Symplectic structure perturbations and continuity of symplectic invariants

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This paper studies how symplectic invariants created from Hamiltonian Floer theory change under the perturbations of symplectic structures, *not necessarily in the same cohomology class*. These symplectic invariants include spectral invariants, boundary depth, and (partial) symplectic quasistates. This paper can split into two parts. In the first part, we prove some energy estimations which control the shifts of symplectic action functionals. These directly imply positive conclusions on the continuity of spectral invariants and boundary depth in some important cases, including any symplectic surface $\Sigma_{g \geq 1}$ and any closed symplectic manifold M with $\dim_{\mathcal{K}} H^2(M; \mathcal{K}) = 1$. This follows by applications on some rigidity of the subsets of a symplectic manifold in terms of heaviness and superheaviness, as well as on the continuity property of some symplectic capacities. In the second part, we generalize the construction in the first part to any closed symplectic manifold. In particular, to deal with the change of Novikov rings from symplectic structure perturbations, we construct a family of variant Floer chain complexes over a common Novikov-type ring. In this setup, we define a new family of spectral invariants called t–spectral invariants, and prove that they are upper semicontinuous under the symplectic structure perturbations. This implies a quasi-isometric embedding from $(\mathbb{R}^\infty, |-|_\infty)$ to $(\widetilde{\mathrm{Ham}}(M, \omega), d_H)$ under some dynamical assumption, imitating the main result of Usher (Ann. Sci. Éc. Norm. Supér. 46 (2013) 57–128).

1 Introduction

In Floer's method [7] of solving Arnold's conjecture (see also Hofer and Salamon [9]) Floer chain complexes were constructed. Let (M, ω) be a symplectic manifold. A Floer chain complex symbolically depends on three parameters: an almost complex structure J on TM, a Hamiltonian function $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M)$ and a symplectic structure ω on M. Conventionally, a Floer chain complex is denoted by $(\mathrm{CF}_*(M, J, H, \omega), \partial_{J,H,\omega})$, where $\partial_{J,H,\omega}$ is the differential. With the help of the symplectic action functional, a Floer chain complex can be viewed as a filtered chain

complex $(\mathrm{CF}_*(M, J, H, \omega), \partial_{J,H,\omega}, \ell_{\omega,H})$, where $\ell_{\omega,H}$ is a filtration function with its values in $\mathbb{R} \cup \{-\infty\}$.

A filtration function provides a "height" for each element in $\mathrm{CF}_*(M, J, H, \omega)$, and it satisfies the non-Archimedean triangle inequality. From this filtered viewpoint, some symplectic invariants were invented from a Floer chain complex or its homology, such as spectral invariants $\rho(a, H; \omega)$ (see Viterbo [30], Schwarz [22], Oh [14] and Usher [23]), boundary depth $\beta(\phi; \omega)$ (see Usher [27] and Usher and Zhang [28]) and (partial) symplectic quasistates $\zeta_a(H; \omega)$ (see Entov and Polterovich [5; 6]). They have played important roles in the study of Hamiltonian dynamics as well as some rigidity properties of the subsets of a symplectic manifold.

It is a natural question how these symplectic invariants change if the three parameters of a Floer chain complex are perturbed. It is well known that the perturbations of almost complex structures do not affect the values of these invariants. Meanwhile, all these three invariants admit a Lipschitz continuity under the perturbations of Hamiltonian functions. The main results of this paper focus on the change of these symplectic invariants under the perturbations of symplectic structures. In this paper, our manifold M is always assumed to be closed.

Denote by (M, ω, H) a Hamiltonian system where ω is a symplectic structure on the manifold M and H is a *nondegenerate* Hamiltonian function on (M, ω). Denote by $\Omega^2_{\mathrm{closed}}(M)$ the set of all closed 2–forms of the manifold M. We call a closed 2–form ω' *a perturbation of* ω if ω' is symplectic. We do not require that ω' and ω be in the same cohomology class. Let ω' be a perturbation of ω. We call ω' *sufficiently close to* ω if $\omega' - \omega \in \Omega^2_{\mathrm{closed}}(M)$ is sufficiently small under a certain norm (see Section 3).

Before we state the main results, it is necessary to point out that some perturbations of a symplectic structure may change the Novikov field associated to (M, ω). Denote by $H_2^S(M)$ the image of $\pi_2(M)$ in $H_2(M; \mathbb{Z})/\mathrm{Tor}$ under the Hurewicz map $\iota \colon \pi_2(M) \to H_2(M; \mathbb{Z})$. Let $\mathcal{K}$ be a fixed field. Recall that an often-used version of Novikov field $\Lambda^{\mathcal{K}, \Gamma_\omega}$ is defined by

$$(1) \qquad \Lambda^{\mathcal{K}, \Gamma_\omega} := \left\{ \sum_{\lambda \in \Gamma_\omega} a_\lambda T^\lambda \ \middle|\ a_\lambda \in \mathcal{K} \text{ and } (\forall C \in \mathbb{R})(\#\{a_\lambda \neq 0 \mid \lambda \leq C\} < \infty) \right\},$$

where $\Gamma_\omega = \{\mathrm{Im}[\omega] \colon H_2^S(M) \to \mathbb{R}\} \leq \mathbb{R}$. Since $\Lambda^{\mathcal{K}, \Gamma_\omega}$ is the coefficient field of the Floer chain complex $(\mathrm{CF}_*(M, J, H, \omega), \partial_{J,H,\omega})$, a negative outcome is that, under a perturbed symplectic structure, the Floer chain complex may be defined over a different

coefficient field. This makes the comparison between the symplectic invariants from this type of perturbation essentially harder than those from the perturbations of almost complex structures and of Hamiltonian functions. For instance, comparing $\rho(a, H; \omega)$ with $\rho(a, H; \omega')$ is in general ambiguous because where the class a is taken is in question. In this paper, we will provide an approach to overcome this difficulty (see Section 7). Finally, note that if our symplectic manifold (M, ω) is aspherical, that is $\pi_2(M) = 0$, then $\Lambda^{\mathcal{K}, \Gamma_\omega} = \mathcal{K}$, which is independent of the symplectic structures. Therefore, depending on whether $\Lambda^{\mathcal{K}, \Gamma_\omega}$ changes or not, we will state our main results in these two different situations.

1.1 Continuity on aspherical manifolds

Here is the first main result in our paper. Recall that $\mathrm{QH}_*(M, \omega)$ denotes the quantum homology of (M, ω), that is $\mathrm{QH}_*(M, \omega) = H_*(M; \mathcal{K}) \otimes_{\mathcal{K}} \Lambda^{\mathcal{K}, \Gamma_\omega}$. In particular, if M is aspherical, then $\mathrm{QH}_*(M, \omega) = H_*(M; \mathcal{K})$.

Theorem 1.1 *Let (M, ω, H) be a Hamiltonian system where M is aspherical. If ω' is a perturbation of ω and is sufficiently close to ω, then there exists a constant C such that*

    (i) $|\rho(a, H; \omega) - \rho(a, H; \omega')| \leq C |\omega - \omega'|$ *for any $a \in H_*(M; \mathcal{K})$;*

    (ii) $|\beta(H; \omega) - \beta(H; \omega')| \leq C |\omega - \omega'|$.

A standard example that satisfies the assumption in Theorem 1.1 is a symplectic surface (Σ_g, ω) with genus $g \geq 1$. Next, from a "rescaling" argument, a direct corollary of Theorem 1.1 is the following result under a slightly relaxed hypothesis on M.

Corollary 1.2 *Let (M, ω, H) be a Hamiltonian system where M satisfies the condition $\dim_{\mathcal{K}} H^2(M; \mathcal{K}) = 1$. Denote by $[M]$ the fundamental class of M. If ω' is a perturbation of ω and is sufficiently close to ω, then there exist constants $\widetilde{C}_1$ and $\widetilde{C}_2$ such that*

    (i) $|\rho([M], H; \omega) - \rho([M], H; \omega')| \leq \widetilde{C}_1 |\omega - \omega'|$;

    (ii) $|\beta(H; \omega) - \beta(H; \omega')| \leq \widetilde{C}_2 |\omega - \omega'|$.

Note that Corollary 1.2 covers some important cases, for instance $\mathbb{C}P^n$ for any $n \in \mathbb{N}$. Roughly speaking, the proof of Theorem 1.1 splits into two steps. First, as elaborated in Section 3, if ω' is a perturbation of ω, then there exists a diffeomorphism on M which "reduces" ω' to $\phi^* \omega'$ in the sense that $\phi^* \omega'$ coincides with ω near any Hamiltonian

1–periodic orbit of the Hamiltonian system (M, H, ω) (see Definition 3.1). This considerably simplifies our discussion. Second, the desired estimation in Theorem 1.1 comes from a Floer-type argument, which is based on some new energy estimations established in Section 4. In this second step, ω' is required to be sufficiently close to ω.

As applications of Theorem 1.1, we can reprove results on some rigidity properties of the subsets of a symplectic surface Σ_g with genus $g \geq 1$. To this end, we need the following definition from Kawasaki [12, Definition 2.9]:

Definition 1.3   For a fixed symplectic manifold (M, ω) and an element $a \in \mathrm{QH}_*(M, \omega)$, a subset $U \subset M$ satisfies the *bounded spectrum condition with respect to a* if there exists a constant $K > 0$ such that $\rho(a, H; \omega) \leq K$ for any Hamiltonian function H supported in $\mathbb{R}/\mathbb{Z} \times U$.

We can prove the following result:

Theorem 1.4   *Let (Σ_g, ω) be a symplectic surface with genus $g \geq 1$. Then the disjoint union of simply connected open subsets satisfies the bounded spectrum condition for any $a \in H_*(\Sigma_g; \mathcal{K})$.*

Recall that the concepts of heaviness and superheaviness of the subsets of a symplectic manifold are introduced in [6] in order to study in a systematical way the nondisplaceability properties (Definition 2.11). The following result is a corollary of Theorem 1.4:

Corollary 1.5   *Let (Σ_g, ω) be a symplectic surface with genus $g \geq 1$.*

   (a) *For a closed subset $X \subset \Sigma_g$, if $\Sigma_g \setminus X$ is a disjoint union of simply connected open subsets, then it is a–superheavy for any $a \in H_*(\Sigma_g, \omega)$.*

   (b) *If a closed subset $X \subset \Sigma_g$ is contained in a disk, then X is not a–heavy for any $a \in H_*(\Sigma_g; \mathcal{K})$.*

Notice that (a) includes the embedding of a wedge of circles $\bigvee_{i=1}^{2g} S^1 \hookrightarrow \Sigma_{g \geq 1}$. This is studied in Ishikawa [11, Example 4.8], which generalizes the main result from [12]. Also, (b) provides a topological obstruction for a closed subset to be a–heavy. When $a = [M]$, a necessary and sufficient condition for a subset to be a–heavy or a–superheavy on a symplectic surface is given in Humilière, Le Roux and Seyfaddini [10].

Another application of Theorem 1.1 and Corollary 1.2 is on the continuity of some symplectic capacities under the perturbations of symplectic structures. Let A be a subset of M. Here we mainly focus on the displacement energy $e^\omega(A)$, Hofer–Zehnder capacity $c_{\mathrm{HZ}}^\omega(A)$ and the spectral capacity $c_\rho^\omega(A)$. Recall that the displacement energy is defined by $e^\omega(A) := \inf\{\|H\|_H \mid \phi_H^1(A) \cap A = \varnothing\}$ and $\|-\|_H$ is the Hofer norm. The capacities $c_{\mathrm{HZ}}^\omega(A)$ and $c_\rho^\omega(A)$ are defined in Definitions 6.3 and 6.4, respectively. Here we put the symplectic structure ω in all the notation to emphasize the dependence on the symplectic structures. It is well known, as the *energy–capacity inequality* (see Frauenfelder, Ginzburg and Schlenk [8, Theorem 1]), that

$$(2) \qquad\qquad c_{\mathrm{HZ}}^\omega(A) \leq c_\rho^\omega(A) \leq e^\omega(A).$$

In general, we should not expect any continuity of $e^\omega(A)$ under the perturbations of symplectic structures, due to the following easy example:

Example 1.6 Take $(M = S^2, \omega_0 = \omega_{\mathrm{std}})$ and let A be the *open* upper hemisphere. Then the subset A is displaceable under ω_0 and $e^{\omega_0}(A) = 2\pi$. Consider a sequence of symplectic structures $\omega_n \to \omega_0$ with $\omega_n = \omega_0 + \alpha_n$, where α_n is *positively* supported over a nonempty open subset of A and *negatively* supported over a nonempty open subset of $M \setminus \bar{A}$ such that the total area of M remains the same for each ω_n. Then A is not displaceable under ω_n by an area consideration. By definition, $e^{\omega_n}(A) = \infty$, which implies that $e^\omega(A)$ is *not* upper semicontinuous at ω_0.

However, we have the following "boundedness" conclusion:

Theorem 1.7 Let (M, ω) be a symplectic manifold and $A \subset M$ be a subset. If M is aspherical or $\dim_{\mathcal{K}} H^2(M; \mathcal{K}) = 1$, then for any $\epsilon > 0$, there exists a neighborhood $U_\omega(\epsilon)$ of ω in $\Omega^2_{\mathrm{closed}}(M)$ such that $e^{\omega'}(A) \geq c_\rho^\omega(A) - \epsilon \geq c_{\mathrm{HZ}}^\omega(A) - \epsilon$ for any $\omega' \in U_\omega(\epsilon)$.

1.2 Semicontinuity on general manifolds

For general symplectic manifolds, we will deal with the coefficient-change problem of Floer chain complexes in the following way. First of all, due to Moser's trick, it is easy to see how the symplectic invariants change when the perturbation is formed from an *exact* 2–form. Then the discussion on perturbations can be simplified to be in $H^2(M; \mathcal{K})$. Since $H^2(M; \mathcal{K})$ is a finite-dimensional vector space, for a fixed symplectic structure ω there exist $[\omega_1], \ldots, [\omega_m]$ as vertices forming a polygon $\Delta(\omega)$

in $H^2(M;\mathcal{K})$ such that each ω_i is a perturbation of ω and $[\omega] \in \Delta^{\circ}(\omega)$, the interior of $\Delta(\omega)$. Then, if a perturbation ω' is sufficiently close to ω,

$$(3) \qquad\qquad [\omega'] = t_0[\omega] + t_1[\omega_1] + \cdots + t_m[\omega_m]$$

for some nonnegative $t_0, \ldots, t_m$ where $\sum_{i=0}^{m} t_i = 1$. To simplify our discussion further, due to Proposition 3.2, all the perturbations of ω here are assumed to coincide with ω near any Hamiltonian 1–periodic orbit of the Hamiltonian system (M, ω, H) (see Definition 3.1). For any Hamiltonian system (M, ω, H), we give the following definition:

Definition 1.8 Define a *Novikov ring with multifiniteness condition* as

$$\Lambda_{\Delta(\omega)} = \left\{ \sum_{A \in H_2^S(M)} a_A T^A \,\middle|\, a_A \in \mathcal{K} \text{ and} \atop (\forall C \in \mathbb{R})(\forall \omega' \in \Delta(\omega))(\#\{a_A \neq 0 \,|\, [\omega'](A) \leq C\} < \infty) \right\}.$$

Proposition 7.6 implies that there exists a well-defined family of (filtered) Floer chain complexes parametrized by $\Delta(\omega)$ over this common coefficient ring $\Lambda_{\Delta(\omega)}$. This family of filtered complexes is denoted by

$$(4) \qquad\qquad \{(\mathrm{CF}_{\Delta(\omega)}(M, H), \partial_{\omega'}, \ell_{\omega'})\}_{\omega' \in \Delta(\omega)}.$$

The construction of (4) takes its inspiration from Ono [16]. Moreover, for every two parameters $\omega_1, \omega_2 \in \Delta(\omega)$, Proposition 7.7 says that their corresponding filtered Floer chain complexes from (4) are homotopy equivalent to each other. Therefore, there exists a well-defined homology $\mathrm{HF}_{\Delta(\omega)}(M, H)$ over $\Lambda_{\Delta(\omega)}$. In other words, we manage to assemble the information of Floer chain complexes or their homologies under various perturbations of ω into a family of filtered chain complexes or a single homology, but over a complicated coefficient ring. It can be shown that $\mathrm{HF}_{\Delta(\omega)}(M, H) \simeq H_*(M;\mathcal{K}) \otimes_{\mathcal{K}} \Lambda_{\Delta\omega} := \mathrm{QH}_{\Delta(\omega)}(M)$. If we choose a class $a \in \mathrm{QH}_{\Delta(\omega)}(M)$, then there will be no ambiguity when we compare spectral invariants under different symplectic structures. In fact, from $\mathrm{HF}_{\Delta(\omega)}(M, H)$, we can define a $\Delta(\omega)$–family of spectral invariants, denoted by $\rho_{\omega'}(a, H)$ for any $a \in \mathrm{QH}_{\Delta(\omega)}(M)$ (see Definition 8.1). Our next main result is on the continuity of this new spectral invariants:

Theorem 1.9 Let (M, ω, H) be a Hamiltonian system, and $a \in \mathrm{QH}_{\Delta(\omega)}(M)$ be a fixed class. Then the map from $\Delta(\omega)$ to $\mathbb{R}$ by $\omega' \to \rho_{\omega'}(a, H)$ is upper semicontinuous at ω.

Meanwhile, Lemma 8.2 compares the standard spectral invariants $\rho(a, H; \omega)$ with $\rho_\omega(a, H)$, and they turn out to be the same. Therefore, we directly get the following corollary on the continuity of the standard spectral invariants:

Corollary 1.10 *Let* (M, ω, H) *be a Hamiltonian system, and* $a \in \mathrm{QH}_{\Delta(\omega)}(M)$ *be a fixed class. Then the map from* $\Delta(\omega)$ *to* $\mathbb{R}$ *by* $\omega' \to \rho(a, H; \omega')$ *is upper semicontinuous at* ω.

Remark 1.11 There is an obvious question on the lower semicontinuity of the spectral invariants under the perturbations of symplectic structures. Unfortunately, the method we provide here can not conclude any positive or negative conclusion on the lower semicontinuity (see Remark 8.5).

Remark 1.12 Recall that boundary depth is defined from the standard Floer chain complex. In the general setup, it is difficult to conclude any quantitative conclusion between two standard Floer chain complexes that are defined from different symplectic structures, so we can appeal to the variant Floer chain complex (4) when comparing two boundary depths. For instance, one can define a $\Delta(\omega)$–parametrized boundary depth from (4). However, our method and estimations are unable to give any positive or negative conclusion on its continuity.

Next, we want to say a few words on (partial) symplectic quasistates. The method we used in this paper can't apply to study the continuity of (partial) symplectic quasistates. The main reason is that in our discussion the Hamiltonian function H should be fixed whenever a symplectic structure is perturbed. In general, one should not expect any continuity result for (partial) symplectic quasistates. We illustrate this by the following easy example, similarly to Example 1.6:

Example 1.13 Take $(M = S^2, \omega_0 = \omega_{\mathrm{std}})$ centered at the origin in xyz–coordinates. The standard equation $L = \{z = 0\}$ is a heavy subset and then it is not displaceable by any Hamiltonian diffeomorphism. For any given $\epsilon > 0$, let $S_\epsilon := \{(x, y, z) \in S^2 \mid z \geq -\epsilon\}$ and H be a time-independent Hamiltonian function supported on S_ϵ such that $H|_L = 1$ and $\omega_0(\mathrm{supp}(H)) = \frac{1}{2}\omega_0(S^2)$. Consider a closed 2–form α only *positively* supported on $S^2 \setminus S_\epsilon$. For any class $a \in \mathrm{QH}_*(S^2, \omega_0)$ and any $\delta > 0$, Definition 2.11 of a heavy subset says that

$$\zeta_a(H; \omega_0) \geq \inf_L H = 1.$$

However, $\zeta_a(H; \omega_0 + \delta\alpha) = 0$ because $(\omega_0 + \delta\alpha)(\mathrm{supp}(H)) < \frac{1}{2}(\omega_0 + \delta\alpha)(S^2)$ implies that the support of H is displaceable in $(S^2, \omega_0 + \delta\alpha)$.

Remark 1.14 Our choice of the perturbation in Example 1.13 is quite special. It is easy to see that there are plenty of other perturbations that we can take so that symplectic quasistates are invariant (in particular change continuously). In general, it would be interesting to systematically study in which way we can perturb the symplectic structure so that symplectic quasistates can satisfy a continuity result.

Recall that d_H denotes Hofer's metric on the universal cover of Hamiltonian diffeomorphism group of (M, ω) denoted by $\widetilde{\mathrm{Ham}}(M, \omega)$. As a standard application of Theorem 1.9, we have the following Theorem 1.15, which is similar to [27, Theorem 1.1]. It concludes a large-scale geometric property of the metric space $(\widetilde{\mathrm{Ham}}(M, \omega), d_H)$ when M satisfies a certain dynamical condition. This dynamical condition imitates the assumption in [27, Theorem 1.1].

Theorem 1.15 *Suppose that a manifold M admits a symplectic structure ω satisfying the saa condition* (2) *in Definition 9.1. Then there exists an embedding $\Phi \colon \mathbb{R}^\infty \to \widetilde{\mathrm{Ham}}(M, \omega)$ such that, for every $\vec{v}, \vec{w} \in \mathbb{R}^\infty$, we have*

$$|\vec{v} - \vec{w}|_\infty \leq d_H(\Phi(\vec{v}), \Phi(\vec{w})) \leq \mathrm{osc}(\vec{v} - \vec{w}),$$

where $|\vec{a}|_\infty = \max_i |a_i|$ and $\mathrm{osc}(\vec{a}) = \max_{i,j} |a_i - a_j|$ for the vector $\vec{a} = (a_1, a_2, \dots)$.

Outline of the paper

In Section 2, the background of Hamiltonian Floer theory and several symplectic invariants derived from this theory are briefly reviewed. In Section 3, we explain how a general perturbation of a symplectic structure can be simplified into a reduced perturbation (see Definition 3.1), and this key step is given by Proposition 3.2. Imitating the standard homotopy argument between two Hamiltonian functions in Hamiltonian Floer theory, Section 4 discusses various energy estimations in terms of the perturbations of symplectic structures. The main results in this section are Proposition 4.3 and Proposition 4.6. In Sections 5 and 6, the main results of this paper, Theorems 1.1 and 1.2 as well as Theorems 1.4 and 1.7 on applications, are proved. In particular, the proof of Theorem 1.4 takes inspiration from Ostrover's trick [17]. Section 7 serves as an algebraic preparation for the discussion of any closed symplectic manifold (in the sense that Novikov fields have to be considered). In this section, we define a parametrized Floer chain complex over an extended version of the Novikov ring (see Proposition 7.6). Based on this discussion, another main result of this paper, Theorem 1.9, is proved in Section 8. Finally, Theorem 1.15 on an application of the Hofer geometry on $\widetilde{\mathrm{Ham}}(M, \omega)$ is proved in Section 9.

Acknowledgement

2 Preliminaries

2.1 Floer chain complex

In this subsection, we will briefly review the constriction of a Floer chain complex. In an abstract language, a Floer chain complex is an example of a filtered complex defined as follows. Denote by $\Lambda^{\mathcal{K},\Gamma}$ an abstract Novikov field (over ground field $\mathcal{K}$), that is,

$$\Lambda^{\mathcal{K},\Gamma} := \left\{ \sum_{\lambda \in \Gamma} a_\lambda T^\lambda \;\middle|\; \Gamma \leq \mathbb{R}, \, a_\lambda \in \mathcal{K} \text{ and } (\forall C \in \mathbb{R})(\#\{a_\lambda \neq 0 \mid \lambda \leq C\} < \infty) \right\}.$$

By this finiteness condition on $\Lambda^{\mathcal{K},\Gamma}$, there exists a well-defined valuation $\nu \colon \Lambda^{\mathcal{K},\Gamma} \to \mathbb{R} \cup \{\infty\}$ which simply takes the minimal exponent λ of any element in $\Lambda^{\mathcal{K},\Gamma}$.

Definition 2.1 We call $(C_*, \partial_C, \ell_C)$ a *filtered complex over* $\Lambda^{\mathcal{K},\Gamma}$ if (C_*, ∂_C) is a chain complex and C_k is a finite-dimensional vector space over $\Lambda^{\mathcal{K},\Gamma}$ for each degree $k \in \mathbb{Z}$. Moreover, $\ell_C \colon C_* \to \mathbb{R} \cup \{-\infty\}$ is a function that satisfies (i) the non-Archimedean triangle inequality; (ii) $\ell_C(\partial_C x) < \ell_C(x)$; and (iii) $\ell_C(\lambda x) = \ell_C(x) - \nu(\lambda)$ for any $x \in C$ and $\lambda \in \Lambda^{\mathcal{K},\Gamma}$.

On the level of chain complexes, we can compare two filtered complexes over the same coefficient field $\Lambda^{\mathcal{K},\Gamma}$. The following definition is taken from Definition 1.3 in [28]:

Definition 2.2 Let $(C_*, \partial_C, \ell_C)$ and $(D_*, \partial_D, \ell_D)$ be filtered complexes over $\Lambda^{\mathcal{K},\Gamma}$ and $\delta \geq 0$. A δ–*quasiequivalence* between C_* and D_* is a quadruple (Φ, Ψ, K_1, K_2) where:

(1) $\Phi \colon C_* \to D_*$ and $\Psi \colon D_* \to C_*$ are chain maps, with $\ell_D(\Phi c) \leq \ell_C(c) + \delta$ and $\ell_C(\Psi d) \leq \ell_D(d) + \delta$ for all $c \in C_*$ and $d \in D_*$.

(2) The maps $K_1\colon C_* \to C_{*+1}$ and $K_2\colon D_* \to D_{*+1}$ obey the homotopy equations $\Psi \circ \Phi - \mathbb{1}_{C_*} = \partial_C K_1 + K_1 \partial_C$ and $\Phi \circ \Psi - \mathbb{1}_{D_*} = \partial_D K_2 + K_2 \partial_D$, and for all $c \in C_*$ and $d \in D_*$ we have $\ell_C(K_1 c) \le \ell_C(c) + 2\delta$ and $\ell_D(K_2 d) \le \ell_D(d) + 2\delta$.

Thus we say $(C_*, \partial_C, \ell_C)$ and $(D_*, \partial_D, \ell_D)$ are δ–*quasiequivalent* if there exists a δ–quasiequivalence between them.

Example 2.3 Recall that the Hofer norm of any function $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M)$ is defined by $\|H\|_H := \int_0^1 (\max_M H_t - \min_M H_t)\, dt$. Then a Floer chain complex, denoted by $(\mathrm{CF}_*(M, J, H, \omega), \partial_{J,H,\omega}, \ell_{H,\omega})$, is a filtered complex over a Novikov field. Moreover, for two pairs (J_-, H_-) and (J_+, H_+), the Floer chain complexes $(\mathrm{CF}_*(M, J_+, H_+, \omega), \partial_{J_+,H_+,\omega}, \ell_{H_+,\omega})$ and $(\mathrm{CF}_*(M, J_-, H_-, \omega), \partial_{J_-,H_-,\omega}, \ell_{H_-,\omega})$ are $\|H_+ - H_-\|_H$–quasiequivalent.

To justify Example 2.3, we elaborate three objects: the generators of $\mathrm{CF}_*(M, J, H, \omega)$, the construction of $\partial_{J,H,\omega}$ and the constructions of homotopy K_1 and K_2 from Definition 2.2. Given a smooth function $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M)$ on a closed symplectic manifold (M, ω), the Hamiltonian flow ϕ_H^t comes from the differential equation

$$\frac{d\phi_t}{dt} = X_H \circ \phi_t, \quad \text{where } \omega(\,\cdot\,, X_H) = d(H(t, \cdot\,)).$$

The generators of $\mathrm{CF}_*(M, J, H, \omega)$ are Hamiltonian contractible loops $\gamma\colon \mathbb{R}/\mathbb{Z} \to M$, where $\gamma(t) = \phi_H^t(\gamma(0))$. With a nondegeneracy condition, there are only finitely many such generators. Set $\Gamma_\omega = \{\mathrm{Im}[\omega]\colon H_2^S(M) \to \mathbb{R}\} \le \mathbb{R}$, where $H_2^S(M)$ is the image of $\pi_2(M)$ in $H_2(M; \mathbb{Z})/\mathrm{Tor}$ under the Hurewicz map $\iota\colon \pi_2(M) \to H_2(M; \mathbb{Z})$; then

$$\mathrm{CF}_*(M, J, H, \omega) := \mathrm{span}_{\Lambda^{\mathcal{K}, \Gamma_\omega}} \langle \text{Hamiltonian contractible loops} \rangle.$$

The grading of each generator is given by the Conley–Zehnder index $\mu_{\mathrm{CZ},\omega}$. Its explicit definition can be found in [19]. Moreover, since each generator γ is a contractible loop, we can *fix* a disk w spanning γ and assign a value to each pair (γ, w) via the symplectic action functional defined as

$$(5) \qquad \mathcal{A}_{H,\omega}((\gamma, w)) = -\int_{D^2} w^* \omega + \int_0^1 H(t, \gamma(t))\, dt.$$

Then define $\ell_{H,\omega}\colon \mathrm{CF}_*(M, J, H, \omega) \to \mathbb{R} \cup \{-\infty\}$ by

$$(6) \qquad \ell_{H,\omega}\left(\sum_i \lambda_i(\gamma_i, w_i)\right) = \max_i\{\mathcal{A}_{H,\omega}(\gamma_i, w_i) - \nu(\lambda_i)\}.$$

Conventionally, to simplify the discussion we usually replace the pair (γ, w) with an equivalence class $[\gamma, w]$. We choose to define that (γ, w) is equivalent to (τ, v) if and only $\gamma(t) = \tau(t)$ and $[w \# (-v)]$ is homologically trivial. Note that the symplectic action functional $\mathcal{A}_{H,\omega}$ and Conley–Zehnder index $\mu_{CZ,\omega}$ are both well defined over $[\gamma, w]$. Thus, $\ell_{H,\omega}$ is well defined over any linear combination of the equivalence classes.

Remark 2.4 In most literature, the equivalence relation between (γ, w) and (τ, v) is weaker than what is given above. Explicitly, (γ, w) is equivalent to (τ, v) if and only if $\gamma(t) = \tau(t)$ and $[w \# (-v)] \in \ker([\omega]) \cap \ker(c_1)$ where c_1 is the first Chern class of (M, ω). Since later in this paper the perturbations of symplectic structures are considered, we need an equivalence relation that is independent of symplectic structures.

The Floer boundary operator $\partial_{J,H,\omega} \colon \mathrm{CF}_*(M, J, H, \omega) \to \mathrm{CF}_{*-1}(M, J, H, \omega)$ is defined by counting solutions (modulo $\mathbb{R}$–translation) of the partial differential equation

$$(7) \qquad \frac{\partial u}{\partial s} + J_t(u(s,t))\left(\frac{\partial u}{\partial t} - X_H(t, u(s,t))\right) = 0,$$

where $\{J_t\}_{0 \leq t \leq 1}$ is a family of almost complex structures that are compatible with ω and $u(s,t) \colon \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to X$ is such that

- u has finite energy $E(u) = \int_{\mathbb{R} \times \mathbb{R}/\mathbb{Z}} \left|\frac{\partial u}{\partial s}\right|^2 dt\, ds$;
- u has asymptotic condition $u(s, \cdot) \to \gamma_\pm(\cdot)$ as $s \to \pm\infty$;
- $\mu_{CZ}([\gamma_-, w_-]) - \mu_{CZ}([\gamma_+, w_+]) = 1$ and $[\gamma_+, w_+] = [\gamma_+, w_- \# u]$.

The celebrated Gromov compactness theorem guarantees that $\partial_{J,H,\omega}$ is well defined over $\Lambda^{\mathcal{K}, \Gamma_\omega}$. Moreover, the Floer chain complex $(\mathrm{CF}_*(M, J, H, \omega), \partial_{J,H,\omega})$ defines a homology $\mathrm{HF}_*(M, J, H, \omega)$, called Floer homology, which only depends on the manifold M itself, so is denoted as $\mathrm{HF}_*(M)$. Explicitly, up to a degree shift, one gets that $\mathrm{HF}_*(M) \simeq H_*(M, \mathcal{K}) \otimes_{\mathcal{K}} \Lambda^{\mathcal{K}, \Gamma_\omega}$, where the right-hand side is called the quantum homology of M and denoted as $\mathrm{QH}_*(M, \omega)$. A standard way to prove this isomorphism is via the PSS–map, denoted as PSS_*. For its explicit construction, see [18].

Finally, given two different pairs (J_-, H_-) and (J_+, H_+), consider a homotopy $(\mathcal{J}, \mathcal{H})$ where $\mathcal{H} = \mathcal{H}_s = \mathcal{H}(s,t,x) \colon \mathbb{R} \times \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ between $H_-(t,x)$ and $H_+(t,x)$ in the form of

$$\mathcal{H}(s,t,x) = (1 - \alpha(s))H_-(t,x) + \alpha(s)H_+(t,x),$$

where $\alpha(s)$ is a cut-off function, ie $\alpha(s) = 0$ for $s \in (-\infty, 0]$, $\alpha(s) = 1$ for $s \in [1, \infty)$ and $0 \leq \alpha'(s) \leq 1$ for $s \in (0, 1)$; $\mathcal{J} = \mathcal{J}_s$ is a homotopy (compatible with ω) between J_- and J_+ by a cut-off function, too. Then, similarly to the boundary operator $\partial_{J,H,\omega}$, the construction of continuation map Φ is by counting solutions $u(s,t): \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ of a parametrized pseudoholomorphic equation

$$(8) \qquad \frac{\partial u}{\partial s} + \mathcal{J}_s(u(s,t))\left(\frac{\partial u}{\partial t} - X_{\mathcal{H}_s}(t, u(s,t))\right) = 0$$

such that

- u has finite energy $E(u) = \int_{\mathbb{R} \times \mathbb{R}/\mathbb{Z}} \left|\frac{\partial u}{\partial s}\right|^2 dt\, ds$;

- u has asymptotic condition $u(s, \cdot) \to \gamma_\pm(\cdot)$ as $s \to \pm\infty$;

- $\mu_{CZ}([\gamma_-, w_-]) - \mu_{CZ}([\gamma_+, w_+]) = 0$ and $[\gamma_+, w_+] = [\gamma_+, w_- \# u]$.

There are two well-known facts [21]. One is that Φ is a chain map by a gluing argument; the other is that, for any other homotopy $(\mathcal{J}', \mathcal{H}')$, the associated chain map Φ' is chain homotopic to Φ. Here we give some details on the shift of symplectic actions. Since the symplectic structures are the same, for brevity, denote the symplectic action functional by $\mathcal{A}_H$ if the Hamiltonian function is H. The standard computation goes as follows:

$$\mathcal{A}_{H_+}([\gamma_+, w_+]) - \mathcal{A}_{H_-}([\gamma_-, w_-])$$
$$= \int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}_{\mathcal{H}_s}([u(s, \cdot), (w_- \# u)(s, \cdot)])\, ds$$
$$= -E(u) + \int_{-\infty}^{\infty} \int_0^1 \alpha'(s)(H_+ - H_-)(t, u(s,t))\, dt\, ds$$
$$\leq -E(u) + \int_0^1 \max_M (H_+ - H_-)(t, u(s,t))\, dt$$
$$\leq \int_0^1 \max_M (H_+ - H_-)\, dt.$$

To pass from the upper bound $\int_0^1 \max_M (H_+ - H_-)\, dt$ to $\|H_+ - H_-\|_H$, strictly speaking we need to normalize H_+ and H_- so that both have mean values zero over M (and this can be done simply by a constant shift due to our assumption that M is closed). This implies that $\int_0^1 \min_M (H_+ - H_-)\, dt \leq 0$ and then $\int_0^1 \max_M (H_+ - H_-)\, dt \leq \|H_+ - H_-\|_H$.

2.2 Some symplectic invariants

2.2.1 Spectral invariant The filtration $\ell_{H,\omega}$ defined in (6) can be used to define a measurement for elements in $\mathrm{HF}_*(M)$, and the outcomes of this measurement are called *spectral invariants*.

Definition 2.5 For any $a \in \mathrm{QH}_*(M, \omega)$, define its spectral invariant with respect to the Hamiltonian system (M, ω, H) by

$$\rho(a, H; \omega) := \inf\{\ell_{H,\omega}(\alpha) \mid \alpha \in \mathrm{CF}_*(M, J, H, \omega) \text{ with } [\alpha] = \mathrm{PSS}_*(a)\}.$$

Recall that PSS_* is the well-known isomorphism from $\mathrm{QH}_*(M, \omega)$ and $\mathrm{HF}_*(M)$. It is easy to see that spectral invariants are independent of the almost complex structures. Also, they enjoy many useful properties. The following result will be used later, which is Theorem 1.4 in [23]:

Theorem 2.6 *Let (M, ω, H) be a Hamiltonian system. Then, for any $a \in \mathrm{QH}_*(M, \omega)$, there exists some $\alpha \in \mathrm{CF}_*(M, J, H, \omega)$ such that $[\alpha] = \mathrm{PSS}_*(a)$ and $\rho(a, H; \omega) = \ell_{H,\omega}(\alpha)$. In other words, define $\mathrm{Spec}(H, \omega) := \{\ell_{H,\omega}(\alpha) \mid \alpha \in \mathrm{CF}_*(M, J, H, \omega)\}$; then $\rho(a, H; \omega) \in \mathrm{Spec}(H, \omega)$.*

2.2.2 Boundary depth Boundary depth is defined on the chain complex level.

Definition 2.7 For a Floer chain complex $(\mathrm{CF}_*(M, J, H, \omega), \partial_{J,H,\omega}, \ell_{H,\omega})$, define its boundary depth by

$$\beta(H; \omega) := \sup_{x \in \mathrm{Im}\, \partial_{J,H,\omega}} \inf\{\ell_{H,\omega}(y) - \ell_{H,\omega}(x) \mid \partial_{J,H,\omega} y = x\}.$$

More generally, the same formula as above defines the boundary depth β for any filtered complex (C, ∂_C, ℓ_C).

By Remark 3.3 in [25], boundary depth is independent of the almost complex structures. What needs to be emphasized is that though boundary depth is even well defined on $\mathrm{Ham}(M, \omega)$ by Corollary 5.4 in [27], we still use the notation $\beta(H; \omega)$. The main reason is that the process that passes from a Hamiltonian function H to its corresponding Hamiltonian diffeomorphism $\phi = \phi_H^1 \in \mathrm{Ham}(M, \omega)$ depends on the symplectic structure ω. Similarly to spectral invariants, boundary depth also enjoys many useful properties. The following ones will be used later. They are [27, Theorem 7.4] and [27, Proposition 3.8], respectively.

Theorem 2.8 *For any Floer chain complex* $(\mathrm{CF}_*(M, J, H, \omega), \partial_{J,H,\omega}, \ell_{H,\omega})$, *there exists* $y \in \mathrm{CF}_*(M, J, H, \omega)$ *such that* $\beta(H; \omega) = \ell_{H,\omega}(y) - \ell_{H,\omega}(\partial y)$. *In other words,* $\beta(H; \omega) \in \mathrm{Spec}^{\Delta}(H, \omega) := \{s - t \mid s, t \in \mathrm{Spec}(H, \omega)\}$.

Theorem 2.9 *Let* $(C_1, \partial_1, \ell_1)$ *and* $(C_2, \partial_2, \ell_2)$ *be two filtered complexes. Denote their boundary depths by* β_1 *and* β_2, *respectively. If these two filtered chain complexes are* δ–*quasiequivalent, then* $|\beta_1 - \beta_2| \leq \delta$.

2.2.3 (Partial) symplectic quasistate In general, any stable homogenous quasimorphism [4; 5] induces a quasistate, ie a functional $\zeta \colon C^{\infty}(M) \to \mathbb{R}$ satisfying the following properties:

- If $\{F, G\} = 0$, then $\zeta(H + aG) = \zeta(H) + a\zeta(G)$ for any $a \in \mathbb{R}$.
- If $H \leq G$, then $\zeta(H) \leq \zeta(K)$.
- $\zeta(1) = 1$.

In particular, if we use *spectral quasimorphism*, which is constructed from spectral invariants [4], we can get (partial) symplectic quasistates. More directly:

Definition 2.10 For any $a \in \mathrm{QH}_*(M, \omega)$, define a function $\zeta_a(-; \omega) \colon C^{\infty}(M) \to \mathbb{R}$ by

$$\zeta_a(H; \omega) = \lim_{k \to \infty} \frac{\rho(a, kH; \omega)}{k}.$$

This function is called the (partial) symplectic quasistate associated to the class a.

Symplectic quasistates are powerful tools to study the rigidity of intersections of subsets in a symplectic manifold. The closely related concepts are *heavy subset* and *superheavy subset*.

Definition 2.11 For a given $a \in \mathrm{QH}_*(M, \omega)$, we call a closed subset $X \subset M$ a–*heavy* if $\zeta_a(H; \omega) \geq \inf_X H$ for all $H \in C^{\infty}(M)$ and a–*superheavy* if $\zeta_a(H; \omega) \leq \sup_X H$ for all $H \in C^{\infty}(M)$.

To end this section, we emphasize that the quasiequivalence conclusion for Floer chain complexes in Example 2.3 readily implies that all three symplectic invariants introduced above satisfy 1–Lipschitz continuity under the perturbations of Hamiltonian functions. Here we summarize them into the following theorem, and they are from Theorem I(5) in [14], Theorem 1.4(iii) in [27] and Theorem 3.2 in [3], respectively.

Theorem 2.12 *We have the following continuity results, where $\|-\|_H$ is the Hofer norm (see Example 2.3):*

(a) *Given any $a \in \mathrm{QH}_*(M, \omega)$, for any $H, G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M)$, we have*

$$|\rho(a, H; \omega) - \rho(a, G; \omega)| \leq \|H - G\|_H.$$

(b) *For any $H, G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M)$, we have*

$$|\beta(H; \omega) - \beta(G; \omega)| \leq \|H - G\|_H.$$

(c) *Given any idempotent element $a \in \mathrm{QH}_*(M, \omega)$, for any $H, G \in C^\infty(M)$,*

$$\min_M(H - G) \leq \zeta_a(H; \omega) - \zeta_a(G; \omega) \leq \max_M(H - G).$$

3 Reduced perturbation

Given a Hamiltonian system (M, ω, H), denote by $\mathrm{Per}(\omega, H)$ the collection of all nonconstant geometrically distinct Hamiltonian 1–periodic orbits, where $x(t)$ denotes a generic element in $\mathrm{Per}(\omega, H)$. Since H is nondegenerate and M is assumed to be closed, $\mathrm{Per}(\omega, H)$ contains only finitely many elements. Consider the following subset of $\Omega^2_{\mathrm{closed}}(M)$:

(9) $\quad \Omega_{\omega,H} = \big\{\alpha \in \Omega^2_{\mathrm{closed}}(M) \mid \alpha$ vanishes in a neighborhood of

$$\text{each } x(t) \in \mathrm{Per}(\omega, H)\big\}.$$

This leads to the following definition:

Definition 3.1 Given a Hamiltonian system (M, ω, H), we call a perturbation ω' a *reduced perturbation of ω* if $\omega' = \omega + \alpha$ for some $\alpha \in \Omega_{\omega,H}$.

In this section, we will explain how any perturbation of a given symplectic structure ω can be reduced to a reduced perturbation as defined in Definition 3.1. Explicitly, we have the following result:

Proposition 3.2 *Let (M, ω, H) be a Hamiltonian system. If ω' is a perturbation of ω, then there exists a diffeomorphism $\phi \in \mathrm{Diff}(M)$ such that $\phi^*\omega' - \omega \in \Omega_{\omega,H}$ and $d_{C^0}(\phi, \mathbb{1}_M) \leq C|\omega' - \omega|$ for some constant C that does not depend on ω'.*

Here let us elaborate on the measurement $|-|$ on the closed 2–forms in Proposition 3.2. It is well known that for any $k \in \mathbb{Z}$, we can associate the k–norm on the space of

closed differential n–forms $\Omega^n_{\text{closed}}(M)$ for any $n \in \mathbb{Z}$. Explicitly, fixing a local chart $\{(U_i, \phi_i)\}^m_{i=1}$ of M, any $\alpha \in \Omega^n(M)$ can be locally expressed as

$$(\phi_i^{-1})^*(\alpha|_{U_i}) = \sum_{(s,t)\in\{1,...,2n\}\times\{1,...,2n\}} f_{i,s,t} \, dx_s \wedge dx_t,$$

where $\{x_1, \ldots, x_{2n}\}$ is the coordinate of $\mathbb{R}^{2n}$ and $f_{i,s,t} : \phi_i(U_i) \to \mathbb{R}$. Then define the k–norm

$$\|\alpha\|_k := \max_{i\in\{1,...,m\}} \max_{(s,t)\in\{1,...,2n\}\times\{1,...,2n\}} \max_{l\leq k} \|f_{i,s,t}\|_{C^l},$$

where $\|-\|_{C^l}$ is the standard C^l–norm defined over the function space. In particular, $(\Omega^n_{\text{closed}}(M), \|\cdot\|_k)$ is a normed vector space. Moreover, there exists a sequence of positive real numbers $\vec{\epsilon} = (\epsilon_k)_{k\geq 0}$ such that, under the $\vec{\epsilon}$–*norm* defined by $\|\alpha\|_{\vec{\epsilon}} := \sum_{k\geq 0} \epsilon_k \|\alpha\|_k$, the space $\Omega^n_{\text{closed}}(M)_{\vec{\epsilon}} = \{\alpha \in \Omega^n_{\text{closed}}(M) \mid \|\alpha\|_{\vec{\epsilon}} < \infty\}$ is a complete normed vector space under $\|-\|_{\vec{\epsilon}}$. For brevity, we denote $\|-\|_{\vec{\epsilon}}$ as $|-|$. In a similar way, we can define a seminorm on $H^n_{\text{dR}}(M;\mathbb{R})$ as follows: given any $a \in H^n_{\text{dR}}(M;\mathbb{R})$, define $|a|_h = \inf\{|\alpha| \mid [\alpha] = a\}$.

Remark 3.3 Due to the $\vec{\epsilon}$–norm, our proposed space $\Omega_{\omega,H}$ defined in (9) should be modified to be $\Omega_{\omega,H} \cap \Omega^2_{\text{closed}}(M)_{\vec{\epsilon}}$. For brevity, we still use the notation $\Omega_{\omega,H}$ as well as the corresponding definition of the reduced perturbations in Definition 3.1.

Before giving the proof of Proposition 3.2, we want to explain how we use it. In order to compare the symplectic invariants under different symplectic structures, we need to compare their associated Floer chain complexes, that is, $(\text{CF}_*(M, J, H, \omega), \partial_{J,H,\omega})$ and $(\text{CF}_*(M, J, H, \omega'), \partial_{J,H,\omega'})$, where ω' is a perturbation of ω. We can simplify this procedure by taking the ϕ concluded from Proposition 3.2 and inserting two intermediate steps,

$$
\begin{array}{ccc}
\text{CF}_*(M, J, H, \omega) & \xrightarrow{\quad(1)\quad} & \text{CF}_*(M, J, H, \omega') \\
\big\vert {\scriptstyle(4)} & & \big\vert {\scriptstyle(2)} \\
\text{CF}_*(M, J, H, \phi^*\omega') & \!\!\!-\!\!\underset{(3)}{-}\!\!-\!\!\! & \text{CF}_*(M, \phi^*J, \phi^*H, \phi^*\omega')
\end{array}
$$

where (1) is the desired comparison. Observe that (2) will not change spectral invariants, boundary depth and (partial) symplectic quasistates. Moreover, (3) only results in a small difference due to the second conclusion of Proposition 3.2 and 1–Lipschitz continuities from Theorem 2.12. In other words, the original comparison (1) can be

replaced by (4) if we forgive the small defect from (3). Hence, by the first conclusion of Proposition 3.2, we only need to consider the reduced perturbations. Moreover, if ω' is any reduced perturbation of ω, the following proposition says that the generators of $\mathrm{CF}_*(M, J, H, \omega)$ and $\mathrm{CF}_*(M, J, H, \omega')$ are the same, and also their degrees are the same. However, their boundary operators are different in general.

Proposition 3.4 *Let (M, ω, H) be a Hamiltonian system. Then, for any reduced perturbation ω' of ω, $\mathrm{Per}(\omega, H) = \mathrm{Per}(\omega', H)$. Moreover, $\mu_{\mathrm{CZ}, \omega}(x(t)) = \mu_{\mathrm{CZ}, \omega'}(x(t))$ for each contractible Hamiltonian 1–periodic orbit $x(t) \in \mathrm{Per}(\omega, H)$.*

Proof By definition, since ω' coincides with ω on $\bigcup_i U_i$, where U_i is a neighborhood of $x_i(t) \in \mathrm{Per}(\omega, H)$, $\mathrm{Per}(\omega', H)$ has at least as many Hamiltonian 1–periodic orbits as $\mathrm{Per}(\omega, H)$ has. Next, we claim that there are no other orbits in $\mathrm{Per}(\omega', H)$ outside $\bigcup_i U_i$. Without loss of generality, we assume $\mathrm{Per}(\omega, H)$ consists of only one element $x(t)$ with a neighborhood U. We will prove our claim by contrapositive. Suppose there exists a sequence of symplectic structures $\{\omega_{1/n}\}$ approaching to ω and, for each $\omega_{1/n}$, there exists some $z_n(t)$ such that $z_n(t) \in \mathrm{Per}(\omega_{1/n}, H)$ and $z_n(t) \subset M \setminus U$. Since M is compact, by the Arzelà–Ascoli theorem, passing to a subsequence, $z_n(t)$ converges to some $z_0(t)$ in $M \setminus U$. Because $X_H^{\omega_{1/n}}$ converges to X_H^ω uniformly on M, it is easy to check $z_0(t)$ is indeed a Hamiltonian 1–periodic orbit under ω. Thus we get a contradiction.

For the second conclusion, fix a disk w spanning $x(t)$; then one gets a trivialization $\Psi\colon w^*TM \to D^2 \times \mathbb{R}^{2n}$. Restricting to the neighborhood of ∂D^2, one gets a symplectic path $\gamma_\Psi(t)$ by the relation $(\Psi \circ d\phi^t_{H,\omega} \circ \Psi^{-1})(t, \vec{v}) = (t, \gamma_\Psi(t)\vec{v})$. Meanwhile, the defining property of a reduced perturbation implies flow $\phi^t_{H,\omega}|_U = \phi^t_{H,\omega'}|_U$. Since Conley–Zehnder indices are computed from the same symplectic path, they have the same indices, as desired. □

Now, let us give the proof of Proposition 3.2.

Proof of Proposition 3.2 For each orbit $x_i(t) \in \mathrm{Per}(\omega, H)$, take a neighborhood U_i of $x_i(t)$ such that U_i deformation retracts to $x_i(t)$. This can be done by taking a union of sufficiently small Darboux neighborhoods of the points on $x_i(t)$. Moreover, if it is necessary we shrink U_i so that $U_i \cap U_j = \varnothing$ whenever $i \neq j$. Consider the long exact sequence

$$(10) \qquad \cdots \to H^2_{\mathrm{dR}}\left(M, \bigcup_i U_i; \mathbb{R}\right) \xrightarrow{\iota_*} H^2_{\mathrm{dR}}(M; \mathbb{R}) \to H^2_{\mathrm{dR}}\left(\bigcup_i U_i; \mathbb{R}\right) \to \cdots,$$

where $\iota\colon \Omega^2_{\text{closed}}\big(M, \bigcup_i U_i\big) \to \Omega^2_{\text{closed}}(M)$ is the inclusion. By our choice of the neighborhood U_i, $\bigcup_i U_i$ deformation retracts to $\bigsqcup_i x_i(t)$, so by a dimension consideration, $H^2_{\text{dR}}\big(\bigcup_i U_i; \mathbb{R}\big) = 0$. Now, let

$$H^2_{\text{dR}}(M; \mathbb{R}) = \bigoplus_{j=1}^{m} \mathbb{R} \cdot c_j \quad \text{for some } c_j \in H^2_{\text{dR}}(M; \mathbb{R}).$$

Then $H^2_{\text{dR}}\big(\bigcup_i U_i; \mathbb{R}\big) = 0$ implies that for each basis element c_j, there exists some $\alpha_j \in \Omega^2_{\text{closed}}\big(M, \bigcup_i U_i\big)$ such that $c_j = \iota_*[\alpha_j]$. By definition, α_j vanishes in every U_i. Meanwhile, for ω and its perturbation ω',

$$[\omega'] - [\omega] = \sum_{j=1}^{m} t_j c_j = \sum_{j=1}^{m} t_j \iota_*[\alpha_j] \quad \text{for some } (t_1, \ldots, t_m) \in \mathbb{R}^m.$$

Therefore, $[\omega_1 - \omega_0] = \big[\sum_{j=1}^{m} t_j \iota(\alpha_j)\big]$. Moreover, by the definition of seminorm $|-|_h$ on $H^*_{\text{dR}}(M; \mathbb{R})$, for any $\delta > 0$, there exists an exact 2–form $d\theta$ such that

$$\alpha := \sum_{j=1}^{m} t_j \iota(\alpha_j) + d\theta$$

and $|\alpha| \leq \|[\alpha]\|_h + \delta = \|[\omega' - \omega]\|_h + \delta \leq |\omega' - \omega| + \delta$. We can choose δ such that $|\omega' - \omega| + \delta \leq C_1 |\omega' - \omega|$ for any preferred constant $C_1 > 1$; then

(11) $$|\alpha| \leq C_1 |\omega' - \omega|.$$

Moreover, by (10) again we know $d\theta = \iota(d\gamma)$ for some $\gamma \in \Omega^2\big(M, \bigcup_i U_i\big)$. Therefore, the α chosen above vanishes near every Hamiltonian 1–periodic orbit, ie $\alpha \in \Omega_{\omega, H}$.

Next, consider the homotopy $h_t = (1 - t)(\omega + \alpha) + t\omega'$, where $t \in [0, 1]$. Note that

$$\left[\frac{dh_t}{dt}\right] = [\omega' - \omega - \alpha] = [\omega' - \omega] - [\alpha] = [\alpha] - [\alpha] = 0.$$

Therefore, h_t represents the same cohomology class for each $t \in [0, 1]$; then, by Moser's trick, there exists some $\phi \in \text{Diff}(M)$ such that

$$\phi^* \omega' = \omega + \alpha.$$

More explicitly, ϕ is the time-one map of the flow ϕ_t of vector field X_t defined as

$$h_t(X_t, -) = -\tau \quad \text{and} \quad d\tau = \omega' - (\omega + \alpha).$$

By the triangle inequality, we know

$$|d\tau| \leq |\omega' - \omega| + |\alpha| \leq (1 + C_1)|\omega' - \omega|.$$

Therefore, using the dual norm of $\vec{\epsilon}$–norm on the space of vector fields, we get $|X_t| \leq C_2 |\omega' - \omega|$ over M for some constant C_2. This implies

$$d_{C^0}(\phi, \mathbb{1}_M) = \sup_{x \in M} \, \mathrm{dist}(\phi(x), x) \leq C_3 \, |\omega' - \omega|$$

for some constant C_3. Here C_2 and C_3 involve the integration of forms and smooth vector fields along the manifold M. Because M is assumed to be closed, both constants are finite and only depend on M. Thus we get the conclusion by setting $C = C_3$. $\quad\square$

4 Energy estimations

Similarly to the analysis from the perturbations of Hamiltonian functions (see Section 2.1), the comparison between two Floer chain complexes with different symplectic structures starts from a homotopy between two symplectic structures. Explicitly, fix a symplectic structure ω_0 and consider a reduced perturbation $\omega_1 = \omega_0 + \alpha$. Take a *smooth* cut-off function $\kappa(s)$ such that $\kappa(s) = 0$ for $s \in (-\infty, 0]$, $\kappa(s) = 1$ for $s \in [1, \infty)$ and $\kappa'(s) > 0$ for $s \in (0, 1)$. Define an *interpolating homotopy* $s \mapsto \omega_s$ between ω_0 and ω_1 by

$$(12) \qquad\qquad \omega_s = (1 - \kappa(s))\omega_0 + \kappa(s)\omega_1.$$

Note that ω_s is also a reduced perturbation of ω_0 for each $s \in \mathbb{R}$. Now take a family of pairs (J_s, ω_s) where, for each $s \in \mathbb{R}$, J_s is an ω_s–compatible almost complex structure. They induce a family of Riemannian metrics $g_s(v, w) := \omega(v, J_s w)$. Then we can give the following definition:

Definition 4.1 Given a Hamiltonian system (M, ω_0, H) and a reduced perturbation ω_1, denote $s \mapsto \omega_s$ as an interpolating homotopy between ω_0 and ω_1. A parametrized Floer operator $\mathcal{F}^s$ is defined as

$$(13) \qquad\qquad \mathcal{F}^s = \frac{\partial}{\partial s} + J_s \left(\frac{\partial}{\partial t} - X_H^{\omega_s} \right),$$

where $X_H^{\omega_s} = J_s \, \mathrm{grad}_{\omega_s} H$ and $\mathrm{grad}_{\omega_s} H$ is the gradient of H with respect to the induced metric g_s. Moreover, we call a map $u(s, t) \colon \mathbb{R} \times S^1 \to M$ an $\mathcal{F}^s$–*trajectory* if u satisfies $\mathcal{F}^s(u) = 0$.

For each $\mathcal{F}^s$–trajectory u, one defines its energy by

$$E(u) = \int_{-\infty}^{\infty} \int_0^1 \left| \frac{\partial u}{\partial s} \right|_{g_s}^2 dt \, ds.$$

Definition 4.2 We call an $\mathcal{F}^s$–trajectory *admissible* if it satisfies the following conditions:

  (a) The energy of u is finite, that is, $E(u) < \infty$.

  (b) $u(s,t)$ satisfies the following asymptotic condition, where $\gamma_-(t)$ and $\gamma_+(t)$ are contractible Hamiltonian 1–periodic orbits:

$$\lim_{s \to -\infty} u(s,t) = \gamma_-(t) \quad \text{and} \quad \lim_{s \to \infty} u(s,t) = \gamma_+(t).$$

  (c) $[\gamma_+ w_+] = [\gamma_+, w_- \# u]$, where the equivalent class is defined by the relation

$$[x, v] = [y, w] \iff x(t) = y(t) \quad \text{and} \quad [v \# (-w)] = 0 \in H_2(M; \mathbb{R}).$$

Note that the symplectic action functional defined in (5) depends on symplectic structures. Along this interpolating homotopy $s \mapsto \omega_s$, the corresponding symplectic action functional at ω_s is

(14) $$\mathcal{A}_{H,\omega_s}([\gamma, w]) = -\int_{D^2} w^* \omega_s + \int_0^1 H(\gamma(t), t)\, dt.$$

Since the Hamiltonian function H remains the same in our perturbation discussion, for brevity, we simply denote $\mathcal{A}_{H,\omega_s}$ by $\mathcal{A}_{\omega_s}$. Our first main result in this section is the following energy estimation. Recall the notation $\alpha = \omega_1 - \omega_0$.

Proposition 4.3 *Suppose that* $u \colon \mathbb{R} \times S^1 \to M$ *is an admissible* $\mathcal{F}^s$–*trajectory from* $[\gamma_-, w_-]$ *to* $[\gamma_+, w_+]$. *Then we have the energy estimation between the symplectic action functionals*

$$-(1 + C\,|\alpha|)E(u) - \int_{D^2} (w_-)^* \alpha \leq \mathcal{A}_{\omega_1}([\gamma_+, w_+]) - \mathcal{A}_{\omega_0}([\gamma_-, w_-])$$

$$\leq -(1 - C\,|\alpha|)E(u) - \int_{D^2} (w_-)^* \alpha$$

for some positive constant C *which is independent of* u.

Here let us state a "local" version of Proposition 4.3. Given an interpolating homotopy between ω_0 and ω_1, for any $s < t$ in $[0, 1]$ the following proposition gives an energy estimation with respect to ω_s and ω_t along this interpolating homotopy between ω_0 and ω_1:

Proposition 4.4 *Suppose that* $u \colon \mathbb{R} \times S^1 \to M$ *is an admissible* $\mathcal{F}^s$–*trajectory from* $[\gamma_-, w_-]$ *to* $[\gamma_+, w_+]$. *Then we have the energy estimation between the symplectic*

action functionals

$$-(1+C_{s,t}|\alpha|)E(u)+(s-t)\int_{D^2}(w_-)^*\alpha \leq \mathcal{A}_{\omega_t}([\gamma_+,w_+])-\mathcal{A}_{\omega_s}([\gamma_-,w_-])$$

$$\leq -(1-C_{s,t}|\alpha|)E(u)+(s-t)\int_{D^2}(w_-)^*\alpha$$

for constant $C_{s,t}=(t-s)\cdot C$, where C is the constant from [Proposition 4.3](#).

The proof of [Proposition 4.4](#) is exactly the same as the proof of [Proposition 4.3](#). For simplicity, we only give the proof of [Proposition 4.3](#).

Proof of [Proposition 4.3](#) First of all,

$$\mathcal{A}_{\omega_1}([\gamma_+,w_+])-\mathcal{A}_{\omega_0}([\gamma_-,w_-])=\int_{-\infty}^{\infty}\frac{d}{ds}\mathcal{A}_{\omega_s}([u(s,\cdot),w_-\#u((-\infty,s]\times S^1)])\,ds.$$

For any $s\in\mathbb{R}$, denote $w_s=w_-\#u((-\infty,s]\times S^1)$ and topologically it is a disk D^2 spanning the loop $\{u(s,t)\}_{t\in S^1}$. By the definition of ω_s and the symplectic action functional $\mathcal{A}_{\omega_s}$,

$$\mathcal{A}_{\omega_s}\big([u(s,\cdot),w_-\#u((-\infty,s]\times S^1)]\big)$$

$$=-\int_{D^2}w_s^*\omega_s+\int_0^1 H(u(s,t),t)\,dt$$

$$=-\int_{D^2}w_s^*(\omega_0+\kappa(s)\alpha)+\int_0^1 H(u(s,t),t)\,dt$$

$$=-\int_{D^2}w_s^*\omega_0-\kappa(s)\int_{D^2}w_s^*\alpha+\int_0^1 H(u(s,t),t)\,dt.$$

It is easy to check that for any *closed* 2–form ω,

$$(15)\qquad\qquad \frac{d}{ds}\int_{D^2}w_s^*\omega=\int_0^1\omega\Big(\frac{\partial u}{\partial s},\frac{\partial u}{\partial t}\Big)\,dt.$$

In particular, for closed 2–forms ω_0 and α, one gets

$$\frac{d}{ds}\Big(-\int_{D^2}w_s^*\omega_0-\kappa(s)\int_{D^2}w_s^*\alpha\Big)$$

$$=-\int_0^1\omega_0\Big(\frac{\partial u}{\partial s},\frac{\partial u}{\partial t}\Big)\,dt-\kappa'(s)\int_{D^2}w_s^*\alpha-\kappa(s)\int_0^1\alpha\Big(\frac{\partial u}{\partial s},\frac{\partial u}{\partial t}\Big)\,dt$$

$$=-\kappa'(s)\int_{D^2}w_s^*\alpha-\int_0^1\omega_s\Big(\frac{\partial u}{\partial s},\frac{\partial u}{\partial t}\Big)\,dt.$$

On the other hand,

$$\frac{d}{ds}\int_0^1 H(u(s,t),t)\,dt = \int_0^1 \frac{d}{ds}H(u(s,t),t)\,dt$$
$$= \int_0^1 dH\Big(\frac{\partial u}{\partial s}\Big)\,dt = -\int_0^1 \omega_s\Big(X_H^{\omega_s},\frac{\partial u}{\partial s}\Big)\,dt.$$

Moreover, since u satisfies $\mathcal{F}^{\omega_s}(u)=0$, we know that the vector field $X_H^{\omega_s}$ is $X_H^{\omega_s} = J_s(u)\operatorname{grad}_{\omega_s}H = J_s(u)\big(-\frac{\partial u}{\partial s}-J_s(u)\frac{\partial u}{\partial t}\big) = \frac{\partial u}{\partial t}-J_s(u)\frac{\partial u}{\partial s}$. Therefore,

$$-\int_0^1 \omega_s\Big(X_H^{\omega_s},\frac{\partial u}{\partial s}\Big)\,dt = -\int_0^1 \omega_s\Big(\frac{\partial u}{\partial t}-J_s(u)\frac{\partial u}{\partial s},\frac{\partial u}{\partial s}\Big)\,dt$$
$$= \int_0^1 \omega_s\Big(\frac{\partial u}{\partial s},\frac{\partial u}{\partial t}\Big)\,dt - \int_0^1 \Big|\frac{\partial u}{\partial s}\Big|_{g_s}^2\,dt.$$

Combining these computations, one gets the equality

$$(16)\qquad \mathcal{A}_{\omega_1}([\gamma_+,w_+]) - \mathcal{A}_{\omega_0}([\gamma_-,w_-]) = -E(u) - \int_{-\infty}^{\infty}\kappa'(s)\int_{D^2}w_s^*\alpha\,ds.$$

For the second term in (16), from an integration by parts,

$$\int_{-\infty}^{\infty}\kappa'(s)\int_{D^2}w_s^*\alpha\,ds = \int_{-\infty}^{\infty}\frac{d}{ds}\Big(\kappa(s)\int_{D^2}w_s^*\alpha\Big)\,ds - \int_{-\infty}^{\infty}\kappa(s)\Big(\frac{d}{ds}\int_{D^2}w_s^*\alpha\Big)\,ds$$
$$= \int_{D^2}(w_-\#u)^*\alpha - \int_{-\infty}^{\infty}\kappa(s)\Big(\frac{d}{ds}\int_{D^2}w_s^*\alpha\Big)\,ds$$
$$= \int_{D^2}(w_-)^*\alpha + \int_{-\infty}^{\infty}\int_0^1 (1-\kappa(s))\alpha\Big(\frac{\partial u}{\partial s},\frac{\partial u}{\partial t}\Big)\,dt\,ds.$$

Notice that the last term satisfies the inequality

$$-\int_{-\infty}^{\infty}\int_0^1 \Big|\alpha\Big(\frac{\partial u}{\partial s},\frac{\partial u}{\partial t}\Big)\Big|\,dt\,ds \le \int_0^1 (1-\kappa(s))\alpha\Big(\frac{\partial u}{\partial s},\frac{\partial u}{\partial t}\Big)\,dt\,ds$$
$$\le \int_{-\infty}^{\infty}\int_0^1 \Big|\alpha\Big(\frac{\partial u}{\partial s},\frac{\partial u}{\partial t}\Big)\Big|\,dt\,ds.$$

Now, we claim that there exist some constant C' and N, independent of the trajectory u, such that

$$(17)\qquad \int_{-\infty}^{\infty}\int_0^1 \Big|\alpha\Big(\frac{\partial u}{\partial s},\frac{\partial u}{\partial t}\Big)\Big|\,dt\,ds \le |\alpha|\cdot C'\cdot\frac{E(u)}{\sqrt{N}} + |\alpha|\cdot E(u).$$

In fact, using the relation $X_H^{\omega_s} + J_s(u)\frac{\partial u}{\partial s} = \frac{\partial u}{\partial t}$, what we want to estimate can be rewritten as

$$\int_{-\infty}^{\infty} \int_0^1 \left| \alpha\left(\frac{\partial u}{\partial s}, X_H^{\omega_s} + J_s(u)\frac{\partial u}{\partial s}\right) \right| dt\, ds$$

$$\leq \int_{-\infty}^{\infty} \int_0^1 \left| \alpha\left(\frac{\partial u}{\partial s}, X_H^{\omega_s}\right) \right| dt\, ds + \int_{-\infty}^{\infty} \int_0^1 \left| \alpha\left(\frac{\partial u}{\partial s}, J(u)\frac{\partial u}{\partial s}\right) \right| dt\, ds.$$

The second term is bounded from above by $|\alpha| \cdot E(u)$. For the first term, by the asymptotic property of u and the definition of α (which vanishes near orbits $\gamma_-(t)$ and $\gamma_+(t)$), we know that there exists some $s_u \in \mathbb{R}$, depending on u, such that $\alpha = 0$ when $s \in (-\infty, -s_u] \cup [s_u, +\infty)$. So

(18) $$\int_{-\infty}^{\infty} \int_0^1 \left| \alpha\left(\frac{\partial u}{\partial s}, X_H^{\omega_s}\right) \right| dt\, ds = \int_{-s_u}^{s_u} \int_0^1 \left| \alpha\left(\frac{\partial u}{\partial s}, X_H^{\omega_s}\right) \right| dt\, ds.$$

Meanwhile, by Lemma 5.2 in [29], there exists some constant positive N, independent of u, such that for any $s_* \in [-s_u, s_u]$ (enlarge s_u if necessary),

$$|\partial_s u(s,t)|_{L^2}^2\big|_{s=s_*} = \int_0^1 \left|\frac{\partial u}{\partial s}\right|^2\bigg|_{s=s_*} dt \geq N.$$

In other words, if we set

$$\mathcal{I} = \left\{ s_* \in \mathbb{R} : |\partial_s u(s,t)|_{L^2}^2\big|_{s=s_*} \geq N \right\}$$

then $[-s_u, s_u] \subset \mathcal{I}$. So, for (18), we can improve it to be integrated over $\mathbb{R}/\mathbb{Z} \times \mathcal{I}$. It will not change the value of the integral by the vanishing property of α. Therefore, we have

$$\int_{-\infty}^{\infty} \int_0^1 \left| \alpha\left(\frac{\partial u}{\partial s}, X_H^{\omega_s}\right) \right| dt\, ds = \int_{\mathcal{I}} \int_0^1 \left| \alpha\left(\frac{\partial u}{\partial s}, X_H^{\omega_s}\right) \right| dt\, ds$$

$$\leq |\alpha| \cdot C' \int_{\mathcal{I}} \int_0^1 \left|\frac{\partial u}{\partial s}\right|_{g_s} dt\, ds,$$

where C' is an upper bound of the uniform norm of the vector field $X_H^{\omega_s}$ for *any* $s \in [0,1]$ on the closed manifold M. On the other hand, due to the energy constraint, the (Lebesgue) measure of $\mathcal{I}$ satisfies $\mu(\mathcal{I}) \leq E(u)/N$. Applying the Cauchy–Schwarz inequality, we get

$$\left(\int_{\mathcal{I}} \int_0^1 1 \cdot \left|\frac{\partial u}{\partial s}\right|_{g_s} dt\, ds \right)^2 \leq \left(\int_{\mathcal{I}} \int_0^1 1^2\, dt\, ds \right) \cdot \left(\int_{\mathcal{I}} \int_0^1 \left|\frac{\partial u}{\partial s}\right|_{g_s}^2 dt\, ds \right) \leq \frac{E(u)^2}{N}.$$

Together, we get the desired conclusion by setting $C := C'/\sqrt{N} + 1$. $\qquad\square$

For later use, we need an energy estimation of another type:

Definition 4.5 Fix a finite number $R \in \mathbb{R}$. Choose a "symmetric" cut-off function as follows. Let $\kappa(s) = 0$ for $s \in (-\infty, -(R + 1)] \cup [R + 1, \infty)$ and $\kappa(s) = 1$ for $s \in [-R, R]$. Moreover, $\kappa'(s) > 0$ for $s \in (-(R + 1), -R)$ and $\kappa'(s) < 0$ for $s \in (R, R + 1)$. Using this $\kappa(s)$, we can define a *symmetric homotopy* $s \mapsto \omega_s$ as in (12) from ω_0 to itself which passes through ω_1. Accordingly, we can define a Floer operator $\mathcal{F}^s_{\text{sym}}$ and an $\mathcal{F}^s_{\text{sym}}$–trajectory as in Definition 4.1, and an admissible $\mathcal{F}^s_{\text{sym}}$–trajectory as in Definition 4.2.

Then we have the following energy estimation:

Proposition 4.6 *Suppose that* $u \colon \mathbb{R} \times S^1 \to M$ *is an admissible* $\mathcal{F}^s_{\text{sym}}$*–trajectory from* $[\gamma_-, w_-]$ *to* $[\gamma_+, w_+]$*. Then we have the energy estimation, between the symplectic action functionals,*

$$-(1 + C\,|\alpha|)E(u) \leq \mathcal{A}_{\omega_0}([\gamma_+, w_+]) - \mathcal{A}_{\omega_0}([\gamma_-, w_-]) \leq -(1 - C\,|\alpha|)E(u),$$

where the constant C *is the one from* Proposition 4.3.

Proof Denote $w_s = w_- \# u((-\infty, s] \times S^1)$. Similarly to (16), one gets

$$\mathcal{A}_{\omega_0}([\gamma_+, w_+]) - \mathcal{A}_{\omega_0}([\gamma_-, w_-]) = -E(u) - \int_{-\infty}^{\infty} \kappa'(s) \int_{D^2} w_s^* \alpha \, ds.$$

Then, integrating by parts, one gets

$$\int_{-\infty}^{\infty} \kappa'(s) \int_{D^2} w_s^* \alpha \, ds = \int_{-\infty}^{\infty} \frac{d}{ds}\left(\kappa(s) \int_{D^2} w_s^* \alpha\right) ds - \int_{-\infty}^{\infty} \kappa(s)\left(\frac{d}{ds} \int_{D^2} w_s^* \alpha\right) ds$$

$$= -\int_{-\infty}^{\infty} \kappa(s)\left(\frac{d}{ds} \int_{D^2} w_s^* \alpha\right) ds$$

$$= -\int_{-\infty}^{\infty} \int_0^1 \kappa(s)\alpha\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\right) dt \, ds.$$

The last equality comes from (15). Moreover,

$$-C\,|\alpha|E(u) \leq \int_{-\infty}^{\infty} \int_0^1 \kappa(s)\alpha\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\right) dt \, ds \leq C\,|\alpha|E(u)$$

by claim (17). Therefore, we get the conclusion. $\qquad\square$

Observe that there is no term $\int_{D^2} w_-^* \alpha$ in the estimation from Proposition 4.6 exactly because the cut-off function $\kappa(s)$ for $\mathcal{F}^s_{sym}$ is symmetric. When working on a general symplectic manifold, the value of $\int_{D^2} w_-^* \alpha$ depends on the disk w_- spanning γ_-, so the action on w_- from $\pi_2(M)$ can possibly change this value. In other words, the estimation from Proposition 4.3 does not provide a uniform bound, which causes an algebraic difficulty involving the Novikov finiteness condition. On a different topic, when using the energy estimation from Proposition 4.3 or Proposition 4.6, we always assume $|\alpha|$ is sufficiently small (which is equivalent to an assumption that the perturbation ω_1 is sufficiently close to ω_0), so that $C|\alpha| < 1$.

5 Proof of Theorem 1.1

The main part of the proof of Theorem 1.1 is from the following quantitative comparison. Recall that the definition of a δ–quasiequivalence is given by Definition 2.2.

Lemma 5.1 *Let* (M, ω, H) *be a Hamiltonian system where* M *is aspherical, and* ω' *is a perturbation of* ω. *Then there exists an* $S(\alpha)$–*quasiequivalence between filtered complexes* $(\mathrm{CF}_*(M, J, H, \omega), \partial_\omega, \ell_\omega)$ *and* $(\mathrm{CF}_*(M, J, H, \omega'), \partial_{\omega'}, \ell_{\omega'})$ *for some constant* $S(\alpha)$ *that depends on* α.

Since M is aspherical, the Novikov field $\Lambda^{\mathcal{K},\Gamma}$ equals $\mathcal{K}$. Therefore, both Floer chain complexes $(\mathrm{CF}_*(M, J, H, \omega), \partial_\omega)$ and $(\mathrm{CF}_*(M, J, H, \omega'), \partial_{\omega'})$ are finite-dimensional over $\mathcal{K}$.

Proof Fix a basis of $\mathrm{CF}_*(M, J, H, \omega)$ over $\mathcal{K}$, say $\{[x_1, w_1], \dots, [x_n, w_n]\}$. They are also a basis for $\mathrm{CF}_*(M, J, H, \omega')$ by Proposition 3.4. We need to find a quadruple (Φ, Ψ, K, K') that satisfies the conditions in Definition 2.2.

First, fix any interpolating homotopy between ω and ω' as in (12). Recall that the Floer operator $\mathcal{F}^s$ and the admissible $\mathcal{F}^s$–trajectories are defined in Definitions 4.1 and 4.2. Consider a map $\Phi\colon \mathrm{CF}_*(M, J, H, \omega) \to \mathrm{CF}_*(M, J, H, \omega')$ defined by

$$(19) \qquad \Phi([x_i, w_i]) = \sum_{\substack{j \in \{1, \dots, n\} \\ \mu_{\mathrm{CZ}}([x_i, w_i]) = \mu_{\mathrm{CZ}}([x_j, w_j])}} n([x_i, w_i], [x_j, w_j])[x_j, w_j],$$

where the number $n([x_i, w_i], [x_j, w_j])$ is defined by counting admissible $\mathcal{F}^s$–trajectories connecting $[x_i, w_i]$ and $[x_j, w_j]$. Due to the index condition, $n([x_i, w_i], [x_j, w_j])$ is a

finite number. Moreover, since there are only finitely many Hamiltonian 1–periodic orbits, the sum in the expression (19) is a finite sum. Finally, by the standard Floer gluing argument (see Sections 3.3 and 3.4 in [21]), Φ is a chain map. Similarly, we can define $\Psi: \mathrm{CF}_*(M, J, H, \omega') \to \mathrm{CF}_*(M, J, H, \omega)$ and it is also a chain map.

Now let us study the change of filtrations. By Proposition 4.3, any admissible $\mathcal{F}^s$–trajectory u connecting $[x_i, w_i]$ and $[x_j, w_j]$ gives rise to an inequality

$$(20) \quad \mathcal{A}_{\omega'}([x_j, w_j]) - \mathcal{A}_\omega([x_i, w_i]) \leq -(1 - C|\alpha|)E(u) - \int_{D^2} w_i^*\alpha \leq -\int_{D^2} w_i^*\alpha$$

since we always assume ω' is sufficiently close to ω (so $1 - C|\alpha| > 0$). By the definition of filtration function (6), for any chain $c \in \mathrm{CF}_*(M, J, H, \omega)$ there exists some $j_0 \in \{1, \dots, n\}$ *depending on* α such that $\ell_{\omega'}(\Phi(c)) = \mathcal{A}_{\omega'}([x_{j_0}, w_{j_0}])$. Meanwhile, there exists some $i_0 \in \{1, \dots, n\}$ such that $[x_{i_0}, w_{i_0}]$ from the chain c connects to $[x_{j_0}, w_{j_0}]$ by an admissible $\mathcal{F}^s$–trajectory u. Therefore,

$$(21) \quad \mathcal{A}_{\omega'}([x_{j_0}, w_{j_0}]) - \mathcal{A}_\omega([x_{i_0}, w_{i_0}]) \leq -\int_{D^2} w_{i_0}^*\alpha.$$

Since $\ell_\omega(c) \geq \mathcal{A}_\omega([x_{i_0}, w_{i_0}])$, we have $\ell_{\omega'}(\Phi(c)) - \ell_\omega(c) \leq -\int_{D^2} w_{i_0}^*\alpha$. Denote

$$S(\alpha) = \max_i \left| \int_{D^2} w_i^*\alpha \right|,$$

and it is nonnegative. Then $\ell_{\omega'}(\Phi(c)) \leq \ell_\omega(c) + S(\alpha)$ for any $c \in \mathrm{CF}_*(M, J, H, \omega)$. A similar argument for Ψ results in the same constant $S(\alpha)$ and a similar filtration inequality.

Next, we construct a homotopy between $\mathbb{1}_\omega$ and $\Psi \circ \Phi$, where $\mathbb{1}_\omega$ is the identity map on the complex $(\mathrm{CF}_*(M, J, H, \omega), \partial_\omega)$. So far, we have two different homotopies of symplectic structure ω: one is the constant homotopy of ω that corresponds to $\mathbb{1}_\omega$, and the other is a symmetric homotopy of ω which passes through ω'. This homotopy is induced by the composition $\Psi \circ \Phi$. Let us denote this symmetric homotopy as $s \mapsto \omega_s$. Take a homotopy $\{\omega_{s,\lambda}\}_{\lambda \in [0,1]}$ between these two homotopies $s \mapsto \omega_s$ and ω such that $\omega_{s,0} = \omega$ and $\omega_{s,1} = \omega_s$. Moreover, we require $\omega_{s,\lambda}$ to be a symmetric homotopy for each $\lambda \in [0, 1]$. Recall that the operator $\mathcal{F}_{\mathrm{sym}}^{s,\lambda}$ and the admissible $\mathcal{F}_{\mathrm{sym}}^{s,\lambda}$–trajectory with respect to $\omega_{s,\lambda}$ are defined in Definition 4.5. Define $K: \mathrm{CF}_*(M, J, H, \omega) \to \mathrm{CF}_{*+1}(M, J, H, \omega)$ as

$$(22) \quad K([x_i, w_i]) = \sum_{\substack{j \in \{1, \dots, n\} \\ \mu_{\mathrm{CZ}}([x_i, w_i]) + 1 = \mu_{\mathrm{CZ}}([x_j, w_j])}} n([x_i, w_i], [x_j, w_j])[x_j, w_j],$$

where the number $n([x_i, w_i], [x_j, w_j])$ is defined by counting pairs (u, λ) where u is an admissible $\mathcal{F}_{\mathrm{sym}}^{s,\lambda}$–trajectory connecting $[x_i, w_i]$ and $[x_j, w_j]$. Again $n([x_i, w_i], [x_j, w_j])$ is finite and (22) is a finite sum. Moreover, by a standard argument considering the corresponding moduli space, one can show that K provides a homotopy between $\mathbb{1}_\omega$ and $\Psi \circ \Phi$. Similarly, we obtain a homotopy K' between $\mathbb{1}_{\omega'}$ and $\Phi \circ \Psi$, where $\mathbb{1}_{\omega'}$ is the identity map on the complex $(\mathrm{CF}_*(M, J, H, \omega'), \partial_{\omega'})$.

Again, let us study the changes of filtrations. For any chain $c \in \mathrm{CF}_*(M, J, H, \omega)$, suppose $\ell_\omega(K(c)) = \mathcal{A}_\omega([x_{q_0}, w_{q_0}])$ for some $q_0 \in \{1, \ldots, n\}$. There exists some $[x_{p_0}, w_{p_0}]$ from the chain c connected to $[x_{q_0}, w_{q_0}]$ by an admissible $\mathcal{F}^{s,\lambda}$–trajectory for some $\lambda \in [0, 1]$. By Proposition 4.6, $\mathcal{A}_\omega([x_{q_0}, w_{q_0}]) \leq \mathcal{A}_\omega([x_{p_0}, w_{p_0}])$. This implies

$$\ell_\omega(K(c)) - \ell_\omega(c) \leq \mathcal{A}_\omega([x_{q_0}, w_{q_0}]) - \mathcal{A}_\omega([x_{p_0}, w_{p_0}]) \leq 0.$$

Since $S(\alpha) \geq 0$, in particular $\ell_\omega(K(c)) \leq \ell_\omega(c) + 2S(\alpha)$ for any $c \in \mathrm{CF}_*(M, J, H, \omega)$. A similar inequality holds for the other homotopy K'. Thus we get the conclusion. $\quad\square$

Remark 5.2 Under the hypothesis of Lemma 5.1, the same argument can prove the following more general result if one uses Proposition 4.4: for any $s \leq t$ in $[0, 1]$, there exists some constant $S_{s,t}(\alpha)$ such that $(\mathrm{CF}_*(M, J, H, \omega_s), \partial_{\omega_s}, \ell_{\omega_s})$ and $(\mathrm{CF}_*(M, J, H, \omega_t), \partial_{\omega_t}, \ell_{\omega_t})$ are $S_{s,t}(\alpha)$–quasiequivalent. Moreover, $S_{s,t}(\alpha) \leq |s - t| \cdot C\, |\alpha|$ for some constant C.

Proof of Theorem 1.1 From Proposition 3.2, if ω' is a perturbation of ω, then there exists a $\phi \in \mathrm{Diff}(M)$ such that $\phi^*\omega'$ is a reduced perturbation of ω. As elaborated in Section 3, if we replace ω' with $\phi^*\omega$ in our discussion, then changes from the perturbation of Hamiltonian functions (from H to $\phi^* H$) need to be considered. Theorem 2.12 implies that the resulting changes on both spectral invariants and boundary depth are no greater than $|\phi^* H - H|_H$. Since M is compact, $|\phi^* H - H|_H \leq C'|\omega' - \omega|$ for some constant C' that involves a C^0–norm of H. In what follows, we will focus on the case when ω' is a reduced perturbation of ω.

The desired conclusion for boundary depth directly comes from Lemma 5.1 and Theorem 2.9. For spectral invariants, by the same idea as the proof of Theorem 3.1(iii) in [18], we have the commutative diagram

$$
\begin{array}{ccc}
 & H_*(M; \mathcal{K}) & \\
{\scriptstyle \mathrm{PSS}_*^\omega} \swarrow & & \searrow {\scriptstyle \mathrm{PSS}_*^{\omega'}} \\
\mathrm{HF}_*(M, J, H, \omega) & \xrightarrow[\Phi_*]{} & \mathrm{HF}_*(M, J, H, \omega')
\end{array}
$$

where Φ_* is the chain map constructed from Lemma 5.1. By Theorem 2.6, there exists some element $c \in \mathrm{CF}_*(M, J, H, \omega)$ such that $\rho(a, H; \omega) = \ell_\omega(c)$ and $[c] = \mathrm{PSS}^\omega_*(a)$. Then

$$[\Phi(c)] = \Phi_*([c]) = \Phi_*(\mathrm{PSS}^\omega_*(a)) = \mathrm{PSS}^{\omega'}_*(a).$$

Therefore,

$$\rho(a, H; \omega') - \rho(a, H; \omega) \leq \ell_{\omega'}(\Phi(c)) - \ell_\omega(c) \leq S(\omega' - \omega).$$

Switch the role of ω and ω'; then we get the other inequality. Therefore,

$$|\rho(a, H; \omega) - \rho(a, H; \omega')| \leq S(\omega' - \omega).$$

Finally, it is easy to see that there exists a constant C such that $S(\omega' - \omega) \leq C |\omega' - \omega|$. Thus we get the conclusion. $\qquad \square$

To end this section, we will give the proof of Corollary 1.2.

Proof of Corollary 1.2 Similarly to the argument in the proof of Theorem 1.1, we will only focus on the case when ω' is a reduced perturbation of ω. By the hypothesis on the dimension of $H^2(M; \mathcal{K})$, there exists a number λ (sufficiently) close to 1 such that $[\omega'] = \lambda[\omega]$. A key observation is that if we rescale λH and $\lambda \omega$, then

$$\beta(\lambda H; \lambda \omega) = \lambda \beta(H; \omega) \quad \text{and} \quad \rho([M], \lambda H; \lambda \omega) = \lambda \rho([M], H; \omega).$$

Hence, by Theorems 2.12(b) and 1.1,

$$
\begin{aligned}
|\beta(H; \omega) &- \beta(H; \omega')| \\
&\leq |\beta(H; \omega) - \beta(\lambda H; \lambda \omega)| + |\beta(\lambda H; \lambda \omega) - \beta(H; \lambda \omega)| + |\beta(H; \lambda \omega) - \beta(H; \omega')| \\
&\leq |1 - \lambda| \beta(H; \omega) + |1 - \lambda| \|H\|_H + C |\lambda \omega - \omega'| \\
&\leq |1 - \lambda| \beta(H; \omega) + |1 - \lambda| \|H\|_H + C(|1 - \lambda| |\omega| + |\omega - \omega'|) \\
&= |1 - \lambda| \cdot A + C |\omega - \omega'|,
\end{aligned}
$$

where $A := \beta(H; \omega) + \|H\|_H + C |\omega|$ and C is the constant from Theorem 1.1. Moreover, $[\omega - \omega'] = (1 - \lambda)[\omega]$ implies

$$(23) \qquad\qquad |1 - \lambda| \|[\omega]\|_h = \|[\omega - \omega']\|_h \leq |\omega - \omega'|.$$

Therefore, $|1 - \lambda| \leq (1/\|[\omega]\|_h) |\omega - \omega'|$. So

$$|1 - \lambda| \cdot A + C |\omega - \omega'| \leq \left(\frac{A}{\|[\omega]\|_h} + C \right) \cdot |\omega - \omega'|.$$

Therefore, set $\widetilde{C}_2 = A/\|[\omega]\|_h + C$ and we get the conclusion.

Similarly, for spectral invariants, by Theorems 2.12(a) and 1.1,

$$|\rho([M], H; \omega) - \rho([M], H; \omega')|$$
$$\leq |\rho([M], H; \omega) - \rho([M], \lambda H; \lambda \omega)| + |\rho([M], \lambda H; \lambda \omega) - \rho([M], H; \lambda \omega)|$$
$$+ |\rho([M], H; \lambda \omega) - \rho([M], H; \omega')|$$
$$\leq |1 - \lambda| \rho([M], H; \omega) + |1 - \lambda| \|H\|_H + C |\lambda \omega - \omega'|$$
$$\leq |1 - \lambda| \cdot B + C |\omega - \omega'|,$$

where $B := \rho([M], H; \omega) + \|H\|_H + C |\omega|$ and C is the constant from Theorem 1.1. The same estimation as in (23) implies the desired conclusion when we set $\widetilde{C}_1 = B/\|[\omega]\|_h + C$. $\square$

Remark 5.3 From the perspective of persistent homology theory in [28], we can associate barcodes, denoted by $\mathcal{B}_\omega$ and $\mathcal{B}_{\omega'}$, to both $\mathcal{C}_\omega := (\mathrm{CF}_*(M, J, H, \omega), \partial_\omega, \ell_\omega)$ and $\mathcal{C}_{\omega'} := (\mathrm{CF}_*(M, J, H, \omega'), \partial_{\omega'}, \ell_{\omega'})$, respectively. Lemma 5.1 implies that their quasiequivalence distance [28, Definition 8.1] satisfies

$$d_Q(\mathcal{C}_\omega, \mathcal{C}_{\omega'}) \leq \tfrac{1}{2} S(\omega - \omega').$$

Recall that d_B denotes the bottleneck distance [28, Definition 8.14]. Then the stability theorem in [28] implies the inequality

$$d_B(\mathcal{B}_\omega, \mathcal{B}_{\omega'}) \leq 2 d_Q(\mathcal{C}_\omega, \mathcal{C}_{\omega'}) \leq S(\omega - \omega') \leq C |\omega - \omega'|.$$

This can be regarded as a generalization of Theorem 1.1.

6 Proofs of Theorems 1.4 and 1.7

Proof of Theorem 1.4 Let $U \subset \Sigma$ be a disjoint union of simply connected open subsets and $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times U)$. Pick an open ball $V \subset \Sigma \setminus \overline{U}$ and fix a closed 2–form α positively supported in V and that vanishes elsewhere. By our choice of H, $\alpha \in \Omega_{\omega, H}$. Fix $\lambda \geq 1$. Consider the isotopy of the reduced perturbations,

$$\omega_s = \omega + s(\lambda \alpha),$$

where $s \in [0, 1]$. Take λ sufficient large such that, on (Σ, ω_1), U can be viewed as a disjoint union of topological balls with total area sufficiently smaller than the total area of Σ. Then it is displaceable. By an energy–capacity inequality, say [24, Corollary 3.3],

$$(24) \qquad \rho(a, H, \omega_1) \leq e^{\omega_1}(U) < +\infty,$$

where $e^{\omega_1}(U)$ denotes the displacement energy of U under the symplectic structure ω_1. Moreover, since ω_s coincides with ω inside U and H vanishes outside U,

$$\rho(a, H; \omega_s) \in \operatorname{Spec}(H, \omega), \quad \text{where } s \in [0, 1],$$

for any $a \in H_*(\Sigma, \mathcal{K})$. Note that in our situation $\operatorname{Spec}(H, \omega_0)$ is just a finite set of $\mathbb{R}$. Then Theorem 1.1 implies that $\{\rho(a, H; \omega_s)\}_{0 \leq t \leq 1}$ is a continuous path over a finite set. So $\rho(a, H; \omega_s)$ is constant for all $s \in [0, 1]$ and then $\rho(a, H; \omega) \leq e^{\omega_1}(U)$. We get the desired conclusion by setting $K = e^{\omega_1}(U)$ in Definition 1.3. $\qquad\square$

Remark 6.1 (i) Corollary 3.3 in [24] only claims an energy–capacity inequality (24) for $a = [\Sigma]$. However, since $[\Sigma]$ is a unit under the quantum product $*$ (here it is just intersection of homology classes), the triangle equality of spectral invariants says

$$\rho(a, H; \omega_1) = \rho(a * [\Sigma], 0 + H; \omega_1) \leq \rho(a, 0; \omega_1) + \rho([\Sigma], H; \omega_1) \leq e^{\omega_1}(U).$$

(ii) Theorem 1.1 only applies locally, so in order to claim that $\{\rho(a, H; \omega_s)\}_{0 \leq t \leq 1}$ is a continuous path, we need to apply Theorem 1.1 inductively. To this end, we can choose a uniform size of the neighborhoods of symplectic structures such that the energy estimation, Proposition 4.3, applies. Then our claim directly follows from the compactness of the interval $[0, 1]$.

(iii) For $\Sigma = S^2$, the argument in the proof of Theorem 1.4 does not apply. Indeed, $s \to \rho([M], H, \omega_s)$ does not provide a function from $[0, 1]$ to $\operatorname{Spec}(H, \omega)$ since the corresponding fundamental classes actually lie in different quantum homologies when ω_s changes.

Before giving the proof of Corollary 1.5, we need to recall equivalent definitions of a heavy subset and a superheavy subset. This is stated as Proposition 4.1 in [6].

Definition 6.2 Given a closed subset $X \subset M$ and $a \in \operatorname{QH}_*(M, \omega)$, if $\zeta_a(H; \omega) = 0$ for any $H \in C^\infty(M)$ such that $H \leq 0$ and $H|_X = 0$, then X is called a–heavy; if $\zeta_a(H; \omega) = 0$ for any $H \in C^\infty(M)$ such that $H \geq 0$ and $H|_X = 0$, then X is called a–superheavy.

Proof of Corollary 1.5 For (a), this is immediate from Definition 6.2. In fact, for any $H \geq 0$ with $H|_X = 0$, it is supported in $\Sigma_g \setminus X$, which is a disjoint union of simply connected regions. By Theorem 1.4, $\rho(a, H, \omega) \leq K$ for some finite $K \geq 0$, so $\rho(a, kH, \omega) < K$ for any $k \in \mathbb{N}$. By the definition of (partial) symplectic quasistates

in Definition 2.10, we know $\zeta_a(H;\omega) = 0$. Therefore, X is a–superheavy. For (b), let U be the simply connected region that X lie in. By Theorem 1.4 and the argument above, for any H supported in U, $\zeta_a(H;\omega) = 0$. Meanwhile, we can always choose H such that $H(x) \geq \delta > 0$ for every $x \in X$. Then Definition 2.11 says that X is not a–heavy. □

Finally, we will study the continuity of some symplectic capacities. Let $A \subset M$. Recall the definitions of $c_{\mathrm{HZ}}^\omega(A)$ and $c_\rho^\omega(A)$:

Definition 6.3 The Hofer–Zehnder capacity $c_{\mathrm{HZ}}^\omega(A)$ is defined as

$$c_{\mathrm{HZ}}^\omega(A) = \sup\{\max H \mid H \in \mathcal{H}(A) \text{ is HZ–admissible}\}.$$

Here $\mathcal{H}(A)$ contains all the autonomous functions on M with compact support in A and $H^{-1}(0)$ and $H^{-1}(\max(H))$ contain nonempty open sets. Moreover, HZ–admissible means that the Hamiltonian flow of H under ω contains no nonconstant periodic orbit of period at most 1. Roughly speaking, it excludes those "fast" Hamiltonian orbits.

Definition 6.4 The spectral capacity $c_\rho^\omega(A)$ is defined as

$$c_\rho^\omega(A) = \sup\{\rho([M], H;\omega) \mid H \in C_c^\infty(\mathbb{R}/\mathbb{Z} \times A)\}.$$

Now, let us give the proof of Theorem 1.7.

Proof of Theorem 1.7 Fix $\epsilon > 0$. By definition, there exists some $H \in C_c^\infty(\mathbb{R}/\mathbb{Z} \times A)$ such that $c_\rho^\omega(A) < \rho([M], H;\omega) + \frac{1}{2}\epsilon$. Under our hypothesis, Theorem 1.1 and Corollary 1.2 state that, near ω, the function $\omega' \to \rho([M], H;\omega')$ is continuous when M satisfies the topological hypothesis in our assumption. In particular, it is lower semicontinuous. Then, for this fixed ϵ, there exists a neighborhood of ω in $\Omega_{\omega,H}$ (so in $\Omega_{\mathrm{closed}}^2(M)$), denoted by $U_\omega(\epsilon)$, such that $\rho([M], H;\omega') - \rho([M], H;\omega) \geq -\frac{1}{2}\epsilon$ for any $\omega' \in U_\omega(\epsilon)$. Hence,

$$c_\rho^{\omega'}(A) - c_\rho^\omega(A) \geq \rho([M], H;\omega') - \rho([M], H;\omega) - \tfrac{1}{2}\epsilon \geq -\epsilon.$$

Therefore, by (2),

$$e^{\omega'}(A) \geq c_\rho^{\omega'}(A) \geq c_\rho^\omega(A) - \epsilon \geq c_{\mathrm{HZ}}^\omega(A) - \epsilon.$$

Thus we get the conclusion. □

Remark 6.5 We can also use another capacity $c_\beta^\omega(A)$ from [27]. It is defined as

$$c_\beta^\omega(A) = \sup\{\beta(H;\omega) \mid H \in C_c^\infty(\mathbb{R}/\mathbb{Z} \times A)\}.$$

Corollary 5.12 in [27] says that $e^\omega(A) \geq \frac{1}{2}c_\beta^\omega(A)$. By the lower semicontinuity of boundary depth from Theorem 1.1 and Corollary 1.2, we can prove that $e^{\omega'}(A)$ is bounded from below by $\frac{1}{2}c_\beta^\omega(A) - \epsilon$ for any $\omega' \in U_\omega(\epsilon)$.

7 Variant Floer chain complexes

7.1 Novikov ring with multifiniteness condition

In this subsection, we will study the Novikov ring with multifiniteness condition defined in Definition 1.8. As explained in the introduction, this is the starting point of comparing Floer chain complexes with different symplectic structures in a general setup. First of all, let us recall the extended version of the Novikov ring considered in [23],

$$\Lambda_\omega = \left\{ \sum_{A \in H_2^S(M)} a_A T^A \;\middle|\; a_A \in \mathcal{K} \text{ and } (\forall C \in \mathbb{R})(\#\{a_A \neq 0 \mid [\omega](A) \leq C\} < \infty) \right\},$$

where $H_2^S(M)$ is the image of $\pi_2(M)$ in $H_2(M;\mathbb{Z})/\text{Tor}$ under the Hurewicz map $\iota\colon \pi_2(M) \to H_2(M;\mathbb{Z})$. By the exact sequence

$$(25) \qquad\qquad 0 \to \ker[\omega] \to H_2^S(M) \xrightarrow{[\omega]} \Gamma_\omega \to 0,$$

where $\Gamma_\omega = \text{Im}[\omega]$, we can write any element $x = \sum_{A \in H_2^S(M)} a_A T^A \in \Lambda_\omega$ as

$$(26) \qquad\qquad x = \sum_{g \in \Gamma} a_g T^g, \quad \text{where } a_g \in \mathcal{K}[\ker[\omega]].$$

In other words, Λ_ω can be rewritten as

$$(27)\quad \Lambda_\omega = \left\{ \sum_{g \in \Gamma_\omega} a_g T^g \;\middle|\; a_g \in \mathcal{K}[\ker[\omega]] \text{ and } (\forall C \in \mathbb{R})(\#\{a_g \neq 0 \mid g \leq C\} < \infty) \right\}.$$

Note that, in general, $\mathcal{K}[\ker[\omega]]$ is not necessarily a PID; therefore, by Theorem 4.2 in [9], Λ_ω is not necessarily a PID. However, since $\mathcal{K}$ is Noetherian, $\mathcal{K}[\ker[\omega]]$ is Noetherian. Compared with the very often used Novikov field $\Lambda^{\mathcal{K},\Gamma}$ defined in (1), there is a natural homomorphism $R_\omega\colon \Lambda_\omega \to \Lambda^{\mathcal{K},\Gamma_\omega}$ defined by

$$(28) \qquad\qquad \sum_{g \in \Gamma_\omega} a_g T^g \xrightarrow{R_\omega} \sum_{g \in \Gamma_\omega} [a_g] T^g,$$

where $[a_g] \in \mathcal{K}$ is defined as follows: if $a_g = \sum_h a_{g,h} S^h$, where $h \in \ker[\omega]$ and S is the formal variable, then $[a_g] = \sum_h a_{g,h}$. In other words, we uniformly weight any $h \in \ker[\omega]$ by the value zero. Then $\Lambda^{\mathcal{K},\Gamma_\omega}$ can be regarded as a Λ_ω–module. The following property of Λ_ω will be useful later:

Lemma 7.1 Λ_ω *is an integral domain.*

Proof First, because $\ker[\omega]$ is a subgroup of $H_2^S(M)$ which is torsion-free and abelian, $\ker[\omega]$ is also torsion-free and abelian. By [2, Proposition 1.3 and Lemma 0.1], $\mathcal{K}[\ker[\omega]]$ is an integral domain. Take two nonzero elements λ_1 and λ_2 in Λ_ω and write them as

$$\lambda_i = \sum_{g_{ij} \in \Gamma_\omega} a_{g_{ij}} T^{g_{ij}}, \quad \text{where } a_{g_{ij}} \in \mathcal{K}[\ker[\omega]] \text{ and } i = 1, 2.$$

Since $\lambda_i \neq 0$, the finiteness condition implies that there exist smallest powers for both $i = 1, 2$. Denote these smallest powers by g_{1j_1} for λ_1 (for some j_1) and g_{2j_2} for λ_2 (by some j_2). Their corresponding coefficients in $\mathcal{K}[\ker[\omega]]$, $a_{g_{1j_1}}$ and $a_{g_{2j_2}}$, are in particular nonzero. Then $a_{g_{1j_1}} \cdot a_{g_{2j_2}} \neq 0$ implies $\lambda_1 \cdot \lambda_2 \neq 0$. Therefore, Λ_ω is an integral domain. $\square$

Recall the construction in Section 1.2. There exist homology classes $[\omega_1], \ldots, [\omega_m]$ in $H^2(M; \mathcal{K})$ forming a polygon $\Delta(\omega)$ containing $[\omega]$ inside such that any reduced perturbation ω' (sufficiently close to ω) has $[\omega']$ written as a convex linear combination as (3). Set $\omega_0 = \omega$. Observe that any such convex linear combination can be inductively constructed from the two-term case

$$[\omega'] = (1-t)[\omega_0] + t[\omega_1].$$

Here, we require $[\omega_0]$ and $[\omega_1]$ to be linearly independent over $\mathcal{K}$ in $H^2(M; \mathcal{K})$. Otherwise, it will be reduced to the rescaling case as studied in Corollary 1.2. This automatically requires that $\dim_{\mathcal{K}} H^2(M; \mathcal{K}) \geq 2$. For simplicity, we will mainly consider this two-term case in the rest of the paper. Moreover, note that ω' and $(1-t)\omega_0 + t\omega_1$ are not necessarily the same. Instead, they differ by an exact 2–form. By Moser's trick, it is easy to deal with the perturbations from exact 2–forms. Therefore, without loss of generality, we will also assume $\omega' = (1-t)\omega_0 + t\omega_1$ for some $t \in [0, 1]$.

Definition 7.2 Fix ω_0 and its reduced perturbation ω_1 such that $[\omega_0]$ and $[\omega_1]$ are linearly independent over $\mathcal{K}$ in $H^2(M;\mathcal{K})$. We call the following $\Lambda_{[0,1]}$ a *Novikov ring with multifiniteness condition*:

$$\Lambda_{[0,1]} = \left\{ \sum_{A \in H_2^S(M)} a_A T^A \,\middle|\, a_A \in \mathcal{K} \text{ and } \right.$$
$$\left. (\forall C \in \mathbb{R})(\forall t \in [0,1])(\#\{a_A \neq 0 \,|\, [\omega_t](A) \leq C\} < \infty) \right\}.$$

By this definition, $\Lambda_{[0,1]} = \bigcap_{t \in [0,1]} \Lambda_{\omega_t}$. Two immediate properties of $\Lambda_{[0,1]}$ follow:

(a) $\Lambda_{[0,1]}$ is nonempty because every finite-length power series, that is polynomial, lies inside.

(b) $\Lambda_{[0,1]}$ is also an integral domain because $\Lambda_{[0,1]}$ is a subring of Λ_{ω_t} for any $t \in [0,1]$ and any subring of an integral domain is also an integral domain (see Lemma 7.1).

In fact, instead of considering uncountably many ω_t for the finiteness condition in Definition 7.2, the following result shows that $\Lambda_{[0,1]}$ has a much easier structure:

Lemma 7.3 $\Lambda_{[0,1]} = \Lambda_{\omega_0} \cap \Lambda_{\omega_1}.$

Proof The inclusion $\Lambda_{[0,1]} \subset \Lambda_{\omega_0} \cap \Lambda_{\omega_1}$ is trivial since the finiteness condition in Definition 7.2 is in particular valid for $t = 0$ and $t = 1$. Now we prove the other inclusion. Take any $x \in \Lambda_{\omega_0} \cap \Lambda_{\omega_1}$, say $x = \sum_{A \in H_2^S(M)} a_A T^A$, that satisfies the finiteness conditions for both ω_0 and ω_1; then, for any $C \in \mathbb{R}$ and for any $t \in (0,1)$, $[\omega_t](A) = (1-t)[\omega_0](A) + t[\omega_1](A) \leq C$ implies that either $(1-t)[\omega_0](A) \leq \frac{1}{2}C$ or $t[\omega_1](A) \leq \frac{1}{2}C$. Therefore,

$$[\omega_0](A) \leq \frac{C}{2(1-t)} \quad \text{or} \quad [\omega_1](A) \leq \frac{C}{2t}.$$

The defining property of the element x implies that, in either case, there are only finitely many A's. Therefore, x also satisfies the finiteness condition in Definition 7.2. □

Example 7.4 Take $x = \sum_{n=0}^{\infty} T^{nA}$, where $A \in \ker[\omega_1] \setminus \ker[\omega_0]$. This is an element in Λ_{ω_0} but not in $\Lambda_{[0,1]}$. In general, $\Lambda_{[0,1]}$ is strictly contained in Λ_{ω_t} for any $t \in [0,1]$. Moreover, the following computation shows that $\Lambda_{[0,1]}$ does not act on $\Lambda_{\omega_0} \setminus \Lambda_{[0,1]}$. Taking x as above,

$$(1 - T^A)x = (1 - T^A)\left(\sum_{n=0}^{\infty} T^{nA} \right) = 1 \in \Lambda_{[0,1]}.$$

This is different from the observation in the proof of Theorem 2.5 in [23] that for any proper subgroup $G \leq \Gamma_\omega$, $\Lambda^{\mathcal{K},G}$ acts on $\Lambda^{\mathcal{K},\Gamma_\omega \backslash G}$.

Remark 7.5 Definition 7.2 can be easily generalized to higher-dimensional situations. Assume that $\Delta(\omega)$ is a polygon in $H^2(M;\mathcal{K})$ that contains $[\omega]$ and has vertices $[\omega_1], \ldots, [\omega_m]$. Define

$$\Lambda_{\Delta(\omega)} = \left\{ \sum_{A \in H_2^S(M)} a_A T^A \,\middle|\, \begin{array}{l} a_A \in \mathcal{K} \text{ and} \\ (\forall C \in \mathbb{R})(\forall [\omega'] \in \Delta(\omega))(\#\{a_A \neq 0 \,|\, [\omega'](A) \leq C\} < \infty) \end{array} \right\}.$$

In particular, $\Lambda_{\Delta(\omega)}$ is an integral domain. Moreover, $\Lambda_{\Delta(\omega)} = \Lambda_\omega \cap \Lambda_{\omega_1} \cap \cdots \cap \Lambda_{\omega_m}$.

Since we have inclusions $i_0 \colon \Lambda_{[0,1]} \to \Lambda_{\omega_0}$ and $i_1 \colon \Lambda_{[0,1]} \to \Lambda_{\omega_1}$, both Λ_{ω_0} and Λ_{ω_1} are $\Lambda_{[0,1]}$–modules. Together with (28), we have the coefficient extensions

$$
\begin{array}{ccc}
& & \Lambda_{\omega_0} \xrightarrow{R_{\omega_0}} \Lambda^{\mathcal{K},\Gamma_{\omega_0}} \\
& \nearrow^{i_0} & \\
\Lambda_{\omega_0} \cap \Lambda_{\omega_1} = \Lambda_{[0,1]} & & \\
& \searrow_{i_1} & \\
& & \Lambda_{\omega_1} \xrightarrow{R_{\omega_1}} \Lambda^{\mathcal{K},\Gamma_{\omega_1}}
\end{array}
$$

Accordingly, we have variant versions of quantum homologies

$$
\begin{array}{ccc}
& \nearrow^{\otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_0}} & \widetilde{\mathrm{QH}}_0 \xrightarrow{\otimes_{\Lambda_{\omega_0}} \Lambda^{\mathcal{K},\Gamma_{\omega_0}}} \mathrm{QH}_0 \\
\mathrm{QH}_{[0,1]} & & \\
& \searrow_{\otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_1}} & \widetilde{\mathrm{QH}}_1 \xrightarrow{\otimes_{\Lambda_{\omega_1}} \Lambda^{\mathcal{K},\Gamma_{\omega_1}}} \mathrm{QH}_1
\end{array}
$$

Here the notation is defined as follows:

- $\mathrm{QH}_{[0,1]} = H_*(M;\mathcal{K}) \otimes_{\mathcal{K}} \Lambda_{[0,1]}$.
- $\widetilde{\mathrm{QH}}_i = \mathrm{QH}_{[0,1]} \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i}$ for $i = 0, 1$.
- $\mathrm{QH}_i = \widetilde{\mathrm{QH}}_i \otimes_{\Lambda_{\omega_i}} \Lambda^{\mathcal{K},\Gamma_{\omega_i}}$ for $i = 0, 1$.

To end this subsection, we consider a similar short exact sequence as (25) but with multivaluation from ω_0 and ω_1. Explicitly, consider the short exact sequence

$$0 \to \ker[\omega_0] \cap \ker[\omega_1] \to H_2^S(M) \xrightarrow{[\omega_0] \times [\omega_1]} \Gamma_{\omega_0} \times \Gamma_{\omega_1} \to 0.$$

This allows us to identify any $x = \sum_{A \in H_2^S(M)} a_A T^A \in \Lambda_{[0,1]}$ with

$$(29) \quad x = \sum_{(g_0,g_1) \in \Gamma_{\omega_0} \times \Gamma_{\omega_1}} a_{(g_0,g_1)} T^{(g_0,g_1)}, \quad \text{where } a_{(g_0,g_1)} \in \mathcal{K}[\ker[\omega_0] \cap \ker[\omega_1]].$$

In other words, each $x \in \Lambda_{[0,1]}$ can be identified with a set of points of $\mathbb{R}^2$ in coordinates g_0, g_1. Moreover, by the multifiniteness condition, this set is discrete in $\mathbb{R}^2$.

7.2 Floer chain complex with multifiniteness condition

Given a Hamiltonian system (M, ω, H), set $\omega_0 = \omega$ and its reduced perturbation ω_1. Similarly to the construction of a Floer chain complex, in this subsection we will construct a variant Floer chain complex over $\Lambda_{[0,1]}$. Define a graded finitely generated free $\Lambda_{[0,1]}$–module $(\mathrm{CF}_{[0,1]})_*$ as

$$(30) \quad (\mathrm{CF}_{[0,1]})_k := \bigoplus_{i=1}^n \Lambda_{[0,1]} \langle [x_i, w_i] \rangle \simeq \bigoplus_{i=1}^n \Lambda_{[0,1]},$$

where n is the number of contractible Hamiltonian 1–periodic orbits $x_i \in \mathrm{Per}(\omega_0, H)$ with CZ–index equal to k and w_i is a disk spanning x_i. Note that (30) is well defined due to Proposition 3.4. Moreover, Lemma 7.3 implies

$$(31) \quad (\mathrm{CF}_{[0,1]})_k = (\widetilde{\mathrm{CF}}_0)_k \cap (\widetilde{\mathrm{CF}}_1)_k = \bigcap_{t \in [0,1]} (\widetilde{\mathrm{CF}}_t)_k$$

for each degree $k \in \mathbb{Z}$. Here $\widetilde{\mathrm{CF}}_t$ is a free Λ_{ω_t}–module for any $t \in [0,1]$.

In order to form a chain complex, we need to choose a boundary operator on $(\mathrm{CF}_{[0,1]})_*$. For any $s \in [0,1]$, let $\omega_s = (1-s)\omega_0 + s\omega_1$, and simply denote by ∂_s the standard Floer boundary operator of $(\mathrm{CF}_*(M, J, H, \omega_s), \partial_{J,H,\omega_s})$. The following proposition shows that we have a family of boundary operators for $(\mathrm{CF}_{[0,1]})_*$.

Proposition 7.6 *For any $s \in [0,1]$, ∂_s is well defined on $(\mathrm{CF}_{[0,1]})_*$ and satisfies $\partial_s^2 = 0$.*

Proof Since ∂_s is already a well-defined boundary operator for the Floer chain complex $(\mathrm{CF}_*(M, J, H, \omega_s), \partial_s)$, the algebraic relation ∂_s^2 holds. In order to show that ∂_s is well defined on $(\mathrm{CF}_{[0,1]})_*$, we need to show that for any $t \in [0,1]$, the output of ∂_s satisfies the finiteness condition in terms of ω_t. Suppose that for a *basis element* $[x, w]$,

$$(32) \quad \partial_s([x, w]) = \sum_{[y,v] \in \text{basis}} \sum_{A \in H_2^S(M)} n_A T^A [y, v].$$

By definition, there exists a Floer trajectory u with respect to the symplectic structure ω_s connecting $[x, w]$ and $[y, v \# A]$. We claim that there exists some constant $C'_{s,t} < 1$ such that

(33) $$\mathcal{A}_{\omega_t}(T^A[y, v]) - \mathcal{A}_{\omega_t}([x, w]) \leq -(1 - C'_{s,t}) E(u).$$

In fact, first we have $\mathcal{A}_{\omega_s}(T^A[y, v]) - \mathcal{A}_{\omega_s}([x, w]) = -E(u)$. Here $E(u)$ is the energy of Floer trajectory under symplectic structure ω_s. By the definition of the symplectic action functional with respect to ω_t,

$$
\begin{aligned}
\mathcal{A}_{\omega_t}([x, w]) &= -\int_{D^2} w^* \omega_t + \int_0^1 H(x(t), t)\, dt \\
&= -\int_{D^2} w^* \omega_s + \int_0^1 H(x(t), t)\, dt + \int_{D^2} w^*(\omega_s - \omega_t) \\
&= \mathcal{A}_{\omega_s}([x, w]) + (s - t) \int_{D^2} w^* \alpha,
\end{aligned}
$$

where $\alpha = \omega_1 - \omega_0$. Similarly,

$$\mathcal{A}_{\omega_t}(T^A[y, v]) = \mathcal{A}_{\omega_s}(T^A[y, v]) + (s - t) \int_{D^2} (v \# A)^* \alpha.$$

Therefore, one gets

$$\mathcal{A}_{\omega_t}(T^A[y, v]) - \mathcal{A}_{\omega_t}([x, w]) = -E(u) + (s - t)\left(\int_{D^2} (v \# A)^* \alpha - \int_{D^2} w^* \alpha \right).$$

By our equivalence relation, $\int_{D^2} (v \# A)^* \alpha = \int_{D^2} (w \# u)^* \alpha$. Hence, by (17), there exists a constant C such that

$$\int_{D^2} (v \# A)^* \alpha - \int_{D^2} w^* \alpha = \int_{D^2} (w \# u)^* \alpha - \int_{D^2} w^* \alpha = \int_{\mathbb{R} \times S^1} u^* \alpha \leq C |\alpha| E(u).$$

Meanwhile, as long as ω_1 is a priori chosen sufficiently close to ω_0, one gets $C |\alpha| < 1$. Therefore, by setting $C'_{s,t} = |s - t| C |\alpha|$, we get the claim (33).

Next, (33) can be rewritten as

$$-\int_{S^2} A^* \omega_t - N_2 \leq -(1 - C'_{s,t}) E(u),$$

where $N_2 = \mathcal{A}_{\omega_t}([x, w]) - \mathcal{A}_{\omega_t}([y, v])$. This is a constant that is independent of the sphere class A. Since $1 - C'_{s,t} > 0$,

$$E(u) \leq \frac{1}{1 - C'_{s,t}} \left(\int_{S^2} A^* \omega_t + N_2 \right).$$

If $\int_{S^2} A^* \omega_t < \lambda$ for $\lambda \in \mathbb{R}$, then

$$E(u) \leq \frac{\lambda + N_2}{1 - C'_{s,t}} < \infty.$$

By the finiteness condition of ω_s, there are only finitely many such sphere classes A. Hence, (32) also satisfies the finiteness condition of ω_t. $\qquad\square$

For each $((\mathrm{CF}_{[0,1]})_*, \partial_s)$, we can associate a filtration function ℓ_{ω_s} using the symplectic action functional $\mathcal{A}_{\omega_s}$ together with the valuation on $\Lambda_{[0,1]}$ with respect to ω_s. Explicitly, for any chain $c = \sum_i \lambda_i [\gamma_i, w_i] \in (\mathrm{CF}_{[0,1]})_*$ with $\lambda_i \in \Lambda_{[0,1]}$,

$$(34) \qquad \ell_{\omega_s}(c) = \max_i \{\mathcal{A}_{H,\omega_s}([x_i, w_i]) - \nu_{\omega_s}(\lambda_i)\},$$

where $\nu_{\omega_s}(\lambda_i)$ denotes the minimal exponent from λ_i after they are evaluated by the symplectic structure ω_s. For brevity, denote ℓ_{ω_s} as ℓ_s. Proposition 7.6 says that there exists a family of variant version of Floer chain complexes $\{((\mathrm{CF}_{[0,1]})_*, \partial_s, \ell_s)\}_{s \in [0,1]}$ which is parametrized by $[0,1]$. It is important to use the filtration function ℓ_s and the boundary operator ∂_s together so that this boundary operator strictly decreases the filtrations. In general, we get a family of Floer chain complexes

$$\{(\mathrm{CF}_{\Delta(\omega)}, \partial_{\omega'}, \ell_{\omega'})\}_{\omega' \in \Delta(\omega)}.$$

A recent paper [13, Theorem 3.12] provides a similar construction of a family of Floer-style chain complexes.

Next, we show that there exists an algebraic relation between any two slices of these variant Floer chain complexes.

Proposition 7.7 *For any $s, t \in [0,1]$, the two complexes $((\mathrm{CF}_{[0,1]})_*, \partial_s, \ell_s)$ and $((\mathrm{CF}_{[0,1]})_*, \partial_t, \ell_t)$ are chain homotopy equivalent.*

Remark 7.8 Different from Lemma 5.1, we can not get any quantitative comparison conclusion in Proposition 7.7. From the proof given below, one can see that this comes from the "non-uniform" estimation from Proposition 4.4, ie the bound depends on the symplectic area of the disk spanning the Hamiltonian 1–periodic orbit at the asymptotic end $s = -\infty$. On the other hand, one can view the family $\{((\mathrm{CF}_{[0,1]})_*, \partial_s, \ell_s)\}_{s \in [0,1]}$ from a different perspective. Since each s–slice $((\mathrm{CF}_{[0,1]})_*, \partial_s, \ell_s)$ provides a persistence module [1], this family provides a 2–dimensional persistence module. Continuity questions studied in this paper might be transferred into a stability problem of the invariants constructed from a higher-dimensional persistence module.

Proof of Proposition 7.7 We need to find a quadruple $(\Phi_{s,t}, \Phi_{t,s}, K_s, K_t)$ such that $\Phi_{s,t}$ and $\Phi_{t,s}$ are chain maps between $(\mathrm{CF}_{[0,1]})_*, \partial_s, \ell_s)$ and $(\mathrm{CF}_{[0,1]})_*, \partial_t, \ell_t)$ and K_s and K_t are homotopies. First, choose a homotopy from ω_s to ω_t as in (12) parametrized by τ, and consider the Floer operator $\mathcal{F}^\tau$ defined in Definition 4.1. For any basis element $[x, w]$, define

$$(35) \qquad \Phi_{s,t}([x, w]) = \sum_{\substack{[y,v]\in\text{basis} \\ \mu_{\mathrm{CZ}}([x,w])=\mu_{\mathrm{CZ}}([y,v\#A])}} \sum_{A\in H_2^S(M)} n_A T^A [y, v],$$

where n_A counts the number of admissible $\mathcal{F}^\tau$–trajectories connecting $[x, w]$ and $[y, v \# A]$. We know that $\Phi_{s,t}$ is a chain map by the standard gluing argument. In order to show $\Phi_{s,t}$ acts on $(\mathrm{CF}_{[0,1]})_*$, we need to check that the output of $\Phi_{s,t}$ satisfies the finiteness condition of ω_r for any $r \in [0, 1]$. Without loss of generality, assume $s < t$. By Proposition 4.4, there exists a constant $C_{s,t}$ such that

$$\mathcal{A}_{\omega_t}(T^A[y, v]) - \mathcal{A}_{\omega_s}([x, w]) \le -(1 - C_{s,t}|\alpha|)E(u) + (s - t)\int_{D^2} w^*\alpha.$$

Meanwhile, evaluate $T^A[y, v]$ by $\mathcal{A}_{\omega_r}$ and one gets the relation

$$\mathcal{A}_{\omega_t}(T^A[y, v]) = \mathcal{A}_{\omega_r}(T^A[y, v]) + (r - t)\int_{S^1\times\mathbb{R}} u^*\alpha + (r - t)\int_{D^2} w^*\alpha,$$

where $\alpha = \omega_1 - \omega_0$. By (17) there exists some constant C such that

$$\mathcal{A}_{\omega_r}(T^A[y, v]) - \mathcal{A}_{\omega_s}([x, w])$$
$$\le -(1 - C_{s,t}|\alpha|)E(u) + (t - r)\int_{S^1\times\mathbb{R}} u^*\alpha + (s - r)\int_{D^2} w^*\alpha$$
$$\le -(1 - C_{s,t}|\alpha|)E(u) + C|\alpha| \cdot (t - r)E(u) + (s - r)\int_{D^2} w^*\alpha$$
$$\le -(1 - C'_{r,s,t})E(u) + (s - r)\int_{D^2} w^*\alpha$$

for some constant $C'_{r,s,t} = (C_{s,t} + C \cdot (t - r))|\alpha| < 1$. Also

$$\mathcal{A}_{\omega_r}(T^A[y, v]) - \mathcal{A}_{\omega_s}([x, w]) = -\int_{S^2} A^*\omega_r - N_3,$$

where $N_3 = \mathcal{A}_{\omega_s}([x, w]) - \mathcal{A}_{\omega_r}([y, v])$, independent of the sphere class A. Therefore,

$$E(u) \le \frac{1}{1 - C'_{r,s,t}}\left(\int_{S^2} A^*\omega_r + N_3 + (s - r)\int_{D^2} w^*\alpha\right).$$

If $\int_{S^2} A^* \omega_r \leq \lambda$ for any given $\lambda \in \mathbb{R}$, then, since $1 - C'_{r,s,t} > 0$ for any $r \in [0,1]$,

$$E(u) \leq \frac{\lambda + N_3 + (s-r) \int_{D^2} w^* \alpha}{1 - C'_{r,s,t}} < \infty.$$

By the Gromov compactness theorem, there are only finitely many sphere classes A. Hence, the output of (35) also satisfies the finiteness condition of ω_r for any $r \in [0,1]$. Symmetrically, we can define $\Phi_{t,s}$ and prove it is a well-defined chain map.

Second, $\Phi_{t,s} \circ \Phi_{s,t}$ is (Floer) homotopic to $\mathbb{1}_s$ on $((CF_{[0,1]})_*, \partial_s)$, by the standard Floer theory. Choose a homotopy between a symmetric homotopy of ω_s passing through ω_t and the identity homotopy ω_s. We can construct a map K_s as in Lemma 5.1 as

$$(36) \qquad K_s([x,w]) = \sum_{\substack{[y,v] \in \text{basis} \\ \mu_{\text{CZ}}([x,w])+1 = \mu_{\text{CZ}}([y,v\#A])}} \sum_{A \in H_2^S(M)} n_A T^A [y,v],$$

where n_A counts the number of pairs (u, λ) where u is an admissible $\mathcal{F}_{\text{sym}}^{\tau,\lambda}$–trajectory connecting $[x,w]$ and $[y, v \# A]$. Again, in order to show K_s acts on $(CF_{[0,1]})_*$, we need to check that the output of K_s satisfies the finiteness condition of ω_r for any $r \in [0,1]$. By Proposition 4.6 (applied to $\omega_0 = \omega_s$ and $\omega_1 = \omega_t$),

$$\mathcal{A}_{\omega_s}(T^A[y,v]) - \mathcal{A}_{\omega_s}([x,w]) \leq -(1 - C_{s,t}|\alpha|) E(u).$$

Evaluate $T^A[y,v]$ by $\mathcal{A}_{\omega_r}$, and one gets the relation

$$\mathcal{A}_{\omega_s}(T^A[y,v]) = \mathcal{A}_{\omega_r}(T^A[y,v]) + (r-s) \int_{S^1 \times \mathbb{R}} u^* \alpha + (r-s) \int_{D^2} w^* \alpha.$$

Hence, by a similar computation as above,

$$(37) \qquad \mathcal{A}_{\omega_r}(T^A[y,v]) - \mathcal{A}_{\omega_s}([x,w]) \leq -(1 - C_{s,r}|\alpha|) E(u) + (s-r) \int_{D^2} w^* \alpha.$$

Then the same argument as above implies the finiteness condition of ω_r. Thus we get the conclusion. $\qquad\square$

7.3 Revised Floer homology

Denote by $CM_{[0,1]}$ the coefficient extension of Morse chain complex $CM_*(M; \mathbb{Z})$ over the ring $\Lambda_{[0,1]}$. In the following diagram, let us summarize the relations between

the various chain complexes that we have encountered so far:

$$
\begin{array}{ccccc}
& & (\mathrm{CF}_{[0,1]}, \partial_0) & \xrightarrow{\iota_0} & (\widetilde{\mathrm{CF}}_0, \partial_0) & \xrightarrow{R_0} & (\mathrm{CF}_*(M, H, J, \omega_0), \partial_0) \\
& \overset{\mathrm{PSS}_0}{\nearrow} & \big\downarrow{\scriptstyle \Phi_{0,t}} & & \big| \text{ no well-defined chain map} \\
(\mathrm{CM}_{[0,1]}, \partial_{\mathrm{Morse}}) & \xrightarrow{\mathrm{PSS}_t} & (\mathrm{CF}_{[0,1]}, \partial_t) & \xrightarrow{\iota_t} & (\widetilde{\mathrm{CF}}_t, \partial_t) \\
& & \big\downarrow & & \big\downarrow \\
& & (\mathrm{CF}_{[0,1]}, \partial_1) & \xrightarrow{\iota_1} & (\widetilde{\mathrm{CF}}_1, \partial_1) & \xrightarrow{R_1} & (\mathrm{CF}_*(M, H, J, \omega_1), \partial_1) \\
& & \big\downarrow{\scriptstyle \wr} \\
& & \Lambda_{[0,1]}\text{–module}
\end{array}
$$

where $\widetilde{\mathrm{CF}}_t = \mathrm{CF}_{[0,1]} \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_t}$ is a free Λ_{ω_t}–module for any $t \in [0,1]$. Take the homology of each chain complex, and one gets the picture

$$
\begin{array}{ccccc}
& & \mathrm{HF}_{[0,1],0} & \xrightarrow{(\iota_0)_*} & \widetilde{\mathrm{HF}}_0 & \xrightarrow{(R_0)_*} & \mathrm{HF}_{\omega_0} \\
& \overset{(\mathrm{PSS}_0)_*}{\nearrow} & \big\downarrow{\scriptstyle (\Phi_{0,t})_*} & & \big| \text{ no well-defined map} \\
\mathrm{QH}_{[0,1]} & \xrightarrow{(\mathrm{PSS}_t)_*} & \mathrm{HF}_{[0,1],t} & \xrightarrow{(\iota_t)_*} & \widetilde{\mathrm{HF}}_t \\
& & \big\downarrow & & \big\downarrow \\
& & \mathrm{HF}_{[0,1],1} & \xrightarrow{(\iota_1)_*} & \widetilde{\mathrm{HF}}_1 & \xrightarrow{(R_1)_*} & \mathrm{HF}_{\omega_1} \\
& & \big\downarrow{\scriptstyle \wr} \\
& & \Lambda_{[0,1]}\text{–module}
\end{array}
$$

(38)

where $\mathrm{QH}_{[0,1]} = H_*(M; \mathcal{K}) \otimes \Lambda_{[0,1]}$. Moreover, $\mathrm{HF}_{[0,1],t}$ is the homology of the Floer chain complex $((\mathrm{CF}_{[0,1]})_*, \partial_t)$, $\widetilde{\mathrm{HF}}_t$ is the homology of the Floer chain complex $((\widetilde{\mathrm{CF}}_t)_*, \partial_t)$ for any $t \in [0,1]$ and HF_{ω_i} is the homology of the Floer chain complex $(\mathrm{CF}_*(M, H, J, \omega_i), \partial_i)$ for $i = 0, 1$.

By Proposition 7.7, $\mathrm{HF}_{[0,1],t}$ are all isomorphic to each other. Moreover, one can show that for any $t \in [0,1]$,

$$
(\Phi_{0,t})_* \circ (\mathrm{PSS}_0)_* = (\mathrm{PSS}_t)_*.
$$

A natural question is how these Floer homologies change when we extend the coefficients in each step. First, the universal coefficient theorem [20, Corollary 7.56(ii) and

Theorem 7.15] says that, for each degree $k \in \mathbb{Z}$, we have the splitting

$$(39) \quad H_k(\widetilde{\mathrm{CF}}_i; \Lambda_{\omega_i}) \simeq H_k(\mathrm{CF}_{[0,1]}; \Lambda_{[0,1]}) \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i} \oplus \mathrm{Tor}^{\Lambda_{[0,1]}}(H_{k-1}(\mathrm{CF}_{[0,1]}), \Lambda_{\omega_i}),$$

where $\mathrm{Tor}^{\Lambda_{[0,1]}}(H_{k-1}(\mathrm{CF}_{[0,1]}), \Lambda_{\omega_i})$ is a torsion module over $\Lambda_{[0,1]}$. It is not easy to see the algebraic relation between $\Lambda_{[0,1]}$ and Λ_{ω_i} if we try to apply some well-known fact, such as that a module over a PID is flat if and only if it is torsion-free (by Lemma 7.1, we only know $\Lambda_{[0,1]}$ is a domain). Fortunately, we still have the following property claiming that the torsion part vanishes, due to the existence of PSS–maps who transfer our discussion back to the Morse homology.

Proposition 7.9 *For any $k \in \mathbb{Z}$,*

$$H_k(\widetilde{\mathrm{CF}}_i; \Lambda_{\omega_i}) \simeq H_k(\mathrm{CF}_{[0,1]}; \Lambda_{[0,1]}) \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i}$$

for $i = 0, 1$, or simply $\widetilde{\mathrm{HF}}_i \simeq \mathrm{HF}_{[0,1]} \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i}$. In particular,

$$\mathrm{rank}_{\Lambda_{\omega_0}} H_k(\widetilde{\mathrm{CF}}_0; \Lambda_{\omega_0}) = \mathrm{rank}_{\Lambda_{\omega_1}} H_k(\widetilde{\mathrm{CF}}_1; \Lambda_{\omega_1}).$$

Proof Recall that $\mathrm{CM}_{[0,1]} = \mathrm{CM}(M; \mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda_{[0,1]}$. By the universal coefficient theorem,

$$H_k(\mathrm{CM}_{[0,1]}; \Lambda_{[0,1]}) \simeq H_k(\mathrm{CM}) \otimes_{\mathbb{Z}} \Lambda_{[0,1]} \oplus \mathrm{Tor}^{\mathbb{Z}}(H_{k-1}(\mathrm{CM}_{[0,1]}), \Lambda_{[0,1]}),$$

where $H_*(\mathrm{CM}) := H_*(\mathrm{CM}; \mathbb{Z})$. By Lemma 7.1, $\Lambda_{[0,1]}$ is an integral domain, so it is torsion-free as a $\mathbb{Z}$–module, which implies the flatness since $\mathbb{Z}$ is a PID. Therefore, the Tor functor vanishes, that is,

$$(40) \qquad\qquad H_*(\mathrm{CM}) \otimes_{\mathbb{Z}} \Lambda_{[0,1]} \simeq H_*(\mathrm{CM}_{[0,1]}; \Lambda_{[0,1]}).$$

By the same argument,

$$(41) \qquad\qquad H_*(\mathrm{CM}) \otimes_{\mathbb{Z}} \Lambda_{\omega_i} \simeq H_*(\mathrm{CM} \otimes_{\mathbb{Z}} \Lambda_{\omega_i}; \Lambda_{\omega_i}).$$

Together, we get the relations

$$\begin{aligned} H_*(\mathrm{CM}) \otimes_{\mathbb{Z}} \Lambda_{\omega_i} &= (H_*(\mathrm{CM}) \otimes_{\mathbb{Z}} \Lambda_{[0,1]}) \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i} \\ &\simeq H_*(\mathrm{CM}_{[0,1]}; \Lambda_{[0,1]}) \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i}. \end{aligned}$$

On the other hand, similarly to (39), we have

$$H_*(\mathrm{CM}_i; \Lambda_{\omega_i}) \simeq H_*(\mathrm{CM}_{[0,1]}; \Lambda_{[0,1]}) \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i} \oplus \mathrm{Tor}^{\Lambda_{[0,1]}}(H_{*-1}(\mathrm{CM}_{[0,1]}), \Lambda_{\omega_i}),$$

where $\mathrm{CM}_i = \mathrm{CM}_{[0,1]} \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i}$. Consider the commutative diagram

$$
\begin{array}{l}
(H_*(\mathrm{CM}) \otimes_{\mathbb{Z}} \Lambda_{[0,1]}) \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i} \\[4pt]
\qquad\Big\downarrow{\scriptstyle j} \\[4pt]
{\scriptstyle t}\quad H_*(\mathrm{CM}) \otimes_{\mathbb{Z}} \Lambda_{\omega_i} \xrightarrow{\ q\ } H_*(\mathrm{CM} \otimes_{\mathbb{Z}} \Lambda_{\omega_i}; \Lambda_{\omega_i}) \\[4pt]
\qquad\Big\downarrow{\scriptstyle s} \qquad\qquad\qquad\qquad \Big\downarrow{\scriptstyle f} \\[4pt]
H_*(\mathrm{CM}_{[0,1]}; \Lambda_{[0,1]}) \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i} \xrightarrow{\ i\ } H_*(\mathrm{CM}_i; \Lambda_{\omega_i}) \to \mathrm{Tor}^{\Lambda_{[0,1]}}(H_{*-1}(\mathrm{CM}_{[0,1]}), \Lambda_{\omega_i}) \\[4pt]
\qquad\Big\downarrow{\scriptstyle g} \qquad\qquad\qquad\qquad \Big\downarrow{\scriptstyle h} \\[4pt]
H_*(\mathrm{CF}_{[0,1]}; \Lambda_{[0,1]}) \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i} \xrightarrow{\ p\ } H_*(\mathrm{CF}_i; \Lambda_{\omega_i}) \to \mathrm{Tor}^{\Lambda_{[0,1]}}(H_{*-1}(\mathrm{CF}_{[0,1]}), \Lambda_{\omega_i})
\end{array}
$$

In this diagram,

- f is an identity map because $\mathrm{CM}_i = \mathrm{CM} \otimes_{\mathbb{Z}} \Lambda_{\omega_i}$;
- q is an isomorphism because of (41);
- g and h are PSS–maps (see [18]), so isomorphisms;
- j is an identity map due to the extension of coefficients;
- t is an isomorphism because of (40).

Therefore, t is an isomorphism, which implies that s is an isomorphism. This implies i is an isomorphism, and then p is an isomorphism. $\qquad\square$

Finally, since $\Lambda^{\mathcal{K},\Gamma_{\omega_i}}$ is a field, any torsion over $\Lambda^{\mathcal{K},\Gamma_{\omega_i}}$ always vanishes. Hence, one gets the following result:

Corollary 7.10 *For any* $k \in \mathbb{Z}$,

$$
H_k(\mathrm{CF}(M, J, H, \omega_i); \Lambda^{\mathcal{K},\Gamma_{\omega_i}}) \simeq H_k(\widetilde{\mathrm{CF}}_i; \Lambda_{\omega_i}) \otimes_{\Lambda_{\omega_i}} \Lambda^{\mathcal{K},\Gamma_{\omega_i}}
$$

for $i = 0, 1$, *or simply* $\mathrm{HF}_{\omega_i} = \widetilde{\mathrm{HF}}_{[0,1]} \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i}$. *In particular,*

$$
\mathrm{rank}_{\Lambda^{\mathcal{K},\Gamma_{\omega_0}}} (\mathrm{HF}_{\omega_0})_k = \mathrm{rank}_{\Lambda^{\mathcal{K},\Gamma_{\omega_1}}} (\mathrm{HF}_{\omega_1})_k.
$$

8 Variant spectral invariants; proof of Theorem 1.9

Fix any class $a \in \mathrm{QH}_{[0,1]}$. Using $(\mathrm{PSS}_t)_*$, one gets a class $(\mathrm{PSS}_t)_*(a)$ in $\mathrm{HF}_{[0,1],t}$. Recall that for each $t \in [0,1]$, $(\mathrm{CF}_{[0,1]}, \partial_t, \ell_t)$ is a filtered complex with respect to the symplectic structure ω_t.

Definition 8.1 Given a Hamiltonian system (M, ω, H), set $\omega_0 = \omega$, and ω_1 is a reduced perturbation of ω_0. For $a \in \mathrm{QH}_{[0,1]}$, we call the following value the t–*spectral invariant* associated to a:

$$\rho_t(a, H) = \inf\{\ell_t(\alpha_t) \mid [\alpha_t] = (\mathrm{PSS}_t)_*(a)\},$$

where $\alpha_t \in (\mathrm{CF}_{[0,1]}, \partial_t)$.

Recall that for each $t \in [0, 1]$, there is a well-defined spectral invariant $\widetilde{\rho}_t(a, H)$ over the coefficient ring Λ_{ω_t} for any $a \in \mathrm{QH}_{[0,1]}$. This is the abstract spectral invariant defined in [23]. Meanwhile, $\rho(a, H; \omega_t)$ denotes the standard spectral invariant over the Novikov *field* $\Lambda^{K, \Gamma_\omega}$. It is readily to see that $\widetilde{\rho}_t(a, H) = \rho(H, a; \omega_t)$ for any $t \in [0, 1]$ and any $a \in \mathrm{QH}_{[0,1]}$. On the other hand, we have the following important property:

Lemma 8.2 Let $t \in [0, 1]$ and a nondegenerate Hamiltonian $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M)$.

(1) **Finiteness** For any nonzero $a \in \mathrm{QH}_{[0,1]}$, $\rho_t(a, H) > -\infty$.

(2) **Realization** For any nonzero $a \in \mathrm{QH}_{[0,1]}$, there exists an $\alpha_t \in \mathrm{CF}_{[0,1]}$ such that

$$\rho_t(a, H) = \ell_t(\alpha_t).$$

(3) **Extension** For any $a \in \mathrm{QH}_{[0,1]}$, $\rho_t(a, H) = \widetilde{\rho}_t(a, H)$.

Note that Lemma 8.2 reduces the comparison between $\rho(a, H; \omega_0)$ and $\rho(a, H; \omega_1)$ to the comparison between $\rho_0(a, H)$ and $\rho_1(a, H)$, which has the advantage that they are over the same coefficient ring $\Lambda_{[0,1]}$.

Proof (1) Regard $a = a \otimes \mathbb{1}$ as an element in $\widetilde{\mathrm{QH}}_t$ over the coefficient ring Λ_{ω_t}, still nonzero. By Theorem 1.3 in [23], we know $\widetilde{\rho}_t(a, H) > -\infty$. On the other hand, by definition, for any $\epsilon > 0$ there exists some $\alpha_t \in (\mathrm{CF}_{[0,1]}, \partial_t)$ representing $[\alpha_t] = (\mathrm{PSS}_t)_*(a)$ such that

$$\ell_t(\alpha_t) \leq \rho_t(a, H) + \epsilon.$$

Then, in $\widetilde{\mathrm{CF}}_t$ over the coefficient ring Λ_{ω_t}, $\alpha_t = \alpha_t \otimes \mathbb{1}$ also represents $(\mathrm{PSS}_t)_*(a)$. By definition, $\widetilde{\rho}_t(a, H) \leq \ell_t(\alpha_t) \leq \rho_t(a, H) + \epsilon$. So $\rho_t(a, H) > -\infty$. Therefore, we get the conclusion (1).

(2)–(3) The same argument works for t–spectral invariants for any $t \in [0, 1]$, so we only prove the case for $t = 0$. Since a is a nonzero element in $\mathrm{QH}_{[0,1]}$, there exists

some chain $\alpha \in \mathrm{CF}_{[0,1]}$ such that $(\mathrm{PSS}_0)_*(a) = [\alpha]$. On the other hand, viewing a as an element in $\widetilde{\mathrm{QH}}_0$ over the coefficient ring Λ_{ω_0}, Theorem 1.4 in [23] says that there exists an optimal boundary $\partial_0 \tilde{y}$ in the sense that

$$\tilde{\rho}_0(a, H) = \ell_0(\alpha - \partial_0 \tilde{y}),$$

where $\tilde{y} \in \widetilde{\mathrm{CF}}_0$ (but not necessarily in $\mathrm{CF}_{[0,1]}$). Decompose y into the two parts

$$(42) \qquad\qquad \tilde{y} = y_f + y_*,$$

where $\ell_0(y_*) \leq \tilde{\rho}_0(a, H)$. Then, by the finiteness condition, the subchain y_f contains only finitely many terms. Also,

$$\ell_0(\alpha - \partial_0 y_f) = \ell_0(\alpha - \partial_0 \tilde{y} + \partial_0 y_*) = \ell_0(\alpha - \partial_0 \tilde{y}),$$

where the final equality comes from the relation $\ell_0(\partial_0 y_*) < \tilde{\rho}_0(a, H) = \ell_0(\alpha - \partial_0 \tilde{y})$. In other words, $\partial_0 y_f$ is also an optimal boundary for the spectral invariant $\tilde{\rho}_0(a, H)$. Since y_f has only finitely many terms, certainly $y_f \in \mathrm{CF}_{[0,1]}$. Proposition 7.6 implies $\partial_0 y_f$ is also in $\mathrm{CF}_{[0,1]}$, and then $\alpha - \partial_0 y_f$ is an element in $\mathrm{CF}_{[0,1]}$. Therefore,

$$(43) \qquad\qquad \tilde{\rho}_0(a, H) = \ell_0(\alpha - \partial_0 y_f) \geq \rho_0(a, H).$$

On the other hand, for any $\epsilon > 0$, there exists some $\alpha' \in \mathrm{CF}_{[0,1]}$ such that $[\alpha'] = (\mathrm{PSS}_0)_*(a)$ and $\ell_0(\alpha') \leq \rho_0(a, H) + \epsilon$. Viewing α' as an element in $\widetilde{\mathrm{CF}}_0$ and a as an element in $\widetilde{\mathrm{QH}}_0$, α' also represents a and hence $\tilde{\rho}_0(a, H) \leq \ell_0(\alpha')$. Therefore,

$$(44) \qquad\qquad \tilde{\rho}_0(a, H) \leq \ell_0(\alpha') \leq \rho_0(a, H) + \epsilon.$$

Since this is true for any $\epsilon > 0$, we get $\tilde{\rho}_0(a, H) \leq \rho_0(a, H)$, which together with (43) finishes the proof. $\square$

Before giving the proof of Theorem 1.9, we will prove the following lemma on the continuity of filtration functions on a fixed chain in $\mathrm{CF}_{[0,1]}$:

Lemma 8.3 *For any fixed chain $c \in \mathrm{CF}_{[0,1]}$, the function $t \mapsto \ell_t(c)$ is continuous on $[0, 1]$.*

Proof We will only prove the continuity at $t = 0$. For any other $t \in (0, 1]$, the proof is exactly the same. First, suppose that $\mathrm{CF}_{[0,1]}$ is a free $\Lambda_{[0,1]}$–module of rank n; we can identify c as an n–tuple $\vec{x}$ in $\Lambda^n_{[0,1]}$. Moreover, by (29), we can write

$$(45) \qquad \vec{x} = \sum_{(g_0, g_1)} \vec{a}_{(g_0, g_1)} T^{(g_0, g_1)}, \quad \text{where } \vec{a}_{(g_0, g_1)} \in (\mathcal{K}[\ker[\omega_0] \cap \ker[\omega_1]])^n.$$

Moreover, $\vec{x}$ can be further identified with a set of points (g_0, g_1) on the $g_0 g_1$–plane. By the finiteness condition of both ω_0 and ω_1, up to a uniform shift on both indices, we can assume that all the points are lying in the first quadrant.

Second, by definition, $\omega_t = (1 - t)\omega_0 + t\omega_1$. Let $t := \frac{1}{1+\lambda}$ for some nonnegative λ and define

$$\omega_\lambda = \lambda\omega_0 + \omega_1 = (1 + \lambda)\omega_t.$$

Note that over a set of homological spheres, ω_t obtains its minimal value if and only if ω_λ obtains its minimal value. One way of viewing ℓ_t is via the perturbation of the valuation function $\bar{v}_t$. Specifically, for any $c \in \mathrm{CF}_{[0,1]}$,

$$\bar{v}_t(c) = \min\left\{ \int_{S^2} A^*\omega_t \;\middle|\; A \text{ is an exponent of } c \right\},$$

and then

$$\ell_t(c) = -\bar{v}_t(c) + p_t(c),$$

where $p_t(c)$ comes from the Hamiltonian actions on Hamiltonian 1–periodic orbits, as well as the symplectic areas of the (fixed) cappings of basis elements. As $p_t(c)$ eventually goes to $p_0(c)$ when $t \to 0$, it suffices to focus on $\bar{v}_t(c)$ when studying the continuity of $\ell_t(c)$. Actually, we will focus on $(1 + \lambda)\bar{v}_t(c)$, that is,

$$\bar{v}_\lambda(c) := \min\left\{ \int_{S^2} A^*\omega_\lambda \;\middle|\; A \text{ is an exponent of } c \right\}$$
$$= \min\{\lambda g_0 + g_1 \mid (g_0, g_1) \text{ is an exponent of } \vec{x}\}.$$

Once rephrased in this way, it suggests a geometric way to view the value $\bar{v}_\lambda(c)$: for any $\lambda \geq 0$ and for any point (g_0, g_1), draw a line passing through (g_0, g_1) with slope $-\lambda$, that is,

$$(46) \qquad\qquad\qquad y = -\lambda(x - g_0) + g_1.$$

Then the minimal y–intercept is just the value $\bar{v}_\lambda(c)$. The nontrivial part is that the optimal point (g_0, g_1) who attains the minimal y–intercept might change along the change of λ (equivalently the change of t). However, we claim that when $\lambda \gg 0$, there exists a point (g_0^*, g_1^*) who serves as the optimal choice for all sufficiently large λ.

The key observation is that for any point $P = (g_0, g_1)$ attaining the value $\bar{v}_\lambda(c)$ for some λ, it fails to attain the value $\bar{v}_\eta(c)$ for any $\eta > \lambda$ if there exists another point Q in the region enclosed by the y–axis, the line (46) passing through (g_0, g_1) with slope $-\lambda$ and the line (46) passing through (g_0, g_1) with slope $-\eta$. When $\lambda \to \infty$,

the width of this closed region goes to zero. Hence, by the discreteness of our points, the choice of an optimal point will be eventually stable. Hence,

$$\bar{v}_t(c) = \frac{\lambda g_0^* + g_1^*}{1 + \lambda} \xrightarrow{\lambda \to \infty} g_0^* = \bar{v}_0(c).$$

The limit $\lambda \to \infty$ is equivalent to the limit $t \to 0$. Thus we get the conclusion.   □

The following proposition is the key step towards the proof of Theorem 1.9. A similar result of this type, but from a different setup, is Proposition 8.4 in [15]. Recall that in (35), we have defined a chain map over $\Lambda_{[0,1]}$, $\Phi_{0,t} \colon ((\mathrm{CF}_{[0,1]})_*, \partial_0, \ell_0) \to ((\mathrm{CF}_{[0,1]})_*, \partial_t, \ell_t)$ for any $t \in [0, 1]$.

Proposition 8.4 *For any chain $c \in (\mathrm{CF}_{[0,1]}, \partial_0, \ell_0)$, the function $t \to \ell_t(\Phi_{0,t}(c))$ is upper semicontinuous at $t = 0$.*

Proof We will prove it by contrapositive. Suppose that there exists a constant $\epsilon_0 > 0$ and a sequence $t_n \to 0$ such that

$$(47) \qquad\qquad \ell_{t_n}(\Phi_{0,t_n}(c)) - \ell_0(c) \geq \epsilon_0.$$

Since $\Phi_{0,t_n}(c) = c + \sigma_n$ where there exist nontrivial Floer trajectories between c and σ_n, by the triangle inequality of ℓ_{t_n},

$$\ell_{t_n}(\Phi_{0,t_n}(c)) = \ell_{t_n}(c + \sigma_n) \leq \max\{\ell_{t_n}(c), \ell_{t_n}(\sigma_n)\}.$$

Then (47) implies that

$$(48) \qquad\qquad \max\{\ell_{t_n}(c) - \ell_0(c), \ell_{t_n}(\sigma_n) - \ell_0(c)\} \geq \epsilon_0 > 0.$$

Lemma 8.3 implies that the first term in (48) will be smaller than ϵ_0 when n is sufficiently large. Therefore, (48) is possible only if $\ell_{t_n}(\sigma_n) - \ell_0(c) \geq \epsilon_0$.

Since there are only finitely many basis elements, by passing to a subsequence, we can assume that for each n, $\ell_{t_n}(\sigma_n) = \mathcal{A}_{\omega_{t_n}}(T^{A_n}[y, v])$, with the same basis element $[y, v]$. By definition, there exists some subchain of c with elements connecting with $T^{A_n}[y, v]$ by Floer trajectories. Again, since there are only finitely many basis generators, by passing to a subsequence, we can assume that

$$T^{A_n}[y, v] \text{ is connected with } T^{B_n}[x, w] \text{ by a Floer trajectory}$$

for a basis element $[x, w]$ and sphere classes B_n. In particular, the set $\{T^{B_n}[x, w]\}_{n=1}^{\infty}$ is a collection of elements from the chain c. Now, we claim that

$$\left| \int_{S^2} B_n^* \alpha \right| \to \infty, \quad \text{where } \alpha = \omega_1 - \omega_0.$$

In fact, since $\ell_0(c) \geq \mathcal{A}_{\omega_0}(T^{B_n}[x, w])$, Proposition 4.4 with $s = 0$ and $t = t_n$ implies the inequalities

$$(49) \quad \ell_{t_n}(x_n) - \ell_0(c) \leq \mathcal{A}_{\omega_{t_n}}(T^{A_n}[y, v]) - \mathcal{A}_{\omega_0}(T^{B_n}[x, w]) \leq -t_n \int_{D^2} (w \# B_n)^* \alpha.$$

Therefore, if $\left| \int_{S^2} B_n^* \alpha \right|$ is bounded, then when t_n is close to 0, this violates (48).

Then, in particular, B_n is not equal to any fixed homotopy class when n is sufficiently large. The finiteness condition of the chain c under the symplectic structure ω_0 implies

$$(50) \qquad\qquad \int_{S^2} B_n^* \omega_0 \to \infty.$$

Returning to the continuation chain map, we know

$$\Phi_{0,t_n}(T^{B_n}[x, w]) = T^{B_n}[x, w] + T^{A_n}[y, v] + \cdots.$$

Meanwhile, by $\Lambda_{[0,1]}$–linearity of $\Phi_{0,t}$, for any $m \in \mathbb{N}$,

$$\Phi_{0,t_n}(T^{B_m}[x, w]) = T^{B_m}[x, w] + T^{A_n + B_m - B_n}[y, v] + \cdots.$$

Since $T^{B_m}[x, w]$ is a generator of the chain c, $T^{A_n + B_m - B_n}[y, v]$ will be a generator of the chain σ_n. However, since $T^{A_n}[y, v]$ attains the maximal filtration with respect to ω_{t_n}, for any m, $\int_{S^2} (A_n + B_m - B_n)^* \omega_{t_n} \geq \int_{S^2} A_n^* \omega_{t_n}$. This implies that

$$(51) \qquad\qquad \int_{S^2} B_m^* \omega_{t_n} \geq \int_{S^2} B_n^* \omega_{t_n}.$$

Rewrite

$$\int_{S^2} B_m^* \omega_{t_n} = \int_{S^2} B_m^* \omega_0 + t_n \int_{S^2} B_m^* \alpha := a_m + t_n b_m,$$

where $a_m = \int_{S^2} B_m^* \omega_0$ and $b_m = \int_{S^2} B_m^* \alpha$. Moreover, denote $c_n = \int_{S^2} B_n^* \omega_{t_n} = a_n + t_n b_n$. Then (51) says that

$$a_m + t_n b_m \geq c_n.$$

Switch the index m and n, then we get

$$a_n + t_m b_n \geq c_m.$$

Since $b_m = (c_m - a_m)/t_m$ and $b_n = (c_n - a_n)/t_n$, the inequalities above are

(52) $$a_m + \frac{t_n}{t_m}(c_m - a_m) \geq c_n \quad \text{and} \quad a_n + \frac{t_m}{t_n}(c_n - a_n) \geq c_m.$$

Solve c_m from the first inequality, and then the second inequality implies that

$$a_n + \frac{t_m}{t_n}(c_n - a_n) \geq a_m + \frac{t_m}{t_n}(c_n - a_m).$$

This is equivalent to the relation

$$\left(\frac{t_m}{t_n} - 1\right)(a_m - a_n) \geq 0.$$

By (50), we know $a_m > a_n$ when $m \gg n$, which is strictly positive due to (50). This implies $t_m > t_n$, which is a contradiction since the t_n converge to 0 (so $t_m < t_n$ when $m \gg n$). $\qquad \square$

Now, we are ready to give the proof of Theorem 1.9.

Proof of Theorem 1.9 For simplicity, we will only prove the case for two symplectic structures ω_0 and ω_1, that is, for any $t \in [0, 1]$ and $a \in \mathrm{QH}_{[0,1]}$, the map $t \mapsto \rho_t(a, H)$ is upper semicontinuous. It is easy to see how this can be generalized to the general case.

By the realization property (2) in Lemma 8.2, there exists some $c \in (\mathrm{CF}_{[0,1]}, \partial_0)$ such that $\ell_0(c) = \rho_0(a, H)$, where $[c] = (\mathrm{PSS}_0)_*(a)$. On the other hand, $\Phi_{0,t}(c)$ represents

$$[\Phi_{0,t}(c)] = (\Phi_{0,t})_*[c] = (\Phi_{0,t})_*((\mathrm{PSS}_0)_*(a)) = (\mathrm{PSS}_t)_*(a).$$

Then, by definition, we know that $\rho_t(a, H) \leq \ell_t(\Phi_{0,t}(c))$. Therefore,

$$\rho_t(a, H) - \rho_0(a, H) \leq \ell_t(\Phi_{0,t}(c)) - \ell_0(c).$$

Upper semicontinuity from Proposition 8.4 implies the desired conclusion. $\qquad \square$

Remark 8.5 There is an obvious question on the lower semicontinuity of t–spectral invariants. A trial of imitating the proof of Theorem 8.3 in [15] can be carried out but some details could not go through deeply due to the non-uniform upper bound from the energy estimation from Proposition 4.3.

9 Applications on quasi-isometric embedding and capacity

9.1 Proof of Theorem 1.15

Definition 9.1 Let us give the following two definitions:

(1) [26, Definition 1.3] We say M admits an *aperiodic symplectic structure* if there exists a symplectic structure ω such that (M, ω) admits an autonomous Hamiltonian function H, not everywhere locally constant, such that its Hamiltonian flow has no nonconstant periodic orbit. Such H is called an *associated aperiodic Hamiltonian* of ω.

(2) For a symplectic structure ω, we call it *strongly aperiodic-approximated* (the saa condition) if there exists a sequence of aperiodic structures ω_n such that (a) $\omega_n \to \omega$ under the norm $|\cdot|$ in Section 3; (b) there exists a sequence of associated aperiodic Hamiltonians H_n of ω_n that C^∞–converges to a differentiable function H on M.

It is well known that the computation of spectral invariants is difficult in general. The following theorem will be helpful in the proof later:

Proposition 9.2 [24, Proposition 4.1] *Let (M, ω) be a symplectic manifold. If H is an autonomous Hamiltonian function on M such that its Hamiltonian flow has no nonconstant contractible periodic orbit with the period at most 1, then*

$$\rho([M], H; \omega) = -\min_M H.$$

Notice that the condition in this proposition is weaker than the assumption of [27, Theorem 1.1]. In [17], for *any* symplectic manifold (M, ω), it is proved that the Hofer diameter of $\widetilde{\mathrm{Ham}}(M, \omega)$ is infinite by using, roughly speaking, a sequence of bump functions on a displaceable subset. Theorem 1.15 shows that under a certain condition (which covers a variety of symplectic manifolds, especially in 4 dimensions) the Hofer diameter of $\widetilde{\mathrm{Ham}}(M, \omega)$ goes to infinity in uncountably many linearly independent directions. The proof of Theorem 1.15 takes its inspiration from the proof of Theorem 1.1 in [27].

Proof of Theorem 1.15 First, note that if ω is already aperiodic, then in particular (M, ω) admits an autonomous Hamiltonian H such that its Hamiltonian flow has no nonconstant contractible periodic orbit. Then, by the compactness of M and Sard's

theorem, there exists a nontrivial closed interval $[a, b]$ (assumed to be $[0, 1]$) such that every $c \in [0, 1]$ is a regular value of H. Now, take a function $g \colon \mathbb{R} \to [0, 1]$ such that its support is in $(0, 1)$, $\max g = 1$ and its only local minimum has value 0. For each $\vec{v} = (v_1, v_2, \dots) \in \mathbb{R}^\infty$, define $f_{\vec{v}} \colon \mathbb{R} \to \mathbb{R}$ by

$$f_{\vec{v}}(s) = \sum_{i=1}^\infty v_i \cdot g\big(2^i(s - (1 - 2^{1-i}))\big).$$

The embedding $\Phi \colon \mathbb{R}^\infty \to \widetilde{\mathrm{Ham}}(M, \omega)$ is constructed as $\Phi(\vec{v}) = [\phi^1_{f_{\vec{v}} \circ H}]$. Then, for any nonzero $\vec{v} \in \mathbb{R}^\infty$, $X_{f_{\vec{v}} \circ H} = f'_{\vec{v}}(H) \cdot X_H$, which implies Φ is a homomorphism. Therefore,

$$
\begin{aligned}
d_H(\Phi(\vec{v}), \Phi(\vec{w})) &= d_H([\phi^1_{f_{\vec{v}} \circ H}], [\phi^1_{f_{\vec{w}} \circ H}]) \\
&= d_H([\phi^1_{f_{\vec{v} - \vec{w}} \circ H}], \mathbb{1}_M) \\
&\leq \|f_{\vec{v} - \vec{w}} \circ H\|_H \\
&= \max(f_{\vec{v}} \circ H) - \min(f_{\vec{w}} \circ H) = \mathrm{osc}(\vec{v} - \vec{w}).
\end{aligned}
$$

Note that this computation is true for any ω without assuming ω to be aperiodic.

On the other hand, $f_{\vec{v}} \circ H$ also satisfies the condition that it has no nonconstant contractible periodic orbit. In particular, it has no nonconstant contractible periodic orbit with period at most 1. By Proposition 9.2, we know

$$\rho([M], f_{\vec{v} - \vec{w}} \circ H; \omega) = -\min_M(f_{\vec{v} - \vec{w}} \circ H) = \max_i(w_i - v_i)$$

and

$$\rho([M], f_{\vec{w} - \vec{v}} \circ H; \omega) = -\min_M(f_{\vec{w} - \vec{v}} \circ H) = \max_i(v_i - w_i).$$

Therefore,

$$d_H(\Phi(\vec{v}), \Phi(\vec{w})) \geq \max\{\max_i(w_i - v_i), \max_i(v_i - w_i)\} = |\vec{v} - \vec{w}|_\infty,$$

where the first inequality comes from (a) in Theorem 2.12.

Next, if ω satisfies the saa condition, then take a sequence of aperiodic symplectic structures $\omega_n \to \omega$. By Corollary 1.10, for each fixed $\vec{w}$ and $\vec{v}$ in $\mathbb{R}^\infty$ and any given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$, we have

$$\max_i(w_i - v_i) = \rho([M], f_{\vec{v} - \vec{w}} \circ H_n; \omega_n) \leq \rho([M], f_{\vec{v} - \vec{w}} \circ H; \omega) + \epsilon$$

and

$$\max_i(v_i - w_i) = \rho([M], f_{\vec{w}-\vec{v}} \circ H_n; \omega_n) \le \rho([M], f_{\vec{w}-\vec{v}} \circ H; \omega) + \epsilon,$$

where the equalities come from the computation above when the symplectic structure is aperiodic. Therefore,

$$d_H(\Phi(\vec{v}), \Phi(\vec{w})) + \epsilon \ge \max\{\max_i(w_i - v_i), \max_i(v_i - w_i)\} = |\vec{v} - \vec{w}|_\infty.$$

Since this result is true for any $\epsilon > 0$, we get the conclusion. □

Remark 9.3 If the continuity result of boundary depth is affirmative (especially the lower semicontinuity), a similar argument as in the proof of Theorem 1.15 implies that if M admits a symplectic structure which satisfies the saa condition, then there is a quasi-isometric embedding from $(\mathbb{R}^\infty, |-|_\infty)$ into $(\mathrm{Ham}(M, \omega), |-|_\infty)$. This can be regarded as an "approximated" version of Theorem 1.1 in [27], which can probably cover more symplectic manifolds.

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