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The motivic Mahowald invariant

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The classical Mahowald invariant is a method for producing nonzero classes in the stable homotopy groups of spheres from classes in lower stems. We study the Mahowald invariant in the setting of motivic stable homotopy theory over $\mathrm{Spec}(\mathbb{C})$. We compute a motivic version of the C_2–Tate construction for various motivic spectra, and show that this construction produces "blueshift" in these cases. We use these computations to show that for $i \geq 1$, the Mahowald invariant of η^i is the first element in Adams filtration i of the w_1–periodic families constructed by Andrews (2018). This provides an exotic periodic analog of the computation of Mahowald and Ravenel (1993) that for $i \geq 1$, the classical Mahowald invariant of 2^i, is the first element in Adams filtration i of the v_1–periodic families constructed by Adams (1966).

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1 Introduction

The classical Mahowald invariant is a method for producing nonzero classes in the stable homotopy groups of spheres from classes in lower stems. The classical Mahowald invariant is defined using Lin's theorem [26], which says that after 2–completion, there is an equivalence of spectra

$$S^0 \simeq \varprojlim_n \Sigma\mathbb{RP}^{\infty}_{-n}$$

between the sphere spectrum and a homotopy limit of stunted real projective spectra. Let $\alpha \in \pi_t(S^0)_{(2)}$ be a class in the 2–primary stable stems. Then by the above equivalence, there is some minimal $N > 0$ such that the map

$$S^t \to S^0 \to \Sigma\mathbb{RP}^{\infty}_{-N}$$

is essential. This gives rise to a nontrivial map $(S^t \to S^{-N+1}) \in \pi_{t+N-1}(S^0)$, where S^{-N+1} is the fiber of the collapse map $\Sigma\mathbb{RP}^{\infty}_{-N} \to \Sigma\mathbb{RP}^{\infty}_{-(N-1)}$. The nontrivial map above is the classical Mahowald invariant of α, which we denote by $M^{\mathrm{cl}}(\alpha)$.

In this paper, we define and analyze an analog of the Mahowald invariant in the setting of motivic stable homotopy theory over $\mathrm{Spec}(\mathbb{C})$. We focus on the interaction between the Mahowald invariant and chromatic homotopy theory. More precisely, we are interested

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in the interaction between the Mahowald invariant and periodic families in the stable stems. This has been studied extensively in the classical setting. In [29], Mahowald and Shick defined a chromatic filtration on the Adams spectral sequence E_2–page, and in [36], Shick showed that an algebraic version (with input and output in the E_2–page of the Adams spectral sequence) of the classical Mahowald invariant took v_n–periodic classes to v_n–torsion classes. This is the algebraic form of a conjecture of Mahowald and Ravenel [27, Conjecture 12] that, roughly speaking, the Mahowald invariant of a v_n–periodic class is v_n–torsion.

The previous conjecture has been verified for several cases. Mahowald and Ravenel showed in [28] that for $i \geq 1$, the classical Mahowald invariant of 2^i is the first element in the stable stems in Adams filtration i. Subsequent work of Sadofsky [35] showed that for $p > 3$ one has $\beta_k \in M^{\mathrm{cl}}(\alpha_k)$. Computations of Behrens [5; 6] at $p = 3$ and later at $p = 2$ provided further evidence that the classical Mahowald invariant of a v_n–periodic class in the stable stems is v_{n+1}–periodic. In this paper, we will produce motivic analogs of Mahowald and Ravenel's computations of the Mahowald invariant of 2^i for $i \geq 1$.

To further explain our goal, we must explain some background on periodicity in classical and motivic stable homotopy theory. By "periodic family" we will mean a family of elements produced from iterating a nonnilpotent self-map on a finite complex. Classically, this method was first used by Adams in [1], where he used the nonnilpotent self-map $v_1^4\colon \Sigma^8 V(0) \to V(0)$ of the mod 2 Moore spectrum to produce v_1–periodic families inside the image of J. The iterated self-map construction was studied at higher heights by Smith [37], Toda [39], and Miller, Ravenel and Wilson [30]. Much has been written about self-maps of finite complexes; see for example the work of Hopkins and Smith [19], and the recent work of Behrens, Hill, Hopkins and Mahowald [7] and Bhattacharya and Egger [8] at $p = 2$.

In the motivic setting over a field of characteristic zero, Levine [25] showed that the classical stable stems sit inside the motivic stable stems. Therefore, all classical periodic families also exist motivically. However, there are nonclassical classes in the motivic stable stems, some of which form "exotic" periodic families. The first instance of exotic periodicity is the nonnilpotence of the algebraic Hopf invariant one element $\eta \in \pi_{1,1}(S^{0,0})$, which was proven by Morel in [31]. In analogy with the work of Adams, one can ask if the cofiber of η admits a nonnilpotent self-map. In [4], Andrews showed that there is a nonnilpotent self-map $w_1^4\colon \Sigma^{20,12} C\eta \to C\eta$, and he used this to produce w_1–periodic families in the motivic stable stems. This suggests that η deserves to be called w_0, and that there should be w_n–periodic families for all n. This particular form of exotic periodicity has been studied further in the work of Gheorghe [13] at $p=2$ and forthcoming work of Krause [24] at all primes.

We can now state our main result. Reinterpreted, Mahowald and Ravenel's computation says that the classical Mahowald invariant of the v_0–periodic class 2^i is the first v_1–periodic class in Adams filtration i. In addition to proving an analogous result in the motivic setting, we prove an exotic analog. Precisely, we show that the motivic Mahowald invariant of the w_0–periodic class η^i is the first w_1–periodic class in Adams filtration i.

Outline Our starting point is the motivic analog of Lin's theorem proven by Gregersen in [16], where he constructed a motivic analog of $\mathbb{RP}^{\infty}_{-\infty}$ which exhibits the motivic sphere spectrum as a homotopy limit of motivic stunted projective spectra. In Section 2, we begin by recalling the necessary background from Gregersen's thesis. We then define the *motivic Mahowald invariant* in analogy with the classical Mahowald invariant and compute the motivic Mahowald invariants of the algebraic Hopf invariant one elements of [12]. We then define the *motivic C_2–Tate construction* of a motivic ring spectrum E, denoted by $E^{t_{\mathrm{gm}}C_2}$, and use this to define some approximations to the motivic Mahowald invariant.

In Section 3, we compute the motivic C_2–Tate construction of several motivic ring spectra by comparing the Atiyah–Hirzebruch spectral sequence and the motivic Adams spectral sequence. In particular, we compute the motivic C_2–Tate construction for two motivic analogs of ko; the analogous classical computations are due to Davis and Mahowald [9]. Recall the motivic ring spectrum kq constructed by Isaksen and Shkembi in [23] with motivic cohomology $H^{**}(\mathrm{kq}) \cong A /\!/ A(1)$. We show that there is an isomorphism in motivic homotopy groups

$$\pi_{**}(\mathrm{kq}^{t_{\mathrm{gm}}C_2}) \simeq \varprojlim_n \bigoplus_{i\geq -n} \pi_{**}(\Sigma^{4i,2i} H\mathbb{Z}_{(2)}),$$

where $H\mathbb{Z}_{(2)}$ is the motivic Eilenberg–Mac Lane spectrum of the integers localized at 2 with mod 2 motivic cohomology $H^{**}(H\mathbb{Z}_{(2)}) \cong A /\!/ A(0)$. This computation is used in later sections to compute the motivic Mahowald invariant of 2^i.

In order to compute the motivic Mahowald invariant of η^i, we construct a motivic ring spectrum that detects w_1–periodic elements. More precisely, we need a spectrum whose Hurewicz image contains some of the w_1–periodic families constructed by Andrews. This spectrum is produced using the forthcoming work of Gheorghe, Wang and Xu [15], where, roughly speaking, they prove that there is an equivalence of categories between $C\tau$–modules and $\mathrm{BP}_*\mathrm{BP}$–comodules. Here, $C\tau$ is the cofiber of $\tau \in \pi_{0,-1}(S^{0,0})$ studied by Gheorghe in [14]. We produce a $C\tau$–module called wko by writing down the corresponding $\mathrm{BP}_*\mathrm{BP}$–comodule. This $C\tau$–module has the property that $\overline{H}^{**}(\mathrm{wko}) \cong \overline{A} /\!/ \overline{A}(1)$, where $\overline{(-)}$ indicates that we are in the $C\tau$–linear setting discussed in [14, Section 5]. We show that there is an isomorphism in motivic homotopy groups

$$\pi_{**}(\mathrm{wko}^{t_{\mathrm{gm}}C_2}) \cong \varprojlim_n \bigoplus_{i \geq -n} \left(\Sigma^{4i,2i}\pi_{**}(\mathrm{wBP}\langle 0\rangle) \oplus \Sigma^{4i-1,2i-1}\pi_{**}(\mathrm{wBP}\langle 0\rangle)\right),$$

where $\mathrm{wBP}\langle 0\rangle$ is a spectrum satisfying $\pi_{**}(\mathrm{wBP}\langle 0\rangle) \cong \mathbb{F}_2[w_0]$ that was constructed by Gheorghe in [13].

In Section 4, we use the motivic C_2–Tate construction computations from the previous section to compute approximations of the motivic Mahowald invariant based on kq and wko. These computations are analogous to those in [28, Theorem 2.16], where they compute these approximations in the classical setting using ko.

In Section 5, we lift the kq– and wko–based approximations to full computations of the motivic Mahowald invariant of 2^i and η^i for all $i \geq 1$. For values of i where the motivic Mahowald invariant lands in the Hurewicz image for kq or wko, this lifting is trivial in view of Proposition 2.20. For the remaining values of i, we pass through a series of approximations obtained by varying the cohomology theory and the filtration of the motivic analog of $\mathbb{RP}^\infty_{-N}$. This passage proceeds by induction on i, with the induction step completed by comparing Adams spectral sequences in the classical, motivic, and $C\tau$–linear settings. In particular, we use Adams' identification of the v_1^4–periodicity operator as a Massey product [2] along with a theorem of Isaksen [22, Theorem 2.1.12] to compare the relevant periodic families at the level of Adams E_2–pages.

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2 Definition and elementary computations

2.1 Background from motivic homotopy theory

We begin by recalling some work of Morel and Voevodsky [40; 32] and Gregersen [16].

We will work in the category of motivic spaces or spectra over $\mathrm{Spec}(\mathbb{C})$ at the prime $p=2$, with everything implicitly completed at 2. Our goal is to sketch the proof of a motivic analog of Lin's theorem (Theorem 2.8 below) which will be used to define the motivic Mahowald invariant.

In [16, Section 3.3], Gregersen defines the *motivic Singer construction* $R_+(-)$ and proves that it associates an Ext_A–equivalent module $R_+(M)$ to any module M over the motivic Steenrod algebra; this is the motivic analog of the classical Singer construction studied in [26] and [3]. In order to prove a motivic version of Lin's theorem, Gregersen constructs a motivic analog of $\mathbb{RP}^{\infty}_{-\infty}$ and shows that its continuous motivic cohomology is isomorphic to the motivic Singer construction of the motivic cohomology of a desuspension of the motivic sphere spectrum. We now recall this construction.

Motivic cohomology with mod 2 coefficients is represented by the motivic Eilenberg–Mac Lane spectrum for $\mathbb{F}_2$. We denote this spectrum by H, and we will denote the motivic cohomology of a point $H^{**}(\mathrm{Spec}(k))$ by $\mathbb{M}_2$. When $k=\mathbb{C}$, we have $\mathbb{M}_2 \cong \mathbb{F}_2[\tau]$ where $|\tau| = (0,1)$.

Let $\mathbb{G}_m$ denote the multiplicative group scheme, $\mathbb{A}^n$ the affine space of rank n, and $\mathbb{P}^n$ the projective space of rank n. Then there is an equivalence $\mathbb{P}^n \simeq (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m$. Let μ_p denote the group scheme of p^{th} roots of unity. The closed inclusion $\mu_p \hookrightarrow \mathbb{G}_m$ defines an action of μ_p on $\mathbb{A}^n \setminus 0$. The motivic lens space is then defined as $L^n := (\mathbb{A}^n \setminus 0)/\mu_p$. The inclusion $\mathbb{A}^n \setminus 0 \hookrightarrow \mathbb{A}^{n+1} \setminus 0$ sending $(x_1,\dots,x_n) \mapsto (x_1,\dots,x_n,0)$ induces a map $L^n \to L^{n+1}$. Taking the colimit over these maps defines a motivic space called L^∞, also known as the geometric classifying space $B(\mu_p)_{\mathrm{gm}}$ of the p^{th} roots of unity.

The motivic spaces L^n are represented by smooth schemes [16, Lemma 4.1.2], so in particular any algebraic bundle $E \to L^n$ is an $\mathbb{A}^1$–homotopy equivalence by [32, Proposition 4.2.3]. The tautological line bundle γ_n^1 over $\mathbb{P}^n$ can be viewed as the closed subset of $\mathbb{A}^{n+1} \times \mathbb{P}^n$ satisfying $x_i y_j = x_j y_i$, where the x_i are coordinates for $\mathbb{A}^{n+1}$ and the y_i are coordinates for $\mathbb{P}^n$. The inclusions $\iota\colon \mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$ satisfy $\iota^*\gamma^1_{n+1} = \gamma^1_n$. There is an identification $L^n \cong E((\gamma^1_{n-1})^{\otimes p} \setminus 0 \downarrow \mathbb{P}^{n-1})$ of L^n with the total space of the complement of the zero section of the p–fold tensor product of the tautological line bundle over $\mathbb{P}^{n-1}$, by [40, Lemma 6.3]. The projection thus induces a map $f_n\colon L^n \to \mathbb{P}^{n-1}$. The tautological line bundle γ^1_n over L^n is the same bundle as the pullback $f_n^*\gamma^1_{n-1}$ of the tautological line bundle over $\mathbb{P}^{n-1}$.

Theorem 2.1 [40, Theorems 4.1 and 6.10] *For any prime p,*

$$H^{**}(\mathbb{P}^n) \cong \mathbb{M}_p[v]/(v^{n+1})$$

with $|v| = (2,1)$, and for $p = 2$,

$$H^{**}(L^n) \cong \mathbb{M}_2[u,v]/(u^2 + \tau v, v^n) \quad \textit{and} \quad H^{**}(L^\infty) \cong \mathbb{M}_2[u,v]/(u^2 + \tau v)$$

where $|u| = (1,1)$ and $|v| = (2,1)$. The action of the motivic Steenrod algebra on $H^{**}(L^\infty)$ is given by

$$\mathrm{Sq}^{2i}(v^k) = \binom{2k}{2i} v^{k+i}, \qquad \mathrm{Sq}^{2i+1}(v^k) = 0,$$
$$\mathrm{Sq}^{2i}(uv^k) = \binom{2k}{2i} uv^{k+i}, \quad \mathrm{Sq}^{2i+1}(uv^k) = \binom{2k}{2i} v^{k+i+1}.$$

The pullback bundle $f_n^* \gamma_{n-1}^1$ over L^n can be identified with $(\mathbb{A}^n \setminus 0) \times_{\mu_p} \mathbb{A}^1$ over L^n by identifying $(\lambda x_1, \dots, \lambda x_n, y) \sim (x_1, \dots, x_n, \lambda y)$ where $\lambda \in \mu_p$. The inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{A}^n$ defines an embedding of γ_{n-1}^1 into the trivial bundle ϵ^n over L^n by sending

$$(x_1, \dots, x_n, y) \mapsto (x_1, \dots, x_n, x_1 y, \dots, x_n y) \in L^n \times \mathbb{A}^n.$$

In analogy with the classical construction of $\mathbb{RP}^\infty_{-\infty}$, one now needs to define the Thom space of the "motivic orthogonal complement". For $\eta \hookrightarrow \xi$ an inclusion of vector bundles, Gregersen defines

$$\mathrm{Th}(\xi, \eta) := \frac{E(\xi)}{E(\xi) \setminus E(\eta)}.$$

By [16, Lemma 4.1.22], if $\eta \oplus \zeta \cong \xi$ is an isomorphism of vector bundles over a smooth scheme, then $\mathrm{Th}(\zeta) \to \mathrm{Th}(\xi, \eta)$ is an $\mathbb{A}^1$–weak equivalence. With this notion of orthogonal complement, one can define motivic stunted projective spectra:

Definition 2.2 [16, Definition 4.1.23] For $n \geq 0$ and $k \geq 0$, let $\underline{L}_{-k}^{n-k}$ be the motivic spectrum

$$\underline{L}_{-k}^{n-k} = \Sigma^{-2kn,-kn} \mathrm{Th}(k\epsilon^n, k\gamma_{n-1}^1).$$

We will frequently refer to (stable) cells of $\underline{L}_{-k}^{n-k}$. The following lemma justifies this terminology.

Lemma 2.3 [16, Lemma 4.2.19] *The spectra $\underline{L}_{-k}^{n-k}$ are stably cellular.*

The following lemma is proven over more general base schemes, but we specialize to the case $k = \mathbb{C}$.

Lemma 2.4 [16, Lemma 4.1.24] *The motivic cohomology of $\underline{L}^{n-k}_{-k}$ is given as a module over $\mathbb{M}_2$ by*

$$H^{**}(\underline{L}^{n-k}_{-k}) = \Sigma^{-2k,-k}\mathbb{M}_2[u,v]/(u^2+\tau v, v^n)$$

with $|u| = (1,1)$ *and* $|v| = (2,1)$.

In order to define the analog of $\mathbb{RP}^{\infty}_{-\infty}$, we need to take a homotopy limit of these motivic lens spaces. By letting n vary, we obtain a spectrum

$$\underline{L}^{\infty}_{-k} := \varinjlim_{n} \underline{L}^{n-k}_{-k}$$

and we note that as in the classical case, $\underline{L}^{\infty}_{0} = \Sigma^{\infty} L^{\infty}_{+}$. The following lemma is proven in forthcoming work of Gregersen, Heller, Kylling, Rognes and Østvær [17]. We thank Paul Arne Østvær for pointing out that the version of this lemma appearing in older versions of this paper was incorrect.

Proposition 2.5 [17] *There exists a class $\tau_k \in \widetilde{H}^{-2k,-k}(\underline{L}^{\infty}_{-k})$ such that multiplication by τ_k induces an isomorphism $\tau_k\colon \Sigma^{-2k,-k}H^{**}(L^{\infty}) \xrightarrow{\cong} \widetilde{H}^{**}(\underline{L}^{\infty}_{-k})$. Moreover, the following diagram commutes:*

$$\begin{tikzcd}
\Sigma^{-2k,-k}H^{**}(L^{\infty}) \arrow[r, "\tau_k"] \arrow[d, "\cdot v"] & \widetilde{H}^{*-2m,*-m}(\underline{L}^{\infty}_{-k}) \arrow[d] \\
\Sigma^{-2k+2,-k+1}H^{**}(L^{\infty}) \arrow[r, "\tau_{k+1}"] & \widetilde{H}^{*-2m,*-m}(\underline{L}^{\infty}_{-k-1})
\end{tikzcd}$$

Now letting k vary, we have $\underline{L}^{\infty}_{-\infty} = \varprojlim_{k} \underline{L}^{\infty}_{-k}$.

Proposition 2.6 [17] *As modules over $\mathbb{M}_2$ we have*

$$H^{**}(\underline{L}^{\infty}_{-k}) \cong \Sigma^{-2k,-k}\mathbb{M}_2[u,v]/(u^2+\tau v),$$
$$H^{**}_c(\underline{L}^{\infty}_{-\infty}) \cong \mathbb{M}_2[u,v,v^{-1}]/(u^2+\tau v).$$

One can extend the isomorphism of $\mathbb{M}_2$–modules in the previous proposition to an isomorphism of A–modules by continuing the periodic action of the motivic Steenrod operations on L^{∞} to the negative cells. For reference, we include a picture of the motivic cohomology of $\underline{L}^{\infty}_{-\infty}$ in a range below; see Figure 1.

The following result implies a motivic analog of Lin's theorem.

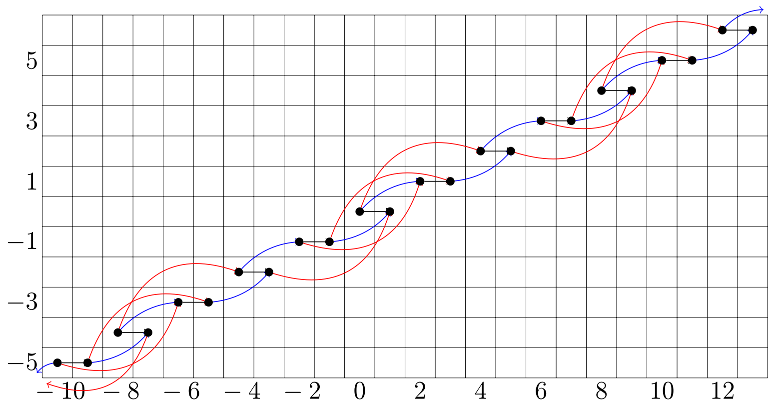


Figure 1: This depicts the continuous motivic cohomology $H_c^{i,j}(\Sigma^{1,0}\underline{L}_{-\infty}^{\infty})$ for $-11 \leq i \leq 12$ and $-5 \leq j \leq 6$. The horizontal axis is topological degree and the vertical axis is motivic weight. A bullet represents $\mathbb{M}_2$, ie it represents an infinite tower of $\mathbb{F}_2$'s connected by τ–multiplication extending downwards. The action of Sq^1 is depicted by black horizontal lines between bullets, the action of Sq^2 is depicted by blue curves with horizontal length 2, and the action of Sq^4 is depicted by red curves with horizontal length 4.

Proposition 2.7 [16, Proposition 4.1.37] *There is an A–module isomorphism*

$$H_c^{**}(\underline{L}_{-\infty}^{\infty}) \cong \Sigma^{1,0}\mathbb{M}_2[u,v,v^{-1}]/(u^2+\tau v) \cong R_+(\mathbb{M}_2),$$

where $R_+(\mathbb{M}_2)$ is the motivic Singer construction of $\mathbb{M}_2$.

The argument sketched at the beginning of this section using the inverse limit motivic Adams spectral sequence proves the following.

Theorem 2.8 [16, Theorem 2.0.2] *There is a π_{**}–isomorphism*

$$S \to \Sigma^{1,0}\underline{L}_{-\infty}^{\infty}$$

after $(2,\eta)$–completion.

Note that over $\mathrm{Spec}(\mathbb{C})$, 2–completion and $(2,\eta)$–completion coincide by [20], so it suffices for us to work in the 2–completed setting.

2.2 Definition of the motivic Mahowald invariant

We begin by recalling the definition of the classical Mahowald invariant.

Definition 2.9 Let $\alpha \in \pi_t(S^0)$. The *classical Mahowald invariant* of α is the coset of completions of the diagram

$$\begin{tikzcd}
S^t \arrow[rr, dashed] \arrow[d, "\alpha"] & & S^{-N+1} \arrow[d] \\
S^0 \arrow[r, "\simeq"] & \Sigma\mathbb{RP}^{\infty}_{-\infty} \arrow[r] & \Sigma\mathbb{RP}^{\infty}_{-N}
\end{tikzcd}$$

where $N > 0$ is the minimal value such that the left-hand composition is nontrivial. The equivalence on the left-hand side is by Lin's theorem [26], and the dashed arrow is the lift to the fiber of the sequence

$$S^{-N+1} \to \Sigma\mathbb{RP}^{\infty}_{-N} \xrightarrow{c} \Sigma\mathbb{RP}^{\infty}_{-N+1},$$

which is nontrivial by choice of N. The classical Mahowald invariant of α will be denoted by $M^{\mathrm{cl}}(\alpha)$.

We now define the motivic analog of the classical Mahowald invariant.

Definition 2.10 Let $\alpha \in \pi_{s,t}(S^{0,0})$. We define the *motivic Mahowald invariant* of α, denoted by $M(\alpha)$, as follows. Consider the coset of completions of the diagram

$$\begin{tikzcd}
S^{s,t} \arrow[rr, dashed] \arrow[d, "\alpha"] & & S^{-2N+1,-N} \vee S^{-2N+2,-N+1} \arrow[d] \\
S^{0,0} \arrow[r, "\simeq"] & \Sigma^{1,0}\underline{L}^{\infty}_{-\infty} \arrow[r] & \Sigma^{1,0}\underline{L}^{\infty}_{-N}
\end{tikzcd}$$

where $N > 0$ is the minimal value such that the left-hand composition is nontrivial. The equivalence on the left-hand side is by Gregersen's theorem, and the dashed arrow is the lift to the fiber of the sequence

$$S^{-2N+1,-N} \vee S^{-2N+2,-N+1} \to \Sigma^{1,0}\underline{L}^{\infty}_{-N} \to \Sigma^{1,0}\underline{L}^{\infty}_{-N+1},$$

which is nontrivial by the choice of N. In contrast with the definition of the classical Mahowald invariant, the target of the dashed arrow is a wedge of spheres. If the composition of the dashed arrow with the projection onto the higher-dimensional sphere is nontrivial, we define the motivic Mahowald invariant $M(\alpha)$ to be the coset of completions composed with the projection onto the higher-dimensional sphere. Otherwise, the composition of the dashed arrow with the projection onto the higher-dimensional sphere is trivial and we define the motivic Mahowald invariant $M(\alpha)$ to be the coset of completions composed with the projection onto the lower-dimensional sphere. We illustrate this convention in the examples later in this section.

Remark 2.11 In the classical setting we could have defined

$$\mathbb{RP}^{\infty}_{-\infty} = \varprojlim_n \mathbb{RP}^{\infty}_{-2n}.$$

Then given $\alpha \in \pi_t(S^0)$ we could consider the coset of completions of the diagram

$$\begin{tikzcd} S^t \arrow[rr, dashed] \arrow[d, "\alpha"] & & S^{-2N+1} \vee S^{-2N+2} \arrow[d] \\ S^0 \arrow[r, "\simeq"] & \Sigma\mathbb{RP}^{\infty}_{-\infty} \arrow[r] & \Sigma\mathbb{RP}^{\infty}_{-2N} \end{tikzcd}$$

where $N > 0$ is the minimal value such that the left-hand composition is nontrivial. If the composition of the dashed arrow with the projection onto the higher-dimensional sphere is nontrivial, we can define $\widetilde{M}^{\mathrm{cl}}(\alpha)$ to be the coset of completions composed with the projection onto the higher-dimensional sphere. Otherwise, we can define $\widetilde{M}^{\mathrm{cl}}(\alpha)$ to be the coset of completions composed with the projection onto the lower-dimensional sphere. Then $\widetilde{M}^{\mathrm{cl}}(\alpha) = M^{\mathrm{cl}}(\alpha)$.

Definition 2.12 Let $\alpha \in \pi_{s,t}(S^{0,0})$. The *Chow degree* of α, denoted by $\mathrm{Ch}(\alpha)$, is given by $\mathrm{Ch}(\alpha) := s - 2t$.

In the sequel, we will use the fact that the motivic Mahowald invariant (almost) preserves Chow degree. We thank the referee for pointing out this approach. The following lemma is immediate from the definition:

Lemma 2.13 *Let $\alpha \in \pi_{s,t}(S^{0,0})$. Then $\mathrm{Ch}(M(\alpha)) = \mathrm{Ch}(\alpha) + \epsilon$ for some $\epsilon \in \{0, 1\}$.*

2.3 Atiyah–Hirzebruch spectral sequence for $\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}$

We can use the Atiyah–Hirzebruch spectral sequence to determine which cell of $\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}$ a class $\alpha \in \pi_{**}(S^{0,0})$ is detected on. In this subsection, we analyze this spectral sequence in a range. These computations will be used in the next subsection to compute some motivic Mahowald invariants.

The Atiyah–Hirzebruch spectral sequence arises from the cellular filtration of $\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}$. This spectral sequence has the form

$$E^1_{s,t,u} = \pi_{t,u}(S^{s,\lfloor s/2\rfloor}) = \pi_{t-s,u-\lfloor s/2\rfloor}(S^{0,0}) \Rightarrow \pi_{t,u}(\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}) \cong \pi_{t,u}(S^{0,0}).$$

We will denote classes in $E^1_{s,t,u}$ by $x[s]$ where $x \in \pi_{**}(S^{0,0})$.

Differentials are induced by attaching maps in $\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}$. Let $\eta_{\mathrm{cl}}, \nu_{\mathrm{cl}}, \sigma_{\mathrm{cl}} \in \pi_*(S^0)$ denote the classical Hopf invariant one elements and let $\eta, \nu, \sigma \in \pi_{**}(S^{0,0})$ denote

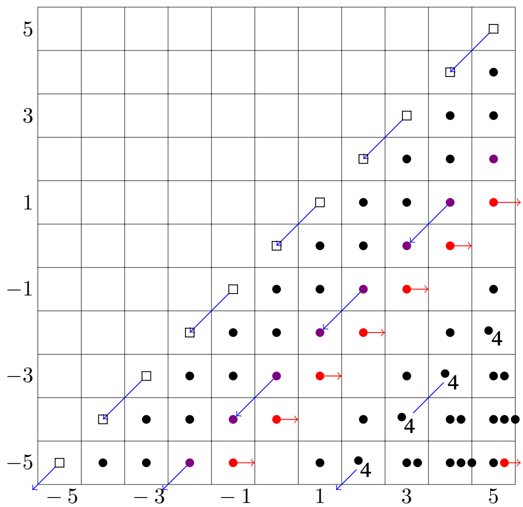


Figure 2: The E^1–page of the Atiyah–Hirzebruch spectral sequence for $-5 \le s \le 5$ and $-5 \le t \le 5$ with d^1–differentials included. A $\square$ represents $\mathbb{Z}_{(2)}[\tau]$, a $\bullet$ represents $\mathbb{F}_2[\tau]$, a violet $\bullet$ represents

$$\pi_{3,*}(S^{0,0}) \cong \frac{\mathbb{Z}/8\{\nu\}[\tau] \oplus \mathbb{Z}/2\{\eta^3\}[\tau]}{4\nu = \tau\eta^3},$$

a $\bullet_n$ represents $\mathbb{Z}/2^n[\tau]$, and a red $\bullet\!\!\rightarrow$ represents $\mathbb{F}_2[\eta]$. Differentials are blue and τ–linear and, in this chart, are multiplication by 2. Motivic weights are suppressed.

the motivic Hopf invariant one elements constructed in [12]. Recall the following correspondence between attaching maps and A–module structure.

Remark 2.14 In $H^*(\mathbb{RP}^{\infty}_{-\infty})$, Sq^1 detects a $(\cdot 2)$–attaching map, Sq^2 detects an η_{cl}–attaching map, Sq^4 detects a ν_{cl}–attaching map, and Sq^8 detects a σ_{cl}–attaching map. In $H^{**}(\underline{L}^{\infty}_{-\infty})$, Sq^1 detects a $(\cdot 2)$–attaching map, Sq^2 detects an η–attaching map, Sq^4 detects a ν–attaching map, and Sq^8 detects a σ–attaching map.

The classical part of this follows from [38, Lemma I.5.3], and the motivic part follows from [12, Remark 4.14]. One can apply the classical part of the remark to show that $\eta_{\mathrm{cl}} \in M^{\mathrm{cl}}(2)$, $\nu_{\mathrm{cl}} \in M^{\mathrm{cl}}(\eta_{\mathrm{cl}})$, and $\sigma_{\mathrm{cl}} \in M^{\mathrm{cl}}(\nu_{\mathrm{cl}})$, recovering a result of Mahowald and Ravenel [28, Proposition 2.3].

In view of the remark, the d^1–differentials in the Atiyah–Hirzebruch spectral sequence have the form $d^1(x[k])) = 2x[k-1]$ for $k \equiv 1 \bmod 2$. Examination of [21, page 8] gives the differentials in Figure 2.

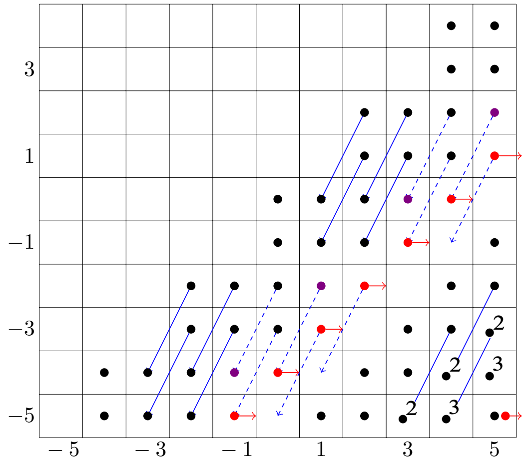


Figure 3: The E^2–page of the Atiyah–Hirzebruch spectral sequence for $-5 \le s \le 5$ and $-5 \le t \le 5$ with d^2–differentials included. A $\bullet$ represents $\mathbb{F}_2[\tau]$, a $\bullet^n$ represents $(\mathbb{F}_2[\tau])^{\oplus n}$, a violet $\bullet$ represents $\mathbb{F}_2[\tau]\{\nu\} \oplus \mathbb{F}_2\{\eta^3\}$, and a red $\bullet$ represents $\mathbb{F}_2$. Differentials are blue and τ–linear; a dashed differential means the source, target, or both are τ–torsion.

The d^2–differentials have the form $d^2(x[k])) = \eta x[k-2]$ for $k \equiv 1, 2 \bmod 4$. Examination of [21, page 8] gives the differentials in Figure 3.

The d^3–differentials have the form $d^3(x[k]) = \langle x, 2, \eta\rangle[k-3]$ for $k \equiv 3 \bmod 8$ or $d^3(x[k]) = \langle x, \eta, 2\rangle[k-3]$ for $k \equiv 5 \bmod 8$. The d^4–differentials have the

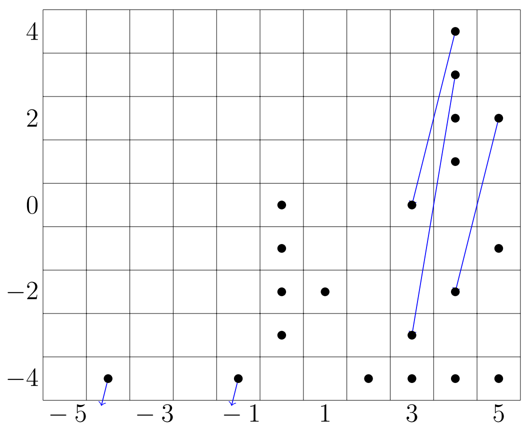


Figure 4: The E^3–page of the Atiyah–Hirzebruch spectral sequence for $-4 \le s \le 5$ and $-5 \le t \le 5$ with some of the d^3– through d^6–differentials included. A $\bullet$ represents $\mathbb{F}_2[\tau]$. Differentials are blue and τ–linear.

form $d^4(x[k]) = \nu[k-4]$ for $k \equiv 1,2,3,4 \bmod 8$. There are no d^5–differentials; these correspond to attaching maps $\eta^4\colon S^{n,m} \to S^{n-4,m-4}$, but every class of the form $\eta^4 x[k]$ is the target of a d^2–differential. The d^6–differentials have the form $d^6(x[k])) = \langle x,\eta,\nu\rangle[k-6]$ for $k \equiv 6 \bmod 16$ or $d^6(x[k]) = \langle x,\nu,\eta\rangle[k-6]$ for $k \equiv 10 \bmod 16$. Examination of [21, page 8] and [22, Table 23] gives the differentials in Figure 4.

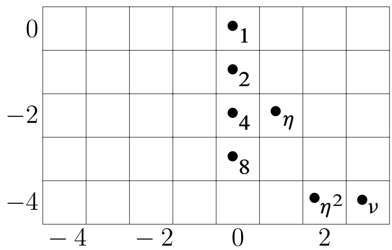


Figure 5: The E^7–page of the Atiyah–Hirzebruch spectral sequence for $-4 \le s \le 0$ and $-4 \le t \le 3$. Each $\bullet$ is labeled with the class it detects in $\pi_{**}(S^{0,0})$.

There could a priori be further differentials in this range. However, we will see in the next subsection that further differentials would produce a contradiction to classical computations of the Atiyah–Hirzebruch spectral sequence for $\Sigma\mathbb{RP}^\infty_{-\infty}$ after applying Betti realization. Therefore we have the following.

Proposition 2.15 *Figure 5 depicts the E^∞–page of the Atiyah–Hirzebruch spectral sequence for $-4 \le s \le 0$ and $-4 \le t \le 3$.*

2.4 Elementary computations

We will now compute the motivic Mahowald invariants of some classes in low degrees.

Proposition 2.16 *We have*

$$\eta \in M(2) \subset \pi_{1,1}(S), \quad \nu \in M(\eta) \subset \pi_{3,2}(S), \quad \sigma \in M(\nu) \subset \pi_{7,4}(S).$$

Proof We present the proof that $\sigma \in M(\nu)$; the other cases are similar. We must determine the minimal $N > 0$ such that the composition

$$S^{3,2} \xrightarrow{\nu} S^{0,0} \simeq \Sigma^{1,0}\underline{L}^\infty_{-\infty} \to \Sigma^{1,0}\underline{L}^\infty_{-N}$$

is nontrivial. First, we observe that the compositions where $N=1$ or $N=2$,

$$S^{3,2} \xrightarrow{\nu} S^{0,0} \simeq \Sigma^{1,0}\underline{L}^\infty_{-\infty} \to \Sigma^{1,0}\underline{L}^\infty_{-1} \quad \text{and} \quad S^{3,2} \xrightarrow{\nu} S^{0,0} \simeq \Sigma^{1,0}\underline{L}^\infty_{-\infty} \to \Sigma^{1,0}\underline{L}^\infty_{-2},$$

are trivial. This can be seen from Figure 5 since taking $N = 1$ or $N = 2$ corresponds to removing everything below $s = -1$ or $s = -3$, respectively. In filtrations $s \geq -3$, there are no classes in E^7 which could detect ν. Therefore the above compositions where $N = 1$ or $N = 2$ are trivial.

Now, we claim that the composition where $N = 3$,

$$S^{3,2} \xrightarrow{\nu} S^{0,0} \simeq \Sigma^{1,0}\underline{L}^{\infty}_{-\infty} \to \Sigma^{1,0}\underline{L}^{\infty}_{-3}, \tag{1}$$

is nontrivial. To show this, we use the Betti realization functor

$$\mathrm{Re}_{\mathbb{C}} \colon \mathrm{SH}_{\mathbb{C}} \to \mathrm{SH}$$

of Morel and Voevodsky [32, Section 3.3.2] from the motivic stable homotopy category over $\mathrm{Spec}(\mathbb{C})$ to the classical stable homotopy category. Betti realization carries motivic spheres $S^{m,n}$ to classical spheres S^m and thus induces a map $\pi_{**}(S^{0,0}) \to \pi_*(S^0)$. This can be used to show that $\mathrm{Re}_{\mathbb{C}}(\underline{L}^{\infty}_{-N}) \simeq \mathbb{RP}^{\infty}_{-2N}$. Applying Betti realization to the composite above gives

$$S^3 \xrightarrow{\eta} S^0 \simeq \Sigma\mathbb{RP}^{\infty}_{-\infty} \to \Sigma\mathbb{RP}^{\infty}_{-6},$$

which is nontrivial by the computation $\sigma_{\mathrm{cl}} \in M^{\mathrm{cl}}(\nu_{\mathrm{cl}})$ from [28, Proposition 2.3]. Thus the composition (1) is nontrivial and $M(\nu)$ is the coset of completions of the diagram

$$\begin{tikzcd}
S^{3,2} \arrow[rr, dashed] \arrow[d, "\nu"] & & S^{-4,-2} \vee S^{-5,-3} \arrow[d] \\
S^{0,0} \arrow[r] & \Sigma^{1,0}\underline{L}^{\infty}_{-\infty} \arrow[r] & \Sigma^{1,0}\underline{L}^{\infty}_{-3}
\end{tikzcd}$$

We claim that composition of the dashed arrow with projection onto the higher-dimensional sphere in the wedge $S^{-4,-2}$ is nontrivial. To see this, note that Atiyah–Hirzebruch filtration $s = -4$ is the only filtration where a class could detect ν without contradicting the upper bound given by Betti realization. Therefore $M(\nu) \subset \pi_{7,4}(S^{0,0}) \cong \mathbb{Z}/16\{\sigma\}$. By examination of the Atiyah–Hirzebruch spectral sequence E^7–page, we can see that the class detecting ν is $\sigma[-4]$. Therefore $\sigma \in M(\nu)$. □

We used a form of compatibility between the motivic Mahowald invariant and the classical Mahowald invariant in the previous argument. The following remark makes this compatibility more precise.

Remark 2.17 Let $\mathrm{Re}_{\mathbb{C}} \colon \mathrm{SH}_{\mathbb{C}} \to \mathrm{SH}$ denote Betti realization. Suppose that $\alpha, \beta \in \pi_*(S^0)$ satisfy $M^{\mathrm{cl}}(\alpha) = \beta$, and suppose that there exist $\alpha', \beta' \in \pi_{**}(S^{0,0})$ such that $\mathrm{Re}_{\mathbb{C}}(\alpha') = \alpha$ and $\mathrm{Re}_{\mathbb{C}}(\beta') = \beta$. Then

$$|M(\alpha')| \leq |\beta'|.$$

This follows from applying $\mathrm{Re}_{\mathbb{C}}$ to the diagram defining the motivic Mahowald invariant and observing that $\mathrm{Re}_{\mathbb{C}}(\underline{L}^{\infty}_{-k}) \simeq \mathbb{RP}^{\infty}_{-2k}$. A generalization of this result appears in forthcoming work of the author.

2.5 Approximations to the motivic Mahowald invariant

As we saw above, computing $M(\alpha)$ from the A–module structure of $H^{**}(\underline{L}^{\infty}_{-N})$ and the Atiyah–Hirzebruch spectral sequence for a class $\alpha \in \pi_{**}(S^{0,0})$ can be somewhat involved. We will use the following approximations to compute the motivic Mahowald invariant. First, we give a definition.

Definition 2.18 The *motivic C_2–Tate construction* of a motivic spectrum E is

$$E^{t_{\mathrm{gm}}C_2} := \varprojlim_k (\Sigma^{1,0} E \wedge \underline{L}^{\infty}_{-k}).$$

Note that this is *not* the categorical C_2–Tate construction in the motivic stable homotopy category, which would use the universal space EC_2 over the simplicial classifying space of C_2. In contrast with the cells of $\underline{L}^{\infty}$, the cells of the simplicial classifying space for C_2 are concentrated in motivic weight zero. In view of the Atiyah–Hirzebruch spectral sequence computations above and in the sequel, this would not work for our purposes.

Definition 2.19 Let E be a motivic spectrum, and let $\alpha \in \pi_{s,t}(E^{t_{\mathrm{gm}}C_2})$. We define the *motivic E–Mahowald invariant*, denoted by $M_E(\alpha)$, as follows. Consider the coset of completions of the diagram

$$\begin{tikzcd}
S^{s,t} \arrow[r, dashed] \arrow[d, "\alpha"] & \Sigma^{-2N+1,-N} E \vee \Sigma^{-2N+2,-N+1} E \arrow[d] \\
E^{t_{\mathrm{gm}}C_2} \arrow[r] & \Sigma^{1,0} E \wedge \underline{L}^{\infty}_{-N}
\end{tikzcd}$$

where $N > 0$ is the minimal value such that the left-hand composition is nontrivial. If the composition of the dashed arrow with the projection onto the more highly suspended wedge summand is nontrivial, we define $M_E(\alpha)$ to be the coset of completions composed with the projection onto this summand. Otherwise, as in Definition 2.10, we define $M_E(\alpha)$ to be the coset of completions composed with the projection onto the less highly suspended wedge summand.

We will often use splittings $\pi_{**}(E^{t_{\mathrm{gm}}C_2}) \cong \bigoplus_{i\in\mathbb{Z}} \pi_{**}(\Sigma^{si,ti}R)$. An element $x \in \pi_{**}(R)$ defines an element $\alpha \in \pi_{**}(E^{t_{\mathrm{gm}}C_2})$ by mapping into the summand with $i=0$, and we will use the notation $M_E(x)$ instead of $M_E(\alpha)$ in this situation.

Note that if $E = S$, this recovers the motivic Mahowald invariant. The following proposition is a motivic analog of [28, Theorem 2.15]. It will allow us to "lift" motivic E–Mahowald invariant computations to motivic Mahowald invariant computations.

Proposition 2.20 *Let E be a unital motivic spectrum with unit map $\eta\colon S^{0,0} \to E$ and let $x \in \pi_{s,t}(S^{0,0})$. If $M_E(x)$ is a coset in $\pi_{**}(E)$ containing a class in the image of a class $y \in \pi_{**}(S^{0,0})$ such that the diagram*

$$\begin{tikzcd} S^{s,t} \arrow[r, "y"] \arrow[d, "x"] & S^{-2N+1,-N} \vee S^{-2N+2,-N+1} \arrow[d, "i"] \\ S^{0,0} \arrow[r, "h"] & \Sigma^{1,0}\underline{L}^{\infty}_{-N} \end{tikzcd}$$

commutes, then $y \in M(x)$.

Proof Consider the commutative diagram

$$\begin{tikzcd} & & \Sigma^{1,0}\underline{L}^{\infty}_{-N+1} \arrow[r] & \Sigma^{1,0}\underline{L}^{\infty}_{-N+1} \wedge E \\ S^{s,t} \arrow[r, "x"] & S^{0,0} \arrow[r, "h"] & \Sigma^{1,0}\underline{L}^{\infty}_{-N} \arrow[r, "\eta"] \arrow[u] & \Sigma^{1,0}\underline{L}^{\infty}_{-N} \wedge E \arrow[u] \\ & & S^{-2N+1,-N} \vee S^{-2N+2,-N} \arrow[r] \arrow[u] & \Sigma^{-2N+1,-N}E \vee \Sigma^{-2N+2,-N}E \arrow[u] \end{tikzcd}$$

If $\eta \circ h \circ x$ is essential, then $h \circ x$ is essential and $y \in M(x)$ by commutativity of the diagram in the hypothesis. $\square$

The motivic Mahowald invariant is calculated by determining the exact cell of $\underline{L}^{\infty}_{-\infty}$ where a class $\alpha \in \pi_{**}(S^{0,0})$ is detected. We will occasionally want a "coarser" invariant which is calculated by determining on which collection of adjacent cells a class is detected. Our definition is modified from [28, Definition 2.12].

Definition 2.21 We say that a finite motivic spectrum X is a *periodic subcomplex of $\underline{L}^{\infty}_{-\infty}$* if the continuous motivic cohomology $H^{**}_c(\underline{L}^{\infty}_{-\infty})$ decomposes additively as a direct sum of the motivic cohomology of suspensions of X, ie we have an additive isomorphism

$$H_c^{**}(\Sigma^{1,0}\underline{L}_{-\infty}^{\infty}) \cong \bigoplus_{i\in\mathbb{Z}} H^{**}(\Sigma^{a+ki,b+\ell i}X)$$

where $a,b\in\mathbb{Z}$ and $k,\ell\geq 0$.

For the periodic subcomplexes X we will consider, there will exist $m,n\in\mathbb{Z}$ with $m<n$ such that X is a wedge summand of a suspension of $\underline{L}_m^n$. Furthermore, X will be an k–cell complex with one cell in each topological dimension $0\leq d\leq k-1$. The following two examples of periodic subcomplexes will be used in the sequel.

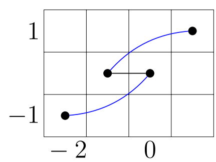
Example 2.22 Let $X=V(0)$ be the mod 2 Moore spectrum. Then examination of Figure 1 shows that

$$H_c^{**}(\Sigma^{1,0}\underline{L}_{-\infty}^{\infty}) \cong \bigoplus_{i\in\mathbb{Z}} H^{**}(\Sigma^{2i,i}V(0)),$$

so we may take $a=b=0$, $k=2$, and $\ell=1$ in Definition 2.21. Furthermore, if we set $m=-1$ and $n=0$, then we have

$$\Sigma^{1,0}\underline{L}_0^1 \simeq S^{-1,-1}\vee V(0)\vee S^{2,1}.$$

Example 2.23 Let X be the 4–cell complex with motivic cohomology depicted below:



Examination of Figure 1 shows that

$$H_c^{**}(\Sigma^{1,0}\underline{L}_{-\infty}^{\infty}) \cong \bigoplus_{i\in\mathbb{Z}} H^{**}(\Sigma^{1+4i,2i}X),$$

so we may take $a=1$, $b=0$, $k=4$, and $\ell=2$ in Definition 2.21. Furthermore, if we set $m=-1$ and $n=0$, then we have

$$\Sigma^{-1,0}\underline{L}_{-1}^{0} \simeq X.$$

Definition 2.24 Let $\alpha\in\pi_{s,t}(S^{0,0})$ and let X be one of the periodic subcomplexes of $\underline{L}_{-\infty}^{\infty}$ defined in Examples 2.22 and 2.23. The *motivic Mahowald invariant based on* X

of α is the coset of completions of the diagram

$$\begin{tikzcd} S^{s,t} \arrow[rr, dashed] \arrow[d, "\alpha"] & & \Sigma^{1,0}\underline{L}_{c-2N}^{c-2N+1} \arrow[d] \\ S^{0,0} \arrow[r, "\simeq"] & \varprojlim_n \Sigma^{1,0}\underline{L}_{c-2n}^{\infty} \arrow[r] & \Sigma^{1,0}\underline{L}_{c-2N}^{\infty} \end{tikzcd}$$

where $c \in \{0,1\}$ and $N > 0$ are minimal values such that the left-hand composition is nontrivial and the composition of the dashed arrow with the projection onto the suspension of X which is a wedge summand of $\Sigma^{1,0}\underline{L}_{c-2N}^{c-2N+1}$ is nontrivial. The motivic Mahowald invariant based on X of α will be denoted by $M(\alpha; X)$.

We can combine Definitions 2.19 and 2.24 to obtain another approximation.

Definition 2.25 Let $\alpha \in \pi_{s,t}(E^{t_{\mathrm{gm}}C_2})$ and let X be one of the periodic subcomplexes of $\underline{L}_{-\infty}^{\infty}$ defined in Examples 2.22 and 2.23. The *motivic E–Mahowald invariant based on X* of α is the coset of completions of the diagram

$$\begin{tikzcd} S^{s,t} \arrow[rr, dashed] \arrow[d, "\alpha"] & & \Sigma^{1,0}\underline{L}_{c-2N}^{c-2N+1} \wedge E \arrow[d] \\ E^{t_{\mathrm{gm}}C_2} \arrow[r, "\simeq"] & \varprojlim_n(\Sigma^{1,0}\underline{L}_{c-2n}^{\infty} \wedge E) \arrow[r] & \Sigma^{1,0}\underline{L}_{c-2N}^{\infty} \wedge E \end{tikzcd}$$

where $c \in \{0,1\}$ and $N > 0$ are minimal values such that the left-hand composition is nontrivial and the composition of the dashed arrow with projection onto the suspension of $X \wedge E$ which is a wedge summand of $\Sigma^{1,0}\underline{L}_{c-2N}^{c-2N+1} \wedge E$ is nontrivial. The motivic E–Mahowald invariant based on X of α will be denoted by $M_E(\alpha; X)$.

Remark 2.26 The approximations defined above are the motivic analogs of ones defined in [28]. For example, they compute the approximation $M_{\mathrm{ko}}^{\mathrm{cl}}(2^i; V(0))$ in order to show that $M^{\mathrm{cl}}(2^i)$ consists of certain v_1–periodic families in the image of J.

3 Motivic C_2–Tate constructions of some motivic ring spectra

In order to compute the motivic E–Mahowald invariant, we need to understand the C_2–Tate constructions of certain motivic ring spectra. The classical result for which we need motivic analogs is the following:

Theorem 3.1 [9] *There is an equivalence of spectra*

$$\mathrm{ko}^{tC_2} \simeq \bigvee_{i\in\mathbb{Z}} \Sigma^{4i} H\mathbb{Z}.$$

This splitting is used in [28] to compute $M^{\mathrm{cl}}(2^i)$. To compute the (motivic) E–Mahowald invariant, it suffices to produce this splitting at the level of homotopy groups. Before proceeding, we make the following convention to simplify notation.

3.1 Splitting needed for computing $M(2^i)$

In the classical setting, Mahowald and Ravenel use the ko–Mahowald invariant to compute $M^{\mathrm{cl}}(2^i)$. We will use the motivic analog of ko, connective Hermitian K–theory kq, to compute $M(2^i)$. The properties of this spectrum we need are due to Isaksen and Shkembi.

Lemma 3.2 [23] *The motivic cohomology of* kq *is given by*

$$H^{**}(\mathrm{kq}) \cong A /\!/ A(1)$$

where $A(1)$ is the subalgebra of the motivic Steenrod algebra generated by Sq^1 and Sq^2.

The motivic homotopy groups of kq *are given by*

$$\pi_{**}(\mathrm{kq}) \cong \mathbb{Z}_2[\tau,\eta,\alpha,\beta]/(2\eta, \tau\eta^3, \eta\alpha, \alpha^2-4\beta)$$

with $|\tau| = (0,-1)$, $|\eta| = (1,1)$, $|\alpha| = (4,2)$, and $|\beta| = (8,4)$.

Proposition 3.3 *There is an isomorphism in homotopy groups*

$$\pi_{**}(\mathrm{kq}^{t_{\mathrm{gm}}C_2}) \simeq \varprojlim_n \bigoplus_{i\geq -n} \pi_{**}(\Sigma^{4i,2i} H\mathbb{Z}_2).$$

Proof Consider the Atiyah–Hirzebruch spectral sequence resulting from the cellular filtration of $\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}$. This spectral sequence has the form

$$E^1_{s,t,u} = \mathrm{kq}_{t,u}(S^{s,\lfloor s/2\rfloor}) = \pi_{t-s,u-\lfloor s/2\rfloor}(\mathrm{kq}) \Rightarrow \pi_{t,u}(\mathrm{kq}^{t_{\mathrm{gm}}C_2}).$$

This spectral sequence is depicted in Figure 6.

Note that η is not nilpotent in $\pi_{**}(\mathrm{kq})$, so in each s–degree, an additional η–tower appears every eight $(t+s)$–degrees. The first two powers of η always support a τ–tower and the remaining elements of the tower are copies of $\mathbb{F}_2$. Further, the motivic weight of each copy of kq_{**} changes by 1 every two s–degrees.

Differentials in this spectral sequence are induced by the attaching maps in $\underline{L}^{\infty}_{-\infty}$ detected by the generators of $A(1)$. There are d^1–differentials between all the

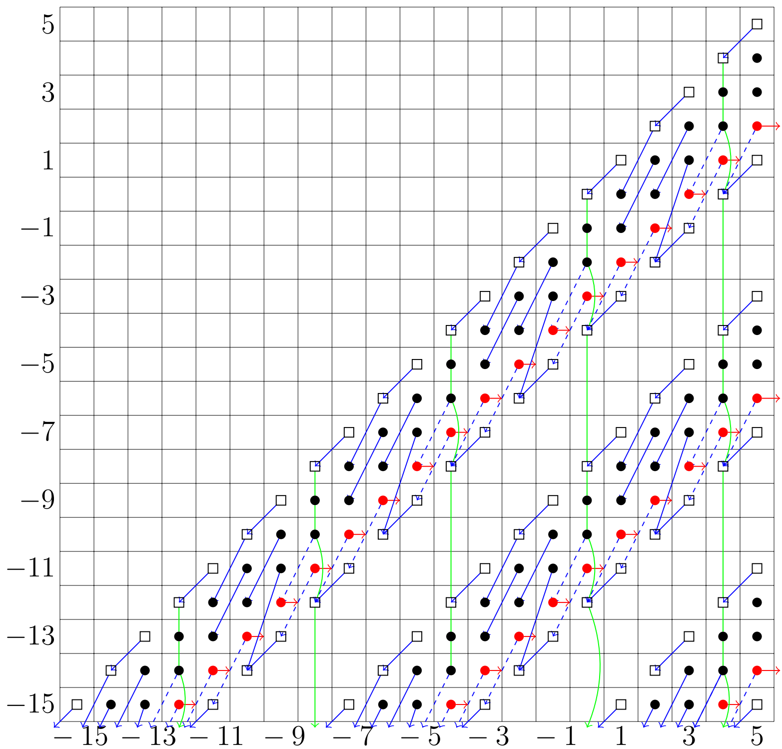


Figure 6: The Atiyah–Hirzebruch spectral sequence for $-9 \leq s \leq 1$ and $-9 \leq t \leq 9$ with some of the differentials drawn in. The differentials are periodic and can be propagated downwards by examining the A–module structure of $H^{**}(\Sigma^{1,0}\underline{L}^2_{-9})$. A $\square$ represents $\mathbb{Z}_{(2)}[\tau]$, a $\bullet$ represents $\mathbb{F}_2[\tau]$, and a red $\bullet$ represents $\mathbb{F}_2$. Differentials are blue and τ–linear; a dashed differential means the source, target, or both are τ–torsion. Green lines indicate hidden extensions.

copies of $\mathbb{Z}_2[\tau]$, so the targets represent $\mathbb{F}_2[\tau]$ on the E^2–page. There are three d^2–differentials which correspond to the classical case; unlike the classical case, there is an additional differential from the element corresponding to η^2 to the element corresponding to η^3 two s–degrees lower. This differential propagates through the entire η–tower, but since $\eta^3\tau = 0$, it only annihilates one copy of $\mathbb{F}_2$ in the copy of $\mathbb{F}_2[\tau]$ contributed by η^2. With this copy of $\mathbb{F}_2$ eliminated, the τ–towers on η^2 in $s = 1$ and the τ–tower on α in $s = -2$ begin in the same motivic weight. The d_3–differentials have the form $d^3(\tau\eta^2[s]) = \alpha[s-3]$ for $s \equiv 1 \bmod 4$, which follows from the Toda bracket $\alpha = \langle \tau\eta^2, \eta, 2\rangle$ in $\pi_{**}(\mathrm{kq})$. Therefore the d^3–differentials between the towers completely annihilate them, and only copies of $\mathbb{F}_2[\tau]$ remain in the abutment.

Comparison with the Adams spectral sequence allows us to determine the hidden multiplication by 2 extensions. The E_2–page of the inverse limit motivic Adams spectral sequence [16, Section 4.2] computing $\pi_{**}(\mathrm{kq} \wedge \Sigma^{1,0}\underline{L}^{\infty}_{-\infty})$ has the form

$$E_2^{***} = \varprojlim_n \mathrm{Ext}_A^{***}(A /\!\!/ A(1) \otimes H^{**}(\Sigma^{1,0}\underline{L}^{\infty}_{-n}), \mathbb{M}_2).$$

By change-of-rings, we can rewrite this E_2–term as

$$\varprojlim_n \mathrm{Ext}_{A(1)}^{***}(H^{**}(\Sigma^{1,0}\underline{L}^{\infty}_{-n}), \mathbb{M}_2).$$

Therefore we just need to understand the action of $A(1)$ on $H^{**}(\Sigma^{1,0}\underline{L}^{\infty}_{-n})$. Recall that Sq^{2^i} acts nontrivially on an element $x \in H^{n,\lceil n/2\rceil}(\underline{L}^{\infty}_{-k})$ if and only if the $(i-1)^{\mathrm{st}}$ digit in the dyadic expansion for n is 1. Using this fact we observe the following:

(1) Sq^1 acts nontrivially on all cells with odd topological dimension since they have dyadic expansion ending in 1. This follows from the relation $\mathrm{Sq}^1(v^i u) = v^{i+1}$.

(2) Sq^2 acts nontrivially on all cells with topological dimension congruent to $2, 3$ mod 4. This follows from the relations $\mathrm{Sq}^2(v^{2i+1}) = v^{2i+2}$ and $\mathrm{Sq}^2(uv^{2i+1}) = uv^{2i+2}$.

We can define a filtration of $\mathrm{Ext}_{A(1)}^{***}(H_c^{**}(\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}), \mathbb{M}_2)$ with associated graded consisting of suspensions of $A(1) /\!\!/ A(0)$ as follows. This filtration arises from a filtration of $H_c^{**}(\underline{L}^{\infty}_{-\infty})$ defined by setting $F_n \subset H_c^{**}(\underline{L}^{\infty}_{-\infty})$ to be the complement of $H_c^{2n,*}(\Sigma^{1,0}\underline{L}^{\infty}_{-\infty})$ inside $H_c^{\leq 2n+1,*}(\underline{L}^{\infty}_{-\infty})$, ie

$$F_n := H_c^{\leq 2n+1,*}(\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}) \setminus H_c^{2n,*}(\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}).$$

Then each bidegree (i,j) of $H_c^{**}(\underline{L}^{\infty}_{-\infty})$ where both generators of $A(1)$ act trivially contributes a copy of $\mathrm{Ext}_{A(0)}(\mathbb{M}_2, \mathbb{M}_2) \cong \pi_{**}H\mathbb{Z}$ in the resulting spectral sequence:

$$\bigoplus_{i \in \mathbb{Z}} \Sigma^{4i,2i}\,\mathrm{Ext}_{A(0)}^{***}(\Sigma^{1,0}\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \mathrm{Ext}_{A(1)}^{***}(H^{**}(\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}), \mathbb{M}_2).$$

The differentials in this spectral sequence change tridegree by $(-1,-1,0)$, so there is no room for differentials. The resulting inverse limit motivic Adams spectral sequence collapses for tridegree reasons, so we obtain the desired extensions. □

The same proof with $E(Q_0, Q_1)$ in place of $A(1)$ and $\pi_{**}(\mathrm{BPGL}\langle 1\rangle) \cong \mathbb{Z}_2[\tau, v_1]$ applies mutatis mutandis to prove the following corollary.

Corollary 3.4 *There is an isomorphism in homotopy groups*

$$\pi_{**}(\mathrm{BPGL}\langle 1\rangle^{t_{\mathrm{gm}}C_2}) \cong \varprojlim_n \bigoplus_{i \geq -n} \pi_{**}(\Sigma^{4i,2i}\mathrm{BPGL}\langle 0\rangle).$$

Remark 3.5 It should be possible to apply the same proof using the Adams spectral sequence with $E(Q_0, \dots, Q_n)$ in place of $A(1)$ and with $\pi_{**}(\mathrm{BPGL}\langle n\rangle) \cong \mathbb{M}_2[v_0, v_1, \dots, v_n]$ in place of $\pi_{**}(\mathrm{kq})$ to produce an analogous splitting in homotopy groups for $\mathrm{BPGL}\langle n\rangle^{t_{\mathrm{gm}} C_2}$. In order to solve extensions, one can use induction on n coupled with the induced maps of spectral sequences from the map $\mathrm{BPGL}\langle n-1\rangle \to \mathrm{BPGL}\langle n\rangle$. We conjecture that for any $n \geq 1$, there is an isomorphism in homotopy groups

$$\pi_{**}(\mathrm{BPGL}\langle n\rangle^{t_{\mathrm{gm}} C_2}) \simeq \varprojlim_n \bigoplus_{i \geq -n} \pi_{**}(\Sigma^{4i,2i}\mathrm{BPGL}\langle n-1\rangle).$$

This is a weak motivic analog of a conjecture of Davis and Mahowald [9, Conjecture 1.6].

3.2 Splitting needed for computing $M(\eta^i)$

The material in this section will be needed to compute $M(\eta^i)$. We begin by recalling essential information about the category of modules over the cofiber of τ studied extensively in [13; 14; 15]. The cofiber of the map $\tau \in \pi_{0,-1}(S^{0,0})$ is a E_∞ motivic ring spectrum [14] and therefore so is the $C\tau$–induced Eilenberg–Mac Lane spectrum $\overline{H} := H\mathbb{F}_2 \wedge C\tau$. The following proposition describes its cooperations and operations in the motivic stable homotopy category.

Proposition 3.6 [14, Propositions 5.4–5.5] *The ring of cooperations of $\overline{H}$ is*

$$\pi_{**}(\overline{H} \wedge \overline{H}) \cong \mathbb{F}_2[\xi_1, \xi_2, \dots] \otimes E(\tau_0, \tau_1, \dots) \otimes E(\beta_\tau)$$

where β_τ is the τ–Bockstein with bidegree $(1,-1)$. The $\overline{H}$–Steenrod algebra is given by

$$\overline{H}^{**}(\overline{H}) \cong A/\tau \otimes E(\beta_\tau).$$

This proposition simplifies when working in the category of $C\tau$–modules. The *$C\tau$–linear $\overline{H}$–homology* of a $C\tau$–module is defined by taking $\overline{H}$–homology in the category of $C\tau$–modules, ie

$$\overline{H}_{**}(X) = \pi_{**}(\overline{H} \wedge_{C\tau} X).$$

By expanding the right-hand side, we see that

$$\overline{H}_{**}(X) \cong \pi_{**}(H \wedge C\tau \wedge_{C\tau} X) \cong H_{**}(X).$$

It turns out that working with the $C\tau$–linear setting simplifies certain computations. For example, the $C\tau$–linear cooperations of $\overline{H}$ do not contain a τ–Bockstein.

Proposition 3.7 [13, Proposition 2.5] *The $C\tau$–linear cooperations of $\overline{H}$ are given by the Hopf algebra*

$$\overline{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \dots] \otimes E(\tau_0, \tau_1, \dots)$$

with bidegrees given by $|\xi_n| = (2^{n+1}-2, 2^n-1)$ *and* $|\tau_n| = (2^{n+1}-1, 2^n-1)$, *and coproduct*

$$\Delta(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \xi_i, \quad \Delta(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \tau_i.$$

Remark 3.8 Any time we use $\overline{H}$ or $\overline{A}$, we will be working in the $C\tau$–linear setting, ie the category of $C\tau$–modules. Note that any $C\tau$–module can be regarded as an $S^{0,0}$–module by composing with the inclusion of the bottom cell $S^{0,0} \to C\tau$.

Following [13, Notation 2.11], let P_i be the dual of ξ_i for $i \geq 1$. Then P_i is exterior and primitive in $\overline{A}$. In [13], Gheorghe constructs an E_∞ motivic ring spectrum wBP satisfying

$$\overline{H}^{**}(\mathrm{wBP}) \cong \overline{A} /\!/ E(P_1, P_2, \dots)$$

which has homotopy groups given by

$$\pi_{**}(\mathrm{wBP}) \cong \mathbb{F}_2[w_0, w_1, \dots]$$

with $|w_i| = (2^{i+2}-3, 2^{i+1}-1)$.

The inclusion of the bottom cell $S^{0,0} \to C\tau$ induces a map $A \to \overline{A}$. Identifying classes in $\overline{A}$ with the images of the motivic Steenrod operations under this map, the P_i can be defined inductively as the images of certain commutators [13, Example 2.12]:

$$P_i = [\mathrm{Sq}^{2^i}, P_{i-1}].$$

In particular, $P_1 = \mathrm{Sq}^2$ detects η–attaching maps in motivic cohomology and detects w_0 in the homotopy of wBP.

In [13], Gheorghe also constructs E_∞ motivic ring spectra $\mathrm{wBP}\langle n\rangle$ such that

$$\pi_{**}(\mathrm{wBP}\langle n\rangle) \cong \mathbb{F}_2[w_0, \dots, w_n].$$

In order to compute $M(\eta^{4i})$, it suffices to produce a splitting of $\pi_{**}(\mathrm{wBP}\langle 1\rangle^{t_{\mathrm{gm}} C_2})$ as a wedge of suspensions of $\pi_{**}(\mathrm{wBP}\langle 0\rangle)$. However, we would like to compute $M(\eta^i)$ for all $i \geq 1$, so we must produce a new motivic spectrum. More precisely, we need a $C\tau$–module analog of classical ko or motivic kq. To build this $C\tau$–module, we use the following result.

Theorem 3.9 [15] *There is an equivalence of stable ∞–categories with t–structures*

$$C\tau\text{–mod}^b_{\text{cell}} \to \mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{–comod})$$

whose restriction to the hearts is taking BPGL*–homology. Here, $C\tau$–$\mathrm{mod}^b_{\text{cell}}$ is the category of cellular module spectra over $C\tau$ whose* BPGL*–homology has bounded Chow degree, and $D^b(\mathrm{BP}_*\mathrm{BP}\text{–comod})$ is the bounded derived category of the abelian category of p–completed* $\mathrm{BP}_*\mathrm{BP}$*–comodules which are concentrated in even degrees.*

Using this theorem, we can define a $C\tau$–module that detects the classes of interest.

Definition 3.10 Recall that $\mathrm{BP}_* \cong \mathbb{Z}_{(2)}[v_1, v_2, \ldots]$ and $\mathrm{BP}_*\mathrm{BP} \cong \mathrm{BP}_*[t_1, t_2, \ldots]$. We define wko to be the $C\tau$–module corresponding to the $\mathrm{BP}_*\mathrm{BP}$–comodule

$$\mathrm{BP}_*[t_1^4, t_2^2, t_3, \ldots]$$

under the above equivalence of categories.

Remark 3.11 Analogously, one can define an exotic motivic analog of the motivic modular forms spectrum mmf [34] by defining wtmf to be the $C\tau$–module corresponding to the $\mathrm{BP}_*\mathrm{BP}$–comodule

$$\mathrm{BP}_*[t_1^8, t_2^4, t_3^2, t_4, \ldots]$$

under the above equivalence of categories.

By construction, we have $\overline{H}^{**}(\mathrm{wko}) \cong \bar{A} /\!/ \bar{A}(1)$ where $\bar{A}(1)$ is the subalgebra of $\bar{A}$ generated by Sq^2 and Sq^4. To calculate its motivic homotopy groups, we will use the $C\tau$–linear $\overline{H}$–based Adams spectral sequence constructed in [13, Section 2.4]. This spectral sequence has the form

$$\mathrm{Ext}^{s,t,w}_{\bar{A}}(\overline{H}^{**}(X), \mathbb{F}_2) \Rightarrow [\Sigma^{t-s,w} C\tau, X]_{C\tau}.$$

Note that $\pi_{**}(X) \cong [\Sigma^{**} C\tau, X]_{C\tau}$ by the usual adjunction, so this spectral sequence computes the motivic homotopy groups of X.

Lemma 3.12 *The homotopy of* wko *is*

$$\pi_{**}(\mathrm{wko}) \cong \mathbb{F}_2[\eta, \nu, \tilde{\alpha}, \tilde{\beta}]/(\eta\nu, \nu^3, \nu\tilde{\alpha}, \tilde{\alpha}^2 - \eta^2\tilde{\beta})$$

where $|\eta| = (1,1)$, $|\nu| = (3,2)$, $|\tilde{\alpha}| = (11,7)$ and $|\tilde{\beta}| = (20,12)$.

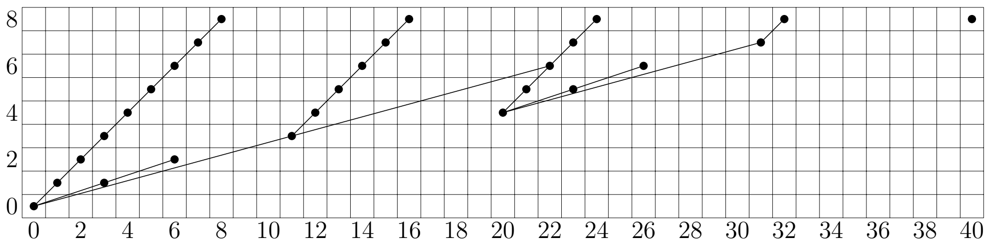


Figure 7: The E_2–page of the $C\tau$–linear $\overline{H}$–based Adams spectral sequence converging to $\pi_{**}(\mathrm{wko})$. A • represents $\mathbb{F}_2$. Lines of slope $\frac{1}{2}$ represent multiplication by η, lines of slope $\frac{1}{3}$ represent multiplication by ν, and lines of slope $\frac{3}{11}$ represent multiplication by $\widetilde{\alpha}$.

Proof The $C\tau$–linear $\overline{H}$–based Adams spectral sequence has the form

$$E_2 = \mathrm{Ext}^{***}_{\overline{A}}(\overline{H}^{**}(\mathrm{wko}), \mathbb{F}_2) \Rightarrow \pi_{**}(\mathrm{wko}).$$

By change-of-rings, the E_2–page is isomorphic to

$$E_2 \cong \mathrm{Ext}^{***}_{\overline{A}(1)}(\mathbb{F}_2, \mathbb{F}_2).$$

The subalgebra $\overline{A}(1)$ is isomorphic to the subalgebra $A(1)$ of the classical Steenrod algebra if we replace Sq^0 by Sq^1 and Sq^1 by Sq^2. Therefore the E_2–page is obtained from the classical $\mathrm{Ext}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$ (which can be computed by using an $A(1)$–resolution) by the following algorithm:

(1) For each classical $\mathbb{F}_2$ in bidegree $(s, t-s)$, place a copy of $\mathbb{F}_2$ in tridegree $(s, 2t-s, t)$.

(2) Replace h_0–extensions between classes in the classical Ext with h_1–extensions between the corresponding classes in the new Ext.

(3) Replace h_1–extensions between classes in the classical Ext with h_2–extensions between the corresponding classes in the new Ext.

Applying this algorithm, we arrive at the the E_2–page depicted in Figure 7.

There is no room for differentials, so we obtain the desired isomorphism. □

Proposition 3.13 *There is an isomorphism*

$$\pi_{**}(\mathrm{wko}^{t_{\mathrm{gm}}C_2}) \cong \varprojlim_n \bigoplus_{i \geq -n} \big(\Sigma^{8i-1,4i-1}\pi_{**}(\mathrm{wBP}\langle 0\rangle) \oplus \Sigma^{8i,4i}\pi_{**}(\mathrm{wBP}\langle 0\rangle)\big).$$

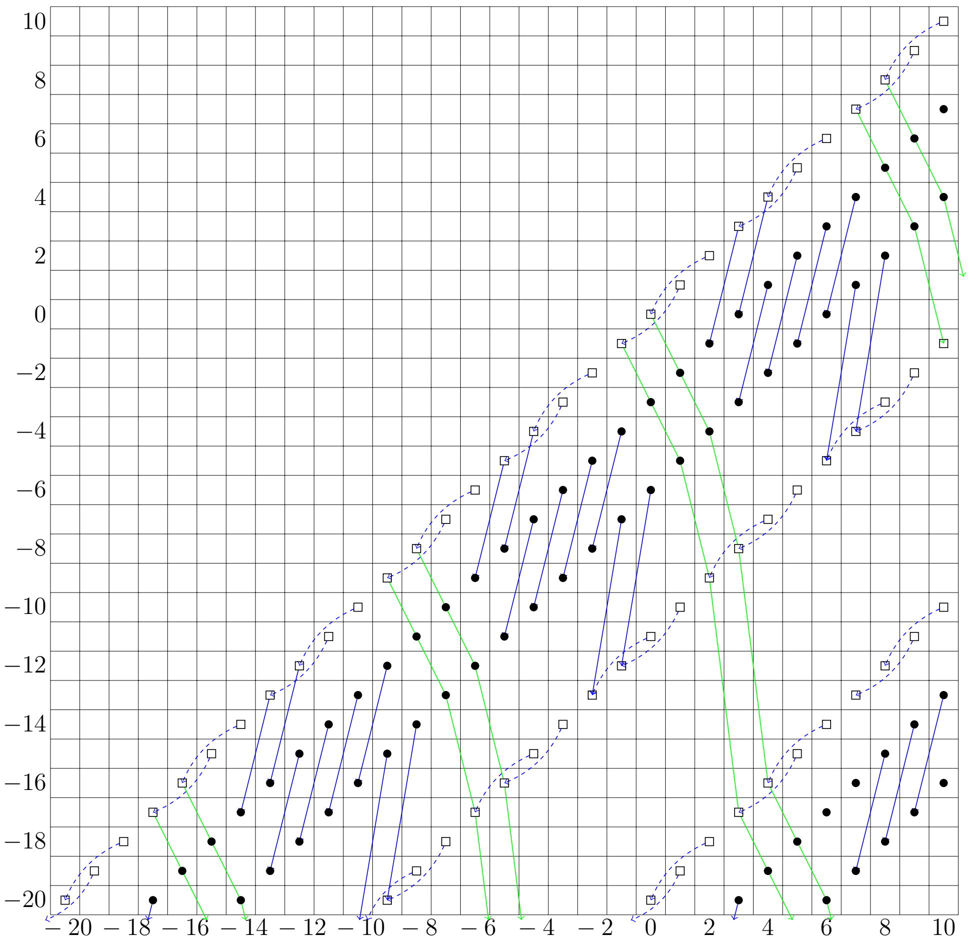


Figure 8: The Atiyah–Hirzebruch spectral sequence for $-20 \leq s \leq 1$ and $-20 \leq t \leq 20$ with some of the differentials drawn in. These differentials are periodic and can be propagated downwards by examining the $\bar{A}(1)$–module structure of $H^{**}(\underline{L}^{5}_{-10})$. A $\square$ represents $\mathbb{F}_2[\eta]$, a $\bullet$ represents $\mathbb{F}_2$, and a horizontal line represents multiplication by η.

Proof Since we need to know which cell of $\underline{L}^{\infty}_{-\infty}$ each class is detected on in the sequel, we begin by proving the stated isomorphism additively using the Atiyah–Hirzebruch spectral sequence. The Atiyah–Hirzebruch spectral sequence arising from the cellular filtration of $\Sigma^{1,0}\underline{L}^{\infty}_{-\infty}$ has the form

$$E^1_{s,t,u} = \mathrm{wko}_{t,u}(S^{s,\lfloor s/2\rfloor}) = \pi_{t+s,u+\lfloor s/2\rfloor}(\mathrm{wko}) \Rightarrow \pi_{s+t,u+\lceil s/2\rceil}(\mathrm{wko}^{t_{\mathrm{gm}}C_2}).$$

This spectral sequence is depicted in Figure 8.

The differentials can be read off from the action of $\bar{A}(1) = \langle \mathrm{Sq}^2, \mathrm{Sq}^4 \rangle$ on $H^{**}(\underline{L}_{-\infty}^{\infty})$. This action is given by $\mathrm{Sq}^{2i}(v^k) = \binom{2k}{2i} v^{k+i}$ and $\mathrm{Sq}^{2i}(uv^k) = \binom{2k}{2i} uv^{k+i}$. In particular, we observe the following:

(1) Sq^2 acts nontrivially on all cells in topological dimension congruent to $2, 3 \bmod 4$.

(2) Sq^4 acts nontrivially on all cells in topological dimension congruent to $4, 5, 6, 7 \bmod 8$.

To solve extensions, we use the inverse limit $C\tau$–linear $\overline{H}$–based Adams spectral sequence, where the inverse limit is taken over the negative skeleta of $\Sigma^{1,0}\underline{L}_{-n}^{\infty}$. Then we have

$$E_2^{***} = \varprojlim_n \mathrm{Ext}_{\bar{A}}^{***}(\bar{A} /\!/ \bar{A}(1) \otimes \overline{H}^{**}(\Sigma^{1,0}\underline{L}_{-n}^{\infty}), \mathbb{F}_2) \Rightarrow \pi_{**}(\mathrm{wko}^{t_{\mathrm{gm}}C_2}).$$

Here we have used that $C\tau$–linear $\overline{H}$–cohomology satisfies the strong Künneth theorem for smash products over $S^{0,0}$ [13, Proposition 2.9]. By change-of-rings, we can rewrite the E_2–term as

$$\varprojlim_n \mathrm{Ext}_{\bar{A}(1)}(\overline{H}^{**}(\Sigma^{1,0}\underline{L}_{-n}^{\infty}), \mathbb{F}_2),$$

so we must understand the action of Sq^2 and Sq^4 on $\overline{H}^{**}(\Sigma^{1,0}\underline{L}_{-n}^{\infty})$. This cohomology is $\overline{H}^{**}(\Sigma^{1,0}\underline{L}_{-n}^{\infty}) = \pi_{**}(H\mathbb{F}_2 \wedge C\tau \wedge \Sigma^{1,0}\underline{L}_{-n}^{\infty}) \cong \Sigma^{-2n+1,-n}\mathbb{F}_2[u,v]/(u^2)$ since smashing with $C\tau$ kills multiplication by τ [14, Lemma 5.3]. The map $\Sigma^{1,0}\underline{L}_{-n}^{\infty} \to C\tau \wedge \Sigma^{1,0}\underline{L}_{-n}^{\infty}$ sends $\Sigma^{1,0}u$ and $\Sigma^{1,0}v$ to the generators of the same name, so we can compute the action of the generators of $\bar{A}(1)$ on $\overline{H}^{**}(\Sigma^{1,0}\underline{L}_{-n}^{\infty})$ using the relations above. Therefore $\mathrm{Sq}^2 \in \bar{A}(1)$ and $\mathrm{Sq}^4 \in \bar{A}(1)$ both act nontrivially on the cells in topological dimension congruent to $6, 7 \bmod 8$. Each such cell corresponds to a direct summand in the following algebraic spectral sequence with associated graded consisting of suspensions of $\bar{A}(1) /\!/ \bar{A}(0)$, which can be constructed like the analogous spectral sequence in the proof of Proposition 3.3:

$$\bigoplus_{i \in \mathbb{Z}} \Sigma^{8i-1,4i-1}\mathrm{Ext}_{\bar{A}(0)}(\mathbb{F}_2) \oplus \Sigma^{8i,4i}\mathrm{Ext}_{\bar{A}(0)}(\mathbb{F}_2) \Rightarrow \mathrm{Ext}_{\bar{A}(1)}(\overline{H}_c^{**}(\Sigma^{1,0}\underline{L}_{-\infty}^{\infty})).$$

The differentials in this spectral sequence change (topological, homological) bidegree by $(-1,-1)$ and preserve motivic weight, so the spectral sequence collapses. The direct sum of resulting Adams spectral sequences also collapses, producing Figure 9, which solves the extensions in the Atiyah–Hirzebruch spectral sequence. □

Remark 3.14 We could have proven the previous proposition using just the $C\tau$–linear $\overline{H}$–based Adams spectral sequence, but we will use the detailed Atiyah–Hirzebruch

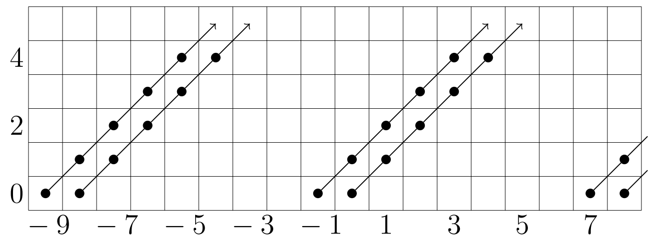


Figure 9: The inverse limit $C\tau$–linear $\overline{H}$–based Adams spectral sequence for $\mathrm{wko}^{t_{\mathrm{gm}} C_2}$ for $-9 \leq t-s \leq 8$ and $0 \leq s \leq 5$.

spectral sequence computations from the proof in order to compute $M_{\mathrm{wko}}(\eta^i)$ in the next section.

As in the case of kq and $\mathrm{BPGL}\langle 1\rangle$, the same proof applies mutatis mutandis to produce a splitting in homotopy groups for $\mathrm{wBP}\langle 1\rangle^{t_{\mathrm{gm}} C_2}$.

Corollary 3.15 *There is an isomorphism in homotopy groups*

$$\pi_{**}(\mathrm{wBP}\langle 1\rangle^{t_{\mathrm{gm}} C_2}) \cong \bigoplus_{i \in \mathbb{Z}} \big(\Sigma^{4i-1,2i-1}\pi_{**}(\mathrm{wBP}\langle 0\rangle) \oplus \Sigma^{4i,2i}\pi_{**}(\mathrm{wBP}\langle 0\rangle)\big).$$

Remark 3.16 An analogous computation replacing $E(P_0, P_1)$ by $E(P_0, P_1, \ldots, P_n)$, along with an induction on n to solve extensions should produce a similar splitting in homotopy groups for $\mathrm{wBP}\langle n\rangle^{t_{\mathrm{gm}} C_2}$. We conjecture in analogy with Remark 3.5 that for any $n \geq 1$, there is an isomorphism in homotopy groups

$$\pi_{**}(\mathrm{wBP}\langle n\rangle^{t_{\mathrm{gm}} C_2}) \cong \bigoplus_{i \in \mathbb{Z}} \big(\Sigma^{4i,2i}\pi_{**}(\mathrm{wBP}\langle n-1\rangle) \oplus \Sigma^{4i-1,2i-1}\pi_{**}(\mathrm{wBP}\langle n-1\rangle)\big).$$

4 Computations of the E–Mahowald invariant

The classical Mahowald invariant is conjectured to produce redshift, ie the Mahowald invariant of a v_n–periodic element $\alpha \in \pi_*(S^0)$ should be v_n–torsion. In this section, we present some evidence that the motivic Mahowald invariant of a v_n–periodic element is v_{n+1}–periodic and the motivic Mahowald invariant of a w_n–periodic element is w_{n+1}–periodic.

The following is a motivic analog of [28, Theorem 2.16]. We will use this in the following section to compute $M(2^i)$.

Proposition 4.1 *For any $a \geq 0$ and $0 \leq b \leq 3$,*

$$M_{\mathrm{kq}}(2^{4a+b}) \ \textit{contains} \begin{cases} \beta^a & \textit{if } b=0, \\ \eta\beta^a & \textit{if } b=1, \\ \tau\eta^2\beta^a & \textit{if } b=2, \\ \alpha\beta^a & \textit{if } b=3. \end{cases}$$

Proof We must determine the minimal $N > 0$ such that the left-hand composition is nontrivial in the following diagram:

$$\begin{tikzcd} S^{0,0} \arrow[r, dashed] \arrow[d, "2^{4a+b}"] & \Sigma^{-2N+1,-N}\mathrm{kq} \vee \Sigma^{-2N+2,-N+1}\mathrm{kq} \arrow[d] \\ \mathrm{kq}^{t_{\mathrm{gm}}C_2} \arrow[r] & \Sigma^{1,0}\mathrm{kq} \wedge \underline{L}^{\infty}_{-N} \end{tikzcd}$$

One can show from the definition that $N = \lceil -s + \frac{1}{2} \rceil$ where s is the cellular filtration degree of the element detecting $2^{4a+b} \in \pi_{**}(H\mathbb{Z})$ in the Atiyah–Hirzebruch spectral sequence computing $\pi_{**}(\mathrm{kq}^{t_{\mathrm{gm}}C_2})$. For small values of a and b, we can read off the desired degrees s from Figure 6. We obtain the following table of values:

x	s	N	$M_{\mathrm{kq}}(x)$
2	-1	1	η
4	-2	2	$\tau\eta^2$
8	-4	3	α
16	-8	5	β

This spectral sequence is 8–periodic in the s–direction, so the desired result follows from this low-degree computation. □

The same proof using the Atiyah–Hirzebruch spectral sequence for $\mathrm{BPGL}\langle 1\rangle$ proves the following:

Corollary 4.2 *For any $i \geq 1$, we have $v_1^i \in M_{\mathrm{BPGL}\langle 1\rangle}(v_0^i)$.*

Remark 4.3 Assuming the conjectured splittings in homotopy groups from the previous section, the previous proof should generalize to prove that redshift for v_{n-1} occurs in the approximation to the motivic Mahowald invariant using $E = \mathrm{BPGL}\langle n\rangle$. In particular, we expect that for any $n \geq 1$, we have $v_n^i \in M_{\mathrm{BPGL}\langle n\rangle}(v_{n-1}^i)$.

The following result will be used to compute $M(\eta^i)$.

Proposition 4.4 *For any $a \geq 0$ and $0 \leq b \leq 3$,*

$$M_{\mathrm{wko}}(\eta^{4a+b}) \text{ contains } \begin{cases} \tilde{\beta}^a & \text{if } b = 0, \\ \nu\tilde{\beta}^a & \text{if } b = 1, \\ \nu^2\tilde{\beta}^a & \text{if } b = 2, \\ \tilde{\alpha}\tilde{\beta}^a & \text{if } b = 3. \end{cases}$$

Proof We must determine the minimal $N > 0$ such that the left-hand composition is nontrivial in the following diagram:

$$\begin{tikzcd} S^{i,i} \arrow[r, dashed] \arrow[d, "\eta^{4a+b}"] & \Sigma^{-2N+1,-N}\mathrm{wko} \vee \Sigma^{-2N+2,-N+1}\mathrm{wko} \arrow[d] \\ \mathrm{wko}^{t_{\mathrm{gm}}C_2} \arrow[r] & \Sigma^{1,0}\mathrm{wko} \wedge \underline{L}_{-N}^{\infty} \end{tikzcd}$$

One can show from the definition that $N = \lceil -s + \frac{1}{2} \rceil$ where s is the cellular filtration degree of the element detecting $\eta^{4a+b} \in \pi_{**}(\mathrm{wBP}\langle 0\rangle)$ in the Atiyah–Hirzebruch spectral sequence computing $\pi_{**}(\mathrm{wko}^{t_{\mathrm{gm}}C_2})$. For small values of a and b, we can read off the desired degrees s from the picture of this Atiyah–Hirzebruch spectral sequence in the previous section. We obtain the following table of values:

x	s	N	$M_{\mathrm{wko}}(x)$
η	-2	2	ν
η^2	-4	3	ν^2
η^3	-8	5	$\tilde{\alpha}$
η^4	-16	9	$\tilde{\beta}$

This spectral sequence is $(16,4)$–periodic in the $(s, t+s)$–direction, so the desired result follows from this low-degree computation. □

The same proof using the Atiyah–Hirzebruch spectral sequence for $\mathrm{wBP}\langle 1\rangle^{t_{\mathrm{gm}}C_2}$ proves the following.

Corollary 4.5 *For any $i \geq 1$, we have $w_1^i \in M_{\mathrm{wBP}\langle 1\rangle}(w_0^i)$.*

Remark 4.6 As in Remark 4.3, this technique should generalize in view of the conjectured splitting of the motivic C_2–Tate construction of $\mathrm{wBP}\langle n\rangle$ to prove that for any $n \geq 1$, we have $w_n^i \in M_{\mathrm{wBP}\langle n\rangle}(w_{n-1}^i)$.

5 Motivic Mahowald invariants of 2^i and η^i

In this section we compute the motivic Mahowald invariants of 2^i and η^i for all $i \geq 1$. We recover the classical computation of Mahowald and Ravenel by showing that the first element of Adams' v_1–periodic family in Adams filtration i is contained in $M(2^i)$, and we show that the first element of Andrews' w_1–periodic family in Adams filtration i is contained in $M(\eta^i)$.

5.1 Atiyah–Hirzebruch–May names

If X is a finite cell complex with at most one cell in each bidegree, we will often refer to the Atiyah–Hirzebruch–May names of elements $\alpha \in \pi_{**}(X)$ in the sequel. In this subsection, we establish terminology and notation for this convention. There are three spectral sequences involved in assigning an Atiyah–Hirzebruch–May name, and these depend on whether or not we are in the $C\tau$–linear setting:

(1) In the usual motivic setting, we use the spectral sequences

$$\text{motivic May SS} \rightsquigarrow \text{motivic Adams SS} \rightsquigarrow \text{Atiyah–Hirzebruch SS}.$$

(2) In the $C\tau$–linear setting, we use the spectral sequences

$$C\tau\text{–linear } \overline{H}\text{–based May SS} \rightsquigarrow C\tau\text{–linear } \overline{H}\text{–based Adams SS} \rightsquigarrow \text{Atiyah–Hirzebruch SS}.$$

First, suppose we are in the usual motivic setting. Let $\alpha \in \pi_{**}(X)$. Then α is detected by some class $x[(m,n)]$ in the E^∞–page of the Atiyah–Hirzebruch spectral sequence arising from the cellular filtration of X. The filtration quotients have the form $\pi_{**}(S^{i,j})$; the notation above indicates that x comes from the filtration quotient $\pi_{**}(S^{m,n})$. Now, $x \in \pi_{**}(S^{m,n}) \cong \pi_{*-m,*-n}(S^{0,0})$ is detected by some class y in the E_∞–page of the motivic Adams spectral sequence. Therefore we can say that α is detected by $y[(i,j)]$. Finally, we can regard y as a class in the E_2–page of the motivic Adams spectral sequence $E_2 = \mathrm{Ext}_A^{***}(\mathbb{M}_2, \mathbb{M}_2)$, so y is detected by some class z in the E^∞–page of the motivic May spectral sequence [11, Section 5]. We say that the *Atiyah–Hirzebruch–May name* of α is $z[(m,n)]$. The E_1–page of the motivic May spectral sequence is generated over $\mathbb{F}_2$ by classes $\{\tau, h_{ij} : i > 0, j \geq 0\}$. Therefore a typical Atiyah–Hirzebruch–May name will have the form $z[(m,n)]$ where z is a polynomial in the h_{ij}. Note that if we replace $\pi_{**}(X)$ by $\mathrm{kq}_{**}(X)$, this naming procedure is still valid since $A(1)_*$ is a quotient of the dual motivic Steenrod algebra so the motivic May spectral sequence makes sense.

Before defining the Atiyah–Hirzebruch–May name in the $C\tau$–linear setting, we need a $C\tau$–linear $\overline{H}$–based May spectral sequence.

Lemma 5.1 *There is a $C\tau$–linear $\overline{H}$–based May spectral sequence with E_1–term given by*

$$E_1 = \mathbb{F}_2[h_{i,j} : i > 0,\, j \geq 0] \Rightarrow \mathrm{Ext}^{s,(b,c)}_{\overline{A}}(\mathbb{F}_2, \mathbb{F}_2).$$

Proof We define an increasing filtration of $\overline{A}_*$ as in [33, Theorem 3.2.3]. Set $|\tau_i| = 2(i+1)-1$ and $|\xi_i^{2^j}| = 2i-1$ and define $h_{i,0} = \tau_{i-1}$ for $i \geq 1$ and $h_{i,j} = \xi_i^{2^{j-1}}$ for $i \geq 1$ and $j \geq 1$. Then each $h_{i,j}$ is primitive in the associated graded algebra under the coproduct described in Proposition 3.7, so we obtain a May spectral sequence with

$$E_1 = \mathbb{F}_2[h_{ij} : i > 0,\, j \geq 0]$$

by [33, Lemma 3.1.9]. The d_1–differentials are given by

$$d_1(h_{i,j}) = \begin{cases} \sum_{k=0}^{i-1} h_{i,0} h_{i-1-k,k} & \text{if } j = 0, \\ \sum_{k=1}^{i} h_{k,j-1} h_{i-k,k+j-1} & \text{otherwise.} \end{cases}$$

□

Remark 5.2 The discussion of the motivic May spectral sequence from [11, Section 5] carries over mutatis mutandis. We will freely use their results when they can be translated to the $C\tau$–linear setting.

Now, suppose we are in the $C\tau$–linear setting. Let $\alpha \in \pi_{**}(X)$. By repeating the procedure from the usual motivic setting above with the $C\tau$–linear $\overline{H}$–based May and Adams spectral sequences, we obtain a well-defined Atiyah–Hirzebruch–May name for α. Note that if we replace $\pi_{**}(X)$ by $\mathrm{wko}_{**}(X)$, this naming procedure is still valid since $\overline{A}(1)_*$ is a quotient of $\overline{A}_*$.

5.2 Periodicity operators

In the sequel, we will use motivic analogs of the homological periodicity operator defined by Adams in [2]. The following is a specialization of [2, Corollary 5.5] to the case $r = 2$.

Lemma 5.3 *In the classical Adams spectral sequence, the Massey product*

$$P_v(-) := \langle h_3, h_0^4, - \rangle$$

induces an isomorphism

$$\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{\cong} \mathrm{Ext}_A^{s+4,t+12}(\mathbb{F}_2, \mathbb{F}_2)$$

when $1 < s < t < \min(4s-2, 2+2^3+T(s-2))$, *where* $T(s)$ *is the numerical function defined by*

$$T(4k) = 12k, \quad T(4k+1) = 12k+2, \quad T(4k+2) = 12k+4, \quad T(4k+3) = 12k+7.$$

This Massey product detects multiplication by v_1^4 *in the classical stable stems.*

Some discussion of Massey products in the motivic Adams spectral sequence can be found in [11, Section 4.4]. The following remark provides a motivic analog of the previous lemma.

Proposition 5.4 *In the motivic Adams spectral sequence, the Massey product*

$$P_v(-) := \langle h_3, h_0^4, - \rangle$$

induces a map

$$\mathrm{Ext}_A^{s,t,u}(\mathbb{M}_2, \mathbb{M}_2) \to \mathrm{Ext}_A^{s+4,t+12,u+4}(\mathbb{M}_2, \mathbb{M}_2)$$

which is an isomorphism modulo indeterminacy in the target when s, t *and* u *satisfy* $1 < s < t < \min(4s-2, 2+2^3+T(s-2))$ *and* $u \leq s$.

Remark 5.5 The part about indeterminacy can probably be removed. In any case, we can avoid checking indeterminacy for the corresponding Toda brackets in the sequel by comparing with classical Toda brackets via Betti realization.

Proof The proof of Adams' classical periodicity result almost carries through, but the proof breaks down when a class $h_1^a x$ in an h_1–tower on some class x exists in the target but there is no corresponding class in the source of P_v, or when $h_1^a x$ is contained in the indeterminacy.

We claim that the restriction $u \leq s$ resolves this issue. To prove this, it suffices to verify that the motivic weight of classes in the target arising from these h_1–towers have motivic weight $w > s$. This follows from the next three observations:

(1) The tridegree of h_1 is $|h_1| = (1,2,1)$, so multiplication by h_1 increases both Adams filtration and weight by one.

(2) With the exception of h_0, each generator h_{ij} in the motivic May spectral sequence satisfies $\mathrm{wt}(h_{ij}) \geq \mathrm{filt}(h_{ij})$. Since $h_0 h_1 = 0$, any class x which can support an h_1–tower necessarily satisfies $w(x) \geq s(x)$. Moreover, this inequality is strict if $x \neq h_{11}$.

(3) If $x h_1^a$ lands in the target or indeterminacy but x is not of the form $P_v(y)$ for some y in the source, then $a \geq 4$. In particular, $\tau x h_1^a = 0$. Define the *weight-slope*

of a periodicity operator to be f/w where f is the filtration shift of the periodicity operator and w is the weight shift of the operator. The third point then follows from the fact that the periodicity operators of greatest weight-slope in $\mathrm{Ext}_A^{***}(\mathbb{M}_2, \mathbb{M}_2)$ are h_1, which changes tridegree by $(1, 2, 1)$, and P_v, which changes tridegree by $(4, 12, 4)$.

Putting these observations together, we see that the only range where interference from h_1–towers can occur is when $u > s$. □

Remark 5.6 Instead of imposing the restriction $u \le s$, one may be able to obtain a similar isomorphism by replacing $\mathbb{M}_2$ with $H^{**}(C\eta)$. In either case, the additional condition is required to avoid certain h_1–local classes described in [18]. For example, the class $h_1^6 d_0$ in $\mathrm{Ext}_A^{10,20,14}(\mathbb{M}_2, \mathbb{M}_2)$ lies in the (s,t)–range stated in the corollary, but is not in the image of $P_v(-)$. We thank Dan Isaksen for pointing out this issue and suggesting a solution.

If $\alpha \in \pi_{**}(S^{0,0})$ is detected by an element $P_v^k(x)$ in the motivic Adams spectral sequence and x detects $\beta \in \pi_{**}(S^{0,0})$, then we will denote α by $P_v^k(\beta)$.

We also need a version of this periodicity operator for detecting w_1^4–periodicity. This arises from the classical periodicity operator via the following lemma.

Lemma 5.7 *The doubling homomorphism*

$$\mathcal{D}\colon \mathrm{Ext}_{A_{\mathrm{cl}}}^{**}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}_{\bar{A}}^{***}(\mathbb{F}_2, \mathbb{F}_2)$$

induced by the map $A_{\mathrm{cl}} \to \bar{A}$ defined by $\mathrm{Sq}^k \mapsto \mathrm{Sq}^{2k}$ is an isomorphism onto the target in degrees $s + f - 2w = 0$ which preserves all higher structure, including products, squaring operations, and Massey products. Here A_{cl} denotes the classical Steenrod algebra, s is the stem, f is the Adams filtration, and w is the motivic weight.

Proof We begin by relating the source with Ext over the motivic Steenrod algebra using some results from [22, Section 2.1.3]. Let A' be the subquotient $\mathbb{M}_2$–algebra of A generated by Sq^{2k} for all $k \ge 0$, subject to the relation $\tau = 0$. Then the doubling homomorphism defined as above is an isomorphism $A_{\mathrm{cl}} \to A'$. By [22, Theorem 2.1.12], this induces an isomorphism between the classical Adams E_2–page $\mathrm{Ext}_{A_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ and the subalgebra of the motivic Adams E_2–page $\mathrm{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$ consisting of elements in degrees (s, f, w) such that $s + f - 2w = 0$. This isomorphism preserves all higher structure.

The lemma now follows from the observation that $\mathrm{Ext}_A^{***}(\mathbb{M}_2,\mathbb{M}_2)$ and $\mathrm{Ext}_{\overline{A}}^{***}(\mathbb{F}_2,\mathbb{F}_2)$ are isomorphic in degrees $s+f-2w=0$. This can be seen by comparing the motivic May spectral sequence and the $C\tau$–linear $\overline{H}$–based May spectral sequence in degrees (m,s,f,w) with $s+f-2w=0$. □

Remark 5.8 By the proof of [22, Theorem 2.1.12], the above map can be described using May names by $h_{i,j-1}\mapsto h_{i,j}$.

Corollary 5.9 *In the $C\tau$–linear $\overline{H}$–based Adams spectral sequence, the Massey product*

$$g(-):=\langle h_4,h_1^4,-\rangle$$

induces a map

$$\mathrm{Ext}_{\overline{A}}^{a,b,c}(\mathbb{F}_2,\mathbb{F}_2)\to\mathrm{Ext}_{\overline{A}}^{a+4,b+24,c+12}(\mathbb{F}_2,\mathbb{F}_2)$$

which is an isomorphism for any triple (a,b,c) such that

$$\mathrm{Ext}_{\overline{A}}^{a,b,c}(\mathbb{F}_2,\mathbb{F}_2)\cong\mathcal{D}(\mathrm{Ext}_{A_{\mathrm{cl}}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2))$$

for some $1<s<t<\min(4s-2,2+2^3+T(s-2))$.

If $P_v(x)$ is defined with no indeterminacy in the classical Adams spectral sequence, then $g(\mathcal{D}(x))$ is defined with no indeterminacy in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence.

We note that the Massey product $g(-)$ detects multiplication by $g:=h_{21}^4$ at the level of $C\tau$–linear $\overline{H}$–based May names in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence. We will discuss the fate of the classes in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence arising from iterating this Massey product when we compute the motivic Mahowald invariant of η^i.

5.3 Motivic Mahowald invariant of 2^i

We now compute $M(2^i)$. We will state our result in terms of iterated Toda brackets; we verify the nontriviality of these Toda brackets in the following lemma.

Lemma 5.10 *The iterated Massey products $P_v^i(x)$ for $x=8\sigma,\eta,\eta^2,\eta^3$ are permanent cycles in the motivic Adams spectral sequence. Moreover, the Betti realizations of the classes which they detect in $\pi_{**}(S^{0,0})$ are the classes in $\pi_*(S^0)$ detected by the iterated Massey products $P^i(x)$ for $x=8\sigma_{\mathrm{cl}},\eta_{\mathrm{cl}},\eta_{\mathrm{cl}}^2,\eta_{\mathrm{cl}}^3$, where $P(-)$ is the classical Adams periodicity operator.*

Proof The iterated Massey products $P_v^i(x)$ are nonvanishing since their classical analogs are nonvanishing by [2]. Denote these classical analogs by $P^i(x)$. The classes $P^i(x)$ are permanent cycles in the classical Adams spectral sequence by [1]. Betti realization induces a map of spectral sequences from the motivic Adams spectral sequence to the classical Adams spectral sequence which sends $h_{ij} \mapsto h_{ij}$ and therefore carries $P_v^i(x)$ to $P^i(x)$. Therefore the images of $P_v^i(x)$ are permanent cycles, so the classes $P_v^i(x)$ are permanent cycles. □

We will need one more proposition before proving our main theorem.

Proposition 5.11 *Let L_m denote the subcomplex of $\underline{L}_{-\infty}^{\infty}$ with cells in topological dimensions $-1+2m \le d \le 6+2m$. Then the degree 16 map is null on L_m for all $m \in \mathbb{Z}$.*

Proof Recall the analogous classical result of Davis and Mahowald [10] which says that the degree 16 map on $\mathbb{RP}_{-1-2m}^{6-2m}$ is null. Let $X = L_m \wedge D^{\mathbb{C}} L_m$, where $D^{\mathbb{C}}(-)$ is the motivic Spanier–Whitehead dual functor $D^{\mathbb{C}}(-) = F(-, S^{0,0})$. Suppose that the degree 16 map on L_m is not null, ie $16 \neq 0 \in \pi_{0,0}(X)$. By Remark 2.17, we have $\mathrm{Re}_{\mathbb{C}}(L_m) \simeq \mathbb{RP}_{-1-2m}^{6-2m}$ and $\mathrm{Re}_{\mathbb{C}}(X) \simeq \mathbb{RP}_{-1-2m}^{6-2m} \wedge D\mathbb{RP}_{-1-2m}^{6-2m} =: Y$, where $D(-)$ is the classical Spanier–Whitehead dual functor $D(-) = F(-, S^0)$. Since $\pi_0(\mathrm{Re}_{\mathbb{C}}(X))$ may be obtained by localizing $\pi_{0,*}(X)$ with respect to τ and then setting $\tau = 1$, we see that $16 \in \pi_{0,0}(X)$ must be τ–torsion.

The motivic homotopy group $\pi_{0,0}(X)$ may be computed via the Atiyah–Hirzebruch spectral sequence arising from the filtration of X by topological dimension. Observe that X is a 64–cell complex with cells concentrated in bidegrees of the form $(2k \pm \epsilon, k \pm \epsilon)$ with $-3 \le k \le 3$ and $\epsilon \in \{0, 1\}$. The possible contributions to $\pi_{0,0}(X)$ in the Atiyah–Hirzebruch spectral sequence have the form $\alpha[2k \pm \epsilon, k \pm \epsilon]$ with $\alpha \in \pi_{-2k\mp\epsilon,-k\mp\epsilon}(S^{0,0})$. Examining $\pi_{**}(S^{0,0})$ in [21, page 8], we see that there are no τ–torsion classes in these bidegrees.

Thus, $16 \in \pi_{0,0}(X)$ must be detected by $\alpha[2k\pm\epsilon, k\pm\epsilon]$ where $\alpha \in \pi_{-2k\mp\epsilon,-k\mp\epsilon}(S^{0,0})$ is τ–torsion free, and $\tau^i \alpha[2k \pm \epsilon, k \pm \epsilon]$ must be killed for all $i \gg 0$. However, the differentials in this Atiyah–Hirzebruch spectral sequence have the same form as those in Section 2.3. In particular, the d_1–differentials may create new τ–torsion classes on the E_2–page, but these new τ–torsion classes will be annihilated by d_2–differentials. Moreover, longer differentials cannot produce τ–torsion in this range. We conclude that 16 must be detected by a τ–torsion free class in the Atiyah–Hirzebruch spectral sequence, contradicting the fact that $\mathrm{Re}_{\mathbb{C}}(16) = 0 \in \pi_0(Y)$. Therefore 16 must be null on L_m. $\square$

Theorem 5.12 *Let $i \geq 1$. Then*

$$M(2^i) \text{ contains } \begin{cases} P_v^{\lfloor i/4 \rfloor}(h_0^3 h_3) & \text{if } i \equiv 0 \bmod 4, \\ P_v^{\lfloor i/4 \rfloor}(h_1) & \text{if } i \equiv 1 \bmod 4, \\ P_v^{\lfloor i/4 \rfloor}(\tau h_1^2) & \text{if } i \equiv 2 \bmod 4, \\ P_v^{\lfloor i/4 \rfloor}(\tau h_1^3) & \text{if } i \equiv 3 \bmod 4. \end{cases}$$

Here we are denoting nontrivial Toda brackets in $\pi_{**}(S^{0,0})$ by the Massey products which detect them.

Proof Nontriviality of the relevant Massey products was proven in the previous lemma. We break the proof apart into two cases depending on the congruence of $i \bmod 4$:

Case 1 $(i \equiv 1, 2 \bmod 4)$ The elements $\beta^k \eta, \beta^k \tau \eta^2 \in \pi_{**}(\mathrm{kq})$ are in the Hurewicz image for kq. Their inverse images are $P^k(\eta), P^k(\tau\eta^2) \in \pi_{**}(S^{0,0})$, so this case is clear by Proposition 2.20 applied to Proposition 4.1. Indeed, the commutativity of the diagram in Proposition 2.20 follows from Proposition 5.11 and the low-dimensional computations of Section 2.3 as in the proof of [28, Theorem 2.17].

Case 2 $(i \equiv 0, 3 \bmod 4)$ The elements $\alpha, \beta \in \pi_{**}(\mathrm{kq})$ are not in the Hurewicz image, so we cannot immediately employ Proposition 2.20. Instead, we must pass through the motivic Mahowald invariant based on the mod 2 Moore spectrum $V(0)$. We obtain the theorem through the following series of approximations:

$$M_{\mathrm{kq}}(2^i) \rightsquigarrow M_{\mathrm{kq}}(2^i; V(0)) \rightsquigarrow M(2^i; V(0)) \rightsquigarrow M(2^i).$$

These approximations are computed in the remainder of the subsection. $\square$

By Proposition 4.1, we have computed $M_{\mathrm{kq}}(2^i)$.

Lemma 5.13 *Let $k \geq 1$. Then*

$$M_{\mathrm{kq}}(2^{4k+j}; V(0)) \text{ contains } \begin{cases} \beta^k[(0,0)] & \text{if } j = 0, \\ \beta^k \alpha[(0,0)] & \text{if } j = 3. \end{cases}$$

Proof By the proof of Proposition 4.1, the coset $M_{\mathrm{kq}}(2^i)$ is detected on the cell of $\Sigma^{1,0} \underline{L}_{-\infty}^{\infty}$ with topological degree $f(i)$ where for each $k \geq 0$ we define

$$f(4k) = -8k, \quad f(4k+3) = -4 - 8k.$$

The topological degree of the cell determines where in the filtration of $\underline{L}_{-\infty}^{\infty}$ by $V(0)$ the coset $M_{\mathrm{kq}}(2^i; V(0))$ is detected, ie it determines the N in Definition 2.25. We obtain the following table of values:

x	s	N	$M_{\mathrm{kq}}(x; V(0))$
2^{4k}	$-8k$	$1+4k$	$\beta^k[(0,0)]$
2^{4k+3}	$-4-8k$	$3+4k$	$\beta^k\alpha[(0,0)]$

To justify that $\beta^k\alpha^\epsilon[(0,0)]$ for $\epsilon \in \{0,1\}$ are the correct Atiyah–Hirzebruch–May names for the classes above, it suffices to note that $\beta^k\alpha^\epsilon \in \mathrm{kq}_{**}$ is not in the image of multiplication by 2 for any k and ϵ. □

Lemma 5.14 *Let $k \geq 1$. Then*

$$M(2^{4k+j}; V(0)) \text{ contains } \begin{cases} P_v^{k-1}(8\sigma)[(1,0)] & \text{if } j=0, \\ P_v^k(\tau\eta^3)[(1,0)] & \text{if } j=3. \end{cases}$$

Proof Let $h\colon S^{0,0} \wedge V(0) \to \mathrm{kq} \wedge V(0)$ be the Hurewicz map smashed with $\mathrm{id}_{V(0)}$. We will establish the values in the following table by induction on k:

x	$h^{-1}(x)$
$\beta^k[(0,0)]$	$P_v^{k-1}(8\sigma)[(1,0)]$
$\beta^k\alpha[(0,0)]$	$P_v^k(\tau\eta^3)[(1,0)]$

We start with the case $x = \beta^k[(0,0)]$ for $k=1$. Recall that $V(0)$ is defined by the cofiber sequence

$$S^{0,0} \xrightarrow{\cdot 2} S^{0,0} \to V(0),$$

and this cofiber sequence gives rise to a long exact sequence in homotopy

$$\cdots \to \pi_{**}(S^{0,0}) \xrightarrow{f_*} \pi_{**}(S^{0,0}) \xrightarrow{j_*} \pi_{**}(V(0)) \xrightarrow{\delta} \pi_{*-1,*}(S^{0,0}) \to \cdots$$

where f_* is induced by $\cdot 2$. There is a motivic May differential $d_4(b_{20}^2) = h_0^4h_3$ which produces the relation $2 \cdot (8\sigma) = 0$ in the motivic Adams spectral sequence converging to $\pi_{**}(S^{0,0})$ [11, Section 5.3]. Therefore $8\sigma \in \ker(f_*) = \mathrm{im}(\delta_*)$, so the Atiyah–Hirzebruch–May name of $\delta^{-1}(8\sigma) \in \pi_{**}(V(0))$ is $h_0^3h_3[(1,0)]$. If we smash the above cofiber sequence with kq and take homotopy groups, the class $h_0^3h_3$ is trivial since h_3 does not appear in the motivic May spectral sequence converging to $\mathrm{Ext}_{A(1)}(\mathbb{M}_2, \mathbb{M}_2)$. Therefore the class $h_0^3h_3[(1,0)]$ does not survive in the motivic Adams spectral sequence converging to $\pi_{**}(\mathrm{kq} \wedge V(0))$. Note that the above May differential implies that b_{20}^2 does not survive in the motivic Adams spectral sequence converging to $\pi_{**}(S^{0,0})$, so $b_{20}^2[(0,0)]$ also does not survive in the motivic Adams spectral sequence converging to $\pi_{**}(V(0))$.

On the other hand, the class b_{20}^2 detects $\beta \in \pi_{**}(\mathrm{kq})$ by the motivic analog of the classical argument that b_{20}^2 detects $\beta \in \pi_*(\mathrm{ko})$. Smashing the cofiber sequence defining $V(0)$ with kq and applying homotopy produces another long exact sequence

$$\cdots \to \mathrm{kq}_{**} \xrightarrow{f_*} \mathrm{kq}_{**} \xrightarrow{j_*} \mathrm{kq}_{**}(V(0)) \xrightarrow{\delta} \mathrm{kq}_{*-1,*} \to \cdots.$$

Since $\alpha \neq 2y$ for any $y \in \pi_{**}(\mathrm{kq})$, we have that $\alpha \notin \mathrm{im}(f_*) = \ker(j_*)$. Therefore the Atiyah–Hirzebruch–May name of $j_*(\alpha) \in \pi_{**}(\mathrm{kq} \wedge V(0))$ is $b_{20}^2[(0,0)]$.

By the previous two paragraphs, we see that the class $b_{20}^2[(0,0)] + h_0^3 h_3[(1,0)]$ detects $8\sigma[(1,0)] \in \pi_{**}(V(0))$ and detects $\beta[(1,0)] \in \pi_{**}(V(0) \wedge \mathrm{kq})$. Therefore we have $h^{-1}(\beta[(0,0)] = 8\sigma[(1,0)]$. This completes the base case of the induction.

Now suppose that we have shown that

$$h^{-1}(\beta^i[(0,0)]) = P_v^{i-1}(8\sigma)[(1,0)]$$

for all $i < n$. We can reformulate the induction hypothesis using the motivic Adams spectral sequence as follows: for $i < n$, the class in $\pi_{**}(V(0))$ detected by

$$b_{20}^{2i}[(0,0)] + P_v^{i-1}(h_0^3 h_3)[(1,0)]$$

maps under h to the class in $\pi_{**}(\mathrm{kq} \wedge V(0))$ detected by $b_{20}^{2i}[(0,0)]$.

To see that this reformulation implies the original induction hypothesis, we need to show that $b_{20}^{2i}[(0,0)]$ does not survive in the motivic Adams spectral sequence converging to $\pi_{**}(V(0))$. Using the algebraic squaring operations described in [11, Section 5], we can produce motivic May differentials

$$d_{2^k}(b_{20}^{2^k}) = d_{2^k}(\mathrm{Sq}^{2^k}(b_{20}^{2^{k-1}})) = h_0^{2^k} h_{1+k}$$

with $k \geq 2$. By the Leibniz rule, we can obtain nontrivial May differentials on b_{20}^{2i} for all $i \geq 1$. These elements could support shorter differentials, but in any case we have shown that $b_{20}^{2i}([0,0])$ does not survive in the motivic Adams spectral sequence converging to $\pi_{**}(V(0))$. Therefore the first class above detects $P_v^{i-1}(8\sigma)[(1,0)]$ in $\pi_{**}(V(0))$. Since $b_{20}^{2i}[(0,0)]$ detects $\beta^i[(0,0)]$ in $\pi_{**}(\mathrm{kq} \wedge V(0))$, we have shown that the reformulation implies the original induction hypothesis.

To complete the induction, consider the diagram

$$\begin{tikzcd}
 & \mathrm{Ext}_A^{***}(V(0)) \arrow[rr, "P_v"] \arrow[dl] & & \mathrm{Ext}_A^{*+4,*+12,*+4}(V(0)) \arrow[dl] \\
\mathrm{Ext}_A^{***}(\mathrm{kq}\wedge V(0)) \arrow[rr, "\cdot b_{20}^2"] & & \mathrm{Ext}_A^{*+4,*+12,*+4}(\mathrm{kq}\wedge V(0)) & \\
 & \mathrm{Ext}_{A_{\mathrm{cl}}}^{**}(V(0)) \arrow[rr, "P_v"] \arrow[dl] \arrow[uu] & & \mathrm{Ext}_{A_{\mathrm{cl}}}^{*+4,*+12}(V(0)) \arrow[dl] \arrow[uu] \\
\mathrm{Ext}_{A_{\mathrm{cl}}}^{**}(\mathrm{ko}\wedge V(0)) \arrow[rr, "\cdot b_{20}^2"] \arrow[uu] & & \mathrm{Ext}_{A_{\mathrm{cl}}}^{*+4,*+12}(\mathrm{ko}\wedge V(0)) \arrow[uu] &
\end{tikzcd}$$

The vertical maps are defined by sending a class with Atiyah–Hirzebruch–May name $h_{ij}[(m)]$ to the element $h_{ij}[(m,0)]$, and similarly for elements detected by products.

Consider the class $b_{20}^{2n-2}[(0)] + P_v^{n-2}(h_0^3h_3)[(1)] \in \mathrm{Ext}_{A_{\mathrm{cl}}}^{**}(V(0))$ in the bottom face of the diagram. This class detects $P_v^{n-2}(8\sigma)[(1)]$. Its image in $\mathrm{Ext}_{A_{\mathrm{cl}}}^{**}(\mathrm{ko}\wedge V(0))$ is $b_{20}^{2n-2}[(0)]$, which detects $\beta^{n-1}[(0)]$; its image in $\mathrm{Ext}_{A_{\mathrm{cl}}}^{*+4*+8}(V(0))$ is $b_{20}^{2n}[(0)] + P_v^{n-1}(h_0^3h_3)[(1)]$, which detects $P_v^{n-1}(8\sigma)[(1)]$; its image in $\mathrm{Ext}_{A_{\mathrm{cl}}}^{*+4,*+8}(\mathrm{ko}\wedge V(0))$ is $b_{20}^{2n}[0]$, which detects $\beta^n[(0)]$. Indeed, since the Adams periodicity operator P_v is well-defined with no indeterminacy for $P_v^j(h_0^3h_3)$ for all $j \geq 0$, we can precisely determine which elements in homotopy are detected by these Massey products by sparseness in the image of J [1].

Now consider the element $P_v^{n-2}(h_0^3h_3)[(1,0)] \in \mathrm{Ext}_A^{***}(V(0))$ in the top face of the cube. By the induction hypothesis, its image in $\mathrm{Ext}_A^{***}(\mathrm{kq}\wedge V(0))$ is $b_{20}^{2(n-1)}[(0,0)]$. Its inverse image under the vertical map is $P_v^{n-2}(h_0^3h_3)[(1)]$. We can calculate the images of $P_v^{n-2}(h_0^3h_3)[(1,0)]$ in the top face using the vertical maps and the previous paragraph. In particular, we conclude that $P_v^{n-1}(h_0^3h_3)[(1,0)] \in \mathrm{Ext}_A^{*+4,*+8,*+4}(V(0))$ maps to $b_{20}^{2n}[(0,0)] \in \mathrm{Ext}_A^{*+4,*+8,*+4}(\mathrm{kq}\wedge V(0))$. This completes the induction step, so we have established the values in the table for $x = \beta^k[(0,0)]$.

The computation of $h^{-1}(\beta^k\alpha[(0,0)])$ is completely analogous. Starting with the differential $d_2(h_0b_{20}) = h_0^3h_2$ which produces the relation $2\cdot(\tau\eta^3)$, we see that the class $h_0b_{20}[(0,0)] + \tau h_1^3[(1,0)]$ detects $\tau\eta^3[(1,0)] \in \pi_{**}(V(0))$ and detects $\alpha \in \pi_{**}(\mathrm{kq}\wedge V(0))$, which shows that $h^{-1}(\alpha[(0,0)]) = \tau\eta^3[(1,0)]$. The same argument using Massey products, algebraic squaring operations, and the commutative cubical diagram completes the induction.

We now apply the obvious analog of Proposition 2.20 for the motivic Mahowald invariant based on a finite complex to obtain the claimed containments. Commutativity of the diagram in Proposition 2.20 again follows from Proposition 5.11 and the low-dimensional computations of Section 2.3 as in the proof of [28, Theorem 2.17]. □

Since we know which cell of $\underline{L}_{-\infty}^{\infty}$ the cosets $M(2^i; V(0))$ were detected on, we obtain the desired refinement to $M(2^i)$.

5.4 Motivic Mahowald invariant of η^i

We now compute $M(\eta^i)$. As in the previous subsection, we will state our result in terms of iterated Toda brackets. We begin by verifying the nontriviality of these Toda brackets; in particular, we show that they detect certain infinite w_1^4–periodic families constructed by Andrews in [4]. We begin by recalling some essential information about these families.

By [4, Theorem 3.4], the finite complex $C\eta$ admits a nonnilpotent w_1^4–self-map $\Sigma^{20,12}C\eta \to C\eta$. Using that $\nu, \nu^2, \nu^3 \in \pi_{**}(S^{0,0})$ lift to classes in $\pi_{**}(C\eta)$, one obtains infinite families of maps $P_w^n(\nu)$, $P_w^n(\nu^2)$, $P_w^n(\nu^3) \in \pi_{**}(S^{0,0})$ by composing with

$$C\eta \xrightarrow{w_1^{4n}} \Sigma^{-20n,-12n} C\eta \xrightarrow{\text{pinch}} S^{2-20n,1-12n}.$$

These are defined precisely in [4, Definition 3.11], where they are denoted by $P^n(\alpha)$ for $\alpha \in \{\nu, \nu^2, \nu^3\}$. Similarly, one obtains an infinite family $P_w^n(\iota) \in \pi_{**}(S^{0,0})$ where $\iota \in \pi_{0,0}(C\eta)$ is the inclusion of the bottom cell $S^{0,0} \to C\eta$. By [4, Theorem 3.12], these compositions are nontrivial for all $n \geq 0$. We note that $P_w(\iota) = \eta^2\eta_4$.

To prove nontriviality of these composites, Andrews uses detection maps from the E_2–page of certain motivic Adams–Novikov spectral sequences to the E_2–page of certain classical Adams spectral sequences [4, Definition 2.8]. For example, he considers the map

$$d\colon \operatorname{Ext}^{***}_{\mathrm{BPGL}_{**}\mathrm{BPGL}}(\mathrm{BPGL}_{**}, \mathrm{BPGL}_{**}) \to \operatorname{Ext}^{**}_{A_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$$

defined by $\tau \mapsto 0$, $v_n \mapsto 0$, and $t_n \mapsto \zeta_n$. This is a graded map if one defines the degree of an element in the source to be its motivic weight. He uses the detection maps to infer the nontriviality of the classes $P_w^n(x)$ in $\operatorname{Ext}^{***}_{\mathrm{BPGL}_{**}\mathrm{BPGL}}(\mathrm{BPGL}_{**}, \mathrm{BPGL}_{**})$ by showing that their images are the infinite families constructed by Adams in [2].

Lemma 5.15 *The iterated Massey products $g^i(x)$ for $x = \eta^2\eta_4, \nu, \nu^2, \nu^3$* (with $g(-)$ as in Corollary 5.9) *are permanent cycles in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence. Moreover, they detect the w_1^4–periodic families $P_w^i(x)$ discussed above.*

Proof By Corollary 5.9 and [2], these iterated Massey products are nontrivial with zero indeterminacy. We claim that they are permanent cycles which detect the infinite w_1^4–periodic families constructed by Andrews in [4]. Composing Andrews' detection map above with the doubling homomorphism gives a map

$$\mathcal{D}\circ d\colon \mathrm{Ext}^{***}_{\mathrm{BPGL}_{**}\mathrm{BPGL}}(\mathrm{BPGL}_{**},\mathrm{BPGL}_{**})\to \mathrm{Ext}^{***}_{\overline{A}}(\mathbb{F}_2,\mathbb{F}_2).$$

By the computations of the images of the detection map in [4, Section 4] along with Remark 5.8, we see that the images of the classes $\alpha^i_{4/4}$ for $1\le i\le 3$ and 1 in the motivic Adams–Novikov spectral sequence under $\mathcal{D}\circ d$ are h_2^i and 1 in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence. Moreover, we can relate the class $x\in \mathrm{Ext}^{4,24,12}_{\mathrm{BPGL}_{**}\mathrm{BPGL}}(\mathrm{BPGL}_{**}(C\eta))$ constructed in [4, Proposition 3.3] which maps to $\alpha_1^2\beta_{4/3}$ under the collapse map to a class in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence. The class $\alpha_1^2\beta_{4/3}$ maps to $h_1^3h_4$ under the composition of the detection map with the doubling map, so the same argument as in the proof of [4, Proposition 3.3] defines an element $x'\in\mathrm{Ext}^{4,24,12}_{\overline{A}}(\overline{H}^{**}(C\eta))$ which maps to $h_1^3h_4$ under the analogous composition. Further, it follows from the $C\tau$–linear $\overline{H}$–based May differential $d_4(b_{21}^2)=h_1^4h_4$ that x' has the Atiyah–Hirzebruch–May name $b_{21}^2[(0,0)]$. This d_4–differential follows from the same argument as the motivic May differential of the same name.

Therefore multiplication by b_{21}^2 detects w_1^4 in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence converging to $\pi_{**}(C\eta)$. Since we have already shown that the Massey products $g(x)$ are nontrivial with zero indeterminacy, we have $g(x)=\langle h_4,h_1^4,x\rangle=b_{21}^2x$. Thus the periodicity operators $g(-)$ defined above and the periodicity operator $P_w(-)$ defined by Andrews both detect w_1^4x. Since Andrews' infinite families are permanent cycles, so are the infinite families $g^i(x)$ for x as stated in the lemma. □

We will need the following proposition to prove the main theorem.

Proposition 5.16 *The map η^4 is null on $\underline{L}^{9+2m}_{-1+2m}$ for all $m\in\mathbb{Z}$.*

Proof The proof is similar to the proof of Proposition 5.11. Let

$$X:=\underline{L}^{9+2m}_{-1+2m}\wedge D^{\mathbb{C}}\underline{L}^{9+2m}_{-1+2m}\quad\text{and}\quad Y:=\mathbb{RP}^{17+4m}_{-2+4m}\wedge D\mathbb{RP}^{17+4m}_{-2+4m},$$

so that $\mathrm{Re}_{\mathbb{C}}(X)\simeq Y$. Since $\mathrm{Re}_{\mathbb{C}}(\eta^4)=\eta^4=0\in\pi_4(Y)$ and $\pi_4(Y)$ may be obtained by localizing $\pi_{4,*}(X)$ with respect to τ and then setting $\tau=1$, we see that $\eta^4\in\pi_{4,4}(X)$ must be τ–torsion.

The motivic homotopy group $\pi_{4,4}(X)$ may be computed via the Atiyah–Hirzebruch spectral sequence arising from the filtration of X by topological dimension. Observe that X is a 400–cell complex with cells concentrated in bidegrees of the form $(2k\pm\epsilon, k)$ with $-9\le k\le 9$ and $\epsilon\in\{0,1\}$. The possible contributions to $\pi_{4,4}(X)$ in the Atiyah–Hirzebruch spectral sequence have the form $\alpha[2k\pm\epsilon,k]$ with $\alpha\in\pi_{4-2k\mp\epsilon,4-k}(S^{0,0})$. Examining $\pi_{**}(S^{0,0})$ in [22, page 8], we see that the only τ–torsion classes in these bidegrees are detected by class of the form xh_1^j for $x\in\{1, c_0, Ph_1, Pc_0, P^2h_1, e_0\}$ with $\tau x h_1^j = 0$. We claim that all of these classes support or are the targets of d_2–differentials. Note that none of these classes are involved in d_1–differentials (ie they are simple h_0–torsion which is not h_0–divisible), and moreover, none of the possible targets (resp. sources) of d_2–differentials which they could support (resp. be targets of) are involved in d_1–differentials.

Let $\{e_{-2}, e_{-1},\dots,e_{17}\}$ denote a basis for $H^{**}(\underline{L}^{9+2m}_{-1+2m})$ (where $|e_i|=i$) and let $\{f_{-17},\dots,f_2\}$ denote a basis for $H^{**}(D^{\mathbb{C}}\underline{L}^{9+2m}_{-1+2m})$ (where $|f_j|=j$), so that $\{e_i\otimes f_{-j}\}_{-2\le i,j\le 17}$ forms a basis for $H^{**}(X)$. By inspection of Figure 1, each cell of $\underline{L}^{9+2m}_{-1+2m}$ is attached to another cell by an η–attaching map. Similarly, each cell of $D^{\mathbb{C}}\underline{L}^{9+m}_{-1+m}$ is attached to another cell by an η–attaching map. Applying the Cartan formula, we see that each cell of X is attached to another cell by an η–attaching map. Therefore each of the above classes is the target or source of a nontrivial d_2–differential in the Atiyah–Hirzebruch spectral sequence.

We have therefore shown that η^4 is not detected by a τ–torsion class in the Atiyah–Hirzebruch spectral sequence. As in the proof of Proposition 5.11, any τ–torsion which is created by d_1–differentials in the Atiyah–Hirzebruch spectral sequence is destroyed by d_2–differentials. Since longer differentials cannot create τ–torsion in this range, we conclude that η^4 must be detected by a τ–torsion free class in the Atiyah–Hirzebruch spectral sequence, contradicting the fact that $\mathrm{Re}_{\mathbb{C}}(\eta^4)=0\in\pi_4(Y)$. Therefore η^4 must be null on $\underline{L}^{9+2m}_{-1+2m}$. □

Theorem 5.17 *Let $i\ge 1$. Then*

$$M(\eta^i)\ \textit{contains}\ \begin{cases} g^{\lfloor i/4\rfloor -1}(h_1^3h_4) & \textit{if } i\equiv 0 \bmod 4,\\ g^{\lfloor i/4\rfloor}(h_2) & \textit{if } i\equiv 1 \bmod 4,\\ g^{\lfloor i/4\rfloor}(h_2^2) & \textit{if } i\equiv 2 \bmod 4,\\ g^{\lfloor i/4\rfloor}(h_2^3) & \textit{if } i\equiv 3 \bmod 4.\end{cases}$$

Here we are denoting nontrivial Toda brackets in $\pi_{**}(S^{0,0})$ by the Massey products which detect them.

Proof We split the proof into two cases depending on the congruence of $i \mod 4$:

Case 1 ($i \equiv 1, 2 \mod 4$) The elements $\widetilde{\beta}^k \nu, \widetilde{\beta}^k \nu^2 \in \pi_{**}(\mathrm{wko})$ are in the Hurewicz image for wko. Their inverse images are $g^k(\nu), g^k(\nu^2) \in \pi_{**}(S^{0,0})$, so this case is clear by Proposition 2.20 applied to Proposition 4.4. Commutativity of the diagram in Proposition 2.20 follows from Proposition 5.16 and the low-dimensional computations in Section 2.3 as in the proof of [28, Theorem 2.17].

Case 2 ($i \equiv 0, 3 \mod 4$) The elements $\widetilde{\alpha}, \widetilde{\beta} \in \pi_{**}(\mathrm{wko})$ are not in the Hurewicz image, so we cannot immediately employ Proposition 2.20. Instead, we must pass through the motivic Mahowald invariant based on the 4–cell complex C defined in Example 2.23. We compute approximations in the following order:

$$M_{\mathrm{wko}}(\eta^i) \rightsquigarrow M_{\mathrm{wko}}(\eta^i; C) \rightsquigarrow M_{V(0)}(\eta^i; C) \rightsquigarrow M_{V(0)}(\eta^i) \rightsquigarrow M(\eta^i).$$

These approximations are computed in the remainder of the subsection. □

By Proposition 4.4, we have computed $M_{\mathrm{wko}}(\eta^i)$.

Lemma 5.18 *Let $k \geq 1$. Then*

$$M_{\mathrm{wko}}(2^{4k+j}; C) \text{ contains } \begin{cases} \widetilde{\beta}^k[(1,1)] & \text{if } j = 0, \\ \widetilde{\beta}^k \widetilde{\alpha}[(1,1)] & \text{if } j = 3. \end{cases}$$

Proof By the proof of Proposition 4.4, we know on which cells of $\underline{L}_{-\infty}^{\infty}$ the elements η^{4k} and η^{4k+3} are first detected. The topological degree of the cell determines where in the filtration of $\underline{L}_{-\infty}^{\infty}$ by C the coset $M_{\mathrm{wko}}(\eta^i; C)$ is detected. We obtain the following table of values:

x	s	N	$M_{\mathrm{wko}}(x; C)$
η^{4k}	$-16k$	$1+8k$	$\widetilde{\beta}^k[(1,1)]$
η^{4k+3}	$-8-16k$	$5+4k$	$\widetilde{\beta}^k\widetilde{\alpha}[(1,1)]$

To justify that $\widetilde{\beta}^k \widetilde{\alpha}^\epsilon[(1,1)]$ for $\epsilon \in \{0,1\}$ are the correct Atiyah–Hirzebruch–May names for the classes above, it suffices to note that $\widetilde{\beta}^k \widetilde{\alpha}^\epsilon \in \mathrm{wko}_{**}$ is not in the image of multiplication by η for any k and ϵ. □

Lemma 5.19 *Let $k \geq 1$. Then*

$$M_{V(0)}(\eta^{4k+j}; C) \text{ contains } \begin{cases} g^{k-1}(\eta^2\eta_4)[(0,0),(3,2)] & \text{if } j = 0, \\ g^k(\nu^3)[(0,0),(3,2)] & \text{if } j = 3. \end{cases}$$

Proof We now use the canonical map $h\colon V(0) \to \mathrm{wko}$ to compute $M_{V(0)}(\eta^i; C)$ from $M_{\mathrm{wko}}(\eta^i; C)$. We need to assign Atiyah–Hirzebruch–May names to elements $\widetilde{\alpha} \in \pi_{**}(V(0) \wedge C)$. Our notation will be that $\widetilde{\alpha}$ is detected by $x[(m_1, n_1), (m_2, n_2)]$ where (m_2, n_2) comes from the Atiyah–Hirzebruch–May name for $\widetilde{\alpha} \in \pi_{**}(C)$ and (m_1, n_1) comes from the Atiyah–Hirzebruch–May name for $\widetilde{\alpha} \in \pi_{**}(V(0) \wedge C)$.

We will establish the values in the following table by induction on k:

x	$h^{-1}(x)$
$\widetilde{\beta}^k[(1,1)]$	$g^{k-1}(\eta^2\eta_4)[(0,0),(3,2)]$
$\widetilde{\beta}^k\widetilde{\alpha}[(1,1)]$	$g^k(\nu^3)[(0,0),(3,2)]$

We start with the case $x = \widetilde{\beta}^k[(1,1)]$. We can obtain the Atiyah–Hirzebruch–May name for an element in $\pi_{**}(V(0) \wedge C)$ arising from $\eta^2\eta_4 \in \pi_{**}(S^{0,0})$ using cofiber sequences as follows. As in the proof of Theorem 5.12, the maps f_*, j_* and δ are the maps in the long exact sequence in homotopy groups associated to a cofibration. First, recall that $C\eta$ is defined by the cofiber sequence

$$S^{1,1} \xrightarrow{\eta} S^{0,0} \to C\eta.$$

By [11, Proposition 5.5], there is a motivic May differential $d_4(b_{21}^2) = h_1^4 h_4$. The same proof implies that there is a $C\tau$–linear $\overline{H}$–based May differential $d_4(b_{21}^2) = h_1^4 h_4$. In the Adams spectral sequence for $\pi_{**}(S^{0,0})$, this differential produces the relation $\eta \cdot (\eta^2\eta_4) = 0$. Then the Atiyah–Hirzebruch–May name for $\delta^{-1}(\eta^2\eta_4) \in \pi_{**}(C\eta)$ is $h_1^3 h_4[(2,1)]$. Now, C is the defined by the cofiber sequence

$$\Sigma^{-1,0} C\eta \xrightarrow{f} \Sigma^{1,1} C\eta \to C$$

where f is the composition of the collapse onto the top cell and multiplication by 2. Therefore C has cells in dimensions $(0,0)$ and $(2,1)$ coming from $\Sigma^{-1,0}C\eta$ and cells in dimensions $(1,1)$ and $(3,2)$ coming from $\Sigma^{1,1}C\eta$. Although $h_0^2 h_2 h_4 = \tau h_1^3 h_4$ in the motivic Adams spectral sequence, the class $\eta^2\eta_4 \in \pi_{**}(S^{0,0})$ detected by $h_1^3 h_4$ is not in the image of $\cdot 2$ [21]. Therefore we see that $\eta^2\eta_4[(3,2)] \in \pi_{**}(\Sigma^{1,1}C\eta)$ is not in the image of f_* and so we obtain the Atiyah–Hirzebruch–May name $h_1^3 h_4[(3,2)]$ for the element $j_*(\eta^2\eta_4[(3,2)]) \in \pi_{**}(C)$. Finally, the complex $V(0) \wedge C$ is defined by the cofiber sequence

$$C \xrightarrow{\cdot 2} C \to V(0) \wedge C.$$

By the same reasoning as above, we see that the Atiyah–Hirzebruch–May name for the element $j_*(\eta^2\eta_4[(3,2)]) \in \pi_{**}(V(0) \wedge C)$ is $h_1^3 h_4[(0,0),(3,2)]$. Note that the above May differential implies that b_{21}^2 does not survive in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence converging to $\pi_{**}(S^{0,0})$, and so it also does not survive in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence converging to $\pi_{**}(V(0)\wedge C)$.

On the other hand, b_{21}^2 detects $\widetilde{\beta}\in\pi_{**}(\mathrm{wko})$ by the $C\tau$–linear version of the classical argument that b_{20}^2 detects $\widetilde{\beta}\in\pi_*(\mathrm{ko})$. Since b_{21}^2 supports multiplication by η, does not support multiplication by 2, and is not in the image of multiplication by η or 2, the same series of cofiber sequences as above shows that there is a class in $\pi_{**}(\mathrm{wko}\wedge V(0)\wedge C)$ with Atiyah–Hirzebruch–May name $b_{21}^2[(0,0),[(1,1)]$. Since h_4 does not appear in the $C\tau$–linear $\overline{H}$–based May spectral sequence converging to $\mathrm{Ext}_{\overline{A}(1)}(\mathbb{F}_2,\mathbb{F}_2)$, we see that $h_1^3h_4$ does not survive in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence converging to $\pi_{**}(\mathrm{wko})$. In particular, this implies that the class above does not survive in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence converging to $\pi_{**}(\mathrm{wko}\wedge V(0)\wedge C)$.

By the previous two paragraphs, we see that the class

$$b_{21}^2[(0,0),(1,1)]+h_1^3h_4[(0,0),(3,2)]$$

detects $\eta^2\eta_4[(0,0),(3,2)]$ in $\pi_{**}(V(0)\wedge C)$, and this class detects $\widetilde{\beta}[(0,0),(1,1)]$ in $\pi_{**}(\mathrm{wko}\wedge V(0)\wedge C)$. Therefore we have $h^{-1}(\widetilde{\beta}[(0,0),(1,1)])=\eta^2\eta_4[(0,0),(3,2)]$. This completes the base case of the induction.

Now suppose that we have shown that

$$h^{-1}(\widetilde{\beta}^i([1,1]))=P^{i-1}(\eta^2\eta_4)[(0,0),(3,2)]$$

for all $i<n$. We can reformulate the induction hypothesis using the $C\tau$–linear $\overline{H}$–based Adams spectral sequence as follows: for $i<n$, the class in $\pi_{**}(V(0)\wedge C)$ detected by

$$b_{21}^{2i}[(0,0),(1,1)]+g^{i-1}(h_1^3h_4)[(0,0),(3,2)]$$

maps under h to the class in $\pi_{**}(\mathrm{wko}\wedge V(0)\wedge C)$ detected by $b_{21}^{2i}[(0,0),(1,1)]$.

To see that this reformulation implies the original induction hypothesis, we need to show that $b_{21}^{2i}[(0,0),(1,1)]$ does not survive in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence converging to $\pi_{**}(V(0)\wedge C)$ for all $i\geq 1$. Using algebraic squaring operations, we can produce May differentials

$$d_{2^k}(b_{21}^{2^k})=d_{2^k}(\mathrm{Sq}^{2^k}(b_{21}^{2^{k-1}}))=h_1^{2^k}h_{2+k}$$

with $k\geq 2$ as described in [11, Remark 5.7]. By the Leibniz rule, we can obtain nontrivial May differentials on b_{21}^{2i} for all $i\geq 1$. These elements could support shorter differentials, but in any case we have shown that b_{21}^{2i} does not survive in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence converging to $\pi_{**}(S^{0,0})$ and so $b_{21}^{2i}[(0,0),(1,1)]$ does not survive in the $C\tau$–linear $\overline{H}$–based Adams spectral sequence converging to $\pi_{**}(V(0)\wedge C)$. So the sum above detects $g^{i-1}(\eta^2\eta_4)[(0,0),(3,2)]$ in $\pi_{**}(C\wedge V(0))$. Since $b_{21}^{2i}[(0,0),(3,2)]$ detects $\widetilde{\beta}^i[(0,0),(1,1)]$ in $\pi_{**}(\mathrm{wko}\wedge V(0)\wedge C)$, we have shown that the reformulation implies the original induction hypothesis.

To complete the induction, consider the diagram

$$\begin{tikzcd}
 & \mathrm{Ext}_A^{***}(V(0)\wedge C) \arrow[rr, "g"] \arrow[dl] & & \mathrm{Ext}_A^{*+4,*+24,*+12}(V(0)\wedge C) \arrow[dl] \\
\mathrm{Ext}_A^{***}(\mathrm{wko}\wedge V(0)\wedge C) \arrow[rr, "\cdot b_{21}^2"] & & \mathrm{Ext}_A^{*+4,*+24,*+12}(\mathrm{wko}\wedge V(0)\wedge C) & \\
 & \mathrm{Ext}_{A_{\mathrm{cl}}}^{**}(V(0)\wedge C) \arrow[rr, "P_v"] \arrow[dl] \arrow[uu] & & \mathrm{Ext}_{A_{\mathrm{cl}}}^{*+4,*+12}(V(0)\wedge C) \arrow[dl] \arrow[uu] \\
\mathrm{Ext}_{A_{\mathrm{cl}}}^{**}(\mathrm{ko}\wedge V(0)\wedge C) \arrow[rr, "\cdot b_{24}^2"] \arrow[uu] & & \mathrm{Ext}_{A_{\mathrm{cl}}}^{*+4,*+12}(\mathrm{ko}\wedge V(0)\wedge C) \arrow[uu] &
\end{tikzcd}$$

The vertical maps in the diagram are induced by sending the class $h_{ij}[(m)(n)]$ to the class $h_{i,j+1}[(m,0),(n,\lfloor\frac{1}{2}(n+1)\rfloor)]$, and similarly for classes detected by products and Massey products.

Consider the element $b_{20}^{2n-2}[(0),(1)]+P_v^{n-2}(h_0^3h_3)[(0),(3)]$ in the bottom face of the diagram. This element detects $P_v^{n-2}(8\sigma)[(0),(3)]\in\pi_*(V(0)\wedge C)$. Its image in $\mathrm{Ext}_{A_{\mathrm{cl}}}^{**}(\mathrm{ko}\wedge V(0)\wedge C)$ is $b_{20}^{2n-2}[(0),(1)]$, which detects $b_{20}^{2(n-1)}[(0),(1)]$, its image in $\mathrm{Ext}_{A_{\mathrm{cl}}}^{*+4,*+8}(V(0)\wedge C)$ is $b_{20}^{2n}[(0),(1)]+P_v^{n-1}(h_0^3h_3)[(0),(3)]$, which detects $P_v^{n-1}(8\sigma)[(0),(1)]$, and its image in $\mathrm{Ext}_{A_{\mathrm{cl}}}^{*+4,*+8}(\mathrm{ko}\wedge V(0)\wedge C)$ is $b_{20}^{2n}[(0),(1)]$, which detects $\widetilde{\beta}^n[(0),(1)]$. This follows from the same argument as in the proof of Theorem 5.12.

Consider the element $g^{n-2}(h_1^3h_4)[(0,0),(3,2)]\in\mathrm{Ext}_{\overline{A}}^{***}(V(0)\wedge C)$ in the top face of the cube. By the induction hypothesis, its image in $\mathrm{Ext}_{\overline{A}}^{***}(\mathrm{wko}\wedge V(0)\wedge C)$ is the class $b_{21}^{2(n-1)}[(0,0),(1,1)]$. Its inverse image under the vertical map is $P_v^{n-2}(h_0^3h_3)[(0),(1)]$. We can calculate the images of $g^{n-2}(h_1^3h_4)[(0,0),(3,2)]$ in the top face using the vertical maps and the previous paragraph. In particular, we are able to conclude that $g^{n-1}(h_1^3h_4)[(0,0),(3,2)]\in\mathrm{Ext}_{\overline{A}}^{*+4,*+20,*+12}(V(0)\wedge C)$ maps to $b_{21}^{2n}[(0,0),(1,1)]\in\mathrm{Ext}_{\overline{A}}^{*+4,*+20,*+12}(\mathrm{wko}\wedge V(0)\wedge C)$. This completes the induction step, so we have established the values in the table for $x=\widetilde{\beta}^k[(1,1)]$.

The computation of $h^{-1}(\widetilde{\beta}^k\widetilde{\alpha}[(1,1)])$ is completely analogous. The May differential $d_4(b_{20}h_1^4) = h_1h_2^3$ produces the relation $\eta\cdot(\nu^3)$ in $\pi_{**}(S^{0,0})$, so we see that the class $b_{20}h_1^4[(0,0),(1,1)]+h_2^3[(0,0),(3,2)]$ detects $\nu^3[(0,0),(3,2)]\in\pi_{**}(V(0)\wedge C)$ and detects $\widetilde{\alpha}\in\pi_{**}(\mathrm{wko}\wedge V(0)\wedge C)$. Thus $h^{-1}(\widetilde{\alpha}[(1,1)]) = \nu^3[(0,0),(3,2)]$. The same argument using Massey products, algebraic squaring operations, and the cubical diagram completes the induction.

The result now follows by applying the obvious analog of Proposition 2.20. Commutativity of the diagram from Proposition 2.20 follows from Proposition 5.16 and the low-dimensional calculations in Section 2.3 as in the proof of [28, Theorem 2.17]. □

Lemma 5.20 *Let $k\geq 1$. Then*

$$M_{V(0)}(\eta^{4k+j}) \text{ contains } \begin{cases} g^{k-1}(\eta^2\eta_4)[(0,0)] & \text{if } j=0,\\ g^k(\nu^3)[(0,0)] & \text{if } j=3.\end{cases}$$

Proof Since we know which cell of $\underline{L}_{-\infty}^{\infty}$ the coset $M_{V(0)}(\eta^i;C)$ is detected on, we obtain the following refinement of the previous computation:

x	$M_{V(0)}(x)$
η^{4k}	$g^{k-1}(\eta^2\eta_4)[(0,0)]$
η^{4k+3}	$g^k(\nu^3)[(0,0)]$

□

Proposition 2.20 applied to the inclusion of the bottom cell $S^{0,0}\hookrightarrow V(0)$ completes the computation.

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