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Poincaré duality complexes with highly connected universal cover

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Turaev conjectured that the classification, realization and splitting results for Poincaré duality complexes of dimension 3 (PD_3–*complexes*) generalize to PD_n–complexes with $(n-2)$–connected universal cover for $n \geq 3$. Baues and Bleile showed that such complexes are classified, up to oriented homotopy equivalence, by the triple consisting of their fundamental group, orientation class and the image of their fundamental class in the homology of the fundamental group, verifying Turaev's conjecture on classification.

We prove Turaev's conjectures on realization and splitting. We show that a triple (G, ω, μ), comprising a group G, a cohomology class $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$ and a homology class $\mu \in H_n(G; \mathbb{Z}^\omega)$, can be realized by a PD_n–complex with $(n-2)$–connected universal cover if and only if the Turaev map applied to μ yields an equivalence. We show that a PD_n–complex with $(n-2)$–connected universal cover is a nontrivial connected sum of two such complexes if and only if its fundamental group is a nontrivial free product of groups.

We then consider the indecomposable PD_n–complexes of this type. When n is odd the results are similar to those for the case $n = 3$. The indecomposables are either aspherical or have virtually free fundamental group. When n is even the indecomposables include manifolds which are neither aspherical nor have virtually free fundamental group, but if the group is virtually free and has no dihedral subgroup of order > 2 then it has two ends.

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1 Introduction

Hendricks classified Poincaré duality complexes of dimension 3 (PD_3–*complexes*) up to oriented homotopy equivalence using the "fundamental triple", comprising the fundamental group, the orientation character and the image of the fundamental class in the homology of the fundamental group.

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Turaev [21] gave an alternative proof of Hendricks' result and provided necessary and sufficient conditions for a triple (G,ω,μ) — comprising a group G, a cohomology class $\omega \in H^1(G;\mathbb{Z}/2\mathbb{Z})$ and a homology class $\mu \in H_3(G;\mathbb{Z}^\omega)$, where $\mathbb{Z}^\omega$ denotes the integers $\mathbb{Z}$ regarded as a right $\mathbb{Z}[G]$–module with respect to the *twisted* structure induced by ω — to be the fundamental triple of a PD_3–complex. Central to this was that the image of μ under a specific homomorphism, which we call *the Turaev map*, be an isomorphism in the stable module category of $\mathbb{Z}[G]$, that is to say, a homotopy equivalence of $\mathbb{Z}[G]$–modules.

The results on classification and splitting allowed Turaev to show that a PD_3–complex is a nontrivial connected sum of two PD_3–complexes if and only if its fundamental group is a nontrivial free product of groups. He conjectured that these results hold for all PD_n–complexes with $(n-2)$–connected universal cover.

Baues and Bleile classified Poincaré duality complexes of dimension 4 in [2]. Their analysis showed that a PD_n–complex X, with $n \geq 3$, is classified up to oriented homotopy equivalence by the triple comprising its $(n-2)$–type $P_{n-2}(X)$, its orientation character $\omega = \omega_X \in H^1(\pi_1(X);\mathbb{Z}/2\mathbb{Z})$ and the image μ of its fundamental class in $H_n(P_{n-2}(X);\mathbb{Z}^\omega)$. (We assume that all spaces have basepoints. Thus, every map $f\colon X \to Y$ has a preferred lift to a map of universal covers. Hence, if $f^*\omega_Y = \omega_X$, there is a well-defined homomorphism $H_n(f;\mathbb{Z}^\omega)$, and it is meaningful to say that f is "oriented", ie that $f_*[X] = [Y]$. See Taylor [20] for a discussion of this issue.)

They called this the *fundamental triple of X*, as it is a generalization of Hendricks' "fundamental triple", for the $(n-2)$–type of X determines its fundamental group. Moreover, when the universal cover of the complex is $(n-2)$–connected — automatically the case when $n = 3$ — the $(n-2)$–type is an Eilenberg–Mac Lane space of type $(\pi_1(X),1)$, so that the $(n-2)$–type and the fundamental group determine each other completely, reducing their fundamental triple to that of Hendricks.

Turaev's conjecture on classification is a direct consequence:

Theorem *There is an oriented homotopy equivalence between two PD_n–complexes with $(n-2)$–connected universal cover if and only if their fundamental triples are isomorphic.*

We prove Turaev's conjectures on realization and splitting. These are, respectively, Theorems A and B below. While our approach to Theorem A is a direct generalization of Turaev's theorem, our proof applies Baues' homotopy systems in detail.

Recall that the group G is of type FP_n if and only if the trivial $\mathbb{Z}[G]$–module $\mathbb{Z}$ has a projective resolution $\boldsymbol{P}$, with P_j finitely generated for $j \leq n$.

Theorem A *Let G be a finitely presentable group, ω a cohomology class in $H^1(G;\mathbb{Z}/2\mathbb{Z})$ and μ a homology class in $H_n(G;\mathbb{Z}^\omega)$, with $n \geq 3$.*

If G is of type FP_{n-1}, with $H^i(G;{}^\omega\mathbb{Z}[G]) = 0$ for $1 < i \leq n-1$, then (G,ω,μ) can be realized as the fundamental triple of a PD_n–complex with $(n{-}2)$–connected universal cover if and only if the Turaev map applied to μ yields an isomorphism in the stable module category of $\mathbb{Z}[G]$.

Theorem B *A PD_n–complex with $(n{-}2)$–connected universal cover decomposes as a nontrivial connected sum of two such PD_n–complexes if only if its fundamental group decomposes as a nontrivial free product of groups.*

Thus, it is enough to investigate PD_n–complexes with $(n{-}2)$–connected universal cover whose fundamental group is indecomposable as free product, and we turn to the analysis of such complexes. Our arguments here exploit the interaction of Poincaré duality with the *Chiswell sequence associated with a graph of groups* (see Crisp [8] and Hillman [15]).

The parity of the dimension n is significant.

When n is odd, indecomposable orientable PD_n–complexes are either aspherical or have virtually free fundamental groups, and the arguments of [15] provide similar constraints on the latter class of groups. (See Section 7.) However, implementing the realization theorem may be difficult, and we do not consider this case further.

When n is even there are indecomposable fundamental groups, G, with virtual cohomological dimension n — $\mathrm{vcd}\, G = n$ — and infinitely many ends. Our strongest results are for groups which are indecomposable and virtually free.

Theorem C *Let X be a PD_{2k}–complex with $(2k{-}2)$–connected universal cover, and such that $G = \pi_1(X)$ is virtually free and indecomposable as a free product. If G is finite then $X \simeq S^{2k}$ or $\mathbb{R}P^{2k}$. If G is infinite and has no dihedral subgroup of order > 2, then G has two ends and its finite subgroups have cohomological period dividing $2k$. Hence, $\widetilde{X} \simeq S^{2k-1}$. If, moreover, X is orientable, then $H^1(G;\mathbb{Z}) \cong \mathbb{Z}$.*

In particular, Theorem C applies to closed 4–manifolds M with $\pi_2(M) = 0$ and such fundamental groups. There is no geometric connected sum decomposition theorem for 4–manifolds currently known that corresponds to Theorem B.

There is also a realization result, when $G \cong F \rtimes_\theta \mathbb{Z}$ with F finite (that is, when $H^1(G;\mathbb{Z}) \cong \mathbb{Z}$).

Theorem D *If $G \cong F \rtimes_\theta \mathbb{Z}$, where F is finite, then $G = \pi_1(X)$ for some* PD_{2k}*–complex X with $\widetilde{X} \simeq S^{2k-1}$ if and only if F has cohomological period dividing $2k$ and $H_{2k-1}(\theta;\mathbb{Z}) = \pm 1$.*

Since putting this paper on arXiv in May 2016 we have learned that Theorem D is a particular case of Proposition 8 of Golasiński and Gonçalves [10]. The paper [10] also gives estimates of the number of homotopy types realizing a given fundamental group. However, we have chosen to retain our independent treatment as it is brief and is a natural complement to our more substantial results.

In general, it is not known when such a PD_n–complex is homotopy equivalent to a closed n–manifold. (This question leads to delicate issues of algebraic number theory; see Hambleton and Madsen [12].) There has been extensive research on the mixed-spherical space form problem, on the fundamental groups of manifolds with universal covering space $S^n \times \mathbb{R}^k$ for $n, k > 0$. A recurring theme is the role of finite dihedral subgroups. See Hambleton and Pedersen [13] for a survey of recent progress.

The main questions left open by our study of indecomposable, virtually free fundamental groups are

(a) what happens when G has dihedral subgroups; and

(b) are there examples with $D_4 = (\mathbb{Z}/2\mathbb{Z})^2$ as a subgroup?

As seems common in topology, there appear to be difficulties associated with 2–torsion!

Section 2 summarizes background material and fixes notation.

Section 3 contains the formulation and proof of the necessity of the condition for the realization of a fundamental triple by a PD_n–complex with $(n-2)$–connected universal cover.

Section 4 completes the proof of Theorem A, with the sufficiency of the condition in Section 3.

Section 5 contains the proof of Theorem B, showing how the fundamental triple detects connected sums.

Section 6 is an interlude outlining the notion of graphs of groups used subsequently.

Section 7 starts the discussion of indecomposable PD_n–complexes with $(n-2)$–connected universal covers, beginning with Crisp's centralizer condition.

In Section 8 we give some supporting results, and construct examples of indecomposable groups G with infinitely many ends and $\operatorname{vcd} G = n$ which are the fundamental groups of closed n–manifolds with $(n-2)$–connected universal cover.

In Section 9 we show that finite subgroups of G of odd order are metacyclic.

In Sections 10 and 11 we prove Theorems C and D.

Section 12 concludes by briefly considering possible examples with D_4 as a subgroup.

2 Background and notation

This section summarizes background material and fixes notation for the rest of the paper. Details and further references can be found in [4; 8; 15].

Let Λ be the integral group ring $\mathbb{Z}[G]$ of the group G. We write I for the *augmentation ideal*, the kernel of the augmentation map

$$\mathrm{aug}\colon \Lambda \to \mathbb{Z}, \quad \sum_{g\in G} n_g g \mapsto \sum_{g\in G} n_g,$$

where $\mathbb{Z}$ is a Λ–bimodule with trivial Λ action. Each cohomology class $\omega \in H^1(G;\mathbb{Z}/2\mathbb{Z})$ may be viewed as a group homomorphism $\omega\colon G \to \mathbb{Z}/2\mathbb{Z} = \{0,1\}$ and yields an anti-isomorphism

$$\overline{}\colon \Lambda \to \Lambda, \quad \lambda = \sum_{g\in G} n_g g \mapsto \bar{\lambda} = \sum_{g\in G} (-1)^{\omega(g)} n_g g^{-1}. \tag{1}$$

Consequently, a right Λ–module A yields the *conjugate left* Λ*–module* ${}^{\omega}A$, with action given by

$$\lambda_{\bullet} a := a.\bar{\lambda}$$

for $\lambda \in \Lambda$ and $a \in A$. Plainly, the conjugate defines a functor from the category of left Λ–modules to the category of right Λ–modules. Similarly, a left Λ–module B yields the conjugate right Λ–module B^{ω}. If M is a Λ–bimodule, then conjugating both the left and the right Λ–module structures leads to ${}^{\omega}M^{\omega}$, with Λ–bimodule structure

$$\lambda_{\bullet} x_{\bullet} \mu := \bar{\mu}.x.\bar{\lambda}.$$

Given left Λ–modules A_j and B_i for $1 \le i \le k$ and $1 \le j \le \ell$, we sometimes write the Λ–morphism $\psi\colon \bigoplus_{j=1}^{\ell} A_j \to \bigoplus_{i=1}^{k} B_i$ in matrix form as $[\psi_{ij}]_{k\times\ell}$ for $\psi_{ij} = \mathrm{pr}_i \circ \psi \circ \mathrm{in}_j\colon A_j \to B_i$, where in_j is the j^{th} natural inclusion and pr_i the i^{th}

natural projection of the direct sum. The composition of such morphisms is given by matrix multiplication.

If B is a left Λ–module and M a Λ–bimodule, then $\mathrm{Hom}_\Lambda(B, M)$ is a right Λ–module with action given by

$$\varphi.\lambda\colon B \to M, \quad b \mapsto \varphi(b).\lambda.$$

The dual of the left Λ–module B is the left Λ–module $B^* = {}^\omega\mathrm{Hom}_\Lambda(B, \Lambda)$. The construction of the dual defines an endofunctor on the category of left Λ–modules.

Evaluation defines a natural transformation ε from the identity functor to the double dual functor, where, for the left Λ–module B,

$$\varepsilon_B\colon B \to B^{**} = {}^\omega\mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(B, \Lambda), \Lambda), \quad b \mapsto \overline{\mathrm{ev}}_b,$$

with $\overline{\mathrm{ev}}_b$ defined by

$$\overline{\mathrm{ev}}_b\colon {}^\omega\mathrm{Hom}(B, \Lambda) \to \Lambda, \quad \psi \mapsto \overline{\psi(b)}.$$

The left Λ–module A defines the natural transformation η from the functor $A^\omega \otimes_\Lambda -$ to the functor $\mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(-, \Lambda), A)$, where, for the left Λ–module B,

$$\eta_B\colon A^\omega \otimes_\Lambda B \to \mathrm{Hom}_\Lambda(B^*, A) = \mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(B, \Lambda), A)$$

is given by

$$\eta_B(a \otimes b)\colon \psi \mapsto \overline{\psi(b)}.a$$

for $a \otimes b \in A^\omega \otimes_\Lambda B$. Both ε and η become natural equivalences when restricted to the category of finitely generated free Λ–modules.

The Λ–morphisms $f, g\colon A_1 \to A_2$ are *homotopic* if and only if the Λ–morphism $f - g\colon A_1 \to A_2$ factors through a projective Λ–module P. Associated with Λ is its *stable module category*, whose objects are all Λ–modules and whose morphisms are all homotopy classes of Λ–morphisms. Thus, an isomorphism in the stable module category of Λ is a homotopy equivalence of Λ–modules.

We work in the category of connected, well-pointed CW–complexes and pointed maps. We write $X^{[k]}$ for the k–skeleton of X, suppressing the basepoint from our notation. The inclusion of the k–skeleton into the $(k+1)$–skeleton induces a homomorphism $\pi_{k+1}(X^{[k]}) \to \pi_{k+1}(X^{[k+1]})$, whose image we denote by $\Gamma_{k+1}(X)$.

From now, we work with the fundamental group $G = \pi_1(X)$ of X and its integral group ring $\Lambda = \mathbb{Z}[G]$. We take X to be a reduced CW–complex, so that $X^{[0]} = \{*\}$, and write $u\colon \widetilde{X} \to X$ for the universal cover of X, fixing a basepoint for $\widetilde{X}$ in $u^{-1}(*)$.

We write $\boldsymbol{C}(\widetilde{X})$ for the cellular chain complex of $\widetilde{X}$ viewed as a complex of left Λ–modules. Since X is reduced, $C_0(\widetilde{X}) = \Lambda$, and the augmentation ideal coincides with the image of the boundary map $C_1(\widetilde{X}) \to C_0(\widetilde{X})$.

The homology and cohomology of X we work with are the abelian groups

$$H_q(X;A) := H_q(A \otimes_\Lambda \boldsymbol{C}(\widetilde{X})),$$
$$H^q(X;B) := H^q\big(\mathrm{Hom}_\Lambda(\boldsymbol{C}(\widetilde{X}), B)\big),$$

where A is a right Λ–module and B is a left Λ–module.

An *n–dimensional Poincaré duality complex* (PD_n–complex) comprises a reduced connected *CW*–complex X whose fundamental group $\pi_1(X)$ is finitely presentable, together with an *orientation character* $\omega = \omega_X \in H^1(\pi_1(X); \mathbb{Z}/2\mathbb{Z})$, viewed as a group homomorphism $\pi_1(X) \to \mathbb{Z}/2\mathbb{Z}$, and a *fundamental class* $[X] \in H_n(X; \mathbb{Z}^\omega)$ such that, for every $r \in \mathbb{Z}$ and left $\mathbb{Z}[\pi_1(X)]$–module M, the *cap product with* $[X]$,

$$- \frown [X]\colon H^r(X;M) \to H_{n-r}(X;M^\omega), \quad \alpha \mapsto \alpha \frown [X],$$

is an isomorphism of abelian groups. We denote this by $(X, \omega, [X])$.

Wall [22; 24] showed that for $n > 3$, every PD_n–complex is *standard*, meaning that it is homotopically equivalent to an n–dimensional *CW*–complex with precisely one n–cell, whereas a PD_3–complex X is either standard or *weakly standard*, the latter meaning that it is homotopically equivalent to one of the form $X' \cup e^3$, where e^3 is a 3–cell and X' is a 3–dimensional *CW*–complex with $H^3(X';B) = 0$ for all coefficient modules B.

In [1], Baues introduced *homotopy systems* to investigate when chain complexes and chain maps of free Λ–modules are realized by *CW*–complexes.

Take an integer $n > 1$. A *homotopy system of order* $(n+1)$ comprises

(a) a reduced n–dimensional *CW*–complex X;

(b) a chain complex $\boldsymbol{C}$ of free Λ–modules coinciding with $\boldsymbol{C}(\widetilde{X})$ in degree q for $q \le n$;

(c) a homomorphism $f_{n+1}\colon C_{n+1} \to \pi_n(X)$ with $f_{n+1} \circ d_{n+2} = 0$ such that the diagram

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{f_{n+1}} & \pi_n(X) \\ {\scriptstyle d_{n+1}}\downarrow & & \downarrow{\scriptstyle j} \\ C_n & \xleftarrow[h_n]{} & \pi_n(X, X^{[n-1]}) \end{array}$$

commutes, where j is induced by the inclusion $(X,*)\to(X,X^{[n-1]})$, and

$$h_n\colon \pi_n(X,X^{[n-1]}) \xrightarrow[\cong]{u_*^{-1}} \pi_n(\widetilde{X},\widetilde{X}^{[n-1]}) \xrightarrow[\cong]{h} H_n(\widetilde{X},\widetilde{X}^{[n-1]})$$

is the Hurewicz isomorphism h composed with u_*^{-1}, the inverse of the isomorphism induced by the universal covering map.

3 Formulation and necessity of the realization conditions

For our generalization of Tuarev's realization condition to PD$_n$–complexes with $n\geq 3$, we introduce a set of functors from the category of chain complexes of projective left Λ–modules to the category of left Λ–modules.

Given $\boldsymbol{f}\colon \boldsymbol{C}\to\boldsymbol{D}$, a map of chain complexes of projective left Λ–modules $\boldsymbol{C}$ and $\boldsymbol{D}$, put $T_q(\boldsymbol{C}) := \operatorname{coker}(d^{\boldsymbol{C}}_{q+1}\colon C_{q+1}\to C_q) = C_q/\operatorname{im}(d^{\boldsymbol{C}}_{q+1})$ and let $T_q(\boldsymbol{f})$ be the induced map of cokernels

$$\begin{tikzcd}
\operatorname{im}(d^{\boldsymbol{C}}_{q+1}) \arrow[r, tail] \arrow[d] & C_q \arrow[r, two heads] \arrow[d, "f_q"'] & T_q(\boldsymbol{C}) \arrow[d, dotted, "\exists!"', "T_q(\boldsymbol{f})"] \\
\operatorname{im}(d^{\boldsymbol{D}}_{q+1}) \arrow[r, tail] & D_q \arrow[r, two heads] & T_q(\boldsymbol{D})
\end{tikzcd}$$

Direct verification shows that each T_q is a functor from the category of chain complexes of left Λ–modules to the category of left Λ–modules.

By Lemma 4.2 in [4], chain-homotopic maps $\boldsymbol{f}\simeq\boldsymbol{g}\colon\boldsymbol{C}\to\boldsymbol{D}$ induce homotopic maps $T_q(\boldsymbol{f})\simeq T_q(\boldsymbol{g})$, that is, $T_q(\boldsymbol{f})-T_q(\boldsymbol{g})$ factors through a projective Λ–module. Hence, for each $q\in\mathbb{Z}$, T_q induces a functor from the category of chain complexes of projective left Λ–modules and chain homotopy classes of chain maps to the stable module category of Λ.

Let X be a PD$_n$–complex with $n\geq 3$, and let $\Lambda=\mathbb{Z}[\pi_1(X)]$. By Remark 2.3 and Lemma 3.6 in [2], we may assume that $X=X'\cup e^n$ is standard (or weakly standard if $n=3$) with

$$C_n(\widetilde{X}) = C_n(\widetilde{X}')\oplus\Lambda e,$$

where e corresponds to e^n, the element $1\otimes e\in\mathbb{Z}^\omega\otimes_\Lambda C_n(\widetilde{X})$ is a cycle representing the fundamental class $[X]$ of X, and e is a generator of $C_n(\widetilde{X})$.

Writing F^q for $T_q({}^{\omega}\mathrm{Hom}_{\Lambda}(-,\Lambda))$, Poincaré duality, together with Lemma 4.3 in [4], provides the homotopy equivalence of Λ–modules

$$T_{-n+1}(-\frown(1\otimes e))\colon F^{n-1}(\boldsymbol{C}(\widetilde{X}))\to T_1(\boldsymbol{C}(\widetilde{X})).$$

Construct the $(n-2)$–type $P=P_{n-2}(X)$ of X by attaching to X cells of dimension n and higher. Then the Postnikov section $p\colon X\to P$ is the identity on the $(n-1)$–skeleta and $C_i(\widetilde{X})=C_i(\widetilde{P})$ for $0\le i<n$. Composing with the isomorphism $\theta\colon T_1(\boldsymbol{C}(\widetilde{X}))\to I$, $[c]\mapsto d_1(c)$, we obtain the homotopy equivalence of left Λ–modules

$$\theta\circ T_{-n+1}(-\frown(1\otimes e))\colon F^{n-1}(\boldsymbol{C}(\widetilde{P}))\to I. \tag{2}$$

We next construct the *Turaev map*, which sends the image of the fundamental class of X in the homology of the Postnikov section to the homotopy class of the homotopy equivalence (2).

Let $\boldsymbol{C}$ be a chain complex of free left Λ–modules. We write $\overline{I}$ for the image of the augmentation ideal I under the anti-isomorphism (1). This gives rise to the short exact sequence of chain complexes $0\to\overline{I}\boldsymbol{C}\to\boldsymbol{C}\to\mathbb{Z}^{\omega}\otimes_{\Lambda}\boldsymbol{C}\to 0$, with associated connecting homomorphism $\delta_r\colon H_r(\mathbb{Z}^{\omega}\otimes_{\Lambda}\boldsymbol{C})\to H_{r-1}(\overline{I}\boldsymbol{C})$.

The set of homotopy classes of module morphisms $A\to B$, written $[A,B]$, is naturally a group and it is straightforward (see [3]) to verify that

$$\hat{\nu}_{\boldsymbol{C},r}\colon H_r(\overline{I}\boldsymbol{C})\to[F^r(\boldsymbol{C}),I],\quad [\lambda.c]\mapsto[F^r\boldsymbol{C}\to I,\ [\varphi]\mapsto\overline{\varphi(\lambda.c)}],$$

is a homomorphism of groups. Composing $\hat{\nu}_{\boldsymbol{C},r-1}$ with δ_r yields the Turaev map

$$\nu_{\boldsymbol{C},r}\colon H_r(\mathbb{Z}^{\omega}\otimes_{\Lambda}\boldsymbol{C})\to[F^{r-1}(\boldsymbol{C}),I].$$

Lemma 1 $\quad \nu_{\boldsymbol{C}(\widetilde{P}),n}(p_*([X]))=[\theta\circ T_{-n+1}(-\frown(1\otimes e))].$

Proof Take a diagonal

$$\Delta\colon\boldsymbol{C}(\widetilde{X})\to\boldsymbol{C}(\widetilde{X})\otimes_{\mathbb{Z}}\boldsymbol{C}(\widetilde{X})$$

and a chain homotopy $\alpha\colon\boldsymbol{C}(\widetilde{X})\to\boldsymbol{C}(\widetilde{X})$ such that $\mathrm{id}-(\mathrm{id}\otimes\mathrm{aug})\Delta=d\alpha+\alpha d$, where we have identified $\boldsymbol{C}\otimes_{\mathbb{Z}}\mathbb{Z}$ with $\boldsymbol{C}$. Let

$$\Delta e=e\otimes\lambda+\sum_{\ell}\sum_{0\le i<n}x_{\ell,i}\otimes y_{\ell,n-i}.$$

Direct calculation shows that $e=\mathrm{aug}(\lambda)e+\alpha de$. Since $[1\otimes e]$ generates the homology $H_n(X;\mathbb{Z}^{\omega})\cong\mathbb{Z}$, this yields $[1\otimes e]=[\mathrm{aug}(\lambda)\otimes e]$, whence $\mathrm{aug}(\lambda-1)=0$. Hence,

$\lambda - 1 \in I = \mathrm{im}(d_1)$, or $\lambda = 1 + d_1(c_1)$ for some $c_1 \in C_1(\widetilde{X})$. Thus, given $\varphi \in \mathrm{Hom}_\Lambda(C_n(\widetilde{X}), \Lambda)$,

$$\varphi \frown (1 \otimes e) = \overline{\varphi(e)}(1 + d_1(c_1)).$$

By direct calculation,

$$\big(\theta \circ T_{-n+1}(- \frown (1 \otimes e))\big)([\varphi]) = \overline{\varphi(d_n(e))}(1 + d_1(c_1))$$

and

$$\nu_{\boldsymbol{C}(\widetilde{P}),n}(p_*([X]))([\varphi]) = \hat{\nu}_{\boldsymbol{C}(\widetilde{P}),n}([d_n(e)])([\varphi]).$$

Hence, by definition, $\nu_{\boldsymbol{C}(\widetilde{P}),n}(p_*([X]))$ is represented by the Λ–morphism

$$F^{n-1}(\boldsymbol{C}(\widetilde{P})) \to I, \quad [\varphi] \mapsto \overline{\varphi(d_n(e))}.$$

To conclude the proof, note that $F^{n-1}(\boldsymbol{C}(\widetilde{P})) \to I$, $[\varphi] \mapsto \overline{\varphi(d_n(e))}.d_1(c_1)$, factors through $C_1(X)$ and is thus null-homotopic. □

As $\theta \circ T_{-n+1}(- \frown (1 \otimes e))$ is a homotopy equivalence of Λ–modules, Lemma 1 provides a necessary condition for realization.

Theorem 2 *Let P be an $(n-2)$–type. Take $\omega \in H^1(P;\mathbb{Z}/2\mathbb{Z})$ and $\mu \in H_n(P;\mathbb{Z}^\omega)$. Then (P,ω,μ) is the fundamental triple of a* PD_n*–complex only if $\nu_{\boldsymbol{C}(\widetilde{P}),n}(\mu)$ is a homotopy equivalence of left $\Lambda = \mathbb{Z}[\pi_1(P)]$–modules.*

Proof Let P be an $(n-2)$–type. Take $\omega \in H^1(P;\mathbb{Z}/2\mathbb{Z})$ and $\mu \in H_n(P;\mathbb{Z}^\omega)$. Suppose that (P,ω,μ) is the fundamental triple of the PD_n–complex X. If P' is an $(n-2)$–type obtained by attaching to X cells of dimension n and higher, then there is a homotopy equivalence $f\colon P \to P'$ with $f_*(\mu) = i_*([X])$, where $i\colon X \to P'$ is the inclusion. By Lemma 1,

$$\nu_{\boldsymbol{C}(\widetilde{P'}),n}(i_*[X]) = [\theta \circ T_{-n+1}(- \frown (1 \otimes e))]$$

and hence $\nu_{\boldsymbol{C}(\widetilde{P}),n}(\mu)$ are homotopy equivalences of Λ–modules. □

Let X now be a PD_n–complex with $(n-2)$–connected universal cover. The $(n-2)$–type of X is an Eilenberg–Mac Lane space $K(\pi_1(X),1)$, and we may identify the fundamental triple of X with $(\pi_1(X),\omega,\mu)$, where μ is the image of $[X]$ in the group homology of $\pi_1(X)$.

Lemma 3 *Let $(X,\omega,[X])$ be a* PD_n*–complex with $(n-2)$–connected universal cover. Then $\pi_1(X)$ is* FP_{n-1}*, and $H^i(\pi_1(X);{}^\omega\Lambda) = 0$ for all $1 < i \le n-1$.*

Proof Since X is a PD_n–complex, it is finitely dominated, and so is homotopy equivalent to a complex with finite $(n-1)$–skeleton. Thus, we may assume that $X^{[n-1]}$ is finite. We construct an Eilenberg–Mac Lane space $K = K(\pi_1(X), 1)$ from X by attaching cells of dimension n and higher. As the universal cover $\widetilde{X}$ of X is $(n-2)$–connected, the cellular chain complexes of the universal covers $\widetilde{X}$ and $\widetilde{K}$ coincide in degrees below n, that is, $C_i(\widetilde{X}) = C_i(\widetilde{K})$ for $0 \le i < n$. In particular, these modules are finitely generated, and so $\pi_1(X)$ is FP_{n-1}.

Moreover, for $1 < i \le n-1$,

$$H^i(\pi_1(X); {}^{\omega}\Lambda) = H^i(X; {}^{\omega}\Lambda) \cong H_{n-i}(X; \Lambda) = 0. \qquad \square$$

We note for later reference that

$$\pi_{n-1}(\widetilde{X}) \cong H_{n-1}(\widetilde{X}; \mathbb{Z}) \cong {}^{\omega}H^1(G; \mathbb{Z}[G]),$$

by Hurewicz's theorem and Poincaré duality, respectively.

Necessary conditions for realization are a corollary to Lemma 3 and Theorem 2.

Corollary 4 (conditions for realizability) *Let G be a group. Take $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$ and $\mu \in H_n(G; \mathbb{Z}^{\omega})$. If (G, ω, μ) is the fundamental triple of a PD_n–complex with $(n-2)$–connected universal cover, then G is a finitely presentable group of type FP_{n-1}, $H^i(G; {}^{\omega}\Lambda) = 0$ for $1 < i \le n-1$ and $\nu_{C(\widetilde{K}),n}(\mu)$ is a homotopy equivalence of $\Lambda = \mathbb{Z}[G]$ modules.*

4 Sufficiency of the realization condition

We now establish the sufficiency of the realization conditions in Corollary 4.

Let G be a finitely presentable group of type FP_{n-1}, with $n \ge 3$. Let K' be an Eilenberg–Mac Lane space of type $(G, 1)$ with universal cover $\widetilde{K}' \to K'$. Identify the (co)homologies of G and K'. Choose $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$ and suppose that $H^i(G; {}^{\omega}\Lambda) = 0$ for $1 < i \le n-1$, where $\Lambda = \mathbb{Z}[G]$. Finally, take $\mu \in H_n(G; \mathbb{Z}^{\omega})$, with $\nu_{C(\widetilde{K}'),n}(\mu)$ a class of homotopy equivalences of Λ–modules.

We construct a PD_n–complex X with $(n-2)$–connected universal cover and fundamental triple (G, ω, μ).

By the hypotheses on G, we may assume that K' has been chosen with finitely many cells in each dimension below n.

Let $h\colon F^{n-1}(\boldsymbol{C}(\widetilde{K}')) \to I$ be a representative of $\nu_{\boldsymbol{C}(\widetilde{K}'),n}(\mu)$. Then h is a homotopy equivalence of Λ–modules. By Theorem 4.1 and Observation 1 in [4], h factors as

$$F^{n-1}(\boldsymbol{C}(\widetilde{K}')) \rightarrowtail F^{n-1}(\boldsymbol{C}(\widetilde{K}')) \oplus \Lambda^m \rightarrowtail I \oplus P \twoheadrightarrow I$$

for some projective Λ–module, P, and $m \in \mathbb{N}$. Let $B = (e^0 \cup e^{n-1}) \cup e^n$ be the n–dimensional ball and replace K' by the Eilenberg–Mac Lane space $K = K' \vee \bigl(\bigvee_{i=1}^m B\bigr)$. Then $F^{n-1}(\boldsymbol{C}(\widetilde{K})) = F^{n-1}(\boldsymbol{C}(\widetilde{K}')) \oplus \Lambda^m$ and the factorization of h becomes

$$h\colon F^{n-1}(\boldsymbol{C}(\widetilde{K})) \overset{j}{\rightarrowtail} I \oplus P \xrightarrow{\mathrm{pr}_I} I$$

with j surjective. Consider the Λ–morphism φ given by the composition

$$C^{n-1}(\widetilde{K}) = {}^{\omega}\mathrm{Hom}_{\Lambda}(C_{n-1}(\widetilde{K}), \Lambda) \xrightarrow{p} F^{n-1}(\boldsymbol{C}(\widetilde{K})) \overset{j}{\rightarrowtail} I \oplus P \xrightarrow{\left[\begin{smallmatrix} i & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right]} \Lambda \oplus P,$$

where p is the projection onto the cokernel and $i\colon I \rightarrowtail \Lambda$ the inclusion. Since $F^{n-1}(\boldsymbol{C}(\widetilde{K})) = C^{n-1}(\widetilde{K})/\operatorname{im}(d_{n-1}^*)$ by definition, $\varphi \circ d_{n-1}^* = 0$. As $C_{n-1}(\widetilde{K})$ is a finitely generated free Λ–module, the natural map

$${}^{\omega}\varepsilon\colon C_{n-1}(\widetilde{K}) \to C_{n-1}(\widetilde{K})^{**}$$

is an isomorphism. Define

$$d_n := ({}^{\omega}\varepsilon)^{-1} \circ \varphi^*\colon (\Lambda \oplus P)^* \to C_{n-1}(\widetilde{K}).$$

It follows from the naturality of ${}^{\omega}\varepsilon$ that $d_{n-1} \circ d_n = 0$.

We first consider the case when P is free, so that $P \cong \Lambda^q$ for some $q \in \mathbb{N}$ and $\Lambda \oplus P \cong \Lambda^{q+1}$.

Since $\widetilde{K}^{[n-1]}$ is $(n-2)$–connected, the Hurewicz homomorphism

$$h_q\colon \pi_q(\widetilde{K}^{[n-1]}) \to H_q(\widetilde{K}^{[n-1]})$$

is an isomorphism for $q \le n-1$ and we obtain the map

$$\begin{aligned} \varphi'\colon \Lambda^{q+1} \cong (\Lambda \oplus P)^* \to \ker(d_{n-1}) = H_{n-1}(\widetilde{K}^{[n-1]}) &\xrightarrow{h_{n-1}^{-1}} \pi_{n-1}(\widetilde{K}^{[n-1]}), \\ x \mapsto h_{n-1}^{-1}([d_n(x)]).& \end{aligned}$$

Let $\boldsymbol{C}$ be the chain complex of Λ–modules

$$\Lambda^{q+1} \cong (\Lambda \oplus P)^* \xrightarrow{d_n} C_{n-1}(\widetilde{K}^{[n-1]}) \xrightarrow{d_{n-1}} \cdots \to C_1(\widetilde{K}^{[n-1]}) \to \Lambda.$$

Then $Y = (\boldsymbol{C}, \varphi', K^{[n-1]})$ is a homotopy system of order n. As $C_i = 0$ for $i > n$, $H^{n+2}(Y; \Gamma_n Y) = 0$ and, by Proposition 8.3 in [2], there is a homotopy system

$(\boldsymbol{C}, 0, X)$ of order $n+1$ realising Y, with X an n–dimensional CW–complex. By construction, $\boldsymbol{C}(\widetilde{X}) = \boldsymbol{C}$ and the universal cover of X is $(n-2)$–connected. Since $X^{[n-1]} = K^{[n-1]}$ and $\pi_q(K) = 0$ for all $q > 1$, the inclusion $i\colon K^{[n-1]} \to K$ extends to a map

$$f\colon X \to K = K(G,1)$$

and we may take $\omega \in H^1(K;\mathbb{Z}/2\mathbb{Z})$ to be an element of $H^1(X;\mathbb{Z}/2\mathbb{Z})$.

Proposition 5 *X is a* PD_n*–complex with fundamental triple* (G,ω,μ)*, that is,*

(i) $\mathbb{Z} \cong H_n(X;\mathbb{Z}^\omega) = \langle [X] \rangle$*;*

(ii) $f_*([X]) = \mu$*;*

(iii) $- \frown [X]\colon H^r(X;{}^\omega\Lambda) \to H_{n-r}(X;\Lambda)$ *is an isomorphism for every* $r \in \mathbb{Z}$*.*

Proof **(i)** As $\boldsymbol{C}(\widetilde{X}) = \boldsymbol{C}$ is a chain complex of finitely generated free Λ–modules, the natural map

$$\eta_{\boldsymbol{C}}\colon \mathbb{Z}^\omega \otimes_\Lambda \boldsymbol{C} \to \mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(\boldsymbol{C},\Lambda),\mathbb{Z})$$

is an isomorphism. Hence, writing ζ^+ for the morphism $\mathrm{Hom}_\Lambda(B,\mathbb{Z}) \to \mathrm{Hom}_\Lambda(A,\mathbb{Z})$ induced by $\zeta\colon A \to B$, we obtain

$$H_n(X;\mathbb{Z}^\omega) = \ker(1 \otimes d_n) \cong \ker(\varphi^+)$$

for $\varphi\colon {}^\omega\mathrm{Hom}_\Lambda(C_{n-1}(\widetilde{K}^{[n-1]}),\Lambda) \to \Lambda \oplus \Lambda^q$ defined above.

Since both p and j are surjective, both p^+ and j^+ are injective, whence

$$\ker(\varphi^+) = \ker\left(\left(\begin{bmatrix} i & 0 \\ 0 & \mathrm{id}\end{bmatrix} \circ j \circ p\right)^+\right) = \ker\left(\begin{bmatrix} i & 0 \\ 0 & \mathrm{id}\end{bmatrix}^+\right) = \ker\left(\begin{bmatrix} i^+ & 0 \\ 0 & \mathrm{id}\end{bmatrix}\right) \cong \ker(i^+).$$

But I is generated by elements $1-g$ for $g \in G$ and $(\psi \circ i)(1-g) = 0$ for $\psi \in \mathrm{Hom}_\Lambda(\Lambda,\mathbb{Z})$. Hence,

$$\ker(\varphi^+) \cong \mathrm{Hom}_\Lambda(\Lambda,\mathbb{Z}) \cong \mathbb{Z},$$

generated by $\mathrm{aug} \circ \mathrm{pr}_\Lambda\colon \Lambda \oplus \Lambda^q \to \mathbb{Z}$, the projection onto the first factor followed by the augmentation map.

Let $[X] = [1 \otimes x] \in H_n(X;\mathbb{Z}^\omega)$ be the homology class corresponding to $\mathrm{aug} \circ \mathrm{pr}_\Lambda$ under the isomorphism $H_n(X;\mathbb{Z}^\omega) = \ker(1 \otimes d_n) \cong \ker(\varphi^+) \cong \mathrm{Hom}_\Lambda(\Lambda,\mathbb{Z})$. Then $x \in (\Lambda \oplus \Lambda^q)^*$ is projection onto the first factor.

(ii) By the proof of Lemma 1, $\nu_{\boldsymbol{C}(\widetilde{X}),n}([X])$ is represented by

$$F^{n-1}(\boldsymbol{C}(\widetilde{X})) \to I, \quad [\psi] \mapsto \overline{\psi(d_n(x))}.$$

Thus, given $\psi \in C_{n-1}(\widetilde{X})^* = C_{n-1}(\widetilde{K})^*$,

$$\begin{aligned}
\overline{\psi(d_n(x))} &= \overline{\psi({}^{\omega}\varepsilon^{-1}(x \circ \varphi))} \\
&= {}^{\omega}\varepsilon({}^{\omega}\varepsilon^{-1}(x \circ \varphi))(\psi) \\
&= (x \circ \varphi)(\psi) \\
&= \left(x \circ \begin{bmatrix} i & 0 \\ 0 & \mathrm{id} \end{bmatrix} \circ j \circ p\right)(\psi) \\
&= (i \circ \mathrm{pr}_I \circ j)([\psi]) \\
&= h([\psi]).
\end{aligned}$$

Hence, $\nu_{\boldsymbol{C}(\widetilde{X}),n}([X])$ is the homotopy class of h, so that

$$\nu_{\boldsymbol{C}(\widetilde{K}),n}(\mu) = \nu_{\boldsymbol{C}(\widetilde{X}),n}([X]) = \nu_{\boldsymbol{C}(\widetilde{K}),n}(f_*([X])).$$

By Lemma 2.5 in [21], $\nu_{\boldsymbol{C}(\widetilde{K}),n}$ is injective, whence $\mu = f_*([X])$.

(iii) First consider $1 \le i < n-1$. Then $H_i(X;\Lambda) = H_i(K^{[n-1]};\Lambda) = 0$.

By the definition of φ,

$$H^{n-1}(X; {}^{\omega}\Lambda^{\omega}) = 0.$$

Moreover, by hypothesis,

$$H^{n-i}(X;\Lambda^{\omega}) = H^{n-i}(K^{[n-1]};\,\Lambda^{\omega}) \cong H^{n-i}(G;\Lambda^{\omega}) = 0$$

for $1 < i < n-1$. Thus,

$$- \frown (1 \otimes [X])\colon H^{n-i}(X; {}^{\omega}\Lambda) \to H_i(X;\Lambda)$$

is an isomorphism for $1 \le i < n-1$.

Next consider $i = 0$. As P and $\Lambda \oplus P$ are free, $\boldsymbol{C}(\widetilde{X})$ is a chain complex of free Λ–modules. Since the (twisted) evaluation map from a finitely generated free Λ–module to its double dual is an isomorphism,

$$\begin{aligned}
H^n(X; {}^{\omega}\Lambda) = {}^{\omega}\mathrm{Hom}_{\Lambda}(C_n(\widetilde{X}), {}^{\omega}\Lambda)/\operatorname{im}(\varphi^*)^* &= (\Lambda \oplus P)^{**}/\operatorname{im}(\varphi^*)^* \\
&\cong (\Lambda \oplus P)/\operatorname{im}(\varphi) \cong \Lambda/I \cong \mathbb{Z}.
\end{aligned}$$

The class $[\gamma]$ of the image of $(1,0) \in \Lambda \oplus P$ under the (twisted) evaluation isomorphism generates $H^n(X; {}^{\omega}\Lambda)$ and so, by Lemma 4.4 of [4],

$$[\gamma] \frown [X] = [\gamma] \frown [1 \otimes x] = [\overline{\gamma(x)}.e_0] = [e_0],$$

where $e_0 \in C_0(\widetilde{X})$ is a chain representing the basepoint. Thus,

$$- \frown [X]\colon\ H^n(X; {}^{\omega}\Lambda) \to H_0(X; \Lambda)$$

is an isomorphism.

Finally, note that by the above, $- \frown (1 \otimes x)$ yields the chain homotopy equivalence

$$\begin{tikzcd}
\operatorname{im} d_{n-1}^* \arrow[r, tail] \arrow[d] & C_{n-1}(\widetilde{X})^* \arrow[r, two heads] \arrow[d, "-\frown(1\otimes x)"'] & C_n(\widetilde{X})^* \arrow[d, "-\frown(1\otimes x)"] \\
\operatorname{im} d_2 \arrow[r, tail] & C_1(\widetilde{X}) \arrow[r, two heads] & C_0(\widetilde{X})
\end{tikzcd}$$

Applying the functor ${}^{\omega}\mathrm{Hom}_{\Lambda}(-, \Lambda)$, we obtain the chain homotopy equivalence $(- \frown (1 \otimes x))^*$, inducing isomorphisms

$$(- \frown [X])^*\colon\ H^0(X; {}^{\omega}\Lambda) \to H_n(X; \Lambda),$$
$$(- \frown [X])^*\colon\ H^1(X; {}^{\omega}\Lambda) \to H_{n-1}(X; \Lambda).$$

By Lemma 2.1 in [4], $(- \frown (1 \otimes x))^*$ induces an isomorphism in homology if and only if $- \frown (1 \otimes x)$ does, whence

$$- \frown [X]\colon\ H^0(X; {}^{\omega}\Lambda) \to H_n(X; \Lambda),$$
$$- \frown [X]\colon\ H^1(X; {}^{\omega}\Lambda) \to H_{n-1}(X; \Lambda)$$

are isomorphisms. □

Suppose now that P is projective, but not free.

Then there are a finitely generated Λ–module Q and a natural number q such that $P^* \oplus Q \cong \Lambda^q$. The natural isomorphisms

$$(\Lambda \oplus P)^* \oplus \Lambda^\infty \cong \Lambda^* \oplus P^* \oplus (Q \oplus P^* \oplus \cdots) \cong \Lambda \oplus (P^* \oplus Q \oplus P^* \oplus Q \oplus \cdots) \cong \Lambda^\infty$$

show that $(\Lambda \oplus P)^* \oplus \Lambda^\infty$ is a free Λ–module.

Consider the chain complex $\boldsymbol{D}$ given by

$$0 \to (\Lambda \oplus P)^* \oplus \Lambda^\infty \xrightarrow{\left[\begin{smallmatrix} d_n & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right]} C_{n-1}(\widetilde{K}^{[n-1]}) \oplus \Lambda^\infty \xrightarrow{[\,d_{n-1}\ 0\,]} C_{n-2}(\widetilde{K}^{[n-1]}) \xrightarrow{d_{n-2}} C_{n-3}(\widetilde{K}^{[n-1]}) \to \cdots .$$

We attach infinitely many n–balls to $K^{[n-1]}$ to obtain a CW–complex, K', whose cellular chain complex coincides with $\boldsymbol{D}$ in dimensions below n. Then $\widetilde{K'}^{[n-1]}$ is $(n-2)$–connected, and the Hurewicz homomorphisms $h_q\colon \pi_q(\widetilde{K'}^{[n-1]}) \to H_q(\widetilde{K'}^{[n-1]})$ are isomorphisms for $q \le n-1$. Defining the map

$$\varphi'\colon (\Lambda \oplus P)^* \oplus \Lambda^\infty \to \ker(d_{n-1}) = H_{n-1}(\widetilde{K'}^{[n-1]}) \xrightarrow{h_{n-1}^{-1}} \pi_{n-1}(\widetilde{K'}^{[n-1]}),$$
$$x \mapsto h_{n-1}^{-1}([d_n(x)]),$$

we obtain the homotopy system $Y' = (\boldsymbol{D}, \varphi', K'^{[n-1]})$ of order n. As $D_i = 0$ for $i > n$, $\widehat{H}^{n+2}(Y'; \Gamma_n Y') = 0$. By Proposition 8.3 in [2], there is then a homotopy system $(\boldsymbol{C}, 0, X')$ of order $n+1$ realising Y', with X' an n–dimensional CW–complex.

Note that $\boldsymbol{D}$, the chain complex of X', is chain homotopy equivalent to the chain complex $\boldsymbol{W}$ given by

$$\cdots \to P^* \oplus Q \xrightarrow{\left[\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{smallmatrix}\right]} P^* \oplus Q \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right]} P^* \oplus Q \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right]} (\Lambda \oplus P)^* \oplus Q \xrightarrow{\left[\begin{smallmatrix} d_n \\ 0 \end{smallmatrix}\right]} C_{n-1}(\widetilde{K}^{[n-1]})$$
$$\xrightarrow{d_{n-1}} C_{n-2}(\widetilde{K}^{[n-1]}) \xrightarrow{d_{n-2}} C_{n-3}(\widetilde{K}^{[n-1]}) \to \cdots .$$

By Theorem 2 of [23], there is a CW–complex X, with cellular chain complex $\boldsymbol{W}$, homotopy equivalent to X'. Since $\boldsymbol{W}$ is finitely generated in each degree, the proof that X realizes (G, ω, μ) is analogous to the proof of Proposition 5.

This completes the proof of Theorem A.

5 Decomposition as connected sum

Wall constructed a new PD_n–complex from given ones using the *connected sum of* PD_n–*complexes* (see [24]). This allows PD_n–complexes to be decomposed as connected sums of other, simpler PD_n–complexes.

Take PD_n–complexes $(X_k, \omega_k, [X_k])$ for $k = 1, 2$. Then we may express X_k as the mapping cone

$$X_k = X'_k \cup_{f_k} e^n_k$$

for suitable $f_k\colon S^{n-1} \to X'_k$. Here, X'_k is an $(n-1)$–dimensional CW–complex when $n > 3$, and when $n = 3$, X'_k is 3–dimensional with $H^3(X'_k; B) = 0$ for all coefficient modules B. For $k = 1, 2$, let $\iota_k\colon X'_k \to X'_1 \vee X'_2$ be the canonical inclusion of the k^{th} summand and put

$$\widehat{f}_k := \iota_k \circ f_k\colon S^{n-1} \to X'_1 \vee X'_2,$$

so that $\widehat{f}_k$ determines an element of $\pi_{n-1}(X_1' \vee X_2')$. Let $f_1 + f_2\colon S^{n-1} \to X_1' \vee X_2'$ represent the homotopy class $[\widehat{f}_1]+[\widehat{f}_2]$. Then the connected sum $X = X_1 \# X_2$ of X_1 and X_2 is the mapping cone of $f_1 + f_2$:

$$X_1 \# X_2 := (X_1' \vee X_2') \cup_{f_1+f_2} e^n.$$

It follows from the Seifert–van Kampen theorem that

$$\pi_1(X) = \pi_1(X_1) * \pi_1(X_2). \tag{3}$$

The canonical inclusion $\mathrm{in}_k\colon \pi_1(X_k) \to \pi_1(X)$ induces a (left or right) $\mathbb{Z}[\pi_1(X_k)]$–module structure on any (left or right) $\Lambda = \mathbb{Z}[\pi_1(X)]$–module. In particular, Λ is a $\pi_1(X_k)$–bimodule. By the universal property of the free product, the group homomorphisms $\omega_{X_k} = \mathrm{in}_k^*(\omega_X)$ uniquely determine a group homomorphism $\omega_X\colon \pi_1(X) \to \mathbb{Z}/2\mathbb{Z}$. For $k = 1, 2$, let L_k be the functor $\Lambda \otimes_{\mathbb{Z}[\pi_1(X_k)]} -$.

Let $\boldsymbol{B}$ be the subcomplex of $\boldsymbol{C}(\widetilde{X})$ containing the n–cells over the n–cell of X attached by $f_1 + f_2$. Then $\boldsymbol{B}$ is a Poincaré duality chain complex [2, page 2361] and it follows from Theorem 2.3 of [4] that $L_k(\boldsymbol{C}(\widetilde{X}_k))$ is also a Poincaré duality chain complex.

Let x denote the chain representing the n–cell attached by $f_1 + f_2$. Repeated application of Theorem 2.3 of [4] shows that $L_1(\boldsymbol{C}(\widetilde{X}_1)) + L_2(\boldsymbol{C}(\widetilde{X}_2))$ is a Poincaré duality chain complex. Hence, $(X, \omega_X, [1 \otimes x])$ is a Poincaré duality complex. This is the *connected sum of* $(X_1, \omega_{X_1}, [X_1])$ *and* $(X_2, \omega_{X_2}, [X_2])$, introduced by Wall [24].

Theorem B *A* PD_n*–complex with* $(n-2)$*–connected universal cover decomposes as a nontrivial connected sum if and only if its fundamental group decomposes as a nontrivial free product of groups.*

Proof Suppose the PD_n–complex, X, is the nontrivial connected sum of the PD_n–complexes, X_1 and X_2. Then, by (3), its fundamental group is the nontrivial free product of the fundamental groups of X_1 and X_2.

For the converse, let $(X, \omega_X, [X])$ be a PD_n–complex with $(n-2)$–connected universal cover and with $\pi_1(X) = G = G_1 * G_2$ for nontrivial groups G_1 and G_2. As $\pi_1(X)$ is finitely presentable, so are G_1 and G_2. For $j = 1, 2$, let $K_j = K(G_j; 1)$ be an Eilenberg–Mac Lane space with finite 2–skeleton. Then $K_1 \vee K_2$ is an Eilenberg–Mac Lane space $K(G_1 * G_2; 1)$, and

$$H_n(K; \mathbb{Z}^{\omega}) = H_n(K_1; \mathbb{Z}^{\omega_1}) \oplus H_n(K_n; \mathbb{Z}^{\omega_2}),$$

where $\omega_j \in H^1(K_j;\mathbb{Z}/2\mathbb{Z})$ for $j=1,2$ is the restriction of the orientation character $\omega \in H^1(K;\mathbb{Z}/2\mathbb{Z})$. Thus, $\mu_X = \mu_1 + \mu_2$, with $\mu_j \in H_n(K_j;\mathbb{Z}^{\omega_j})$ for $j=1,2$.

By the discussion above, if the PD_n–complex X_j, with $(n-2)$–connected universal cover realizes the fundamental triple (G_j,ω_j,μ_j), then the connected sum of X_1 and X_2 realizes the fundamental triple of X, whence, by the classification theorem in [2], X is orientedly homotopy equivalent to $X_1 \# X_2$. Hence, it is sufficient to construct realizations of (G_j,ω_j,μ_j) for $j=1,2$.

Let S_j be the functor $\Lambda \otimes_{\mathbb{Z}[G_j]} -$, so that, for $i \geq 1$,

$$C_i(\widetilde{K}) = S_1(C_i(\widetilde{K}_1)) \oplus S_2(C_i(\widetilde{K}_2)).$$

It follows that

$$F^{n-1}(\boldsymbol{C}(\widetilde{K})) = S_1\big(F^{n-1}(\boldsymbol{C}(\widetilde{K}_1))\big) \oplus S_2\big(F^{n-1}(\boldsymbol{C}(\widetilde{K}_2))\big)$$

and

$$I(\pi_1(X)) = S_1(I(G_1)) \oplus S_2(I(G_2)),$$

where the canonical inclusion is given by

$$S_j(I(G_j)) \to I(G_1 * G_2), \quad \sigma \otimes \lambda \mapsto \sigma\lambda,$$

for $\sigma \in \mathbb{Z}[\pi_1(X)]$ and $\lambda \in I(G_1 * G_2)$ viewed as an element of $I(\pi_1(X))$.

Let

$$\varphi_j\colon F^{n-1}(\boldsymbol{C}(\widetilde{K}_i)) \to I(G_j)$$

be a $\mathbb{Z}[G_j]$–morphism representing the class $\nu_{C(\widetilde{K}_j),n}(\mu_j)$. Then the class, $\nu_{C(\widetilde{K}),n}(\mu)$ of homotopy equivalences is represented by

$$\begin{array}{ccc} S_1\big(F^{n-1}(\boldsymbol{C}(\widetilde{K}_1))\big) \oplus S_2\big(F^{n-1}(\boldsymbol{C}(\widetilde{K}_2))\big) & = & F^{n-1}(\boldsymbol{C}(\widetilde{K})) \\ \Big\downarrow{\scriptstyle S_1(\varphi_1)\oplus S_2(\varphi_2)} & & \\ S_1(I(G_1)) \oplus S_2(I(G_2)) & = & I(G_1 * G_2) \end{array}$$

and it follows from the proof of the analogous proposition for $n=3$ [21, pages 269–270] that φ_j is a homotopy equivalence of modules. By Theorem A, (G_j,ω_j,μ_j) is realized by a PD_n–complex X_j with $(n-2)$–connected universal cover. $\square$

As in [21], Theorem B implies that when $\pi_1(X)$ is torsion-free the indecomposable summands of X are either aspherical or copies of $S^{n-1}\times S^1$ or $S^{n-1}\tilde{\times} S^1$, where

$S^k \tilde{\times} S^1$ is the mapping cylinder of an orientation-reversing self-homeomorphism of S^k.

In the next sections we shall consider what may happen when we allow $\pi_1(X)$ to have torsion.

6 An interlude on graphs of groups

Our arguments in the second part of this paper use the notion of graph of groups, for which our main references are [9; 18]. In particular, we rely on the fact that every finitely presentable group is accessible: it is the fundamental group of a finite graph of groups in which all edge groups are finite and all vertex groups are either finite or have one end. (See Theorem VI.6.3 of [9].) A *graph of groups* $(\mathcal{G}, \Gamma)$ consists of a graph Γ with origin and target functions o and t from the set of edges $E = E(\Gamma)$ to the set of vertices $V = V(\Gamma)$, and a family $\mathcal{G}$ of groups G_v for each vertex v and subgroups $G_e \le G_{o(e)}$ for each edge e, with monomorphisms $\phi_e\colon G_e \to G_{t(e)}$. (We shall usually suppress the maps ϕ_e from our notation.) All edges are oriented, but we do not use this, and in considering paths or circuits in Γ we shall not require that the edges be compatibly oriented. The *fundamental group* of $(\mathcal{G}, \Gamma)$ is the group $\pi\mathcal{G}$ with presentation

$$\bigl\langle G_v, t_e \ \forall v \in V(\Gamma),\ e \in E(\Gamma) \bigm| t_e g t_e^{-1} = \phi_e(g) \ \forall g \in G_e,\ e \in E(\Gamma), \\ t_f = 1 \ \forall f \in E(\Upsilon) \bigr\rangle,$$

where Υ is some maximal tree for Γ. The generator t_e is the *stable letter* associated to the edge e. Different choices of maximal tree give isomorphic groups. We may assume that $(\mathcal{G}, \Gamma)$ is *reduced*: if an edge joins distinct vertices then the edge group is isomorphic to a proper subgroup of each of these vertex groups. If $\pi\mathcal{G}$ is indecomposable as a free product then $(\mathcal{G}, \Gamma)$ is *indecomposable*: all edge groups are nontrivial. An edge e is a *loop isomorphism* at v if $o(e) = t(e) = v$ and the inclusions induce isomorphisms $G_e \cong G_v$. It is an *MC-tie* if $o(e) \ne t(e)$ and G_e has index 2 in each of $G_{o(e)}$ and $G_{t(e)}$. (We shall give the motivation for this name later.)

In an alternative formulation, the graph of groups $(\mathcal{G}, \Gamma)$ determines a tree T on which $\pi\mathcal{G}$ acts, such that the stabilizers of edges are the conjugates of the edge groups and the stabilizers of the vertices are the conjugates of the vertex groups. A $\pi\mathcal{G}$–tree T is *terminal* if each edge stabilizer is finite and each vertex stabilizer is finite or has one end. If $(\mathcal{G}, \Gamma)$ is reduced, the corresponding $\pi\mathcal{G}$–tree T is incompressible in the terminology of [9]. If G is a finitely generated accessible group then there is an essentially unique incompressible terminal G–tree, by Proposition IV.7.4 of [9]. Following [8], we shall say that a vertex of T is finite or infinite if its stabilizer in $\pi\mathcal{G}$ is finite or infinite, respectively. Let V_f be the subset of vertices v such that G_v is finite. Every finite subgroup of $\pi\mathcal{G}$ fixes a vertex of T, by Corollary I.4.9 of [9], and so is conjugate to a subgroup of G_v for some $v \in V$. (See Proposition I.7.11 of [9].) Thus vertex subgroups are *maximal* finite subgroups of $\pi\mathcal{G}$.

The two most important special cases are when Γ has a single edge e. If the vertices are distinct then $\pi\mathcal{G} \cong A *_C B$ is the generalized free product of $A = G_{o(e)}$ and $B = G_{t(e)}$ with amalgamation over $C = G_e$. If $o(e) = t(e)$ then $\pi\mathcal{G} \cong A*_\varphi$ is the HNN extension with base $A = G_{o(e)}$, associated subgroups G_e and $\phi_e(G_e)$ and characteristic isomorphism $\varphi = \phi_e$.

If σ is a subgroup of finite index in $\pi\mathcal{G}$ and T is a terminal $\pi\mathcal{G}$–tree then the stabilizers of the natural action of σ on T are finite or one-ended. Hence, σ is the fundamental group of a finite graph of groups $(\mathcal{G}_\sigma, \Gamma_\sigma)$, where $\Gamma_\sigma = \sigma\backslash T$ projects naturally onto Γ. However, if σ is a proper subgroup of $\pi\mathcal{G}$ then $(\mathcal{G}_\sigma, \Gamma_\sigma)$ need not be reduced or indecomposable.

A finitely generated group is virtually free if and only if it is the fundamental group of a finite graph of finite groups. (See Corollary IV.1.9 of [9].) It is virtually $\mathbb{Z}$ if and only if it has two ends if and only if it has a (maximal) finite normal subgroup F such that the quotient is infinite cyclic or is isomorphic to the infinite dihedral group $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. (See pages 129–130 of [9].)

If H is a subgroup of a group G, let $C_G(H)$ and $N_G(H)$ denote the centralizer and normalizer of H in G, respectively. If $x \in G$, let $\langle x\rangle$ be the cyclic subgroup generated by x, and let $C_G(x) = C_G(\langle x\rangle)$.

Lemma 6 *Let $(\mathcal{G}, \Gamma)$ be a reduced finite graph of groups in which all edge groups are finite and all vertex groups are either finite or have one end. If an edge e is a loop isomorphism or an MC-tie then $N_{\pi\mathcal{G}}(G_e)$ is infinite. If a vertex group G_v is finite then $N_{\pi\mathcal{G}}(G_v)$ is infinite if and only if there is a loop isomorphism at v.*

Proof If e is a loop isomorphism at v then the stable letter t_e associated with the edge normalizes $G_e = G_v$. If e is an MC-tie with ends u and v then G_e is normal in each of G_u and G_v. Hence, if $\alpha \in G_u \setminus G_e$ and $\beta \in G_v \setminus G_e$ then $\alpha\beta$ is an element of infinite order in $N_{\pi\mathcal{G}}(G_e)$.

Suppose that G_v is finite and $N_{\pi\mathcal{G}}(G_v)$ is infinite. The fixed-point set of the action of G_v on a terminal $\pi\mathcal{G}$–tree is a nonempty subtree which is preserved by $N_{\pi\mathcal{G}}(G_v)$. Since $N_{\pi\mathcal{G}}(G_v)$ is infinite this subtree must have a nontrivial edge, with image e in Γ having v as one vertex. Then $G_e = G_v$, since this edge is fixed by G_v, and so e must be a loop isomorphism at v, since $\mathcal{G}$ is reduced. □

Let $(\mathcal{G}, \Gamma)$ be a graph of groups in which all edge groups are finite and all vertex groups are either finite or have one end. There is an associated "Chiswell" exact sequence of right $\mathbb{Z}[\pi\mathcal{G}]$–modules

$$0 \to \bigoplus_{v\in V_f} \mathbb{Z}[G_v\backslash\pi\mathcal{G}] \xrightarrow{\Delta} \bigoplus_{e\in E} \mathbb{Z}[G_e\backslash\pi\mathcal{G}] \to H^1(\pi\mathcal{G};\mathbb{Z}[\pi\mathcal{G}]) \to 0,$$

in which the image of a coset $G_v g$ of G_v in $\pi\mathcal{G}$ under Δ is

$$\Delta(G_v g) = \sum_{o(e)=v} \Bigl(\sum_{G_e h\subset G_v} G_e hg \Bigr) - \sum_{t(e)=v} \Bigl(\sum_{G_e h\subset G_v} G_e hg \Bigr).$$

The outer sums are over edges e and the inner sums are over cosets of G_e in G_v. (This follows from part (1) of Theorem 2 of [7], with $i = 0$, for if $G < \pi$ then $H^0(G;\mathbb{Z}[\pi]) \cong \mathbb{Z}[G\backslash\pi]$ if G is finite and is 0 if G is infinite. The extreme terms in the sequence in the cited theorem are 0, since the vertex groups are finite or one-ended.) If C is a finite subgroup of G then the summands $\mathbb{Z}[G_v\backslash\pi\mathcal{G}]$ and $\mathbb{Z}[G_e\backslash\pi\mathcal{G}]$ are themselves direct sums of permutation modules, when considered as right $\mathbb{Z}[C]$–modules.

7 The centralizer condition of Crisp

In the remainder of this paper we shall consider indecomposable PD_n–complexes with $(n-2)$–connected universal covers. The arguments of [8] for the case $n = 3$ apply equally well in higher dimensions. When n is odd they imply that the indecomposable PD_n–complexes of this type are either aspherical or have virtually free fundamental group. Theorem 17 of [8] leads to strong constraints on the possible groups when the fundamental group is virtually free, as in [15]. The consequences are different when n is even. In particular, there may be no simple characterization of the indecomposables. However, if the PD_n–complex is indecomposable and its fundamental group is virtually free then in all known cases the fundamental group either has two ends or has order ≤ 2.

Let X be an indecomposable PD_n–complex with $(n-2)$–connected universal cover, and let $G=\pi_1(X)$ and $\omega=w_1(X)$. Let $G^+=\mathrm{Ker}(\omega)$ and let X^+ be the corresponding orientable covering space. Since G is finitely presentable, it is the fundamental group of a finite graph of groups $(\mathcal{G},\Gamma)$, where all vertex groups are finite or have one end and all edge groups are finite, by Theorem VI.6.3 of [9]. Since G is indecomposable as a proper free product, by Theorem B, all edge groups are nontrivial.

The first nontrivial higher homotopy group of X is $\pi_{n-1}(X)$. As observed in Section 3, this is isomorphic to $H_{n-1}(\widetilde{X};\mathbb{Z})$ and then to ${}^{\omega}H^1(G;\mathbb{Z}[G])$, by the Hurewicz theorem and Poincaré duality, respectively. A homological argument by devissage gives isomorphisms

$$H_s(C;{}^{\omega}H^1(G;\mathbb{Z}[G]))\cong H_s(C;H_{n+1}(\widetilde{X};\mathbb{Z}))\cong H_{s+n}(C;\mathbb{Z})$$

for all subgroups $C\le G$ and all $s\ge 1$. (See Lemma 2.10 of [14].) The work in [8] relates these homological properties of $H^1(G;\mathbb{Z}[G])$ to the presentation of $H^1(G;\mathbb{Z}[G])$ via the Chiswell exact sequence, when $n=3$. We shall see that this connection extends to all dimensions n, with due consideration of the parity of n.

Suppose first that n is odd, and that C is a finite cyclic subgroup of $G=\pi_1(X)$. Then $H_{n+1}(C;\mathbb{Z})=0$ and $H_{n+2}(C;\mathbb{Z})\cong C$. The arguments of Theorems 14 and 17 of [8] extend immediately to show that (i) if X is orientable and indecomposable then either X is aspherical or G is virtually free; and (ii) if $g\in G$ has prime order $p>1$ and $C_G(g)$ is infinite then $p=2$, $w(g)=-1$ and $C_G(g)$ has two ends. We may then apply the analysis of [15] to further constrain the possibilities. However, implementing the realization theorem may be difficult, since it involves the module $F^{n-1}(C(\widetilde{K}))=\mathrm{coker}(d^*_{n-2})$. As there is no algorithm for computing the homology of a finitely presentable group in degrees >1 [11], there may be no algorithm to provide an explicit matrix for d_{n-2} if $n>3$, in general. This may not be a problem when G is virtually free. In particular, is $S_3*_{\mathbb{Z}/2\mathbb{Z}}S_3$ the fundamental group of a PD_{2k+1}–complex with $(2k-1)$–connected universal cover for any $k>1$? (It is the group of a finite PD_3–complex [15].)

When n is even and C is finite cyclic, $H_{n+1}(C;\mathbb{Z})\cong C$ and $H_{n+2}(C;\mathbb{Z})=0$. In this case, Lemma 2.10 of [14] gives

$$H_1(C;{}^{\omega}H^1(G;\mathbb{Z}[G]))\cong H_{n+1}(C;\mathbb{Z})\cong C.$$

Let T be a terminal G–tree, with $e(T)$ ends and $\infty(T)$ vertices with infinite stabilizers, and let $\xi(T)=e(T)+\infty(T)-1$. If $g\in G$ has prime order then (since n is even)

Remark 13 of [8] gives either

$$\omega(g) = 1 \quad \text{and} \quad \xi(T^{\langle g \rangle}) = 1$$

or

$$\omega(g) = -1 \quad \text{and} \quad \xi(T^{\langle g \rangle}) = -1.$$

(The G–tree T is denoted by X in [8]). The argument of Theorem 17 of [8] then gives the following:

Theorem 7 *Let X be an indecomposable PD_n–complex with $(n-2)$–connected universal cover, and let $G = \pi_1(X)$ and $\omega = w_1(X)$. If n is even, $x \in G$ has order $m > 1$ and $C_G(x)$ is infinite, then $C_G(x)$ is virtually $\mathbb{Z}$ and either $\omega(x) = 1$ or $4|m$. Moreover, no conjugate of x is in any infinite vertex group.*

Proof If $g \in G$ has prime order p and $\omega(g) = -1$ then $p = 2$ and $\xi(T^{\langle g \rangle}) = -1$. Hence, g does not fix any end or infinite vertex, and so $T^{\langle g \rangle}$ is a nonempty finite tree with all vertices finite. Since $C_G(g)$ leaves $T^{\langle g \rangle}$ invariant, it is finite. Hence, if $C_G(g)$ is infinite then $\omega(g) = +1$ and $\xi(T^{\langle g \rangle}) = 1$. As in [8], it follows that g fixes a ray $(\varepsilon, \varepsilon')$, but fixes no infinite vertex, and $C_G(g)$ is virtually $\mathbb{Z}$.

Suppose now that $x \in G$ has finite order m and $C_G(x)$ is infinite. If $m = 2k$ then x^k has order 2 and $C_G(x) \le C_G(x^k)$, so $C_G(x^k)$ is infinite. Hence, $\omega(x^k) = 1$, and so either $\omega(x) = 1$ or $4|m$.

If p is a prime factor of m then $x^{m/p}$ has order p, and so does not fix any infinite vertex of T. Hence, the same is true of x, and so no conjugate of x is in any infinite vertex group. □

We shall apply Theorem 7 together with the *normalizer condition — a proper subgroup of a nilpotent group is properly contained in its normalizer* [17, Proposition 5.2.4] — and the next lemma.

Lemma 8 *Let G be a group with a finite subgroup C.*

(1) *$C_G(C)$ has finite index in $N_G(C)$.*

(2) *If G has a subgroup isomorphic to $A *_C B$ and $N_G(C)$ is finite or has two ends, then either $N_A(C) = C$ or $N_B(C) = C$ or $[N_A(C) : C] = [N_B(C) : C] = 2$.*

(3) *If G has a subgroup isomorphic to $A *_C \varphi$ and $N_G(C)$ is finite or has two ends, then either $N_A(C) = C$ or $N_A(\varphi(C)) = \varphi(C)$ or*

$$[N_A(C) : C] = [N_A(\varphi(C)) : \varphi(C)] = 2.$$

Proof The first assertion is clear, since $\mathrm{Aut}(C)$ is finite.

For the second assertion, we may assume that $G = A *_C B$. The image of the subgroup generated by $N_A(C) \cup N_B(C)$ in the quotient $N_G(C)/C$ is isomorphic to $N_A(C)/C * N_B(C)/C$. Hence, if $N_G(C)$ is finite then either $N_A(C) = C$ (and $N_B(C)$ is finite) or $N_B(C) = C$ (and $N_A(C)$ is finite). If $N_G(C)$ has two ends then so does $N_G(C)/C$, and so $N_A(C)/C = N_B(C)/C = \mathbb{Z}/2\mathbb{Z}$.

If G has a subgroup isomorphic to $A *_C \varphi$ and t is the stable letter of the HNN extension, let $B = tAt^{-1}$. Then G has a subgroup isomorphic to $A *_C B$, where $C \le A$ is identified with $\varphi(C) = tCt^{-1} \le B$, and so (3) follows from (2). □

Lemma 9 *Let $(\mathcal{G}, \Gamma)$ be a reduced finite graph of groups. If e is an edge such that $G_{o(e)}$ and $G_{t(e)}$ are finite nilpotent groups, then either e is a loop isomorphism or it is an MC-tie. In particular, if $G_{o(e)}$ or $G_{t(e)}$ has odd order, then e must be a loop isomorphism.*

Proof This follows from the normalizer condition, Lemma 8 and Theorem 7. □

When $\pi_1(X)$ is virtually free, the following lemma complements Theorem 7.

Lemma 10 *Let X be an indecomposable* PD_n*–complex with $(n-2)$–connected universal cover, with n even. If $G = \pi_1(X)$ is virtually free and $g \in G^+$ has prime order $p \ge 2$, then $N_G(\langle g \rangle)$ has two ends.*

Proof We may assume that $G \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is an indecomposable, reduced finite graph of finite groups. Let F be a free normal subgroup of finite index in G, and let ρ be the indecomposable factor of FC containing C. Then $N_\rho(C) = N_{FC}(C)$, and so has finite index in $N_G(C)$, since FC has finite index in G. Thus, we may assume that $G = \rho$. Since p is prime, the nontrivial edge stabilizers for the action of $C = \langle g \rangle$ on the terminal G–tree T are isomorphic to C. Hence, Γ has just one vertex and all the edges are loop isomorphisms, so G is a semidirect product $C \rtimes F(r)$, with $r \ge 0$. Clearly C is normal in this group. If $r = 0$ then G is finite and so $\widetilde{X} \simeq S^n$. But then $|G|\chi(X) = \chi(S^n) = 2$, and $G^+ = 1$, contrary to hypothesis. Therefore, $r > 0$, and so $N_G(C)$ is infinite, since it contains $C \rtimes F(r)$. Hence, $N_G(C)$ has two ends, by Theorem 7 and Lemma 8. □

8 Other consequences of the Chiswell sequence and Poincaré duality

We shall assume henceforth that n is even, and that X is a PD_n–complex with $(n-2)$–connected universal cover. If $G=\pi_1(X)$ is finite then $\widetilde{X}\simeq S^n$, and so $|G|\leq 2$. Hence $X\simeq S^n$ or $\mathbb{R}P^n$. If G has one end then $\widetilde{X}$ is contractible, and so X is aspherical. Hence, we may also assume that G has more than one end.

While our main concerns shall be with the case when all vertex groups are finite, elementary considerations give some complementary results.

Lemma 11 *Let X be an indecomposable PD_n–complex with $(n-2)$–connected universal cover, and let $(\mathcal{G},\Gamma)$ be a reduced finite graph of groups in which all edge groups are finite and all vertex groups are either finite or have one end, and such that $\pi\mathcal{G}\cong G=\pi_1(X)$. Let $g\in G$ have order q and $\omega(g)=1$, where $\omega=w_1(X)$. Then there is an exact sequence*

$$0\to\bigoplus_{v\in V_f}H_1(\langle g\rangle;{}^{\omega}(\mathbb{Z}[G_v\backslash G]))\to\bigoplus_{e\in E}H_1(\langle g\rangle;{}^{\omega}(\mathbb{Z}[G_e\backslash G]))\to\mathbb{Z}/q\mathbb{Z}\to 0.$$

Proof Since $\mathbb{Z}[G_v\backslash G]$ is a permutation $\mathbb{Z}[\langle g\rangle]$–module and $\omega(g)=1$,

$$H_0(\langle g\rangle;{}^{\omega}(\mathbb{Z}[G_v\backslash G]))$$

is a free abelian group for all $v\in V_f$. Since $H_1(\langle g\rangle;{}^{\omega}H^1(G;\mathbb{Z}[G]))\cong\mathbb{Z}/q\mathbb{Z}$ and $H_2(\langle g\rangle;{}^{\omega}H^1(G;\mathbb{Z}[G]))=0$ by Lemma 2.10 of [14], the result follows from the long exact sequence of homology for $\langle g\rangle$ associated to the short exact sequence of left $\mathbb{Z}[\pi]$–modules obtained by conjugating the Chiswell sequence. □

The result holds also if $\omega(g)=-1$ and no odd power of g is conjugate to an element of a finite vertex group. Otherwise, the righthand term of the short exact sequence may be $\mathbb{Z}/q'\mathbb{Z}$, where $q'=q$ or $\frac{1}{2}q$. However, we shall not need to consider the orientation-reversing case more closely.

Theorem 12 *Let X be an indecomposable PD_n–complex with $(n-2)$–connected universal cover, and let $(\mathcal{G},\Gamma)$ be a reduced finite graph of groups in which all edge groups are finite and all vertex groups are either finite or have one end, and such that $\pi\mathcal{G}\cong G=\pi_1(X)$. Let $\omega=w_1(X)$. Let $g\in G$ have order $q>1$. Then:*

(1) *If $q = p^r$ for some prime p and $r \geq 1$, and $\omega(g) = 1$, then g is conjugate to an element of an edge group. If g is in a finite vertex group G_v, then g is conjugate to an element of G_e for some edge e with $v \in \{o(e), t(e)\}$.*

(2) *Let $g \in G_e$, where e is an edge such that $G_{o(e)}$ and $G_{t(e)}$ each have one end, and suppose that $\omega(g) = 1$. If $xgx^{-1} \in G_{e'}$ for some $x \in G$ and edge e' such that $G_{o(e')}$ and $G_{t(e')}$ each have one end, then $x \in G_e$. Hence, $N_G(G_e) = G_e$.*

(3) *If G_v has one end for all $v \in V$ and $g \in G$ has finite order, then $\omega(g) = 1$.*

Proof If g has order p^r for some prime p and $\omega(g) = 1$, then

$$H_1(\langle g\rangle; {}^{\omega}(\mathbb{Z}[G_e\backslash G])) \cong \mathbb{Z}/p^r\mathbb{Z}$$

for at least one edge e, by Lemma 11, for otherwise $\bigoplus_{e\in E} H_1(\langle g\rangle; {}^{\omega}(\mathbb{Z}[G_e\backslash G]))$ has exponent dividing p^{r-1}. Therefore, g must fix some coset $G_e x$, and so $xgx^{-1} \leq G_e$. If $g \in G_v$ but has no conjugate in G_e for any edge e with $v \in \{o(e), t(e)\}$, then the map from $H_1(\langle g\rangle; {}^{\omega}(\mathbb{Z}[G_v\backslash G]))$ to $\bigoplus_{e\in E} H_1(\langle g\rangle; {}^{\omega}(\mathbb{Z}[G_e\backslash G]))$ has nontrivial kernel.

If $xgx^{-1} \in G_e'$ for some $x \notin G_e$ and edge e' with both adjacent vertex groups having one end, then $H^1(G; \mathbb{Z}[G])$ has more than one copy of the augmentation $\mathbb{Z}[\langle g\rangle]$–module $\mathbb{Z}$ as a direct summand. But then $H_1(\langle g\rangle; {}^{\omega}H^1(G; \mathbb{Z}[G]))$ would have at least two copies of $\mathbb{Z}/q\mathbb{Z}$ as direct summands, which would contradict Lemma 2.10 of [14].

If all vertex groups have one end, the Chiswell sequence reduces to an isomorphism $H^1(G; \mathbb{Z}[G]) \cong \bigoplus_{e\in E} \mathbb{Z}[G_e\backslash G]$. If g has order $2k$ and $\omega(g) = -1$, then $\omega(xgx^{-1}) = -1$ for all $x \in G$, and so $H_1(\langle g\rangle; {}^{\omega}(\mathbb{Z}[G_e\backslash G]))$ has exponent dividing k for all edges $e \in E$. Hence, $H_1(\langle g\rangle; {}^{\omega}H^1(G; \mathbb{Z}[G]))$ has exponent dividing k. This contradicts Lemma 2.10 of [14]. □

There are easy counterexamples to part (1) if n is odd or if $\omega(g) = -1$. Using Lemma 10, it can be shown that (1) holds if q is odd. (We do not need to know this below.) However, it does not always hold when q is even. The simplest counterexample is given by the fundamental group of the double of the nontrivial I–bundle over the lens space $L(6,1)$, which is an amalgam of two copies of $\mathbb{Z}/6\mathbb{Z}$ over $\mathbb{Z}/3\mathbb{Z}$.

If $G \cong N \rtimes \mathbb{Z}/p\mathbb{Z}$, where N is torsion-free and p is an odd prime, then all edge groups are $\mathbb{Z}/p\mathbb{Z}$. Since $(\mathcal{G}, \Gamma)$ is reduced and indecomposable, either all vertex groups have one end or Γ has just one vertex and $G \cong \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ (by Lemma 10).

Example Let $n \geq 4$ be even, and let M be an orientable n–manifold such that $\widetilde{M}$ is $(n-2)$–connected. Suppose that M has a self-homeomorphism g of prime order p and with nonempty finite fixed-point set. Then g is orientation-preserving, since n is even. Let s be the number of fixed points, and let $U = M \setminus N$, where N is a $\langle g\rangle$–invariant regular neighbourhood of the fixed-point set. Then $\mu = \pi_1(U) \cong \pi_1(M)$, since $n > 2$. Let $V = U/\langle g\rangle$ and $X = D(V) = V \cup_{\partial V} V$. Then $\widetilde{X}$ is $(n-2)$–connected, by a Mayer–Vietoris argument, and $\pi_1(X) \cong (\mu * \mu * F(s-1)) \rtimes \mathbb{Z}/p\mathbb{Z}$. If μ has one end then $\pi_1(X) \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is a graph of groups, with Γ having two vertices and s edges, both vertex groups $\mu \rtimes \mathbb{Z}/p\mathbb{Z}$ and all edge groups $\mathbb{Z}/p\mathbb{Z}$.

This construction can be generalized, by starting with a finite group F which acts semifreely and with finite fixed-point set on one or several closed n–manifolds with $(n-2)$–connected universal covers. After deleting regular neighbourhoods of the fixed points, we may hope to assemble the pieces along pairs of boundary components with equivalent F–actions. Note that F must have cohomological period dividing n, since it acts freely on the boundary spheres. The analogous construction in the odd-dimensional cases gives only nonorientable examples (with F of order 2).

Explicitly: The 4–dimensional torus $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ has such self-maps, of orders 2, 3, 4, 5, 6 and 8. (It also has a semifree action of $Q(8)$ with finite fixed-point set.) The group $\mathbb{Z}/k\mathbb{Z}$ acts semifreely, with two fixed points, on T_k, the closed orientable surface of genus k. The corresponding diagonal action on $T_k \times T_k$ is semifree, with four fixed points. Similarly, $S^1 \times S^3$ has an orientation-preserving involution with four fixed points. (In the latter case, doubling the complement of the fixed-point set gives a virtually free group which is the free product of three two-ended factors.)

If p is prime, a locally smoothable $\mathbb{Z}/p\mathbb{Z}$–action on a closed manifold which is orientable over $\mathbb{F}_p$ cannot have exactly one fixed point. (See Corollary IV.2.3 of [5].) Thus, the above construction always leads to groups of the form $\sigma \rtimes \mathbb{Z}/p\mathbb{Z}$, where σ has a nontrivial free factor. Is there an example with $\pi_1(X)$ indecomposable and virtually a free product of PD_n–groups?

If $\pi_1(X)$ is virtually torsion-free, must the edge groups have cohomological period dividing n? In general, must $\pi_1(X)$ be virtually torsion-free? We suspect no, but have no counterexamples.

9 Virtually free fundamental group

We shall now restrict further to the class of (infinite) virtually free groups. The known indecomposable examples among manifolds with such groups are mapping tori of self-homeomorphisms of $(n-1)$–dimensional spherical space forms and unions of mapping cylinders of double coverings of two such space forms (twisted I–bundles) with homeomorphic boundary. The fundamental groups have two ends, and graph of group structures with just one edge, which is a loop isomorphism or an MC-tie, respectively. (The examples involving mapping cylinders suggested the latter term.) There are similar constructions involving PD_{n-1}–complexes with universal cover $\simeq S^{n-1}$.

Our goal is to show that these examples are essentially all, provided that the fundamental group has no dihedral subgroup of order > 2. There are examples with dihedral subgroups and two ends, and there may still be indecomposable examples with infinitely many ends. (See Section 11 below.)

Lemma 13 *Let X be a PD_n–complex with $(n-2)$–connected universal cover, and such that $G = \pi_1(X)$ is virtually free. Let H be a nontrivial subgroup of $G_v \cap G^+$. Then there is an edge e with v as a vertex and such that $G_e \cap H \neq 1$.*

Proof Let F be a free normal subgroup of finite index in G^+. Then FH is the fundamental group of a finite orientable cover of X. If $G_e \cap H = 1$ for all edges e with v as a vertex, then the induced graph of groups structure for FH has a vertex group H with all adjacent edge groups trivial, and so H is a free factor of FH. Therefore, H is the fundamental group of an orientable PD_n–complex with $(n-2)$–connected universal cover, by Theorem B. But this is impossible, since n is even and $H \neq 1$. □

Theorem 14 *Let X be a PD_n–complex with $(n-2)$–connected universal cover, and such that $G = \pi_1(X)$ is virtually free. Then finite nilpotent subgroups of G of odd order are cyclic, and so finite subgroups of G of odd order are metacyclic.*

Proof Let F be a free normal subgroup of finite index in G and let $p\colon G \to G/F$ be the natural epimorphism. If S is a finite subgroup of G then $FS = p^{-1}p(S) \cong F \rtimes S$. On replacing FS by an indecomposable factor, if necessary, we may assume that $FS \cong \pi\mathcal{G}_S$, where $(\mathcal{G}_S, \Gamma_S)$ is an indecomposable reduced finite graph of groups, with vertex groups isomorphic to subgroups of S, and with at least one edge, since $|S| > 2$.

Suppose first that S is nilpotent, of odd order. Then Γ_S has just one vertex v and one edge, which is a loop isomorphism, by Lemma 9. Hence, $G \cong S \rtimes \mathbb{Z}$ and so has two ends. Therefore, $\widetilde{X} \simeq S^{n-1}$ and so S has periodic cohomology. Since S is nilpotent of odd order, it is cyclic.

In general, S is metacyclic, by Proposition 10.1.10 of [17], since all its Sylow subgroups are cyclic. □

This does not extend to subgroups of even order, as it stands. However, for groups of odd order "metacyclic with cyclic Sylow subgroups" is equivalent to "having periodic cohomology", and in all known examples the vertex groups have the latter property.

Corollary 15 *If G has no subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ then all finite subgroups of G have periodic cohomology.*

Proof The exclusion of $(\mathbb{Z}/2\mathbb{Z})^2$ implies that finite 2–groups in G are cyclic or quaternionic. (See Proposition 5.3.6 of [17].) Since all finite p–groups of odd order in G are cyclic by Theorem 14, it follows that all finite subgroups have periodic cohomology. (See Proposition VI.9.3 of [6].) □

Finite groups with periodic cohomology fall into six families:

(I) $\mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$.

(II) $\mathbb{Z}/m\mathbb{Z} \rtimes (\mathbb{Z}/q\mathbb{Z} \times Q(2^i))$ for $i \geq 3$.

(III) $\mathbb{Z}/m\mathbb{Z} \rtimes (\mathbb{Z}/q\mathbb{Z} \times T_k^*)$ for $k \geq 1$.

(IV) $\mathbb{Z}/m\mathbb{Z} \rtimes (\mathbb{Z}/q\mathbb{Z} \times O_k^*)$ for $k \geq 1$.

(V) $(\mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}) \times \mathrm{SL}(2,p)$ for $p \geq 5$ prime.

(VI) $\mathbb{Z}/m\mathbb{Z} \rtimes (\mathbb{Z}/q\mathbb{Z} \times \mathrm{TL}(2,p))$ for $p \geq 5$ prime.

Here m is odd, and m, q and the order of the quotient by the metacyclic subgroup $\mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$ are relatively prime. The first family includes cyclic groups, dihedral groups $D_{2m} = \mathbb{Z}/m\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$ with m odd, and the groups of odd order with periodic cohomology. The group $Q(2^i)$ is the quaternionic group of order 2^i, with presentation

$$\langle x, y \mid x^{2^{i-1}} = 1,\ x^{2^{i-2}} = y^2,\ yxy^{-1} = x^{-1}\rangle,$$

and T_k^* and O_k^* are the generalized binary tetrahedral and octahedral groups, respectively. Then $T_k^* \cong O_k^{*\prime} \cong Q(8) \rtimes \mathbb{Z}/3^k\mathbb{Z}$ and has index 2 in O_k^*. If p is an odd prime then $\mathrm{TL}(2,p)$ may be defined as follows. Choose a nonsquare $\rho \in \mathbb{F}_p^\times$, and let $\mathrm{TL}(2,p) \subset \mathrm{GL}(2,p)$ be the subset of matrices with determinant 1 or ρ. The multiplication $\star$ is given by $A \star B = AB$ if A or B has determinant 1, and $A \star B = \rho^{-1}AB$ otherwise. Then $\mathrm{SL}(2,p) = \mathrm{TL}(2,p)'$ and has index 2. (Note also that $\mathrm{SL}(2,3) \cong T_1^*$ and $\mathrm{TL}(2,3) \cong O_1^*$.)

We shall not specify the actions in the semidirect products here, as these play no role in our arguments. We shall only need the following simple facts about such groups, which may easily be checked by inspecting the terms of the above list. Let P be a finite group of even order with periodic cohomology. If P is not metacyclic then its Sylow 2–subgroup is quaternionic, and P has a unique element of order 2, which is central. Hence, if P has a dihedral subgroup $D_{2\ell}$ then it is metacyclic. Moreover, $D'_{2\ell}$ is then normal in P. If P is metacyclic or of type IV or VI it has a unique subgroup of index 2, while if it is of type III or V there is no subgroup. Groups of type II have three such subgroups. (See [25] for more on these groups.)

10 Virtually free groups without dihedral subgroups

Our strategy for proving Theorem 20 (the main part of Theorem C) is to use Theorem 7, the normalizer condition and Lemma 13 to show that the graph has just one edge, which is either a loop isomorphism or an MC-tie. Lemma 16 implies that if G_v is a vertex group of maximal order then there is either a loop isomorphism or an MC-tie with v as one vertex. Lemmas 17 and 19 show that if π has no dihedral subgroup then all edge groups have index ≤ 2 in adjacent vertex groups. The main result then follows fairly easily. We shall assume that X is a PD$_n$–complex and $G=\pi_1(X)\cong\pi\mathcal{G}$, where $(\mathcal{G},\Gamma)$ is a reduced, indecomposable finite graph of finite groups, with at least one edge, and that G does not have D_4 as a subgroup. We shall not state these conditions explicitly in the lemmas.

Lemma 16 *At each vertex v there is either a loop isomorphism or an edge e with distinct vertices v and w and such that $[G_v:G_e]=2$ and $N_G(G_e)$ has two ends.*

Proof Let F be a free normal subgroup of finite index in G. After replacing G by an indecomposable factor of FG_v, if necessary, we may assume that $G=FG_v$. Since G is infinite and indecomposable, there is at least one edge with v as a vertex. If G_v has prime order then each such edge must be a loop isomorphism. Thus, we may assume henceforth that $|G_v|$ is not prime.

If G_v is metacyclic but $|G_v|$ is not a power of 2 then G_v has a cyclic normal subgroup S of odd prime order p. If G_v has order $2^k\geq 4$ or if G is not metacyclic then it has a central element g of order 2 such that $\omega(g)=1$, and we let $S=\langle g\rangle$. In each case, $S\leq G^+$.

By Lemma 13, there is an edge e with v as one vertex and such that $S\leq G_e$. If both vertices are v then S is normalized by G_v and by t_e, the stable letter associated to e, since S is the unique subgroup of G_v of order p. The subgroup $\langle G_v, t_e\rangle$ has infinitely many ends unless $G_e = G_v$. Hence, $G_e = G_v$, and so e is a loop isomorphism, by Theorem 7.

If e has distinct vertices $v \neq w$ then G_w is isomorphic to its image in $FG_v/F \cong G_v$. Hence, S is also normal in G_w, and $|G_w| \leq |G_v|$. Therefore, S is normal in $G_v *_{G_e} G_w$. Theorem 7 and Lemma 8 together imply that the normalizer of any finite subgroup of G is finite or has two ends. Hence, $[G_v : G_e] \leq 2$ and $[G_w : G_e] \leq 2$. Since e is not a loop isomorphism, $[G_v : G_e] = [G_w : G_e] = 2$. Hence, G_e is normal in $G_v *_{G_e} G_w$, and so $N_G(G_e)$ has two ends. □

If G_v has no subgroup of index 2 (eg if it is metacyclic of odd order or is of type III or V) then Lemma 16 ensures that there is a loop isomorphism at v. If G_v has maximal order among finite subgroups of G and $[G_v : G_e] = 2$, then e is an MC-tie. The argument for Lemma 16 shows that if e is not a loop isomorphism then it an MC-tie for the induced graph of groups structure for FG_v. However, it is not otherwise obvious that it must be an MC-tie for the original graph of groups $(\mathcal{G}, \Gamma)$.

Lemma 17 *Let f be an edge with both vertices v. If f is not a loop isomorphism then $|G_f| = 2$ and G_v is dihedral.*

Proof Suppose that $|G_f| > 2$. Let g be an element of G_f of prime order p. Since the Sylow subgroups of G_v are cyclic or quaternionic, each Sylow p–subgroup has a unique subgroup S of order p. Therefore, if t_f is the stable letter associated to f then there is an $a \in G_v$ such that $at_f g t_f^{-1} a^{-1} = g^s$ for some $0 < s < p$. Hence, $(at_f)^{p-1}$ centralizes S. By Lemma 16, there is another edge e which is either a loop isomorphism at v or has distinct vertices u and v, and such that $[G_v : G_e] = 2$ and $N_G(G_e)$ has two ends.

If e is a loop isomorphism, then S is also centralized by some power of t_e. If $[G_v : G_e] = 2$ and $N_G(G_e)$ has two ends, we may assume that $S \leq G_e$, since $|G_f| > 2$, and then S is centralized by an element of infinite order in $N_G(G_e)$. In each case, we find that S is centralized by a nonabelian free subgroup, contradicting Theorem 7.

Therefore, we must have $G_f \cong \mathbb{Z}/2\mathbb{Z}$. If G_v is not dihedral then G_f is central in G_v. But the subgroup generated by G_v and t_f contains a nonabelian free group, and so we again contradict Theorem 7. Since $G_v \not\cong D_4$, it has order at least 6. □

This lemma indicates why we could require an MC-tie to have distinct vertices:

Lemma 18 *Let e and f be distinct edges with vertices u, v and v, w, respectively. If e is a loop isomorphism at v or an MC-tie, then $w \neq u$ or v, and f is neither a loop isomorphism nor an MC-tie.*

Proof Suppose that e is a loop isomorphism at v and that f also has both vertices v. If f is a loop isomorphism then G_v is normalized by the free group generated by the stable letters t_e and t_f, which contradicts Theorem 7. Therefore, f is not a loop isomorphism, and so Lemma 17 applies.

If e is a loop isomorphism and f is an MC-tie with vertices v, w, then G_f is normalized by $G_v *_{G_f} G_w$ and by some power of t_e (since G_v has only a finite number of subgroups of index 2). This again leads to a contradiction with Theorem 7.

Finally, if $u \neq v$ and e is an MC-tie then similar arguments show that $w \neq u$ or v, and that f is not an MC-tie. □

In particular, if every edge with v as one vertex is either a loop isomorphism or an MC-tie, then there is just one edge, and so G has two ends.

Lemma 19 *Let f be an edge with vertices $v \neq w$. If $[G_w : G_f] > 2$ then G_f has order 2, and G_v or G_w is dihedral.*

Proof In order to show that G_f has order 2, we may assume without loss of generality that $G = FG_w$, where F is a free normal subgroup of finite index. Then every finite subgroup of G is isomorphic to a subgroup of G_w, and so G_w has maximal order among such subgroups. We may also assume that $o(f) = v$ and $t(f) = w$. Clearly f is neither a loop isomorphism nor an MC-tie.

There are edges e and g, with vertices u, v and w, x, respectively, which are loop isomorphisms or for which $[G_v : G_e] = 2$ or $[G_w : G_g] = 2$, by Lemma 16. In the latter case, g must be an MC-tie, by the maximality of $|G_w|$. Hence, $v \neq w$ or x, by Lemma 18, and so $g \neq f$. The subgroups G_e and G_g are centralized by elements of infinite order. Hence, G_f has a subgroup H which is the intersection of two subgroups of index ≤ 2 in G_f, and which is centralized by these elements. We shall show that we may assume that they generate a nonabelian free subgroup of $C_G(H)$.

If e and g are each loop isomorphisms then $H = G_f$ is centralized by powers of the stable letters t_e and t_f. If e is a loop isomorphism and g is an MC-tie then

$H = G_f \cap G_g$ is centralized by powers of t_e and $\alpha'\beta'$, where $\alpha' \in G_w \setminus G_g$ and $\beta' \in G_x \setminus G_g$ do not involve t_e.

If e is not a loop isomorphism then $u \neq v$, by Lemma 17, and G_e is normalized by some $\alpha\beta$, where $\alpha \in G_u \setminus G_e$ and $\beta \in G_u \setminus G_e$. Suppose that $u \neq w$ or x. If g is a loop isomorphism then $H = G_f \cap G_e$ is normalized by powers of $\alpha\beta$ and t_g, If g is an MC-tie then $H = G_f \cap G_e \cap G_g$ is normalized by powers of $\alpha\beta$ and of $\alpha'\beta'$. A similar argument applies if $u = w$ or x.

In each case, these pairs generate a nonabelian free subgroup of $C_G(H)$, which contradicts Theorem 7, unless $H = 1$. Since G_f is nontrivial and is not D_4, we then have $G_f = \mathbb{Z}/2\mathbb{Z}$.

We now return to the general case (ie we do not assume that $G = FG_w$). Since $G_f = \mathbb{Z}/2\mathbb{Z}$, it is central in the Sylow 2–subgroups of G_v and G_w, and $C_G(G_f)$ has two ends or is finite. Hence, either G_f is its own centralizer in one vertex group, in which case the vertex group is dihedral, or these Sylow subgroups both have order 4, and no element of odd order in either vertex group commutes with G_f. In the latter case, the vertex groups are metacyclic groups of the form $\mathbb{Z}/m\mathbb{Z} \rtimes_\theta \mathbb{Z}/4\mathbb{Z}$, where m is odd and $\theta\colon \mathbb{Z}/4\mathbb{Z} \to (\mathbb{Z}/m\mathbb{Z})^\times$ is injective. Such groups have a unique subgroup of index 2, and so $G_f \leq G_e$ and $G_f \leq G_g$. But then the earlier argument applies with $H = G_f$ to show that $C_G(H)$ has a nonabelian free subgroup, contradicting Theorem 7. Hence, G_v or G_w is dihedral. □

Theorem 20 *Let X be a PD_n–complex with $(n-2)$–connected universal cover, where n is even. If $G = \pi_1(X)$ is infinite, virtually free and indecomposable, and no maximal finite subgroup is dihedral, then G has two ends, and its finite subgroups have cohomological period dividing n.*

Proof Let G_v be a vertex group of maximal order. Suppose first that G_v has odd order. Then G_v has no subgroup of index 2 and none of order 2, and so every edge e with v as a vertex must be a loop isomorphism, by Lemmas 17 and 19.

Since no maximal finite subgroup is dihedral, there are no dihedral vertex groups. Therefore, if G_v has even order > 4 and f is an edge with vertices v, w, then $[G_v : G_f] \leq 2$, by Lemmas 17 and 19. Since $|G_v|$ is maximal, f is either a loop isomorphism or an MC-tie.

In each of these cases there must be just one edge, by Lemma 18, and so π has two ends.

Finally, if all vertex groups have order 4 then they are cyclic, and all proper edge groups have order 2. Hence, there is a unique subgroup S of order 2. Clearly $S \leq G^+$, and so $G = C_G(S)$ has two ends, by Lemma 10.

Since G has two ends, $\widetilde{X} \simeq S^{n-1}$, and so finite subgroups of G have cohomological period dividing n. □

In the final case there are three possibilities: $G \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \rtimes_{-1} \mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$.

Corollary 21 *If G has no element of even order then $G \cong S \rtimes \mathbb{Z}$, where S is a finite metacyclic group of odd order and of cohomological period dividing n.* □

In particular, if n is a power of 2 then S must be cyclic. (See Exercise VI.9.6 of [6].)

When there is 2–torsion, G need not be an extension of $\mathbb{Z}$ by a finite normal subgroup. For example, if MC is the mapping cylinder of the double cover of a lens space $L = L(2m, q)$ and $X = D(\mathrm{MC}) = \mathrm{MC} \cup_L \mathrm{MC}$ is the double, then G is an extension of the infinite dihedral group D_∞ by $\mathbb{Z}/m\mathbb{Z}$, and $\widetilde{X} \cong S^3 \times \mathbb{R}$.

Theorem 22 *Let X be an orientable PD_n–complex such that $\widetilde{X} \simeq S^{n-1}$. Then $G = \pi_1(X) \cong F \rtimes_\theta \mathbb{Z}$, where F is the maximal finite normal subgroup of G, and X is a mapping torus.*

Proof Since $\widetilde{X} \simeq S^{n-1}$, the group G has two ends. Hence, it has a maximal finite normal subgroup F and a subgroup σ of index ≤ 2 which contains F and is such that $\sigma/F \cong \mathbb{Z}$. The covering space $X_F = \widetilde{X}/F \simeq S^{n-1}/F$ associated to F is a PD_{n-1}–complex, and so the covering space X_σ associated to σ is the mapping torus of a self-homotopy-equivalence of F. Hence, $\chi(X_\sigma) = 0$, and so $\chi(X) = 0$ also. But if $\sigma \neq G$ then G/G' is finite. Since $cd_{\mathbb{Q}} G = 1$, it follows from the spectral sequence for the universal covering that $H_q(X; \mathbb{Q}) = 0$ for $0 < q < n-1$. This is also the case when $q = n-1$, by Poincaré duality. Hence, $\chi(X) = 2$ (since n is even). This is a contradiction. Therefore, $\sigma = G \cong F \rtimes \mathbb{Z}$ and X is a mapping torus. □

We shall now restate and prove Theorem C of the introduction.

Theorem C *Let X be a PD_{2k}–complex with $(2k-2)$–connected universal cover, and such that $G = \pi_1(X)$ is virtually free and indecomposable as a free product. If*

G is finite then $X \simeq S^{2k}$ or $\mathbb{R}P^{2k}$. If G is infinite and has no dihedral subgroup of order > 2 then G has two ends and its finite subgroups have cohomological period dividing $2k$. Hence, $\widetilde{X} \simeq S^{2k-1}$. If, moreover, X is orientable, then $H^1(G;\mathbb{Z}) \cong \mathbb{Z}$.

Proof If G is finite then $\widetilde{X} \simeq S^{2k}$, and so $|G|\chi(X) = \chi(S^{2k}) = 2$. Hence, either $G = 1$ and $X \simeq S^{2k}$, or $G = \mathbb{Z}/2\mathbb{Z}$ and $X \simeq \mathbb{R}P^{2k}$.

If G is infinite and has no dihedral subgroup of order > 2, then G has two ends, and its finite subgroups have cohomological period dividing $2k$, by Theorem 20.

The final assertion follows from Theorem 22. □

It remains an open question whether the conclusion of Theorem C must hold if G has a dihedral maximal finite subgroup. (There are such examples with $G = \pi_1(X)$ having two ends — see Theorem 23 below.) Lemmas 18 and 19 impose some restrictions, but leave open the possibility that, for instance, there might be a PD_{2k}–complex X with $(2k-2)$–connected universal cover and $\pi_1(X) \cong \pi\mathcal{G}$, where the underlying graph Γ is a cycle of length four, the vertex groups are dihedral and the edges are alternately MC-ties or have edge group of order 2.

11 Construction of examples

Every finite group F with cohomological period q is the fundamental group of an orientable PD_{q-1}–complex with universal cover $\simeq S^{kq-1}$ for all $k \geq 1$ [19; 24]. Since q is even, such complexes are odd-dimensional. (In fact, the only nonorientable quotients of finite group actions on spheres are the even-dimensional real projective spaces $\mathbb{R}P^{2k}$.) We may use such complexes to realize groups with two ends.

Theorem 23 *Let F be a finite group. If $G \cong F \rtimes_\theta \mathbb{Z}$ then there is a PD_{2k}–complex X with $\pi_1(X) \cong G$ and $\widetilde{X} \simeq S^{2k-1}$ if and only if F has cohomological period dividing $2k$ and $H_{2k-1}(\theta;\mathbb{Z})$ is multiplication by ± 1. If $|F| > 2$, then X is orientable if and only if $H_{2k-1}(\theta;\mathbb{Z}) = 1$.*

Proof If a PD_{2k}–complex X has fundamental group G and universal cover $\widetilde{X} \simeq S^{2k-1}$ then F has cohomological period dividing $2k$, since it acts freely on $\widetilde{X}$, and the condition $H_{2k-1}(\theta;\mathbb{Z}) = \pm 1$ follows from the Wang sequence for the projection of X onto S^1 corresponding to the epimorphism $G \to G/F \cong \mathbb{Z}$. (See Theorem 11.1 of [14] for the case $k = 2$.)

Suppose, conversely, that F has cohomological period dividing $2k$. Then there is a based orientable PD_{2k-1}–complex X_F with fundamental group F and $\widetilde{X}_F \simeq S^{2k-1}$. If $H_{2k-1}(\theta;\mathbb{Z}) = \pm 1$ then there is a self-homotopy-equivalence f of X_F which induces θ [16]. The mapping torus of f is then a PD_{2k}–complex with fundamental group G and universal cover $\simeq S^{2k-1}$.

If $|F| > 2$ then $H_{2k-1}(\theta;\mathbb{Z}) = 1$ (as an automorphism of $H_{2k-1}(F;\mathbb{Z}) \cong \mathbb{Z}/|F|\mathbb{Z}$) if and only if $H_{2k-1}(f;\mathbb{Z}) = 1$ (as an automorphism of $H_{2k-1}(X;\mathbb{Z}) \cong \mathbb{Z}$) if and only if X is orientable. □

In particular, when the dimension $2k$ is divisible by 4, there are examples X with $\pi_1(X) \cong D_{2m} \times \mathbb{Z}$ for odd $m > 1$. These do not satisfy the hypotheses of Theorem C.

Suppose now that $G \cong E *_F H$, where E and H are finite groups with periodic cohomology and $[E:F] = [H:F] = 2$. Let n be a multiple of the cohomological periods of E and H. However, there is one subtlety: We must be able to choose PD_{2k-1}–complexes X_E and X_H with fundamental groups E and H and universal covers $\simeq S^{2k-1}$ in such a way that the double covers associated to the subgroups F are homotopy equivalent. For then we may construct a PD_{2k}–complex X with fundamental group G and $\widetilde{X} \simeq S^{n-1}$ by gluing together two mapping cylinders via a homotopy equivalence of their "boundaries". See Chapter 11 of [14] for an example with $k = 2$, $E = Q(24)$, $H = \mathbb{Z}/3\mathbb{Z} \times Q(8)$ and $F = \mathbb{Z}/12\mathbb{Z}$ where this construction cannot be carried through. (The difficulty is that PD_3–complexes with fundamental group E or H have unique homotopy types: the double covers corresponding to F are lens spaces which are not homotopy equivalent. Similar examples should exist in higher dimensions.)

Theorem 23 is essentially Theorem D of the introduction, and is the case $m = 1$ of Proposition 8 of [10]. Part of the discussion of the case $G \cong E *_F H$ in the previous paragraph may also be found in the final section of [10].

12 Must finite subgroups have periodic cohomology?

There remains the key question of whether the group $(\mathbb{Z}/2\mathbb{Z})^2$ ever arises in this context. Suppose that Y is a PD_{2k}–complex with $(2k-2)$–connected universal cover and virtually free fundamental group, and that $\pi_1(Y)$ has a subgroup $C \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let F be a free normal subgroup of finite index in $\pi_1(Y)$. We may assume that

$F < \pi_1(Y)^+$. Then Y has a finite cover Y_{FC} with fundamental group $FC \cong F \rtimes C$. As in Theorem 14, some indecomposable factor X of Y_{FC} has fundamental group $G = F(r) \rtimes C$ for some $r > 1$. (Since G cannot be finite of order 4 and C does not have periodic cohomology, $r \neq 0$ or 1.) We may assume that $G \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is an indecomposable reduced finite graph of groups, with all vertex groups $G_v \cong C$ and all edge groups G_e of order 2. In particular, every edge group is orientable, by Theorem 7.

Since G is virtually free, it has a well-defined virtual Euler characteristic

$$\chi^{\mathrm{virt}}(G) = \frac{\chi(F(r))}{|C|} = \frac{1-r}{4}.$$

We also have $\chi^{\mathrm{virt}}(G) = \frac{1}{4}|V| - \frac{1}{2}|E|$, since $(\mathcal{G}, \Gamma)$ is indecomposable and reduced. Moreover, $\chi(X) = 2\chi^{\mathrm{virt}}(G)$, by the multiplicativity of (virtual) Euler characteristic for finite covers (passage to finite-index subgroups), and so $\chi^{\mathrm{virt}}(G) \in \frac{1}{2}\mathbb{Z}$. Hence, $|V|$ is even.

Suppose first that X is not orientable. Then $\omega|_C \neq 1$, since $\omega(F) = 1$. Hence, if e and f are two edges with $o(e) = o(f) = v$ then $G_e = G_f = \mathrm{Ker}(\omega|_{G_v})$. If $e \neq f$ then $C_G(G_e)$ contains a nonabelian free subgroup. Hence, there is at most one edge at each vertex. Since Γ is connected, there is just one edge e, which must have distinct vertices, for otherwise $C_G(G_e)$ would have a nonabelian free subgroup. Hence, $G \cong G_u *_{G_e} G_v \cong (\mathbb{Z}/2\mathbb{Z}) \times D_\infty$ has two ends, and so $\widetilde{X} \simeq S^{2k-1}$. But then C has periodic cohomology, which is false. Therefore, X must be orientable.

Since X is finitely covered by $\#^r(S^3 \times S^1)$, $H_2(X; \mathbb{Q}) = 0$, and so $\chi(X)$ is even. Hence, $\chi^{\mathrm{virt}}(G)$ is integral. Moreover, if there is a vertex v of valency ≤ 2 then there is an epimorphism $f \colon G \to \mathbb{Z}/2\mathbb{Z}$ which is nontrivial on G_e for all edges e with v as one vertex. But then $\mathrm{Ker}(f) \cong \pi\widetilde{\mathcal{G}}$, where $(\widetilde{\mathcal{G}}, \Gamma)$ is a graph of groups with all vertex groups of order 2 and with trivial edge groups for the edges with v as one vertex. This is impossible if the double cover is orientable. If there is a vertex w with valency > 3 then two edges with w as one vertex have the same edge group $G_e < G_w$, and $C_G(G_e)$ contains a nonabelian free subgroup. Therefore each vertex of Γ has valency 3, so $2|E| = 3|V|$. In summary,

$$\begin{cases} X \text{ is orientable,} \\ \text{vertices of } \Gamma \text{ have valence } 3, \\ |V| \text{ is even,} \\ r = 1 + 2|V| \equiv 1 \bmod 4. \end{cases}$$

The simplest example meeting these criteria has $V = \{v, w\}$ and $E = \{a, b, c\}$, with each edge having origin v and target w. Then

$$G_v = \langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ c = ab\rangle$$

and

$$G_w = \langle a', b', c' \mid (a')^2 = (b')^2 = (c')^2 = 1,\ c' = a'b'\rangle.$$

The edge groups are $G_a = \langle a\rangle$, $G_b = \langle b\rangle$ and $G_c = \langle ab\rangle$, as subgroups of G_v, and for each edge x the monomorphism $\phi_x\colon G_x \to G_w$ is given by $\phi_x(x) = x'$. The edge a is a maximal tree in Γ. Let t and u be stable letters corresponding to the other edges. Then $\pi\mathcal{G}$ has the presentation

$$\langle G_v, G_w, t, u \mid a' = a,\ b' = tbt^{-1},\ a'b' = uabu^{-1}\rangle,$$

which simplifies to

$$\langle a, b, t, u \mid a^2 = b^2 = (ab)^2 = 1,\ atbt^{-1} = tbt^{-1}a = uabu^{-1}\rangle.$$

Is this the fundamental group of an orientable PD_{2k}–complex with $(2k-2)$–connected universal cover? Can the arguments of Section 2 of [8] be tweaked to rule this out?

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