# ANNALS OF K-THEORY

Equivariant vector bundles, their derived category
and *K*-theory on affine schemes

Amalendu Krishna and Charanya Ravi

**msp**

# Equivariant vector bundles, their derived category and *K*-theory on affine schemes

Amalendu Krishna and Charanya Ravi

Let $G$ be an affine group scheme over a noetherian commutative ring $R$. We show that every $G$-equivariant vector bundle on an affine toric scheme over $R$ with $G$-action is equivariantly extended from $\mathrm{Spec}(R)$ for several cases of $R$ and $G$.

We show that, given two affine schemes with group scheme actions, an equivalence of the equivariant derived categories implies isomorphism of the equivariant $K$-theories as well as equivariant $K'$-theories.

## 1. Introduction

The goal of this paper is to answer some well-known questions related to group scheme actions on affine schemes over a fixed affine base scheme. Our particular interest is to explore when are the equivariant vector bundles on such schemes equivariantly trivial and when does an equivalence of their derived categories imply homotopy equivalence of the equivariant $K$-theory. Both questions have been extensively studied and are now satisfactorily answered in the nonequivariant case (see [Lindel 1981; Rickard 1989; Dugger and Shipley 2004]).

**1A.** *Equivariant Bass–Quillen question.* The starting point for the first question is the following classical problem from [Bass 1973, Problem IX]:

**Conjecture 1.1** (Bass–Quillen). *Let $R$ be a regular commutative noetherian ring of finite Krull dimension. Then every finitely generated projective module over the polynomial ring $R[x_1, \ldots, x_n]$ is extended from $R$.*

The most complete answer to this conjecture was given by Lindel [1981], who showed (based on the earlier solutions by Quillen and Suslin when $R$ is a field) that the above conjecture has an affirmative solution when $R$ is essentially of finite type over a field. For regular rings which are not of this type, some cases have been solved (see [Rao 1988], for example), but the complete answer is still unknown.

In this paper, we are interested in the equivariant version of this conjecture, which can be loosely phrased as follows.

Let $R$ be a noetherian regular ring and let $G$ be a flat affine group scheme over $R$. Let $A = R[x_1, \ldots, x_n]$ be a polynomial $R$-algebra with a linear $G$-action and let $P$ be a finitely generated $G$-equivariant projective $A$-module. The equivariant version of the above conjecture asks:

**Question 1.2.** *Is $P$ an equivariant extension of a $G$-equivariant projective module over $R$?*

The equivariant Bass–Quillen question was studied, for example, in [Knop 1991; Kraft and Schwarz 1992; 1995; Masuda et al. 1996] when $R = \mathbb{C}$ is the field of complex numbers. This question is known to be very closely related to the linearization problem for reductive group action on affine spaces.

The first breakthrough was achieved by Knop [1991], who found counterexamples to this question when $G$ is a nonabelian reductive group over $\mathbb{C}$. In fact, he showed that every connected reductive nonabelian group over $\mathbb{C}$ admits a linear action on a polynomial ring for which the equivariant Bass–Quillen conjecture fails. Later, such counterexamples were found by Masuda and Petrie [1995] when $G$ is a finite nonabelian group. Thus the only hope to prove this conjecture is when $G$ is diagonalizable. It was subsequently shown by Masuda, Moser-Jauslin and Petrie [Masuda et al. 1996] that the equivariant Bass–Quillen conjecture indeed has a positive solution when $R = \mathbb{C}$ and $G$ is diagonalizable. This was independently shown also by Kraft and Schwarz [1995].

It is not yet known if the equivariant Bass–Quillen conjecture has a positive solution over any field other than $\mathbb{C}$. One of the two goals of this paper is to solve the general case of the equivariant Bass–Quillen question for diagonalizable group schemes over an arbitrary ring or field. Our approach to solving this problem in fact allows us to prove the stronger assertion that such a phenomenon holds over all affine toric schemes over an affine base. This approach was motivated by a similar result of Masuda [1998] over the field of complex numbers.

Let $R$ be a commutative noetherian ring. Recall from [SGA 3$_{\mathrm{II}}$ 1970, Exposé VIII] that an affine group scheme $G$ over $R$ is called *diagonalizable* if there is a finitely generated abelian group $P$ such that $G = \mathrm{Spec}(R[P])$, where $R[P]$ is the group algebra of $P$ over $R$.

Let $L$ be a lattice and let $\sigma \subseteq L_{\mathbb{Q}}$ be a strongly convex, polyhedral, rational cone. Let $\Delta$ denote the set of all faces of $\sigma$. Let $A = R[\sigma \cap L]$ be the monoid algebra over $R$. Let $\psi : L \to P$ be a homomorphism which makes $\mathrm{Spec}(A)$ a scheme with $G$-action. Let $A^G$ denote the subring of $G$-invariant elements in $A$. Let us assume that every finitely generated projective module over $R[Q]$ is extended from $R$ if $Q$ is torsion-free (see Theorem 5.2).

Our main result can now be stated as follows (see Theorem 7.8). The underlying terms and notations can be found in the body of this text.

**Theorem 1.3.** *Let R and A be as above. Assume that all finitely generated projective modules over $A_\tau$ and $(A_\tau)^G$ are extended from R for every $\tau \in \Delta$. Then every finitely generated G-equivariant projective A-module is equivariantly extended from R.*

For examples of rings satisfying the hypothesis of the theorem, see Sections 5, 6 and 7.

Let us now assume that *R* is either a PID, or a regular local ring of dimension at most 2, or a regular local ring containing a field. As a consequence of the above theorem, we obtain the following solution to the equivariant Bass–Quillen question:

**Theorem 1.4.** *Let R be as above and let G be a diagonalizable group scheme over R acting linearly on a polynomial algebra $R[x_1, \ldots, x_n, y_1, \ldots, y_r]$. Then the following hold*:

(1) *If $A = R[x_1, \ldots, x_n]$, then every finitely generated equivariant projective A-module is equivariantly extended from R.*

(2) *If R is a PID and $A = R[x_1, \ldots, x_n, y_1^{\pm 1}, \ldots, y_r^{\pm 1}]$, then every finitely generated equivariant projective A-module is equivariantly extended from R.*

This theorem is generalized to the case of nonlocal regular rings in Theorem 8.4. We note here that, previously, it was not even known whether every *G*-equivariant bundle on a polynomial ring over *R* is "stably" extended from *R*.

The above results were motivated in part by the following important classification problem for equivariant vector bundles over smooth affine schemes. One of the most notable (among many) recent applications of the nonequivariant Bass–Quillen conjecture is Morel's classification [2012, Theorem 8.1] of vector bundles over smooth affine schemes. He showed, using Lindel's theorem [1981], that all isomorphism classes of rank-*n* vector bundles on a smooth affine scheme *X* over a field *k* are in bijection with the set of $\mathbb{A}^1$-homotopy classes of maps from *X* to the classifying space of $\mathrm{GL}_{n,k}$. It is important to note here that, even though Morel's final result is over a field, its proof crucially depends on Lindel's theorem for geometric regular local rings.

The equivariant version of the Morel–Voevodsky $\mathbb{A}^1$-homotopy category was constructed in [Heller et al. 2015]. One can make sense of the equivariant classifying space in this category, analogous to the one in the topological setting [May 1996]. The equivariant analogue (Theorem 1.4) of Lindel's theorem now completes one very important step in solving the classification problem for equivariant vector bundles. It remains to see how one can use Theorem 1.4 to complete the proof

of the equivariant version of Morel's classification theorem. This will be taken up elsewhere.

**1B.** *Equivariant derived category and K-theory.*  We now turn to the second question. To motivate this, recall that it is a classical question in algebraic $K$-theory to determine if it is possible that two schemes with equivalent derived categories of quasicoherent sheaves (or vector bundles) have (homotopy) equivalent algebraic $K$-theories. This question gained prominence when Thomason and Trobaugh [1990] showed that the equivalence of $K$-theories is true if the given equivalence of derived categories is induced by a morphism between the underlying schemes. There has been no improvement of this result for the general case of schemes to date.

However, Dugger and Shipley [2004] (see also [Rickard 1989]) showed a remarkable improvement over the result of Thomason and Trobaugh for affine schemes. They showed more generally that any two (possibly noncommutative) noetherian rings with equivalent derived categories (which may not be induced by a map of rings!) have equivalent $K$-theories.

Parallel to the equivariant analogue of the Bass–Quillen question, one can now ask if it is true that two affine schemes with group scheme actions have equivalent equivariant $K$-theories if their equivariant derived categories are equivalent. No case of this problem has been known yet.

In this paper, we show that the general results of Dugger and Shipley [2004] apply in the equivariant setup too, which allows us to solve the above question. More precisely, we combine Dugger and Shipley's results and Proposition 4.6 to prove the following theorem.

Let $R$ be a commutative noetherian ring and let $G$ be an affine group scheme over $R$. Assume that either $G$ is diagonalizable or $R$ contains a field of characteristic zero and $G$ is a split reductive group scheme over $R$. Given a finitely generated $R$-algebra $A$ with $G$-action, let us denote this datum by $(R, G, A)$. Let $D^G(A)$ and $D^G(\mathrm{proj}/A)$ denote the derived categories of $G$-equivariant $A$-modules and $G$-equivariant (finitely generated) projective $A$-modules, respectively. Let $K^G(A)$ and $K'_G(A)$ denote the $K$-theory spectra of $G$-equivariant (finitely generated) projective $A$-modules and $G$-equivariant $A$-modules, respectively.

**Theorem 1.5.** *Let $(R_1, G_1, A_1)$ and $(R_2, G_2, A_2)$ be two data of the above type. Then $D^{G_1}(A_1)$ and $D^{G_2}(A_2)$ are equivalent as triangulated categories if and only if $D^{G_1}(\mathrm{proj}/A_1)$ and $D^{G_2}(\mathrm{proj}/A_2)$ are equivalent as triangulated categories.*

*In either case, there are homotopy equivalences of spectra $K^{G_1}(A_1) \simeq K^{G_2}(A_2)$ and $K'_{G_1}(A_1) \simeq K'_{G_2}(A_2)$.*

In other words, this theorem says that the equivariant $K$-theory as well as the $K'$-theory of affine schemes with group action can be completely determined by

the equivariant derived category, which is much simpler to study than the full equivariant geometry of the scheme.

*Brief outline of the proofs.* We end this section with an outline of our methods. Our proof of Theorem 1.3 is based on the techniques used in [Kraft and Schwarz 1995] to solve the equivariant Bass–Quillen question over $\mathbb{C}$. As in [loc. cit.], we show that all equivariant vector bundles actually descend to bundles on the quotient scheme for the group action. This allows us then to use the solution to the nonequivariant Bass–Quillen question to conclude the final proof.

In order to do this, one runs into several technical ring-theoretic issues and one has to find algebraic replacements for the geometric techniques available only over $\mathbb{C}$. Another problem is that the approach of [Masuda et al. 1996] to solve Question 1.2 for $R = \mathbb{C}$ crucially uses the result of [Bass and Haboush 1985] that every equivariant vector bundle over $\mathbb{C}[x_1, \ldots, x_n]$ is stably extended from $\mathbb{C}$. But we do not know this over other rings.

Our effort is to resolve these issues by a careful analysis of group scheme actions on affine schemes. Instead of working with schemes, we translate the problem into studying comodules over some Hopf algebras. Sections 2 and 3 are meant to do this. In Section 4, we prove some crucial properties of equivariant vector bundles on affine schemes, which play a very important role in proving Theorem 1.5. These sections generalize several results of [Bass and Haboush 1985] to more general rings.

In Section 5, we prove some properties of equivariant projective modules over monoid algebras, which are the main object of study. In Section 6, we show how to descend an equivariant vector bundle to the quotient scheme and then we use the solution to the Bass–Quillen conjecture in the nonequivariant case to complete the proof of Theorem 1.3 in Section 7. Theorem 1.4 and its generalization are proven in Section 8.

We prove Theorem 1.5 in Section 9 by combining the results of Section 4, [Dugger and Shipley 2004] and a generalization of a theorem of Rickard [1989]. This generalization is shown in the Appendix.

## 2. Recollection of group scheme action and invariants

In this section, we recall some aspects of group schemes and their actions over a given affine scheme from [SGA $3_\mathrm{I}$ 1970, Exposé III; SGA $3_\mathrm{II}$ 1970, Exposé VIII]. We prove some elementary results about these actions which are of relevance to the proofs of our main results. In this text, a *ring* will always mean a commutative noetherian ring with unit.

Let $S = \mathrm{Spec}(R)$ be a noetherian affine scheme and let $\mathbf{Sch}_S$ denote the category of schemes which are separated and of finite type over $S$. Let $\mathbf{Alg}_R$ denote the

category of finite-type $R$-algebras. We shall assume throughout this text that $S$ is connected. If $R$ and $S$ are clear in a context, the fiber product $X \times_S Y$ and tensor product $A \otimes_R B$ will be simply written as $X \times Y$ and $A \otimes B$, respectively. For an $R$-module $M$ and an $R$-algebra $A$, the base extension $M \otimes_R A$ will be denoted by $M_A$.

## 2A. *Group schemes and Hopf algebras.*

Recall that a group scheme $G$ over $S$ (equivalently, over $R$) is an object of $\mathbf{Sch}_S$ which is equipped with morphisms $\mu_G : G \times G \to G$ (multiplication), $\eta : S \to G$ (unit) and $\tau : G \to G$ (inverse) that satisfy the known associativity, unit and symmetry axioms. These axioms are equivalent to saying that the presheaf $X \mapsto h_G(X) := \operatorname{Hom}_{\mathbf{Sch}_S}(X, G)$ on $\mathbf{Sch}_S$ is a group-valued (contravariant) functor.

If $G$ is an affine group scheme over $S$, one can represent it algebraically in terms of Hopf algebras over $R$. As this Hopf algebra representation will be a crucial part of our proofs, we recall it briefly.

Let us assume that $G$ is an affine group scheme with coordinate ring $R[G]$. Then the multiplication, unit section and inverse maps above are equivalent to having the morphisms $\Delta : R[G] \to R[G] \otimes R[G]$, $\epsilon : R[G] \to R$ and $\sigma : R[G] \to R[G]$ in $\mathbf{Alg}_R$ such that $\mu_G = \operatorname{Spec}(\Delta)$, $\eta = \operatorname{Spec}(\epsilon)$ and $\tau = \operatorname{Spec}(\sigma)$. The associativity, unit and symmetry axioms are equivalent to the commutative diagrams

$$
\begin{array}{ccc}
R[G] \xrightarrow{\ \Delta\ } R[G] \otimes R[G] \xrightarrow{\ \Delta \otimes \mathrm{Id}\ } (R[G] \otimes R[G]) \otimes R[G] \\
\quad\searrow{\scriptstyle \Delta} \qquad\qquad\qquad\qquad\qquad \Big\uparrow {\scriptstyle \text{can. iso.}} \\
R[G] \otimes R[G] \xrightarrow{\ \mathrm{Id} \otimes \Delta\ } R[G] \otimes (R[G] \otimes R[G])
\end{array}
\tag{2.1}
$$

$$
\begin{array}{cc}
R[G] \xleftarrow{\ \mathrm{Id} \otimes \epsilon\ } R[G] \otimes R[G] \xrightarrow{\ \epsilon \otimes \mathrm{Id}\ } R[G] & \qquad R[G] \otimes R[G] \xrightarrow{\ \sigma \cdot \mathrm{Id}\ } R[G] \\
\ {\scriptstyle \mathrm{Id}}\nwarrow \ \ \uparrow{\scriptstyle \Delta} \ \ \nearrow{\scriptstyle \mathrm{Id}} & \qquad {\scriptstyle \Delta}\uparrow \qquad\qquad\quad \uparrow \\
R[G] & \qquad R[G] \xrightarrow{\ \ \epsilon\ \ } R
\end{array}
\tag{2.2}
$$

In other words, $(R[G], \Delta, \epsilon, \sigma)$ is a Hopf algebra over $R$ and it is well known that the transformation $(G, \mu_G, \eta, \tau) \mapsto (R[G], \Delta, \epsilon, \sigma)$ gives an equivalence between the categories of affine group schemes over $S$ and finite-type Hopf algebras over $R$ (see [Waterhouse 1979, Chapter 1]).

## 2A1. *$R$-$G$-modules.*

Let $G$ be an affine group scheme over $R$. An $R$-$G$-module is an $R$-module $M$ equipped with a natural transformation $h_G(\operatorname{Spec}(A)) \to \operatorname{GL}(M)(A)$ of group functors, where the functor $\operatorname{GL}(M)$ associates the group $\operatorname{Aut}_A(A \otimes_R M)$ to an $R$-algebra $A$.

Equivalently, an $R$-$G$-module is an $R$-module $M$ which is also a comodule over the Hopf algebra $R[G]$, in the sense that there is an $R$-linear map $\rho : M \to R[G] \otimes_R M$

such that the following diagrams commute:

$$
\begin{array}{ccc}
M & \xrightarrow{\ \ \rho\ \ } & R[G] \otimes M \\
\rho \downarrow & & \downarrow \Delta \otimes \mathrm{Id}_M \\
R[G] \otimes M & \xrightarrow{\mathrm{Id}_{R[G]} \otimes \rho} & R[G] \otimes R[G] \otimes M
\end{array}
\qquad
\begin{array}{ccc}
M & \xrightarrow{\ \rho\ } & R[G] \otimes M \\
\mathrm{Id}_M \downarrow & & \downarrow \epsilon \otimes \mathrm{Id}_M \\
M & \xrightarrow{\ \simeq\ } & R \otimes M
\end{array}
\qquad (2.3)
$$

The reader can check that the comodule structure on $M$ associated to a natural transformation of functors $h_G \to \mathrm{GL}(M)$ is given by the map $\rho : M \to R[G] \otimes M$ with $\rho(m) = h_G(R[G])(\mathrm{Id}_{R[G]})(1 \otimes m)$. We shall denote an $R$-$G$-module $M$ in the sequel in terms of an $R[G]$-comodule by $(M, \rho)$.

A morphism $f : (M, \rho) \to (M', \rho')$ between $R$-$G$-modules is an $R$-linear map $f : M \to M'$ such that $\rho' \circ f = (\mathrm{Id}_{R[G]} \otimes f) \circ \rho$. We say that $M$ is an $R$-$G$-submodule of $M'$ if $f$ is injective. The set of all $R$-$G$-module homomorphisms from $M$ to $M'$ will be denoted by $\mathrm{Hom}_{RG}(M, M')$.

We shall say that an $R$-$G$-module $M$ is finitely generated (resp. projective) if it is finitely generated (resp. projective) as an $R$-module. The categories of $R$-$G$-modules will be denoted by $(R$-$G)$-Mod. The category of finitely generated projective $R$-$G$-modules will be denoted by $(R$-$G)$-proj. The category of not necessarily finitely generated projective $R$-$G$-modules will be denoted by $(R$-$G)$-Proj.

If $G$ is an affine group scheme which is flat over $R$, then it is easy to check that $(R$-$G)$-Mod is an abelian category and $(R$-$G)$-proj is an exact category. The flatness is essential here because in its absence the kernel of an $R$-$G$-module map $f : M \to M'$ may not acquire a $G$-action as $R[G] \otimes_R \mathrm{Ker}(f)$ may fail to be a submodule of $R[G] \otimes_R M$.

**2A2.** *Submodule of invariants.* Let $G$ be an affine group scheme over $R$ and let $(M, \rho)$ be an $R$-$G$-module. An element $m \in M$ is said to be *G-invariant* under the action of $G$ if $\rho(m) = 1 \otimes m$. The $R$-submodule of $G$-invariant elements of $M$ will be denoted by $M^G$.

Given an element $\lambda \in R[G]$, we say that $m \in M$ is *semi-invariant* of weight $\lambda$ under the $G$-action if $\rho(m) = \lambda \otimes m$. The following is a straightforward consequence of the definitions and $R$-linearity of $\rho$.

The group scheme $G$ is called *linearly reductive* if $\mathrm{Inv} : (R$-$G)$-Mod $\to R$-Mod sending $M$ to $M^G$ is an exact functor.

**Lemma 2.4.** *Given an $R$-$G$-module $(M, \rho)$ and character $\lambda \in R[G]$, the set*

$$
M_\lambda := \{ m \in M \mid \rho(m) = \lambda \otimes m \}
$$

*is an $R$-$G$-submodule of $M$. In particular, $M^G$ is an $R$-$G$-submodule of $M$. Every $R$-submodule of $M_\lambda$ is an $R$-$G$-submodule of $M_\lambda$.*

**Example 2.5.** Let $k$ be an algebraically closed field and let $G$ be a linear algebraic group over $k$. In this case, a (finite) $k$-$G$-module is the same as a finite-dimensional representation $V$ of $G$. We can now check that the above notion of $G$-invariants is same as the classical definition of $V^G$, given by $V^G = \{v \in V \mid g \cdot v = v \text{ for all } g \in G\}$. Choose a $k$-basis $\{v_1, \ldots, v_n\}$ for $V$ and suppose that

$$\rho(v_i) = \sum_{j=1}^{n} a_{ij} \otimes v_j. \tag{2.6}$$

One can use (2.3) to see that $V$ becomes a $G$-representation via the homomorphism $\rho' : G \to \mathrm{GL}(V)$ given by $\rho'(g) = (a_{ij}(g))$. Recall here that an element of $k[G]$ is the same as a morphism $G \to \mathbb{A}_k^1$. If we write an element of $V$ in terms of a row vector $x = (x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i v_i$, then it follows easily from (2.6) that $\rho(x) = 1 \otimes x$ if and only if $(a_{ij}(g))x = x$ for $g \in G$. But this is the same as saying that $\rho'(g)(x) = x$ for all $g \in G$.

**2A3.** *Group scheme action.* Let $G$ be a group scheme over $S = \mathrm{Spec}(R)$ and let $X \in \mathbf{Sch}_S$. Recall that a $G$-action on $X$ is a morphism $\mu_X : G \times_S X \to X$ which satisfies the usual associative and unital identities for an action.

If $G$ is an affine group scheme over $S$ and $X = \mathrm{Spec}(A)$ is an affine $S$-scheme, then a $G$-action on $X$ as above is equivalent to a map $\phi : A \to R[G] \otimes_R A$ in $\mathbf{Alg}_R$ such that $\phi$ defines an $R[G]$-comodule structure on $A$. In this case, one has $\mu_X = \mathrm{Spec}(\phi)$. We shall denote this $G$-action on $X$ by the pair $(A, \phi)$ and call $A$ an $R$-$G$-algebra. Note that this notion of $R$-$G$-algebra makes sense for any (possibly noncommutative) $R$-algebra $R \to A$ such that the image of $R$ is contained in the center of $A$. We shall use this $R$-$G$-algebra structure on the endomorphism rings (see Lemma 3.8).

We also recall, in the language of Hopf algebras, the $G$-action on an $R$-$G$-algebra $A$ is *free* if the map $\Phi : A \otimes_R A \to R[G] \otimes_R A$ given by $\Phi(a_1 \otimes a_2) = \phi(a_1)(1 \otimes a_2)$ is surjective.

## 3. Equivariant quasicoherent sheaves on affine schemes

Recall from [Thomason 1987, §1.2] that if $X \in \mathbf{Sch}_S$ has a $G$-action $\mu_X : G \times_S X \to X$ then a $G$-equivariant quasicoherent sheaf on $X$ is a quasicoherent sheaf $\mathcal{F}$ on $X$ together with an isomorphism of sheaves of $\mathcal{O}_{G \times_S X}$-modules on $G \times_S X$

$$\theta : p^*(\mathcal{F}) \xrightarrow{\simeq} \mu_X^*(\mathcal{F}), \tag{3.1}$$

where $p : G \times_S X \to X$ is the projection map. This isomorphism satisfies the cocycle condition on $G \times_S G \times_S X$

$$(1 \times \mu_X)^*(\theta) \circ p_{23}^*(\theta) = (\mu_G \times 1)^*(\theta), \tag{3.2}$$

where $p_{23} : G \times_S G \times_S X \to G \times_S X$ is the projection to the last two factors.

A morphism of $G$-equivariant sheaves $f : (\mathcal{F}_1, \theta_1) \to (\mathcal{F}_2, \theta_2)$ is a map of sheaves $f : \mathcal{F}_1 \to \mathcal{F}_2$ such that $\mu_X^*(f) \circ \theta_1 = \theta_2 \circ p^*(f)$.

## 3A. *A-G-modules.*

Let us now assume that $G$ is an affine group scheme over $S = \mathrm{Spec}(R)$ which acts on an affine $S$-scheme $X = \mathrm{Spec}(A)$ with $A \in \mathbf{Alg}_R$. Let $\phi : A \to R[G] \otimes_R A$ be the action map such that $\mu_X = \mathrm{Spec}(\phi)$.

**Definition 3.3.** An $A$-module $M$ is an $A$-$G$-module if $(M, \rho)$ is an $R$-$G$-module such that

$$\rho(a.m) = \phi(a).\rho(m) \quad \text{for all } a \in A \text{ and } m \in M. \tag{3.4}$$

An $A$-$G$-module homomorphism is an $A$-module homomorphism which is also an $R$-$G$-module homomorphism. Given a pair of $A$-$G$-modules, the set of $A$-$G$-module homomorphisms will be denoted by $\mathrm{Hom}_{AG}(\_, \_)$.

We shall denote the category of $A$-$G$-modules by $(A$-$G)$-Mod. An $A$-$G$-module $M$ will be called projective, if it is projective as an $A$-module. We shall denote the category of finitely generated projective $A$-$G$-modules by $(A$-$G)$-proj. The category of (not necessarily finitely generated) projective $A$-$G$-modules will be denoted by $(A$-$G)$-Proj. Notice that, given a morphism of $R$-$G$ algebras $f : (A, \phi_A) \to (B, \phi_B)$, there is a pull-back map $f^* : (A$-$G)$-Mod $\to (B$-$G)$-Mod which preserves projective modules. It is easy to check that, given an $R$-$G$-module $M$ and an $A$-$G$-module $N$, the extension of scalars gives an isomorphism

$$\mathrm{Hom}_{RG}(M, N) \xrightarrow{\simeq} \mathrm{Hom}_{AG}(M_A, N). \tag{3.5}$$

**Proposition 3.6.** *There is an equivalence between the category of $G$-equivariant quasicoherent $\mathcal{O}_X$-modules and the category of $A$-$G$-modules.*

*Proof.* Let $M$ be an $A$-module which defines a $G$-equivariant quasicoherent sheaf on $X$ and let $\theta : R[G] \otimes_R M \xrightarrow{\simeq} R[G] \otimes_R M$ be an isomorphism of $R[G] \otimes_R A$-modules as in (3.1) satisfying (3.2).

We define an $A$-$G$-module structure on $M$ by setting $\rho : M \to R[G] \otimes_R M$ to be the map $\rho(m) = \theta(1 \otimes m)$. The map $\rho$ is clearly $R$-linear and one checks that

$$\rho(a \cdot m) = \theta(1 \otimes a \cdot m) = \theta(a \cdot (1 \otimes m)) = \phi(a) \cdot \theta(1 \otimes m) = \phi(a) \cdot \rho(m).$$

Since the map $\phi : A \to R[G] \otimes_R A$ is just the inclusion map $a \mapsto 1 \otimes a$ when restricted to $R$, one checks easily from (3.2) that

$$(1 \times \mu_X)^*(\theta) \circ p_{23}^*(\theta)(1 \otimes 1 \otimes m) = (1 \times \mu_S)^*(\theta) \circ p_{23}^*(\theta)(1 \otimes 1 \otimes m)$$

$$= (\mathrm{Id}_{R[G]} \otimes \rho) \circ \rho(m)$$

and it is also immediate that $(\mu_G \times 1)^*(\theta)(1 \otimes 1 \otimes m) = (\Delta \otimes \mathrm{Id}_{R[G]}) \circ \rho(m)$. This is the first square of (2.3). The second square of (2.3) is obtained at once by

applying the map $(\eta \times \eta \times 1)^*$ to (3.2), where $\eta : S \to G$ is the unit map. We have thus shown that $M$ is an $A$-$G$-module.

Conversely, suppose that $M$ is an $A$-$G$-module. We define $\theta : R[G] \otimes_R M \to R[G] \otimes_R M$ by setting $\theta(x \otimes m) = x \cdot \rho(m)$. In other words, we have

$$\theta = (\alpha \otimes \mathrm{Id}_M) \circ (\mathrm{Id}_{R[G]} \otimes \rho), \tag{3.7}$$

where $\alpha : R[G] \otimes_R R[G] \to R[G]$ is the multiplication of the ring $R[G]$.

Since $\rho$ is $R$-linear, we see that $\theta$ is $R[G]$-linear. To show that $\theta$ is $(R[G] \otimes_R A)$-linear, it is thus enough to show that it is $A$-linear. This is standard and can be checked as follows: For any $a \in A$, $x \in R[G]$ and $m \in M$, we get, inside $R[G] \otimes_R M = R[G] \otimes_R A \otimes_A M$,

$$\begin{aligned}
\theta(a \cdot (x \otimes m)) &= \theta(x \otimes a \otimes m) \\
&= \theta(x \otimes 1 \otimes a \cdot m) \\
&= (x \otimes 1) \cdot \rho(a \cdot m) \\
&= (x \otimes 1) \cdot (\phi(a) \cdot \rho(m)) \\
&= (x \otimes 1) \cdot \phi(a) \cdot \rho(m).
\end{aligned}$$

The fourth equality above follows from (3.4). On the other hand, we have

$$\begin{aligned}
a \cdot \theta(x \otimes m) &= \phi(a) \cdot \theta(x \otimes 1 \otimes m) \\
&= \phi(a) \cdot (x \otimes 1) \cdot \theta(1 \otimes m) \\
&= \phi(a) \cdot (x \otimes 1) \cdot \rho(m).
\end{aligned}$$

The two sets of identities above show that $\theta$ is $(R[G] \otimes_R A)$-linear. To show that $\theta$ is an isomorphism, we define $\theta^{-1} : R[G] \otimes_R M \to R[G] \otimes_R M$ by

$$\theta^{-1} = (\alpha \otimes \mathrm{Id}_M) \circ (\mathrm{Id}_{R[G]} \otimes \sigma \otimes \mathrm{Id}_M) \circ (\mathrm{Id}_{R[G]} \otimes \rho),$$

where $\sigma : R[G] \to R[G]$ is the inverse map of its Hopf algebra structure.

It is easy to check using (2.2) and (2.3) that $\theta \circ \theta^{-1} = \theta^{-1} \circ \theta = \mathrm{Id}_{R[G] \otimes_R M}$. The cocycle condition (3.2) is a formal consequence of the left square in (2.3). It is also straightforward to check that the two constructions given above yield the desired equivalence between the categories of $G$-equivariant quasicoherent sheaves on $X$ and $A$-$G$-modules on $A$. We leave these verifications as an exercise. $\square$

**Lemma 3.8.** *Assume that $G$ is flat over $R$ and let $(A, \phi)$ be an $R$-$G$-algebra. Let $(L, \rho_L)$, $(M, \rho_M)$ and $(N, \rho_N)$ be $A$-$G$-modules and let $p : (M, \rho_M) \to (N, \rho_N)$ be an $A$-$G$-linear map. Assume that $(L, \rho_L)$ is finitely generated. Then $\mathrm{Hom}_A(L, N)$ has a natural $A$-$G$-module structure and $\mathrm{Hom}_A(L, L)$ has a natural $A$-$G$-algebra structure such that the following hold*:

(1) *The induced map $\mathrm{Hom}_A(L, M) \xrightarrow{p \circ} \mathrm{Hom}_A(L, N)$ is $A$-$G$-linear.*

(2) $\mathrm{Hom}_{AG}(L, N) = \mathrm{Hom}_A(L, N)^G$.

(3) *If* $(M, \rho_M)$ *and* $(N, \rho_N)$ *are finitely generated, then*

$$\mathrm{Hom}_A(N, L) \xrightarrow{-\circ p} \mathrm{Hom}_A(M, L)$$

*is A-G-linear.*

*Proof.* To define an *A-G*-module structure on $\mathrm{Hom}_A(L, N)$, we need to define an *R*-linear map $\psi_{LN} : \mathrm{Hom}_A(L, N) \to R[G] \otimes_R \mathrm{Hom}_A(L, N)$ satisfying (2.3) and (3.4).

Since $R[G]$ is flat over $R$ and $L$ is a finitely generated *A*-module, it is well known (see [Eisenbud 1995, Proposition 2.10], for example) that there is a canonical isomorphism of $(R[G] \otimes_R A)$-modules:

$$\beta : R[G] \otimes_R \mathrm{Hom}_A(L, N) \to \mathrm{Hom}_{R[G] \otimes_R A}(R[G] \otimes_R L, R[G] \otimes_R N).$$

Using $\beta$, we can define $\psi_{LN}(f)$ for any $f \in \mathrm{Hom}_A(L, N)$ to be the composition

$$R[G] \otimes_R L \xrightarrow{\theta_L^{-1}} R[G] \otimes_R L \xrightarrow{\mathrm{Id} \otimes f} R[G] \otimes_R N \xrightarrow{\theta_N} R[G] \otimes_R N, \qquad (3.9)$$

where $\theta_L$ and $\theta_N$ are as in (3.7). One checks using (3.2), (3.4) and (3.7) that $\psi_{LN}$ defines an *A-G*-module structure on $\mathrm{Hom}_A(L, N)$. To show that $\mathrm{Hom}_A(L, L)$ has an *A-G*-algebra structure, we need to show that $\psi_{LL}(f \circ g) = \psi_{LL}(f) \circ \psi_{LL}(g)$. But this is immediate from (3.9).

The map $\mathrm{Hom}_A(L, M) \xrightarrow{p\circ} \mathrm{Hom}_A(L, N)$ is known to be *A*-linear. Thus we only need to show that it is *R-G*-linear in order to prove (1). Using (3.9), this is equivalent to showing that, for any $f \in \mathrm{Hom}_A(L, M)$, the identity

$$(\mathrm{Id}_{R[G]} \otimes p) \circ \theta_M \circ (\mathrm{Id}_{R[G]} \otimes f) \circ \theta_L^{-1} = \theta_N \circ (\mathrm{Id}_{R[G]} \otimes (p \circ f)) \circ \theta_L^{-1} \quad (3.10)$$

holds in $\mathrm{Hom}_{R[G] \otimes_R A}(R[G] \otimes_R L, R[G] \otimes_R N)$. In order to prove this identity, it suffices to show that $(\mathrm{Id}_{R[G]} \otimes p) \circ \theta_M = \theta_N \circ (\mathrm{Id}_{R[G]} \otimes p)$. But this is equivalent to saying that $p$ is *R-G*-linear (see the definition of morphism of *G*-equivariant sheaves below (3.2)). This proves (1), and the proof of (3) is similar.

To prove (2), recall that $f \in \mathrm{Hom}_A(L, N)^G$ if and only if

$$\psi_{LN}(f) = \theta_N \circ (\mathrm{Id} \otimes f) \circ \theta_L^{-1} = \mathrm{Id} \otimes f$$

(see Section 2A2), or equivalently if $\theta_N \circ (\mathrm{Id} \otimes f) = (\mathrm{Id} \otimes f) \circ \theta_L$. We are thus left with showing that $\theta_N \circ (\mathrm{Id} \otimes f) = (\mathrm{Id} \otimes f) \circ \theta_L$ if and only if $\rho_N \circ f = (\mathrm{Id} \otimes f) \circ \rho_L$. But the "if" part follows directly from (3.7) and the "only if" part follows by evaluating $\theta_L$ on $1 \otimes L \hookrightarrow R[G] \otimes_R L$. $\qquad \square$

**3B.** *Diagonalizable group schemes.* Recall from [SGA 3$_{\text{II}}$ 1970, Exposé VIII] that an affine group scheme $G$ over $R$ is called *diagonalizable* if there is a finitely generated abelian group $P$ such that $G = \mathrm{Spec}(R[P])$, where $R[P]$ is the group algebra of $P$ over $R$. Recall that there is a group homomorphism (the exponential map) $e : P \to (R[P])^{\times}$ and the $R$-algebra $R[P]$ carries the following Hopf algebra structure: $\Delta(e_a) = e_a \otimes e_a$, $\sigma(e_a) = e_{-a}$ and $\epsilon(e_a) = 1$ for $a \in P$, where we write $e_a$ for $e(a)$. As $R[P]$ is a free $R$-module with basis $P$, we see that $G$ is a commutative group scheme which is flat over $R$. It is smooth over $R$ if and only if the order of the finite part of $P$ is prime to all residue characteristics of $R$.

Taking $P = \mathbb{Z}$, we get the group scheme $\mathbb{G}_m = \mathrm{Spec}(R[\mathbb{Z}]) = \mathrm{Spec}(R[t^{\pm 1}])$. For an affine group scheme $G$ over $R$, its group of characters is the set $X(G) := \mathrm{Hom}(G, \mathbb{G}_m)$, whose elements are the morphisms $f : G \to \mathbb{G}_m$ in the category of affine group schemes over $R$. Every element of $P$ defines a unique homomorphism of abelian groups $\mathbb{Z} \to P$ and defines a unique morphism of group schemes $\mathrm{Spec}(R[P]) \to \mathbb{G}_m$. One checks that this defines an isomorphism $P \xrightarrow{\simeq} X(G)$ and yields an antiequivalence of categories from finitely generated abelian groups to diagonalizable group schemes over $R$. In particular, the category $\mathbf{Diag}_R$ of diagonalizable group schemes over $R$ is abelian. We shall use the following known facts about the diagonalizable group schemes and quasicoherent sheaves for the action of such group schemes.

**Proposition 3.11** [SGA 3$_{\text{II}}$ 1970, Exposé VIII, §3]. *Let $\phi : G \to G'$ be a morphism of diagonalizable group schemes. Then there are diagonalizable group schemes $H, G/H$ and $G'/G$ together with exact sequences in $\mathbf{Diag}_R$*

$$0 \to H \to G \xrightarrow{\phi} G/H \to 0 \quad and \quad 0 \to G/H \to G' \to G'/G \to 0.$$

**Proposition 3.12** [SGA 3$_{\text{I}}$ 1970, Exposé I, Proposition 4.7.3]. *Let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme. Then the category of $R$-$G$-modules is equivalent to the category of $P$-graded $R$-modules. The equivalence is given by associating to every $R$-$G$-module $(M, \rho)$ the $P$-graded $R$-module $M = \bigoplus_{a \in P} M_a$, where $M_a := \{m \in M \mid \rho(m) = e_a \otimes m\}$ is the subspace of $M$ containing elements of weight $e_a$ (see Section 2A2). To every $P$-graded $R$-module $M = \bigoplus_{a \in P} M_a$, we associate the $R$-$G$-module $(M, \rho)$, where $\rho(m) := (e_a \otimes m)$ for all $m \in M_a$ and $a \in P$.*

**Corollary 3.13.** *Let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme and let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

*be an exact sequence of $R$-$G$-modules. Then the following hold:*

(1) *For each $a \in P$, there is an exact sequence $0 \to (M_1)_a \to (M_2)_a \to (M_3)_a \to 0$ of $R$-$G$-modules.*

(2) $0 \to M_1^G \to M_2^G \to M_3^G \to 0$ *is an exact sequence of R-G-modules.*

(3) *The sequence* $0 \to M_1 \to M_2 \to M_3 \to 0$ *splits as a sequence of R-G-modules if and only if it splits as a sequence of R-modules.*

*Proof.* Assertions (1) and (2) follow directly from Lemma 2.4 and Proposition 3.12. The "only if" part of (3) is immediate and, to prove the "if" part, it is enough, using (1) and Proposition 3.12, to give a splitting of the *R-G*-linear map $t_a :$ $(M_2)_a \to (M_3)_a$ for $a \in P$.

Let $s : M_3 \to M_2$ be an *R*-linear splitting of $t : M_2 \to M_3$. For $a \in P$, consider the composite map $u_a : (M_3)_a \xrightarrow{i_a} M_3 \xrightarrow{s} M_2 \xrightarrow{p_a} (M_2)_a$, where $i_a$ and $p_a$ are the inclusion and the projection maps, respectively. As $t = \bigoplus_{a \in P} t_a$ and hence $t_a \circ p_a = p_a \circ t$, one checks at once that $t_a \circ u_a$ is the identity on $(M_3)_a$. Moreover, for each $m \in (M_3)_a$, one has

$$(\mathrm{Id}_{R[G]} \otimes u_a) \circ \rho_3(m) = e_a \otimes u_a(m) = \rho_2 \circ u_a(m)$$

and this shows that $u_a : (M_3)_a \to (M_2)_a$ is an *R-G*-linear splitting of $t_a$.     □

Given any $v \in P$, we shall denote the free *R-G*-module of rank one with constant weight $e_v$ by $R_v$ (see Section 2A2).

**Lemma 3.14.** *Let* $G = \mathrm{Spec}(R[P])$ *be a diagonalizable group scheme and let* $(A, \phi)$ *be an R-G-algebra. Given two free R-G-modules* $(V, \rho_V)$ *and* $(W, \rho_W)$ *of rank one and respective constant weights* $e_v$ *and* $e_w$, *the A-G-module structure on* $\mathrm{Hom}_A(V_A, W_A)$ *is given by*

$$\mathrm{Hom}_A(V_A, W_A) \simeq (R_{w-v}) \otimes_R A.$$

*In particular,* $\mathrm{Hom}_{AG}(V_A, W_A) \simeq A_{v-w}$ *and* $\mathrm{End}_{AG}((V, \rho_V)) \simeq A^G$.

*Proof.* This follows directly from Lemma 3.8 by unraveling the *A-G*-module structure defined on $\mathrm{Hom}_A(V_A, W_A)$.     □

**Lemma 3.15.** *Let*

$$0 \to P_1 \xrightarrow{\phi_1} P_2 \xrightarrow{\phi_2} P_3 \to 0$$

*be an exact sequence of finitely generated abelian groups and set* $G_i = \mathrm{Spec}(R[P_i])$. *Let* $\phi_i^* : R[P_i] \to R[P_{i+1}]$ *denote the corresponding map of group algebras. Let* $(A, \theta)$ *be an R-$G_1$-algebra.*

(1) $(A, (\phi_1^* \otimes \mathrm{Id}_A) \circ \theta)$ *is an R-$G_2$-algebra.*

(2) *If* $(E, \rho) \in (A\text{-}G_2)\text{-Mod}$, *then* $E_b := \{\lambda \in E \mid (\phi_2^* \otimes \mathrm{Id}_E) \circ \rho(\lambda) = e_b \otimes \lambda\} \subseteq E$ *is an A-$G_2$-submodule for each* $b \in P_3$.

(3) *If* $E \in (A\text{-}G_2)\text{-proj}$, *then so does* $E_b$.

*Proof.* The item (1) is clear. For (2), we can write $E = \bigoplus_{a \in P_2} E_a$, where each $E_a$ is an $R$-$G_2$-submodule. In particular, each $E_b$ is an $R$-$G_2$-submodule. To see that it is an $A$-$G_2$-submodule, it suffices to know that $E_b$ is an $A$-submodule of $E$. Setting $A = \bigoplus_{c \in P_1} A_c$, it suffices to check that $x\lambda \in E_b$ for $x \in A_c$ and $\lambda \in E_b$. But this is a straightforward verification using the fact that $(\phi_2^* \circ \phi_1^*)(e_c) = 1$ and we skip it. The item (3) is clear as each $E_b$ is a direct factor of $E$ as an $A$-module. □

**Corollary 3.16.** *With the assumptions of Lemma 3.15, assume furthermore that the action of $G_1$ on $A$ is free and that every finitely generated projective module over $A^{G_1}$ is extended from $R$. Given any finitely generated projective $A$-$G_2$-module $E$, we have $E \simeq F_A$ for some $R$-$G_2$-module $F$.*

*Proof.* We can use Lemma 3.15 to assume that $E = E_b$ for some $b \in P_3$. For any $a \in \phi_2^{-1}(b)$, it is easy to check that the evaluation map

$$\operatorname{Hom}_{AG_3}(R_a \otimes_R A, E) \otimes_A (R_a \otimes_R A) \to E \tag{3.17}$$

is an isomorphism of $A$-$G_3$-modules. Lemma 3.15, however, says that $E' := \operatorname{Hom}_{AG_3}(R_a \otimes_R A, E) = (\operatorname{Hom}_A(R_a \otimes_R A, E))^{G_3}$ is an $A$-$G_2$-module. It follows that (3.17) is an $A$-$G_2$-linear isomorphism.

As $E'$ has trivial $G_3$-action, it can be viewed as a projective $A$-$G_1$-module. It follows from our assumption and [Vistoli 2005, Theorem 4.46] that this is the pullback of a finitely generated projective module over $A^{G_1}$. Since every such module over $A^{G_1}$ is extended from $R$, we conclude that $E' \simeq F' \otimes_R A$ as an $A$-$G_2$-module for some finitely generated projective $R$-module $F'$. Taking $F = F' \otimes_R R_a$, we get $E \simeq F_A$. □

## 4. Structure of ringoid modules on $(A$-$G)$-Mod

Let $R$ be a commutative noetherian ring and let $G$ be a flat affine group scheme over $R$. Let $(A, \phi)$ be an $R$-$G$-algebra. We have observed in Section 2A1 that the flatness of $G$ ensures that $(A$-$G)$-Mod is an abelian category. In this section, we show that $A$-$G$-modules have the structure of modules over a ringoid (defined below) for various cases of $G$. We shall say that an $A$-$G$-module is *$A$-$G$-projective* if it is a projective object of the abelian category $(A$-$G)$-Mod.

**Lemma 4.1** (resolution property). *Let $G = \operatorname{Spec}(R[P])$ be a diagonalizable group scheme over $R$. Then every finitely generated $A$-$G$-module is a quotient of a finitely generated, free $A$-$G$-module in the category $(A$-$G)$-Mod.*

*Proof.* Let $M$ be a finitely generated $A$-$G$-module. As an $R$-$G$-module, we can write $M = \bigoplus_{a \in P} M_a$, where each $M_a$ is an $R$-module and has constant weight "$e_a$". We can find a finite set of elements $S = \{m_{a_1}^1, \ldots, m_{a_1}^{k_1}, \ldots, m_{a_m}^1, \ldots, m_{a_m}^{k_m}\} \subset M$ which generates $M$ as an $A$-module, with $a_1, \ldots, a_m \in P$, $k_i \in \mathbb{N}$ and $m_{a_i}^j \in M_{a_i}$.

Consider the free $R$-$G$-module $F = \bigoplus_{i=1}^{m} R_{a_i}^{k_i}$, where $R_{a_i}$ denotes the free rank-1 $R$-$G$-module with constant weight $e_{a_i}$. Then we have an $R$-$G$-module map $F \to M$ such that the set $S$ lies in its image. Therefore, (3.5) yields a unique $A$-$G$-module surjection $F_A \twoheadrightarrow M$, where $F_A$ is a free $A$-$G$-module of finite rank. $\qquad \square$

**Remark 4.2.** A similar argument shows that every $A$-$G$-module (not necessarily finitely generated) has an $A$-$G$-linear epimorphism from a direct sum of (possibly infinite) rank-1 free $A$-$G$-modules.

**Lemma 4.3.** *Let $G$ be as above. Then a finitely generated $A$-$G$-module is $A$-$G$-projective if and only if it is projective as an $A$-module. In particular, the category $(A$-$G)$-Mod has enough projectives.*

*Proof.* Suppose $L$ is a finitely generated projective $A$-$G$-module. Let $M \xrightarrow{\phi} N$ be a surjective $A$-$G$-module homomorphism. Then $\mathrm{Hom}_A(L, M) \xrightarrow{\phi \circ -} \mathrm{Hom}_A(L, N)$ is an $A$-$G$-linear map by Lemma 3.8 and is surjective as $L$ is a projective $A$-module. By Corollary 3.13(2), the map $\mathrm{Hom}_A(L, M)^G \xrightarrow{\phi \circ -} \mathrm{Hom}_A(L, N)^G$ is also surjective and, therefore, $\mathrm{Hom}_{AG}(L, M) \xrightarrow{\phi \circ -} \mathrm{Hom}_{AG}(L, N)$ is surjective by Lemma 3.8. Hence, $L$ is $A$-$G$-projective.

Conversely, suppose $L$ is $A$-$G$-projective. By Lemma 4.1, there exists a finitely generated free $A$-$G$-module $F$ and an $A$-$G$-module surjection $F \twoheadrightarrow L$. Since $L$ is $A$-$G$-projective, there is a splitting and hence it is a direct summand of $F$. Since $F$ is a projective $A$-module, $L$ is $A$-projective as well. The existence of enough projectives in $(A$-$G)$-Mod now follows from this, Lemma 4.1 and Remark 4.2 since any direct sum of $A$-$G$-projectives is also $A$-$G$-projective. $\qquad \square$

Let us now consider more general situations. Recall from [SGA 3$_{\mathrm{III}}$ 1970, Exposé XIX] that an affine group scheme $G$ over $R$ is called *reductive* if it is smooth over $R$ and, for every point $x \in S = \mathrm{Spec}(R)$, the geometric fiber $G \times_S \mathrm{Spec}(\overline{k(x)})$ is a reductive linear algebraic group over $\mathrm{Spec}(\overline{k(x)})$. We say that $G$ is split reductive if it is a connected and reductive group scheme over $R$ and it admits a maximal torus $T \simeq \mathbb{G}_{m,R}^r$ such that the pair $(G, T)$ corresponds to a (reduced) root system $(A, \mathcal{R}, A^\vee, \mathcal{R}^\vee)$ defined over $\mathbb{Z}$ (see [SGA 3$_{\mathrm{III}}$ 1970, Exposé XXII]). It is known that all Chevalley groups, such as $\mathrm{GL}_n$, $\mathrm{SL}_n$, $\mathrm{PGL}_n$, $\mathrm{Sp}_{2n}$ and $\mathrm{SO}_n$, are split reductive group schemes over $R$.

Using similar techniques, we can now extend Lemmas 4.1 and 4.3 to the class of split reductive group schemes over $R$, as follows:

**Lemma 4.4.** *Let $R$ be a unique factorization domain containing a field of characteristic zero. Let $G$ be a connected reductive group scheme over $R$ which contains a split maximal torus $\mathbb{G}_{m,R}^r$. Let $(A, \phi)$ be an $R$-$G$-algebra. Then:*

(1) *Every finitely generated $A$-$G$-module is a quotient of a finitely generated, free $A$-$G$-module in the category $(A$-$G)$-Mod.*

(2) *A finitely generated A-G-module is A-G-projective if and only if it is projective as an A-module.*

*Proof.* Let $k \hookrightarrow R$ be a field of characteristic zero. Since $R$ is a UFD and $G$ contains a split maximal torus, it is known in this case (see [SGA 3$_{\text{III}}$ 1970, Exposé XXII, Proposition 2.2], for example) that $G$ is in fact a split reductive group scheme over $R$. In particular, it is defined over the ring $\mathbb{Z}$ and hence over $k$. Let $G_0$ be a $k$-form for $G$. In other words, $G_0$ is a connected reductive group over $k$ such that $k[G_0] \otimes_k R \simeq R[G]$.

Let $M$ be a finitely generated $A$-$G$-module. Since $G_0$ is reductive and $\text{char}(k) = 0$, we see that it is linearly reductive (see Section 2A2). Since $R[G] = k[G_0] \otimes_k R$, we see that the $R$-$G$-module structure on $M$ given by $(M, \rho)$ is same thing as the $k$-$G_0$-module structure $(M, \rho)$ (see Section 2A1). With this $k$-$G_0$-module structure, we can write $M$ as a (possibly infinite) direct sum of irreducible $k$-$G_0$-modules. Let $S = \{m_1, \ldots, m_s\}$ be a generating set of $M$ as an $A$-module. Then we can find finitely many irreducible $k$-$G_0$-submodules of $M$ whose direct sum contains $S$. Letting $F$ denote this direct sum, we get a $k$-$G_0$-linear map $F \to M$ whose image contains $S$. This map uniquely defines an $R$-$G$-linear map $F_R \to M$. Extending this further to $A$ using (3.5), we get a unique $A$-$G$-linear map $F_A \to M$, which is clearly surjective. This proves (1).

Suppose $L$ is a finitely generated projective $A$-$G$-module. Let $M \xrightarrow{\phi} N$ be a surjective $A$-$G$-module homomorphism. Then $\text{Hom}_A(L, M) \xrightarrow{\phi \circ} \text{Hom}_A(L, N)$ is an $A$-$G$-linear map by Lemma 3.8 and is surjective as $L$ is a projective $A$-module. Using the linear reductivity of $G_0$ and arguing as in the proof of Lemma 4.3, we see that the map $\text{Hom}_A(L, M)^{G_0} \xrightarrow{\phi \circ} \text{Hom}_A(L, N)^{G_0}$ is surjective. As argued in the proof of (1) above, it is easy to see from the identification of $(M, \rho_R)$ with $(M, \rho_k)$ and Section 2A2 that $E^G = E^{G_0}$ for any $R$-$G$-module $E$. We conclude that the map $\text{Hom}_A(L, M)^G \xrightarrow{\phi \circ} \text{Hom}_A(L, N)^G$ is surjective. Therefore, $\text{Hom}_{AG}(L, M) \xrightarrow{\phi \circ} \text{Hom}_{AG}(L, N)$ is surjective. Hence $L$ is $A$-$G$-projective. The converse follows exactly as in the diagonalizable group case using (1). $\square$

We recall a few definitions in category theory:

**Definition 4.5.** Let $\mathcal{A}$ be a cocomplete abelian category. We say that a set of objects $\{P_\alpha\}_\alpha$ is a *set of strong generators* for $\mathcal{A}$ if for every object $X$ in $\mathcal{A}$ we have $X = 0$ whenever $\text{Hom}_{\mathcal{A}}(P_\alpha, X) = 0$ for all $\alpha$.

An object $P$ is called *small* if $\bigoplus_\lambda \text{Hom}_{\mathcal{A}}(P, X_\lambda) \to \text{Hom}_{\mathcal{A}}(P, \bigoplus_\lambda X_\lambda)$ is a bijection for every set of objects $\{X_\lambda\}_\lambda$.

Recall that a *ringoid* $\mathcal{R}$ is a small category which is enriched over the category **Ab** of abelian groups. This means that the hom-sets in $\mathcal{R}$ are abelian groups and the compositions of morphisms are bilinear maps of abelian groups. A ringoid with

only one object can be easily seen to be equivalent to a (possibly noncommutative) ring $R$.

A (right) $\mathcal{R}$-module is a contravariant functor $M : (\mathcal{R})^{\mathrm{op}} \to \mathbf{Ab}$. It is known that the category $\mathcal{R}$-Mod of (right) $\mathcal{R}$-modules is a complete and cocomplete abelian category, where the limits and colimits are defined objectwise. An $\mathcal{R}$-module is called free of rank one if it is of the form $B \mapsto \mathrm{Hom}_{\mathcal{R}}(B, A)$ for some $A \in \mathcal{R}$. Such modules are denoted by $H_A$. We say that an $\mathcal{R}$-module is finitely generated if it is a quotient of a finite coproduct of rank-one free $\mathcal{R}$-modules. It is known that $\mathcal{R}$-Mod is a Grothendieck category which has a set of small and projective strong generators. This set is given by the collection $\{H_A \mid A \in \mathrm{Obj}(\mathcal{R})\}$. We refer to [Mitchell 1972] for more details about ringoids.

A combination of the previous few results gives us the following conclusion:

**Proposition 4.6.** *Given a commutative noetherian ring $R$, an affine group scheme $G$ over $R$ and an $R$-$G$-algebra $(A, \phi)$, the following hold:*

(1) *If $G = \mathrm{Spec}(R[P])$ is a diagonalizable group scheme, then the category $(A$-$G)$-Mod has a set of small and projective strong generators.*

(2) *If $R$ is a UFD containing a field of characteristic zero and $G$ is a split reductive group scheme, then the category $(A$-$G)$-Mod has a set of small and projective strong generators.*

*In either case, the category $(A$-$G)$-Mod is equivalent to the category $\mathcal{R}$-mod for some ringoid $\mathcal{R}$ and this equivalence preserves finitely generated projective objects.*

*Proof.* If $G = \mathrm{Spec}(R[P])$ is diagonalizable, we set $S = \{A \otimes_R R_a \mid a \in P\}$, and if $G$ is split reductive, we set $S = \{A \otimes_k V_a\}_a$, where $\{V_a\}_a$ is the set of isomorphism classes of all irreducible $k$-$G_0$-modules. The proposition now follows from Lemmas 4.1, 4.3 and 4.4 and Remark 4.2. It is shown as part of the proofs of these lemmas that $S$ is a set of strong generators for $(A$-$G)$-Mod.

The last part follows from (1) and (2) and [Freyd 1964, Exercise 5.3H], which says that the functor

$$\mathrm{Hom}(S, \_) : (A\text{-}G)\text{-Mod} \to \mathrm{End}(S)\text{-Mod}$$

is an equivalence of categories, where $\mathrm{End}(S)$ is the full subcategory of $(A$-$G)$-Mod consisting of objects in $S$. To show that this equivalence preserves finitely generated projective objects, we only need to show that it preserves finitely generated objects, since any equivalence of abelian categories preserves projective objects. Suppose now that $M$ is a finitely generated $A$-$G$-module in case (1).

It was shown in the proof of Lemma 4.1 that there is a finite set $\{a_1, \ldots, a_m\} \subseteq P$ and a surjective $A$-$G$-linear map $\bigoplus_{i=1}^{m}(A \otimes_R R_{a_i}) \twoheadrightarrow M$. But this precisely means

that $\bigoplus_{i=1}^{m} H_{a_i}(A \otimes_R R_a) \twoheadrightarrow \mathrm{Hom}(S, M)(A \otimes_R R_a)$ for all $a \in P$ and this means $\mathrm{Hom}(S, M)$ is a finitely generated object of $\mathrm{End}(S)$-Mod. The case (2) follows similarly. $\qquad\square$

**Remark 4.7.** If $G$ is a finite constant group scheme over $R$ whose order is invertible in $R$, then one can show using the same argument as above that the category $(A\text{-}G)$-Mod has a single small and projective generator given by $A \otimes_R R[G]$. In particular, a variant of Freyd's theorem implies that $(A\text{-}G)$-Mod is equivalent to the category of right $S$-modules, where $S$ is the endomorphism ring of $A \otimes_R R[G]$.

## 5. Group action on monoid algebras

In this section, we prove some properties of projective modules over the ring of invariants when a diagonalizable group acts on a monoid algebra. We fix a commutative noetherian ring $R$ and a diagonalizable group scheme $G = \mathrm{Spec}(R[P])$ over $R$.

Let $Q$ be a monoid, i.e., a commutative semigroup with unit. Let $G(Q)$ be the Grothendieck group associated to $Q$.

**Definition 5.1.** We say that $Q$ is

- *cancellative* if $ax = ay$ implies $x = y$ in $Q$;
- *seminormal* if $x \in G(Q)$ and $x^2, x^3 \in Q$ implies $x \in Q$;
- *normal* if $x \in G(Q)$ and $x^n \in Q$ for any $n > 0$ implies $x \in Q$;
- *torsion-free* if $x^n = y^n$ for some $n > 0$ implies $x = y$;
- *having no nontrivial unit* if $x, y \in Q$ and $xy = 1$ imply that $x$ is the unit of $Q$.

Given a monoid $Q$, we can form the monoid algebra $R[Q]$. As an $R$-module, $R[Q]$ is free with a basis consisting of the symbols $\{e_a \mid a \in Q\}$, and the multiplication on $R[Q]$ is defined by the $R$-bilinear extension of $e_a \cdot e_b = e_{ab}$. The elements $e_a$ are called the monomials of $R[Q]$. For example, polynomial ring $R[x_1, \ldots, x_n]$ is a monoid algebra defined by the monoid $\mathbb{Z}_+^n$, and the monomials of $R[\mathbb{Z}_+^n]$ are exactly the monomials of the polynomial ring.

**5A.** *Projective modules over monoid algebras.* For $R$ as above, consider the following conditions.

(†) Every (not necessarily finitely generated) projective $R$-module is free and every finitely generated projective $R[Q]$-module is extended from $R$ if $Q$ is a torsion-free abelian group.

(††) Every (not necessarily finitely generated) projective $R$-module is free and every (finitely generated) projective module over $R[Q \times \mathbb{Z}^n]$ is extended

from $R$ if $Q$ is a torsion-free, seminormal and cancellative monoid which has no nontrivial unit and $n \geq 0$ is an integer.

**Theorem 5.2.** *Let $R$ be a commutative noetherian ring that is any of the following*:

(1) *A principal ideal domain.*

(2) *A regular local ring of dimension $\leq 2$.*

(3) *A regular local ring containing a field.*

*Then $R$ satisfies* (†) *and* (††).

*Proof.* The first part of (†) holds more generally for any commutative noetherian ring $R$ which is either local or a principal ideal domain. This follows from [Kaplansky 1958, Theorem 2; Bass 1973].

That the principal ideal domains satisfy (†) and (††) follows from [Bruns and Gubeladze 2009, Theorem 8.4]. These conditions for (2) follow from [Swan 1992, Theorem 1.2, Corollary 3.5]. To show (†) and (††) for (3), we first reduce to the case when $R$ is essentially of finite type over a field, using the methods of [Swan 1998, Theorem 2.1] and Neron–Popescu desingularization. In the special case when $R$ is essentially of finite type over a field, (3) follows from [Swan 1992, Theorem 1.2, Corollary 3.5]. □

**5B. *Projective modules over the ring of invariants.*** Let $Q$ be a monoid and let $u : Q \to P$ be a homomorphism of monoids. Consider the graph homomorphism $\gamma_u : Q \to P \times Q$ given by $\gamma_u(a) = (u(a), a)$. This defines a unique morphism $\phi : R[Q] \to R[P \times Q] \simeq R[P] \otimes_R R[Q]$ of monoid $R$-algebras, given by $\phi(f_a) = g_{\gamma_u(a)} = e_{u(a)} \otimes f_a$, where $e : P \to (R[P])^\times$, $f : Q \to (R[Q])^\times$ and $g : P \times Q \to (R[P \times Q])^\times$ are the exponential maps (see Section 3B). Notice that these exponential maps are injective. Setting $A = R[Q]$, we thus get a canonical map of $R$-algebras

$$\phi : A \to R[P] \otimes_R A. \qquad (5.3)$$

One checks at once that this makes $(A, \phi)$ into an $R$-$G$-algebra.

**Proposition 5.4.** *Let $Q' = \mathrm{Ker}(u)$ be the submonoid of $Q$. Assume that $Q$ satisfies any of the properties listed in Definition 5.1. Then $Q'$ also satisfies the same property. In each case, there is an isomorphism of $R$-algebras $R[Q'] \xrightarrow{\simeq} A^G$.*

*Proof.* Since we work with (commutative) monoids, we shall write their elements additively. It is immediate from the definition that the properties of being cancellative, torsion-free and having no nontrivial units are shared by all submonoids of $Q$. The only issue is to show that $Q'$ is seminormal (resp. normal) if $Q$ is so.

So let us assume that $Q$ is seminormal and let $x \in G(Q')$ be such that $2x, 3x \in Q'$. Since $G(Q') \subseteq G(Q)$, we see that $x \in Q$. Setting $y = u(x)$, we get $2y = u(2x) = 0 = u(3x) = 3y$. Since $P = G(P)$, we get $y = 3y - 2y = 0$ and this means $x \in Q'$.

Suppose now that $Q$ is normal and $x \in G(Q')$ is such that $nx \in Q'$ for some $n > 0$. As $G(Q') \subseteq G(Q)$ and $Q$ is normal, we get $x \in Q$. The commutative diagram

$$
\begin{array}{ccc}
Q' & \longrightarrow & Q \\
\downarrow & & \downarrow \quad\searrow{\scriptstyle u} \\
G(Q') & \longrightarrow G(Q) & \xrightarrow[G(u)]{} P
\end{array}
$$

now shows that $u(x) = G(u)(x) = 0$ and hence $x \in Q'$.

It is clear from the definition that $R[Q'] \subseteq A^G$ and so we only need to show the reverse inclusion to prove the second part of the proposition. Let $p = \sum_a r_a f_a \in A^G$ with $0 \neq r_a \in R$. This means that $\phi(p) = 1 \otimes p = e_0 \otimes p$. Equivalently, we get

$$
\sum_a r_a(e_{u(a)} \otimes f_a) = \sum_a r_a(e_0 \otimes f_a) \iff \sum_a r_a(e_{u(a)} - e_0) \otimes f_a = 0
$$

$$
\iff r_a(e_{u(a)} - e_0) = 0 \quad \text{for all } a
$$

$$
\iff e_{u(a)} = e_0 \qquad \text{for all } a
$$

$$
\iff u(a) = 0 \qquad \text{for all } a
$$

$$
\iff a \in Q' \qquad \text{for all } a.
$$

The second equivalence follows from the fact that $R[P] \otimes_R R[Q]$ is a free $R[P]$-module with basis $\{f_a \mid a \in Q\}$ and the third follows from the fact that $R[P]$ is a free $R$-module with basis $\{e_b \mid b \in P\}$ and $r_a \neq 0$. The last statement implies that each summand of $p$ belongs to $R[Q']$ and so does $p$. This proves the proposition. $\qquad \square$

**Corollary 5.5.** *Assume that $R$ satisfies* (††). *Let $Q$ be a monoid which is cancellative, torsion-free, seminormal and has no nontrivial unit. Let $A = R[Q]$ be the monoid algebra having the $R$-$G$-algebra structure given by* (5.3). *Then finitely generated projective modules over $A$ and $A^G$ are free.*

**Corollary 5.6.** *Let $R$ be a principal ideal domain and let $Q$ be a monoid which is cancellative, torsion-free and seminormal* (*possibly having nontrivial units*). *Let $A = R[Q]$ be the monoid algebra having the $R$-$G$-algebra structure given by* (5.3). *Then finitely generated projective modules over $A$ and $A^G$ are free.*

*Proof.* This follows from Proposition 5.4 and the main result of [Gubeladze 1988]. $\qquad \square$

We end this section with the following description of finitely generated free $R$-$G$-modules when $R$ satisfies (†) and its consequence:

**Lemma 5.7.** *Assume that $R$ satisfies* (†). *Then every finitely generated free $R$-$G$-module is a direct sum of free $R$-$G$-modules of rank one. Every free $R$-$G$-module of rank one has constant weight of the form $e_a$ for some $a \in P$.*

*Proof.* Let $M$ be a finitely generated free $R$-$G$-module. By Proposition 3.12, we can write $M = \bigoplus_{a \in P} M_a$. Lemma 2.4 says that this is a direct sum decomposition as $R$-$G$-modules. Moreover, each $M_a$ is a direct factor of the free $R$-module $M$ and hence is projective and thus free as $R$ satisfies (†).

Therefore, it is enough to show that, if $M$ is a free $R$-$G$-module of constant weight $e_a$, then every $R$-submodule of $M$ is an $R$-$G$-submodule. But this follows directly from Lemma 2.4. The decomposition $M = \bigoplus_{a \in P} M_a$ also shows that a free rank-one $R$-$G$=module must have a constant weight of the form $e_a$ with $a \in P$. $\square$

**Corollary 5.8.** *Assume that $R$ satisfies* (†). *Under the assumptions of Corollary 3.16 suppose that $F$, $F' \in (R\text{-}G_2)$-proj are isomorphic as $R$-$G_3$-modules. Then $F_A \simeq F_A'$ as $A$-$G_2$-modules.*

*Proof.* By Lemma 5.7 and Proposition 3.12, it is enough to prove that, if $F$ and $F'$ are one-dimensional free $R$-$G_2$-modules of constant weights $e_a$ and $e_{a'}$, where $a, a' \in P_2$ with $\phi_2(a) = \phi_2(a')$, then $F_A \simeq F_A'$ as $A$-$G_2$-modules.

As $G_3$ acts trivially on $A$ and $\phi_2(a) = \phi_2(a')$, we have $\mathrm{Hom}_{AG_3}(F_A, F_A') = \mathrm{Hom}_A(F_A, F_A')$. By Lemma 3.15, $\mathrm{Hom}_{AG_3}(F_A, F_A') = \mathrm{Hom}_A(F_A, F_A')$ as $A$-$G_2$-modules and $\mathrm{Hom}_A(F_A, F_A') \simeq R_{a'-a} \otimes A$ as an $A$-$G_2$-module by Lemma 3.14. The argument of Corollary 3.16 shows that $\mathrm{Hom}_{AG_3}(F_A, F_A') \simeq A$ as an $A$-$G_2$-module. Therefore, $R_{a'-a} \otimes A \simeq A$ and hence $R_a \otimes A \simeq R_{a'} \otimes A$ as $A$-$G_2$-modules. $\square$

## 6. Toric schemes and their quotients

Let $R$ be a commutative noetherian ring and let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme over $R$. In this section, we recall the notion of affine $G$-toric schemes and study their quotients for the $G$-action.

**6A.** *Toric schemes.* Let $L$ be a lattice (a free abelian group of finite rank). A subset of $L_{\mathbb{Q}}$ of the form $l^{-1}(\mathbb{Q}_+)$, where $l : L_{\mathbb{Q}} \to \mathbb{Q}$ is a nonzero linear functional and $\mathbb{Q}_+ = \{r \in \mathbb{Q} \mid r \geq 0\}$, is called a half-space of $L_{\mathbb{Q}}$. A *cone* of $L_{\mathbb{Q}}$ is an intersection of a finite number of half-spaces. A cone is always assumed to be convex, polyhedral and rational ("rational" means that it is generated by vectors in the lattice). The *dimension* of a cone $\sigma$ is defined to be the dimension of the smallest subspace of $L_{\mathbb{Q}}$ containing $\sigma$. We say that $\sigma$ is *strongly convex* in $L_{\mathbb{Q}}$ if it spans $L_{\mathbb{Q}}$. By replacing $L_{\mathbb{Q}}$ by its subspace $\sigma + (-1)\sigma$, there is no loss of generality in assuming that $\sigma$ is a strongly convex cone in $L_{\mathbb{Q}}$.

The intersection $\sigma \cap L$ is clearly a cancellative, torsion-free monoid. Moreover, $L_{\sigma} = \sigma \cap L$ is known to be finitely generated and normal (see [Danilov 1978, Lemma 1.3; Bruns and Gubeladze 2009, Corollary 2.24]). It follows from [Bruns and Gubeladze 2009, Theorem 4.40] that the monoidal $R$-algebra $A = R[L_{\sigma}]$ is

a normal integral domain if $R$ is so. The scheme $X_\sigma = \mathrm{Spec}(R[L_\sigma])$ is called an *affine toric scheme* over $R$. The inclusion $\iota_\sigma : L_\sigma \hookrightarrow L$ defines a Hopf algebra map $\phi_\sigma : A \to R[L] \otimes_R A$ (the graph of $\iota_\sigma$), which is equivalent to giving an action of the "big torus" $T_\sigma = \mathrm{Spec}(R[L])$ on $X_\sigma$. The inclusion $R[L_\sigma] \hookrightarrow R[L]$ embeds $T_\sigma$ as a $T_\sigma$-invariant affine open subset of $X_\sigma$, where $T_\sigma$ acts on itself by multiplication.

A *face* of $\sigma$ is its subset of the form $\sigma \cap l^{-1}(0)$, where $l : L_\mathbb{Q} \to \mathbb{Q}$ is a linear functional that is positive on $\sigma$. A face of a cone is again a cone, so for each face $\tau$ of $\sigma$, we have a toric scheme $X_\tau$ which has an action of $T_\sigma$ given by the inclusion $L_\tau \hookrightarrow L$ and this action factors through the action of the big torus $T_\tau = \mathrm{Spec}(R[M])$ of $X_\tau$ (where $M$ is the smallest sublattice of $L$ such that $M_\mathbb{Q}$ is a subspace containing $\tau$) on $X_\tau$. Let $\chi$ be the characteristic function of the face $\tau$, i.e., the function which is 1 on $\tau$ and 0 outside $\tau$. The assignment $e_m \mapsto \chi(m)e_m$ (for $m \in L_\sigma$) extends to a surjective homomorphism of $R$-algebras $i_\tau : R[L_\sigma] \twoheadrightarrow R[L_\tau]$, which defines a closed embedding of $X_\tau$ in $X_\sigma$. The natural inclusion $L_\tau \hookrightarrow L_\sigma$ defines a retraction morphism $\pi_\tau : R[L_\tau] \to R[L_\sigma]$. Both $i_\tau$ and $\pi_\tau$ are $R$-$T_\sigma$-algebra morphisms such that the composition $i_\tau \circ \pi_\tau$ is the identity.

If $\tau' \subseteq \sigma$ is another face different from $\tau$ and $\eta$ is their intersection, then we get a commutative diagram

$$\begin{array}{ccccc}
R[L_\tau] & \xrightarrow{\pi_\tau} & R[L_\sigma] & \xrightarrow{\iota_\tau} & R[L_\tau] \\
{\scriptstyle \iota_\eta}\downarrow & & {\scriptstyle \iota_{\tau'}}\downarrow & & \downarrow{\scriptstyle \iota_\eta} \\
R[L_\eta] & \xrightarrow{\pi_\eta} & R[L_{\tau'}] & \xrightarrow{\iota_\eta} & R[L_\eta]
\end{array} \qquad (6.1)$$

in which the composite horizontal maps are the identity.

Let $J$ denote the ideal of $R[L_\sigma]$ generated by all the monomials $e_m$ with $m$ strictly inside $\sigma$. Then $J$ is a $T_\sigma$-invariant ideal of $R[L_\sigma]$ such that $X_\sigma \setminus Y = T_\sigma$, where $Y = \mathrm{Spec}(R[L_\sigma]/J)$ (see [Danilov 1978, Section 2.6.1], for example).

**Lemma 6.2.** *Let* $\Delta^1$ *denote the set of codimension* 1 *faces of* $X_\sigma$. *Then the ideal* $J$ *is the ideal defining the closed subscheme* $\bigcup_{\tau \in \Delta^1} X_\tau$ *of* $X_\sigma$, *i.e.,* $Y = \bigcup_{\tau \in \Delta^1} X_\tau$.

*Proof.* The ideal $\mathcal{I}(X_\tau)$ that defines $X_\tau$ is generated by all monomials $e_m$ with $m \in (\sigma \setminus \tau) \cap L$. Since $\mathcal{I}\left(\bigcup_{\tau \in \Delta^1} X_\tau\right) = \bigcap_{\tau \in \Delta^1} \mathcal{I}(X_\tau)$, the lemma follows. $\qquad\square$

**Lemma 6.3.** *For any* $m \in L$, *there is a sufficiently large integer* $N$ *such that* $f/e_m \in R[L_\sigma]$ *for any* $f \in J^N$.

*Proof.* It is enough to prove the lemma when $f = \prod_{k=1}^N e_{m_k}$ with $m_k$ strictly inside $\sigma$. Let $v_1, \ldots, v_p$ be generators of $L_\sigma$ and let $l_1, \ldots, l_q$ be linear functionals defining $\sigma$. Set $s = \min_{i,j}\{l_i(v_j) > 0\}$. Since $m_k$ lies strictly inside $\sigma$, $l_i(m_k) > 0$ for any $i$. Since $m_k$ is a linear combination of the $v_j$ with nonnegative integer coefficients, we get $l_i(m_k) \geq s$ for any $i$. Therefore, $l_i\left(\sum_{k=1}^N m_k - m\right) \geq Ns - l_i(m)$ for any $i$. Since $s$ is positive, we must have $l_i\left(\sum_{k=1}^N m_k - m\right) \geq 0$ for any $i$ if $N$ is

sufficiently large. That is, $\sum_{k=1}^{N} m_k - m \in L_\sigma$ independent of the choice of the $m_k$. $\qquad\square$

**6B. $G$-toric schemes and their quotients.** Let $\sigma$ be a strongly convex, rational, polyhedral cone in $L_\mathbb{Q}$, where $L$ is a lattice of finite rank. Let $A = R[L_\sigma]$ and $X = X_\sigma = \mathrm{Spec}(A)$. Let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme over $R$.

**Definition 6.4.** An *affine $G$-toric scheme* is an affine toric scheme $X_\sigma$ as above with a $G$-action such that the action of $G$ on $X_\sigma$ factors through the action of $T_\sigma$.

Since $\mathrm{Spec}(R)$ is connected, a $G$-toric scheme structure on $X_\sigma$ is equivalent to having a map of monoids $\psi : L \to P$ such that the $R$-$G$-algebra structure on $A = R[L_\sigma]$ is defined by the composite action map

$$\phi_P : A \to R[L] \otimes_R A \xrightarrow{\psi \otimes \mathrm{Id}} R[P] \otimes_R A. \qquad (6.5)$$

**Examples 6.6.** We shall say that $G$ acts linearly on a polynomial algebra $A = R[t_1, \ldots, t_n]$ if there is a free $R$-$G$-module $(V, \rho)$ of rank $n$ such that $A = \mathrm{Sym}_R(V)$. In this case, we also say that $G$ acts linearly on $\mathrm{Spec}(A) = \mathbb{A}_R^n$.

Assume that $R$ satisfies (†). Let $A = R[x_1, \ldots, x_n, y_1, \ldots, y_r]$ be a polynomial $R$-algebra with a linear $G$-action, with $n, r \geq 0$. Using Lemma 5.7, we can assume that the $G$-action on $A$ is given by $\phi(x_i) = e_{\lambda_i} \otimes x_i$ for $1 \leq i \leq n$ and $\phi(y_j) = e_{\gamma_j} \otimes y_j$ for $1 \leq j \leq r$.

(1) Let $A = R[x_1, \ldots, x_n]$. Consider the cone $\sigma = \mathbb{Q}_+^n$ of $L_\mathbb{Q}$, where $L$ is the lattice $\mathbb{Z}^n$. Then $A = R[\sigma \cap L]$ and $\mathrm{Spec}(A)$ is an affine $G$-toric scheme via the morphism $\psi : \mathbb{Z}_+^n \to P$ given by $\psi(\alpha_i) = \lambda_i$, where $\{\alpha_1, \ldots, \alpha_n\}$ is the standard basis of $\mathbb{Z}_+^n$.

(2) Let $A = R[x_1, \ldots, x_n, y_1^{\pm 1}, \ldots, y_r^{\pm 1}]$. Then it can be seen, as in (1) above, that $\mathrm{Spec}(A)$ is an affine $G$-toric scheme by considering the lattice $L = \mathbb{Z}^{n+r}$ and the cone $\sigma = \mathbb{Q}_+^n \oplus \mathbb{Q}^r$ in $L_\mathbb{Q}$.

**Lemma 6.7.** *Let $\theta : L \to P$ be a homomorphism from $L$ to a finitely generated abelian group and let $M = \mathrm{Ker}(\theta)$. Then $R[\sigma \cap M]$ is a finitely generated $R$-algebra.*

*Proof.* By replacing $P$ by the image of $\theta$, we can assume that $\theta$ is an epimorphism. This yields an exact sequence

$$0 \to M_\mathbb{Q} \xrightarrow{i_M} L_\mathbb{Q} \to P_\mathbb{Q} \to 0. \qquad (6.8)$$

We write $\sigma = \bigcap_{i=1}^{r} \sigma_i$, where $\sigma_i = l_i^{-1}(\mathbb{Q}_+)$ is a half-space. By taking repeated intersections of $M$ with these $\sigma_i$ and using induction, we easily reduce to the case when $r = 1$. We set $\tau = \sigma \cap M_\mathbb{Q}$. Then $\tau = l^{-1}(\mathbb{Q}_+) \cap M_\mathbb{Q} = m^{-1}(\mathbb{Q}_+)$,

where $m = l \circ i_M$. In particular, $\tau$ is a cone in $M_{\mathbb{Q}}$. Furthermore, as $M \hookrightarrow L$, it is a free abelian group and hence a lattice in $M_{\mathbb{Q}}$. It follows from Gordon's lemma (see [Fulton 1993, Proposition 1.2.1], for instance) that $\tau \cap M$ is a finitely generated monoid. Therefore, $\sigma \cap M = \sigma \cap M_{\mathbb{Q}} \cap M = \tau \cap M$ is a finitely generated monoid. Since any generating set of $\sigma \cap M$ generates $R[\sigma \cap M]$ as an $R$-algebra, the lemma follows.                                                                                                    $\square$

Combining Lemma 6.7 and Proposition 5.4, we get:

**Corollary 6.9.** *Let $A^G$ denote the ring of $G$-invariants of $A$ with respect to $\phi_P$. Then $A^G$ is a finitely generated $R$-algebra.*

**Lemma 6.10.** *Let $B$ be any flat $A^G$-algebra. Then $B = (A \otimes_{A^G} B)^G$.*

*Proof.* Set $B' = A \otimes_{A^G} B$. To prove this lemma, we need to recall how $G$ acts on $B'$. The map $\phi_P : A \to R[P] \otimes_R A$ induces a $B'$-algebra map

$$B' = A \otimes_{A^G} B \xrightarrow{\phi_P \otimes 1_B} (R[P] \otimes_R A) \otimes_{A^G} B.$$

This can also be written as $\phi_{P,B} : B' \to R[P] \otimes_R B'$ with $\phi_{P,B} = \phi_P \otimes 1_B$, which gives a $G$-action on $\mathrm{Spec}(B')$.

Let $\gamma_P : A \to R[P] \otimes_R A$ be the ring homomorphism $\gamma_P(a) = 1 \otimes a$, which gives the projection map $G \times X \to X$. Set $\gamma_{P,B} = \gamma_P \otimes 1_B : B' \to R[P] \otimes_R B'$. It is clear that

$$\gamma_{P,B}(a \otimes b) = \gamma_P(a) \otimes b = 1 \otimes a \otimes b = 1 \otimes (a \otimes b).$$

Since $B'$ is generated by elements of the form $a \otimes b$ with $a \in A$ and $b \in B$, we see that $\gamma_{P,B}(\alpha) = 1 \otimes \alpha$ for all $\alpha \in B'$.

Since $A = R[L_\sigma]$ is flat (in fact free) over $R$ (see Lemma 6.7), the map $\gamma_P : A \to R[P] \otimes_R A$ is injective. Furthermore, there is an exact sequence (by definition of $A^G$)

$$0 \to A^G \to A \xrightarrow{\phi_P - \gamma_P} R[P] \otimes_R A. \tag{6.11}$$

As $B$ is flat over $A^G$, the tensor product with $B$ over $A^G$ yields an exact sequence

$$0 \to B \to B' \xrightarrow{(\phi_P \otimes 1_B) - (\gamma_P \otimes 1_B)} R[P] \otimes_R B'. \tag{6.12}$$

Since $\phi_P \otimes 1_B = \phi_{P,B}$ and $\gamma_P \otimes 1_B = \gamma_{P,B}$, we get an exact sequence

$$0 \to B \to B' \xrightarrow{\phi_{P,B} - \gamma_{P,B}} R[P] \otimes_R B'. \tag{6.13}$$

But this is equivalent to saying that $B = (B')^G$.                                        $\square$

**Lemma 6.14.** *Let $I$, $I' \subseteq A$ be inclusions of $A$-$G$-modules such that $I + I' = A$. Then the sequence*

$$0 \to I^G \to A^G \to (A/I)^G \to 0$$

*is exact and $I^G + I'^G = A^G$. In particular, the map $\mathrm{Spec}((A/I)^G) \hookrightarrow \mathrm{Spec}(A^G)$ is a closed immersion and $\mathrm{Spec}((A/I)^G) \cap \mathrm{Spec}((A/I')^G) = \varnothing$ in $\mathrm{Spec}(A^G)$.*

*Proof.* The assumption $I + I' = A$ is equivalent to saying that the map $I \oplus I' \to A$ is surjective. The lemma is now an immediate consequence of Corollary 3.13(2). $\square$

Combining the above lemmas, we obtain the following. We refer to [Mumford et al. 1994, §0.1] for the terms used in this result.

**Proposition 6.15.** *Let $X = X_\sigma$ be a $G$-toric scheme over $R$ as above. Then a categorical quotient in $\mathbf{Sch}_S$, $p : X \to X'$, for $G$-action (in the sense of [Mumford et al. 1994, Definition 0.5]) exists. Moreover, the following hold*:

(1) *If $Z \subseteq X$ is a $G$-invariant closed subscheme, then $p(Z)$ is a closed subscheme of $Y$.*

(2) *If $Z_1, Z_2 \subseteq X$ are $G$-invariant closed subschemes with $Z_1 \cap Z_2 = \varnothing$, then $p(Z_1) \cap p(Z_2) = \varnothing$.*

(3) *The map $p : X \to X'$ is a uniform categorical quotient in $\mathbf{Sch}_S$.*

(4) *The quotient map $p$ is submersive.*

*Proof.* We take $X' = \mathrm{Spec}(A^G)$. It follows from Lemma 6.7 that $X'$ is an affine scheme of finite type over $R$. The fact that $p : X \to X'$, given by the inclusion $A^G \hookrightarrow A$, is a categorical quotient follows at once from the exact sequence (6.11). The universality of $p$ with respect to $G$-invariant maps $p' : Y' \to X'$ of affine $G$-schemes with trivial $G$-action on $Y'$ also follows immediately from (6.11). The properties (1) and (2) are direct consequences of Lemma 6.14. To prove (3), let $Y' \to X'$ be a flat morphism between finite type $R$-schemes. To show that $p' : Y' \times_{X'} X \to Y'$ is a categorical quotient, we can use the descent argument of [Mumford et al. 1994, §0.2, Remark 8] to reduce to the case when $Y'$ is affine. In this case, the desired property follows at once from Lemma 6.10. Item (4) follows from (1)–(3) and [Mumford et al. 1994, §0.2, Remark 6]. $\square$

**Corollary 6.16.** *Let $X = \mathrm{Spec}(A)$ be a $G$-toric scheme as above and let $p : X \to X'$ be the quotient map. Let $Y \subsetneq X$ be a closed subscheme defined by a $G$-invariant ideal $J$. Let $h \in A^G$ be a nonunit such that $h \equiv 1 \pmod{J}$ and set $V' = \mathrm{Spec}(A^G[h^{-1}])$. Then we can find an open subscheme $U'$ of $X'$ such that $X' = U' \cup V'$ and $p^{-1}(U') \cap Y = \varnothing$.*

*Proof.* Our assumption says that $V' \subsetneq X'$ is a proper open subset of $X'$ and $Y \subset V = p^{-1}(V')$ is a $G$-invariant closed subset. Setting $Y' = p(Y)$, it follows from Proposition 6.15 that $Y' \subsetneq X'$ is a closed subset contained in $V'$. In particular, $Y_1 = p^{-1}(Y')$ is a $G$-invariant closed subscheme of $X$ such that $Y \subseteq Y_1 \subsetneq V \subsetneq X$. The open subset $U' = X' \setminus Y'$ now satisfies our requirements. $\square$

## 7. Equivariant vector bundles on *G*-toric schemes

In this section, we prove our main result about equivariant vector bundles on affine *G*-toric schemes.

**7A.** ***The setup.*** We shall prove Theorem 7.8 under the following setup. Let $R$ be a commutative noetherian ring and let $S = \mathrm{Spec}(R)$. Let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme over $R$. Let $L$ be a lattice of finite rank and let $\sigma$ be a strongly convex, polyhedral, rational cone in $L_{\mathbb{Q}}$. Let $\Delta$ denote the set of all faces of $\sigma$.

Let $A = R[L_\sigma]$ be such that $X = \mathrm{Spec}(A)$ is a *G*-toric scheme via a homomorphism $\psi : L \to P$ (see (6.5)). Set $Y = \bigcup_{\tau \in \Delta^1} X_\tau$. Let $X' = \mathrm{Spec}(A^G)$ and let $p : X \to X'$ denote the uniform categorical quotient in **Sch**$_S$ defined by the inclusion $A^G \hookrightarrow A$.

**7B.** ***Reduction to faithful action.*** We set $Q = \psi(L)$ and $H = \mathrm{Spec}(R[Q])$. Then $H$ is a diagonalizable closed subgroup of $T_\sigma$ which acts faithfully on $X$ and $G$ acts on $X$ via the quotient $G \twoheadrightarrow H$ (see Proposition 3.11). The following lemma reduces the proof of the main theorem of this section to the case of faithful action of $G$ on $X$.

We shall say that a finitely generated projective *A*-*G*-module $M$ over an *R*-*G*-algebra $A$ is *trivial* if it can be equivariantly extended from $R$, that is, there is a finitely generated projective *R*-*G*-module $F$ such that $M \simeq F_A$.

**Lemma 7.1.** *If every finitely generated projective A-H-module is trivial, then so is every finitely generated projective A-G-module.*

*Proof.* Given any $E \in (A\text{-}G)$-proj, we can write $E = \bigoplus_{b \in P/Q} E_b$ with $E_b = \bigoplus_{\{a \mid b = a \bmod Q\}} E_a$. Lemma 3.15 says that each $E_b \in (A\text{-}G)$-proj. It suffices to show that each $E_b$ is trivial.

Now, $E_b$ is trivial if and only if $E_b \otimes_R R_{-a}$ is trivial for any $a$ with $b = a \bmod Q$. But $E_b \otimes_R R_{-a}$ is a projective *A*-*H*-module and so we can find an *A*-*H*-module isomorphism $\phi : E_b \otimes_R R_{-a} \xrightarrow{\simeq} F_A$ for some $F \in (R\text{-}H)$-proj. This is then an *A*-*G*-module isomorphism as well. $\square$

**7C.** ***Trivialization in a neighborhood of Y.*** Note that, if $X = \mathrm{Spec}(A)$ is an affine G-toric scheme and $\tau$ is any face of the cone $\sigma$, then $X_\tau$ is a *G*-invariant closed subscheme of $X$. Moreover, the map $\pi_\tau : R[L_\tau] \to A = R[L_\sigma]$ defined before is *G*-equivariant (because it is $T_\sigma$-equivariant).

**Lemma 7.2.** *Let $\tau_1, \ldots, \tau_k$ denote the codimension-1 faces of $\sigma$ and let $I_j$ denote the ideal of A defining the closed subscheme $X_{\tau_j}$ associated to the face $\tau_j$. Let E be an A-G-module and F be an R-G-module such that $E/I_j \simeq F_{A/I_j}$ for all $1 \le j \le k$. Then $E/J \simeq F_{A/J}$, where J denotes the ideal defining $Y = \bigcup_{i=1}^{k} X_{\tau_i}$.*

*Proof.* Let $J_r$ be the ideal defining the $G$-invariant closed subscheme $Y_r = \bigcup_{i=1}^{r} X_{\tau_i}$ for $1 \leq r \leq k$. We prove by induction on $r$ that $E/J_r \simeq F_{A/J_r}$. Assume that $\phi : E/J_r \simeq F_{A/J_r}$ and $\eta : E/I_{r+1} \simeq F_{A/I_{r+1}}$ are given isomorphisms. This gives us a $G$-equivariant automorphism $\eta \circ \phi^{-1}$ of $F_{A/(J_r+I_{r+1})}$. Under the $G$-equivariant retraction $\Pi_{r+1} : X_\sigma \to X_{\tau_{r+1}} = \mathrm{Spec}(A/I_{r+1})$ (where $\Pi_i = \mathrm{Spec}(\pi_i)$), we have $\Pi_{r+1}(Y_r) \subset Y_r \cap X_{\tau_{r+1}}$ (see (6.1)).

Therefore $\phi' = (\Pi_{r+1}|_{Y_r})^*(\eta \circ \phi^{-1})$ defines an $A/(J_r)$-$G$-linear automorphism of $F_{A/J_r}$. Replacing $\phi$ by the isomorphism $\phi' \circ \phi$, we can arrange that $\phi$ and $\eta$ agree modulo $(J_r + I_{r+1})$. So they define a unique isomorphism $E/J_{r+1} \to F_{A/J_{r+1}}$. To see this, use the exact sequence

$$0 \to E/J_{r+1} \to E/J_r \times E/I_{r+1} \to E/(J_r + I_{r+1}) \to 0. \qquad \square$$

**Lemma 7.3.** *Let $P \in M_m(A^G)$ be a rank-m matrix with entries in $A^G$ such that $P$ is invertible modulo $I_j$ for all $1 \leq j \leq k$, where $I_j$ and $J$ are as in Lemma 7.2. Then, for any positive integer $N$, there is $\widetilde{P}_N \in \mathrm{GL}_m(A^G)$ such that $(P\widetilde{P}_N)_{ij} \in J^N$ for all $i \neq j$.*

*Proof.* For $1 \leq i \leq k$, we consider the commutative diagram of retractions

$$
\begin{array}{ccc}
(A/I_i)^G & \longrightarrow & A/I_i \\
\pi_{\tau_i}^G \downarrow & & \downarrow \pi_{\tau_i} \\
A^G & \longrightarrow & A
\end{array}
\tag{7.4}
$$

Since $P \bmod I_1$ is invertible, $P_1 := \pi_{\tau_1}(P \bmod I_1) \in \mathrm{GL}_m(A^G)$ and hence $P P_1^{-1} \equiv \mathrm{Id}_m \pmod{I_1}$. We now let $P_2$ denote the image of $P P_1^{-1} \bmod I_2$ under the $G$-equivariant retraction $\pi_{\tau_2}$. This yields $P_2 \equiv \mathrm{Id}_m \pmod{I_1}$ (see (6.1)) and so $P P_1^{-1} P_2^{-1} \equiv \mathrm{Id}_m \pmod{I_1 \cap I_2}$. Repeating this procedure and using Lemma 6.2, we can find $\widetilde{P}_1 \in \mathrm{GL}_m(A^G)$ such that $P \widetilde{P}_1 \equiv \mathrm{Id}_m \pmod{J}$, which proves the lemma for $N = 1$.

Assume now that there exists $\widetilde{P}_N \in \mathrm{GL}_m(A^G)$ such that $(P\widetilde{P}_N)_{ij} \equiv 0 \pmod{J^N}$ for $i \neq j$ and $(P\widetilde{P}_N)_{ii} \equiv 1 \pmod{J}$. By elementary column operations

$$C_i \mapsto C_i - (P\widetilde{P}_N)_{ji} C_j \quad \text{for } i > j = 1, \ldots, m-1$$

and

$$C_i \mapsto C_i - (P\widetilde{P}_N)_{ji} C_j \quad \text{for } i < j = 2, \ldots, m$$

on $P\widetilde{P}_N$, we get a matrix whose off-diagonal elements are 0 $\pmod{J^{N+1}}$ and diagonal elements are 1 $\pmod{J}$. These operations correspond to right multiplication by some $P' \in \mathrm{GL}_m(A^G)$. Taking $\widetilde{P}_{N+1} = \widetilde{P}_N P'$ completes the induction step. $\square$

**Lemma 7.5.** *Assume that $R$ satisfies (†) and let $I$ be a $G$-invariant ideal of $A$. Let $F$ and $E$ be finitely generated free $R$-$G$- and $A$-$G$-modules, respectively. Given*

*any $(A/I)$-$G$-module isomorphism $\phi : E/I \xrightarrow{\simeq} F_{A/I}$, there exists $h \in A^G$ such that $h \equiv 1$ modulo $I$ and $\phi$ extends to an $A_h$-$G$-module isomorphism $\phi_h : E_h \xrightarrow{\simeq} F_{A_h}$.*

*Proof.* Let $\phi'$ denote the inverse of $\phi$. Since $E$ and $F_A$ are projective $A$-$G$-modules, $\phi$ and $\phi'$ extend to $A$-$G$-module homomorphisms $T : E \to F_A$ and $T' : F_A \to E$ by Lemma 4.3. As $R$ satisfies (†), $F$ is a direct sum of rank-1 free $R$-$G$-modules by Lemma 5.7. Since $E$ and $F_A$ are isomorphic modulo $I$, they have the same rank, say $m$. Fix an $R$-basis $\{v_1, \ldots, v_m\}$ of $F$ consisting of elements of constant weights $e_{w_1}, \ldots, e_{w_m}$ ($w_i \in P$) and fix any $A$-basis of $E$.

With respect to the chosen bases, $T$ and $T'$ define matrices in $M_m(A)$ which are invertible modulo $I$. Moreover, as $TT' = (a_{ij})$ defines an $A$-$G$-module endomorphism of $F_A$, it can be easily checked using Lemma 3.14 that $a_{ij} \in A_{w_i - w_j}$ and, using the Leibniz formula for the determinant, one checks that $\det(TT') \in A^G$. We take $h = \det(TT')$ to finish the proof. $\square$

## 7D. *Descent to the quotient scheme.*

The following unique "descent to the quotient" property of the $G$-equivariant maps will be crucial for proving our main results on equivariant vector bundles:

**Lemma 7.6.** *Assume that $R$ satisfies (†). Let $q : W \to W'$ be a uniform categorical quotient in $\mathbf{Sch}_S$ for a $G$-action on $W$, where $w : W \to S$ and $w' : W' \to S$ are structure maps. Assume that $q$ is an affine morphism. Let $F$ be a finitely generated projective $R$-module. Given any $G$-equivariant endomorphism $f$ of $w^*(F)$, there exists a unique endomorphism $\tilde{f}$ of $w'^*(F)$ such that $f = q^*(\tilde{f})$. In particular, $\tilde{f}$ is an automorphism if $f$ is so.*

*Proof.* The second part follows from the uniqueness assertion in the first part, so we only have to prove the existence of a unique $\tilde{f}$. Since $W'$ is noetherian, we can write $W' = \bigcup_{i=1}^{r} U'_i$, where each $U'_i$ is affine open. We prove the lemma by induction on $r$. If $r = 1$, then $W'$ is affine and hence so is $W$. We can write $W = \mathrm{Spec}(B)$ and $W' = \mathrm{Spec}(B^G)$ for some finite-type $R$-$G$-algebra $B$ (see Proposition 6.15). As $F$ is a free $R$-$G$-module of constant weight $e_0$, it follows from Lemma 3.14 that $f \in M_n(B^G)$ with $n = \mathrm{rank}(F)$. In particular, it defines a unique endomorphism $\tilde{f}$ of $w'^*(F)$ such that $f = q^*(\tilde{f})$.

We now assume $r \geq 2$ and set $U' = \bigcup_{i=2}^{r} U'_i$. Then $q : U_1 := q^{-1}(U'_1) \to U'_1$ and $q : U := q^{-1}(U') \to U'$ are uniform categorical quotients. As $U'_1$ is affine, there exists a unique $\tilde{f}_{U'_1} : F_{U'_1} \to F_{U'_1}$ such that $q^*(\tilde{f}_{U'_1}) = f|_{U_1}$. By the induction hypothesis, there exists a unique $\tilde{f}_{U'} : F_{U'} \to F_{U'}$ such that $q^*(\tilde{f}_{U'}) = f|_U$. As $V' := U'_1 \cap U'$ has a cover by $r - 1$ affine opens, the induction hypothesis and uniqueness imply that $\tilde{f}_{U'_1}|_V = \tilde{f}_{U'}|_V$. The reader can check that $\tilde{f}_{U'_1}$ and $\tilde{f}_{U'}$ glue together to define the desired unique endomorphism $\tilde{f} : w'^*(F) \to w'^*(F)$. $\square$

**7E. *The main theorem.*** We now use the above reduction steps to prove our main result of this section. We first consider the case of faithful action.

**Lemma 7.7.** *Suppose $\psi : L \twoheadrightarrow P$. Assume that $R$ satisfies (†) and that every finitely generated projective $A^G$-module is extended from $R$. Let $E \in (A\text{-}G)$-proj and $F \in (R\text{-}G)$-proj. Suppose there exist $G$-equivariant isomorphisms $\eta : E|_U \xrightarrow{\simeq} F_A|_U$ and $\phi : E|_V \xrightarrow{\simeq} F_A|_V$, where $U = X \setminus Y$ is the big torus of $X$ and $V = \mathrm{Spec}(A[h^{-1}])$ for some $h \in A^G$ such that $h \equiv 1 \pmod{J}$, where $J$ is the defining ideal of the inclusion $Y \hookrightarrow X$. Then $E \simeq F_A$ as $A$-$G$-modules.*

*Proof.* If $h$ is a unit in $A^G$, we have $V = X$ and we are done. So assume that $h$ is not a unit in $A^G$. Let $p : X \to X'$ denote the quotient map as in Proposition 6.15. Set $V' = \mathrm{Spec}(A^G[h^{-1}])$ so that $V = p^{-1}(V')$ and let $U' \subseteq X'$ be as obtained in Corollary 6.16 so that $U_1 := p^{-1}(U') \subseteq U$. Set $W' = U' \cap V'$ and $W = p^{-1}(W')$. Then $\eta : E|_{U_1} \to F_A|_{U_1}$ is a $G$-equivariant isomorphism. Let $\Phi = \phi \circ \eta^{-1}$ denote the $G$-equivariant automorphism of $F_A|_W$.

By Lemma 5.7, we can write $F = \bigoplus_{i=1}^m \widetilde{F}_{\lambda_i}$, where $\lambda_i \in P$ are not necessarily distinct and $\widetilde{F}_{\lambda_i}$ are free $R$-$G$-modules of rank 1 and constant weight $e_{\lambda_i}$. Since $L \twoheadrightarrow P$, there exist monomials in $R[L]$ of any given weight. Suppose $d_i \in R[L]$ is a monomial having weight $e_{\lambda_i}$. Let $D$ be the diagonal matrix with diagonal entries $d_1, \ldots, d_m$. Then $D \in \mathrm{Hom}_{R[L]G}(F_{R[L]}, F'_{R[L]})$ is an isomorphism of $R[L]$-$G$-modules, where $F'$ is a free $R$-$G$-module of rank $m$ and constant weight $e_0$. Thus $\widetilde{\Phi} := D\Phi D^{-1}$ is a $G$-equivariant automorphism of $F'_A|_W$.

Since $p : W \to W'$ is a uniform categorical quotient which is an affine morphism, we can apply Lemma 7.6 to find a unique automorphism $f$ of $F'_{A^G}|_{W'}$ such that $\widetilde{\Phi} = p^*(f)$. As $X' = U' \cup V'$, such an automorphism defines a locally free sheaf on $X'$ by gluing of sheaves [Hartshorne 1977, Exercise II.1.22]. Since every such locally free sheaf on $X'$ is free by assumption, we have [loc. cit.] $f = f_2 \circ f_1$ for some automorphisms $f_1$ and $f_2$ of $F'_{A^G}|_{U'}$ and $F'_{A^G}|_{V'}$, respectively. Then $\widetilde{\Phi} = p^*(f) = p^*(f_2) \circ p^*(f_1)$ and hence we get $\Phi = (D^{-1}p^*(f_2)D)(D^{-1}p^*(f_1)D)$. As $p^*(f_2)$ defines a matrix $P_1$ in $\mathrm{GL}_m(A^G[h^{-1}])$, by an appropriate choice of basis we can find $s \geq 0$ such that $P := h^s P_1 \in M_m(A^G)$.

By Lemma 7.3, we can find $\widetilde{P}_N \in \mathrm{GL}_m(A^G)$ such that $(P\widetilde{P}_N)_{ij} \in J^N$ for $i \neq j$. The $(ij)$-th entry of $D^{-1}P\widetilde{P}_N D$ is $d_i^{-1}d_j(P\widetilde{P}_N)_{ij}$. Taking $N$ sufficiently large, we may assume that $d_i^{-1}d_j(P\widetilde{P}_N)_{ij} \in A$ by Lemma 6.3.

Setting $\theta_1 = (D^{-1}\widetilde{P}_N^{-1}p^*(f_1)D)$ and $\theta_2 = (D^{-1}h^{-s}P\widetilde{P}_N D)$, we see that $\theta_1$ and $\theta_2$ define G-equivariant automorphisms of $F_A|_{U_1}$ and $F_A|_V$, respectively, such that $\theta_2 \circ \theta_1 = \Phi = \phi \circ \eta^{-1}$.

If we set $\eta' = \theta_1 \circ \eta$ and $\phi' = \theta_2^{-1} \circ \phi$, we see that $\eta' : E|_{U_1} \to F_A|_{U_1}$ and $\phi' : E|_V \to F_A|_V$ are $G$-equivariant isomorphisms such that $\eta'|_W = \phi'|_W$. By gluing therefore, we get a $G$-equivariant isomorphism $E \to F_A$ on $X$. $\square$

**Theorem 7.8.** *Consider the setup of Section 7A. Assume that $R$ satisfies (†) and that finitely generated projective modules over $A_\tau$ and $(A_\tau)^G$ are extended from $R$ for every $\tau \in \Delta$. Then every finitely generated projective $A$-$G$-module is trivial.*

*Proof.* We can assume that the map $\psi : L \to P$ is surjective by Lemma 7.1. Let $E \in (A\text{-}G)$-proj. Since $R$ satisfies (†) and every finitely generated projective $A$-module is extended from $R$, we see that $E$ is a free $A$-module of finite rank. In particular, Lemma 7.5 applies.

Let $\tilde{\tau}$ denote the face of $\sigma$ of smallest dimension. Then $X_{\tilde{\tau}}$ is a torus whose dimension is that of the largest subspace of $L_\mathbb{Q}$ contained in $\sigma$. Let $M$ denote the smallest sublattice of $L$ such that $\tilde{\tau} = M_\mathbb{Q}$. Let $\phi : M \hookrightarrow L \to P$ denote the composite map. Consider the abelian groups $Q_1 := \text{Im}(\phi)$ and $Q_2 := P/Q_1$. Fix a finitely generated projective $R$-$G$-module $F$ such that $E|_{X_{\tilde{\tau}}} \simeq F \otimes_R R[L_{\tilde{\tau}}]$. This exists by Corollary 3.16, applied to the sequence

$$0 \to Q_1 \to P \to P/Q_1 \to 0.$$

We prove by induction on the dimension of the cone $\sigma$ that $E \simeq F_{R[L_\sigma]}$. Assume that $E|_{X_\tau} \simeq F_{R[L_\tau]}$ for all codimension-1 faces $\tau$ of $\sigma$. Let $Y = \bigcup_{\tau \in \Delta^1} X_\tau$ be as before. We first apply Lemma 7.2 to get an isomorphism $\tilde{\phi} : E/J \simeq F_{A/J}$. We next apply Lemma 7.5 to find $h \in A^G$ such that $\tilde{\phi}$ extends to an isomorphism $\phi$ on $V = \text{Spec}(A_h) \supseteq Y$.

Applying Corollaries 3.16 and 5.8 to the torus $T_\sigma = \text{Spec}(R[L])$, there exists an $R[L]$-$G$-module isomorphism $\eta : E|_{T_\sigma} \xrightarrow{\simeq} F_{R[L]} = F_A|_{T_\sigma}$ (consider the exact sequence $0 \to P \to P \to 0 \to 0$ and note that the action of $G$ on $T_\sigma$ is free). We now apply Lemma 7.7 to conclude that $E \simeq F_A$. This completes the induction step and proves the theorem. $\square$

As an easy consequence of Corollary 5.6 and Theorem 7.8, we obtain:

**Corollary 7.9.** *Consider the setup of Section 7A and assume that $R$ is a principal ideal domain. Then every finitely generated projective $A$-$G$-module is trivial.*

# 8. Vector bundles over $\mathbb{A}_R^n \times \mathbb{G}_{m,R}^r$

In this section, we apply Theorem 7.8 to prove triviality of $G$-equivariant projective modules over polynomial and Laurent polynomial rings. When $R$ satisfies (††), we have the following answer to the equivariant Bass–Quillen question:

**Theorem 8.1.** *Let $R$ be a regular ring and let $R[x_1, \ldots, x_n, y_1, \ldots, y_r]$ be a polynomial $R$-algebra with a linear $G$-action with $n, r \geq 0$. Then the following hold:*

(1) *If $R$ satisfies (††) and $A = R[x_1, \ldots, x_n]$, then every finitely generated projective $A$-$G$-module is trivial.*

(2) *If $R$ is a PID and $A = R[x_1, \ldots, x_n, y_1^{\pm 1}, \ldots, y_r^{\pm 1}]$, then every finitely generated projective $A$-$G$-module is trivial.*

*Proof.* As shown in Examples 6.6, $\mathrm{Spec}(A)$ is an affine toric $G$-scheme in both cases. To prove (1), note that $R$ satisfies the hypotheses of Theorem 7.8, by Corollary 5.5. Therefore, (1) follows from Theorem 7.8. Similarly, (2) is a special case of Corollary 7.9.                                                     $\square$

**8A.** *Vector bundles over $\mathbb{A}_R^n$ without condition* (††). Let $R$ be a noetherian ring and let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme over $R$. We now show that if the localizations of $R$ satisfy (††) then the equivariant vector bundles over $\mathbb{A}_R^n$ can be extended from $\mathrm{Spec}(R)$. In order to show this, we shall need the following equivariant version of Quillen's patching lemma [1976, Lemma 1]. In this section, we shall allow our $R$-$G$-algebras to be noncommutative (see Section 2A3).

Given a (possibly noncommutative) $R$-$G$-algebra $A$, a polynomial $A$-$G$-algebra is an $R$-$G$-algebra $A[t]$ which is a polynomial algebra over $A$ with indeterminate $t$ such that the inclusion $A \hookrightarrow A[t]$ is a morphism of $R$-$G$-algebras and $t \in A[t]$ is semi-invariant (see Section 2A2). For a polynomial $A$-$G$-algebra $A[t]$, let $(1 + t A[t])^\times$ denote the (possibly noncommutative) group of units $\phi(t) \in A[t]$ such that $\phi(0) = 1$.

Given an $A[t]$-$G$-module $M$ (with $A$ commutative), we shall say that $M$ is extended from $A$ if there is an $A$-$G$-module $N$ and an $A[t]$-$G$-linear isomorphism $\theta : N \otimes_A A[t] \xrightarrow{\cong} M$. It is easy to check that this condition is equivalent to saying that there is an $A[t]$-$G$-linear isomorphism $\theta : (M/tM) \otimes_A A[t] \xrightarrow{\cong} M$.

**Lemma 8.2** (equivariant patching lemma). *Let $(A, \phi)$ be an $R$-$G$-algebra and let $(A[t], \tilde{\phi})$ be a polynomial $A$-$G$-algebra as above. Let $0 \neq f \in R$ and let $\theta(t) \in (1 + t A_f[t])^\times$ be a $G$-invariant polynomial. Then there exists $k \geq 0$ such that, for any $a, b \in R$ with $a - b \in f^k R$, we can find a $G$-invariant element $\psi(t) \in (1 + t A[t])^\times$ with $\psi_f(t) = \theta(at)\theta(bt)^{-1}$.*

*Proof.* This is a straightforward generalization of [Quillen 1976, Lemma 1] with the same proof almost verbatim. The only extra thing we need to check is that if $\theta(t) \in (1 + t A_f[t])^\times \cap (A[t])^G$ then $\psi(t)$ (as constructed in [loc. cit.]) is also $G$-invariant. But this can be checked directly, using the fact that $t$ is semi-invariant. We leave the details to the reader.                                            $\square$

**Lemma 8.3.** *Let $(A, \phi)$ and $(A[t], \tilde{\phi})$ be as in Lemma 8.2. Assume that $A$ is commutative. Let $M$ be a finitely generated $A[t]$-$G$-module and let $Q(M) = \{f \in R \mid M_f \text{ is an extended } A_f[t]$-$G$-module$\}$. Then $Q(M) \cup \{0\}$ is an ideal of $R$.*

*Proof.* We only need to check that if $f_0, f_1 \in Q(M)$ then $f_0 + f_1 \in Q(M)$. We can assume that $f_0 + f_1$ is invertible in $R$. In particular, $(f_0, f_1) = R$. Set

$$A_i = A_{f_i}, \quad M_i = M_{f_i} \quad \text{for } i = 0, 1, \quad N = M/tM \quad \text{and} \quad E = \mathrm{Hom}_A(N, N).$$

Given isomorphisms $u_i : N \otimes_A A_i[t] \xrightarrow{\simeq} M_i$, Quillen [1976, Theorem 1] constructs automorphisms $\psi_i(t) \in \mathrm{Hom}_{A_i[t]}(N \otimes_A A_i[t], N \otimes_A A_i[t]) = E_i[t]$ for $i = 0, 1$ with the following properties:

$$u'_i := u_i \cdot \psi_i(t) : N \otimes_A A_i[t] \xrightarrow{\simeq} M_i \quad \text{and} \quad (u'_0)_{f_1} = (u'_1)_{f_0}.$$

One should observe here that the isomorphism

$$E_i[t] \xrightarrow{\simeq} \mathrm{Hom}_{A_i[t]}(N \otimes_A A_i[t], N \otimes_A A_i[t]), \quad f \otimes t^i \mapsto (n \otimes a \mapsto f(n) \otimes at^i),$$

is $R$-$G$-linear (see Lemma 3.8).

To prove the lemma, we only need to show that each $\psi_i(t)$ is $G$-equivariant. By Lemma 3.8, this is equivalent to showing that $\psi_i(t) \in (E_i[t])^G$ for $i = 0, 1$. But this follows at once (as the reader can check by hand) by observing that each $u_i$ is $G$-invariant and subsequently applying Lemma 8.2 to $E_{f_0}$ and $E_{f_1}$, which are (possibly noncommutative) $R$-$G$-algebras by Lemma 3.8. $\square$

The following result generalizes Theorem 8.1 to the case when the base ring $R$ does not necessarily satisfy (††), but whose local rings satisfy (††). For examples of local rings satisfying (††), see Theorem 5.2.

**Theorem 8.4.** *Let $R$ be a noetherian integral domain such that its localizations at all maximal ideals satisfy* (††). *Let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme over $R$. Let $V = \bigoplus_{i=1}^{n} R x_i$ be a direct sum of one-dimensional free $R$-$G$-modules and let $A = R[x_1, \ldots, x_n] = \mathrm{Sym}_R(V)$. Then every finitely generated projective $A$-$G$-module is extended from $R$.*

*Proof.* We prove the theorem by induction on $n$. There is nothing to prove when $n = 0$ and the case $n = 1$ is an easy consequence of Theorem 8.1 and Lemma 8.3. Suppose now that $n \geq 2$ and every projective $R[x_1, \ldots, x_{n-1}]$-$G$-module is extended from $R$.

Let $M$ be a finitely generated projective $A$-$G$-module and set $A_i = R[x_1, \ldots, x_i]$. It follows from Theorem 8.1 that $M_{\mathfrak{m}}$ is extended from $(A_{n-1})_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of $R$. We now apply Lemma 8.3 to $(A_{n-1}, \phi_{n-1})$ and $(A_{n-1}[x_n], \tilde{\phi}_{n-1}) = (A, \phi)$ to conclude that $M$ is extended from $A_{n-1}$. It follows by induction that $M$ is extended from $R$. $\square$

## 9. Derived equivalence and equivariant *K*-theory

In this section, we shall apply the results of Section 4 to show that the derived equivalence of equivariant quasicoherent sheaves on affine schemes with group action implies the equivalence of the equivariant $K$-theory of these schemes. When the underlying group is trivial, this was shown by Dugger and Shipley [2004]. In the equivariant setup too, we make essential use of some general results of Dugger and Shipley, which we now recall.

**9A.** *Some results of Dugger and Shipley.* Recall that an object $X$ in a cocomplete triangulated category $\mathcal{T}$ is called *compact* if the natural map $\varinjlim_\alpha \operatorname{Hom}_{\mathcal{T}}(X, Z_\alpha) \to \operatorname{Hom}_{\mathcal{T}}(X, \varinjlim_\alpha Z_\alpha)$ is a bijection for every direct system $\{Z_\alpha\}$ of objects in $\mathcal{T}$.

If $\mathcal{A}$ is an abelian category, then an object of the category $\operatorname{Ch}_{\mathcal{A}}$ of chain complexes over $\mathcal{A}$ is called *compact* if its image in the derived category $D(\mathcal{A})$ is compact in the above sense.

The key steps in the proof of our main theorem of this section are Propositions 4.6 and A.1 and the following general results of [Dugger and Shipley 2004]:

**Theorem 9.1** [Dugger and Shipley 2004, Theorem D]. *Let $\mathcal{A}$ and $\mathcal{B}$ be cocomplete abelian categories which have sets of small, projective, strong generators. Let $K_c(\mathcal{A})$ (resp. $K_c(\mathcal{B})$) denote the Waldhausen K-theory of the compact objects in $\operatorname{Ch}(\mathcal{A})$ (resp. $\operatorname{Ch}(\mathcal{B})$). Then:*

(1) *$\mathcal{A}$ and $\mathcal{B}$ are derived equivalent if and only if $\operatorname{Ch}(\mathcal{A})$ and $\operatorname{Ch}(\mathcal{B})$ are equivalent as pointed model categories.*

(2) *If $\mathcal{A}$ and $\mathcal{B}$ are derived equivalent, then $K_c(\mathcal{A}) \simeq K_c(\mathcal{B})$.*

**Theorem 9.2** [Dugger and Shipley 2004, Corollary 3.9]. *Let $\mathcal{M}$ and $\mathcal{N}$ be pointed model categories connected by a zigzag of Quillen equivalences. Let $\mathcal{U}$ be a complete Waldhausen subcategory of $\mathcal{M}$, and let $\mathcal{V}$ consist of all cofibrant objects in $\mathcal{N}$ which are carried into $\mathcal{U}$ by the composite of the derived functors of the Quillen equivalences. Then $\mathcal{V}$ is a complete Waldhausen subcategory of $\mathcal{N}$, and there is an induced zigzag of weak equivalences between $K(\mathcal{U})$ and $K(\mathcal{V})$.*

**Theorem 9.3** [Dugger and Shipley 2004, Theorems 4.2 and 7.5]. *Let $\mathcal{R}$ and $\mathcal{S}$ be two ringoids (see Section 4). Then the following conditions are equivalent:*

(1) *There is a zigzag of Quillen equivalences between $\operatorname{Ch}(\operatorname{Mod}\text{-}\mathcal{R})$ and $\operatorname{Ch}(\operatorname{Mod}\text{-}\mathcal{S})$.*

(2) *$D(\mathcal{R}) \simeq D(\mathcal{S})$ are triangulated equivalent.*

(3) *The bounded derived categories of finitely generated projective $\mathcal{R}$- and $\mathcal{S}$-modules are triangulated equivalent.*

**9B.** *Derived equivalence and K-theory under group action.* Let $R$ be a commutative noetherian ring and let $G$ be an affine group scheme over $R$. Let $(A, \phi)$ be an $R$-$G$-algebra and let $X = \operatorname{Spec}(A)$ be the associated affine $S$-scheme with $G$-action, where $S = \operatorname{Spec}(R)$. We shall denote this datum in this section by $(R, G, A)$. Let $\operatorname{Ch}^G(A)$ denote the abelian category of unbounded chain complexes of $A$-$G$-modules and let $D^G(A)$ denote the associated derived category. One knows that $D^G(A)$ is a cocomplete triangulated category.

We shall say that $(R, G, A)$ has the *resolution property* if for every finitely generated $A$-$G$-module $M$ there is a finitely generated projective $A$-$G$-module $E$ and a $G$-equivariant epimorphism $E \twoheadrightarrow M$.

Recall that a bounded chain complex of finitely generated, projective $A$-$G$-modules is called a *strict perfect complex*. A (possibly unbounded) chain complex of $A$-$G$-modules is called a *perfect complex* if it is isomorphic to a strict perfect complex in $D^G(A)$. We shall denote the categories of strict perfect and perfect complexes of $A$-$G$-modules by $\mathrm{Sperf}^G(A)$ and $\mathrm{Perf}^G(A)$, respectively. It is known that $\mathrm{Sperf}^G(A)$ and $\mathrm{Perf}^G(A)$ are both complicial bi-Waldhausen categories in the sense of [Thomason and Trobaugh 1990] and there is a natural inclusion $\mathrm{Sperf}^G(A) \hookrightarrow \mathrm{Perf}^G(A)$ of complicial bi-Waldhausen categories. As this inclusion induces an equivalence of the associated derived categories, it follows from [Thomason and Trobaugh 1990, Theorem 1.9.8] that this induces a homotopy equivalence of the associated Waldhausen $K$-theory spectra. We shall denote the common derived category by $D^G(\mathrm{Perf}/A)$ and the common $K$-theory spectrum by $K^G(A)$. It follows from [Thomason and Trobaugh 1990, Theorem 1.11.7] that $K^G(A)$ is homotopy equivalent to the $K$-theory spectrum of the exact category of finitely generated projective $A$-$G$-modules.

Let $\mathrm{Ch}_b^G(A)$ denote the category of bounded chain complexes of finitely generated $A$-$G$-modules and let $D_b^G(A)$ denote its derived category. The Waldhausen $K$-theory spectrum of $\mathrm{Ch}_b^G(A)$ will be denoted by $K'_G(A)$. Let $\mathrm{Ch}^{\mathrm{hb},-}(A\text{-}G\text{-proj})$ be the category of chain complexes of finitely generated projective $A$-$G$-modules which are bounded above and cohomologically bounded. Let $D^{\mathrm{hb},-}(A\text{-}G\text{-proj})$ denote the associated derived category.

If $(R, G, A)$ has the resolution property, then every complex of $\mathrm{Ch}_b^G(A)$ is quasi-isomorphic to a complex of $\mathrm{Ch}^{\mathrm{hb},-}(A\text{-}G\text{-proj})$ and vice versa. It follows from [Thomason and Trobaugh 1990, Theorem 1.9.8] that they have homotopy equivalent Waldhausen $K$-theory spectra:

$$K'_G(A) \simeq K(\mathrm{Ch}^{\mathrm{hb},-}(A\text{-}G\text{-proj})). \tag{9.4}$$

**Lemma 9.5.** *Assume that $(R, G, A)$ has the resolution property. Given any complex $K \in \mathrm{Ch}^G(A)$, there exists a direct system of strict perfect complexes $F_\alpha$, and a quasi-isomorphism*

$$\varinjlim_\alpha F_\alpha \xrightarrow{\sim} K.$$

*Proof.* The nonequivariant case of this result was proven in [Thomason and Trobaugh 1990, Proposition 2.3.2] and a similar proof applies here as well once we verify that [ibid., Hypothesis 1.9.5.1] holds for $\mathcal{A} = (A\text{-}G)\text{-Mod}$, $\mathcal{D}$ the category of (possibly infinite) direct sums of finitely generated projective A-G-modules and $\mathcal{C}$ the category of cohomologically bounded above complexes in $\mathrm{Ch}^G(A)$. For this, it is enough to show that if $M \to N$ is a surjective map of $A$-$G$-modules then there is a (possibly infinite) direct sum $F$ of finitely generated projective $A$-$G$ modules and an $A$-$G$-linear map $F \to M$ such that the composite $F \to M \to N$ is surjective. But $M$

is the direct limit of its finitely generated $A$-$G$-submodules, as shown in [Laumon and Moret-Bailly 2000, Proposition 15.4] (see also [Thomason 1987, Lemma 2.1] when $G$ is faithfully flat over $S$). Therefore, it follows from the resolution property that $M$ is a quotient of a direct sum of finitely generated projective $A$-$G$-modules. □

In order to lift the derived equivalence to an equivalence of Waldhausen categories, we need to use model structures on the category of chain complexes of $A$-$G$-modules. We refer to [Hovey 1999] for model structures and various related terms that we shall use here. Let $\mathcal{A}$ be a Grothendieck abelian category with enough projective objects and let $\mathrm{Ch}_{\mathcal{A}}$ denote the category of unbounded chain complexes over $\mathcal{A}$. Recall from [Hovey 2007, Proposition 7.4] that $\mathrm{Ch}_{\mathcal{A}}$ has the *projective model structure*, in which the weak equivalences are the quasi-isomorphisms, fibrations are termwise surjections and the cofibrations are the maps having the left lifting property with respect to fibrations which are also weak equivalences.

**Lemma 9.6.** *Let $E$ be a bounded above complex of projective objects in a Grothendieck abelian category $\mathcal{A}$ with enough projective objects. Then $E$ is cofibrant in the projective model structure on $\mathrm{Ch}_{\mathcal{A}}$.*

*Proof.* This is proved in [Hovey 1999, Lemma 2.3.6] in the case when $\mathcal{A}$ is the category of modules over a ring. The same proof goes through for any abelian category for which the projective model structure exists. □

Given a datum $(R, G, A)$ as above, let $\mathrm{Ch}_{\mathrm{cc}}^G(A)$ denote the full subcategory of $\mathrm{Ch}^G(A)$ consisting of chain complexes which are compact and cofibrant (in projective model structure). For the notion of Waldhausen subcategories of a model category, see [Dugger and Shipley 2004, §3].

**Proposition 9.7.** *Let $(R, G, A)$ be as in Proposition 4.6. Then there is an inclusion $\mathrm{Sperf}^G(A) \hookrightarrow \mathrm{Ch}_{\mathrm{cc}}^G(A)$ of Waldhausen subcategories of $\mathrm{Ch}^G(A)$ such that the induced map on the K-theory spectra is a homotopy equivalence.*

*Proof.* It follows from the results of Section 4 that $\mathrm{Sperf}^G(A)$ is same as the category of bounded chain complexes of finitely generated projective objects of $(A$-$G)$-Mod. To check now that $\mathrm{Sperf}^G(A)$ and $\mathrm{Ch}_{\mathrm{cc}}^G(A)$ are Waldhausen subcategories of $\mathrm{Ch}^G(A)$, we only need to check that they are closed under taking push-outs. But this is true for the first category because every cofibration in $\mathrm{Ch}^G(A)$ is a termwise split injection with projective cokernels (see [Hovey 1999, Theorem 2.3.11]) and this is true for the second category because of the well-known fact that the cofibrations are closed under push-out and, if two vertices of a triangle in a triangulated category are compact, then so is the third.

To show that $\mathrm{Sperf}^G(A)$ is a subcategory of $\mathrm{Ch}_{\mathrm{cc}}^G(A)$, we have to show that every object of $\mathrm{Sperf}^G(A)$ is cofibrant and compact. The first property follows

from Lemmas 4.3, 4.4 and 9.6. To prove compactness, we can use Proposition 4.6 to replace $(A\text{-}G)$-Mod by $\mathcal{R}$-Mod, where $\mathcal{R}$ is a ringoid. But, in this case, it is shown in [Keller 1994, §4.2] that a bounded complex of finitely generated projective objects of $\mathcal{R}$-Mod is compact.

To show that the inclusion $\mathrm{Sperf}^G(A) \hookrightarrow \mathrm{Ch}^G_{\mathrm{cc}}(A)$ induces a homotopy equivalence of $K$-theory spectra, we can use [Blumberg and Mandell 2011, Theorem 1.3] to reduce to showing that this inclusion induces an equivalence of the associated derived subcategories of $D^G(A)$. To do this, all we need to show is that every compact object of $D^G(A)$ is isomorphic to an object of $\mathrm{Sperf}^G(A)$. We have just shown above that every object of $\mathrm{Sperf}^G(A)$ is compact. It follows now from Lemma 9.5 and [Neeman 1996, Theorem 2.1] that every compact object of $D^G(A)$ comes from $\mathrm{Sperf}^G(A)$. Notice that we have shown in Lemmas 4.1 and 4.4 that the hypothesis of Lemma 9.5 is satisfied in our case. The proof of the proposition is now complete. ☐

For $i = 1, 2$, let $R_i$ be a commutative noetherian ring, $G_i$ an affine group scheme over $R_i$ and $A_i$ an $R_i$-$G_i$-algebra such that one of the following holds:

(1) $G_i$ is a diagonalizable group scheme over $R_i$.

(2) $R_i$ is a UFD containing a field of characteristic zero and $G_i$ is a split reductive group scheme over $R_i$.

We are now ready to prove the main result of this section.

**Theorem 9.8.** *Let* $(R_1, G_1, A_1)$ *and* $(R_2, G_2, A_2)$ *be as above. Then* $D^{G_1}(A_1)$ *and* $D^{G_2}(A_2)$ *are equivalent as triangulated categories if and only if* $D^{G_1}(\mathrm{Perf}/A_1)$ *and* $D^{G_2}(\mathrm{Perf}/A_2)$ *are equivalent as triangulated categories. In either case, the following hold*:

(1) *There is a homotopy equivalence of spectra* $K^{G_1}(A_1) \simeq K^{G_2}(A_2)$.

(2) *There is a homotopy equivalence of spectra* $K'_{G_1}(A_1) \simeq K'_{G_2}(A_2)$.

*Proof.* It follows from Lemmas 4.3 and 4.4 that the derived categories of perfect complexes are the same as the bounded derived categories of finitely generated projective objects. The first assertion of the theorem is now an immediate consequence of Proposition 4.6 and Theorem 9.3.

If $D^{G_1}(A_1)$ and $D^{G_2}(A_2)$ are equivalent as triangulated categories, it follows from Theorem 9.1 and Propositions 4.6 and 9.7 that there is a homotopy equivalence of spectra $K^{G_1}(A_1) \simeq K^{G_2}(A_2)$.

To prove (2), we first conclude from Proposition 4.6 and Theorem 9.3 that the equivalence of the derived categories is induced by a zigzag of Quillen equivalences between $\mathrm{Ch}^{G_1}(A_1)$ and $\mathrm{Ch}^{G_2}(A_2)$. It follows from Propositions 4.6 and A.1 that this derived equivalence induces an equivalence between the triangulated subcategories $D^{\mathrm{hb},-}(A_1\text{-}G_1\text{-proj})$ and $D^{\mathrm{hb},-}(A_2\text{-}G_2\text{-proj})$ of the corresponding derived

categories. It follows that this zigzag of Quillen equivalences carries the Waldhausen subcategory $\mathrm{Ch}^{\mathrm{hb},-}(A_1\text{-}G_1\text{-proj})$ of $\mathrm{Ch}^{G_1}(A_1)$ onto the Waldhausen subcategory $\mathrm{Ch}^{\mathrm{hb},-}(A_2\text{-}G_2\text{-proj})$ of $\mathrm{Ch}^{G_2}(A_2)$. Furthermore, it follows from Proposition 4.6 and Lemma 9.6 that the objects of $\mathrm{Ch}^{\mathrm{hb},-}(A_1\text{-}G_1\text{-proj})$ and $\mathrm{Ch}^{\mathrm{hb},-}(A_2\text{-}G_2\text{-proj})$ are cofibrant objects for the projective model structure on the chain complexes. We can therefore apply Theorem 9.2 and (9.4) to conclude that there is a homotopy equivalence of spectra $K'_{G_1}(A_1)$ and $K'_{G_2}(A_2)$. This finishes the proof.    $\square$

**Remark 9.9.** If $G$ is a finite constant group scheme whose order is invertible in the base ring $R$, then one can check that the analogue of Theorem 9.8 is a direct consequence of Remark 4.7 and the main results of [Dugger and Shipley 2004].

## Appendix: Ringoid version of Rickard's theorem

In the proof of Theorem 9.8, we used the following ringoid (see Section 4) version of a theorem of Rickard [1989, Proposition 8.1] for rings. We shall say that a ringoid $\mathcal{R}$ is (right) coherent if every submodule of a finitely generated (right) $\mathcal{R}$-module is finitely generated. We say that $\mathcal{R}$ is complete if every $\mathcal{R}$-module is a filtered direct limit of its finitely generated submodules. We shall assume in our discussion that the ringoids are complete and right coherent. Given a ringoid $\mathcal{R}$, we have the following categories: Mod-$\mathcal{R}$ is the category of $\mathcal{R}$-modules; mod-$\mathcal{R}$ is the category of finitely generated $\mathcal{R}$-modules; Free-$\mathcal{R}$ (resp. free-$\mathcal{R}$) is the category of free (resp. finitely generated free) $\mathcal{R}$-modules; Proj-$\mathcal{R}$ (resp. proj-$\mathcal{R}$) is the category of projective (resp. finitely generated projective) $\mathcal{R}$-modules. Let $\mathrm{Ch}(-)$ denote the category of chain complexes and $D(-)$ denote the derived category of unbounded chain complexes. The superscripts $-$, b and hb denote the full subcategories of bounded above, bounded and cohomologically bounded chain complexes, respectively. $D(\mathrm{Mod}\text{-}\mathcal{R})$ is denoted by $D(\mathcal{R})$.

Since every bounded above complex of finitely generated projective $\mathcal{R}$-modules has a resolution by a bounded above complex of finitely generated free modules, we see that there are equivalences of subcategories $D^-(\mathrm{free}\text{-}\mathcal{R}) \simeq D^-(\mathrm{proj}\text{-}\mathcal{R})$ and $D^{\mathrm{b}}(\mathrm{mod}\text{-}\mathcal{R}) \simeq D^{\mathrm{hb},-}(\mathrm{proj}\text{-}\mathcal{R})$. We shall say that two ringoids $\mathcal{R}$ and $\mathcal{S}$ are derived equivalent if there is an equivalence $D(\mathcal{R}) \simeq D(\mathcal{S})$ of triangulated categories. We shall say that a set $\mathbb{T}$ of objects in $D^{\mathrm{b}}(\mathrm{proj}\text{-}\mathcal{R})$ is a set of *tiltors* if it generates $D(\mathcal{R})$ and $\mathrm{Hom}_{D(\mathcal{R})}(T, T'[n]) = 0$ unless $n = 0$ for any $T, T' \in \mathbb{T}$.

**Proposition A.1.** *Let $\mathcal{R}$ and $\mathcal{S}$ be ringoids which are derived equivalent. Then $D^{\mathrm{hb},-}(\mathrm{proj}\text{-}\mathcal{R})$ and $D^{\mathrm{hb},-}(\mathrm{proj}\text{-}\mathcal{S})$ are equivalent as triangulated categories.*

*Proof.* Any equivalence of triangulated categories $D(\mathcal{R})$ and $D(\mathcal{S})$ induces an equivalence of its compact objects and hence induces an equivalence between $D^{\mathrm{hb}}(\mathrm{Mod}\text{-}\mathcal{R})$ and $D^{\mathrm{hb}}(\mathrm{Mod}\text{-}\mathcal{S})$, because an object $X$ of $D(\mathcal{R})$ is in $D^{\mathrm{hb}}(\mathrm{Mod}\text{-}\mathcal{R})$ if and only if, for every compact object $A$, one has $\mathrm{Hom}_{D(\mathcal{R})}(A, X[n]) = 0$ for all but

finitely many $n$. Since $D^{hb,-}(\text{proj-}\mathcal{R}) = D^-(\text{proj-}\mathcal{R}) \cap D^{hb}(\text{Mod-}\mathcal{R})$, the proposition is about showing that the triangulated categories $D^-(\text{proj-}\mathcal{R})$ and $D^-(\text{proj-}\mathcal{S})$ are equivalent.

This result was proven by Rickard [1989, Proposition 8.1] when $\mathcal{R}$ and $\mathcal{S}$ are both rings. We only explain here how Rickard's proof goes through even for ringoids without further changes. The completeness assumption and our hypotheses together imply that the triangulated categories $D^-(\text{Proj-}\mathcal{R})$ and $D^-(\text{Proj-}\mathcal{S})$ are equivalent. It follows from [Dugger and Shipley 2004, Theorem 7.5] that this induces an equivalence of the triangulated subcategories $D^b(\text{proj-}\mathcal{R})$ and $D^b(\text{proj-}\mathcal{S})$. Let $S$ denote the set of images of the objects of $\mathcal{S}$ (the representable objects of $\mathcal{S}$-Mod) under this equivalence and let $\mathcal{T} := \text{End}(S)$ denote the full subcategory of $D^b(\text{proj-}\mathcal{R})$ consisting of objects in $S$. One easily checks that $S$ is a set of tiltors such that $\text{End}(S) \simeq \mathcal{S}$ as ringoids (see [Dugger and Shipley 2004, Theorem 7.5]).

Rickard constructs (in the case of rings) a functor $F : D^-(\text{Proj-}\mathcal{T}) \to D^-(\text{Proj-}\mathcal{R})$ of triangulated categories which is an equivalence and shows that it induces an equivalence between $D^-(\text{proj-}\mathcal{T})$ and $D^-(\text{proj-}\mathcal{R})$. We recall his construction, which works for ringoids as well. The functor $\text{Hom}_{D(\mathcal{R})}(\mathcal{T}, -)$ from $D^-(\text{Proj-}\mathcal{R})$ to $\mathcal{T}$-Mod induces an equivalence between the direct sums of objects of $\mathcal{T}$ and free objects of $\mathcal{T}$-Mod. Moreover, the completeness assumption on $\mathcal{S}$ implies that the inclusion $\text{Ch}^-(\text{Free-}\mathcal{T}) \to \text{Ch}^-(\text{Proj-}\mathcal{T})$ induces an equivalence of their homotopy categories. One is thus reduced to constructing a functor from the category $D^-(\text{Free-}\mathcal{T})$ of bounded above chain complexes of direct sums of copies of objects in $S$ to $D^-(\text{Proj-}\mathcal{R})$ with the requisite properties.

An object $X$ of $D^-(\text{Free-}\mathcal{T})$ consists of a bigraded object $X = (X^{**}, d, \delta)$ of projective $\mathcal{R}$-modules such that each row is a chain complex of objects which are direct sums of objects in $S$ but the columns are not necessarily chain complexes. The goal is then to modify the differentials of $X^{**}$ so that it becomes a double complex and then one defines $F(X)$ to be the total complex of $X^{**}$, which is an object of $D^-(\text{Proj-}\mathcal{R})$.

In order to modify the differentials of $X^{**}$, Rickard uses his Lemma 2.3, whose proof works in the ringoid case if we know that $\text{Hom}_{D(\mathcal{R})}(T, T'[n]) = 0$ unless $n = 0$ for any $T, T' \in S$. But this is true in our case as $S$ is a set of tiltors. The rest of [Rickard 1989, §2] shows how one can indeed modify $X^{**}$ to get a double complex under this assumption. The point of the other sections is to show how this yields an equivalence of triangulated categories, which only uses the requirement that $S$ is a set of tiltors and, in particular, it generates $D^b(proj-\mathcal{R})$ and hence $D(\mathcal{R})$.

Finally, the functor $F$ will take $D^-(\text{proj-}\mathcal{T})$ to $D^-(\text{proj-}\mathcal{R})$ if $F(\text{Tot}(X^{**}))$ is a bounded above complex of finitely generated projective $\mathcal{R}$-modules whenever each row of $X^{**}$ is a finite direct sum of objects in $S$. But this is obvious because each object of $S$ is a bounded complex of finitely generated projective $\mathcal{R}$-modules. $\quad\square$

## Acknowledgements

Krishna would like to thank the Mathematics Department of KAIST, Korea for invitation and support, where part of this work was carried out. The authors would like to thank the referees for carefully reading the paper and suggesting many improvements.

## References

[Bass 1973] H. Bass, "Some problems in "classical" algebraic *K*-theory", pp. 3–73 in *Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic* (Seattle, WA, 1972)), edited by H. Bass, Lecture Notes in Math. **342**, Springer, Berlin, 1973. MR Zbl

[Bass and Haboush 1985] H. Bass and W. Haboush, "Linearizing certain reductive group actions", *Trans. Amer. Math. Soc.* **292**:2 (1985), 463–482. MR Zbl

[Blumberg and Mandell 2011] A. J. Blumberg and M. A. Mandell, "Algebraic *K*-theory and abstract homotopy theory", *Adv. Math.* **226**:4 (2011), 3760–3812. MR Zbl

[Bruns and Gubeladze 2009] W. Bruns and J. Gubeladze, *Polytopes, rings, and K-theory*, Springer, Dordrecht, 2009. MR Zbl

[Danilov 1978] V. I. Danilov, "The geometry of toric varieties", *Uspekhi Mat. Nauk* **33**:2(200) (1978), 85–134. In Russian; translated in *Russian Math. Surveys*, **33**:97 (1978) 97–154. MR Zbl

[Dugger and Shipley 2004] D. Dugger and B. Shipley, "*K*-theory and derived equivalences", *Duke Math. J.* **124**:3 (2004), 587–617. MR Zbl

[Eisenbud 1995] D. Eisenbud, *Commutative algebra: with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer, New York, 1995. MR Zbl

[Freyd 1964] P. Freyd, *Abelian categories: an introduction to the theory of functors*, Harper & Row, New York, 1964. MR Zbl

[Fulton 1993] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies **131**, Princeton Univ. Press, 1993. MR Zbl

[Gubeladze 1988] I. D. Gubeladze, "The Anderson conjecture and a maximal class of monoids over which projective modules are free", *Mat. Sb.* (*N.S.*) **135(177)**:2 (1988), 169–185. In Russian; translated in *Math. USSR-Sb.* **63**:1 (1989) 165–180. MR Zbl

[Hartshorne 1977] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977. MR Zbl

[Heller et al. 2015] J. Heller, A. Krishna, and P. A. Østvær, "Motivic homotopy theory of group scheme actions", *J. Topol.* **8**:4 (2015), 1202–1236. MR Zbl

[Hovey 1999] M. Hovey, *Model categories*, Mathematical Surveys and Monographs **63**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[Hovey 2007] M. Hovey, "Cotorsion pairs and model categories", pp. 277–296 in *Interactions between homotopy theory and algebra* (Chicago, IL, 2004), edited by L. L. Avramov et al., Contemp. Math. **436**, Amer. Math. Soc., Providence, RI, 2007. MR Zbl

[Kaplansky 1958] I. Kaplansky, "Projective modules", *Ann. of Math* (2) **68** (1958), 372–377. MR Zbl

[Keller 1994] B. Keller, "Deriving DG categories", *Ann. Sci. École Norm. Sup.* (4) **27**:1 (1994), 63–102. MR Zbl

[Knop 1991] F. Knop, "Nichtlinearisierbare Operationen halbeinfacher Gruppen auf affinen Räumen", *Invent. Math.* **105**:1 (1991), 217–220. MR Zbl

[Kraft and Schwarz 1992] H. Kraft and G. W. Schwarz, "Reductive group actions with one-dimensional quotient", *Inst. Hautes Études Sci. Publ. Math.* **76** (1992), 1–97. MR Zbl

[Kraft and Schwarz 1995] H. Kraft and G. Schwarz, "Finite automorphisms of affine *N*-space", pp. 55–66 in *Automorphisms of affine spaces* (Curaçao, 1994), edited by A. van den Essen, Kluwer, Dordrecht, 1995. MR Zbl

[Laumon and Moret-Bailly 2000] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik (3) **39**, Springer, Berlin, 2000. MR Zbl

[Lindel 1981] H. Lindel, "On the Bass–Quillen conjecture concerning projective modules over polynomial rings", *Invent. Math.* **65**:2 (1981), 319–323. MR Zbl

[Masuda 1998] M. Masuda, "Equivariant algebraic vector bundles over affine toric varieties", pp. 76–84 in *Transformation group theory* (Taejŏn, 1996), edited by D. Y. Suh, Korea Advanced Institute of Science and Technology, Taejŏn, 1998. MR

[Masuda and Petrie 1995] M. Masuda and T. Petrie, "Stably trivial equivariant algebraic vector bundles", *J. Amer. Math. Soc.* **8**:3 (1995), 687–714. MR Zbl

[Masuda et al. 1996] M. Masuda, L. Moser-Jauslin, and T. Petrie, "The equivariant Serre problem for abelian groups", *Topology* **35**:2 (1996), 329–334. MR Zbl

[May 1996] J. P. May, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics **91**, Amer. Math. Soc., Providence, RI, 1996. MR Zbl

[Mitchell 1972] B. Mitchell, "Rings with several objects", *Advances in Math.* **8** (1972), 1–161. MR Zbl

[Morel 2012] F. Morel, $\mathbb{A}^1$*-algebraic topology over a field*, Lecture Notes in Mathematics **2052**, Springer, Heidelberg, 2012. MR Zbl

[Mumford et al. 1994] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik (2) **34**, Springer, Berlin, 1994. MR Zbl

[Neeman 1996] A. Neeman, "The Grothendieck duality theorem via Bousfield's techniques and Brown representability", *J. Amer. Math. Soc.* **9**:1 (1996), 205–236. MR Zbl

[Quillen 1976] D. Quillen, "Projective modules over polynomial rings", *Invent. Math.* **36** (1976), 167–171. MR Zbl

[Rao 1988] R. A. Rao, "The Bass–Quillen conjecture in dimension three but characteristic $\neq 2, 3$ via a question of A. Suslin", *Invent. Math.* **93**:3 (1988), 609–618. MR Zbl

[Rickard 1989] J. Rickard, "Morita theory for derived categories", *J. London Math. Soc.* (2) **39**:3 (1989), 436–456. MR Zbl

[SGA 3$_\text{I}$ 1970] M. Demazure and A. Grothendieck, *Schémas en groupes, Tome I: Propriétés générales des schémas en groupes, Exposés I–VII* (Séminaire de Géométrie Algébrique du Bois Marie 1962–1964), Lecture Notes in Math. **151**, Springer, Berlin, 1970. MR Zbl

[SGA 3$_\text{II}$ 1970] M. Demazure and A. Grothendieck, *Schémas en groupes, Tome II: Groupes de type multiplicatif, et structure des schémas en groupes généraux, Exposés VIII–XVIII* (Séminaire de Géométrie Algébrique du Bois Marie 1962–1964), Lecture Notes in Math. **152**, Springer, Berlin, 1970. MR Zbl

[SGA 3$_\text{III}$ 1970] M. Demazure and A. Grothendieck, *Schémas en groupes, Tome III: Structure des schémas en groupes réductifs, Exposés XIX–XXVI* (Séminaire de Géométrie Algébrique du Bois Marie 1962–1964), Lecture Notes in Math. **153**, Springer, Berlin, 1970. MR Zbl

[Swan 1992] R. G. Swan, "Gubeladze's proof of Anderson's conjecture", pp. 215–250 in *Azumaya algebras, actions, and modules* (Bloomington, IN, 1990), edited by D. Haile and J. Osterburg, Contemp. Math. **124**, Amer. Math. Soc., Providence, RI, 1992. MR Zbl

[Swan 1998] R. G. Swan, "Néron–Popescu desingularization", pp. 135–192 in *Algebra and geometry* (Taipei, 1995), edited by M.-C. Kang, Lect. Algebra Geom. **2**, International, Cambridge, MA, 1998. MR Zbl

[Thomason 1987] R. W. Thomason, "Algebraic *K*-theory of group scheme actions", pp. 539–563 in *Algebraic topology and algebraic K-theory* (Princeton, NJ, 1983), edited by W. Browder, Ann. of Math. Stud. **113**, Princeton Univ. Press, 1987. MR Zbl

[Thomason and Trobaugh 1990] R. W. Thomason and T. Trobaugh, "Higher algebraic *K*-theory of schemes and of derived categories", pp. 247–435 in *The Grothendieck Festschrift*, vol. III, edited by P. Cartier et al., Progr. Math. **88**, Birkhäuser, Boston, 1990. MR Zbl

[Vistoli 2005] A. Vistoli, "Grothendieck topologies, fibered categories and descent theory", pp. 1–104 in *Fundamental algebraic geometry: Grothendieck's FGA explained*, Math. Surveys Monogr. **123**, Amer. Math. Soc., Providence, RI, 2005. MR Zbl

[Waterhouse 1979] W. C. Waterhouse, *Introduction to affine group schemes*, Graduate Texts in Mathematics **66**, Springer, New York, 1979. MR Zbl

AMALENDU KRISHNA: amal@math.tifr.res.in
*School of Mathematics, Tata Institute of Fundamental Research, 1 Homi Bhabha Road, Colaba, Mumbai 400005, India*

CHARANYA RAVI: charanya@math.tifr.res.in
*School of Mathematics, Tata Institute of Fundamental Research, 1 Homi Bhabha Road, Colaba, Mumbai 400005, India*

# ANNALS OF K-THEORY

msp.org/akt

PUBLISHED BY

**mathematical sciences publishers**
**nonprofit scientific publishing**

http://msp.org/

# ANNALS OF K-THEORY