Algebraic & Geometric Topology 18 (2018) 1259–1279

Combinatorial spin structures on triangulated manifolds

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We give a combinatorial description of spin and spin^c–structures on triangulated manifolds of arbitrary dimension. These encodings of spin and spin^c–structures are established primarily for the purpose of aiding in computations. The novelty of the approach is that we rely heavily on the naturality of binary symmetric groups to avoid lengthy explicit constructions of smoothings of PL manifolds.

57R15; 55S35, 57R05

1 Introduction

In this paper a framework for combinatorially representing spin and spin^c–structures on triangulated manifolds in a manner suitable for computer implementation is built. This should be seen as part of a general effort to merge the techniques of algorithmic 3–manifold theory, such as triangulations, normal surface theory and geometrization, with elements of 4–manifold theory, where gauge-theoretic invariants often require additional structures.

The governing perspective on spin and spin^c–structures in this paper comes from the obstruction-theoretic approach to spin structures of Milnor [16]. Although Milnor's approach is fundamentally combinatorial in nature, there is some nontrivial work to translate Milnor's language into a language a modern computer can use. To this end, we put combinatorial spin structures in a formalism perhaps most comparable to Forman's discrete Morse theory [8]. It is assumed the reader is familiar with obstruction theory on manifolds along the lines of Milnor and Stasheff [17]. Other references like Whitehead [21] and Gompf and Stipicz [11] are also excellent resources for basic obstruction theory.

Relatively flexible triangulations are allowed in this article. For example, *unordered delta complexes* (see Hatcher [12]) suffice. The *ideal triangulations* of Thurston [19], a further weakening of unordered delta complexes, are also perfectly acceptable. Ideal triangulations are unordered delta complexes, such that if one removes a finite collection of vertices, one obtains a manifold. In short, a *triangulation* in this paper is a

Published: 3 April 2018 DOI: 10.2140/agt.2018.18.1259

space constructed by gluing simplices together via affine-linear identifications of their boundary facets, and where we demand that the characteristic maps of every simplex is an embedding when restricted to the interior of the simplex.

Readers comfortable with the basics of triangulations, spin structures and obstruction theory can jump to Section 4 for the primary constructions of this paper. In the literature, there are several available tools for combinatorially representing 3– and 4–manifolds with additional structure on their tangent bundles. The Kaplan algorithm [14] was perhaps the first (see [11, Sections 5.6–5.7] for a modern exposition). Kaplan's algorithm gives a simple framework to represent spin structures on a 3–manifold given by an integral surgery presentation, and provides a simple tool to determine when such spin structures extend over the bounding 4–manifold. Another combinatorial representation of 3–manifolds comes from *spines*, popularized by Matveev [15]. Techniques to represent spin structures on 3–manifold and 4–manifold spines were developed by Benedetti and Petronio [1; 2; 3; 4]. The techniques in this paper would be described as being in the language of the "frame along the dual 1–skeleton" in [4]. Spin^c–structures on simplicially triangulated 3–manifolds can be described as the combinatorial Euler structures of Turaev [20]. Étienne Gallais [9] has recently used this technique to study combinatorial Euler structures on triangulated 3–manifolds using Forman's combinatorial vector fields to represent Euler structures. One of Gallais's observations is that with these techniques, not all combinatorial Euler structures are represented on delta complexes. Simplicial triangulations are required to capture *all* spin^c–structures using this technique. We wish to avoid simplicial triangulations, as unordered delta complexes have shown themselves to be rather efficient means for describing interesting manifold types in both 3–manifold theory (see Burton, Budney and Pettersson [7] and Thurston [19]) and 4–manifold theory (see Budney and Hillman [6] and Budney, Burton and Hillman [5]).

2 Notation, obstruction theory

Throughout this paper, N will be a PL n–manifold that will be endowed with a triangulation or a CW–structure, often both. If the cell structure is unambiguous, the i–skeleton will be denoted by N^i.

Given a fibre bundle $\psi\colon E \to B$ with fibre F, and a subspace $X \subset B$, the *restriction bundle* is the map $\psi|_{\psi^{-1}(X)}\colon \psi^{-1}(X) \to X$ which also has fibre F. We abbreviate $\psi|_{\psi^{-1}(X)}$ by $\psi|_X$.

A *trivialization* of a vector bundle $\psi\colon E \to B$ is an ordered k–tuple of vector fields that forms a basis for each and every fibre. Trivializations correspond to vector bundle isomorphisms $B \times \mathbb{R}^k \to E$ via the map $(b, x_1, \dots, x_k) \mapsto \sum_{i=1}^k x_i \vec{v}_i(b)$, where $(\vec{v}_i\colon B \to E)_{i \in \{1,2,\dots,k\}}$ is the trivialisation.

A vector bundle $\psi\colon E \to N$ is *orientable* if and only if there is a trivialization of $\psi|_{N^1}$. Given a trivialization of $\psi|_{N^1}$, the homotopy class of its restriction to N^0 is called an *orientation* of ψ. If a vector bundle $\psi\colon E \to N$ is orientable, its set of orientations admits a free transitive action of $H^0(N, \mathbb{Z}_2)$ — the action is given by flipping orientations on path components of N.

In the language of classifying maps, a vector bundle $\psi\colon E \to N$ is *orientable* if and only if its classifying map $N \to \mathrm{Gr}_{\infty,k} \equiv B\mathrm{O}_k$ lifts to the Grassmannian $\mathrm{Gr}^+_{\infty,k} \equiv B\mathrm{SO}_k$ of oriented k–subspaces of $\mathbb{R}^\infty$:

$$\begin{tikzcd} & B\mathrm{SO}_k \arrow[d] \\ N \arrow[ur] \arrow[r] & B\mathrm{O}_k \end{tikzcd}$$

An *orientation* of N is the homotopy class of this lift. The fact that this is equivalent to the previous definition is described in [16; 17]. The key ingredient in this interpretation is that SO_k is the path component of the identity in O_k. This implies that the exact CW–structure on the space N is not relevant to the existence of orientations, which is one reason to prefer this formalism. If N is a smooth manifold, *orientability and orientations of* N refer to orientability and orientations of the tangent bundle $\pi\colon TN \to N$.

We denote the n^{th} *spin group* by Spin_n. This is defined as is the unique connected Lie group which admits an onto 2-to-1 Lie group homomorphism $\mathrm{Spin}_n \to \mathrm{SO}_n$. Since $\pi_1 \mathrm{SO}_n$ is cyclic of order 2 or infinite cyclic, this is well defined. A vector bundle $\psi\colon E \to N$ admits a spin structure if the classifying map $N \to B\mathrm{O}_k$ admits a lift $N \to B\mathrm{Spin}_k$. A spin structure is a homotopy class of a map $N \to B\mathrm{Spin}_k$ such that the composite with $B\mathrm{Spin}_k \to B\mathrm{O}_k$ is a classifying map for the bundle ψ. Since the homomorphism $\mathrm{Spin}_k \to \mathrm{O}_k$ factors as a composition $\mathrm{Spin}_k \to \mathrm{SO}_k \to \mathrm{O}_k$, spin structures induce orientations:

$$\begin{tikzcd} & B\mathrm{Spin}_k \arrow[d] \\ N \arrow[ur] \arrow[r] \arrow[dr] & B\mathrm{SO}_k \arrow[d] \\ & B\mathrm{O}_k \end{tikzcd}$$

Since $\pi_1 \mathrm{SO}_k \simeq \mathbb{Z}_2$ for $k \geq 3$, the corresponding description for spin structures in the obstruction-theoretic setting is that $\psi\colon E \to N$ *admits a spin structure* if and only if there exists a trivialization of $\psi|_{N^2}$. Given such a trivialization, the homotopy class of its restriction to N^1 is a *spin structure*. The case $k = 2$ is special since $\pi_1 \mathrm{SO}_2$ is infinite cyclic. Typically in the literature people phrase the obstruction-theoretic formulation as saying $\psi \oplus \epsilon^1$ admits a spin structure, where $\epsilon^1\colon N \times \mathbb{R} \to N$ is the trivial 1–dimensional bundle over N, but one could just as easily describe it in terms of trivializations of $\psi|_{N^1}$ such that the obstructions to extending over N^2 are all even.

The k^{th} *complex spin group* Spin^c_k is the group $(\mathrm{Spin}_k \times \mathrm{Spin}_2)/\mathbb{Z}_2 \equiv \mathrm{Spin}_k \times_{\mathbb{Z}_2} \mathrm{Spin}_2$. This means we are taking the product of the k^{th} spin group with the 2^{nd} spin group, and modding out by one copy of $\mathbb{Z}_2$ acting diagonally on the product via the covering action on the respective spin groups. Via projection to the right and left factor, respectively, this group admits two extensions: $\mathrm{Spin}_k \to \mathrm{Spin}^c_k \to \mathrm{SO}_2$ and $\mathrm{Spin}_2 \to \mathrm{Spin}^c_k \to \mathrm{SO}_k \equiv \mathrm{Spin}_k/\mathbb{Z}_2$. The latter extension is used to define Spin^c–structures, and the former gives the inclusion $\mathrm{Spin}_k \to \mathrm{Spin}^c_k$:

$$\begin{tikzcd}
 & B\mathrm{Spin}_k \arrow[d] \\
 & B\mathrm{Spin}^c_k \arrow[d] \\
N \arrow[ruu] \arrow[ru] \arrow[r] \arrow[rd] & B\mathrm{SO}_k \arrow[d] \\
 & B\mathrm{O}_k
\end{tikzcd}$$

A vector bundle $\psi\colon E \to N$ admits a spin^c–structure if the classifying map $N \to B\mathrm{O}_k$ admits a lift to $B\mathrm{Spin}^c_k$ [10]. A spin^c*–structure* is a homotopy class of a map $N \to B\mathrm{Spin}^c_k$ such that the composition with $B\mathrm{Spin}^c_k \to \mathrm{O}_k$ classifies the bundle ψ. To interpret a spin^c–structure, notice that if one composes with the former extension, one gets a map $N \to B\mathrm{SO}_2$ which classifies an oriented 2–dimensional vector bundle over N. Alternatively this is a 1–dimensional $\mathbb{C}$–bundle over N. If $\nu\colon E' \to N$ is the 1–dimensional $\mathbb{C}$–bundle over N classified by this map, then $\psi \oplus \nu\colon E \oplus E' \to N$ is classified by the corresponding map $N \to B\mathrm{SO}_k \times B\mathrm{SO}_2 \equiv B(\mathrm{SO}_k \times \mathrm{SO}_2)$. Consider $\mathrm{SO}_k \times \mathrm{SO}_2$ as a subgroup of SO_{k+2}. This group is covered by some subgroup of Spin_{k+2}, and by design this group is isomorphic to $\mathrm{Spin}_k \times_{\mathbb{Z}_2} \mathrm{Spin}_2$. Thus a Spin^c–structure on a bundle $\psi\colon E \to N$ consists of two things: a complex line bundle

$\nu\colon E' \to N$ and a spin structure on $\psi \oplus \nu$. Given this, spin^c–structures can be readily transcribed into an obstruction-theoretic formalism. A complex line bundle is classified by a map $N \to BSO_2 \equiv K(\mathbb{Z}, 2)$ and homotopy classes of maps $N \to K(\mathbb{Z}, 2)$ are in bijective correspondence with elements of $H^2(N, \mathbb{Z})$. Thus a spin^c–structure on N is prescribed by such a cohomology class, together with a homotopy class of a trivialization of $(\psi \oplus \nu)|_{N^1}$ which extends to N^2.

When working with a triangulation T of a manifold N, we will make heavy use of the *dual polyhedral decomposition*. This construction originated in the work of Poincaré, and is available in [18]. Since these ideas are no longer in wide circulation and we need some fixed notation to refer to this decomposition, a brief sketch is given. Denote the standard n–simplex by

$$\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ for all } i \text{ and } x_0 + x_1 + \dots + x_n = 1\}.$$

For $i \in \{0, 1, \dots, n\}$ the i^{th} face map of Δ^n is $f_i\colon \Delta^{n-1} \to \Delta^n$ given by

$$f_i(x_0, \dots, x_{n-1}) = (x_0, x_1, \dots, x_{i-1}, 0, x_i, x_{i+1}, \dots, x_{n-1}).$$

Given a permutation $\sigma \in \Sigma_{n+1} \equiv \Sigma(\{0, 1, \dots, n\})$, the induced automorphism of Δ^n is the map $\sigma_*\colon \Delta^n \to \Delta^n$ given by $\sigma_*(x_0, x_1, \dots, x_n) = (x_{\sigma^{-1}(0)}, x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$. An *unordered delta complex* is a CW–complex X such that the domains of the attaching maps are the boundaries of simplices (rather than discs), $\phi\colon \partial\Delta^n \to X^{(n-1)}$, and for each i, the composite satisfies $\phi \circ f_i = \Phi \circ \sigma_*$, where $\Phi\colon \Delta^{n-1} \to X^{(n-1)}$ is the characteristic map of some $(n-1)$–simplex and $\sigma \in \Sigma_n$ is some permutation. If all the permutations σ were the identity, X would be an *ordered delta complex*.

Let $[0, n] = \{0, 1, \dots, n\}$, and let I denote a subset of $[0, n]$. The *dual polyhedral bit* δ_I of Δ^n is the convex hull of the barycentres of all faces of Δ^n with vertex set a superset of I. Thus, $\delta_{[0,n]}$ is the barycentre of Δ^n and $\delta_{[0,n]\setminus\{i\}}$ is the convex hull of the barycentre of Δ^n together with the barycentre of the i^{th} face of Δ^n. One can define δ_I via a system of equations, or

$$\delta_I = \{(x_0, x_1, \dots, x_n) \in \Delta^n : x_i \geq x_j \text{ for all } i \in I \text{ and } j \in [0, n]\}.$$

If T is a triangulation of a manifold N and $\chi\colon \Delta^n \to N$ the characteristic map of a simplex, $\chi(\delta_I)$ is defined to be a *dual polyhedral bit of the triangulation* T. Given an i–dimensional simplex σ of T, *the closed dual* $(n-i)$*–cell* corresponding to σ is the union of all $(n-i)$–dimensional dual polyhedral bits corresponding to σ in all the top-dimensional simplices containing σ. The collection of all dual cells forms

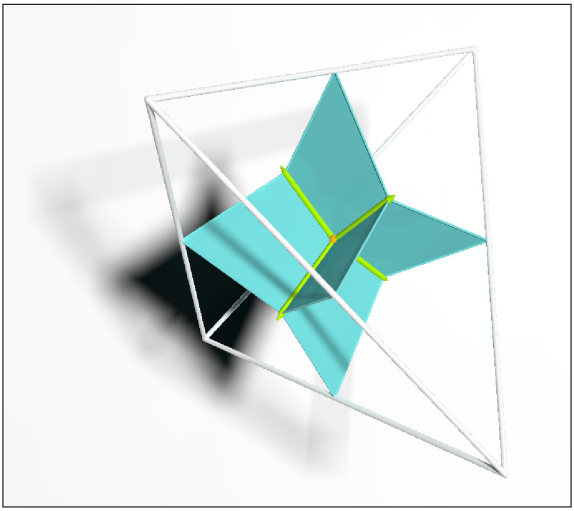

Figure 1: Dual polyhedral bits inside a tetrahedron Δ^3

a CW–decomposition of N, called *the polyhedral decomposition of N dual to T*. We denote this dual polyhedral decomposition by P throughout the paper. Given a triangulation T or CW–complex P, we denote the set of k–cells by T_k and P_k, respectively, while the k–skeleton we continue to denote by T^k and P^k, respectively. The key feature of the dual decomposition is that for every i–simplex $\sigma \in T_i$ there is one and only one dual $(n-i)$–cell $e^{n-i} \in P_{n-i}$ with $\sigma \cap e^{n-i} \neq \varnothing$. The nonempty intersection is the barycentre of σ.

Generally speaking, if N is a triangulated PL manifold, the tangent bundle TN is *not defined*; moreover, it is frequently not unique when it *is* defined [13]. Thankfully, nonsmoothable PL structures and distinct smoothings of PL structures do not appear below dimension 7. Thus the regular neighbourhoods of the dual 2–skeleton of a triangulated PL manifold *do* have unique smoothings as the links of faces of codimension 1 and 2 are 0–spheres and 1–spheres, respectively, which have unique smooth structures; see [13]. In particular, $TN|_{P^2}$ can be referred to without ambiguity and we can discuss spin structures on PL manifolds.

3 Geometry of simplices

This section describes some group-theoretic preliminaries related to the geometry of simplices. Let $\mathrm{Sym}(X)$ be the full group of isometries of an object X and let

$\mathrm{Sym}^+(X) \subset \mathrm{Sym}(X)$ be the orientation-preserving subgroup, provided these concepts make sense. Let $\mathcal{D} = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0 = x_1 = \cdots = x_n\}$ be the "thin" diagonal and let $\mathcal{A} = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0 + x_1 + \cdots + x_n = 0\}$ be the antidiagonal.

Symmetries of Δ^n are determined by how they permute the vertices, thus there is an identification $\mathrm{Sym}(\Delta^n) \equiv \Sigma_{n+1}$ and $\mathrm{Sym}^+(\Delta^n) \equiv A_{n+1}$. If we translate Δ^n to the origin

$$\Delta_0^n = \left\{(x_0, \ldots, x_n) \in \mathcal{A} \,\middle|\, x_i \geq \frac{-1}{n+1} \text{ for all } i\right\},$$

a linear extension gives an embedding $\mathrm{Sym}(\Delta_0^n) \to \mathrm{SO}_{n+1}$. The set $\mathcal{D}$ is an eigenspace relative to an eigenvalue $+1$ when the symmetry preserves orientation, and an eigenspace relative to an eigenvalue -1 when it reverses the orientation of Δ_0^n.

We will now examine the relations between the symmetric group and the *group of motions* of an n–simplex. Let $\mathrm{Emb}(\Delta^n, \mathbb{R}^{n+1})$ be the space of affine-linear embeddings of the n–simplex in $(n+1)$–dimensional Euclidean space. The space $\mathrm{Emb}(\Delta^n, \mathbb{R}^{n+1})$ has the homotopy type of a Stiefel manifold — the displacement vectors from one vertex to the remaining vertices give such a map. This Stiefel manifold in turn has the homotopy type of SO_{n+1} by Gram–Schmidt.

The group Σ_{n+1} acts freely on the right on $\mathrm{Emb}(\Delta^n, \mathbb{R}^{n+1})$ by relabelling the vertices of the simplex. The group Σ_{n+1} also acts on the left on $\mathrm{Emb}(\Delta^n, \mathbb{R}^{n+1})$ by relabelling the coordinate axes of $\mathbb{R}^{n+1}$ but we will not need this action. The *motion group of the n–simplex* is defined to be $\pi_1(\mathrm{Emb}(\Delta^n, \mathbb{R}^{n+1})/\Sigma_{n+1})$. Since $n \geq 2$ is always assumed, the homotopy long exact sequence of the bundle

$$\Sigma_{n+1} \to \mathrm{Emb}(\Delta^n, \mathbb{R}^{n+1}) \to \mathrm{Emb}(\Delta^n, \mathbb{R}^{n+1})/\Sigma_{n+1}$$

gives us the $\mathbb{Z}_2$–central extension

$$0 \to \mathbb{Z}_2 \to \pi_1(\mathrm{Emb}(\Delta^n, \mathbb{R}^{n+1})/\Sigma_n) \to \Sigma_{n+1} \to 0.$$

If G is a group and K an abelian group, it is a standard theorem of group cohomology that the central extensions of G with kernel K, taken up to extension-preserving isomorphism, are in bijective correspondence with $H^2(G, K)$. It turns out that $H^2(A_n, \mathbb{Z}_2)$ is a group of order two provided $n \geq 4$. Thus, there is only one nontrivial $\mathbb{Z}_2$–central extension of A_n. Schur called it the *double cover* of A_n, also called the *binary alternating group* and denoted by either $2A_n$ or $\widetilde{A}_n$. We use the latter notation. Schur also went on to show that $H^2(\Sigma_n, \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2^2$ for $n \geq 4$; moreover, the restriction map $H^2(\Sigma_n,\mathbb{Z}_2)\to H^2(A_n,\mathbb{Z}_2)$ is onto, thus there are two nonisomorphic $\mathbb{Z}_2$–central extensions of Σ_n which contain $\widetilde{A}_n$. We will give a geometric interpretation for one of these extensions. A convenient notation for elements in these extensions is given by Proposition 3.1.

Proposition 3.1 [22] *For all $n\geq 2$, there exist groups $\widetilde{\Sigma}_n^+$ and $\widetilde{\Sigma}_n^-$ which are $\mathbb{Z}_2$–central extensions of Σ_n such that:*

(1) *Given a k–tuple $(a_1,\dots,a_k)$ of distinct elements of $\{0,1,\dots,n\}$ there is an element $[a_1a_2\cdots a_k]\in\widetilde{\Sigma}^{\pm}_{n+1}$ called a k–cycle.*
(2) *The homomorphism $\widetilde{\Sigma}_n^{\pm}\to\Sigma_n$ sends $[a_1a_2\cdots a_k]$ to $(a_1a_2\cdots a_k)$ for all k–cycles.*
(3) *$[a_1a_2\cdots a_k]=[a_1a_2\cdots a_i][a_ia_{i+1}\cdots a_k]$ for all k and all $1<i<k$.*
(4) *If $\{a_1a_2\cdots a_k\}$ and $\{b_1b_2\cdots b_j\}$ are disjoint then $[a_1a_2\cdots a_k][b_1b_2\cdots b_j]=(-1)^{(k-1)(j-1)}[b_1b_2\cdots b_j][a_1a_2\cdots a_k]$.*
(5) *$[a_1a_2\cdots a_k]^{[b_1b_2\cdots b_j]}=(-1)^{(k-1)(j-1)}[\phi^{-1}(a_1)\phi^{-1}(a_2)\cdots\phi^{-1}(a_k)]$, where $\phi\in\Sigma_n$ is the cycle $(b_1b_2\cdots b_j)$.* (We use the notation $g^h=h^{-1}gh$ for conjugation.)
(6) *$[a_1a_2\cdots a_k]^k=\varepsilon$ for all $k\geq 2$ provided $[a_1a_2\cdots a_k]\in\Sigma^{\varepsilon}_{n+1}$.*

We call an element of $\widetilde{\Sigma}_n^{\pm}$ *odd* or *even* if its projection to Σ_n is odd or even, respectively.

Given that $\mathrm{Sym}(\Delta^n)\equiv\mathrm{Sym}(\Delta^n_0)\subset\mathrm{SO}_{n+1}$, there is a canonical lift of $\mathrm{Sym}(\Delta^n)$ along the 2-to-1 covering map $\mathrm{Spin}_{n+1}\to\mathrm{SO}_{n+1}$. We denote this 2-to-1 cover by $\widetilde{\mathrm{Sym}}(\Delta^n)\to\mathrm{Sym}(\Delta^n)$. It is a $\mathbb{Z}_2$–central extension, since the kernel of the map $\mathrm{Spin}_{n+1}\to\mathrm{SO}_{n+1}$ is central.

Proposition 3.2 *$\widetilde{\mathrm{Sym}}(\Delta^n)$ is canonically isomorphic to the motion group of Δ^n in $\mathbb{R}^{n+1}$, $\pi_1(\mathrm{Emb}(\Delta^n,\mathbb{R}^{n+1})/\Sigma_{n+1})$. It is also the $\mathbb{Z}_2$ central extension of Σ_{n+1}, denoted by $\widetilde{\Sigma}^-_{n+1}$. Under this isomorphism, $\widetilde{\mathrm{Sym}}^+(\Delta^n)$ corresponds to $\widetilde{A}_{n+1}$.*

Proof The isomorphism between the motion group $\pi_1(\mathrm{Emb}(\Delta^n,\mathbb{R}^{n+1})/\Sigma_{n+1})$ and the spin cover $\widetilde{\mathrm{Sym}}(\Delta^n)$ follows from the path-lifting property of the covering maps

$$\Sigma_{n+1}\to\mathrm{Emb}(\Delta^n,\mathbb{R}^{n+1})\to\mathrm{Emb}(\Delta^n,\mathbb{R}^{n+1})/\Sigma_{n+1}.$$

Given an element of $\pi_1(\mathrm{Emb}(\Delta^n,\mathbb{R}^{n+1})/\Sigma_{n+1})$, lift a representative to a path in $\mathrm{Emb}(\Delta^n,\mathbb{R}^{n+1})$ such that the endpoints differ by the action of Σ_{n+1}. For such a lift, the initial embedding starts at the standard embedding of the simplex Δ^n in $\mathbb{R}^{n+1}$. Such a path extends to a path of affine linear automorphisms of $\mathbb{R}^{n+1}$, starting at $\mathrm{Id}_{\mathbb{R}^{n+1}}$.

Using that O_{n+1} is a deformation retract of $\mathrm{GL}(\mathbb{R}^{n+1})$ we can homotope this path (rel endpoints) to a path in SO_{n+1}, which therefore lifts to a path in Spin_{n+1}, starting at the identity element. This describes the endpoint of the path as an element of $\widetilde{\mathrm{Sym}}(\Delta^n)$.

To verify that $\widetilde{\mathrm{Sym}}(\Delta^n)$ is isomorphic to $\widetilde{\Sigma}^-_{n+1}$, we will use the model Δ^n_0 for the n–simplex. This has the advantage that the symmetries of Δ^n_0 are linear. In this model, notice that either lift of the transposition $(a\ b)$ to Spin_{n+1} has order 4. This is because in SO_{n+1} the transposition $(a\ b)$ has a 2–dimensional (-1)–eigenspace whose orthogonal complement is fixed pointwise. The (-1)–eigenspace is spanned by the vectors $e_a - e_b$ and $\sum_{i=0}^n e_i$. Thus if we denote any lift of $(a\ b)$ by $[a\ b]$ then $[a\ b]^2 = -1$. This is a proof by reduction to a universal example, as it is a direct computation to identify the spin cover $\mathrm{Spin}_2 \to \mathrm{SO}_2$ with the map of the unit circle in the complex plane $S^1 \to S^1$ given by $z \mapsto z^2$. Relation (6) of Proposition 3.1 holds for $k = 2$, and therefore for all $k \geq 2$. □

Consider the subgroup of A_{n+1} which preserves the set $\{n-1, n\}$, ie its elements either fix $n-1$ and n pointwise or transpose them. Since an element of A_{n+1} is determined by its value on $n-1$ points, this subgroup is isomorphic to Σ_{n-1}. Thus, corresponding to a codimension-2 face of Δ^n there is an associated inclusion $\Sigma_{n-1} \to A_{n+1}$. The lift of this Σ_{n-1} to $\widetilde{A}_{n+1}$ is isomorphic to $\widetilde{\Sigma}^-_{n-1}$. Using the notation of Proposition 3.1 one can verify that an embedding $\widetilde{\Sigma}^-_{n-1} \to \widetilde{A}_{n+1}$ is given by

$$A \mapsto \begin{cases} A & \text{if } A \text{ is even,} \\ A[n-1\ n] & \text{if } A \text{ is odd.} \end{cases}$$

There are precisely two embeddings $\widetilde{\Sigma}^-_{n-1} \to \widetilde{A}_{n+1}$ which cover the standard inclusion $\Sigma_{n-1} \to A_{n+1}$, since $\{0, 1, 2, \ldots, n-2\} \subset \{0, 1, 2, \ldots, n\}$. These two inclusions are essentially the same, as they differ by a precomposition with an automorphism of $\widetilde{\Sigma}^-_{n-1}$ that fixes $\widetilde{A}_{n-1}$ pointwise. The automorphism is given by

$$\widetilde{\Sigma}^-_{n-1} \to \widetilde{\Sigma}^-_{n-1}, \quad \sigma \mapsto (-1)^{|\sigma|}\sigma,$$

where $|\sigma|$ is the parity of σ. This is the unique nontrivial automorphism of $\widetilde{\Sigma}^-_{n-1}$ that fixes $\widetilde{A}_{n-1}$ pointwise.

4 Representing spin structures on triangulated manifolds

As in Section 2, let P be the dual polyhedral decomposition to T, a triangulation of an n–manifold N. We remind that the k–skeleton of P is denoted by P^k while the set of k–cells is denoted by P_k.

This section gives a combinatorial technique to encode homotopy classes of sections of the Stiefel manifold of $(n-1)$–frames of N over the dual 1–skeleton P^1. It also gives a combinatorial technique to determine which of these sections extend over the 2–skeleton P^2. Once N is oriented, this is our formalism for encoding spin structures as an $(n-1)$–frame extends to an oriented n–frame uniquely up to homotopy.

To encode sections of the Stiefel bundle over P^1, we make the sections as close to *simplicial* as possible. That is:

(1) Evaluated at a point of $p \in P_0$, the vectors of each section should point towards some of the vertices of the top-dimensional simplex containing p.

(2) We demand that the vectors of each section, evaluated at the barycentre of each dual edge $e \in P_1$, point to some of the vertices of the codimension-1 facet $F \in T_{n-1}$ dual to e.

The total space of the Stiefel manifold of linearly independent $(n-1)$–frames over P^1 will be denoted by $V_{n-1}TN|_{P^1}$. The set of sections of the bundle $V_{n-1}TN|_{P^1} \to P^1$ is written $\Gamma_1 VN$, and the subspace satisfying (1) and (2) above is abbreviated by $\Gamma_1^{\mathbb{S}} VN$.

Every section in $\Gamma_1 VN$ is homotopic to one satisfying conditions (1) and (2). Given any section, perform the homotopy along the finite set $P^0 \cup (T^{n-1} \cap P^1)$, and extend to P^1 via the homotopy extension property. To do this we use only that the Stiefel space $V_{n-1}\mathbb{R}^n$ is connected, and that the "vertex-pointing" subset of $V_{n-1}\mathbb{R}^n$ is a nonempty set. Thus the inclusion $\pi_0\Gamma_1^{\mathbb{S}} VN \to \pi_0\Gamma_1 VN$ is onto.

We explain below how the set $\pi_0\Gamma_1^{\mathbb{S}} VN$ can be thought of as a subset of the product

$$\prod_{S \in T_n} A_{n+1} \times \prod_{F \in \bigsqcup_2 T_{n-1}} \widetilde{A}_{n+1} :$$

- The A_{n+1} corresponding to $S \in T_n$ factor encodes the section at the barycentre of a top-dimensional simplex S. The injections $\{0, 1, 2, \dots, n-2\} \to \{0, 1, 2, \dots, n\}$ have unique extensions to alternating bijections, so can be considered as elements of A_{n+1}.
- $\bigsqcup_2 T_{n-1}$ denotes two copies of T_{n-1}, one for each side of a dual edge $e \in P_1$ split at its barycentre. Our conditions (1) and (2) above fix the behaviour of the section at the endpoints of half of a dual edge e. Thus, corresponding to every dual edge there are two elements of $\widetilde{A}_{n+1}$ that determine the element of $\pi_0\Gamma_1^{\mathbb{S}} VN$. We use the characteristic map of $e \in P_1$ to determine which half of e is the "first half" and which is the "second half".

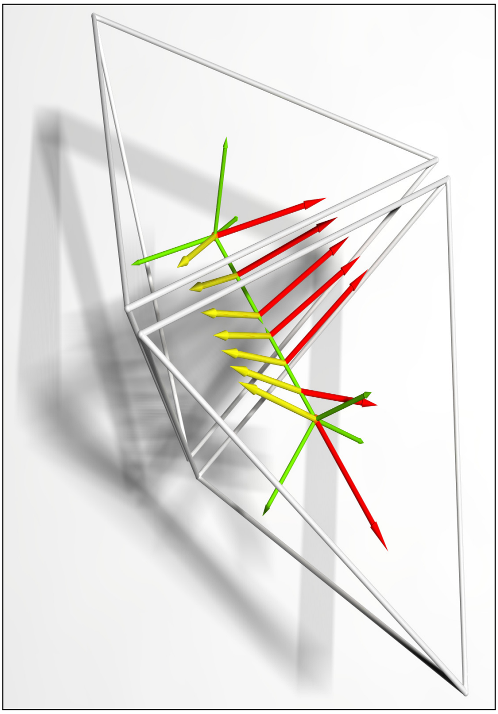

Figure 2: A section of $V_{n-1}(TN)|_{P^1}$ satisfying (1)–(2) with the dual 1–skeleton P^1 in green, $n = 3$

The condition that sections are "face-pointing" at the barycentre of a dual edge e gives a constraint — ie $\pi_0\Gamma_1^{\mathbb{S}}VN$ is a *proper subset* of the above product. We call this first constraint the *face constraint*. The continuity of our sections (at the barycentre of each edge $e \in P_1$) forces one further constraint, which we call the *continuity constraint*:

$$\pi_0\Gamma_1^{\mathbb{S}}VN \subsetneq \prod_{S\in T_n} A_{n+1} \times \prod_{F\in\bigsqcup_2 T_{n-1}} \widetilde{A}_{n+1}.$$

We call the set $\Gamma_1^{\mathbb{S}}VN$ the *simplicial sections* of the bundle $V_{n-1}(TN)|_{P^1}$. We express the face and continuity constraints as formulas involving the characteristic maps of the triangulation, below.

Given a codimension-1 face F incident to a top-dimensional simplex S, let

$$\chi_F\colon \Delta^{n-1} \to N \quad \text{and} \quad \chi_S\colon \Delta^n \to N$$

be the characteristic maps, respectively, and let $\iota \in \Sigma_{n+1}$ be the *characteristic inclusion* of F in S, ie $\chi_F = \chi_S \circ \iota$. If e is the edge dual to F in S, let $\beta \in \widetilde{A}_{n+1}$ be the motion corresponding to the half-edge $e \cap S$. The *face constraint* can be expressed as

$$\beta\big(\alpha(\{0,1,\dots,n-2\})\big) \subset \iota(\{0,1,\dots,n-1\}),$$

where $\alpha \in A_{n+1}$ represents the section at the barycentre of S. This is the formula that says directly that the vertices pointed to by the vectors fields at $e \cap F$ are the vertices of F. Equivalently, we could say the vector field does not point to the vertex opposite to the face, ie $\iota(n) \notin \beta\big(\alpha(\{0,1,\dots,n-2\})\big)$.

Given a codimension-1 face F incident to two top-dimensional simplices S_1 and S_2, let $\iota_1, \iota_2 \in \Sigma_{n+1}$ be the respective characteristic inclusions of F in S_1 and S_2, respectively. The *continuity constraint* can be expressed as $(\iota_2^{-1}\beta_2\iota_2)^{-1}\iota_1^{-1}\beta_1\iota_1 \in \langle[n-1,n]\rangle$, where $\langle[n-1,n]\rangle$ is the subgroup generated by $[n-1,n]$ in $\widetilde{A}_{n+1}$. This statement is equivalent to the statement that the two functions $\iota_k^{-1}\beta_k\iota_k$, as bijections of the set $\{0,1,2,\dots,n\}$, agree when restricted to the subset $\{0,1,2,\dots,n-2\}$. The group Σ_{n+1} acts naturally by conjugation on the group $\widetilde{\Sigma}^{-}_{n+1}$, and the symbol $\iota_k^{-1}\beta_k\iota_k$ indicates the right action of ι_k on β_k.

To encode the homotopy relation, we proceed by induction on the skeleton of P^1, ie first we perform the homotopy on the 0–skeleton P^0, and then we extend to P^1 using the homotopy extension property. Finally we perform the homotopy on the edges of P^1, *leaving the endpoints fixed*. The advantage of this perspective is that it allows us to see that the homotopy relation as the orbit space of a group action on the simplicial sections $\pi_0\Gamma_1^{\mathbb{S}}VN$.

The motions of an n–simplex Δ^n are given by $\widetilde{A}_{n+1}$ (see Section 3). We represent a simplicial section as an element $\big(\prod_S \alpha_S, \prod_F \beta_F\big) \in \prod_{S\in T_n} A_{n+1} \times \prod_{F\in\bigsqcup_2 T_{n-1}} \widetilde{A}_{n+1}$. Let $A \in \widetilde{A}_{n+1}$ correspond to a motion of the n–simplex S; then the result of applying the motion A to the simplicial sections at the barycentre of S, and extending to the entire simplicial section gives

$$A.\Big(\prod_{S'}\alpha_{S'}, \prod_F \beta_F\Big) = \Big(A\alpha_S \times \prod_{S'\neq S}\alpha_{S'}, \prod_{F\mid S}\beta_F A^{-1} \times \prod_{F\nmid S}\beta_F\Big),$$

where $F \mid S$ means "F is a boundary facet of S" or, equivalently, "S is incident to F". This is a group action of $\widetilde{A}_{n+1}$ on $\pi_0\Gamma_1^{\mathbb{S}}VN$. Moreover, observe that if S_1 and S_2 are distinct top-dimensional simplices of the triangulation T, then the two actions commute.

To complete the description of the homotopy relation on simplicial sections, we describe the result of a homotopy of the section on the interior of an edge (fixed on the complement of the edge's interior). In principle, the justification for the formula below is the same as above, ie the standard algebra of obstruction theory, but the formula is made more complicated due to a change-of-coordinates issue. We have chosen to store all our motion data in the coordinates of the ambient top-dimensional simplex where the motion occurs. But the edges of the dual cell complex P_1 *cross* from one top-dimensional simplex to another, across a face $F \in T_{n-1}$. The group of motions of F (fixing its barycentre) in the ambient triangulation is $\widetilde{\Sigma}_n^-$, so we must provide the formalism for converting from the motions of F to motions in the adjacent top-dimensional simplices. Given $A \in \widetilde{\Sigma}_n^-$ representing a motion of F in the ambient triangulation (fixing the barycentre), the result of performing that homotopy on an element of $\pi_0\Gamma_1^{\mathbb{S}} VN$ at the barycentre of the edge, fixing the section outside the edge, is given by

$$A.\Big(\prod_S \alpha_S, \prod_{F'} \beta_{F'}\Big) = \Big(\prod_S \alpha_S, \, \xi_{A,F} \times \prod_{F' \neq F} \beta_{F'}\Big),$$

where

$$\xi_{A,F} = \begin{cases} \iota A \iota^{-1} \beta_F & \text{if } A \text{ is even,} \\ \iota [n \ A(k)] A \iota^{-1} \beta_F & \text{if } A \text{ is odd.} \end{cases}$$

In the above formula, ι is the characteristic inclusion of F in S and k is the index of the vertex in F that is missed by the vector fields, ie $\{n, k\} = \iota^{-1}\beta_{F'}\alpha_S(\{n-1, n\})$.

There are a variety of ways to justify this formula; perhaps the most pragmatic is to consider the two n–dimensional simplices S_1 and S_2 incident to F as two faces of some abstract $(n+1)$–dimensional simplex $\mathcal{S}$ that is not part of the triangulation T, ie $S_i \subset \partial\mathcal{S}$ for $i = 1, 2$. If $\{0, 1, 2, \dots, n-1\}$ is the vertex set for F, let $\{0, 1, 2, \dots, n\}$ be the vertex set for S_1, $\{0, 1, 2, \dots, n-1, n+1\}$ the vertex set for S_2 and $\{0, 1, 2, \dots, n, n+1\}$ the vertex set for $\mathcal{S}$. In Section 3 we defined the inclusion $\widetilde{\Sigma}_n^- \to \widetilde{A}_{n+2}$ via the formula $A \mapsto [n+1 \ n]A$ provided A is an odd permutation. Notice that $[n+1 \ n]A$ maps $k \mapsto A(k)$ and maps $n+1 \mapsto n$. So if we postcompose $[n+1 \ n]A$ with the 3–cycle $[n \ n+1 \ A(k)]$ (which is the minimal motion in $\mathcal{S}$ returning S_i to its initial position or, stated another way, this motion applied to F projects into N as an embedding), we get

$$[n \ n+1 \ A(k)][n+1 \ n]A = [A(k) \ n][n \ n+1]^2 A = [n \ A(k)]A.$$

We can replace $[n \ A(k)]$ with $-[n \ A(k)] = [A(k) \ n]$ in the above formula, as it simply corresponds to the opposite embedding $\widetilde{\Sigma}_{n-1}^- \to \widetilde{A}_{n+1}$, which is just a convention for how lower-dimensional motions convert to higher-dimensional motions.

Proposition 4.1 *The maps defined above,*

$$\widetilde{A}_{n+1} \times \pi_0 \Gamma_1^{\mathbb{S}} VN \to \pi_0 \Gamma_1^{\mathbb{S}} VN$$

for every $S \in T_n$ and

$$\widetilde{\Sigma}_n^- \times \pi_0 \Gamma_1^{\mathbb{S}} VN \to \pi_0 \Gamma_1^{\mathbb{S}} VN$$

for every $F \in T_{n-1}$, are group actions; moreover, the actions commute. This gives us a group action

$$\left(\prod_S \widetilde{A}_{n+1} \times \prod_F \widetilde{\Sigma}_n^- \right) \times \pi_0 \Gamma_1^{\mathbb{S}} VN \to \pi_0 \Gamma_1^{\mathbb{S}} VN$$

whose stabilizers are isomorphic to $\mathbb{Z}_2^m$, where m is the number of path components of N. The orbits of this action correspond to $\pi_0 \Gamma_1 VN$ via the map $\pi_0 \Gamma_1^{\mathbb{S}} VN \to \pi_0 \Gamma_1 VN$; thus, if N is oriented, the orbits correspond canonically to spin structures.

Proof The maps for the top-dimensional simplices $\widetilde{A}_{n+1} \times \Gamma_1^S VN$ are group actions as they are essentially the canonical left and right actions of $\widetilde{A}_{n+1}$ on A_{n+1} and $\widetilde{A}_{n+1}$, respectively.

The maps involving $\widetilde{\Sigma}_n^-$ corresponding to the codimension-1 faces F require a more subtle argument. In the special case of even permutations, this is again the standard action of $\widetilde{A}_n$ on $\widetilde{A}_{n+1}$, after conjugation by ι.

Let us consider the case where $A \in \widetilde{A}_{n+1}$ can be an odd permutation. There are two nontrivial cases, $A_1.A_2.\beta$, where both A_1 and A_2 are odd, and the case A_1 odd and A_2 even.

(1) Let us first face the case where both A_1 and A_2 are odd. If we let k_2 satisfy $\{n, k_2\} = \iota^{-1}\beta\alpha\{n-1, n\}$ then we have

$$A_1.(A_2.\beta) = A_1.(\iota[n\ A_2 k_2] A_2 \iota^{-1} \beta),$$

then $\{n, k_1\} = \iota^{-1}\iota[n\ A_2k_2]A_2\iota^{-1}\beta\alpha\{n-1\ n\} = \{n\ A_2k_2\}$, giving

$$\begin{aligned} A_1.A_2.\beta &= \iota[n\ A_1k_1]A_1\iota^{-1}\iota[n\ A_2k_2]A_2\iota^{-1}\beta \\ &= -\iota A_1[n\ A_2k_2][n\ A_2k_2]A_2\iota^{-1}\beta = \iota A_1A_2\iota^{-1}\beta = (A_1A_2).\beta. \end{aligned}$$

(2) Now consider the case where A_1 is odd and A_2 is even. The argument is simpler:

$$\begin{aligned} A_1.(A_2.\beta) = A_1.(\iota A_2\iota^{-1}\beta) &= \iota[n\ A_1A_2k]A_1\iota^{-1}\iota A_2\iota^{-1}\beta \\ &= \iota[n\ A_1A_2k]A_1A_2\beta = (A_1A_2).\beta. \end{aligned}$$

We now establish that the kernel is $\mathbb{Z}_2^m$. If an element is stabilized under the action, the underlying section at P^0 and motions along P^1 are fixed. This forces the components of the $\prod_S \widetilde{A}_{n+1}$ factor to be ± 1. Similarly the components of the product $\prod_F \widetilde{\Sigma}_n^-$ are all ± 1. Thus we can think of the elements of the stabilizers as 0–dimensional mod-2 cocycles. Such objects correspond to $H^0(N, \mathbb{Z}_2)$, which is isomorphic to $\mathbb{Z}_2^m$. □

We turn our attention to spin structures — the issue of determining which elements of $\Gamma_1^S VN$ admit extensions to P^2. Let $W \in T_{n-2}$ be a codimension-2 simplex of T. Let $S_0, S_1, \dots, S_{m-1}$ be the circuit of n–simplices about W. Let $F_0, F_1, \dots, F_{m-1}$ be the corresponding circuit of codimension-1 simplices. We choose these coherently — the normal sphere to W in T is a triangulated circle and we index the S_i and the F_i in accord with that cyclic order.

Given an element of $\pi_0 \Gamma_1^{\mathbb{S}} VN$, we will set up a formula representing the obstruction to extending it over the 2–cell dual to W. Let β_{1i} and $\beta_{2i} \in \widetilde{A}_{n+1}$ be the motions of the simplex S_i as one travels from the barycentre of S_i to the barycentres of the faces F_{i-1} and F_i, respectively (the index i taken mod m). Our perspective will be to *cut* the normal circle to W in T and align the simplices $S_0, S_1, \dots, S_{m-1}$ as if they were parallel. We compute the motion of the vector fields as one traverses the circuit of simplices, in these parallelized coordinates. Let $w_i\colon \Delta^{n-2} \to \Delta^n$ be the characteristic inclusion corresponding to $W \hookrightarrow S_i$, extending uniquely to be an element $w_i \in A_{n+1}$ via the condition that $w_i(n-1)$ and $w_i(n)$ represent the vertices of an edge of the normal circle to W, with its cyclic orientation. The *parallelized total motion* in the simplex S_i (about W) will be denoted by S_i^w, and is defined as

$$S_i^w = \begin{cases} \beta_{2i}^{w_i} (\beta_{1i}^{w_i})^{-1} & \text{if } [0, n-2] \setminus w_i^{-1} \beta_{1i} \alpha_i = \varnothing, \\ [a\ n\ n{-}1] \beta_{2i}^{w_i} (\beta_{1i}^{w_i})^{-1} & \text{if } [0, n-2] \setminus w_i^{-1} \beta_{1i} \alpha_i = \{a\}. \end{cases}$$

Proposition 4.2 *An element of $\pi_0 \Gamma_1^{\mathbb{S}} VN$ extends over the 2–cell dual to $W \in T_{n-2}$ if and only if the product of the parallelized total motions is the nontrivial central element of $\widetilde{A}_{n+1}$, namely*

$$S_{m-1}^w S_{m-2}^w \cdots S_1^w S_0^w = -1.$$

The explanation for this formula is in a similar spirit to Proposition 4.1. Imagine two consecutive simplices S_i and S_{i+1} stuck together along their common face F_i, and imagine the simplices pulled apart so that they are parallel. The motion $\beta_{2i}^{w_i} (\beta_{1i}^{w_i})^{-1}$ is what one applies to the vector fields in S_i as one travels along P^1 from the face F_{i-1}

to F_i in the simplex S_i. Consider what additional motion we need to apply to these vector fields as we rotate S_{i+1} to be parallel to S_i. If our vector fields at F_i are pointing into W, we would be done because the motion that makes the simplex S_{i+1} parallel to S_i has no effect on the vector fields. This is the case when $[0,n-2]\setminus w_i^{-1}\beta_{1i}\alpha_i=\varnothing$. If the vector fields hit the vertex of F_i not in W, ie they miss a vertex of W, then $[0,n-2]\setminus w_i^{-1}\beta_{1i}\alpha_i=\{a\}$ and our motion to rotate S_{i+1} to be parallel with S_i affects the vector fields. The motion can be expressed as $[a\ n\ n-1]$ in the coordinates of F_i, hence the formulas for S_i^w. Our formulas for S_i^w with $i=0,1,2,\dots,m-1$ are now in a common "parallel" coordinate system and can be concatenated. We demand the product be -1, since the act of "closing" the parallel simplices contributes an extra 2π rotation into the product.

Thus, a *combinatorial spin structure* on a triangulated, oriented n–manifold N is an orbit of $\prod_S \widetilde{A}_{n+1}\times\prod_F \widetilde{\Sigma}_n^-$ acting on $\pi_0\Gamma_1^{\mathbb{S}}VN$ whose elements extend over all dual 2–cells $W\in P_2$.

Example 4.3 In Figure 3, the red arrows indicate the vector field over the 0–skeleton, given by α_i, as well as the vector field when pushed into the faces F_i. The blue arrows indicate our convention that our motions are specified as motions as one travels *from* the barycentres of top-dimensional simplices *to* the barycentres of the codimension-1 simplices F_i. We have chosen to embed the triangles in the plane so that $(0\ 1\ 2)$ represents a counterclockwise $\frac{2\pi}{3}$ rotation.

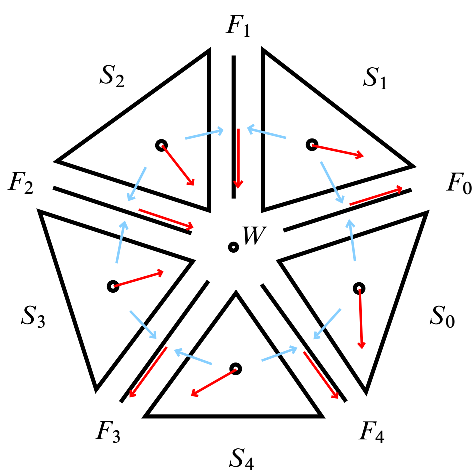

Figure 3: $S_0^w=\pm1$, $S_1^w=\pm[2\ 1\ 0]$, $S_2^w=\pm1$, $S_3^w=\pm[0\ 1\ 2]$, $S_4^w=\pm1$

In Example 4.3, we choose the "short" motions consistent with the figure, ie all nontrivial turns are either clockwise or counterclockwise by $\frac{2\pi}{3}$. We complete the computation with

$$S_0^w = 1, \quad S_1^w = [0\ 2\ 1], \quad S_2^w = 1, \quad S_3^w = -[0\ 1\ 2], \quad S_4^w = 1,$$

giving

$$S_4^w S_3^w S_2^w S_1^w S_0^w = -[0\ 1\ 2][0\ 2\ 1] = -1.$$

By Proposition 4.2 (or by visual inspection), the vector field extends over the 2–cell dual to W.

Although the parameter space for $\pi_0\Gamma_1^{\mathbb{S}}VN$ has order

$$|A_{n+1}|^{|T_n|}|\Sigma_n|^{|T_{n-1}|}4^{|T_{n-1}|},$$

one could implement this formalism by assuming the vector fields over P^0 are induced by the characteristic maps, and similarly for the vector fields on the barycentres of the dual edges $e \in P_1$. One can assume that on one half of each edge the vector fields chosen are given by some canonical path. In this setup, $\pi_0\Gamma_1^{\mathbb{S}}VN$ is a subset of a set parametrized by $|F|$ bits. This is analogous to orientations: orientation can be thought of as a plus or minus sign ($\pm$) associated to every top-dimensional simplex, satisfying a coherence condition. Spin structures are similarly parametrized by a $\pm$ sign on every dual edge $e \in P_1$ and satisfy an analogous coherence condition.

The first Stiefel–Whitney class $\omega_1 \in H^1(N, \mathbb{Z}_2)$ of a manifold N is the obstruction to orientability. From the perspective of triangulations, the 1–cocycle representing ω_1 is given by comparing the orientations of top-dimensional simplices adjacent across a face F. If they are oriented compatibly, meaning the transition function ϕ satisfies $\phi \in \Sigma_{n+1} \setminus A_{n+1}$, then $\omega_1(F) = 1$; otherwise, $\omega_1(F) = -1$.

There is a similar computation of ω_2, the second Stiefel–Whitney class. As a 2–cocycle, w_2 is computed by constructing $n-1$ everywhere-linearly independent sections on P^1. Its value on a 2–cell dual to $W \in T_{n-2}$ is precisely our extension obstruction $-S_{m-1}^w S_{m-2}^w \cdots S_1^w S_0^w$.

5 Combinatorial complex spin structures

As described in Section 2, a spin^c–structure on an n–manifold N consists of a homotopy class of a lift of the tangent bundle classifying map $N \to B\mathrm{O}_n$ to the Spin^c classifying space $B\mathrm{Spin}_n^c$. We take the perspective of Section 2 and consider a spin^c–structure on N as a 1–dimensional complex bundle over N — call it ν — together with a spin structure on the sum of the two bundles $TN \oplus \nu$. One-dimensional complex bundles over N are classified by maps $N \to B\mathrm{SO}_2$, which correspond precisely (via obstruction theory) to elements of $H^2(N,\mathbb{Z})$.

Let β be a cochain representing an element of $H^2(N,\mathbb{Z})$ and ν the complex line bundle associated to β, and consider the problem of finding a spin structure on $TN \oplus \nu$. Since $\pi_1\mathrm{SO}_3 \to \pi_1\mathrm{SO}_5$ is an isomorphism, we can demand that our trivialization of $TN \oplus \nu$ over P^1 is the direct sum of a trivialization of TN over P^1 with a fixed trivialization of ν over P^1. Since SO_2 is connected, ν is trivial when restricted to P^1.

Checking whether or not such a trivialization of $TN \oplus \nu$ over P^1 extends to a trivialization over P^2, we get the condition

$$S^w_{m-1} S^w_{m-2} \cdots S^w_1 S^w_0 = (-1)^{1+\beta(W)},$$

where $\beta(W)$ is the value of β on the 2–cell dual to W, and the remainder of the formula is as in Section 4.

Thus, by design, N has a combinatorial spin^c–structure if and only if ω_2 is the mod-2 reduction of a class in $H^2(N,\mathbb{Z})$. More specifically, there exists a spin structure on $TN \oplus \nu$ if and only if ω_2 is the mod-2 reduction of β.

Appendix

This section collects a few lesser-known facts related to the paper. These results are useful to anyone interested in implementing these techniques in software, and they are available in the software package Regina [7].

When n is odd, $\mathrm{Sym}(\Delta^n)$ has an alternative interpretation. There is a canonical isomorphism $\mathrm{Sym}(\Delta^n_0) \simeq \mathrm{Sym}^+(\Delta^n_0 \cup -\Delta^n_0)$, where $-\Delta^n_0$ is the antipodal simplex. The isomorphism is given by

$$\mathrm{Sym}(\Delta^n_0) \to \mathrm{Sym}^+(\Delta^n_0 \cup -\Delta^n_0), \quad A \mapsto (-1)^{|A|} A.$$

Given a subgroup G of SO_n, let $\widetilde{G} \subset \mathrm{Spin}_n$ be the preimage of G under the covering map $\mathrm{Spin}_n \to \mathrm{SO}_n$. By design, $\widetilde{G}$ is a $\mathbb{Z}_2$–central extension of the group G:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathrm{Spin}_n \arrow[r] & \mathrm{SO}_n \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[u, "\simeq"] & \widetilde{G} \arrow[r] \arrow[u] & G \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

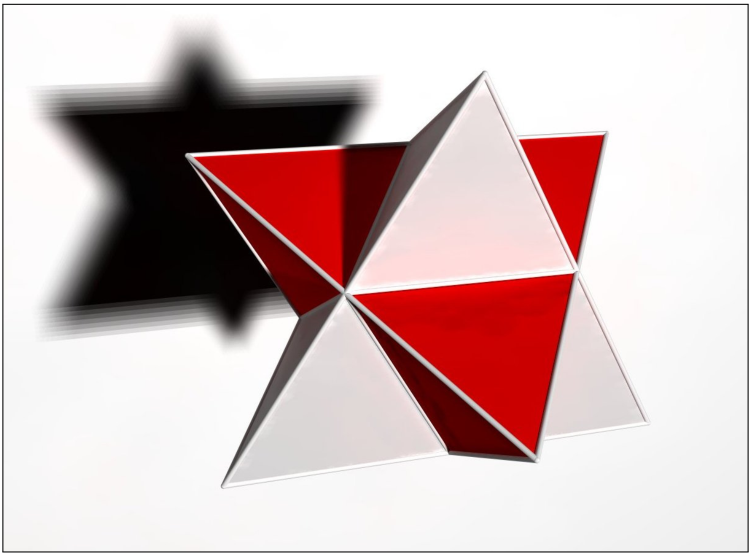

Figure 4: $\Delta^3_0 \cup -\Delta^3_0$

We give some concrete descriptions of the low-dimensional groups $\widetilde{A}_5$ and $\widetilde{\Sigma}_4^-$, respectively. Although not required for the main results of the paper, we compile this information here for easy reference.

Proposition A.1 *The group* Spin_3 *has a natural identification with the unit sphere in the quaternions,* S^3*, which acts on the quaternions by conjugation. This action is an orthogonal linear map and it fixes the real line pointwise. If we call the orthogonal complement of* $\mathbb{R}$ *in the quaternions the* purely imaginary *quaternions, we can identify the purely imaginary quaternions with* $\mathbb{R}^3$*. Thus our action can be interpreted as a Lie group homomorphism* $S^3 \to \mathrm{SO}_3$.

Consider Δ^3 *to be the convex hull of the four points*

$$\left\{\left(\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right),\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)\right\}\subset\mathbb{R}^3.$$

Then $\widetilde{\Sigma}_4^-$ *is isomorphic to the subgroup of* S^3 *which preserves* $\Delta^3 \cup -\Delta^3$*. It consists of the elements*

$$\left\{\pm 1,\ \pm a,\ \frac{1}{\sqrt{2}}(\pm 1\pm a),\ \frac{1}{\sqrt{2}}(\pm a\pm b),\ \frac{1}{2}(\pm 1\pm a\pm b\pm c)\ \middle|\ \{a,b,c\}=\{i,j,k\}\right\}\subset S^3.$$

The group Spin_4 *has a natural identification with* $S^3 \times S^3$*, the homomorphism* $S^3 \times S^3 \to \mathrm{SO}_4$ *given by left and right multiplication by unit quaternions.*

Consider Δ^4 *to be the convex hull of the points*

$$\{(1,0,0,0),\ \tfrac{1}{4}(-1,-\sqrt{5},\sqrt{5},\sqrt{5}),\ \tfrac{1}{4}(-1,\sqrt{5},-\sqrt{5},\sqrt{5}),\\ \tfrac{1}{4}(-1,\sqrt{5},\sqrt{5},-\sqrt{5}),\ \tfrac{1}{4}(-1,-\sqrt{5},-\sqrt{5},-\sqrt{5})\}.$$

The 120 *elements of* $\widetilde{A}_5 \subset S^3 \times S^3$ *are given by* $\pm(1,1)$, *having orders* 1 *and* 2, *respectively,* (α,α), *where* $\alpha = \frac{1}{2}(\pm 1 \pm i \pm j \pm k)$, *having orders* 3 *and* 6, *respectively, together with* $\pm\big(\frac{1}{2}+a\alpha+b\beta,\ \frac{1}{2}+\overline{a}\alpha+\overline{b}\beta\big)$, *where* $a=\pm\frac{1}{4}(\sqrt{5}-1)$, $b=\pm\frac{1}{4}(\sqrt{5}+1)$, $\{\alpha,\beta,\gamma\}=\{i,j,k\}$ *and* $\alpha\beta=\gamma$, *where* $\overline{a}$ *indicates the image of* a *under the automorphism of* $\mathbb{Q}[\sqrt{5}]$ *given by* $\sqrt{5}\mapsto -\sqrt{5}$. *These elements have order* 3 *and* 6, *respectively. There are also the elements* $\pm\big(a\pm\overline{a}\alpha\pm\frac{\beta}{2},\ \overline{a}\pm a\alpha\pm\frac{\beta}{2}\big)$, *where* $\{\alpha,\beta,\gamma\}=\{i,j,k\}$. *If* $\alpha\beta=\gamma$, *then* $a=\frac{1}{4}(1-\sqrt{5})$; *otherwise,* $a=\frac{1}{4}(1+\sqrt{5})$. *These elements have order* 5 *and* 10. *There are the elements* $\pm(\alpha,\alpha)$, *where* $\alpha\in\{i,j,k\}$. *These elements have order* 4. *Finally there are the elements* $(a\alpha+b\beta+c\gamma, \overline{a}\alpha+\overline{b}\beta+\overline{c}\gamma)$, *where* $\{\alpha,\beta,\gamma\}=\{i,j,k\}$, $\alpha\beta=\gamma$, $a=\pm\frac{1}{4}(1+\sqrt{5})$, $b=\pm\frac{1}{4}(1-\sqrt{5})$ *and* $c=\pm\frac{1}{2}$. *These elements have order* 4.

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Received: 11 October 2014 Revised: 30 October 2017