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Germs of fibrations of spheres by great circles always extend to the whole sphere

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We prove that every germ of a smooth fibration of an odd-dimensional round sphere by great circles extends to such a fibration of the entire sphere, a result previously known only in dimension three.

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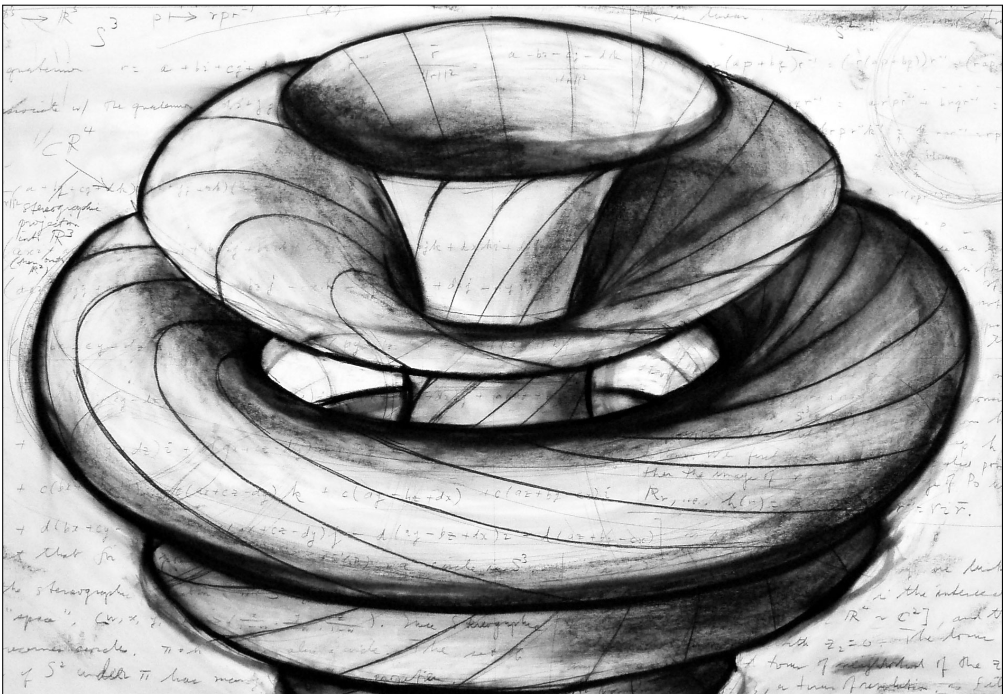

Hopf fibration of 3*–sphere by great circles*
Lun-Yi Tsai, charcoal and graphite on paper, 2007

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1 Introduction

Consider a fibration F of the unit $2n{+}1$–sphere S^{2n+1} by oriented great circles, and focus on one of the fibers P, as shown below. All fibrations in this paper are fiber bundles.

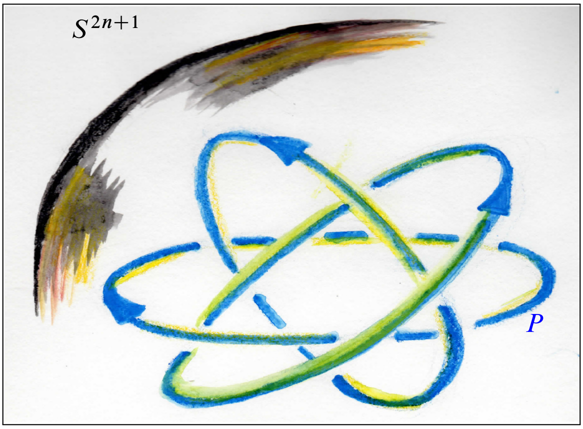

Figure 1: A fibration of S^{2n+1} by oriented great circles

The oriented great circle P spans an oriented 2–plane through the origin in $\mathbb{R}^{2n+2}$, which we also denote by P, and so appears as a single point in the Grassmann manifold $G_2\mathbb{R}^{2n+2}$ of all such oriented 2–planes. If the fibration F is smooth, then its base space M_F appears as a smooth $2n$–dimensional submanifold of this Grassmann manifold, and we can focus on the tangent $2n$–plane $T_P M_F$ to M_F at P.

Theorem A *Every germ of a smooth fibration of S^{2n+1} by oriented great circles extends to such a fibration of all of S^{2n+1}.*

A *germ* of a fibration of S^{2n+1} by oriented great circles consists of such a fibration in an open neighborhood of a given fiber P, with two germs equivalent if they agree on some smaller neighborhood of P. To *extend* such a germ to a fibration of S^{2n+1} means to find a fibration of S^{2n+1} which agrees with the given germ on some neighborhood of P.

The main tool for proving Theorem A is the following, also previously known only in dimension three:

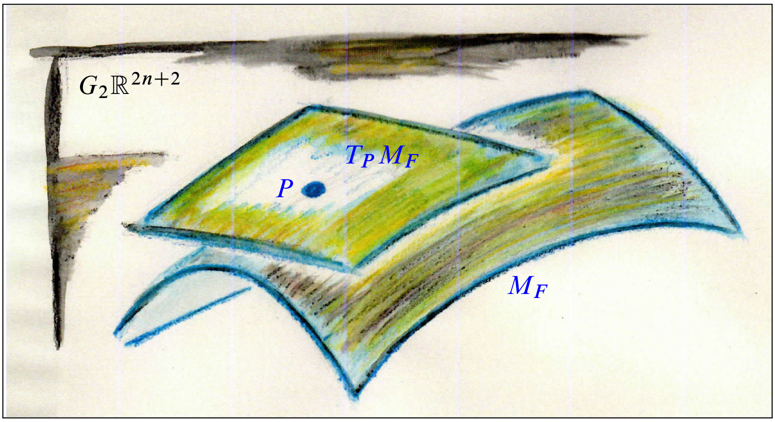

Figure 2: The base space M_F of the fibration F and its tangent plane $T_P M_F$ at P

Theorem B *The space $\{T_P M_F\}$ of tangent $2n$–planes at P to the base spaces M_F of all smooth oriented great circle fibrations F of S^{2n+1} containing P deformation retracts to its subspace $\{T_P M_H\}$ of tangent $2n$–planes to such Hopf fibrations H of S^{2n+1}.*

The path to Theorem B consists of the following steps.

First, two definitions. The *bad set* $\mathrm{BS}(P) \subset G_2\mathbb{R}^{2n+2}$ consists of all oriented 2–planes through the origin in $\mathbb{R}^{2n+2}$ which meet P in at least a line, and the *bad cone* $\mathrm{BC}(P) \subset T_P(G_2\mathbb{R}^{2n+2})$ is its tangent cone at P.

Proposition 1 *A closed, connected, smooth, $2n$–dimensional submanifold M of $G_2\mathbb{R}^{2n+2}$ is the base space of a smooth fibration of S^{2n+1} by great circles if and only if it is transverse to the bad cone at each of its points.*

Next we focus in on the tangent space $T_P(G_2\mathbb{R}^{2n+2})$ to the Grassmannian at the point P, see how to regard it as the $4n$–dimensional vector space $\mathrm{Hom}(P, P^\perp)$, and show that a $2n$–plane through the origin there is transverse to the bad cone $\mathrm{BC}(P)$ if and only if it is the graph of a linear transformation $T\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with no real eigenvalues, with the role of $\mathbb{R}^{2n}$ played by two copies of $P^\perp$.

Proposition 2 *There is a $\mathrm{GL}(2n, \mathbb{R})$–equivariant deformation retraction of the space of linear transformations $T\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with no real eigenvalues to its subspace of linear complex structures $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$.*

This is due to Benjamin McKay [15].

By a *linear complex structure* we mean a linear map $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $J^2 = -I$. For an *orthogonal complex structure*, we require in addition that J be orthogonal.

Proposition 3 *There is an $O(2n)$–equivariant deformation retraction of the space of linear complex structures on $\mathbb{R}^{2n}$ to its subspace of orthogonal complex structures.*

These results then help us prove:

Proposition 4 *There exists a smooth fibration F of S^{2n+1} by oriented great circles whose base space M_F is tangent at P to any preassigned $2n$–plane transverse to the bad cone* $\mathrm{BC}(P)$.

We then assemble these results to prove Theorems A and B.

2 Background

We give some brief background information here, and refer the reader to the arXiv version of this paper [3] for more details.

Hopf fibrations

In 1931, Heinz Hopf [11] gave a remarkable example of a map f from the unit 3–sphere S^3 to the unit 2–sphere S^2. In coordinates,

$$\begin{aligned} y_1 &= 2(x_1x_3 + x_2x_4), \\ y_2 &= 2(x_2x_3 - x_1x_4), \\ y_3 &= x_1^2 + x_2^2 - x_3^2 - x_4^2; \end{aligned}$$

see Figure 3.

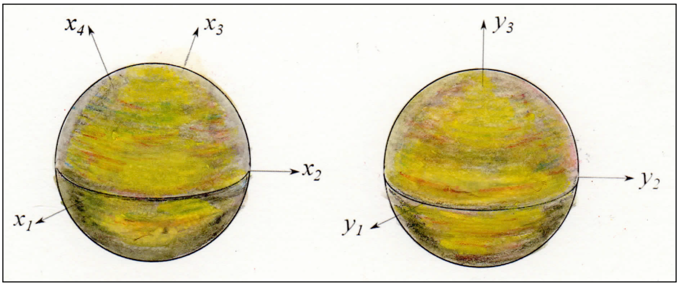

Figure 3: Hopf's map f from S^3 to S^2

This was the first example of a homotopically nontrivial map from a sphere to another sphere of lower dimension, signaling the birth of homotopy theory. Although Hopf presented his map via the above formulas early in his paper, he commented later in the same paper that the great circle fibers of his map were the intersections of the 3–sphere with the complex lines in $\mathbb{C}^2$.

In a follow-up paper in 1935, Hopf [12] gave higher-dimensional analogues of his first map, using complex numbers, quaternions, and Cayley numbers, with the nonassociativity of the Cayley numbers responsible for the truncation of the third series:

$$S^1 \subset S^3 \to S^2 = \mathbb{C}P^1, \quad S^1 \subset S^5 \to \mathbb{C}P^2, \quad \dots, \quad S^1 \subset S^{2n+1} \to \mathbb{C}P^n, \quad \dots,$$
$$S^3 \subset S^7 \to S^4 = \mathbb{H}P^1, \quad S^3 \subset S^{11} \to \mathbb{H}P^2, \quad \dots, \quad S^3 \subset S^{4n+3} \to \mathbb{H}P^n, \quad \dots,$$
$$S^7 \subset S^{15} \to S^8.$$

Blaschke manifolds

Let M be a closed (compact, no boundary) Riemannian manifold. On each geodesic α from the point p on M, the *cut point* is the last point to which α minimizes distance, and the *cut locus* $C(p)$ is the set of these. For example, on a round sphere, the cut locus of each point is just its antipodal point.

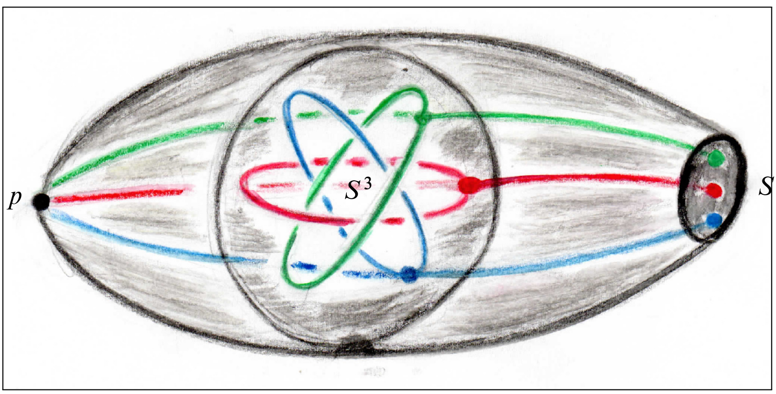

Figure 4: The complex projective plane $\mathbb{C}P^2$

In the picture of $\mathbb{C}P^2$ in Figure 4, focus on the point p at the left, and on the geodesics which begin there and eventually coalesce along its cut locus $C(p)$, a round 2–sphere at the right. If we go out along these geodesics any fixed intermediate distance, we come to a 3–sphere on which we record that a circle's worth of geodesics from p will coalesce along each point of $C(p)$. If this intermediate distance is very small, then the 3–sphere is almost round, and its fibration by these circles is almost a Hopf fibration. But as the 3–sphere moves towards the cut locus at the right, these circles will eventually shrink until in the limit they become points, and the 3–sphere collapses to a 2–sphere. The complex projective plane itself is homeomorphic to the mapping cone of this collapsing map $S^3 \to S^2$.

Given the closed Riemannian manifold M, if the distance from p to its cut point along α depends neither on the choice of α nor on the choice of p, then M is called a *Blaschke manifold*, the term coined by Marcel Berger [1] in 1978.

Examples of Blaschke manifolds are the standard spheres and projective spaces S^n, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathrm{Ca}P^2$, on which all the geodesics from any point come together again after the same distance, independent of direction and point of origin.

The terminology honors Wilhelm Blaschke, who asked, in the first, 1921, edition [2] of his *Vorlesungen über Differentialgeometrie*, whether such a surface must be isometric to a round S^2 or round $\mathbb{R}P^2$.

In 1963, Leon Green [9] proved that a Blaschke surface can only be a round S^2 or $\mathbb{R}P^2$.

By 1980, the combined work of Berger [1], Jerry Kazdan [14], Alan Weinstein [25] and C T Yang [26] showed that Blaschke manifolds "modeled on" S^n and $\mathbb{R}P^n$ must, up to scale, be isometric to them. Quite a lot is known about Blaschke manifolds in general, but isometry is known in no other cases.

Once again, please see the arXiv version of this paper [3] for more details, as well as the survey by Benjamin McKay [16] of what is known to date.

In another direction, the study of fibrations of round spheres by great circles has led to the study of fibrations of Euclidean space by straight lines, at the hands of Marcos Salvai [21; 22] and Michael Harrison [10].

Salvai studied the metric structure of the space of all oriented lines in $\mathbb{R}^n$, and then characterized, within this metric structure, the base spaces of suitably nondegenerate fibrations of $\mathbb{R}^n$ by straight lines.

Harrison focused on the condition that the straight line fibers should be *skew* (nonintersecting and nonparallel), and pointed out that a fibration of a round sphere by great circles leads, via radial projection onto any tangent hyperplane, to a fibration of that hyperplane by pairwise skew straight lines. He proved that the space of such skew fibrations of $\mathbb{R}^3$ by straight lines deformation retracts through such fibrations to its subspace of projected Hopf fibrations, and so has the homotopy type of a pair of disjoint copies of S^2, a nice follow-up to the corresponding theorem for fibrations of S^3 by great circles.

How do Blaschke manifolds determine fibrations of round spheres by great subspheres?

Let M be a Blaschke manifold, p a point of M, and T_pM the tangent space to M at p. Let $B(p)$ denote a round ball of radius r in T_pM, where r is the common distance from each point of M to its cut locus. See Figure 5.

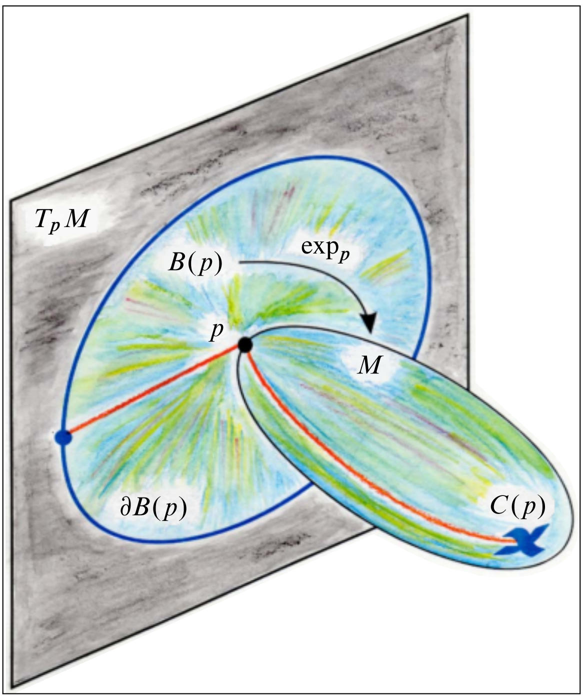

Figure 5: The exponential map $\exp_p\colon T_pM \to M$ takes a round ball $B(p)$ onto the Blaschke manifold M and takes $\partial B(p)$ to the cut locus $C(p)$.

Theorem (Omori [20], Nakagawa and Shiohama [18; 19]) *If M is a Blaschke manifold, then the cut locus $C(p)$ to any point p in M is a smooth submanifold of M, and $\exp_p\colon \partial B(p) \to C(p)$ is a smooth fiber bundle with great subsphere fibers.*

By the above theorem, any Blaschke manifold leads to a smooth fibration of a round sphere by great subspheres. The Blaschke manifold M can be recovered topologically from the fibration $\exp_p\colon \partial B(p) \to C(p)$, since M is homeomorphic to its mapping cone. Thus to understand Blaschke manifolds topologically, one should understand the topological classification of fibrations of spheres by great subspheres.

Conjecture *Any smooth fibration of a sphere by great subspheres is topologically equivalent to a Hopf fibration.*

Caution *There are many inequivalent fibrations of S^7 by 3–spheres* (Milnor [17], Eells and Kuiper [4]), *but in general their fibers are not great 3–spheres.*

To prove the conjecture, one must figure out how to capitalize on the hypothesis of *great* sphere fibers.

The conjecture is known to be true in the following cases:

- Any fibration of S^3 by simple closed curves is topologically equivalent to the Hopf fibration [24].
- Any smooth fibration of S^7 by great 3–spheres or of S^{15} by great 7–spheres is topologically equivalent to a Hopf fibration [8].
- Any smooth fibration of S^{2n+1} by great circles is smoothly equivalent to a Hopf fibration [27; 15].

3 The Grassmann manifold

Coordinates in the Grassmann manifold $G_2\mathbb{R}^{2n+2}$

Given a fibration F of S^{2n+1} by oriented great circles, each fiber P of F lies in and orients some 2–plane through the origin in $\mathbb{R}^{2n+2}$, which we denote by P as well, and so appears as a single point in the Grassmann manifold $G_2\mathbb{R}^{2n+2}$ of all such oriented 2–planes.

The base space M_F of F then appears as a $2n$–dimensional topological submanifold of $G_2\mathbb{R}^{2n+2}$, and if the fibration F is smooth, then the submanifold M_F is also smooth.

Let P be an oriented great circle on S^{2n+1}, equivalently, an oriented 2–plane through the origin in $\mathbb{R}^{2n+2}$, and let $P^\perp$ be its orthogonal complement.

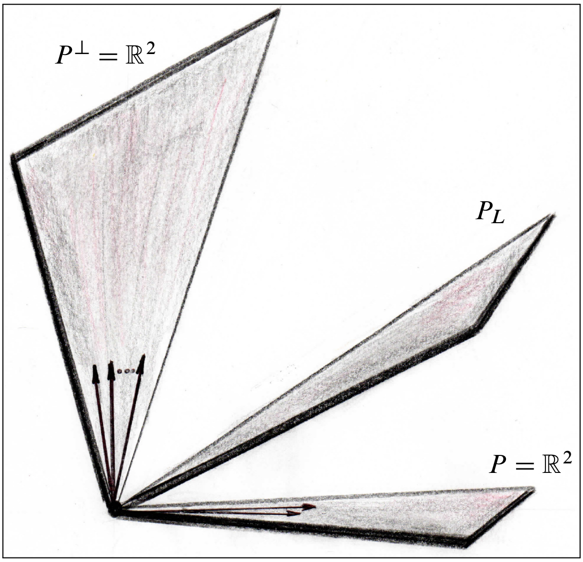

Figure 6: P_L is the graph of $L\colon P \to P^\perp$ in $P + P^\perp = \mathbb{R}^{2n+2}$

The $4n$–dimensional vector space $\mathrm{Hom}(P, P^\perp)$ will serve simultaneously as a large coordinate neighborhood about P in $G_2\mathbb{R}^{2n+2}$, and as the tangent space $T_P(G_2\mathbb{R}^{2n+2})$ to this Grassmann manifold at P, as follows.

Suppose that the oriented 2–plane Q in $\mathbb{R}^{2n+2}$ contains no vector orthogonal to P, and suppose that its orthogonal projection to P is orientation-preserving. Let $N(P)$ be the collection of all such 2–planes Q. This set $N(P)$ is the domain of our coordinate chart

$$G_2\mathbb{R}^{2n+2} \supset N(P) \xrightarrow{\phi} \mathrm{Hom}(P, P^\perp),$$

defined as follows.

Given $Q \in N(P)$, we can view Q as the graph of a linear transformation $L_Q\colon P \to P^\perp$ and we set $\phi(Q) = L_Q$. Note that P is itself the graph of the zero transformation, so $\phi(P) = 0$.

Conversely, given a linear transformation $L\colon P \to P^\perp$, the graph of L is a 2–plane P_L in $\mathbb{R}^{2n+2}$, which we may orient via orthogonal projection back to P, allowing us to view this graph as an element of $N(P)$.

Since $\mathrm{Hom}(P, P^\perp)$ is a vector space, the differential ϕ_* of

$$\phi\colon N(P) \to \mathrm{Hom}(P, P^\perp)$$

is an isomorphism of the tangent space $T_P(G_2\mathbb{R}^{2n+2})$ with $\mathrm{Hom}(P, P^\perp)$.

Thus we may view $\mathrm{Hom}(P, P^\perp)$ simultaneously as a coordinate neighborhood of P in $G_2\mathbb{R}^{2n+2}$ and as the tangent space $T_P(G_2\mathbb{R}^{2n+2})$ to this Grassmannian at P. To connect these two roles, we consider the "identity map"

$$I\colon T_P(G_2\mathbb{R}^{2n+2}) = \mathrm{Hom}(P, P^\perp) \to \mathrm{Hom}(P, P^\perp) = N(P) \subset G_2\mathbb{R}^{2n+2}.$$

Caution *The map I is **not** the exponential map. Like the exponential map, it takes some lines through the origin in $T_P(G_2\mathbb{R}^{2n+2}) = \mathrm{Hom}(P, P^\perp)$ to geodesics through P in $G_2\mathbb{R}^{2n+2}$, but with distortion of parametrization. It takes other lines through the origin to non-geodesics through P.*

Next we fix bases of P and $P^\perp$ in order to write elements of $\mathrm{Hom}(P, P^\perp)$ as $2n \times 2$ matrices. Let $\{e_1, e_2\}$ be an orthonormal basis for P, consistent with its orientation. Now orient $P^\perp$ so that the orientations on P and $P^\perp$ together give the orientation on $\mathbb{R}^{2n+2}$. Finally, choose an orthonormal basis $\{f_1, f_2, \ldots, f_{2n}\}$ for $P^\perp$ consistent with its orientation.

We write elements of $\mathrm{Hom}(P, P^\perp)$ as $2n\times 2$ matrices $A = A_1 \mid A_2$, where A_1 and A_2 are column $2n$–vectors. We see that $\mathrm{Hom}(P, P^\perp)$ is the sum of two copies of $P^\perp$, since we may write

$$\mathrm{Hom}(P, P^\perp) = \{A_1 \mid A_2\} = \{A_1 \mid 0\} + \{0 \mid A_2\} = P^\perp + P^\perp,$$

with the identifications

$$P^\perp = \{A_1 \mid 0\} = \{0 \mid A_2\}.$$

Geometrically, the columns A_1 and A_2 have the following meaning.

Let $P(t)$ be the oriented 2–plane in $\mathbb{R}^{2n+2} = P + P^\perp$ spanned by the frame

$$\{e_1 + tA_1,\ e_2 + tA_2\}.$$

For $-\infty < t < \infty$, this gives us a path $t \mapsto P(t)$ in $G_2\mathbb{R}^{2n+2}$ which runs within the domain $N(P)$ of our coordinate chart $\phi\colon N(P) \to \mathrm{Hom}(P, P^\perp)$. The corresponding path in $\mathrm{Hom}(P, P^\perp)$ is the line $t \mapsto tA_1 \mid tA_2$, and the tangent vector to this path at $t = 0$ is

$$A_1 \mid A_2 \in \mathrm{Hom}(P, P^\perp) = T_P(G_2\mathbb{R}^{2n+2}).$$

The "bad set" and the "bad cone"

Consider oriented great circle fibrations F of S^{2n+1} which contain a fixed great circle fiber P. Because the fibers of F are disjoint, the base space M_F certainly cannot also pass through Q in $G_2\mathbb{R}^{2n+2}$ if the corresponding great circles P and Q intersect on S^{2n+1}.

This motivates the following definitions.

The *bad set* $\mathrm{BS}(P) \subset G_2\mathbb{R}^{2n+2}$ consists of all oriented 2–planes through the origin in $\mathbb{R}^{2n+2}$ which meet P in at least a line. If M_F contains the great circle fiber P, then M_F intersects the bad set $\mathrm{BS}(P)$ only at P and nowhere else.

The *bad cone* $\mathrm{BC}(P) \subset T_P(G_2\mathbb{R}^{2n+2})$ is the tangent cone to the bad set at P.

Within the coordinate neighborhood $N(P) = \mathrm{Hom}(P, P^\perp)$ of P in $G_2\mathbb{R}^{2n+2}$, the bad set $\mathrm{BS}(P)$ consists of linear transformations $L\colon P \to P^\perp$ with nontrivial kernel, because the graphs of such linear transformations intersect P in at least a line. Equivalently, these are the $2n \times 2$ matrices $A = A_1 \mid A_2$ of rank 0 or 1. They all have the form

$$A = A_1 \cos t \mid A_1 \sin t,$$

where A_1 is a column $2n$–vector.

We note that, in the $\mathrm{Hom}(P, P^\perp)$ coordinates on $N(P)$, the portion of the bad set within that neighborhood is a union of lines through the origin $0 = \phi(P)$, namely

$$sA = sA_1 \cos t \mid sA_1 \sin t,$$

with $-\infty < s < \infty$.

It follows from this that the tangent cone to the bad set at P coincides with this portion of the bad set, that is,

$$I(\mathrm{BC}(P)) = \mathrm{BS}(P) \cap N(P).$$

With abuse of language, we may simply write $\mathrm{BC}(P) \subset \mathrm{BS}(P)$, and view the bad cone at P as a portion of the bad set at P.

Properties of the bad cone

(1) In the $\mathrm{Hom}(P, P^\perp)$ coordinates on $N(P)$, the bad cone at Q contains the translate of the bad cone at P, namely

$$\mathrm{BC}(P) + L_Q \subset \mathrm{BC}(Q),$$

where $L_Q = \phi(Q)$ in our chart $\phi\colon N(P) \to \mathrm{Hom}(P, P^\perp)$ centered at P.

That's because the linear transformations $L_{Q^*}\colon P \to P^\perp$ which correspond to points of $\mathrm{BC}(Q)$ are those which agree with L_Q on some nonzero vector u in P. Thus $L_{Q^*} - L_Q$ contains u in its kernel, and hence belongs to $\mathrm{BC}(P)$.

(2) The bad cone $\mathrm{BC}(P)$ is homeomorphic to a cone over $S^1 \times S^{2n-1}$.

We see this as follows. If $L\colon P \to P^\perp$ is a linear transformation with a nontrivial kernel, then its $2n \times 2$ matrix A has the form

$$A = A_1 \cos t \mid A_1 \sin t,$$

where A_1 is some column $2n$–vector.

If we fix t and let A_1 vary, we get a $2n$–plane which is part of the bad cone.

If we then let t vary, we fill out the bad cone with a circle's worth of such $2n$–planes, modulo the involution $(t, A_1) \mapsto (t+\pi, -A_1)$.

Equivalently, $\mathrm{BC}(P)$ is a cone over the quotient of $S^1 \times S^{2n-1}$ by this involution. But this quotient is homeomorphic to $S^1 \times S^{2n-1}$, since the antipodal map on an odd-dimensional sphere is isotopic to the identity.

In similar fashion, the bad set $\mathrm{BS}(P)$ is homeomorphic to the suspension of $S^1 \times S^{2n-1}$.

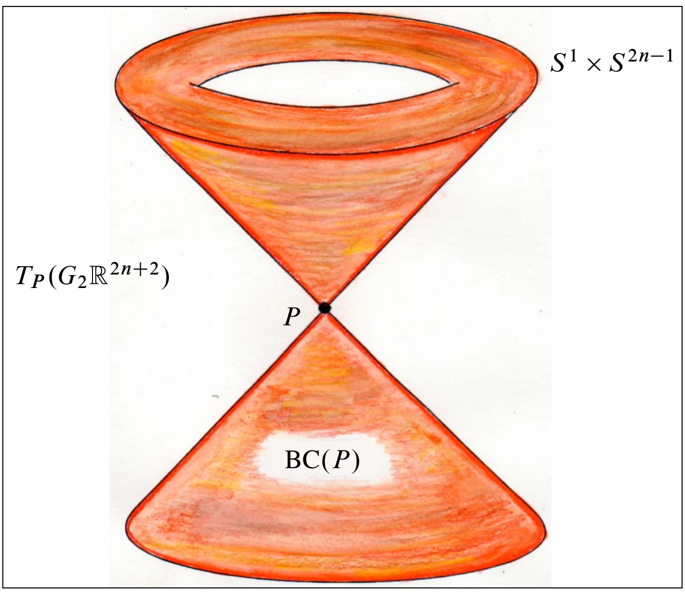

Figure 7: The bad cone $\mathrm{BC}(P)$

When we come to Proposition 1, we will visualize the Grassmann manifold $G_2\mathbb{R}^{2n+2}$ with a bad cone $\mathrm{BC}(P)$ inside the tangent space $T_P(G_2\mathbb{R}^{2n+2})$ at *each* of its points P, thus giving us a *field of bad cones,* as shown in Figure 8.

4 Proof of Proposition 1

Now we characterize the submanifolds of $G_2\mathbb{R}^{2n+2}$ which correspond to the base space of some smooth fibration of S^{2n+1} by great circles.

Proposition 1 *A closed connected smooth $2n$–dimensional submanifold of $G_2\mathbb{R}^{2n+2}$ is the base space of a fibration of S^{2n+1} by great circles if and only if it is transverse to the bad cone at each of its points.*

Proof Suppose first that F is a smooth fibration of S^{2n+1} by oriented great circles. We want to show that its base space M_F in $G_2\mathbb{R}^{2n+2}$ is a smooth submanifold transverse to the field of bad cones there.

For S^3, this is Theorem B of [7].

For smooth fibrations of spheres by great subspheres of any dimension, this is [8, Theorem 4.1].

This was proved again for all great circle fibrations of S^{2n+1} by McKay [15], from a different point of view.

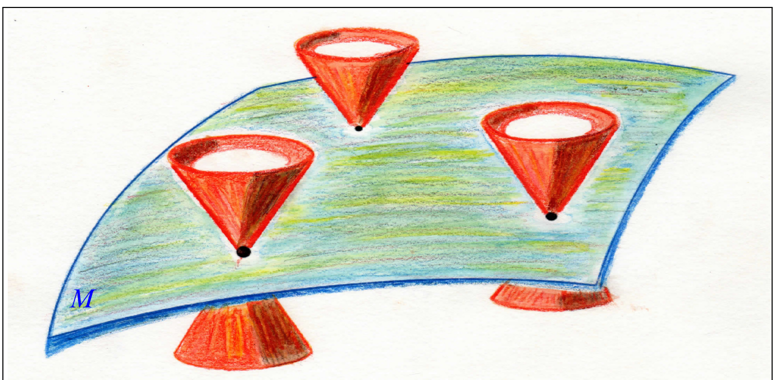

Figure 8: M is like a submarine negotiating a mine field

Suppose, conversely, that M is a closed, smooth, $2n$–dimensional submanifold of $G_2\mathbb{R}^{2n+2}$ which is transverse to the field of bad cones.

There is a canonical S^1 bundle E over $G_2\mathbb{R}^{2n+2}$, whose fiber over P is the great circle in the 2–plane P. Let E_M be the restriction of this bundle to the submanifold M,

$$E_M = \{(P, v) : P \in M,\ v \in P,\ \|v\| = 1\}.$$

Let ρ: $E_M \to M$ be the projection map, and let g: $E_M \to S^{2n+1}$ be the map which includes each great circle fiber into S^{2n+1}, that is, $g(P, v) = v$.

Our task is to show that g is a diffeomorphism.

First, we claim that transversality of M to the bad cone through each of its points implies that the map g is an immersion.

Suppose, to the contrary, that dg has a nontrivial kernel at some point v in the fiber P. Consider a path γ: $(-1, 1) \to E_M$, and write $\gamma(t) = (P(t), v(t))$, such that $\gamma(0) = (P, v)$ and $\gamma'(0) \neq 0$. We will show that if $\gamma'(0)$ is in the kernel of the derivative dg_v, then M must be tangent to the bad cone $\mathrm{BC}(P)$ at P.

Consider the path $P(t) = \rho\gamma(t)$ in M, with $P(0) = P$.

Using the coordinate neighborhood $\mathrm{Hom}(P, P^\perp)$ about P in $G_2\mathbb{R}^{2n+2}$, the path $P(t)$ corresponds to a path $L(t)$ in $\mathrm{Hom}(P, P^\perp)$.

Since $P(0) = P$, we have $L(0) = 0$.

Now $g\gamma(t) = v(t)$ lies in $P(t)$, which is the graph of $L(t)$, so we can write

$$g\gamma(t) = (w(t), L(t)w(t))$$

as an ordered pair of vectors in $P \times P^\perp$, with $w(0) \neq 0$.

We differentiate with respect to t and set $t = 0$ to get

$$(g\gamma)'(0) = (w'(0), L'(0)w(0) + L(0)w'(0)) \in P \times P^\perp.$$

Now we are assuming that $(g\gamma)'(0) = 0$ in $\mathbb{R}^{2n+2} = P + P^\perp$ and we know that $L(0) = 0$, so we conclude that $L'(0)w(0) = 0$.

Since $w(0) \neq 0$, this tells us that $L'(0)$ has a nontrivial kernel, and hence lies in the bad cone $\mathrm{BC}(P)$ at P.

Therefore the path $P(t)$ in M is tangent to the bad cone at $P(0) = P$, contrary to the assumption that M is transverse to the field of bad cones.

So we have just shown that the map $g\colon E_M \to S^{2n+1}$ is an immersion.

But E_M is compact, and so the map g is both open and closed, and hence its image $g(E_M)$ must be all of S^{2n+1}.

Thus g is a covering map, and since S^{2n+1} is simply connected for $n \geq 1$, g must be a diffeomorphism.

Thus E_M gives a smooth fibration of S^{2n+1} by great circles, with M as its base space, completing the proof of the lemma. □

Remarks (1) The proofs in [7; 8] that the base space M_F of a smooth fibration F by great subspheres is transverse to the field of bad cones use the fact that the local trivializations of F are diffeomorphisms.

One can have a topological fibration F of S^{2n+1} by great circles whose base space M_F is a smooth submanifold of $G_2\mathbb{R}^{2n+2}$ occasionally tangent to a bad cone, and then the local trivializations of F will be smooth homeomorphisms, but not diffeomorphisms.

(2) A small, smooth $2n$–disk in $G_2\mathbb{R}^{2n+2}$ which is transverse to the field of bad cones gives a fibration of an open tube in S^{2n+1} by great circles.

5 Proof of Proposition 2

$2n \times 2n$ matrices with no real eigenvalues

In this section, we see how $2n \times 2n$ matrices with no real eigenvalues arise in our study of $2n$–planes tangent to the base space of a smooth fibration of S^{2n+1} by great circles.

In the $4n$–dimensional vector space $\mathrm{Hom}(P, P^\perp) = P^\perp + P^\perp$, we need to recognize those $2n$–dimensional subspaces which are transverse to the bad cone $\mathrm{BC}(P)$, since they will be precisely those, according to Propositions 1 and 4, which can serve as tangent spaces to the base spaces of fibrations of S^{2n+1} by great circles.

Lemma *A $2n$–dimensional subspace of* $\mathrm{Hom}(P, P^\perp) = P^\perp + P^\perp$ *is transverse to the bad cone* $\mathrm{BC}(P)$ *if and only if it is the graph of a linear map with no real eigenvalues from one $P^\perp$ summand to the other.*

Proof A $2n$–dimensional subspace T of $\mathrm{Hom}(P, P^\perp)$ transverse to the bad cone can meet each of the two summands $P^\perp = \{A_1 \mid 0\}$ and $P^\perp = \{0 \mid A_2\}$ only at the

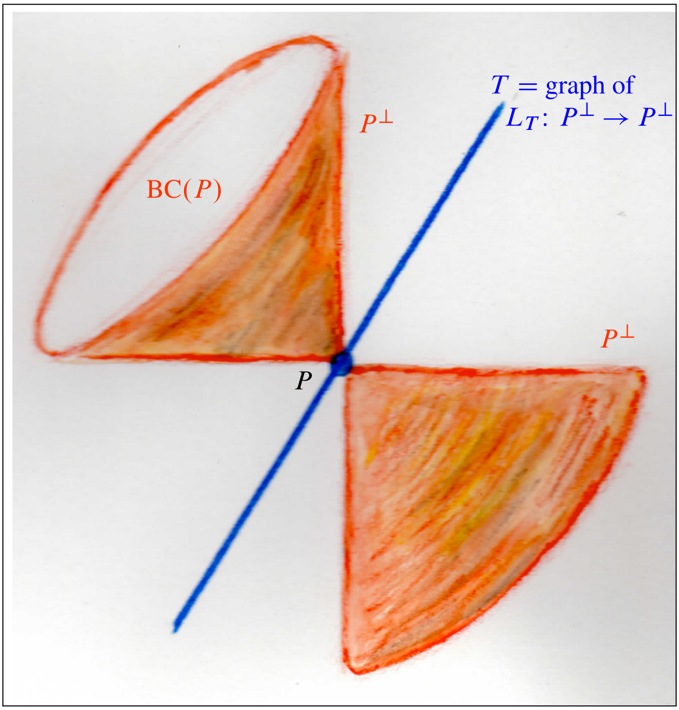

Figure 9: T is transverse to the bad cone $\mathrm{BC}(P)$ if and only if it is the graph of a linear map $L_T\colon P^\perp \to P^\perp$ with no real eigenvalues.

origin, since these summands lie entirely in the bad cone. Hence T is the graph of a linear map $L_T\colon P^\perp \to P^\perp$ between these subspaces, in either order.

If L_T has a real eigenvalue λ with eigenvector A_1, then its graph T contains the vector $A_1 \mid \lambda A_1$, a $2n \times 2$ matrix of rank 1, hence in the bad cone $\mathrm{BC}(P)$.

Thus a $2n$–dimensional subspace T of $\mathrm{Hom}(P, P^\perp)$ which is transverse to the bad cone is the graph of a linear map L_T as above with no real eigenvalues.

Conversely, if T is a $2n$–dimensional subspace of $\mathrm{Hom}(P, P^\perp)$ which is the graph of a linear map $L_T\colon P^\perp \to P^\perp$ with no real eigenvalues, then T contains no $2n \times 2$ matrices of rank 1, and so is transverse to the bad cone $\mathrm{BC}(P)$, proving the lemma. $\square$

Improving maps with no real eigenvalues

Recall that by a *linear complex structure* we mean a linear map $J\colon \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$ such that $J^2 = -I$, and that for an *orthogonal complex structure* we require in addition that the map J be orthogonal.

Given any orthogonal complex structure $J\colon \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$, the unit circles in the J–complex lines yield a Hopf fibration H of S^{2n+1} by oriented great circles.

Lemma *The tangent $2n$–plane to the base space M_H at a complex line P is the graph of $J|_{P^\perp}\colon P^\perp \to P^\perp$.*

Proof The points L in the large coordinate neighborhood $\mathrm{Hom}(P, P^\perp)$ of P in $G_2\mathbb{R}^{2n+2}$ are represented by $2n \times 2$ matrices $A = A_1 \mid A_2$, where the two columns are the L–images in $P^\perp$ of an ON basis e_1, e_2 for P with $J(e_1) = e_2$.

The points Q in this neighborhood which lie in the base space M_H of the fibration H are J–complex lines, meaning images of a J–complex linear map $L\colon P \to P^\perp$. Since $L(e_1) = A_1$ and $L(e_2) = A_2$ and $L \circ J = J \circ L$, we have

$$A_2 = L(e_2) = L(J(e_1)) = J(L(e_1)) = J(A_1).$$

Thus the points of M_H in this coordinate neighborhood lie on the graph of the map $J|_{P^\perp}\colon P^\perp \to P^\perp$.

Since the coordinate neighborhood $\mathrm{Hom}(P, P^\perp)$ of P serves as its own tangent space at P, the graph of $J|_{P^\perp}\colon P^\perp \to P^\perp$ serves as the tangent $2n$–plane to M_H at P, as claimed. $\square$

Remarks (1) The portion of M_H within the large open neighborhood $\mathrm{Hom}(P, P^\perp)$ of P in $G_2\mathbb{R}^{2n+2}$ appears as a $2n$–plane through the origin there.

(2) The above lemma and remark hold equally well if $J\colon \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$ is only a linear complex structure such that $J(P^\perp) = P^\perp$.

Proposition 2 *There is a $\mathrm{GL}(2n, \mathbb{R})$–equivariant deformation retraction of the space of linear transformations $T\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with no real eigenvalues to its subspace of linear complex structures $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$.*

Proof See McKay [15, pages 1163–1166].

Let $T\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear transformation with no real eigenvalues. Complexify $\mathbb{R}^{2n}$ to get $\mathbb{C}^{2n}$, and regard $T\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$. Since T is real, its eigenvalues λ occur in conjugate pairs.

Split $\mathbb{C}^{2n}$ into a direct sum $\sum_\lambda E_\lambda T$ of the generalized eigenspaces of T, where

$$E_\lambda T = \{v \in \mathbb{C}^{2n} : (T - \lambda I)^k v = 0 \text{ for some } k > 0\},$$

with $\dim(E_\lambda T)$ = multiplicity of the eigenvalue λ. Complex conjugation in $\mathbb{C}^{2n}$ takes $E_\lambda T$ to $E_{\bar\lambda} T$ since T is real.

Reorganize the direct sum as

$$\mathbb{C}^{2n} = \sum_{\operatorname{Im}\lambda>0} E_\lambda T + \sum_{\operatorname{Im}\lambda<0} E_\lambda T = V_C^+ + V_C^-,$$

and note that complex conjugation interchanges V_C^+ and V_C^-.

Now define a complex linear map $J_T\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ by $J_T(v) = iv$ if $v \in V_C^+$ and $J_T(v) = -iv$ if $v \in V_C^-$. This map J_T commutes with complex conjugation, and hence takes real vectors to real vectors, so that $J_T\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a linear complex structure.

It is clear from construction that the correspondence $T \to J_T$ is $\mathrm{GL}(2n,\mathbb{R})$–equivariant.

Our desired deformation retraction is given by the formula

$$T_t = (1-t)T + tJ_T.$$

One easily checks by looking at the blocks in the Jordan normal form for T that each of the transformations T_t has no real eigenvalues.

Since T and J_T each commute with complex conjugation, the same is true for T_t, and hence it also takes real vectors to real vectors.

To confirm that the proposed deformation retraction T_t depends continuously on T, we must check that J_T itself depends continuously on T.

Since J_T is defined as multiplication by i on V_C^+ and by $-i$ on V_C^-, this amounts to checking that the subspaces V_C^+ and V_C^- depend continuously on the choice of T from among the linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with no real eigenvalues.

This is implied by Lemma 2 on page 1164 of McKay [15], where he shows that the map $T \to J_T$ is the projection of a smooth fiber bundle.

We give a different argument here.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of T with positive imaginary part, with complex conjugates $\bar\lambda_1, \bar\lambda_2, \dots, \bar\lambda_n$, which are the eigenvalues of T with negative imaginary part. In each case, an eigenvalue may be listed several times according to its multiplicity.

Consider the complex polynomials

$$p_T^+(z) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n) \quad \text{and} \quad p_T^-(z) = (z-\bar\lambda_1)(z-\bar\lambda_2)\cdots(z-\bar\lambda_n),$$

which are the characteristic polynomials of the restrictions of T to V_C^+ and V_C^-, respectively. Their product $p_T(z) = p_T^+(z)p_T^-(z)$ is the characteristic polynomial of T on all of V_C.

By the Cayley–Hamilton theorem, the linear transformation $p_T^+(T)$ vanishes on V_C^+, the linear transformation $p_T^-(T)$ vanishes on V_C^-, while their product (composition) $p_T(T) = p_T^+(T)p_T^-(T)$ vanishes on all of V_C.

Since $p_T^+(z)$ and $p_T^-(z)$ have no roots in common, they are relatively prime, and hence there are polynomials $a_T^+(z)$ and $a_T^-(z)$ such that

$$a_T^+(z)p_T^+(z) + a_T^-(z)p_T^-(z) = 1.$$

Inserting T in place of z, we get

$$(\star) \qquad a_T^+(T)p_T^+(T) + a_T^-(T)p_T^-(T) = I.$$

Lemma *The kernels of the linear maps $p_T^+(T)$ and $p_T^-(T)$: $V_C \to V_C$ are precisely*

$$\ker p_T^+(T) = V_C^+ \quad \textit{and} \quad \ker p_T^-(T) = V_C^-.$$

Proof We already know $p_T^+(T)$ vanishes on V_C^+, so that $\ker p_T^+(T)$ contains V_C^+, and likewise $\ker p_T^-(T)$ contains V_C^-. Now V_C^+ and V_C^- are complex n–dimensional subspaces of the complex $2n$–dimensional space V_C. If either $\ker p_T^+(T)$ is larger than V_C^+ or $\ker p_T^-(T)$ is larger than V_C^-, then there would have to be a nonzero vector v in V_C which lies in both kernels. But then applying formula $(\star)$ above to v would give a contradiction, because the left side would kill v, while the right side would preserve it. This completes the proof of the lemma. □

Now as T varies continuously among linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with no real eigenvalues, the roots of its characteristic polynomial also vary continuously (with multiple roots permitted to split into simpler ones), and so by the above lemma, the subspaces V_C^+ and V_C^- also vary continuously.

This completes the proof of Proposition 2. □

6 Proof of Proposition 3

Now we discuss the second step of our deformation retraction.

Proposition 3 *There is an $O(2n)$–equivariant deformation retraction of the space of linear complex structures on $\mathbb{R}^{2n}$ to its subspace of orthogonal complex structures.*

To prove this, we will use the one-to-one correspondence between linear complex structures $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and direct sum decompositions of $\mathbb{C}^{2n} = V_C^+ + V_C^-$ into a pair of conjugate complex subspaces, the $+i$ and $-i$ eigenspaces of J on $\mathbb{C}^{2n}$, as described in the proof of Proposition 2.

We will check that the complex structure J is orthogonal if and only if V_C^+ and V_C^- are orthogonal to one another.

Our goal will then be to describe a deformation retraction from the set of all pairs V_C^+ and V_C^- of complex n–dimensional conjugate subspaces of $\mathbb{C}^{2n}$ to its subset of orthogonal such pairs. Intuitively, this deformation retraction is given by opening up all the angles between V_C^+ and V_C^- in a coordinated fashion until they become orthogonal.

We now turn to providing the details.

Characterization of orthogonal complex structures

Lemma *A linear complex structure $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is orthogonal if and only if v and $J(v)$ are orthogonal to one another for all vectors v in $\mathbb{R}^{2n}$.*

Proof If J is an orthogonal complex structure, it is easy to check that v and $J(v)$ are orthogonal to one another for all vectors v in $\mathbb{R}^{2n}$.

In the other direction, suppose that $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a linear complex structure for which v and $J(v)$ are orthogonal for all vectors v in $\mathbb{R}^{2n}$.

Apply this statement to the vector $w = u + J(v)$ to learn that

$$\begin{aligned} 0 &= w \cdot J(w) = (u + J(v)) \cdot J(u + J(v)) \\ &= (u + J(v)) \cdot (J(u) + J^2(v)) = (u + J(v)) \cdot (J(u) - v) \\ &= u \cdot J(u) - u \cdot v + J(v) \cdot J(u) - J(v) \cdot v \\ &= -u \cdot v + J(u) \cdot J(v), \end{aligned}$$

whence $J(u) \cdot J(v) = u \cdot v$, confirming that J is an orthogonal transformation. □

Lemma *A linear complex structure $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is orthogonal if and only if the conjugate complex subspaces V_C^+ and V_C^- of $\mathbb{C}^{2n}$ are orthogonal to one another.*

Proof We start with $\mathbb{R}^2$, and let $J\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by the matrix

$$\begin{bmatrix} 0 & b \\ -1/b & 0 \end{bmatrix}.$$

It is easy to see by continuity that every complex structure J on $\mathbb{R}^2$ moves some nonzero vector orthogonal to itself, so that it can be expressed in the above matrix form for some orthonormal basis.

The above map J is orthogonal if and only if $b = \pm 1$.

The eigenvalues of J are i and $-i$, and corresponding eigenvectors of J on C^2 are the column vectors $u = [b\ i]$ and $v = [b\ \ -i]$.

The complex subspaces V_C^+ and V_C^- of $\mathbb{C}^2$ are generated in this case by the i and $-i$ eigenvectors above. That is,

$$\begin{aligned} V_C^+ &= \mathbb{C}\{u = [b\ i]\} = \mathbb{R}\{u = [b\ i],\ u' = iu = [ib\ \ -1]\}, \\ V_C^- &= \mathbb{C}\{v = [b\ \ -i]\} = \mathbb{R}\{v = [b\ \ -i],\ v' = iv = [ib\ 1]\}. \end{aligned}$$

We compute the dot products of these vectors and learn that

$$u \cdot v = b^2 - 1, \quad u \cdot v' = 0, \quad u' \cdot v = 0, \quad u' \cdot v' = b^2 - 1.$$

Hence the $+i$ and $-i$ eigenspaces V_C^+ and V_C^- are orthogonal to one another if and only if $b = \pm 1$, which is precisely the condition that the complex structure J be orthogonal.

This completes the argument for $\mathbb{R}^2$.

With this in hand, we carry out the general argument for $\mathbb{R}^{2n}$.

If $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is an orthogonal complex structure, then we can choose an orthonormal basis for $\mathbb{R}^{2n}$ with respect to which the matrix for J is in block diagonal form, with 2×2 blocks

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

down the diagonal.

Then V_C^+ and V_C^- are each complex n–dimensional subspaces of $\mathbb{C}^{2n}$. Each is an orthogonal direct sum of complex lines. The r^{th} complex lines in each direct sum are orthogonal to one another by the completed task in $\mathbb{R}^2$, whereas the r^{th} complex line in one sum is automatically orthogonal to the s^{th} complex line in the other sum when $r \neq s$. It follows that the complex subspaces V_C^+ and V_C^- are orthogonal to one another in $\mathbb{C}^{2n}$.

If $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is *not* an orthogonal complex structure, then it follows from our earlier lemma characterizing orthogonal complex structures that there is some vector v in $\mathbb{R}^{2n}$ for which $J(v)$ is not orthogonal to v.

The 2–plane spanned by this v and $J(v)$ is invariant under J, but on it J is *not* a rotation by $90°$, as we saw in $\mathbb{R}^2$, and hence V_C^+ and V_C^- are *not* orthogonal to one another.

This completes the proof of the lemma. □

Principal angles

We discuss the notion of *principal angles* in three settings:

(1) between a pair of real linear subspaces in $\mathbb{R}^n$,

(2) between a pair of complex linear subspaces in $\mathbb{C}^n$,

(3) between a complex linear subspace and its complex conjugate subspace in $\mathbb{C}^{2n}$.

The intention is to characterize the *relative position* of the two subspaces, up to the action of an appropriate group of isometries of the ambient space, which in the three cases above are the groups $O(n)$, $U(n)$, and $O(2n)$.

The notion and use of principal angles in the real setting (1) is familiar in geometry, and goes back at least to Camille Jordan [13]; see also Gluck [6]. But the extension to the complex settings (2) and (3) appears to be much less familiar, though we note the papers by Scharnhorst [23] and by Galántai and Hegedüs [5], the latter having a very nice set of references.

(1) Principal angles between a pair of linear subspaces in $\mathbb{R}^n$

Let P and Q be k–planes through the origin in $\mathbb{R}^n$. Then the relative position of P and Q in $\mathbb{R}^n$ is characterized up to the action of $O(n)$ by k principal angles $\theta_1, \theta_2, \dots, \theta_k$, obtained as follows.

The angle θ_1 is the smallest that any vector in P makes with any vector in Q. Pick such unit vectors v_1 in P and w_1 in Q. Let P_2 be the orthogonal complement of v_1 in $P = P_1$ and let Q_2 be the orthogonal complement of w_1 in $Q = Q_1$. Thus P_2 and Q_2 are $k-1$–planes through the origin in $\mathbb{R}^n$.

Remark It follows easily from the minimality of θ_1 that P_2 is also orthogonal to w_1, and that Q_2 is also orthogonal to v_1.

We move to the induction step. If $\theta_1 = 0$, then $v_1 = w_1$ and we replace $\mathbb{R}^n$ by the $\mathbb{R}^{n-1}$ orthogonal to $v_1 = w_1$, and replace the k–planes P and Q by the $k-1$–planes P_2 and Q_2.

If $\theta_1 > 0$, then v_1 and w_1 are independent and span a 2–plane through the origin. We replace $\mathbb{R}^n$ by the $\mathbb{R}^{n-2}$ orthogonal to this 2–plane, and replace the k–planes P and Q by the $k-1$–planes P_2 and Q_2. In this case we need the above remark, to guarantee that P_2 and Q_2 lie in this $\mathbb{R}^{n-2}$.

Now we iterate the construction, with $\mathbb{R}^n$ replaced by either $\mathbb{R}^{n-1}$ or $\mathbb{R}^{n-2}$ as detailed above, and with P and Q replaced by P_2 and Q_2.

Following through to the end, we get orthonormal bases

$$v_1, v_2, \dots, v_k \quad \text{and} \quad w_1, w_2, \dots, w_k$$

for the k–planes P and Q, respectively, with *principal angles*

$$\theta_1 \le \theta_2 \le \cdots \le \theta_k \le \tfrac{\pi}{2}$$

between the vectors v_1 and w_1, v_2 and w_2, ..., v_k and w_k, and with v_r orthogonal to w_s for $r \neq s$.

The principal angles between P and Q characterize their relative position in $\mathbb{R}^n$ as follows.

(1) Theorem (principal angles in $\mathbb{R}^n$) *Let P and Q be a pair of k–planes through the origin in $\mathbb{R}^n$, and likewise for P' and Q'. Then there is a rigid motion (element of $O(n)$) taking P to P' and simultaneously taking Q to Q' if and only if the principal angles between P and Q are the same as those between P' and Q'.*

Proof The condition of matching principal angles is clearly necessary for the existence of such a rigid motion.

Conversely, if the principal angles $\theta_1 \le \theta_2 \le \cdots \le \theta_k$ between P and Q match the principal angles $\theta_1' \le \theta_2' \le \cdots \le \theta_k'$ between P' and Q', then we easily obtain a rigid motion of $\mathbb{R}^n$ which takes the orthonormal bases $v_1, v_2, \dots, v_k$ and $w_1, w_2, \dots, w_k$ for P and Q to the orthonormal bases $v_1', v_2', \dots, v_k'$ and $w_1', w_2', \dots, w_k'$ for P' and Q'. $\square$

(2) Principal angles between a pair of complex linear subspaces of $\mathbb{C}^n$

Let P and Q be complex k–dimensional linear subspaces of $\mathbb{C}^n$, which to real eyes look like $2k$–planes through the origin in $\mathbb{R}^{2n}$.

To get principal angles between P and Q, and corresponding orthonormal bases for each of them, we begin as in the real case. Let θ_1 be the smallest angle that any vector in P makes with any vector in Q, and pick such unit vectors v_1 in P and w_1 in Q.

Then consider iv_1 and iw_1. These will be another pair of unit vectors in P and Q, respectively, since each of these is a *complex* linear subspace. The angle between iv_1 and iw_1 is also θ_1, because multiplication by i is an isometry of $\mathbb{C}^n$ which takes P to itself and Q to itself.

The list of principal angles begins with θ_1, θ_1, while our orthonormal bases for P and Q over the reals begins with v_1, iv_1 for P and w_1, iw_1 for Q.

We economize and list angles and bases from a complex point of view, so that our principal angles begin with just θ_1, while our orthonormal bases for P and Q over the complex numbers begins with v_1 for P and w_1 for Q.

We then iterate, as in the real case, and end with complex orthonormal bases

$$v_1, v_2, \dots, v_k \quad \text{and} \quad w_1, w_2, \dots, w_k$$

for the k–planes P and Q, with *principal angles*

$$\theta_1 \le \theta_2 \le \cdots \le \theta_k \le \tfrac{\pi}{2}$$

between the vectors v_1 and w_1, v_2 and w_2, ..., v_k and w_k, and with v_r orthogonal to w_s for $r \ne s$.

(2) Theorem (principal angles in $\mathbb{C}^n$) *Let P and Q be a pair of complex k–planes through the origin in $\mathbb{C}^n$, and likewise for P' and Q'. Then there is an element of $U(n)$ taking P to P' and simultaneously taking Q to Q' if and only if the principal angles between P and Q are the same as those between P' and Q'.*

We omit the proof, which is basically the same as in the real case.

(3) Principal angles between conjugate complex linear subspaces in $\mathbb{C}^{2n}$

Let P^k and $\overline{P}^k$ be conjugate complex subspaces of $\mathbb{C}^{2n}$ which meet only at the origin. We want to define the principal angles between them.

Let θ_1 be the smallest angle that any complex line L in P^k makes with its conjugate complex line $\overline{L}$ in $\overline{P}^k$. We claim that there will be a unit vector v_1 in L which makes that angle θ_1 with its complex conjugate $\overline{v}_1$ in $\overline{L}$.

The reason for this is that the nearest neighbor map from the unit circle in L to the unit circle in $\overline{L}$ is orientation-preserving, while the complex conjugation map between these unit circles is orientation-reversing. So there is sure to be a coincidence between these two maps, meaning a unit vector v_1 in L whose nearest neighbor in $\overline{L}$ is its own conjugate $\overline{v}_1$.

Thus v_1 makes the angle θ_1 with $\overline{v}_1$, and likewise iv_1 makes that same angle θ_1 with $i\overline{v}_1$. We note that iv_1 and $i\overline{v}_1$, though nearest neighbors in L and $\overline{L}$, are *not* complex conjugates of one another.

Now let P_2 be the orthogonal complement of the complex line $L = \mathbb{C}v_1$ in P^k, and then $\overline{P}_2$ will automatically be the orthogonal complement of the complex line $\overline{L} = \mathbb{C}\overline{v}_1$ in $\overline{P}^k$.

Remark As in the previous two cases, we find that P_2 is also orthogonal to $\overline{L} = \mathbb{C}\overline{v}_1$, and then (automatically) $\overline{P}_2$ is also orthogonal to $L = \mathbb{C}v_1$, and omit the details.

Then, since P^k and $\overline{P}^k$ meet only at the origin, we have $\theta_1 > 0$.

So we replace $\mathbb{C}^{2n}$ by the $\mathbb{C}^{2n-2}$ orthogonal to $\mathbb{C}v_1 + \mathbb{C}\overline{v}_1$, and replace P and $\overline{P}$ by the complex $k-1$–dimensional subspaces P_2 and $\overline{P}_2$, both lying in this $\mathbb{C}^{2n-2}$, thanks to the above remark.

As before, we iterate the construction, with $\mathbb{C}^{2n}$ replaced by $\mathbb{C}^{2n-2}$ and with P and $\overline{P}$ replaced by P_2 and $\overline{P}_2$.

Following through to the end, we get complex orthonormal bases

$$v_1, v_2, \dots, v_k \quad \text{and} \quad \overline{v}_1, \overline{v}_2, \dots, \overline{v}_k$$

for the k–planes P^k and $\overline{P}^k$, respectively, with *constrained principal angles*

$$0 < \theta_1 \le \theta_2 \le \cdots \le \theta_k \le \tfrac{\pi}{2}$$

between the vectors v_1 and $\overline{v}_1$, v_2 and $\overline{v}_2$, …, v_k and $\overline{v}_k$, and with $\mathbb{C}v_r$ orthogonal to $\mathbb{C}v_s$ for $r \ne s$.

Remark The "constraint" on these principal angles is seen at the beginning, when we minimize the angle θ_1 between a complex line L in P^k and its conjugate $\overline{L}$ in $\overline{P}^k$, and then likewise throughout the construction. But it is an easy exercise to check that the constrained principal angles between P^k and $\overline{P}^k$ coincide with the ordinary principal angles between these complex subspaces of $\mathbb{C}^{2n}$. We leave this to the reader, henceforth drop the adjective "constrained", and use this information in what follows.

(3) Theorem (principal angles for conjugate complex subspaces of $\mathbb{C}^{2n}$) *Let P^k and $\overline{P}^k$ be a pair of conjugate complex subspaces of $\mathbb{C}^{2n}$ which meet only at the origin, and Q^k and $\overline{Q}^k$ another such pair. Then there is an element of $O(2n)$ taking P^k to Q^k (and automatically taking $\overline{P}^k$ to $\overline{Q}^k$) if and only if the principal angles between P^k and $\overline{P}^k$ coincide with the principal angles between Q^k and $\overline{Q}^k$.*

Proof Let P^k and $\overline{P}^k$ be a pair of conjugate complex subspaces of $\mathbb{C}^{2n}$ which meet only at the origin, and Q^k and $\overline{Q}^k$ another such pair. The condition of matching principal angles is clearly necessary for the existence of an element of $O(2n)$ taking P^k to Q^k and $\overline{P}^k$ to $\overline{Q}^k$.

Suppose, conversely, that the principal angles between P^k and $\overline{P}^k$ coincide with the principal angles between Q^k and $\overline{Q}^k$.

Then by Theorem (2) there is an element F of $U(2n)$ which takes the orthonormal bases

$$v_1, v_2, \dots, v_k \quad \text{and} \quad \overline{v}_1, \overline{v}_2, \dots, \overline{v}_k$$

for P^k and $\overline{P}^k$ to the orthonormal bases

$$w_1, w_2, \dots, w_k \quad \text{and} \quad \overline{w}_1, \overline{w}_2, \dots, \overline{w}_k$$

for Q^k and $\overline{Q}^k$.

We claim that F commutes with complex conjugation, and hence takes real points of $\mathbb{C}^{2n}$ to real points of $\mathbb{C}^{2n}$.

Any unit vector in $\mathbb{C}v_1$ can be written as $e^{i\theta}v_1$, and since F is complex linear, $F(e^{i\theta}v_1) = e^{i\theta}w_1$. Likewise, $F(e^{i\theta}\overline{v}_1) = e^{i\theta}\overline{w}_1$. This last equality is also true with θ replaced by $-\theta$, hence $F(e^{-i\theta}\overline{v}_1) = e^{-i\theta}\overline{w}_1$. But $e^{-i\theta}\overline{v}_1$ is the complex conjugate of $e^{i\theta}v_1$, and $e^{-i\theta}\overline{w}_1$ is the complex conjugate of $e^{i\theta}w_1$. Thus F commutes with complex conjugation on $\mathbb{C}v_1$, and it likewise commutes with complex conjugation on $\mathbb{C}\overline{v}_1$, so it commutes with complex conjugation on $\mathbb{C}v_1 + \mathbb{C}\overline{v}_1$. Similarly, it commutes with complex conjugation on $\mathbb{C}v_r + \mathbb{C}\overline{v}_r$, and hence on all of $P^k + \overline{P}^k \to Q^k + \overline{Q}^k$.

If $k = n$, then $P^k + \overline{P}^k$ is all of $\mathbb{C}^{2n}$ and so F commutes with complex conjugation on all of $\mathbb{C}^{2n}$. If $k < n$, then we can easily modify F on the orthogonal complement of $P^k + \overline{P}^k$ so that it commutes with complex conjugation there as well.

Finally, since F commutes with complex conjugation on all of $\mathbb{C}^{2n}$, it takes the real points $\mathbb{R}^{2n}$ of $\mathbb{C}^{2n}$ to themselves, and is hence an element of the subgroup $O(2n)$ of $U(2n)$.

This completes the proof of Theorem (3). □

Proof of Proposition 3 We will exhibit an $O(2n)$–equivariant deformation retraction of the space of linear complex structures on $\mathbb{R}^{2n}$ to its subspace of orthogonal complex structures.

We start with a linear complex structure $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and the corresponding direct sum decomposition of the complexification $\mathbb{C}^{2n} = V_C^+ + V_C^-$ into a pair of conjugate complex subspaces, the $+i$ and $-i$ eigenspaces of $J\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$.

We want to move V_C^+ and V_C^- apart until they are orthogonal, keeping the intermediate positions as complex conjugates of one another, so as to deform the linear complex structure J through other linear complex structures, until we arrive at the orthogonal complex structure corresponding to the terminal positions of V_C^+ and V_C^- in this deformation, as shown in Figure 10 for $\mathbb{C}^4$. In $\mathbb{C}^{2n}$, we open up V_C^+ and V_C^- like $2n$ pairs of scissors in the real 2–planes spanned by

$$v_1 \text{ and } \overline{v}_1, \qquad i v_1 \text{ and } i\overline{v}_1, \qquad \dots, \qquad v_n \text{ and } \overline{v}_n, \qquad i v_n \text{ and } i\overline{v}_n$$

at rates proportional to the complementary angles $\frac{\pi}{2} - \theta_i$, so that they all open up to angle $\frac{\pi}{2}$ at the same time.

Each of these 2–planes contains a line of real vectors and an orthogonal line of purely imaginary vectors.

As the $2n$ pairs of scissors open up, the opening vectors v_k and $\overline{v}_k$ remain symmetric with respect to reflection in the real line in their 2–plane, and hence remain conjugates of one another.

By contrast, the opening vectors $i v_k$ and $i\overline{v}_k$ remain symmetric with respect to reflection in the purely imaginary line in their 2–plane, and hence remain *negative* conjugates of one another.

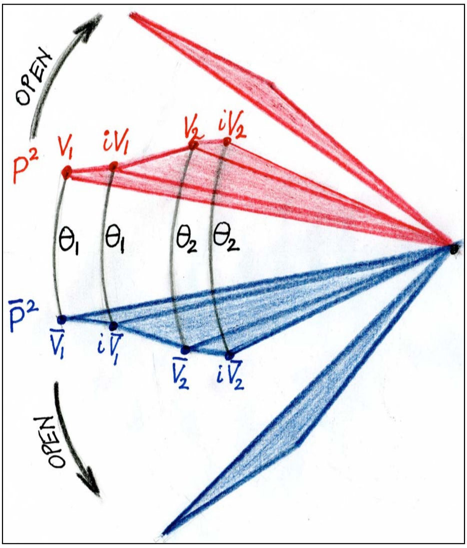

Figure 10: Opening up a pair of complex 2–dimensional conjugate subspaces in $\mathbb{C}^4$, guided by the principal angles, until they become orthogonal.

It follows that the complex $2n$–dimensional subspaces V_C^+ and V_C^- remain complex conjugates of one another as they open up, until they are finally orthogonal to one another.

This opening up of V_C^+ and V_C^- is not affected by the ambiguity in the choice of the above bases for these subspaces, even if several successive principal angles are equal.

During this opening, all the complex structures on $\mathbb{C}^{2n}$ commute with complex conjugation, and hence take the subspace $\mathbb{R}^{2n}$ of real points to itself.

The result is a deformation retraction of the space of linear complex structures on $\mathbb{R}^{2n}$ to its subspace of orthogonal complex structures, and the geometric naturality of all the constructions testifies to the $O(2n)$–equivariance of this procedure.

This completes the proof of Proposition 3. □

7 Proof of Proposition 4

Proposition 4 *There exists a smooth fibration F of S^{2n+1} by oriented great circles whose base space M_F is tangent at P to any preassigned $2n$–plane transverse to the bad cone* $\mathrm{BC}(P)$.

We begin with a sketch of the proof.

We start in the tangent space $\mathrm{Hom}(P, P^{\perp})$ to $G_2\mathbb{R}^{2n+2}$ at P with a given $2n$–plane which is transverse to the bad cone $\mathrm{BC}(P)$, and hence the graph of a linear map $A\colon P^{\perp} \to P^{\perp}$ with no real eigenvalues. See Figure 11.

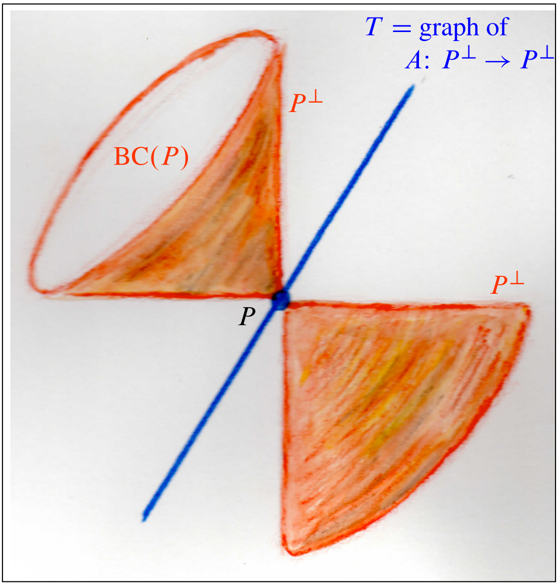

Figure 11: The graph of $A\colon P^{\perp} \to P^{\perp}$ is transverse to the bad cone

We must find a fibration F of S^{2n+1} by great circles including P, with this preassigned tangent $2n$–plane to its base space M_F at P.

To do this, let $J_A\colon P^{\perp} \to P^{\perp}$ be the linear complex structure with the same generalized eigenspaces as A, the one to which we deformed A in Proposition 2.

Extend J_A to a complex structure on $\mathbb{R}^{2n+2} = P + P^{\perp}$ which rotates the oriented 2–plane P within itself by $90°$.

This complex structure J_A on $\mathbb{R}^{2n+2}$ determines a Hopf-like fibration H_{J_A} of S^{2n+1} by the oriented unit circles on the J_A–complex lines.

The graph of $J_A\colon P^{\perp} \to P^{\perp}$ is a $2n$–plane in $\mathrm{Hom}(P, P^{\perp})$, which can be regarded as part of the base space of this fibration H_{J_A}, and also as its tangent space at P.

We will interpolate between the graphs of A and J_A, using the fact that they have the same generalized eigenspaces, to construct the base space M_F of a fibration F

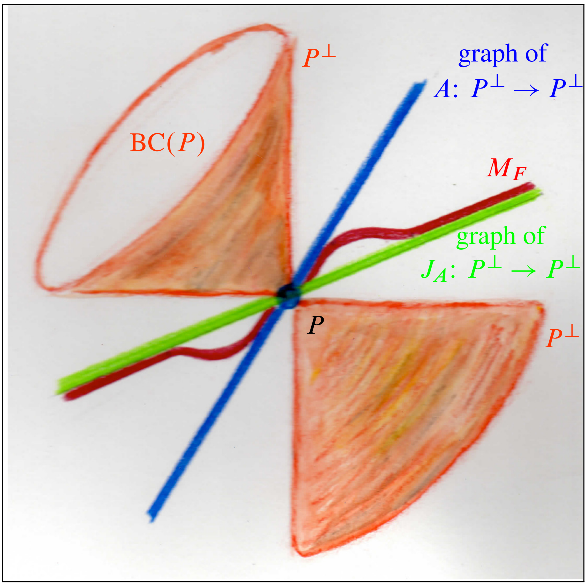

Figure 12: Interpolating between the graph of A and the graph of the corresponding linear complex structure J_A

of S^{2n+1} by great circles which is tangent at P to the graph of A, and which agrees with the fibration H_{J_A} outside a small neighborhood of P.

The details of the interpolation are given in the full proof, which we begin now.

Proof of Proposition 4. Recall that the $4n$–dimensional vector space $\mathrm{Hom}(P, P^\perp)$ serves both as a coordinate neighborhood about P in $G_2\mathbb{R}^{2n+2}$, and as the tangent space to this Grassmannian at P.

We start with a $2n$–dimensional subspace of $\mathrm{Hom}(P, P^\perp)$, which is the graph of a linear transformation $A\colon P^\perp \to P^\perp$ with no real eigenvalues. Our goal is to construct a smooth fibration F of S^{2n+1} by oriented great circles, whose base space M_F can be viewed within this neighborhood as the graph of the smooth nonlinear function $N\colon P^\perp \to P^\perp$, defined by

$$N(x) = f(|x|)A(x) + (1 - f(|x|))J(x)$$

for all $x \in P^\perp$; see Figure 12. Here, $f\colon [0,\infty) \to [0,1]$ is a smooth bump function which will be defined shortly, and $J = J_A$ is the linear complex structure corresponding to A which was defined in the proof of Proposition 2.

Our task is to choose f so that the differential dN_x of N at each point $x \in P^\perp$ has no real eigenvalues.

We compute dN_x applied to a vector v in $P^\perp$, keeping in mind that the linear functions A and J serve as their own differentials at all points x:

$$\begin{aligned} dN_x(v) = f(|x|)A(v) + (1 - f(|x|))J(v) \\ + f'(|x|)\Big(\frac{x}{|x|}\cdot v\Big)A(x) - f'(|x|)\Big(\frac{x}{|x|}\cdot v\Big)J(x). \end{aligned}$$

Suppose that $dN_x(v) = \lambda v$ at some point $x \in P^\perp$, for some unit vector v, and for some real number λ.

We will insert this into the previous equation, and then choose the bump function f to prevent this from happening at any point x and for any λ.

We get

$$\lambda v = f(|x|)A(v) + (1 - f(|x|))J(v) + f'(|x|)\Big(\frac{x}{|x|}\cdot v\Big)A(x) - f'(|x|)\Big(\frac{x}{|x|}\cdot v\Big)J(x),$$

and rewrite this as

$$(\star\star)\qquad \lambda v - \big[f(|x|)A + (1 - f(|x|))J\big](v) = f'(|x|)\Big(\frac{x}{|x|}\cdot v\Big)[A(x) - J(x)].$$

Next we will find an $\epsilon > 0$ so that the left-hand side of $(\star\star)$ has norm $\geq \epsilon$, independent of the bump function f and the point $x \in P^\perp$. Then we will choose f so that the right-hand side has norm $< \epsilon$.

Suppose first that we cannot find a positive lower bound for the norm of the left-hand side.

The left-hand side cannot be zero at any $x \in P^\perp$, since the linear maps $tA + (1-t)J$ from $P^\perp$ to $P^\perp$ have no real eigenvalues for $0 \leq t \leq 1$, as we showed in the proof of Proposition 2.

Now suppose that as we vary $x \in P^\perp$ among those x for which dN_x has a real eigenvalue, the norm of the left-hand side of $(\star\star)$ becomes arbitrarily close to zero. Note that as we vary x, the eigenvalue λ of dN_x, if it exists, might change.

So we suppose for each integer n there is a real number λ_n, a unit vector v_n, and a real number $t_n \in [0,1]$ such that

$$\big|\lambda_n v_n - [t_n A + (1 - t_n)J](v_n)\big| < \frac{1}{n}.$$

We note that the real numbers λ_n are bounded in size, since

$$|tA + (1-t)J| \leq |A| + |J|$$

is bounded and v_n is a unit vector.

Then, due to the compactness of this bounded interval of real numbers, compactness of the unit $2n-1$–sphere in $P^{\perp}$, and compactness of the interval $[0,1]$, there is a subsequence (n_k) of the integers with

$$\lambda_{n_k} \to \lambda, \quad v_{n_k} \to v \quad \text{and} \quad t_{n_k} \to t,$$

so that in the limit we have

$$\lambda v - [tA + (1-t)J](v) = 0,$$

which contradicts the fact that $tA + (1-t)J$ has no real eigenvalues.

Thus, independent of our choice of f (yet to be made), there is an $\epsilon > 0$ such that

$$\left|\lambda v - \left[f(|x|)A + (1 - f(|x|))J\right](v)\right| \geq \epsilon.$$

We fix this $\epsilon > 0$ and consider the right-hand side of $(\star\star)$,

$$f'(|x|)\left(\frac{x}{|x|} \cdot v\right)[A(x) - J(x)],$$

which has norm $\leq |f'(|x|)||A - J||x|$.

We will determine how to choose f so that

$$|f'(s)|s < \frac{\epsilon}{|A-J|}$$

for any real number s in $[0,\infty)$.

Let $S(f) = \sup\{s|f'(s)| : s \geq 0\}$. We want to choose the bump function f so that $S(f) \leq \epsilon/|A - J|$, thus making $S(f)$ as small as necessary.

Start by choosing any smooth bump function $f\colon [0,\infty) \to [0,1]$ so that $f(s) = 1$ for s near 0 and $f(s) = 0$ for s sufficiently large.

Then define $f_n(s) = f(s^{1/n})$ for $n = 1, 2, 3, \ldots$.

A quick check shows that $S(f_n) = S(f)/n$, hence, for sufficiently large n, the bump function f_n can be used in place of f, so that the right-hand side of $(\star\star)$ has norm $< \epsilon$.

This contradicts our supposition that $dN_x(v) = \lambda v$ at some point $x \in P^\perp$, for some unit vector v and for some real number λ, and therefore confirms that the differential dN_x of N at each point $x \in P^\perp$ has no real eigenvalues.

Now we want to define the fibration F of S^{2n+1} by oriented great circles so that its base space M_F within the coordinate neighborhood $\mathrm{Hom}(P, P^\perp)$ is the graph of N, and outside that neighborhood coincides with the base space M_J of the fibration of S^{2n+1} by the unit circles on the J–complex lines.

Since the differential dN_x at each $x \in P^\perp$ has no real eigenvalues, the base space M_F is everywhere transverse to the field of bad cones, and so, by Proposition 1, is indeed the base space of a smooth fibration F of S^{2n+1} by oriented great circles.

By construction, M_F agrees with the graph of A near the fiber P, so that we certainly have $T_P M_F = A$, as required.

This completes the proof of Proposition 4. □

8 Proof of Theorem B

Theorem B *The space $\{T_P M_F\}$ of tangent $2n$–planes at P to the base spaces M_F of all smooth oriented great circle fibrations F of S^{2n+1} containing P, deformation retracts to its subspace $\{T_P M_H\}$ of tangent $2n$–planes to such Hopf fibrations H of S^{2n+1}.*

That is, the set of $2n$–planes in $T_P G_2\mathbb{R}^{2n+2}$ tangent to the base space of a fibration of S^{2n+1} by great circles deformation retracts to its subspace of $2n$–planes tangent to Hopf fibrations.

Proof Start with the space $\{T_P M_F\}$ of tangent $2n$–planes at P to the base spaces M_F of all smooth great circle fibrations F of S^{2n+1}.

Use Propositions 1 and 4 to write

$$\begin{aligned}\{T_P M_F\} &= \{2n\text{–planes in } T_P(G_2\mathbb{R}^{2n+2}) \text{ transverse to } \mathrm{BC}(P)\}\\ &= \{\text{linear maps } T\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n} \text{ with no real eigenvalues}\},\end{aligned}$$

with $P^\perp$ playing the role of $\mathbb{R}^{2n}$.

Then by Propositions 2 and 3, the above space deformation retracts to its subspace

$$\{\text{orthogonal complex structures } J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}\},$$

which is in one-to-one correspondence with the space $\{T_P M_H\}$ of tangent $2n$–planes at P to the base spaces M_H of Hopf fibrations H of S^{2n+1} containing the fiber P by the second lemma in Section 5.

This proves Theorem B. $\square$

9 Proof of Theorem A

Theorem A *Every germ of a smooth fibration of S^{2n+1} by oriented great circles extends to such a fibration of all of S^{2n+1}.*

Proof Let F be a germ of a smooth fibration of S^{2n+1} by great circles containing the fiber P, and $M_F \subset G_2\mathbb{R}^{2n+2}$ its base space.

We must produce a smooth fibration F'' of all of S^{2n+1} by great circles which agrees with F in a neighborhood of P.

Let $T_P M_F$ be the tangent $2n$–plane to M_F at P.

We know that $T_P M_F$ is transverse to $\mathrm{BC}(P)$, so by Proposition 4, there is a smooth fibration F' of all of S^{2n+1} by great circles with $T_P M_{F'} = T_P M_F$.

By routine interpolation, we get a smooth submanifold M'' of $G_2\mathbb{R}^{2n+2}$ which agrees with M_F in a small neighborhood of P, and then agrees with M'_F outside a slightly larger neighborhood of P, and whose tangent planes are all as close as desired to $T_P M_{F'} = T_P M_F$. See Figure 13.

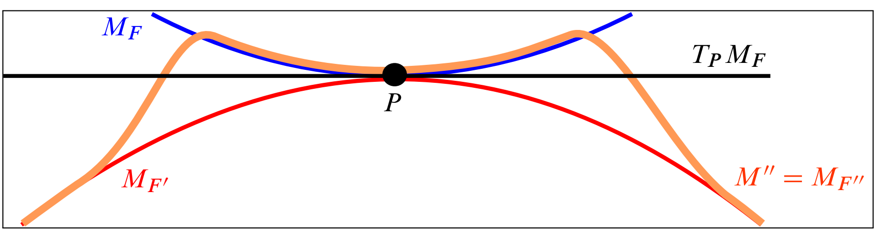

Figure 13: Interpolation between the base space M_F of the germ and the base space $M_{F'}$ of an entire fibration which is tangent to the germ

Thanks to this closeness, the tangent planes to M'' are transverse to the bad cones at all points, and hence $M'' = M_{F''}$ is the base space of a fibration F'' of *all* of S^{2n+1} by great circles. This fibration F'' agrees with F in a neighborhood of P, completing the proof of Theorem A. $\square$

10 Why is Theorem A easy to prove in dimension 3?

There are two special features in this lowest dimension:

(1) The Grassmann manifold $G_2\mathbb{R}^4$ of oriented 2–planes through the origin in $\mathbb{R}^4$ is isometric (up to scale) to $S^2 \times S^2$.

(2) There is a moduli space for the family of all fibrations of S^3 by oriented great circles: two copies of the space of distance-decreasing maps of $S^2 \to S^2$; see Gluck and Warner [7].

In particular, the base space M_F of a fibration F of S^3 by oriented great circles appears in the $S^2 \times S^2$ structure of $G_2\mathbb{R}^4$ as the graph of a distance-decreasing map from either S^2 factor to the other.

Then in this lowest dimension, Theorem A says that every germ G of a (smooth) fibration of S^3 by oriented great circles extends to a fibration F of all of S^3, and the proof is contained in Figure 14.

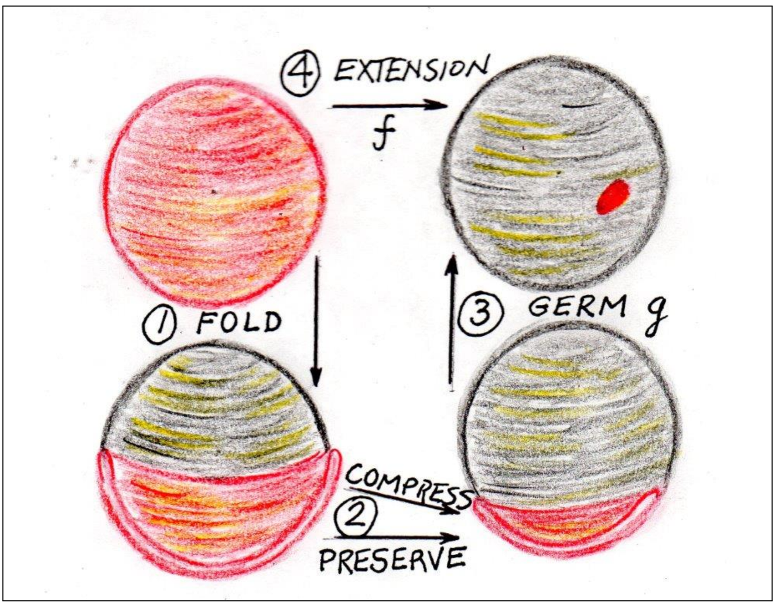

Figure 14: Proof of Theorem A in dimension 3

Let the germ G correspond to a distance-decreasing map g from a neighborhood N of the south pole on one S^2 factor of $G_2\mathbb{R}^4$ to the other S^2 factor.

Then a distance-decreasing map $f\colon S^2 \to S^2$ which extends g is constructed by folding S^2 in half, so that the northern hemisphere goes to the southern hemisphere, then compressing the southern hemisphere into the neighborhood N of the south pole (with no compression near the south pole), and finally composing this fold-compression with the given distance-decreasing map g.

The graph of the resulting distance-decreasing map $f\colon S^2 \to S^2$ inside $S^2 \times S^2 = G_2\mathbb{R}^4$ is the base space M_F of the desired fibration F of S^3 by oriented great circles which agrees with G in a neighborhood of a fiber.

The best unsolved problem for great circle fibrations of spheres

Prove that the space of all (smooth) fibrations of S^{2n+1} by great circles deformation retracts to its subspace of Hopf fibrations.

This was proved for S^3 in Gluck and Warner [7]. Theorem B of the present paper can be regarded as an infinitesimal version of this desired theorem.

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