Algebraic & Geometric Topology 17 (2017) 2753–2762

On growth of systole along congruence coverings of Hilbert modular varieties

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We study how the systole of principal congruence coverings of a Hilbert modular variety grows when the degree of the covering goes to infinity. We prove that, given a Hilbert modular variety M_k of real dimension $2n$ defined over a number field k, the sequence of principal congruence coverings M_I eventually satisfies

$$\mathrm{sys}_1(M_I) \geq \frac{4}{3\sqrt{n}} \log(\mathrm{vol}(M_I)) - c,$$

where c is a constant independent of M_I.

22E40, 11R80; 53C22

1 Introduction

The *systole* of a riemannian manifold is the least length of a noncontractible closed geodesic in M and it is denoted by $\mathrm{sys}_1(M)$. In 1994, P Buser and P Sarnak [2] constructed the first explicit examples of surfaces with systole growing logarithmically with the genus using a sequence of principal congruence coverings of an arithmetic compact Riemann surface. These sequences of surfaces $\{S_p\}$ satisfy the inequality

$$\mathrm{sys}_1(S_p) \geq \tfrac{4}{3} \log(\mathrm{genus}(S_p)) - c,$$

where c is a constant independent of p. This result was generalized in 2007 by M Katz, M Schaps and U Vishne [6] to principal congruence coverings of any compact arithmetic Riemann surface and arithmetic hyperbolic 3–manifolds. It is known that a sequence of principal congruence coverings of a compact arithmetic hyperbolic manifold asymptotically attains the logarithmic growth of the systole (see Gromov [4, 3.C.6]) but the examples above are the only cases where the explicit constant in the systole growth is known so far. In particular, it would be interesting to understand how the asymptotic constant depends on the dimension.

The purpose of this paper is to generalize the construction of Buser and Sarnak to Hilbert modular varieties which are noncompact riemannian manifolds of dimension $2n$. We will show that the sequence of principal congruence coverings $M_I \to M_k$ of a

Published: 19 September 2017 DOI: 10.2140/agt.2017.17.2753

Hilbert modular variety eventually satisfies

$$\operatorname{sys}_1(M_I) \geq \frac{4}{3\sqrt{n}} \log(\operatorname{vol}(M_I)) - c, \tag{1}$$

where c is a constant independent of I. We also prove that inequality (1) is asymptotically sharp. We refer to Theorem 4.2 and Theorem 4.3 for the precise statement of the results.

Since M_k is noncompact, it is a priori not clear if the systole of M_I is bounded above by a logarithmic function of its volume. In fact, an interesting more general question is to understand if the systole of a sequence of congruence coverings of a noncompact finite-volume arithmetic manifold of nonpositive curvature and which is not flat grows logarithmically in its volume. An affirmative answer seems very plausible but, to our knowledge, it has not been established in the literature. In this regard we will prove that the sequence of principal congruence coverings $M_I \to M_k$ of a Hilbert modular variety eventually satisfies

$$\operatorname{sys}_1(M_I) \leq \frac{4\sqrt{n}}{3} \log(\operatorname{vol}(M_I)) - d \tag{2}$$

for some constant d independent of M_I. These results give us the first examples of explicit constants for the growth of systole of a sequences of congruence coverings of arithmetic manifolds in dimensions greater than three.

We will begin in Section 2 recalling basic aspects of the action of $(\mathrm{PSL}_2(\mathbb{R}))^n$ on $(\mathbb{H}^2)^n$. We then define the congruence coverings M_I of a Hilbert modular variety M_k, and we prove inequality (2). In Section 3 we estimate the length of closed geodesics of M_I in terms on the norm of the ideal I. In Section 4 we relate the norm of the ideal I to $\operatorname{vol}(M_I)$, and we prove inequality (1) and the sharpness of the constant $4/(3\sqrt{n})$.

Acknowledgements I would like to thank Mikhail Belolipetsky for his advice, support and many valuable suggestions on the preliminary versions of this work. I want to thank Cayo Doria for helpful discussions which significantly improved the quality of this paper. I also thank the referee for valuable comments and corrections. This work was supported by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior-CAPES.

2 Preliminaries

2.1 The action of $(\mathrm{PSL}_2(\mathbb{R}))^n$ on $(\mathbb{H}^2)^n$

The group $\mathrm{PSL}_2(\mathbb{R})$ acts on the upper half plane model of the hyperbolic plane $\mathbb{H}^2$ by fractional linear transformations via

$$Bz = \frac{az+b}{cz+d} \quad \text{if } B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } z \in \mathbb{H}^2.$$

An element $B \in \mathrm{PSL}_2(\mathbb{R})$ is called *elliptic* if it has a fixed point in $\mathbb{H}^2$, *parabolic* if it has no fixed points in $\mathbb{H}^2$ and has only one fixed point in $\partial\mathbb{H}^2$, and *hyperbolic* if it has no fixed points in $\mathbb{H}^2$ and has two fixed points in $\partial\mathbb{H}^2$. An equivalent description is the following:

- B is *elliptic* if and only if $|\mathrm{tr}(B)| < 2$;
- B is *parabolic* if and only if $|\mathrm{tr}(B)| = 2$;
- B is *hyperbolic* if and only if $|\mathrm{tr}(B)| > 2$.

Here $\mathrm{tr}(B)$ denotes the trace of the matrix B.

Given a hyperbolic transformation B, the *translation length* of B, denoted by ℓ_B, is defined by

$$\ell_B = \inf\{d_{\mathbb{H}^2}(z, Bz) \mid z \in \mathbb{H}^2\}.$$

This infimum is attained at points on the unique geodesic $\overline{\alpha}_B$ in $\mathbb{H}^2$ joining the fixed points of B in $\partial\mathbb{H}^2$. The transformation B leaves $\overline{\alpha}_B$ invariant and acts on it as a translation. In particular, if a subgroup $\Lambda \subset \mathrm{PSL}_2(\mathbb{R})$ acts properly discontinuously and freely on $\mathbb{H}^2$, every hyperbolic element $B \in \Lambda$ determines a noncontractible closed geodesic α on the Riemann surface $\mathbb{H}^2/\Lambda$, whose length is equal to the translation length ℓ_B of B. Reciprocally, any closed geodesic α in $\mathbb{H}^2/\Lambda$ lifts to a geodesic $\overline{\alpha}_B$ in $\mathbb{H}^2$ fixed by a hyperbolic matrix $B \in \Lambda$.

On the other hand, since B is hyperbolic, B is conjugate to a matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

where $|\lambda| = e^{\ell_B/2}$. Hence $2\cosh(\ell_B/2) = |\mathrm{tr}(B)|$ and for any $z \in \mathbb{H}^2$ we have

$$d_{\mathbb{H}^2}(z, Bz) \geq 2\log(|\mathrm{tr}(B)| - 1) > 0. \tag{3}$$

We refer to [1, Chapter 7] for further details about the geometry of the isometries of the hyperbolic plane $\mathbb{H}^2$.

The action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}^2$ extends to an action of the n–fold product $(\mathrm{PSL}_2(\mathbb{R}))^n$ on the n–fold product $(\mathbb{H}^2)^n$ in a natural way: if $z = (z_1, \dots, z_n) \in (\mathbb{H}^2)^n$ and $B = (B_1, \dots, B_n) \in (\mathrm{PSL}_2(\mathbb{R}))^n$, then

$$Bz := (B_1 z_1, \dots, B_n z_n),$$

where the action in every factor is the action by fractional linear transformations.

Let us recall the definition of a Hilbert modular variety (see [3]). Let k be a totally real number field of degree n, $\mathcal{O}_k$ the ring of integers of k and $\sigma_1, \dots, \sigma_n$ the n embeddings of k into the real numbers $\mathbb{R}$. The group $\mathrm{PSL}_2(\mathcal{O}_k)$ becomes an arithmetic noncocompact irreducible lattice of the semisimple Lie group $(\mathrm{PSL}_2(\mathbb{R}))^n$ via the map $\Delta(B) = (\sigma_1(B), \dots, \sigma_n(B))$, where $\sigma_i(B)$ denotes the matrix obtained by applying σ_i to the entries of B (see [7, Proposition 5.5.8]). Via this embedding, $\mathrm{PSL}_2(\mathcal{O}_k)$ acts on the n–fold product of hyperbolic planes $(\mathbb{H}^2)^n$ with finite covolume. The quotient $M_k = (\mathbb{H}^2)^n / \mathrm{PSL}_2(\mathcal{O}_k)$ is called a *Hilbert modular variety* and the group $\Gamma = \mathrm{PSL}_2(\mathcal{O}_k)$ is called a *Hilbert modular group*.

2.2 Congruence coverings of M_k

Let $I \subset \mathcal{O}_k$ be an ideal, the *principal congruence subgroup* $\Gamma(I) \subset \Gamma$ *at level* I is defined by

$$\Gamma(I) = \{A \in \mathrm{SL}_2(\mathcal{O}_k) \mid A \equiv \mathrm{Id} \mod I\}/\{1, -1\},$$

where Id denotes the identity 2×2 matrix. Since $\mathcal{O}_k / I$ is finite, $\Gamma(I)$ is a finite-index subgroup of Γ for any ideal I of $\mathcal{O}_k$. We associate to $\Gamma(I)$ a *congruence cover* $M_I = (\mathbb{H}^2)^n / \Gamma(I) \to M_k$. Note that Γ is an irreducible lattice in $(\mathrm{PSL}_2(\mathbb{R}))^n$ and so the varieties M_k and M_I do not split into products. We remark that M_k has quotient singularities, so the covering $M_I \to M_k$ should be interpreted in the orbifold sense. For large enough I the varieties M_I are manifolds by Selberg's lemma (see also Corollary 3.3).

This construction is a particular case of a more general situation: if G is a semisimple Lie group, a discrete subgroup $\Lambda \subset G$ is called arithmetic if there exists a number field K, a algebraic K–group H, and a surjective continuous homomorphism $\varphi\colon \mathrm{H}(K \otimes_{\mathbb{Q}} \mathbb{R}) \to G$ with compact kernel such that $\varphi(\mathrm{H}(\mathcal{O}_K))$ is commensurable to Λ, where $\mathrm{H}(\mathcal{O}_K)$ denotes the $\mathcal{O}_K$–points of H with respect to some fixed embedding of H into GL_m. For any ideal $I \subset \mathcal{O}_K$ the principal congruence subgroup of $\mathrm{H}(\mathcal{O}_K)$ at level I is defined by

$$\mathrm{H}(I) := \ker\bigl(\mathrm{H}(\mathcal{O}_K) \xrightarrow{\pi_I} \mathrm{H}(\mathcal{O}_K/I)\bigr),$$

where π_I is the reduction map modulo I. Any discrete subgroup of G containing some of these subgroups $\mathrm{H}(I)$ is called a *congruence subgroup of* G.

By Margulis' arithmeticity theorem (see [7, Chapter 5]), for $n \geq 2$ any irreducible lattice in $(\mathrm{PSL}_2(\mathbb{R}))^n$ is arithmetic. A conjecture of Serre, proved to be true in the nonuniform case, shows that any nonuniform lattice of $(\mathrm{PSL}_2(\mathbb{R}))^n$ is a congruence subgroup.

The coverings $M_I \to M_k$ are regular coverings because the subgroups $\Gamma(I)$ are normal subgroups of Γ. It is worth noting that in a sequence of nonregular congruence coverings of an arithmetic manifold the systole could grow slower than logarithmically with respect to the volume (see [5, Section 4.1]).

2.3 Upper bound for the systole growth of M_I

As was explained above, if Λ is any discrete group of isometries of $\mathbb{H}^2$ acting freely on $\mathbb{H}^2$, every hyperbolic element $\gamma \in \Lambda$ produces a noncontractible closed geodesic on $\mathbb{H}^2/\Lambda$. We can use this idea to see that the quotients M_I which we are interested in have closed geodesics, and subsequently we find a upper bound for $\mathrm{sys}_1(M_I)$.

We denote by $\mathrm{N}(I)$ the norm of an ideal $I \subset \mathcal{O}_k$, which is the cardinality of the quotient ring $\mathcal{O}_k/I$, and similarly $\mathrm{N}(r)$ denotes the field norm of an element r of the number field k.

Suppose $I \subset \mathcal{O}_k$ is an ideal with $\mathrm{N}(I) > 2$ and such that M_I is a riemannian manifold (see Corollary 3.3). The norm $\mathrm{N}(I)$ is a rational integer with $\mathrm{N}(I) \in I$, so if we take the matrix

$$B = \begin{pmatrix} 1-\mathrm{N}(I)^2 & \mathrm{N}(I) \\ -\mathrm{N}(I) & 1 \end{pmatrix},$$

then $B \in \Gamma(I)$ and $|\mathrm{tr}(\sigma_i(B))| > 2$ for any $i = 1, \dots, n$. This means that the matrices $\sigma_1(B) = \sigma_2(B) = \cdots = \sigma_n(B)$ are hyperbolic and if we take $\overline{\alpha}$ to be the only geodesic in $\mathbb{H}^2$ fixed by B, the curve $\overline{\beta} = \overline{\alpha} \times \cdots \times \overline{\alpha}$ is a geodesic in $(\mathbb{H}^2)^n$ that is fixed by $(\sigma_1(B), \dots, \sigma_n(B))$, and $\overline{\beta}$ projects to a noncontractible closed geodesic β in M_I. Note that this geodesic might not be the shortest one, so $\mathrm{sys}_1(M_I) \le \ell(\beta) = \sqrt{n}\,\ell_B$, where ℓ_B denotes the translation length of B along $\overline{\alpha}$.

We know that $2\cosh(\ell_B/2) = |\mathrm{tr}(B)| = \mathrm{N}(I)^2 - 2 < \mathrm{N}(I)^2$, and so

$$\mathrm{sys}_1(M_I) \le 4\sqrt{n}\log \mathrm{N}(I).$$

Now, as we will see in Section 4, there exists a constant C_k independent of I such that $[\Gamma : \Gamma(I)] \ge C_k\,\mathrm{N}(I)^3$ (Lemma 4.1), and then

$$\mathrm{sys}_1(M_I) \le \frac{4\sqrt{n}}{3}\log[\Gamma : \Gamma(I)] - \frac{4\sqrt{n}}{3}\log C_k. \tag{4}$$

This proves inequality (2) since $\mathrm{vol}(M_I) = [\Gamma : \Gamma(I)]\,\mathrm{vol}(M)$.

3 Distance estimate for congruence subgroups

In this section we will prove that the congruence subgroups $\Gamma(I)$ act freely on $(\mathbb{H}^2)^n$ when the norm of the ideal I is big enough and we will relate the length of closed geodesics in M_I to the norm of the ideal I. The first fact follows from Selberg's lemma [7, Section 4.8] but in our case the proof gives an explicit bound in terms of the norm of I. Some of the ideas are inspired by [6], where the authors studied the systole of compact arithmetic hyperbolic surfaces and 3–manifolds.

In this section, sometimes we will use the notation A or $(\sigma_1(A),\dots,\sigma_n(A))$ for the same element in Γ or its image in $(\mathrm{PSL}_2(\mathbb{R}))^n$ via the map Δ defined in Section 2.

For our purpose, it is convenient to express any element $A=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ of Γ in the form

$$A=\begin{pmatrix} x_0+x_1 & x_2+x_3\\ x_2-x_3 & x_0-x_1\end{pmatrix},$$

where

$$x_0=\frac{a+d}{2},\quad x_1=\frac{a-d}{2},\quad x_2=\frac{b+c}{2},\quad x_3=\frac{b-c}{2}$$

are elements of the field K. We have $x_0^2-x_1^2-x_2^2+x_3^2=1$ and we write $y_0=x_0-1$. With this notation, if $I\subset\mathcal{O}_k$ is an ideal and $A\in\Gamma(I)$ then $2x_0-2\in I$ and $2x_i\in I$ for $i=1,2,3$. In terms of fractional ideals it means that y_0,x_1,x_2 and x_3 lie in $I/2$.

Lemma 3.1 *If $A\in\Gamma(I)$, then $y_0\in I^2/8$. In particular, if $y_0\neq 0$ then $|\mathrm{N}(y_0)|\geq \mathrm{N}(I)^2/8^n$.*

Proof We know that $A\in\Gamma(I)$ implies $x_0-1,x_1,x_2,x_3\in I/2$. Now, by replacing $x_0=1+y_0$ in the equation $x_0^2-x_1^2-x_2^2+x_3^2=1$ we obtain

$$2y_0=-y_0^2+x_1^2+x_2^2-x_3^2\in I^2/4.$$

Hence $y_0\in I^2/8$. □

Lemma 3.2 *If $A\in\Gamma(I)$ with $y_0\neq 0$ then $|\mathrm{tr}(\sigma_j(A))|\geq \mathrm{N}(I)^{2/n}/4-2$ for some $j\in\{1,\dots,n\}$.*

Proof By definition we have $\mathrm{N}(y_0)=\prod_{j=1}^n\sigma_j(y_0)$, so by Lemma 3.1, for some $j\in\{1,\dots,n\}$, we have $|\sigma_j(y_0)|\geq \mathrm{N}(I)^{2/n}/8$. Therefore

$$|\mathrm{tr}(\sigma_j(A))|=|2\sigma_j(x_0)|=|2\sigma_j(y_0)+2|\geq\frac{\mathrm{N}(I)^{2/n}}{4}-2.$$ □

With this we can guarantee the riemannian structure for M_I:

Corollary 3.3 *For any ideal $I\subset\mathcal{O}_k$ with $\mathrm{N}(I)\geq 4^n$, the subgroup $\Gamma(I)$ acts freely on $(\mathbb{H}^2)^n$ and so $M_I=(\mathbb{H}^2)^n/\,\Gamma(I)$ admits a structure of a riemannian manifold with nonpositive sectional curvature.*

Proof The element $A=(\sigma_1(A),\dots,\sigma_n(A))\in\Gamma(I)$ has a fixed point on $(\mathbb{H}^2)^n$ if and only if $\sigma_i(A)$ has a fixed point in $\mathbb{H}^2$ for any $i=1,\dots,n$, but this happens if and only if $|\mathrm{tr}(\sigma_i(A))|<2$, which, by Lemma 3.2, is impossible if $\mathrm{N}(I)\geq 4^n$. □

Now observe that for $i=1,\dots,n$ and $A\in\Gamma$,

$$2|\sigma_i(y_0)|-2\leq|\mathrm{tr}(\sigma_i(A))|\leq 2+2|\sigma_i(y_0)|. \tag{5}$$

Proposition 3.4 *Let $I \subset \mathcal{O}_k$ be an ideal with $\mathrm{N}(I) \geq 40^{n/2}$ and $A \in \Gamma(I)$ with $y_0 \neq 0$. Then for any point $z = (z_1, \dots, z_n) \in (\mathbb{H}^2)^n$ we have*

$$d_{(\mathbb{H}^2)^n}(z, Az) \geq \frac{4}{\sqrt{n}} \log \mathrm{N}(I) - 2\sqrt{n} \log 40.$$

Proof By Lemma 3.2, $|\mathrm{tr}(\sigma_j(A))| \geq 8$ for some $j \in \{1, \dots, n\}$, hence we can subdivide our analysis into two different cases:

Case 1 $|\mathrm{tr}(\sigma_i(A))| \geq 8$ **for any** $i = 1, \dots, n$ In this case all of the matrices $\sigma_i(A)$ are hyperbolic and the right-hand side of (5) implies that $|\sigma_i(y_0)| \geq 3$ for $i = 1, \dots, n$.

Using (3), the left-hand side of (5), the fact that $|\sigma_i(y_0)| \geq 3$ for $i = 1, \dots, n$, the convexity of the function x^2 and Lemma 3.1 we obtain

$$\begin{aligned}
d_{(\mathbb{H}^2)^n}(z, Az) &= \sqrt{d^2_{\mathbb{H}^2}(z_1, \sigma_1(A)z_1) + \cdots + d^2_{\mathbb{H}^2}(z_n, \sigma_n(A)z_n))} \\
&\geq 2\sqrt{\log^2(|\mathrm{tr}(\sigma_1(A))| - 1) + \cdots + \log^2(|\mathrm{tr}(\sigma_n(A))| - 1)} \\
&\geq 2\sqrt{\log^2(2|\sigma_1(y_0)| - 3) + \cdots + \log^2(2|\sigma_n(y_0)| - 3)} \\
&\geq 2\sqrt{\log^2|\sigma_1(y_0)| + \cdots + \log^2|\sigma_n(y_0)|} \\
&\geq \frac{2}{\sqrt{n}}\bigl(\log|\sigma_1(y_0)| + \cdots + \log|\sigma_n(y_0)|\bigr) \\
&= \frac{2}{\sqrt{n}} \log|\mathrm{N}(y_0)| \geq \frac{4}{\sqrt{n}} \log \mathrm{N}(I) - 2\sqrt{n} \log 8.
\end{aligned}$$

Case 2 **There are exactly** $k < n$ **of the indices** $1, \dots, n$ **such that** $|\mathrm{tr}(\sigma_j(A))| < 8$ Without loss of generality we assume that $|\mathrm{tr}(\sigma_j(A))| < 8$ for $j = 1, \dots, k$. By the left-hand side of (5), $|\sigma_j(y_0)| < 5$ for any such j and by Lemma 3.1 we have

$$\prod_{i=k+1}^{n} |\sigma_i(y_0)| = \frac{|\mathrm{N}(y_0)|}{\prod_{i=1}^{k} |\sigma_i(y_0)|} > \frac{1}{5^n \cdot 8^n} \mathrm{N}(I)^2.$$

Now, as $|\mathrm{tr}(\sigma_i(A))| \geq 8$ for $i = k+1, \dots, n$, for these indices $\sigma_i(A)$ is hyperbolic and $|\sigma_i(y_0)| \geq 3$ by the left-hand side of (5). By using (3) and the previous facts we obtain

$$\begin{aligned}
d_{(\mathbb{H}^2)^n}(z, Az) &= \sqrt{d^2_{\mathbb{H}^2}(z_1, \sigma_1(A)z_1) + \cdots + d^2_{\mathbb{H}^2}(z_n, \sigma_n(A)z_n))} \\
&\geq \sqrt{d^2_{\mathbb{H}^2}(z_{k+1}, \sigma_{k+1}(A)z_{k+1}) + \cdots + d^2_{\mathbb{H}^2}(z_n, \sigma_n(A)z_n))} \\
&\geq 2\sqrt{\log^2(|\mathrm{tr}(\sigma_{k+1}(A))| - 1) + \cdots + \log^2(|\mathrm{tr}(\sigma_n(A))| - 1)} \\
&\geq 2\sqrt{\log^2(2|\sigma_{k+1}(y_0)| - 3) + \cdots + \log^2(2|\sigma_n(y_0)| - 3)}
\end{aligned}$$

$$\begin{aligned}
&\geq 2\sqrt{\log^2|\sigma_{k+1}(y_0)| + \cdots + \log^2|\sigma_n(y_0)|} \\
&\geq \frac{2}{\sqrt{n-k}}\big(\log|\sigma_{k+1}(y_0)| + \cdots + \log|\sigma_n(y_0)|\big) \\
&= \frac{2}{\sqrt{n-k}} \log \prod_{i=k+1}^{n} |\sigma_i(y_0)| \geq \frac{4}{\sqrt{n}} \log \mathrm{N}(I) - 2\sqrt{n}\log 40.
\end{aligned}$$

In both cases we get

$$d_{(\mathbb{H}^2)^n}(z, Az) \geq \frac{4}{\sqrt{n}} \log(\mathrm{N}(I)) - 2\sqrt{n}\log(40). \qquad \square$$

Corollary 3.5 *For any ideal $I \subset \mathcal{O}_k$ with $\mathrm{N}(I) \geq 40^{n/2}$, the length of any noncontractible closed geodesic α in M_I satisfies*

$$\ell(\alpha) \geq \frac{4}{\sqrt{n}} \log \mathrm{N}(I) - 2\sqrt{n}\log 40.$$

Proof By Corollary 3.3, M_I is a riemannian manifold with the metric induced from $(\mathbb{H}^2)^n$. If we lift α to a geodesic $\widetilde{\alpha} = (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n)$ in its universal cover $(\mathbb{H}^2)^n$ there is an element $A \in \Gamma(I)$ acting on $\widetilde{\alpha}$ as a translation and for any z in the graph of $\widetilde{\alpha}$ we have $\ell(\alpha) = d_{(\mathbb{H}^2)^n}(z, Az)$. Since α is noncontractible, $\widetilde{\alpha}$ is not a point, then for some $i \in \{1, \ldots, n\}$ $\widetilde{\alpha}_i$ is a nontrivial geodesic in $\mathbb{H}^2$, and so $\sigma_i(A)$ acts on it as a translation. This implies that $\sigma_i(A)$ is hyperbolic and, in particular, $|\mathrm{tr}(A)| \neq 2$. Since $|\mathrm{tr}(A)| \neq 2$ implies $y_0 \neq 0$, the result now follows from Proposition 3.4. $\square$

4 Proof of the main results

To finish the proofs of the theorems we need to find uniform bounds for the quotient $[\Gamma : \Gamma(I)]/\mathrm{N}(I)^3$, for ideals $I \subset \mathcal{O}_k$ with norm sufficiently large.

Lemma 4.1 *For almost any ideal $I \subset \mathcal{O}_k$ we have*

$$\zeta_k(2)^{-1}\,\mathrm{N}(I)^3 \leq [\Gamma : \Gamma(I)] < \mathrm{N}(I)^3, \tag{6}$$

where ζ_k denotes the Dedekind zeta function of k.

Proof A well-known corollary of the strong approximation theorem (see Theorem 7.15 of [8]) implies that for almost all ideals $I \subset \mathcal{O}_k$ the reduction map

$$\mathrm{SL}_2(\mathcal{O}_k) \xrightarrow{\pi_I} \mathrm{SL}_2(\mathcal{O}_k/I)$$

is surjective. For those ideals the index $[\Gamma : \Gamma(I)]$ is equal to the cardinality of $\mathrm{SL}_2(\mathcal{O}_k/I)$, which is given by the formula

$$\mathrm{N}(I)^3 \prod_{\mathfrak{p}|I} \Big(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^2}\Big).$$

From this the right-hand side of inequality (6) follows easily. On the other hand, the product formula for the Dedekind zeta function of k says that

$$\zeta_k(2) = \prod_{\mathfrak{p}\subset\mathcal{O}_k} \frac{1}{1-\mathrm{N}(\mathfrak{p})^{-2}} \geq \prod_{\mathfrak{p}|I} \frac{1}{1-\mathrm{N}(\mathfrak{p})^{-2}}.$$

This proves the second inequality. □

Theorem 4.2 *Let k be a totally real number field of degree n and $\mathcal{O}_k$ be the ring of integers of k. Any sequence of ideals in $\mathcal{O}_k$ with $\mathrm{N}(I) \to \infty$ eventually satisfies*

$$\mathrm{sys}_1(M_I) \geq \frac{4}{3\sqrt{n}} \log(\mathrm{vol}(M_I)) - c,$$

where $\Gamma(I)$ is the principal congruence subgroup of $\Gamma = \mathrm{PSL}_2(\mathcal{O}_k)$ at level I, $M_I = (\mathbb{H}^2)^n/\Gamma(I)$ and c is a constant independent of I.

Proof For any ideal I with $\mathrm{N}(I) \geq 40^{n/2}$, Corollary 3.3 implies that M_I is a riemannian manifold with the metric induced by the product metric on $(\mathbb{H}^2)^n$. Now, by Corollary 3.5 and Lemma 4.1, we conclude that

$$\mathrm{sys}_1(M_I) \geq \frac{4}{3\sqrt{n}} \log[\Gamma : \Gamma(I)] - 2\sqrt{n}\log 40$$

when $\mathrm{N}(I) \to \infty$. □

To finish, we prove that among congruence coverings of Hilbert modular varieties the constant $4/(3\sqrt{n})$ in the growth of the systole in general cannot be improved to any $\gamma > 4/(3\sqrt{n})$.

Theorem 4.3 *Let k be a totally real number field of degree n and $\mathcal{O}_k$ be the ring of integers of k. Then there exists a sequence of ideals in $\mathcal{O}_k$ with $\mathrm{N}(I) \to \infty$ such that*

$$\mathrm{sys}_1(M_I) \leq \frac{4}{3\sqrt{n}} \log(\mathrm{vol}(M_I)) + c_1,$$

where $\Gamma(I)$ is the principal congruence subgroup of $\Gamma = \mathrm{PSL}_2(\mathcal{O}_k)$ at level I, $M_I = (\mathbb{H}^2)^n/\Gamma(I)$ and c_1 is a constant independent of M_I.

Proof Let p be a rational integer and consider the ideal $I_p = p\mathcal{O}_k$ in $\mathcal{O}_k$. Since $\mathrm{N}(I_p) = p^n$, by following the same argument as in Section 2.3 with the matrix

$$B = \begin{pmatrix} 1-p^2 & p \\ -p & 1 \end{pmatrix},$$

we obtain that $\mathrm{sys}_1(M_{I_p}) \leq 4\sqrt{n}\log(p)$ when p is large enough. Therefore, Lemma 4.1 implies that

$$\mathrm{sys}_1(M_{I_p}) \leq \frac{4}{3\sqrt{n}}\log[\Gamma : \Gamma(I_p)] + \frac{4}{3\sqrt{n}}\log\zeta_k(2)$$

when $p \to \infty$, and then we obtain the result with

$$c_1 = \frac{4}{3\sqrt{n}}\log\frac{\zeta_k(2)}{\mathrm{vol}(M_k)},$$

where $M_k = (\mathbb{H}^2)^n/\Gamma$. □

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Received: 9 June 2016 Revised: 6 April 2017