by the change of variables $t_i \to 1/t_i$. We show that these two are both parametrizations of the same cycle in Section 3E, Corollary 3.61.

Definition 3.52. There is a subalgebra

$$\mathcal{G}_{1L} \subset \mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}}, \sim_{\text{ori}})$$

generated over $\mathbb{Q}$ by admissible graphs.

$$\mathcal{G}_{1L} = \mathbb{Q}[G \mid G \in \mathcal{G} \text{ admissible}]/(\sim_{\text{ord}}, \sim_{\text{v}}, \sim_{\text{ori}}).$$

Lemma 3.53. *The differential operator ∂ restricts to a differential operator on $\mathcal{G}_{1L}$.*

Proof. By the Leibniz rule, it is sufficient to consider connected graphs. We show that if G is an admissible graph, then so is $\partial_e(G)$ for any $e \in E(G)$.

First, we check that if G is strongly connected, then G/e is as well. If $v, w \in V(G)$ are in the same connected component of G/e, then the paths between v and w are either shortened by the contraction of the edge e, or unaffected. Therefore, the connected components of G/e are strongly connected, as desired.

As taking the derivative along any edge does not affect the loop coefficient of any loop in G, we have $\partial_e(G) \in \mathcal{G}_{1L}{}^{\bullet}_{\star+1}$ for $G \in \mathcal{G}_{1L}{}^{\bullet}_{\star}$. □

Therefore, $\mathcal{G}_{1L}$ is a sub-DGA of $\mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}}, \sim_{\text{ori}})$. We show that the homomorphism Z defined in Section 3B is well-defined on $\mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}}, \sim_{\text{ori}})$.

Theorem 3.54. *Let $\overline{G}$ be as in Definition 3.50. The graphs $G, \overline{G} \in \mathbb{Q}[\mathcal{G}]^{\star}_{\bullet}$ map to the same algebraic cycle in $\operatorname{Alt}\mathcal{Z}^{\bullet}_{1L}(\square^{2\bullet-\star})$ under Z.*

Proof. Recall from Theorem 3.19 and (3.25) that, given a basis $\beta = \{L_1, \dots, L_\bullet\}$ of $H^1(G)$, the cycle $Z(G)$ is defined by the set of equations

$$\left\{1 = \chi_G(L_i) \prod_{e \in E(L_i)} (1 - \phi_{\omega(e)})^{\epsilon(e, L_i)}\right\}_{L_i \in \beta}.$$

Note that the set β also defines a basis of $H^1(\overline{G})$, and that $\chi_{\overline{G}}(L_i) = (\chi_G(L_i))^{-1}$ for each $L_i \in \beta$, as the only difference between G and $\overline{G}$ is the orientation of the edges. Similarly, the function $\epsilon(e, L_i)$ defined on G is the negative of the same defined on $\overline{G}$. Therefore, the cycle $Z(\overline{G})$ is defined by the set of equations

$$\left\{1 = (\chi_G(L_i))^{-1} \prod_{e \in E(L_i)} (1 - \phi_{\omega(e)})^{-\epsilon(e, L_i)}\right\}_{L_i \in \beta}.$$

That is, $Z(G)$ and $Z(\overline{G})$ are defined by the same algebraic cycles. □

Therefore, $Z : \mathbb{Q}[\mathcal{G}]^{\star}_{\bullet}/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}, \sim_{\mathrm{ori}}) \to \mathrm{Alt}\,\mathcal{Z}^{\bullet}_{1L}(\square^{2\bullet-\star})$ is a well-defined algebra homomorphism. In the following section, we show that this sub-DGA is isomorphic to $\mathcal{A}^{\times}_{1L}$.

3E. ***From graphs to admissible cycles.*** We now return to the homomorphism defined in Section 3B. In this section, we show that the map Z defined in (3.17), restricts to an isomorphism of DGAs between $\mathcal{G}_{1L}$ and $(\mathcal{A}^{\times}_{1L})$.

To compare the DGA of admissible cycles to the DGA of admissible graphs, we show that the homomorphism Z, when restricted to $\mathcal{G}_{1L}$ is compatible with both the differential on $(\mathcal{A}^{\times}_{1L})$, defined in (2.6), and the differential on $\mathcal{G}_{1L}{}^{\bullet}_{\star}$, defined in Theorem 3.35.

Recall from Definition 2.3 the faces $F_{I,J}$ of $\square^n$.

Lemma 3.55. *For $G \in \mathcal{G}_{1L}$, the derivative is*

$$Z(\partial_e(G)) = \begin{cases} Z(G) \cap F_{\omega(e),\varnothing} & \text{if } \mathrm{sgn}_{\omega(e)} = +, \\ Z(G) \cap F_{\varnothing,\omega(e)} & \text{if } \mathrm{sgn}_{\omega(e)} = -\,. \end{cases}$$

Proof. Consider G to be a connected graph. We consider two cases, when $\partial_e(G)$ is connected, and when it is a disconnected graph.

The cycle $Z(G)$ is equipped with a parametrization

$$\phi : \mathbb{P}_k^{|V(G)|-1} \to (\mathbb{P}^1_k)^{|E(G)|},$$

where the coordinate of $Z(G)$ corresponding to the $\omega(e)$-th edge is

$$\phi_{\omega(e)} = \Big(1 - x a_{\omega(e)} y\Big)^{\mathrm{sgn}_{\omega(e)}}.$$

Recall from Definition 2.12 that $Z(G)$ is the cycle defined by intersecting the image of ϕ with $\square^{|E(G)|}$. In other words, $Z(G) = i^*\phi_*$, where $i : \square^{|E(G)|} \hookrightarrow (\mathbb{P}^1_k)^{|E(G)|}$.

Let $\iota_{I,J} : F_{I,J} \to \square^n$ be the injection into the appropriate face of codimension $|I \cup J|$. If $\mathrm{sgn}_{\omega(e)} = +$ (resp. $\mathrm{sgn}_{\omega(e)} = -$), the intersection $Z(G) \cap F_{\omega(e),\varnothing}$ (resp. $Z(G) \cap F_{\varnothing,\omega(e)}$) is the further pullback $\iota^*_{\omega(e),\varnothing}(i^*\phi_*)$ (resp. $\iota^*_{\varnothing,\omega(e)}(i^*\phi_*)$).

For the remainder of this proof, we assume that $\mathrm{sgn}_{\omega(e)} = +$. The calculation for $\mathrm{sgn}_{\omega(e)} = -$ is similar, and left to the reader.

The intersection $Z(G) \cap F_{\omega(e),\varnothing}$ imposes the restriction $x = a_e y$. Therefore, it can be parametrized by

$$\phi_{\partial_e} : \mathbb{P}_k^{|V(G)|-2} \to (\mathbb{P}^1_k)^{|E(G)|-1}, \tag{3.56}$$

formed by removing the $\omega(e)$-th coordinate of ϕ and replacing each instance of x with $a_e y$. If $\partial_e(G)$ is connected, this is exactly the parametrization defined by the contracted graph. Therefore, the lemma holds when $\partial_e G$ is connected.

If $\partial_e(G) = \prod_{i=1}^k G_i$ is disconnected, then the parametrization defined by this disconnected graph,

$$\phi' : \prod_{i=1}^{k} \mathbb{P}_k^{|V(G_i)|-1} \to (\mathbb{P}_k^1)^{|E(G)|-1},$$

is different from the parametrization, ϕ_{∂_e}, defined by the contraction ∂_e in (3.56). However, consider the affine space $\mathbb{A}_k^{|V(G)|-2}$ defined by setting $x = a_e y = 1$ in $\mathbb{P}_k^{|V(G)|-1}$. Then there is a product of corresponding affine spaces, $\prod_{i=1}^k \mathbb{A}_k^{|V(G_i)|-1}$, associated to the disconnected parametrization, each formed by setting the variable of the new vertex defined by the contraction to 1. The two parametrizations ϕ_{∂_e} and ϕ' agree on these affine spaces. On the hyperplanes at infinity, at least one of the parametrizing variables is 0. Since G is strongly connected, none of the coordinates correspond to purely sink vertices in either G or $\partial_e(G)$. Therefore, setting a parametrization variable to 0 corresponds to setting a coordinate of the image of ϕ' or ϕ_{∂_e} to 1. However, $\square^{|E(G)|-1}$ omits precisely the points of $\mathbb{P}_k^{|E(G)|-1}$ where one of the coordinates is set to 1. Therefore, the parametrized cycles $Z(\partial_e G) = (i^*\phi'_*)$ and $\partial_e Z(G) = i^*\phi_{\partial_e *}$ agree on the pullback to $\mathbb{P}_k^{|E(G)|-1}$, as desired. □

This is the key step to understanding the relationship between the differential on graphs and the differential on cycles.

Theorem 3.57. *If $G \in \mathcal{G}_{1L}$, then*

$$\partial Z(G) = Z(\partial(G)).$$

Proof. Recall from (2.6) that

$$\partial Z(G) = \sum_{e \in E(G)} (-1)^{\omega(e)-1} (\partial_{\omega(e),\varnothing} - \partial_{\varnothing,\omega(e)}) Z(G).$$

From Lemma 3.55,

$$\partial Z(G) = \sum_{\mathrm{sgn}(e)=+} (-1)^{\omega(e)-1} (Z(\partial_e G) - \partial_{\varnothing,\omega(e)} Z(G)) + \sum_{\mathrm{sgn}(e)=-} (-1)^{\omega(e)-1} (Z(\partial_e G) - \partial_{\omega(e),\varnothing} Z(G)).$$

The theorem follows from the fact that $\partial_{\varnothing,\omega(e)} Z(G)$ is empty if $\mathrm{sgn}(e) = +$ and $\partial_{\omega(e),\varnothing} Z(G) = \varnothing$ if $\mathrm{sgn}(e) = -$.

As above, we only do the calculation for $\mathrm{sgn}(e) = +$, as the calculation for $\mathrm{sgn}(e) = -$ is similar. By definition,

$$\partial_{\varnothing,\omega(e)} Z(G) = Z(G) \cap F_{\varnothing,\omega(e)}.$$

That is, the coordinate $\phi_{\omega(e)} = 1 - x/(a_e y) = \infty$. This implies that $x/y = \infty$. Since G is strongly connected, there is another edge e' such that $t_e = s_{e'}$. Then $\phi_{\omega(e')} = 1 - y/(a_{e'} x) = 1$. Therefore, $\partial_{\varnothing,\omega(e')} Z(G) = \varnothing$. □

For any two edges $e, e' \in E(G)$, with $G \in \mathcal{G}_{1L}$, the derivatives ∂_e and $\partial_{e'}$ commute, by Lemma 3.46. Therefore, we can talk about contracting a subgraph of another graph, without noting the order in which the edges are contracted.

Definition 3.58. Let $G' \subset G$, with $E(G') = \{e_1, \dots, e_n\}$. We write

$$\partial_{G'}(G) = \partial_{e_n}(\cdots(\partial_{e_1}(G))\cdots),$$

where $e_i \in E(G')$.

Notice that if the contracted graph G' is not a subtree of G, then $\partial_{G'}(G) = 0$.

We use this shorthand to show that the graphs in $\mathcal{G}_{1L}{}^{\bullet}_{\star}$ correspond exactly to admissible cycles in $(\mathcal{A}_{1L}^{\times})^{\bullet}_{\star}$. Recall that an algebraic cycle in $\mathcal{Z}^{\bullet}(\operatorname{Spec} k, \star)$ is admissible if it intersects all faces of $\square^{2\bullet-\star}$ in codimension $\bullet$ or not at all.

Theorem 3.59. *For $G \in \mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}}, \sim_{\text{ori}})$, the cycle $Z(G)$ is admissible if and only if $G \in \mathcal{G}_{1L}$.*

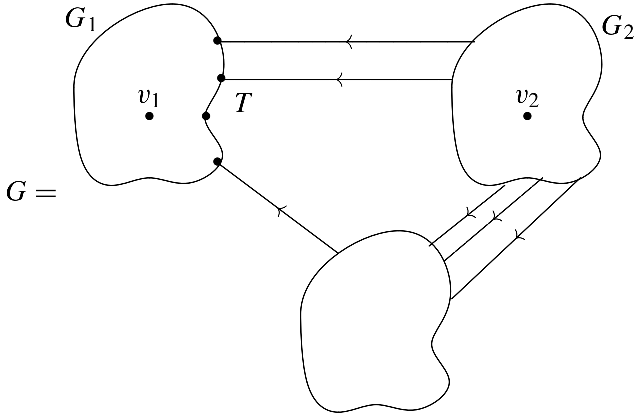
Proof. It is sufficient to look at connected graphs.

Consider a $G \in \mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}}, \sim_{\text{ori}})$ such that there exists a loop, L with loop coefficient 1 in G. Specifically, chose a graph $G \notin \mathcal{G}_{1L}{}^{\bullet}_{\star}$. By Lemma 3.40, we can label the edges of any spanning tree of L by ones. Since rescaling does not change the loop coefficient by Lemma 3.31, all edges of L can be labeled by ones. Let $T \subset L$ be a subgraph of the loop L consisting of all but two of the edges of L. Suppose $E(L \setminus T) = \{e_1, e_2\}$. Let $I = \{e \in E(T) \mid \operatorname{sgn}_e = +\}$ and $J = \{e \in E(T) \mid \operatorname{sgn}_e = -\}$. The graph $\partial_T(G)$, formed by taking the derivative of G along the edges in T, corresponds to intersecting $Z(G)$ with the face $F_{I,J}$. The $\omega(e_1)$-th and $\omega(e_2)$-th coordinate of $Z(\partial_T(G))$ are of the form $\operatorname{sgn}(e_i)(1 - x/y)^{\operatorname{sgn}(e_i)}$ for $i \in \{1, 2\}$. This cycle is not admissible.

To see this, notice that the intersection of $Z(\partial_T(G))$ with the face $F_{\omega(e_1),\varnothing}$ (if $\operatorname{sgn}(e_1) = +$) or $F_{\varnothing,\omega(e_1)}$ (if $\operatorname{sgn}(e_1) = -$) also sets the $\omega(e_2)$-th coordinate to 0, giving it the wrong codimension.

Conversely, suppose $G \in \mathcal{G}_{1L}{}^{\bullet}_{\star}$. Specifically, G is strongly connected. Let G' be a (not necessarily connected) subgraph of G. Let $I = \{e \in E(G') \mid \operatorname{sgn}_e = +\}$ and $J = \{e \in E(G') \mid \operatorname{sgn}_e = -\}$. Consider $D_{G'}(G)$. By Lemma 3.53, $D_{G'}(G)$ is also in $\mathcal{G}_{1L}{}^{\bullet}_{\star}$. If G' is not a forest, then $D_{G'}(G) = 0$. Therefore, we only consider the case when G' is a forest (possibly consisting of a single tree). By Lemma 3.55 $D_{G'}(G)$ amounts to intersecting $Z(G)$ with the face $F_{I,J}$. Since G' is a forest, $h_1(G') = 0$, and $h_1(G) = h_1(D_{G'}(G))$. Therefore $Z(G) \cap F_{I,J}$ has codimension $\star$ in F, making it admissible.

Finally, if G is not strongly connected, then there exists two vertices v_1 and v_2 such that there is not an orientation-respecting path in G from v_1 to v_2. Let G_1 be the largest subgraph of G defined by the vertices that can be reached by orientation-respecting paths from v_1. Let G_2 be the largest subgraph of G defined by the vertices that can reach v_2 by orientation-preserving paths in G. By construction, G_1 and G_2 are disjoint subgraphs:



In particular, the subgraph G_1 has i edges flowing into its vertices from the rest of the graph, $G \setminus G_1$. Let T be a subtree of G_1 connecting all the sink vertices of these incoming edges. The derivative $\partial_T(G)$ has at least two connected components. Write

$$\partial_T(G) = \pm G' \partial_T(G_1),$$

with G' the (possibly disconnected) subgraph of $\partial_T(G)$ that contains G_2 as a subgraph. The graph G' has a sink vertex in the connected component containing (G_2). Therefore, the cycle $Z(G')$ has at least two coordinates of the form $\phi_i = (1 - x/(ay))^{\mathrm{sgn}_i}$ and $\phi_j = (1 - z/(by))^{\mathrm{sgn}_j}$. Setting the coordinate $\phi_i = 0(\infty)$ sets the coordinate $\phi_j = 0(\infty)$, by the arguments above. Since the derivative $\partial_T(G)$ has the wrong codimension intersecting the face $F_{i,\varnothing(\varnothing,i)}$, the cycle $Z(G)$ is not admissible. $\square$

It follows from Theorems 3.26, 3.32, 3.54 and 3.59, that the homomorphism Z is surjective:

Corollary 3.60. *The homomorphism*

$$Z : \mathscr{G}_{1L}{}^{\bullet}_{\star} \to (\mathscr{A}^{\times}_{1L})^{\bullet}_{\star}$$

is a surjection of DGAs.

Proof. By Theorems 3.26, 3.32, 3.54 and 3.59, we see that Z is a homomorphism of DGAs with image contained in $(\mathscr{A}^{\times}_{1L})^{\bullet}_{\star}$. We check surjection of this map. By definition, if $Z \in (\mathscr{A}^{\times}_{1L})^{\bullet}_{\star}$, there is a parametrization $\phi : \mathbb{P}_k^{\bullet-\star} \to (\mathbb{P}_k^1)^{2\bullet-\star}$, with $\phi_i = 1 - x_i/(a_i y_i)$. Assuming that Z is reducible, Corollary 3.27 states that this defines a connected graph G with $2\bullet - \star$ edges and $\bullet - \star + 1$ vertices. Since Z is admissible, by Theorem 3.59, $G \in \mathscr{G}_{1L}$. $\square$

It remains to show that Z is an isomorphism.

Corollary 3.61. *Any cycle in $(\mathscr{A}^{\times}_{1L})$ remains invariant under inverting all the parametrizing variables, or scaling some of them by a constant multiple.*

Proof. This follows from Theorem 3.19.

Let $\mathscr{Z} \in (\mathscr{A}^{\times}_{1L})$ be the cycle parametrized by the variables $\{v_1, \dots, v_n\}$ such that each coordinate is of the form $\phi_i^{\mathrm{sgn}_i}$ with

$$\phi_i = 1 - \frac{v_{i_s}}{a_i v_{i_t}},$$

and $v_{i_s}, v_{i_t} \in \{v_1, \dots, v_n\}$. Let $\mathscr{Z}' \in (\mathscr{A}^{\times}_{1L})$ be the cycle with coordinates

$$\phi_i = 1 - \frac{b_{i_s} v_{i_s}}{a_i b_{i_t} v_{i_t}}$$

for $b_{i_j} \in k^{\times}$, and $\hat{\mathscr{Z}} \in (\mathscr{A}^{\times}_{1L})$ be the cycle with coordinates

$$\phi_i = 1 - \frac{v_{i_t}}{a_i v_{i_s}}.$$

The claim of this corollary is that

$$\mathscr{Z}' = \mathscr{Z} = \hat{\mathscr{Z}}. \tag{3.62}$$

Algebra and writing the cycles out in the form of (3.20) shows that these equalities hold. $\square$

In terms of graphs, the first equality in (3.62) corresponds to rescaling at vertices to pass from G to $v_{1b_1}(\cdots(v_{nb_n}(G))\cdots)$. The second equality corresponds to changing the orientations of all the edges in the graph.

We are now ready to show that the two algebras $\mathscr{G}_{1L}{}^{\bullet}_{\star}$ and $(\mathscr{A}^{\times}_{1L})^{\bullet}_{\star}$ are isomorphic.

Theorem 3.63. *The map $Z : \mathscr{G}_{1L}{}^{\bullet}_{\star} \to (\mathscr{A}^{\times}_{1L})^{\bullet}_{\star}$ defined in (3.17) is an isomorphism of DGAs.*

Proof. Theorem 3.57 shows that Z is a homomorphism of DGAs. Corollary 3.60 shows that this map is surjective.

Rescaling a vertex on a graph G, that is passing from G to $v_\alpha(G)$, corresponds to rescaling the corresponding parametrizing variable in $Z(G)$. Similarly, inverting the orientations of all the edges, passing from G to $\overline{G}$, corresponds to inverting all the parametrizing variables in $Z(G)$. Since, by Corollary 3.61, neither of these reparametrizations changes the underlying cycle, the map Z is one-to-one.

Explicitly, define a map

$$G : (\mathscr{A}^{\times}_{1L})^{\bullet}_{\star} \to \mathscr{G}_{1L}{}^{\bullet}_{\star}$$

that is a left inverse of Z. For any cycle parametrized in a $\mathbb{P}^1_k$-linear form,

$$G(\mathrm{Alt}[\phi_1^{\mathrm{sgn}_1}, \dots, \phi_n^{\mathrm{sgn}_n}])$$

is a graph constructed as follows: Write each ϕ_i as $1-x/(a_i y)$. If ϕ_i is a constant, write it as $1-1/(a_i)$. Each independent variable in $\mathrm{Alt}[\phi_1,\dots,\phi_n]$ corresponds to a vertex. For each ϕ_i, draw an oriented edge of G, oriented from the numerator variable to the denominator variable, labeled by a_i. In this scheme, constant coordinates correspond to one edge loops. The term ω is defined by the ordering and signs of the ϕ_i. □

4. Elements of $H^0(B(\mathscr{G}_{1L}))$

In the previous section, we establish an isomorphism between the DGA of $\mathbb{P}^1_k$-linear cycles, $(\mathscr{A}^{\times}_{1L})^{\bullet}_{\star}$, and the DGA of admissible graphs $\mathscr{G}_{1L}{}^{\bullet}_{\star}$. We use this to establish that everything that needs to be done for $(\mathscr{A}^{\times}_{1L})^{\bullet}_{\star}$ cycles can be done on the algebra of graphs $\mathscr{G}_{1L}{}^{\bullet}_{\star}$. For the rest of this paper, we restrict our attention to the DGA of graphs.

In particular, to define the category of motives, we are interested in studying the Hopf algebra

$$H_0(B(\mathscr{G}_{1L}))\simeq H_0(B(\mathscr{A}^{\times}_{1L})).$$

We maintain the definition of the bar construction $B(\mathscr{G}_{1L})$ as in Definition 2.1, with $A=\mathscr{G}_{1L}$. Following convention, we indicate the tensor product in the bar construction by $|$.

As in Definition 2.1, write the degree and tensor graded components of $B(\mathscr{G}_{1L})$ as

$$B(\mathscr{G}_{1L})^n_m=\bigoplus_{\sum_1^n(w_i-1)=m}[\mathscr{G}_{1L}{}^{\bullet}_{w_1}\,|\cdots|\,\mathscr{G}_{1L}{}^{\bullet}_{w_n}]. \tag{4.1}$$

Note that, as in Definition 2.1, the degree of a graph in the bar construction is shifted from the degree of a graph in the algebra. That is, if $G\in\mathscr{G}_{1L}{}^{\bullet}_{j}$, then $G\in B(\mathscr{G}_{1L})^1_{j-1}$.

Definition 4.2. Due to the multiple degrees assigned to graphs in an algebraic and bar construction context, we write $\deg_B$ (as opposed to simply deg) for the shifted degree of a graph as it contributes to the total degree in the bar construction.

Explicitly, if $G\in\mathscr{G}_{1L}{}^{\bullet}_{j}$, $\deg(G)=j$ then $\deg_B(G)=j-1$.

To set notation, we define differentials that make the bar complex $(B(\mathscr{G}_{1L}),\partial+\mu)$ a bicomplex. Write $\partial_{\mathscr{G}}$ and $\mu_{\mathscr{G}}$ for the derivatives and product on the graphs. Then ∂ and μ are the degree-one operators on $B(\mathscr{G}_{1L})$ induced by $\partial_{\mathscr{G}}$ and $\mu_{\mathscr{G}}$, calculated by the degree of graphs in the bar construction under the Leibnitz rule. Let ∂_j be the differential operator that acts by $(-1)^{\deg_B G_i}\,\mathrm{id}$ on the first $j-1$ tensor components, by $\partial_{\mathscr{G}}$ on the j-th tensor component, and by id on the remaining tensor components. Then for $[G_1\,|\cdots|\,G_n]\in B(\mathscr{G}_{1L})^n_m$, write

$$\begin{aligned}\partial[G_1 \mid \cdots \mid G_n] &:= \sum_{j=1} \partial_j [G_1 \mid \cdots \mid G_n] \\ &= \sum_{j=1}^{n} (-1)^{\sum_{k=1}^{j-1} \deg_B(G_k)} [G_1 \mid \cdots \mid \partial_{\mathcal{G}}(G_j) \mid \cdots \mid G_n], \end{aligned} \tag{4.3}$$

is a degree-one differential operator $\partial : B(\mathcal{G}_{1L})^n_m \to B(\mathcal{G}_{1L})^n_{m+1}$. Similarly, let μ_j be the differential operator that acts by $(-1)^{\deg_B G_i}$ id on the first $j-1$ tensor components, by $(-1)^{\deg_B G_j}\mu$ on the j-th and $(j+1)$-st components, and as id on the remaining components. Then

$$\begin{aligned}\mu[G_1 \mid \cdots \mid G_n] &:= \sum_{j=1} \mu_j [G_1 \mid \cdots \mid G_n] \\ &= \sum_{j=1}^{n-1} (-1)^{\sum_{i=1}^{j} \deg G_i} [G_1 \mid \cdots \mid G_j \cdot G_{j+1} \mid \cdots \mid G_n]. \end{aligned} \tag{4.4}$$

This is a degree-one differential operator, as $[G_1 \mid G_2] \in B(\mathcal{G}_{1L})^2_{m_1+m_2-2}$ while $\mu[G_1 \mid G_2] = [G_1 G_2] \in B(\mathcal{G}_{1L})^1_{m_1+m_2-1}$ for $G_i \in \mathcal{G}_{1L}{}^{r_i}_{m_i}$.

In order to study elements of $H^i(B(\mathcal{G}_{1L}))$, identify elements in the kernel of

$$D + \mu : \bigoplus_{n \geq 1} B(\mathcal{G}_{1L})^n_i \to \bigoplus_{n \geq 1} B(\mathcal{G}_{1L})^n_{i+1}.$$

By Definition 2.8, we see that elements of this kernel are exactly the elements with completely decomposable boundaries.

Remark 4.5. Very few generators of $\mathcal{G}_{1L}{}^{\bullet}_{\star}$ as an algebra have a decomposable boundary. The completely decomposable objects in $B(\mathcal{G}_{1L})$ correspond to linear combinations of tensor products of graphs.

In this paper, we wish to study $H^0(B(\mathcal{G}_{1L}))$. Therefore, we study completely decomposable elements of $\bigoplus_{i \geq 1} B(\mathcal{G}_{1L})^i_0$ defined by completely decomposable elements of $B(\mathcal{G}_{1L})^1_0$. From Definition 2.8, a completely decomposable element, ε, of $B(\mathcal{G}_{1L})^1_0$ defines a trivial cycle in $H^0(B(\mathcal{G}_{1L}))$ if it can be written as the coboundary of another sum of graphs $\sum_i G_i \in \mathcal{G}_{1L}{}^{\bullet}_2$,

$$\partial \sum_i G_i = \varepsilon,$$

or if it can be written as the sum of a product of graphs,

$$\mu \sum_i [G_{1,i} \mid G_{2,i}] = \varepsilon.$$

In this section, we first give a result that greatly reduces the number of algebraic cycles in $\mathscr{A}^{\times}_{1L}{}^{\bullet}_{1}$ one needs to construct $H^0(B(\mathcal{G}_{1L}))$.

Theorem 4.6. *If $\varepsilon \in \mathcal{A}^{\times\,\bullet}_{1L\,i}$ is a completely decomposable algebraic cycle which can be written as $Z(\sum G_j)$, where each $G_j \in \mathcal{G}_{1L}{}^{\bullet}_{i}$, and some G_j have valence-two vertices, then ε defines a coboundary element of $B(\mathcal{A}^{\times}_{1L})$.*

In particular, taking $i = 1$, we see that sums of graphs involving valence-two vertices have trivial motivic contributions. This is a major calculational aid in that it identifies a large class of cycles that we need not consider for motivic content. The proof of this theorem is the subject of Section 4A. See Theorem 4.16 for the graphical version of this statement. In Section 4B1 we give examples of some completely decomposable graphs.

Since we are only interested in the zeroth cohomology henceforth, for the remainder of this paper we only consider graphs in $\mathcal{G}_{1L}{}^{\bullet}_{1}$, that is, cycles in $\mathcal{A}^{\times\,\bullet}_{1L\,1}$.

4A. ***Valence-two vertices.*** In this section we show that there is a large class of graphs in $\mathcal{G}_{1L}$ that correspond to the trivial cycles in $H^i(B(\mathcal{G}_{1L}))$. Namely, we show that completely decomposable sums of graphs with two valent vertices can be written as the coboundary of an element of $\mathcal{G}_{1L}{}^{\bullet}_{i-1}$. We start by studying the properties of decomposable graphs in $\mathcal{G}_{1L}$ with two valent vertices.

Definition 4.7. A handle of length $n > 1$ is a linear subgraph $h \in G$ defined by n edges and $n+1$ vertices $\{v_0, \ldots, v_n\}$ labeled as follows: The vertex v_i has valence 2 if $1 \le i < n$. The vertex v_0 and v_n have valence 1 in the handle h, but strictly greater than 2 in G. Write $E(h) = \{e^1, \ldots, e^n\}$, with e^i the edge in h connecting v_{i-1} and v_i. Write $H(G)$ to the be set of handles of a graph G. Further, write $H_{\text{odd}}(G)$ for the set of handles of odd lengths and $H_{\text{even}}(G)$ for the set of handles of even length.

Minimally decomposable sums of graphs can be classified by the number of handles they have.

Lemma 4.8. *Consider $G \in \mathcal{G}_{1L}$, a connected graph with handles, $H(G) \neq 0$. Then*

$$\sum_{e \in E(h)} (-1)^{\omega(e)-1} \partial_e G = \begin{cases} 0 & \text{if } h \in H_{\text{even}}(G), \\ (-1)^{\omega(e^1)-1} \partial_{e^1} G & \text{if } h \in H_{\text{odd}}(G). \end{cases}$$

Proof. The essence of this proof comes from showing the relation

$$(-1)^{\omega(e^i)} \partial_{e^i} G = -(-1)^{\omega(e^{i+1})} \partial_{e^{i+1}} G. \tag{4.9}$$

To see this, choose a representation of G such that the edges of h are labeled by ones.

Write $c(e^i, e^{i+1}) \in \mathfrak{S}_{|E(G)|}$ as the cyclic element of order $|\omega(e^{i+1}) - \omega(e^i)|$. Write this as

$$c(e^i, e^{i+1}) := (\omega(e^i))(\omega(e^i)+1) \ldots (\omega(e^{i+1})-1)(\omega(e^{i+1})).$$

The sign of this permutation is given by $\operatorname{sgn}(c(e^i, e^{i+1})) = (-1)^{\omega(e^{i+1})-\omega(e^i)+1}$. In this notation, the orderings of the contracted graphs can be related by

$$\hat{\omega}_{e^{i+1}} = \begin{cases} c(e^i, e^{i+1})\hat{\omega}_{e^i} & \text{if } \omega(e^i) < \omega(e^{i+1}), \\ c(e^i, e^{i+1})^{-1}\hat{\omega}_{e^i} & \text{if } \omega(e^i) > \omega(e^{i+1}). \end{cases}$$

Since the underlying contracted graphs, $G/e^i = G/e^{i+1}$, are the same, we have, by Lemma 3.8,

$$(-1)^{\omega(e^{i+1})-\omega(e^i)+1}\partial_{e^{i+1}}(G) = \partial_{e^i} G,$$

which is equivalent to (4.9).

Summing over all edges in a fixed handle h gives

$$\sum_{e \in E(h)} (-1)^{\omega(e)}\partial_e(G) = \begin{cases} 0 & \text{if } n \text{ even}, \\ (-1)^{\omega(e^1)}\partial_{e^1}(G) & \text{if } n \text{ odd}. \end{cases} \qquad \square$$

Call edges of G that are not handles, interior edges of G.

Definition 4.10. By abuse of notation, write $\mathring{G}$ to indicate the interior graph of G. This is the graph G with all its handles removed (not contracted). More precisely,

$$\mathring{G} = G \setminus \{e \mid e \in E(h),\ h \in H(G)\}.$$

In this section, we write

$$\partial|_H(\omega, G) = \sum_{e \in H(G)} (-1)^{\omega(e)-1}\partial_e(G), \tag{4.11}$$

$$\partial|_{\mathring{G}}(G) = \sum_{e \in \mathring{G}} (-1)^{\omega(e)-1}\partial_e(G), \tag{4.12}$$

so that $\partial = \partial|_H + \partial|_{\mathring{G}}$. This allows for a neat reorganizing of the terms in the derivative ∂G by interior edges and edges with valence-two endpoints.

Corollary 4.13. *The derivative is*

$$\partial(G) = \sum_{e \in E(\mathring{G})} (-1)^{\omega(e)-1}\partial_e(G) + \sum_{e \in h,\, h \in H_{\text{odd}}(G)} (-1)^{\omega(e_1(h))-1}\partial_{e_1(h)}(G).$$

As a direct corollary, we see that graphs with valence-two vertices form a separate class of graphs in themselves. If $\varepsilon \in B(\mathcal{G}_{1L})$ is a minimally decomposable sum of graphs, then either all the summands involve a valence-two vertex, or none of them do. In fact, one can be more specific than this.

Corollary 4.14. *Consider a minimally decomposable sum of graphs $\varepsilon = \sum_j G_j$ in $\mathcal{G}_{1L}{}_i^\bullet$ of fixed degree. The summand graph G_j has a valence-two vertex if and only if the graphs in each of the summand have the same number of handles*:

$$|H(G_j)| = |H(G_{j'})| \quad \textit{for all } j \neq j'.$$

Proof. If $\partial_e(G_j)$ is not decomposable, then it must cancel with a sum of another derivative $\partial_{e'}(G_{j'})$. By Lemma 4.8 and Corollary 4.13, applying ∂ does not change the number of handles on a graph. Since ε has a minimally completely decomposable boundary, there are no summands that do not contribute to the cancellation of the terms in ∂_e. Therefore, G_j and G'_j must have the same number of handles. □

Finally, we show that sums of graphs with decomposable boundaries and valence-two vertices characterize trivial classes in $H^0(B(\mathcal{G}_{1L}))$. In the proof of this theorem, we work up to products of graphs. For this, we establish some notation.

Definition 4.15. For G a connected graph in $\mathcal{G}_{1L}$, if $\partial_e G$ is decomposable, we write

$$\partial_e G \doteq 0.$$

In general, we write $\partial G \doteq G'$, where G' is a linear sum of connected graphs, that is, G' is the sum of graphs corresponding to edge differentials that do not split the graph into two connected components.

Theorem 4.16. *Let $\varepsilon = \sum_j (G_j) \in \mathcal{G}_{1L}{}_i^\bullet$ be a sum of graphs with minimally completely decomposable boundary, such that each graph has bivalent vertices. Then there exists a sum of graphs $\eta \in \mathcal{G}_{1L}{}_{i-1}^\bullet$ such that $[\partial\eta] = [\varepsilon]$. In other words, $[\varepsilon]$ is exact.*

Translated into the language of algebraic cycles, instead of graphs, this theorem gives Theorem 4.6.

Proof. Let ε be a minimal completely decomposable sum of graphs. Write $\varepsilon = \sum_{G\in S} G$, where S is the set of summands (not including multiplicity). By Corollary 4.14, each $G \in S$ has the same number of total handles. That is, $H(G) = m$ for all $G \in S$. It suffices to work with sums of connected graphs.

We can partition the underlying set of graphs S by the number of odd handles they have. Write $\varepsilon_i = \sum_{G\in S_i} G$, where $S_i = \{G \in S \mid H_{\mathrm{odd}}(G) = i\}$. In this way, we may write

$$\varepsilon = \sum_{i=j}^{n} \varepsilon_i.$$

In other words, while every graph in S has m handles, they all have between j and n handles of odd length. From equations (4.11) and (4.12), write the differential operator as $\partial = \partial|_H + \partial|_{\mathring{G}}$. Then the sum $\partial(\varepsilon)$ decomposes into $n - j + 1$ sums that evaluate to 0, up to decomposable elements. By collecting terms according to the number of odd handles are present in the graph:

$$\begin{aligned} \partial|_{\mathring{G}}\varepsilon_n &\doteq 0, \\ \partial|_H\varepsilon_n + \partial|_{\mathring{G}}\varepsilon_{n-1} &\doteq 0, \\ &\vdots \\ \partial|_H\varepsilon_j &\doteq 0, \end{aligned} \tag{4.17}$$

with $j \geq 0$.

In order to better understand the cancellations represented by the equations in (4.17), further classify the handles with odd length of the graphs $G \in S_i$. Write

$$H_R(G) = \{h \in H_{\text{odd}}(G) \mid \partial|_h(G) + \partial|_{h'}(G') = 0 \text{ for some } G' \in S_i,\ h' \in H_{\text{odd}}(G')\}.$$

This is the set of handles that cancel with other handles of other graphs in S_i. Notice that G and G' must be different graphs, otherwise G would be a torsion element and thus 0. Similarly, write

$$H_I(G) = \{h \in H_{\text{odd}}(G) \mid \partial|_h(G) + (-1)^{\omega(e)-1}\partial e(G') = 0 \text{ for some } G' \in S_{i-1},\ e \in \mathring{G}'\}.$$

This is the set of handles that cancel with interior edges of graphs in S_{i-1}. Thus defined, $H_{\text{odd}}(G) = H_R(G) \cup H_I G$.

By construction, no graph in S has fewer than j handles of odd length. Since $\partial|_H \varepsilon_j \doteq 0$, for every $G \in S_j$ we have $H_R(G) = H_{\text{odd}}(G)$.

We define the $\eta = \sum_{i=j}^{n} \eta_{i+1}$ desired in the theorem by extending specific even handles, $h \in H_{\text{even}}(G)$, of summands of ε $(G \in S)$. The specifics of which handles are extended is described below.

The construction proceeds by induction on the number of odd handles. For $j > 0$, $H_{\text{odd}} = H_R(G) \neq \varnothing$ for all $G \in S_j$. For every graph handle pair, (G, h) and (G', h'), for $G, G' \in S_j$ and $h \in H_R(G)$, $h' \in H_R(G')$ such that $\partial|_h(G) + \partial|_{h'}(G') = 0$, there is a $\tilde{G} \in \tilde{S}_{j+1}$ with odd handles corresponding both to h and h'. Namely, this is the graph constructed by extending the even handle of G correspond to the odd handle $h' \in H_{\text{odd}}(G')$ to an odd handle. Order the edges of h and $h' \in H_{\text{odd}}(\tilde{G})$ so that $\partial|_h(\tilde{G}) = (-1)^j G$ and $\partial|_{h'}(\tilde{G}) = (-1)^j G'$.

Write

$$\eta_{j+1} = \sum_{\tilde{G} \in \tilde{S}_{j+1}} \tilde{G}$$

with $\tilde{S}_{j+1}$ the set of summands of η_{j+1}. This is a minimal set of the $\tilde{G}$s constructed above to satisfy $\partial|_H \eta_{j+1} = \varepsilon_j$.

If $j = n$, this concludes the proof, as, by construction,

$$\partial|_{\mathring{G}} \eta_{n+1} \doteq 0.$$

Therefore, $\partial \eta_{n+1} = \varepsilon_n$, as desired.

If $j = 0$, then $H_R(G) = \varnothing$ for all $G \in S_0$, and $\partial|_H \varepsilon_0 = 0$. Therefore, we may construct η_1 by extending an arbitrary even handle per graph. This construction is not unique. However, there is a restriction on the choice of edge to extend, as outlined towards the end of this proof. As above, $\partial|_H \eta_1 = \varepsilon_0$.

To understand $\partial|_{\mathring{G}} \eta_{j+1}$ for $n > j \geq 0$, note that

$$\partial|_H \partial|_{\mathring{G}} \eta_{j+1} = -\partial|_{\mathring{G}} \partial|_H \eta_{j+1} = -\partial|_{\mathring{G}} \varepsilon_j \doteq \partial|_H \varepsilon_{j+1}.$$

The first equality comes from the anticommuting of derivatives (as argued in the proof of Theorem 3.35, the second from the construction of $\partial|_H \eta_{j+1}$ and the last from (4.17). This implies that

$$\partial|_H \partial|_{\mathring{G}} \eta_{j+1} = \sum_{\tilde{G} \in \tilde{S}_{j+1}} \sum_{h \in H_I(\tilde{G})} \partial|_h \partial|_{\mathring{\tilde{G}}} \tilde{G} \doteq \sum_{G \in S_{j+1}} \sum_{h \in H_I(G)} \partial|_h G = \partial|_H \varepsilon_{j+1}.$$

By the middle equality, we may divide the summands of $\partial|_{\mathring{G}} \eta_{j+1}$ in two groups: those that correspond to elements of S_{j+1} ($\varepsilon_{j+1}(1)$) (the $G \in S_{j+1}$ such that $H_I(G) \neq \varnothing$), and those that differ from elements of S_{j+1} by the position of one handle of odd length R_{j+1}. Write

$$\partial|_{\mathring{G}} \eta_{j+1} \doteq R_{j+1} + \varepsilon_{j+1}(1),$$

where R_{j+1} is a sum of terms that differ from summands of ε_j by the placement of one odd handle, and $\varepsilon_j(1)$ are the summands of $\partial|_{\mathring{G}} \eta_{j+1}$ that are also summands of ε_j.

We continue constructing η_i by induction on i.

Define $\varepsilon_i(2) = \varepsilon_i - \varepsilon_i(1)$ to be the difference between ε_i and the quantity $\varepsilon_i(1)$ defined in the previous inductive step. Consider the sum of graphs $-R_i + \varepsilon_i(2)$, with R_i as defined in the previous inductive step. Let T_i be the underlying set of graphs in the sum $-R_i + \varepsilon_i(2)$. By construction each summand in R_i differs from a summand of $\varepsilon_i(2)$ by the placement of one odd handle. The remaining summands of $\varepsilon_i(2)$ (those that do not have a corresponding summand in R_i) are precisely the $G \in S_i$ such that $H_I(G) = \varnothing$. Construct η_{i+1} as before, comparing graphs in T_i instead of S_j.

As above, we have

$$\begin{aligned}\partial|_H \partial|_{\mathring{G}} \eta_i &= -\partial|_{\mathring{G}} \partial|_H \eta_i = -\partial|_{\mathring{G}}(-R_{i-1} + \varepsilon_{i-1}(2)) \\ &= -\partial|_{\mathring{G}}\big(-(R_{i-1} + \varepsilon_{i-1}(1)) + \varepsilon_{i-1}\big) \doteq -\partial|_{\mathring{G}}(-\partial|_{\mathring{G}} \eta_{i-1} + \varepsilon_{i-1}) \doteq \partial|_H \varepsilon_i.\end{aligned}$$

Therefore, we may write

$$\partial|_{\mathring{G}} \eta_i \doteq R_i + \varepsilon_i(1).$$

For $n > i > 1$, R_i cannot be 0. If $R_i = 0$, then $\varepsilon(1) = \varepsilon_i$, as every graph $G \in S_i$ is such that $H_R(G) \neq \varnothing$. In this case, $\partial \eta_i = \varepsilon_{i+1} + \varepsilon_i(2) - R_i$. Therefore, we may write $\partial \sum_{j+1}^{i} \eta_k \doteq \sum_j^i \varepsilon_k$, which contradict that ε is a minimal sum. However, for $n > i = 1$, one must be careful to extend even handles of ε_0 so that $R_1 \neq 0$, otherwise the induction can't continue. This choice can always be made by comparing ε_0 to ε_1.

Finally, if $i = n+1$, note that

$$\partial|_{\mathring{G}} \eta_{n+1} \doteq 0.$$

Therefore, the process terminates. ∎

So far we have shown a class of minimally decomposable sums of graphs (algebraic cycles) that give rise to trivial motives. We have said nothing about how to find such minimally decomposable sums. There is as yet a short yet significant selection of literature on trying to understand this structure [Gangl et al. 2007; 2009; Soudères 2015; 2016a]. In the next section we give some examples of minimally decomposable sums in degree 4, only one of which has been previously studied [Gangl et al. 2009]. As of yet, we do not claim to add to the existing knowledge about the structure of, and relations between, minimally decomposable sums, other than identifying further examples. In future work, we hope to return to this larger class of example to better understand which sums of graphs define classes in $H^0(B(\mathcal{G}_{1L}))$.

4B. ***Examples.*** In this section, we give several examples of classes of $H^0(B(\mathcal{G}_{1L}))$. Generally speaking, it is nontrivial to find linear combinations of graphs which define classes in $H^0(B(\mathcal{G}_{1L}))$. Individual graphs do not have decomposable boundaries. It is only when summed with appropriate graphs with whom the boundaries cancel does one find classes in $H^0(B(\mathcal{G}_{1L}))$.

In the following subsection, we give examples of several sums in weight four.

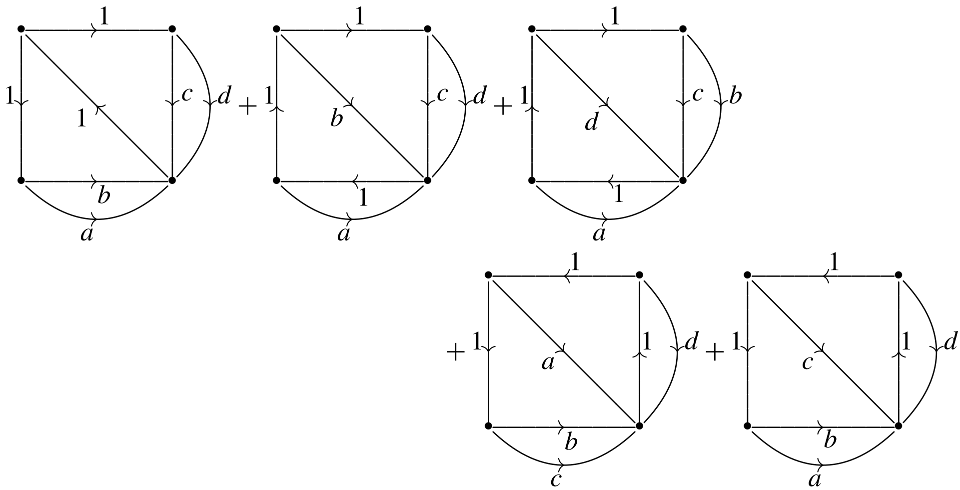
Remark 4.18. In all of these examples in this section, we write only a sum of graphs in ${\mathcal{G}_{1L}}^4_1$, and not the full representative in $B(\mathcal{G}_{1L})$. We can do this since the indecomposable graphs in a completely decomposable sum of graphs determines its class in $B(\mathcal{G}_{1L})$ (see Remark 2.10).

After giving examples in weight 4, we turn our attention to an particularly nice infinite family of graphs for which we compute the Hodge realization functor in Section 5.

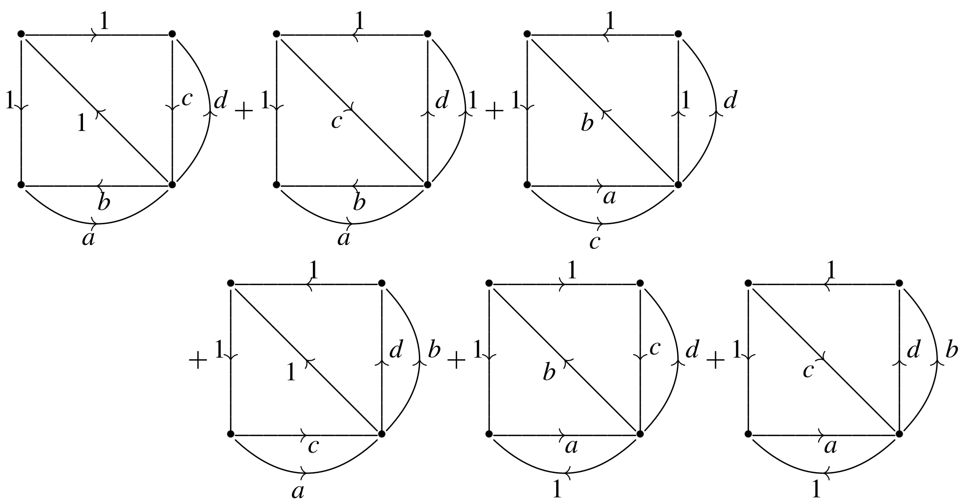
4B1. *Some minimally decomposable examples in degree* 4. In this section we give several examples of minimally decomposable sums of graphs in weight four. One of these, Example 4.19, corresponds exactly to the decomposable cycles identified in [Gangl et al. 2009] that correspond with $\mathrm{Li}_{1,1,1,1}\big(\frac{b}{a}, \frac{c}{b}, \frac{d}{c}, \frac{1}{d}\big)$. We also find a different minimally decomposable sum of graphs that involves the same unoriented graphs, but with different coefficients and orientations on the edges. In Example 4.22 we give two minimally decomposable sums that involve a different underlying graph, though closely related to the underlying graph of the previous example. Example 4.23 gives the degree-four example of the family of graphs studied in detail in Section 4B2. (In Section 5C we calculate the Hodge realization of these graphs.) Finally, Example 4.24 gives a more complicated minimally decomposable sum in degree four involving several distinct underlying graphs.

The reader is encouraged to play with these examples and construct others. There seems to be a lot of variety as to the type and number of underlying graphs in a sum that is decomposable. It would be very interesting to understand this structure better.

Example 4.19. Gangl et al. [2009] define a family of five binary graphs that correspond to $\mathrm{Li}_{1,1,1,1}\left(\frac{b}{a}, \frac{c}{b}, \frac{d}{c}, \frac{1}{d}\right)$. In the notation developed here, we depict this same minimally decomposable sum of trees as



Example 4.20. There is another decomposable sum of graphs involving the same underlying unoriented graphs:



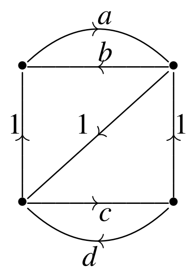
Remark 4.21. For $G \in \mathcal{G}_{1L_\star^\bullet}$ a connected graph, and $\beta = \{L_1, \dots, L_\bullet\}$ a loop basis of $H_1(G)$, let β index the system of polynomial equations f_{L_i} that define the admissible cycle $Z(G)$ in Theorem 3.19. Namely, f_L is the equation

$$1 = \prod_{e \in E(L)} a_e (1 - \phi_{w(e)})^{\epsilon(e,L)}.$$

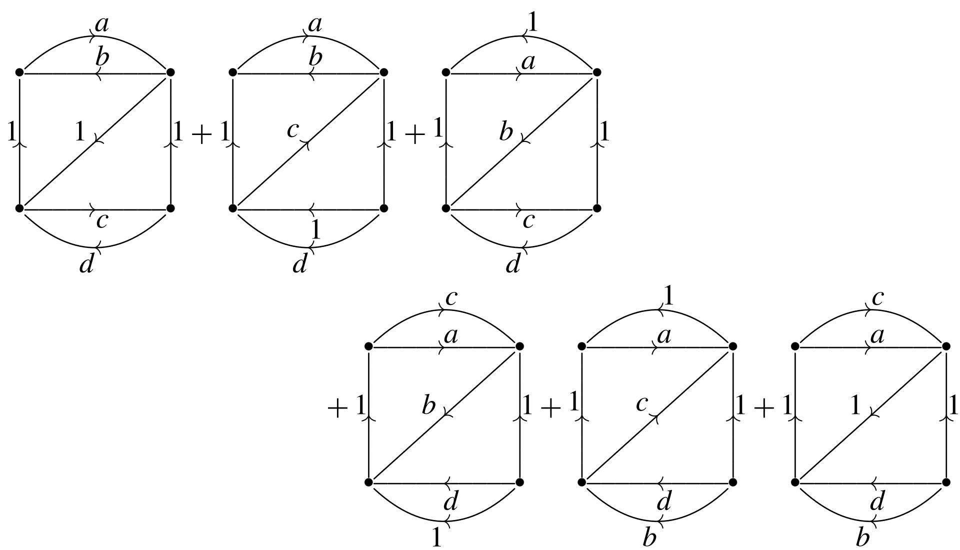
Then reversing the orientation of an edge e in graph G without changing its label replaces every factor of $a_e(1-\phi_{w(e)})$ with $(a_e(1-\phi_{w(e)}))^{-1}$. In other words, such graphs represent closely related algebraic cycles. For instance, in the above example, the first graph in Example 4.19 and the first graph in Example 4.20 differ by changing the orientations of the edge labeled b and the edge labeled d. This is also true of the last graph in the first sum and the second graph in the second sum. The second graph in the first sum and the fifth graph in the second sum differ by the orientation of the edges labeled b and d, along with the orientation of two of the edges labeled 1. Presumably these two sums of graphs give rise to closely related sums of algebraic cycles.

While the motive associated to the first sum has been studied (see [Gangl et al. 2009], for example) the other appears to be new. We suspect that they define dependent classes in $H^0(B(\mathcal{G}_{1L}))$. It would be interesting to use the Hodge realization techniques developed in Section 5 and/or other graphical tools to analyze the motives they represent.

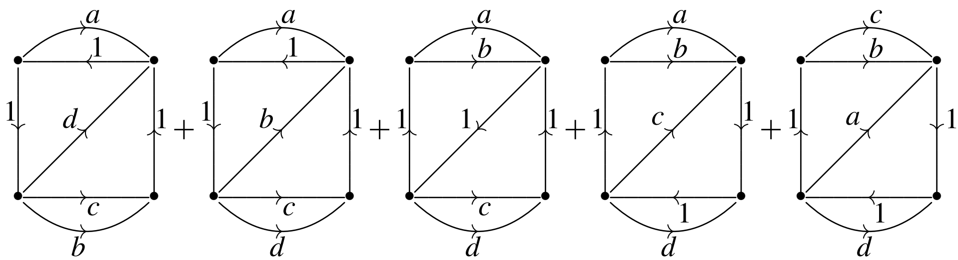
There is a related family of graphs, defined by changing the labelings and orientations of



Example 4.22. The following sum of six diagrams is minimally decomposable:

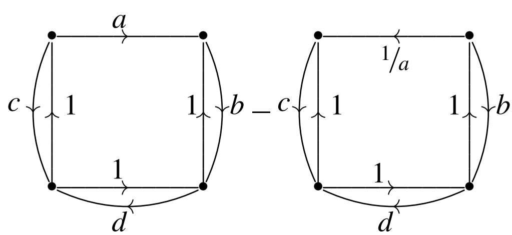


as is this sum of five related diagrams:

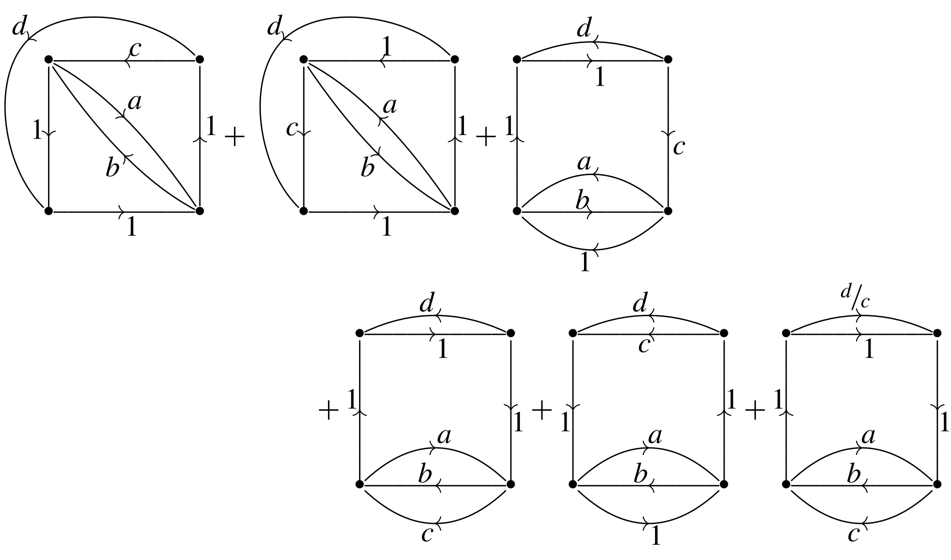


Next we present the weight-four example of the necklace graphs that are the subject of (4.26).

Example 4.23. The following sum of graphs is minimally decomposable:



Example 4.24. We end this section with a complicated minimally decomposable sum that, unlike the previous examples, involves several different types of unoriented graphs:



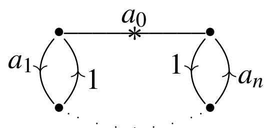
It is highly likely that the classes defined by all of the above examples are related. It would be very interesting to work out the precise dependencies.

These examples illustrate that, even in the vastly simplified case of $\mathscr{A}_{1L}^{\times}$, there is a richness and complexity amongst the minimally decomposable classes of $B(\mathscr{G}_{1L})$. By further studying these minimally decomposable sums of graphs, we hope to gain a better understanding of the structure of (our subcategory of) mixed Tate motives.

4B2. *The n-beaded necklace graph.* In this section, we introduce an infinite family of terms in $H^0(B(\mathscr{G}_{1L}))$, which we refer to as necklace diagrams. In Section 5, we show that these correspond to trivial classes.

Definition 4.25. The necklace graph with n beads is the graph of the form

$$G^*(a_0, \dots, a_n) = \tag{4.26}$$



with $* \in \{L, R\}$ (left, right) to indicate the orientation of the marked edge. The ordering is given as follows: each edge labeled a_i is in the $(2i+1)$-st position; for $i > 0$ the "parallel edge" labeled 1 (which shares vertices with that labeled a_i) is in the $2i$-th position. The signs associated to the edges are all positive.

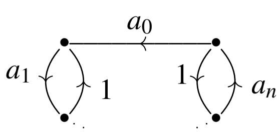
When $n = 0$, we write

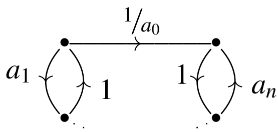
$$G(a) = G^R(a) = G^L(a) = \tag{4.27}$$

We consider the following linear combination of n-beaded necklace graphs:

$$\varepsilon^n(a_0, \dots, a_n) = G^L(a_0, a_1, \dots, a_n) - G^R\Big(\frac{1}{a_0}, a_1, \dots, a_n\Big)$$

$$= \quad - \tag{4.28}$$





To avoid extreme notational complexity in keeping track of labels of graphs, we introduce some notation.

Definition 4.29. Define a set $\mathbb{n} = \{1, \dots, n\}$. We define $\mathbb{a}_{\mathbb{n}}$ to be the n-tuple $(a_1, \dots, a_n)$. For any $S \subset \mathbb{n}$, $\mathbb{a}_{\mathbb{n}\setminus S} = (a_1, \dots, \hat{a}_S, \dots, a_n)$ is the $n-|S|$-tuple with the elements labeled by $s \in S$ removed.

Lemma 4.30. *The sum of graphs $\varepsilon^n(a_0, \mathbb{a}_{\mathbb{n}})$ is completely decomposable.*

Proof. By direct calculation,

$$\partial\varepsilon^n(a_0, \mathbb{a}_{\mathbb{n}}) = \sum_{i=1}^{n} \big(\varepsilon^{n-1}(a_0, \mathbb{a}_{\mathbb{n}\setminus i}) - \varepsilon^{n-1}(a_0 a_i, \mathbb{a}_{\mathbb{n}\setminus i})\big) \cdot G(a_i).$$

The proof follows by induction. $\square$

We explicitly write the entire minimally decomposable element of $B(\mathcal{G}_{1L})$ defined by $\varepsilon^n(a_0, \mathbb{a}_\mathbb{n})$.

Recall that

$$[a_1 \mid \cdots \mid a_n] ⧢ [b_1 \mid \cdots \mid b_m]$$

is the shuffle product of the ordered sets $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_m)$.

In particular, for $a, b \in {\mathcal{G}_{1L}}_1$,

$$a ⧢ b = [a \mid b] + [b \mid a].$$

The shuffle product $a ⧢ b$ is in $\ker \mu$. That is

$$\mu(a ⧢ b) = 0. \tag{4.31}$$

Lemma 4.32. *The element*

$$\varepsilon^\mathbb{n}(a_0, \mathbb{a}_\mathbb{n}) = \sum_{S \subset \mathbb{n}} (-1)^{|S|} \sum_{J \subseteq S} (-1)^{|J|} \Bigg[\varepsilon^{n-|S|}\Bigg(a_0 \prod_{j \in J} a_j, \mathbb{a}_{\mathbb{n}\setminus S} \Bigg) \,\Bigg|\, ⧢_{s \in S} G(a_s) \Bigg]$$

is in $H^0(B(\mathcal{G}_{1L}))$.

Proof. Recall that, since $\varepsilon^n \in {\mathcal{G}_{1L}}_1^{n+1}$, it defines an element of degree 0 in $B(\mathcal{G}_{1L})$.

Consider the component of $\varepsilon^\mathbb{n}(a_0, \mathbb{a}_\mathbb{n})$ in $B(\mathcal{G}_{1L})^0_{k+1}$. We compute $\partial + \mu$ on this term. By Lemma 4.30,

$$\begin{aligned}
&\partial \sum_{\substack{|S|=k \\ J \subseteq S}} (-1)^{|J|} \Bigg[\varepsilon^{n-k}\Bigg(a_0 \prod_{j \in J} a_j, \mathbb{a}_{\mathbb{n}\setminus S} \Bigg) \,\Bigg|\, ⧢_{s \in S} G(a_s) \Bigg] \\
&\quad = \sum_{\substack{|S|=k \\ i \in n\setminus S \\ J \subseteq S}} (-1)^{|J|} \Bigg[\Bigg(\varepsilon^{n-k-1}\Bigg(a_0 \prod_{j \in J} a_j, \mathbb{a}_{\mathbb{n}\setminus (S\cup i)} \Bigg) \\
&\qquad\qquad - \varepsilon^{n-k-1}\Bigg(\Bigg(a_0 \prod_{j \in J} a_j \Bigg) a_i, \mathbb{a}_{\mathbb{n}\setminus (S\cup i)} \Bigg) \Bigg) \cdot G(a_i) \,\Bigg|\, ⧢_{s \in S} G(a_s) \Bigg].
\end{aligned}$$

Collecting terms, the right-hand side becomes

$$\sum_{\substack{|S|=k \\ i \in n\setminus S \\ J \subseteq S\cup i}} (-1)^{|J|} \Bigg[\varepsilon^{n-k-1}\Bigg(a_0 \prod_{j \in J} a_j, \mathbb{a}_{\mathbb{n}\setminus (S\cup i)} \Bigg) \cdot G(a_i) \,\Bigg|\, ⧢_{s \in S} G(a_s) \Bigg].$$

However, by (4.31), this is

$$\mu \sum_{\substack{|S|=k+1 \\ J \subseteq S}} (-1)^{|J|} \Bigg[\varepsilon^{n-k-1}\Bigg(a_0 \prod_{j \in J} a_j, \mathbb{a}_{\mathbb{n}\setminus (S\cup i)} \Bigg) \,\Bigg|\, G(a_i) \,\Bigg|\, ⧢_{s \in S} G(a_s) \Bigg].$$

Therefore,

$$(\partial + \mu)(\varepsilon^\mathbb{n}(a_0, \mathbb{a}_\mathbb{n})) = 0. \qquad \square$$

Therefore, $\varepsilon^{\mathbb{m}}(a_0, \mathbb{a}_{\mathbb{m}})$ defines a class in $H^0(B(\mathscr{G}_{1L}))$, as stated in Remark 2.10.

Definition 4.33. Let $[\varepsilon^{\mathbb{m}}(a_0, \mathbb{a}_{\mathbb{m}})] \in H^0(B(\mathscr{G}_{1L}))$ be the class defined by $\varepsilon^{\mathbb{m}}(a_0, \mathbb{a}_{\mathbb{m}})$.

This choice of notation emphasizes that this is the class in $H^0(B(\mathscr{G}_{1L}))$ associated to an element in $\mathscr{G}_{1L}{}^{\bullet}_{1}$ with completely decomposable boundary.

5. Hodge realization

In this section we describe the Hodge realization for a number field k for our category and compute some examples. We follow the approach to constructing a Hodge realization described in [Bloch and Kriz 1994, Sections 7 and 8; Kimura 2013]. Namely, we first note that the Hodge realization as constructed in Section 7 of [Bloch and Kriz 1994] can be defined independently of choice. However, as noted at the beginning of [ibid., Section 8], this construction is not very amenable to computation, and a second description of the Hodge realization functor is given. Here we restrict to this second description of the Hodge realization. Namely, we explicitly construct a comodule J of $\mathscr{H}_T = H^0(B(\mathscr{G}_{1L}))$ and construct a natural mixed Tate Hodge structure on J. This, as in [Gangl et al. 2009], provides the Hodge realization for our graphical structure as J associates a natural mixed Tate Hodge structure on any graded comodule $\mathscr{M}$ of $\mathscr{H}_T$.

In the context of the graphs, the $\mathbb{Q}$ mixed Tate Hodge structure is given by the rational lattice

$$H_{\mathbb{Q}} = H^0(B(\mathscr{T}_{1L}^{\text{twist}}, \mathscr{G}_{1L})),$$

where $\mathscr{T}_{1L}^{\text{twist}}$ is a right $\mathscr{G}_{1L}$ module, and $B(\mathscr{T}_{1L}^{\text{twist}}, \mathscr{G}_{1L})$ is the corresponding cyclic bar construction. Both filtrations are induced from the weights of graphs (or the codimension of the corresponding cycles), as defined in Section 3A. These are introduced in detail in Section 5A.

5A. *Topologically augmented admissible graphs.*

As in [Bloch and Kriz 1994], in order to create the construction outlined above, one must define a set of topologically supported cycles in $\square^n$.

Definition 5.1. Let $\mathscr{X}^{\bullet}_{\text{top}}(\Delta_{\bullet}, \square^{2\bullet-\star})$ be the free abelian group (vector space) generated by admissible algebraic cycles supported on the image of a smooth map $\sigma : \Delta_{\bullet} \to \mathbb{P}_k(\mathbb{C})^{2\bullet-\star}$ of codimension $\bullet$ and algebraic degree $\star$. Then define a vector space $\mathscr{X}_{\text{top}} = \bigoplus_{\bullet,\star} \operatorname{Alt} \mathscr{X}^{\bullet}_{\text{top}}(\Delta_{\bullet}, \square^{2\bullet-\star})$.

These topological cycles define a means of passing from the algebraic cycles to integrals by considering the supports. In particular, given a completely decomposable element $\varepsilon \in B(\mathscr{A}^{\times}_{1L})$, with $[\varepsilon] \in H^0(B(\mathscr{A}^{\times}_{1L}))$, one considers the element $1 \otimes \varepsilon$ in the circular bar construction, $B(\mathscr{X}_{\text{top}}, \mathscr{A}^{\times}_{1L})$. This does not define a cohomology class. Namely, it is not completely decomposable. The task then is to find an element $\xi \in B(\mathscr{Z}_{\text{top}}, \mathscr{A}_{1L}^{\times})$ such that $1 \otimes \varepsilon + \xi$ is completely decomposable, that is,

$$[1 \otimes \varepsilon + \xi] \in H^0(B(\mathscr{Z}_{\text{top}}, \mathscr{A}_{1L}^{\times})).$$

It is worth noting that, while the cohomology class thus defined is unique, the element ξ need not be. In particular, in the example worked out in Section 5C2, the given ξ is by no means the only possible construction.

In the context of graphs, we parallel this construction by defining topologically augmented admissible graphs, which, under a natural extension of the homomorphism Z defined in Section 3B, correspond to elements of $\mathscr{Z}^{\bullet}_{\text{top}}(\Delta_{\bullet}, \square^{2\bullet-\star})$. These topologically augmented graphs generate a $\mathscr{G}_{1L}$ module, which we develop in this section. First we establish some notation.

Let $\Delta_n \subset \mathbb{R}^n$ be the standard real n-simplex. Let $C^{\infty}(n, m)$ be the set of smooth maps from Δ_n to $(\mathbb{P}_k(\mathbb{C})^1)^N$ of dimension m. Here N is an arbitrary integer $N \geq n$.

Definition 5.2. We say that m is the simplicial dimension of maps in $C^{\infty}(n, m)$.

Note that σ need not be injective, that is, m may be less than n. In particular, $\mathbb{C}^{\infty}(n, 0)$ consists of all constant maps from Δ_n. We view $C^{\infty}(n, m)$ as a chain complex, $C(n)_m$.

We parametrize Δ_n by an ordered set as usual, $0 \leq t_1 \leq \cdots \leq t_n \leq 1$, sometimes writing $0 = t_0$ and $1 = t_{n+1}$. Then any $\sigma \in C(n)_m$ is a continuous function of $\{t_1, \ldots, t_n\}$.

Definition 5.3. Given the standard face maps s_i and degeneracy maps d_i on Δ_n, for any subset $I \in \{t_0, \ldots, t_n\}$ of size $|I| = p$ we write d_I for the standard codimension-p degeneracy map.

Let $\mathbb{n} = \{1, \ldots, n\}$ as before. Any continuous map $\sigma \in C(n)_m$ can be written in terms of codimension-$n{-}m$ face maps. That is, there is a set $I \in \mathbb{n}$ with $|I| = m$ and $\sigma' \in C(n)_n$ such that

$$\sigma = d_{I*}\sigma'. \tag{5.4}$$

The degeneracy maps define a differential on the chain complex $C(n)_m$. In particular, we write

$$\delta_i : C(m)_m \to C(m)_{m-1}, \quad \sigma \mapsto d_{i*}s_{i*}\sigma,$$

with $\delta = \sum_{i=0}^{m} (-1)^i \delta_i$. More generally, for $\sigma \in C(n)_m$, where $\sigma = d_{I*}\sigma'$, write

$$\delta_i : C(n)_m \to C(n)_{m-1}, \quad \sigma \mapsto \delta_i d_{I*}\sigma'. \tag{5.5}$$

Therefore, we have shown:

Lemma 5.6. *For a fixed n, $(C(n)_*, \delta)$ is a chain complex.*

Remark 5.7. As in the prequel, the symbol $\bullet$ will always correspond to the codimension of a cycle (loop number of a graph). The symbol $\star$ will always correspond to the algebraic degree, and the symbol $*$ always the simplicial dimension of the graph.

Given this notation, we define the right module of topologically augmented admissible graphs. Generators of this algebra are given by the pair $\sigma \in C(n)_*$ and an admissible graph $G \in \mathcal{G}_{1}L_i^n$. In particular, the topologically augmented graph (G, σ) has edges labeled, not by elements of $k^\times$ as usual, but by the image of σ. For $t \in \Delta_n$, write $\sigma(t)$ as the $(2n-i)$-tuple $\sigma(t) = (\sigma_1(t), \dots, \sigma_{2n-i}(t))$. The coordinate $\sigma_i(t)$ labels the edge $e \in E(G)$ that is in the i-th position, that is, such that $\omega(e) = i$. There is a natural extension of the vector space homomorphism Z defined in Section 3B to the topologically augmented admissible graphs such that each graph maps to a topologically supported cycle in $\mathcal{X}_{\text{top}}$.

For each $G \in \mathcal{G}_{1}L_\star^\bullet$, $\sigma \in C(\bullet)_*$ and $t \in \Delta_\bullet$ such that $\sigma_{\omega(e)}(t) \neq 0, \infty$ for any $e \in E(G)$, the pair $(G, \sigma(t))$ defines a graph in $\mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{ori}}, \sim_{\text{v}})$. If $\sigma_{\omega(e)}(t) = 0, \infty$, we say that $(G, \sigma(t))$ is the trivial graph. As we show below, in Lemma 5.15, graphs with such labels correspond to algebraic cycles with 1 in the appropriate coordinate. In particular, for a general σ, the labels $\sigma(t)$ need not correspond to an admissible labeling of the underlying graph G. We wish to consider pairs $(G, \sigma(t))$ which evaluate to admissible graphs almost everywhere on Δ_n. Such $\sigma \in C(\bullet)_*$ are called admissible simplices for G.

Definition 5.8. A map $\sigma \in C(\bullet)_*$ is admissible for a graph G if the following hold:

(1) Let $\delta_J(\sigma)$ indicate the degeneracy map onto the face opposite that defined by J in $\Delta_\bullet$. For all J, each loop of the augmented graph $(G, \delta_J\sigma)$ does not have loop coefficient 1 almost everywhere on $\Delta_\bullet$.

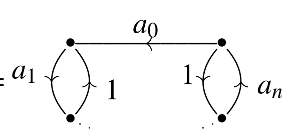
(2) For all $e \in E(G)$, if there exists a $t \in \Delta_\bullet$ such that $\sigma_{\omega(e)}(t) = 0$, there exists an $e' \in E(G)$ such that $\sigma_{\omega(e')}(t) = \infty$. Therefore, the cycle $Z(G, \sigma(t))$ is trivial.

(3) Writing $\delta\sigma = \sum_{i=0}^{\bullet}(-1)^i \delta_i\sigma$, there is some i for which no coordinate of $\delta_i\sigma$ is ∞.

We are now ready to define the vector space of admissible topologically augmented graphs.

Definition 5.9. Let $\mathcal{T}_{1}L_{2\bullet-*}^\bullet$ be the vector space of topologically augmented graphs (G, σ), with $h_1(G) = \bullet$ and $\sigma \in C(\bullet)_*$ an admissible labeling.

Example 5.10. Consider the necklace graph $G^L(a_0, \dots, a_n) \in \mathcal{G}_{1}L_1^{n+1}$

$G^L(a_0, \dots, a_n) =$ 

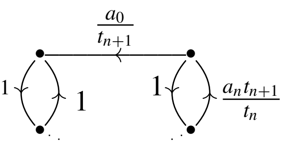
There is a constant map $\sigma \in C(n+1)_0$ of the form $\sigma(\Delta_{n+1}) = (a_0, 1, a_1, \dots, 1, a_n)$. As this has 0-dimensional topological support, this is the constant map. The pair $(G^L, \sigma) \in \mathcal{T}_{1L}{}^{n+1}_{2n+2}$ is a trivially topologically augmented graph. That is, $(G^L, \sigma) = G^L \in \mathcal{G}_{1L}{}^{n+1}_{1}$.

Consider a different map, $\sigma' \in C(n+1)_2$, of the form

$$\sigma'(\Delta_{n+1}) = \left(\frac{a_0}{t_{n+1}}, 1, a_1, \dots, 1, \frac{t_{n+1}a_n}{t_n}\right).$$

Then the pair

$$(G^L, \sigma') =$$



is an element of $\mathcal{T}_{1L}{}^{n+1}_{2n}$.

Note that $\mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*}$ is not an algebra. In particular, there is no natural product structure on $C(n)_*$. For general $(G, \sigma) \in \mathcal{T}_{1L}{}^{n}_{2n-*}$ and $(G', \sigma') \in \mathcal{T}_{1L}{}^{n'}_{2n'-*'}$, the product is given by the graph $(GG', \sigma \times \sigma')$. As in (5.4), write σ and σ' as degeneracies $d_{I*}\bar{\sigma}$ and $d_{I*}\bar{\sigma}'$ for some $\bar{\sigma} \in C(n)_n$ and $\bar{\sigma}' \in C(n')_{n'}$. However, $d_{I*}\bar{\sigma} \times d_{I*}\bar{\sigma}'$ does not correspond to a smooth map restricted to some face of $\Delta_{n\times n'}$. Therefore, we consider $\mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*}$ as a $\mathcal{G}_{1L}{}^{\bullet}_{\star}$ module.

There is an inclusion of the algebra of admissible nonaugmented graphs into $\mathcal{T}_{1L}$:

Example 5.11. There is an inclusion $\mathcal{G}_{1L}{}^{\bullet}_{\star} \hookrightarrow \mathcal{T}_{1L}{}^{\bullet}_{2\bullet}$. Any graph $G \in \mathcal{G}_{1L}{}^{\bullet}_{\star}$ can be written as $(G, d_{I*}\sigma_0)$ via the constant map

$$\sigma_0(\Delta_\bullet) = (a_1, \dots, a_{|E(G)|}),$$

where $a_{\omega(e)}$ is the label of edge $e \in E(G)$.

Proposition 5.12. *The vector space $\mathcal{T}_{1L}$ is a $\mathcal{G}_{1L}$ module.*

Proof. As done in Example 5.11, write $G \in \mathcal{G}_{1L}{}^{\bullet}_{\star}$, as $(G, \sigma) \in \mathcal{T}_{1L}{}^{\bullet}_{2\bullet}$ with $\sigma \in C(\bullet)_0$. Further consider $(G', \sigma') \in \mathcal{T}_{1L}{}^{\bullet'}_{2\bullet'-m}$ with $\sigma' \in C(\bullet')_m$.

In general, we cannot write $(G, \sigma)(G', \sigma') = (GG', \sigma \times \sigma')$ as an element in $\mathcal{T}_{1L}$. However, since $\sigma \in C(\bullet)_0$, we can rewrite this as $(GG', d_{(I'')*}\sigma_m)$, where I'' is the appropriate face in $\Delta_{\bullet+\bullet'}$.

Therefore, the product of a non-topologically augmented graph $G \in \mathcal{G}_{1L}{}^{\bullet}_{\star}$ with an augmented one $(G', \sigma') \in \mathcal{T}_{1L}{}^{\bullet'}_{2\bullet'-m}$ is

$$(G, \sigma_0) \cdot (G', \sigma') = (G \cdot G', (\sigma_0, \sigma')) \in \mathcal{T}_{1L}{}^{\bullet+\bullet'}_{2(\bullet+\bullet')-m}.$$

This gives the module structure. □

The vector space $\mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*}$ is a bigraded vector space. We may write

$$\mathcal{T}_{1L} = \bigoplus_{0\le\bullet,*} \mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*}.$$

Finally, we consider $\mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*}$ as a complex. The module has two natural differential structures on it, induced by the topological differential δ on Δ_n and the algebraic differential ∂ on $\mathcal{G}_{1L}$. Before defining these explicitly and the associated bicomplex structure on augmented graphs, it is necessary to introduce a shifted vector space, $\mathcal{T}_{1L}^{\text{twist}}$.

Definition 5.13. For $(G,\sigma) \in \mathcal{T}_{1L}$ we define a twisted module $\mathcal{T}_{1L}^{\text{twist}}$, where the grading of each element is shifted from that of $\mathcal{T}_{1L}$ by the dimension of the range of σ, i.e, the number of edges of the graph G. That is, for $G \in \mathcal{G}_{1L}{}^{\bullet}_{\star}$ and $(G,\sigma) \in \mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*}$, the same element is in $(\mathcal{T}_{1L}^{\text{twist}})^{\bullet}_{\star_t} := \mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*-n}$ for $n = 2\bullet - \star$. Henceforth define a topologically twisted degree $\star_t := \star - *$ to be the difference between the algebraic degree and topological dimension. Write

$$\mathcal{T}_{1L}^{\text{twist}} = \bigoplus_{\bullet,\star_t} (\mathcal{T}_{1L}^{\text{twist}})^{\bullet}_{\star_t}.$$

For $\sigma_m \in C(n)_m$, write $\sigma_m = d_{I*}\sigma'$ for some $\sigma' \in C(m)_m$. The topological differential, δ is induced by the differential on the chain $C(n)_m$ defined in (5.5):

$$\delta : (\mathcal{T}_{1L}^{\text{twist}})^{\bullet}_{\star_t} \to (\mathcal{T}_{1L}^{\text{twist}})^{\bullet}_{\star_t+1}, \quad (G,\sigma_m) \mapsto \sum_{i=0}^{m} (-1)^i (G, \delta_i \sigma_m). \tag{5.14}$$

This is a degree-one differential operator on $(\mathcal{T}_{1L}^{\text{twist}})$.

The algebraic differential ∂ is induced from the differential ∂ on $\mathcal{G}_{1L}$. On $\mathcal{T}_{1L}^{\text{twist}}$, vertex rescaling is a direct generalization of rescaling on $\mathcal{G}_{1L}$, allowing one to rescale by functions $\sigma \in C^\infty(|E(G)|, |E(G)|)$. For s_e and t_e the source and terminal vertices of $e \in G$, write

$$(\partial_e \sigma)_{\omega(e')} = \begin{cases} 1 & \text{if } e = e', \\ \sigma_{\omega(e')} & \text{if } s_e \text{ is not a vertex of } e', \\ \sigma_{\omega(e')}\sigma_{\omega(e)} & \text{if } s_e = t_{e'}, \\ \sigma_{\omega(e')}/\sigma_{\omega(e)} & \text{if } s_e = s_{e'}, \end{cases}$$

as one expects from vertex rescaling and Definition 3.38. Then

$$\partial : (\mathcal{T}_{1L}^{\text{twist}})^{\bullet}_{\star_t} \to (\mathcal{T}_{1L}^{\text{twist}})^{\bullet}_{\star_t+1}, \quad (G,\sigma_m) \mapsto \sum_{e \in E(G)} (-1)^{\omega(e)-1} (\partial_e G, \partial_e \sigma),$$

which is a degree-one differential operator on $\mathcal{T}_{1L}^{\text{twist}}$.

The topologically augmented graphs correspond to the vector space of topologically supported admissible algebraic cycles $\mathcal{Z}^{\bullet}_{\text{top}}(\Delta_*, \square^{2\bullet-\star})$.

Lemma 5.15. *The map Z defined in Section 3B extends to a module homomorphism*

$$Z : (\mathcal{T}_{1L}^{\text{twist}}) \to \mathcal{Z}_{\text{top}},$$

as defined in Definition 5.1.

Proof. Each edge of the augmented graph (G, σ_m) defines a coordinate $\phi_{\omega(e)} = 1 - x_e/(\sigma_{m,\omega(e)} y_e)$, where x_e and y_e are the variables associated to the source and the target vertices of the edge e as usual. Then $\phi = (\phi_1, \dots, \phi_n)$ parametrizes an algebraic cycle supported on an m-simplex in $\square^n$.

It remains to check that $Z(G, \sigma_m)$ is an admissible topologically supported cycle. By Definition 5.8, the loop number of any loop in (G, σ_m) is not 1 almost everywhere in Δ_m or on any of its faces. If $\sigma_{\omega(e)}(t) = 0$ for some $t \in \sigma_m$, then the cycle $Z(G, \sigma_m(t))$ is trivial, as the corresponding coordinate is 1. Therefore, by condition (2) of Definition 5.8, if there is some $t \in \sigma_m$ and an edge $e \in E(G)$, $Z(G, \sigma_m(t))$ is trivial. Therefore, by Theorem 3.59, $Z(G, \sigma_m)$ is admissible almost everywhere on Δ_m. $\square$

The third condition in Definition 5.8 gives rise to the following statement:

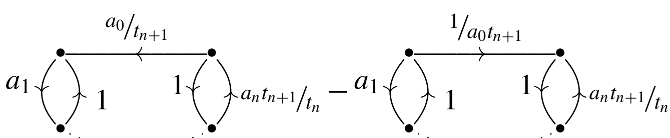
Lemma 5.16. *The image of $\mathcal{T}^{\mathrm{twist}}_{1L}$ under Z is an acyclic chain complex under δ.*

Proof. Equation (5.14) shows that $\mathcal{T}^{\mathrm{twist}}_{1L}$ is a chain complex under δ. In particular,

$$\delta : (\mathcal{T}^{\mathrm{twist}}_{1L})^{\bullet}_{\star_t} \to (\mathcal{T}^{\mathrm{twist}}_{1L})^{\bullet}_{\star_t+1}.$$

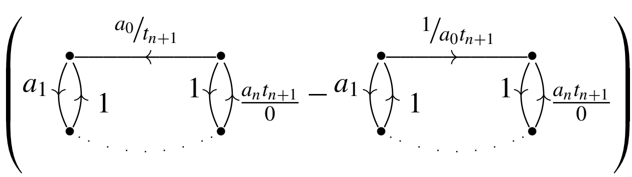
The third condition of Definition 5.8 imposes acyclicity. By Lemma 5.15, if $\delta_i\sigma$ has a coordinate set at ∞, then $Z(G, \delta_i\sigma)$ is a trivial cycle. Requiring that there is some face of $\Delta_\bullet$ such that $(\delta_i\sigma)_{\omega(e)} \neq \infty$ for all $e \in E(G)$ implies that $\delta Z(G, \sigma) \neq 0$. In other words, the image of $Z(\mathcal{T}^{\mathrm{twist}}_{1L})$ is an acyclic chain complex under δ. $\square$

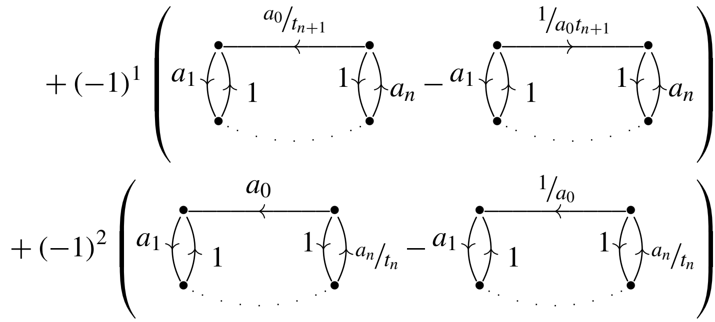
Example 5.17. In this example, we augment the sum of graphs $\varepsilon^n(a_0, \dots, a_n)$ defined in (4.28) by a 2-dimensional support, as in Example 5.10. First, recall notation from Definition 4.29. Writing $\mathbb{n} = \{1, \dots, n\}$, define an n-tuple $\mathbb{a}_{\mathbb{n}} = (a_1, \dots, a_n)$. Similarly, for any $S \subset \mathbb{n}$, write $\mathbb{a}_{\mathbb{n}\setminus S} = (a_1, \dots, \hat{a}_S, \dots, a_n)$ for the same n-tuple with the elements $\{a_s \mid s \in S\}$ removed. Then write the augmented sum of graphs

$(\varepsilon^n, \sigma(a_0, \mathbb{a}_{\mathbb{n}})_2) =$ 

Here, $\sigma(a_0, \mathbb{a}_{\mathbb{n}\setminus S})_2 \in C(n - |S| + 1)_2$ is a labeling on the decomposable sum of the $n-|S|$-beaded necklace.

Then the topological differential is

$\delta(\varepsilon^n, \sigma(a_0, \mathbb{a}_{\mathbb{n}})_2) = (-1)^0$ 

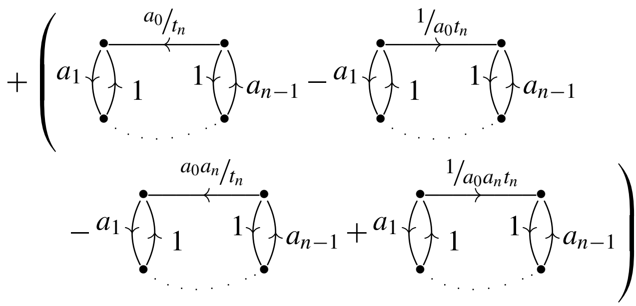


The first two terms in this sum correspond to trivial graphs.

Recall from (4.27) that $G(a)$ is the graph with a single edge and a single loop labeled by a.

The algebraic differential on this graph is

$$\partial(\varepsilon^n, \sigma(a_0, \mathbb{a}_{\mathbb{n}})_2) = \sum_{i=1}^{n-1} \big((\varepsilon^{n-1}, \sigma(a_0, \mathbb{a}_{\mathbb{n}\setminus i})_2) - (\varepsilon^{n-1}, \sigma(a_0 a_i, \mathbb{a}_{\mathbb{n}\setminus i})_2)\big)\, G(a_i)$$



$G(a_n)$.

Due to the form of the augmentation $\sigma(n+1)_2$ chosen in this example, we may write the second and third lines above as

$$\big((\varepsilon^{n-1}, \delta_{n-1}\sigma(a_0, \mathbb{a}_{\mathbb{n}-\mathbb{1}})_2) - (\varepsilon^{n-1}, \delta_{n-1}(a_0 a_n, \mathbb{a}_{\mathbb{n}-\mathbb{1}})_2)\big) \cdot G(a_n).$$

5B. ***A comodule and Hodge structure.*** We are now ready to define the Hodge comodule J.

First we build the circular bar construction $B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L}, \mathbb{Q})$. In the sequel, we take the last entry as given, and simply write $B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})$. As in [Bloch and Kriz 1994; Kriz and May 1995] and references therein, we define the $B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})$ on the tensor algebra $\mathcal{T}_{1L}^{\text{twist}} \otimes T(\mathcal{G}_{1L})/D(\mathcal{G}_{1L})$ as in Definition 2.1.

Consider $(G_0, \sigma) \otimes G_1 \otimes \cdots \otimes G_k \in B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})_w^k$ with $G_i \in {\mathcal{G}_{1L}}_{w_i}^{r_i}$ for $0 \le i \le k$, and $\sigma \in C(r_0)_m$. The total degree of this bar element is $w = \sum_{i=0}^k w_i - (k+1) - m$.

We define the bicomplex structure on it by extending the differentials from (4.3) and (4.4) for the bar construction $(B(\mathcal{G}_{1L}), \mu, \partial)$.

As before, for $j > 0$ write ∂_j to indicate the operator on $B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})$ that acts as ∂ on the j-th tensor component of $T(\mathcal{G}_{1L})$, as $(-1)^{\deg_B G_i}$ id on $\mathcal{T}_{1L}^{\text{twist}}$ and the first $j-1$ tensor components of $T(\mathcal{G}_{1L})$, and as id on the rest. As before, $\deg_B G_i$ refers to the *graphical* bar degree of the component, excluding any topological considerations. Hence, for $(G_0,\sigma)\in(\mathcal{T}_{1L}^{\text{twist}})^{\bullet}_{\star_t}$ with $\sigma\in C(\bullet)_*$, we have $\deg_B(G_0,\sigma)=\star_t+*-1=\star-1$. Define ∂_0 as $\partial+\delta$ on $\mathcal{T}_{1L}^{\text{twist}}$ and the identity on the other tensor components of the bar element. In this shifted notation, ∂_0 is a degree-one operator on $\mathcal{T}_{1L}^{\text{twist}}$.

For the product, with (G_0,σ) as above, define μ_j as the degree-one operator on $B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})$ that acts as $(-1)^{\deg_B G_0-m}\,\text{id}$ on the zeroth tensor component and as $(-1)^{\deg_B G_i}\text{id}$ on the next $j-1$ tensor components of $T(\mathcal{G}_{1L})$, as $(-1)^{\deg_B G_j}\mu$ on the j-th and $(j+1)$-st components, and as the identity on the remaining elements.

Then, in parallel to (4.4), for $\sigma\in C(\bullet)_m$ write

$$\begin{aligned}\mu&[(G_0,\sigma)\mid G_1\mid\cdots\mid G_n]\\ &:=\sum_{j=0}\mu_j[(G_0,\sigma)\mid G_1\mid\cdots\mid G_n]\\ &=\sum_{j=0}^{n-1}(-1)^{(\sum_{i=0}^{j}\deg_B G_i)-m}[(G_0,\sigma)\mid G_1\mid\cdots\mid G_j\cdot G_{j+1}\mid\cdots\mid G_n].\end{aligned}\tag{5.18}$$

Similarly, in parallel to (4.3), write

$$\begin{aligned}\partial&[(G_0,\sigma)\mid G_1\mid\cdots\mid G_n]\\ &:=\sum_{j=0}\partial_j[(G_0,\sigma)\mid G_1\mid\cdots\mid G_n]\\ &=\sum_{j=0}^{n-1}(-1)^{\sum_{i=0}^{j-1}\deg_B G_i}[(G_0,\sigma)\mid G_1\mid\cdots\mid \partial G_j\mid\cdots\mid G_n].\end{aligned}\tag{5.19}$$

In parallel to (2.2), we explicitly draw a few terms of $(B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L}),\mu,\partial)$ (recall that $\bigoplus_n(\mathcal{T}_{1L}^{\text{twist}})_i^n=B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})_i^0$):

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots \\
\cdots \arrow[r, "\mu"] & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})_0^3 \arrow[r, "\mu"] \arrow[u, "\partial"] & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})_1^2 \arrow[r, "\mu"] \arrow[u, "\partial"] & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})_2^1 \arrow[r, "\mu"] \arrow[u, "\partial"] & \bigoplus_n(\mathcal{T}_{1L}^{\text{twist}})_3^n \arrow[u, "\partial"] \\
\cdots \arrow[r, "\mu"] & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})_{-1}^3 \arrow[r, "\mu"] \arrow[u, "\partial"] & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})_0^2 \arrow[r, "\mu"] \arrow[u, "\partial"] & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})_1^1 \arrow[r, "\mu"] \arrow[u, "\partial"] & \bigoplus_n(\mathcal{T}_{1L}^{\text{twist}})_2^n \arrow[u, "\partial"] \\
\cdots \arrow[r, "\mu"] & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})_{-2}^3 \arrow[r, "\mu"] \arrow[u, "\partial"] & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})_{-1}^2 \arrow[r, "\mu"] \arrow[u, "\partial"] & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})_0^1 \arrow[r, "\mu"] \arrow[u, "\partial"] & \bigoplus_n(\mathcal{T}_{1L}^{\text{twist}})_1^n \arrow[u, "\partial"] \\
\cdots \arrow[r, "\mu"] & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})_{-3}^3 \arrow[r, "\mu"] \arrow[u, "\partial"] & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})_{-2}^2 \arrow[r, "\mu"] \arrow[u, "\partial"] & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})_{-1}^1 \arrow[r, "\mu"] \arrow[u, "\partial"] & \bigoplus_n(\mathcal{T}_{1L}^{\text{twist}})_0^n \arrow[u, "\partial"] \\
 & \vdots \arrow[u, "\partial"] & \vdots \arrow[u, "\partial"] & \vdots \arrow[u, "\partial"] & \vdots \arrow[u, "\partial"]
\end{tikzcd}$$

Definition 5.20. We may now define the comodule $J = H^0(B(\mathcal{T}_{1L}^{\mathrm{twist}}, \mathcal{G}_{1L}))$ and $J_{\mathbb{C}} = J \otimes \mathbb{C}$.

Following [Kimura 2013, Proposition 3.3], the weight filtration $W_{2r} = W_{2r-1}$ is induced by the algebraic weight (codimension) filtration on $B(\mathcal{T}_{1L}^{\mathrm{twist}}, \mathcal{G}_{1L})$. Write

$$B(\mathcal{T}_{1L}^{\mathrm{twist}}, \mathcal{G}_{1L})(r) = \mathcal{T}_{1L}^{\mathrm{twist}} \otimes B(\mathcal{G}_{1L})(r) = \mathcal{T}_{1L}^{\mathrm{twist}} \otimes \bigoplus_{k\geq 1} \bigoplus_{\sum_{i=1}^k r_i = r} {\mathcal{G}_{1L}}_\star^{r_1} \otimes \cdots \otimes {\mathcal{G}_{1L}}_\star^{r_k}.$$

Here, $B(\mathcal{G}_{1L})(r)$ is the tensor product of unaugmented graphs with total codimension r. That is, we may write

$$W_r(B(\mathcal{T}_{1L}^{\mathrm{twist}}, \mathcal{G}_{1L})) = \bigoplus_{q\leq r} B(\mathcal{T}_{1L}^{\mathrm{twist}}, \mathcal{G}_{1L})(q).$$

This induces the weight filtration on J in the usual way, $\mathrm{gr}_{2r}^W J = \mathrm{gr}_{2r-1}^W J = H^0(B(\mathcal{T}_{1L}^{\mathrm{twist}}, \mathcal{G}_{1L})(r))$. Similarly, $\mathrm{gr}_{2r}^W J_{\mathbb{C}} = \mathrm{gr}_{2r-1}^W J_{\mathbb{C}} = H^0(B(\mathcal{T}_{1L}^{\mathrm{twist}}, \mathcal{G}_{1L})(r))$.

Definition 5.21. Let

$$\Omega_n = \frac{1}{(2\pi i)^n} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$$

be the logarithmic n-form on $\square^n$.

Definition 5.22. For $(G, \sigma) \in (\mathcal{T}_{1L}^{\mathrm{twist}})_{\star_t}^{\bullet}$ and $\sigma \in C(\bullet)_*$, we define an evaluation map

$$\mathcal{I} : \mathcal{T}_{1L} \to \mathbb{C}, \quad (G, \sigma) \mapsto \int_{(G,\sigma)} \Omega_{2\bullet-\star}.$$

This integral is only well-defined if $* = 2\bullet - \star$. That is, $\sigma \in C(\bullet)_{2\bullet-\star}$. However, since $* \leq \bullet \leq 2\bullet - \star$, this implies that $* = \bullet = \star$.

Explicitly,

$$\int_{(G,\sigma)} \Omega_n = \int_{\Delta_n} \sigma_*(\Omega_m) = \frac{1}{(2\pi i)^m} \int_{\Delta_m} \frac{d(1-1/\sigma_1)}{1-1/\sigma_1} \wedge \cdots \wedge \frac{d(1-1/\sigma_m)}{1-1/\sigma_m},$$

where the ordering of the coordinates of σ are given by the ordering of the edges of G.

We call $\mathcal{I}(G, \sigma_m)$ the period associated to (G, σ_m). The evaluation map induces a quasiisomorphism between $B(\mathcal{T}_{1L}^{\mathrm{twist}}, \mathcal{G}_{1L}) \otimes \mathbb{C}$ and $B(\mathcal{G}_{1L}) \otimes \mathbb{C}$:

$$\begin{aligned} \mathcal{I} \otimes \mathrm{id} : B(\mathcal{T}_{1L}^{\mathrm{twist}}, \mathcal{G}_{1L}) \otimes \mathbb{C} &\to \mathbb{C} \otimes B(\mathcal{G}_{1L}), \\ [(G_0, \sigma) \mid G_1 \mid \cdots \mid G_n] &\mapsto \mathcal{I}(G_0, \sigma)[G_1 \mid \cdots \mid G_n]. \end{aligned} \tag{5.23}$$

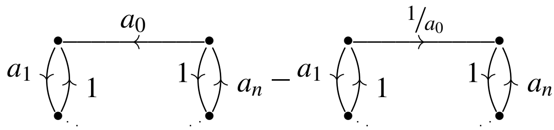
Thus (again following [Kimura 2013]) we can define the Hodge filtration by

$$F^k J_{\mathbb{C}} = \bigoplus_{r\geq k} H^0(B(\mathcal{G}_{1L}))(r) \otimes \mathbb{C}.$$

Remark 5.24. The realization functor appears to depend on choices of simplices. However, it is in fact well-defined and independent of choice, as our complex is isomorphic to a subcomplex (via the equivalence with algebraic cycles) of the full realization map on the category of mixed Tate motives as defined in Section 7 of [Bloch and Kriz 1994].

5C. ***Hodge realization for necklace diagrams.*** For the remainder of this paper, we study the Hodge realization of the specific class $[\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})] \in H^0(B(\mathcal{G}_{1L}))$. This is defined in Definition 4.33 by the sum of graphs

$$\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}}) =$$



As always, $\mathbb{a}_{\mathbb{n}}$ is the n-tuple $(a_1, \dots, a_n)$ that labels the beads of the completely decomposable sum of necklace graphs. Section 5C calculates the period of the class $[\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})] \in H^0(B(\mathcal{G}_{1L}))$ defined by this graph. In Section 5C1, we construct an element $[\xi^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}}) + 1 \otimes \varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})] \in H^0(B(\mathcal{T}^{\text{twist}}_{1L}, \mathcal{G}_{1L}))$ that defines the period. For ease of notation, we drop the arguments $(a_0, \mathbb{a}_{\mathbb{n}})$ whenever possible.

The current state of art for Hodge realization functor calculates the periods associated to elements of $H^0(B(A_{1L}))$ that can be represented by binary trees. See [Bloch and Kriz 1994; Kimura 2013] for cycles that map to classical polylogarithms, and [Gangl et al. 2007; 2009] for cycles that map to multiple polylogarithms. In this section, we compute the period associated to an algebraic cycle that is not in this small family of $\mathbb{P}^1_k$ linear cycles.

5C1. *Corresponding element of* $B(\mathcal{T}^{\text{twist}}_{1L}, \mathcal{G}_{1L})$. By Lemma 4.30, the sum of graphs ε^n is completely decomposable. Therefore, by Lemma 4.32, the sum

$$\varepsilon^{\mathbb{n}} = \sum_{S \subset \mathbb{n}} (-1)^{|S|} \sum_{J \subseteq S} (-1)^{|J|} \left[\varepsilon^{n-|S|}\left(a_0 \prod_{j \in J} a_j, \mathbb{a}_{n \setminus S} \right) \,\middle|\, \coprod_{s \in S} G(a_s) \right]$$

is a representative element defining the class $[\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})] \in H^0(B(\mathcal{G}_{1L}))$.

In this section, we define an element $\xi^{\mathbb{n}} \in \bigoplus_{i=1}^{n+1} B(\mathcal{T}^{\text{twist}}_{1L}, \mathcal{G}_{1L})^i_0$ such that $\xi^{\mathbb{n}} + 1 \otimes \varepsilon^{\mathbb{n}}$ defines a class in $H^0(B(\mathcal{T}^{\text{twist}}_{1L}, \mathcal{G}_{1L}))$. Since $(\mu + \partial)\varepsilon^{\mathbb{n}} = 0$ in $B(\mathcal{G}_{1L})$, we see that $(\mu + \partial) 1 \otimes \varepsilon^{\mathbb{n}} = \varepsilon^{\mathbb{n}}$, seen as an element in $\bigoplus_{i=1}^{n} B(\mathcal{T}^{\text{twist}}_{1L}, \mathcal{G}_{1L})^i_1$. Here, as in Example 5.11, we write

$$\varepsilon^n = (\varepsilon^n, \sigma(a_0, \mathbb{a}_{\mathbb{n}})_0) \in \mathcal{T}^{\text{twist}}_{1L}(n)^1.$$

It is sufficient to identify an element $\xi^{\mathbb{n}} \in \bigoplus_{i=1}^{n+1} B(\mathcal{T}^{\text{twist}}_{1L}, \mathcal{G}_{1L})^i_0$ such that

$$(\partial + \mu)\xi^{\mathbb{n}} = -\varepsilon^{\mathbb{n}}. \tag{5.25}$$

The remainder of this section is devoted to identifying $\xi^{\mathbb{n}}$, which is a complicated sum of elements in the circular bar construction. We introduce it in stages, starting with the easiest to state, then breaking each sum into component pieces in order to demonstrate the appropriate properties. We state what criteria these summands need to satisfy, and provide proofs along the way.

Write $\xi^{\mathbb{n}} = \sum_{k=0}^{n}(-1)^k \xi^{n-k}$, with $\xi^{n-k} \in B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})_0^k$ defined as

$$\xi^{n-k} = \sum_{\substack{S\subset \mathbb{n}\\ |S|=k}} \xi_{\text{top}}^{n-k}(a_0, \mathbb{a}_{\mathbb{n}\setminus S}) \otimes \coprod_{i\in S} G(a_i).$$

Here $\xi_{\text{top}}^{n-k}(a_0, \mathbb{a}_{\mathbb{n}\setminus S})$ is a topologically augmented graph in $(\mathcal{T}_{1L}^{\text{twist}})_0^{n-k+1}$ such that

$$(\partial+\delta)\partial\xi_{\text{top}}^{n-k}(a_0, \mathbb{a}_{\mathbb{n}\setminus S}) + \mu\Bigg(\sum_{i\in\mathbb{n}\setminus S} \xi_{\text{top}}^{n-k-1}(a_0, \mathbb{a}_{\mathbb{n}\setminus S\cup i}) \otimes G(a_i)\Bigg) = -\varepsilon^{n-k}(a_0, \mathbb{a}_{\mathbb{n}\setminus S}). \quad (5.26)$$

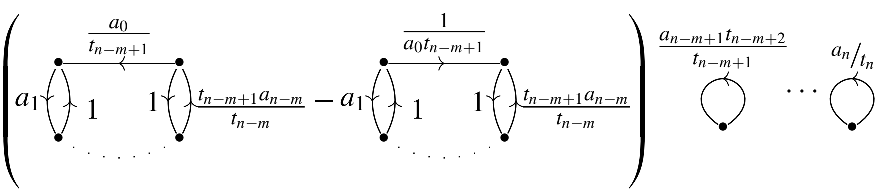
This is the key condition that we prove explicitly in Theorem 5.30.

In order to define ξ_{top}^n, we begin with a family of disconnected sums of unaugmented graphs

$$\xi_m^n(a_0, \mathbb{a}_{\mathbb{n}}) = \varepsilon^{n-m}(a_0, \mathbb{a}_{\mathbb{n}-\mathbb{m}})G(a_{n-m+1})\cdots G(a_n).$$

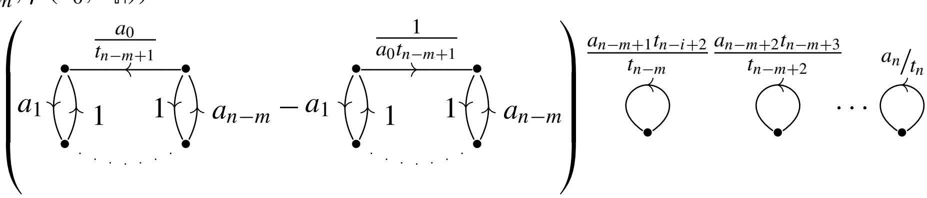
Each graph $\xi_m^n \in \mathcal{G}_1{L}_{m+1}^{n+1}$ consists of $m+1$ connected components, with graphical degree $m+1$. We impose upon this family of graphs two topological augmentations $\sigma(a_0, \mathbb{a}_{\mathbb{n}})$ and $\rho(a_0, \mathbb{a}_{\mathbb{n}}) \in C(n+1)_{m+1}$ of the form

$(\xi_m^n, \sigma(a_0, \mathbb{a}_{\mathbb{n}}))$

$=$ 

and

$(\xi_m^n, \rho(a_0, \mathbb{a}_{\mathbb{n}}))$

$=$ 

Note that the only difference between the labeling $\sigma(a_0, \mathbb{a}_\mathbb{n})_m$ and $\rho(a_0, \mathbb{a}_\mathbb{n})_m$ is the label on the last bead of the first connected component, $\varepsilon^{n-m}(a_0, \mathbb{a}_{\mathbb{n}-\mathbb{m}})$, and that of the second connected component. This distinction is necessary for the appropriate cancellations between algebraic and topological differentials needed to satisfy condition (5.26). Before writing down the expression for ξ^{n-k}_{top}, we introduce some further notation to simplify the expression.

We define two new terms as sums of ξ^n_m with variants of σ and ρ:

$$\lambda^n_m(a_0, \mathbb{a}_\mathbb{n}) = \sum_{J \subset \{n-m+1,\dots,n\}} (-1)^{|J|} \left(\xi^n_m, \sigma\left(a_0 \prod_{j \in J} a_j, \mathbb{a}_\mathbb{n} \right)\right), \tag{5.27}$$

$$\chi^n_m(a_0, \mathbb{a}_\mathbb{n}) = \sum_{I \subset \{n-m+2,\dots,n\}} (-1)^{|I|} \left(\xi^n_m, \rho\left(a_0 \prod_{i \in I} a_i, \mathbb{a}_\mathbb{n} \right)\right). \tag{5.28}$$

Under this notation, we write

$$\xi^n_{\mathrm{top}} = \sum_{m=0}^{n} \lambda^n_m - \sum_{m=1}^{n} \chi^n_m.$$

Note that sum for χ^n_m starts at $m = 1$ whilst the sum for λ^n_m starts at $m = 0$. Furthermore, the sets I and J differ. Namely, the first argument for ρ, augmenting χ^n_m, never contains a_{n-m+1}, while this label appears in the first argument of σ summands of λ^n_m. The terms λ^n_m, χ^n_m and ξ^n_{top} are constructed so that the summands of the differentials of λ^n_m cancel with terms in the differentials of χ^n_{m+1} and terms of the form $\sum_{i \in \mathbb{n}} \xi^{n-1}_{\mathrm{top}}(a_0, \mathbb{a}_{\mathbb{n}\setminus i})$ leaving the term ε^n. This is how ξ^{n-k}_{top} satisfies (5.26). We show this cancellation explicitly in Theorem 5.30.

The unaugmented graphs ξ^n_m are in $\mathscr{G}_1{L^{n+1}_{m+1}}$. Therefore the augmented graphs λ^n_m and χ^m_n are in $\mathscr{T}^{\mathrm{twist}}_{1L}(n+1)^0$. Furthermore, ξ^n_{top} is a sum of admissible augmented graphs. If $t_{n-m+k} = 0$, then $t_{n-m+i} = 0$ for all $i < k$. Therefore, the edges labeled a_0/t_{n-m+1} and $1/(a_0 t_{n-m+1})$ are labeled by ∞, making the graphs $(\xi^n_m, \sigma(n+1)_{m+1})(a_0, \dots, a_n)$ and $(\xi^n_m, \rho(n+1)_{m+1})(a_0, \dots, a_n)$ trivial at this point.

Remark 5.29. Recall that, as shown in Lemma 5.15, the labels on the edges of these graphs correspond to the coefficients of the coordinates of the cycles. That is, the augmented cycle is parametrized $\phi_{2n+1} = 1 - x a_0 t_{n-m+1}/y$. Therefore, if $t_{n-m+1} = 0$, then $\phi_{2n+1} = 1$.

It remains to check that $\xi^{\mathbb{n}}$ defined above satisfies the necessary conditions.

Theorem 5.30. *The element $\xi^{\mathbb{n}} + 1 \otimes \varepsilon^{\mathbb{n}} \in \bigoplus_{i=1}^{n+1} B(\mathscr{T}^{\mathrm{twist}}_{1L}, \mathscr{G}_{1L})^i_0$ defines a class in $H^0(B(\mathscr{T}^{\mathrm{twist}}_{1L}, \mathscr{G}_{1L}))$.*

Proof. By the arguments presented in this section, it is sufficient to check that ξ^n_{top} satisfies (5.26). It is enough to show this for $k = 0$.

We proceed by computing the four terms of $(\delta+\partial)(\lambda_m^n - \chi_m^n)$ to show that

$$(\delta+\partial)\xi_{\text{top}}^n = -\varepsilon^n - \mu\Bigg(\sum_{i\in\mathbb{n}} \xi_{\text{top}}^{n-1}(a_0, \mathbb{a}_{\mathbb{n}\setminus i}) \otimes G(a_i)\Bigg),$$

as required.

When $m=0$, the graph $(\xi_0^n, \sigma(a_0, \mathbb{a}_{\mathbb{n}})) = \lambda_0^n$ is augmented by a 1-simplex with topological boundary

$$\delta\lambda_0^n = -\delta^1\lambda_0^n = -\varepsilon^n.$$

For more general m, the algebraic boundary of the augmented sum of graphs λ_m^n is

$$\partial\lambda_m^n = -\mu\Bigg(\sum_{i=1}^{n-m-1} \lambda_m^{n-1}(a_0, \mathbb{a}_{\mathbb{n}\setminus i}) - \lambda_m^{n-1}(a_0a_i, \mathbb{a}_{\mathbb{n}\setminus i}) \otimes G(a_i)\Bigg) + \delta^2\chi_{m+1}^n(a_0, \mathbb{a}_{\mathbb{n}}) - \delta^1\chi_{m+1}^n(a_0a_{n-m}, \mathbb{a}_{\mathbb{n}}). \tag{5.31}$$

The algebraic boundary of the augmented sum of graphs χ_m^n is

$$-\partial\chi_m^n = \mu\Bigg(\sum_{i=1}^{n-m} \chi_m^{n-1}(a_0, \mathbb{a}_{\mathbb{n}\setminus i}) - \chi_m^{n-1}(a_0a_i, \mathbb{a}_{\mathbb{n}\setminus i}) \otimes G(a_i)\Bigg). \tag{5.32}$$

For $m \geq 1$, the topological boundary of the augmented sum of graphs λ_m^n is

$$\delta\lambda_m^n = -\delta^1\chi_m^n(a_0, \mathbb{a}_{\mathbb{n}}) + \delta^1\chi_m^n(a_0a_{n-m+1}, \mathbb{a}_{\mathbb{n}}) - \mu\Bigg(\sum_{i=n-m+1}^{n} \lambda_{m-1}^{n-1}(a_0, \mathbb{a}_{\mathbb{n}\setminus i}) \otimes G(a_i)\Bigg). \tag{5.33}$$

The topological boundary of the augmented sum of graphs χ_m^n is

$$-\delta\chi_m^n = \delta^1\chi_m^n(a_0, \mathbb{a}_{\mathbb{n}}) - \delta^2\chi_{m+1}^n(a_0, \mathbb{a}_{\mathbb{n}}) + \mu\Bigg(\sum_{i=n-m+1}^{n} \chi_{m-1}^{n-1}(a_0, \mathbb{a}_{\mathbb{n}\setminus i}) \otimes G(a_i)\Bigg). \tag{5.34}$$

Adding up equations (5.31), (5.32), (5.33) and (5.34), we see that

$$(\delta+\partial)\xi_{\text{top}}^n = -\varepsilon_n - \mu\Bigg(\sum_{i\in\mathbb{n}} \xi_{\text{top}}^{n-1}(a_0, \mathbb{a}_{\mathbb{n}\setminus i}) \otimes G(a_i)\Bigg),$$

which matches (5.26). □

5C2. *Integrals associated to necklace diagrams.* This section is devoted to calculating the period associated to $\varepsilon^{\mathbb{n}}$. We show that this is 0 for $n \geq 1$.

By abuse of notation, in this section we write the augmented graphs λ_m^n and χ_m^n as

$$\lambda_m^n = \varepsilon^{n-m}\left(\frac{a_0}{t_{n-m+1}}, \mathbb{a}_{\mathbb{n}-\mathbb{m}-\mathbb{1}}, a_{n-m}\frac{t_{n-m+1}}{t_{n-m}}\right)G\left(a_{n-m+1}\frac{t_{n-m+2}}{t_{n-m+1}}\right)\cdots G\left(\frac{a_n}{t_n}\right),$$
$$\chi_m^n = \varepsilon^{n-m}\left(\frac{a_0}{t_{n-m+1}}, \mathbb{a}_{\mathbb{n}-\mathbb{m}}\right)G\left(a_{n-m+1}\frac{t_{n-m+2}}{t_{n-m}}\right)G\left(a_{n-m+2}\frac{t_{n-m+3}}{t_{n-m+2}}\right)\cdots G\left(\frac{a_n}{t_n}\right).$$

Theorem 5.35. *The period associated to $\xi^{\mathbb{n}} + 1\otimes\varepsilon^{\mathbb{n}}$ is 0 for all n. Therefore, $[\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})] \in H^0(B(\mathcal{G}_{1L}))$ defines a trivial cohomology class.*

Proof. We apply the map $\mathcal{I}\otimes\mathrm{id}$ from (5.23) to the element $\xi^{\mathbb{n}} + 1\otimes\varepsilon^{\mathbb{n}}$. This integral is only well defined when m, the simplicial dimension of the augmented graph, is equal to n, the loop number of the graph. Therefore,

$$\mathcal{I}\xi^n = \sum_{k=0}^{n}(-1)^k \sum_{\substack{S\subset\mathbb{n}\\|S|=k}} \mathcal{I}\otimes\mathrm{id}\big(\lambda_{n-k}^{n-k}(a_0, \mathbb{a}_{\mathbb{n}\setminus S}) - \chi_{n-k}^{n-k}(a_0, \mathbb{a}_{\mathbb{n}\setminus S})\big)\Big[\coprod_{s\in S} G(a_s)\Big].$$

Since $\mathcal{I}(1) = 0$, the evaluation map is $(\mathcal{I}\otimes\mathrm{id})(1\otimes\varepsilon^{\mathbb{n}}) = 0$.

Recall that $\varepsilon_0(a_0) = G(a_0) - G(1/a_0)$. Therefore, from equations (5.27) and (5.28), we have

$$\lambda_n^n(a_0, \mathbb{a}_{\mathbb{n}}) = \sum_{J\subset\{1,\dots,n\}} (-1)^{|J|}\varepsilon_0\left(a_0\prod_{j\in J} a_j\frac{1}{t_0}\right)G\left(a_1\frac{t_2}{t_1}\right)\cdots G\left(a_n\frac{1}{t_n}\right),$$
$$\chi_n^n(a_0, \mathbb{a}_{\mathbb{n}}) = \sum_{I\subset\{2,\dots,n\}} (-1)^{|I|}\varepsilon_0\left(a_0\prod_{i\in I} a_i\frac{1}{t_1}\right)G\left(a_1\frac{t_2}{t_0}\right)\cdots G\left(a_n\frac{1}{t_n}\right).$$

Collecting like terms, we write

$$\begin{aligned}\lambda_n^n - \chi_n^n = \Bigg(&\sum_{J\subset\{1,\dots,n\}} (-1)^{|J|}\varepsilon_0\left(a_0\prod_{j\in J} a_j\frac{1}{t_0}\right)G\left(a_1\frac{t_2}{t_1}\right)\\ &- \sum_{I\subset\{2,\dots,n\}} (-1)^{|I|}\varepsilon_0\left(a_0\prod_{i\in I} a_i\frac{1}{t_1}\right)G\left(a_1\frac{t_2}{t_0}\right)\Bigg)\\ &\times G\left(a_2\frac{t_3}{t_2}\right)\cdots G\left(a_n\frac{1}{t_n}\right). \end{aligned} \tag{5.36}$$

To evaluate this integral, we recall a few facts about the iterated integrals associated to multiple polylogarithms. First of all, for a constant cycle supported on a 1-simplex,

$$\mathcal{I}(G(a/t)) = \int_0^1 \frac{d(1-t/a)}{1-t/a} = -\int_0^1 \frac{dt}{a-t} = -\mathrm{Li}\left(\frac{1}{a}\right).$$

Inverting the label of the edge gives

$$\mathcal{I}(G(t/a)) = \int_0^1 \frac{d(1-a/t)}{1-a/t} = \int_0^1 \frac{a\,dt}{t^2-at} = -\int_0^1 \frac{dt}{t} - \int_0^1 \frac{dt}{a-t}.$$

Subtracting the second expression from the first gives $\mathcal{I}(\varepsilon_0(a/t)) = \int_0^1 (1/t)\,dt$. We may write this as $\mathrm{Li}_1(1) = 0$, by standard renormalization of polylogarithms [Goncharov 2001b].

Similarly, for the cycle supported on a two-simplex,

$$\mathcal{I}(G(a/t_0)G(b/t_1)) = -\int_0^1 \frac{1}{b-t_1}\Bigl(\int_0^{t_1} \frac{dt_0}{t_0}\Bigr)\,dt_1 = \mathrm{Li}_2\Bigl(\frac{1}{b}\Bigr).$$

The last equality in this equation comes from the shuffle product on iterated integrals:

$$\Bigl(\int_0^z \frac{dt}{b-t}\Bigr)\Bigl(\int_0^z \frac{ds}{s}\Bigr) = \int_0^z \frac{1}{b-t}\Bigl(\int_0^t \frac{ds}{s}\Bigr)\,dt + \int_0^z \frac{1}{s}\Bigl(\int_0^s \frac{dt}{b-t}\Bigr)\,ds.$$

By the standard regularization arguments above, the left-hand side is 0. Therefore,

$$\int_0^z \frac{1}{t-b}\Bigl(\int_0^t \frac{ds}{s}\Bigr)\,dt = \mathrm{Li}_2\Bigl(\frac{1}{b}\Bigr). \tag{5.37}$$

This does not depend on the first argument, a. Therefore, the alternating signs in the sums for λ_1^1 and χ_1^1 force $\mathcal{I}(\lambda_1^1(a,b)) = \mathcal{I}(\chi_1^1(a,b)) = 0$.

For cycles supported on a three-simplex, there are two terms to check:

$$\mathcal{I}(\lambda_2^2(a,b,c)) = (-1)^2\int_0^1 \frac{1}{c-t_2}\Bigl(\int_0^{t_2} \frac{1}{bt_2-t_1}\Bigl(\int_0^{t_1} \frac{dt_0}{t_0}\Bigr)\,dt_1\Bigr)\,dt_2 = \mathrm{Li}_1\Bigl(\frac{1}{c}\Bigr)\mathrm{Li}_2\Bigl(\frac{1}{b}\Bigr)$$

and

$$\mathcal{I}(\chi_2^2(a,b,c)) = (-1)^2\int_0^1 \frac{1}{c-t_2}\Bigl(\int_0^{t_2} \frac{1}{t_1}\Bigl(\int_0^{t_1} \frac{dt_0}{bt_2-t_0}\Bigr)\,dt_1\Bigr)\,dt_2 = \mathrm{Li}_1\Bigl(\frac{1}{c}\Bigr)\mathrm{Li}_2\Bigl(\frac{1}{b}\Bigr).$$

As before, since neither integral depends on a, the alternating signs in the sums for λ_2^2 and χ_2^2 force both $\mathcal{I}(\lambda_2^2(a,b,c)) = \mathcal{I}(\chi_2^2(a,b,c)) = 0$.

For a general $n+1$-simplex, we have

$$\mathcal{I}(\xi_n^n, \sigma(a_0, \mathbb{a}_{\mathbb{n}})_{n+1}) = (-1)^n \prod_{i=2}^n \mathrm{Li}_1\Bigl(\frac{1}{a_i}\Bigr)\mathrm{Li}_2\Bigl(\frac{1}{a_1}\Bigr).$$

Similarly,

$$\mathcal{I}(\xi_n^n, \rho(a_0, \mathbb{a}_{\mathbb{n}})_{n+1}) = (-1)^n \prod_{i=2}^n \mathrm{Li}_1\Bigl(\frac{1}{a_i}\Bigr)\mathrm{Li}_2\Bigl(\frac{1}{a_1}\Bigr).$$

Since neither of these expressions depend on a_0 we have that $\mathcal{I}(\lambda_n^n)(a_0, \mathbb{a}_{\mathbb{n}}) = \mathcal{I}(\chi_n^n)(a_0, \mathbb{a}_{\mathbb{n}}) = 0$. Therefore, $\mathcal{I}(\xi_n^{\mathrm{top}}) = 0$ for all n. This is the period associated to ε^n.

This implies that

$$\mathcal{I} \otimes \mathrm{id}(\xi^n + 1 \otimes \varepsilon^n) = 0$$

for all n. Therefore this defines a trivial class in $H^0(B(\mathcal{G}_{1L}))$. $\square$

6. Outlook and future work

This paper is a first step in a program to understand the cohomology of (part of) the Bloch–Kriz cycle complex, and by extension to understand the motives associated to these cycles. By introducing a graphical representation of certain cycles, we pave the way for graph-theoretic methods to be added to the list of tools used to tackle the problem of understanding mixed Tate motives, the algebra of multiple zeta values, and the relations between such values.

Some topics for future study include:

(1) We have not yet dealt systematically with relations between closely related minimally decomposable sums. In particular, we expect the examples listed in Section 4B1 to all be related. A further analysis of these classes, their Hodge realizations, and generalizations of these classes, should give insight into constructing relations among motives and hopefully corresponding relations among the associated periods.

(2) We excluded graphs with edges labeled by 0, i.e., precisely the graphs needed to correspond to the classical polylogarithms. There is an unwritten conjecture of Brown and Gangl that only the multiple logarithms are necessary to generate the entire space of multiple polylogarithms (including the standard polylogarithms). If one assumes this conjecture, then our inability to label our edges with 0 is not a significant setback. However, in future work, we hope to devise a way of encoding edges labeled by 0s, possibly by including colored, unoriented edges, so that all the results of this paper hold in the new general setting.

(3) The graphs we study lend themselves easily to study via the language of matroids. Roughly speaking, a matroid is a combinatorial way of encoding the independence data of a matrix or graph (in this case, the subtrees of a graph). While simple to define, this is a powerful tool when it comes to studying boundaries of geometric objects. We hope that this will lead to some insight for an algorithm for finding, or a classification of, sums of algebraic cycles that lead to elements with completely decomposable boundary.

(4) The Hodge realization functor is admittedly difficult to compute explicitly. The computation of the Hodge realization in the section above, though comparable to previous computations using algebraic cycles, does not really use the graphical machinery developed earlier. We hope in future work to give a simpler and more graphically intuitive description of the Hodge realization.

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