msp

Algebraic & Geometric Topology 17 (2017) 2125–2144

On mod p A_p–spaces

RUIZHI HUANG
JIE WU

We prove a necessary condition for the existence of an A_p–structure on mod p spaces, and also derive a simple proof for the finiteness of the number of mod p A_p–spaces of given rank. As a direct application, we compute a list of possible types of rank 3 mod p homotopy associative H–spaces.

55P45, 55S25; 55N15, 55P15, 55S05

1 Introduction

A longstanding problem in algebraic topology is to classify finite H–spaces. However, this problem is rather complicated, and has only been solved in few cases. There is Zabrodsky's localization and mixing theorem [27] yielding that a simply connected finite complex is an H–space if and only if each of its p–localizations is an H–space. One would also like to know for which primes p the localization at p fails to be an H–space, so it is natural to consider the p–local version of H–spaces.

Let X be a CW-complex whose cohomology is an exterior algebra generated by r elements of odd dimension; we call r the rank of X. For $r=1$, J F Adams [1; 2] has determined that S^1, S^3, S^7 are the only H–spaces localized at 2 by solving the famous Hopf invariant one problem, and all odd spheres are H–spaces localized at any odd prime p. For $r=2$, the case $p=2$ (then the integral case) has been solved in a series of papers: see Adams [3], Hubbuck [15], Zabrodsky [28; 29], Douglas and Sigrist [7], Mimura, Nishida and Toda [19], as well as the case $p>3$ by N Hagelgans [9]. The remaining case $p=3$ is challenging and has been an open question for decades; recent progress on it can be found in Grbić, Harper, Mimura, Theriault and Wu [8].

The phenomenon that the H–structures are largely controlled by the prime $p=2$ appears similarly when we consider higher homotopy associative structures. Namely, if we consider A_p–spaces in the sense of J Stasheff [21; 22], the A_p–structure is controlled by that of the localization at p, where a connected A_2–space is just an H–space. In general, for any A_n–space X, Stasheff suggests an n–projective space $P_n(X)$ over X, which is analogous to Milnor's classifying space for topological groups.

Published: 3 August 2017 DOI: 10.2140/agt.2017.17.2125

(See Definition 3.5 and the paragraph before that for the explicit definition of A_n–spaces and related comments.)

Let $n = p$. It is well-known that there exists some nontrivial p^{th} power in the cohomology of p–stage projective space $P_p(X)$ which exactly detects the A_p–structure. Furthermore, Hemmi [12] has defined a modified projective space $R_n(X)$ for a special family of A_n–spaces, which is our main concern in this paper. Based on these ideas and constructions we prove the following theorem, which generalizes the result of Wilkerson [25] for local spheres:

Theorem 1.1 *Fix an odd prime $p \geq 3$ and let X be a connected p–local A_p–space with cohomology ring $H^*(X, \mathbb{Z}/p\mathbb{Z}) \cong \bigwedge(x_{2m_1-1}, \dots, x_{2m_r-1})$, where $m_1 \leq m_j$ for all j. Define*

$$m = \gcd\{m_i \mid m_i \leq pm_1\}.$$

Then $m \mid p-1$.

For the converse of the theorem, we recall that Stasheff [23] has constructed a realization for polynomial algebras $\mathbb{Z}/p\mathbb{Z}[x_{2m}, x_{4m}, \dots, x_{2km}]$ with $m \mid p-1$ using a theorem of Quillen. Here, our proof of this theorem is based on a generalization of a method of Adams and Atiyah [4]; (see also Section 2), using which we also derive a simple proof of a finiteness theorem of Hubbuck and Mimura [16] (also see Theorem 3.7) which claims that there are only finitely many possible homotopy types of spaces with fixed rank r which are A_p–spaces.

For the special case when $p = 3$, a mod 3 A_3–space is a usual 3–local homotopy associative H–space. The only simply connected homotopy associative H–space at 3 of rank 1 is S^3. If we define the increasing sequence $(m_1, \dots, m_r)$ to be the type of X in Theorem 1.1, then the complete list of types for rank 2 3–local simply connected homotopy associative H–spaces are $(2,3)$, $(2,4)$, $(2,6)$ and $(6,8)$; see Wilkerson [24, Theorem 5.1]. It is clear that

$$S^3 \times S^5 \overset{3}{\simeq} \mathrm{SU}(3)$$

provides an example for $(2,3)$, Sp(2) for $(2,4)$, and G_2 for $(2,6)$. Harper [10] gives a decomposition

$$F_4 \overset{3}{\simeq} K \times B_5(3),$$

where $B_5(3)$ is the S^{11} bundle over S^{15} classified by α_1, and, further, Zabrodsky [30] shows that $B_5(3)$ is a loop space, which provides an example for $(6,8)$. In this paper, we consider the case of rank 3. With the help of the method of Adams and Atiyah, and some results of Wilkerson (see [24] or Theorem 4.2), we prove the following theorem by careful analysis of the effect of both Steenrod operations and Adams' ψ–operations.

Theorem 1.2 *Let X be an indecomposable 3–local homotopy associative H–space with cohomology ring $H^*(X,\mathbb{Z}/3\mathbb{Z})\cong\bigwedge(x_{2r-1},x_{2n-1},x_{2m-1})$, where $\deg(x_k)=k$ and $1<r<n<m$. Then the type of X (r,n,m) can only be one of*

$$(2,4,6),\ (2,6,8),\ (3,5,7),\ (3,6,8),\ (6,8,10),\ (6,8,12).$$

In this list, the only known example is $\mathrm{Sp}(3)$, which is of type $(2,4,6)$. Here are a few things we know about potential examples of rank 3 3–local A_3–spaces of the remaining five types. For $(2,6,8)$, we can form a space X as the total space of a G_2–principal fibration over S^{15}, which is classified by the generator of

$$\pi_{15}(BG_2)\cong\pi_{14}(G_2)\overset{3}{\cong}\pi_{14}(S^3)\overset{3}{\cong}\mathbb{Z}/3\mathbb{Z}.$$

Then the classifying map factors as $S^{15}\xrightarrow{f}BS^3\to BG_2$, and we get $X\overset{3}{\simeq}(G_2\times Y)/S^3$, where Y is the total space of the fibration classified by f and also an H–space by Theorem 7.1 of Grbić, Harper, Mimura, Theriault and Wu [8]. However, we still do not know whether X is an H–space or not. For the case $(3,5,7)$ we have Nishida's $B_2^3(3)$, which is a 3–component of $\mathrm{SU}(7)$ (see Mimura, Nishida and Toda [20]). Still, we do not know whether $B_2^3(3)$ is homotopy associative. If X is of type $(3,6,8)$, then X has a generating complex of the form $S^5\vee A$ by the knowledge of the homotopy groups of spheres, where A is of type $(6,8)$. For $(6,8,10)$, Harper and Zabrodsky [11] have proved that if the exterior algebra of rank p generated by $\{x_{2n-1},\mathscr{P}^1x_{2n-1},\dots,\mathscr{P}^{p-1}x_{2n-1}\}$ can be realized by an H–space, then $p\mid n$, and the converse is still open for $n>p$. For the last possible case of type $(6,8,12)$, we have $\mathscr{P}^1(x_{11})=x_{15}$ and $\mathscr{P}^3(x_{11})=x_{23}$.

The article is organized as follows. In Section 2 we will introduce a refined version of Adams and Atiyah's method from [4]. In Section 3 we use number theory to prove Theorem 1.1 and the finiteness theorem of Hubbuck and Mimura. Section 4 is devoted to the proof of Theorem 1.2.

2 A method of Adams and Atiyah

In [4], Adams and Atiyah develop a method to detect the p^{th} power of cohomology elements using Adams' ψ–operations. For our purpose, we need to modify it slightly.

Given a connected CW-complex X with no p–torsion in $H^*(X,\mathbb{Z})$, suppose there exists a subalgebra $\bar{\mathcal{H}}$ of $H^*(X;\mathbb{Z}/p\mathbb{Z})$ such that

$$\bar{\mathcal{H}}\cong\bar{A}\oplus\bar{B}$$

as rings, where $\bar{A}$ contains $\bar{\mathcal{H}}^0$, $\bar{B}$ is an ideal and also $\bar{\mathcal{H}}$ and $\bar{B}$ are closed under the action of the mod p Steenrod algebra $\mathscr{A}_p$. Then by the Atiyah–Hirzebruch–Whitehead spectral sequence and [5, Theorem 6.5], we have the corresponding filtered subalgebra $\mathcal{H}$ of $K(X)\otimes\mathbb{Z}_{(p)}$ such that

$$\mathcal{H}\cong A\oplus B,$$

as filtered rings, and also $\mathcal{H}$ and B are closed under ψ^p–action. Write the Chern character of an element $x\in K(X)\otimes\mathbb{Z}_{(p)}$ as

$$\mathrm{ch}(x)=a_0+\sum\nolimits_i a_{2i}+\sum\nolimits_j b_{2j},$$

with $a_0\in\mathbb{Q}$, $a_{2i}\in\bar{A}^{>0}\otimes\mathbb{Q}$ and $b_{2j}\in\bar{B}^{>0}\otimes\mathbb{Q}$ (the subscripts refer to the degree). Then we have

$$\mathrm{ch}(\psi^k(x))=a_0+\sum\nolimits_i k^i a_{2i}+\sum\nolimits_j k^j b_{2j}.$$

Hence ψ^k is indeed a semisimple linear transformation if we use the Chern character to identify $K(X)\otimes\mathbb{Q}$ with $H^{\mathrm{even}}(X;\mathbb{Q})$, and the eigenspace decomposition of $\widetilde{K}(X)\otimes\mathbb{Q}$ is independent of the choice of ψ^k. In particular, $\mathcal{H}\otimes\mathbb{Q}$ and $B\otimes\mathbb{Q}$ are invariant under ψ^k for any k, as they are invariant under ψ^p, and then $\mathcal{H}$ and B are also invariant under each ψ^k. Then, as in [4], we get a (partial) eigenspace decomposition

$$\widetilde{\mathcal{H}}\cong\bigoplus_{i=1}^{r}V_i\oplus W,\qquad B^{>0}\otimes\mathbb{Q}\cong W,$$

where $\widetilde{\mathcal{H}}=\mathcal{H}^{>0}\otimes\mathbb{Q}$, $\deg(V_i)=2m_i$ (which means the degree of its elements) and V_i is allowed to be the 0 vector space. For each ψ^k, V_i is the eigenspace corresponding to the eigenvalue k^{m_i}. We also notice that $A^{>0}\otimes\mathbb{Q}\cong\bigoplus_{i=1}^r V_i$ but only as vector spaces. Now define a linear transformation on $\widetilde{K}(X)\otimes\mathbb{Q}$ by

$$\pi_i=\prod_{\substack{1\le j\le r\\ j\neq i}}\frac{\psi^{k_j}-k_j^{m_j}}{k_j^{m_i}-k_j^{m_j}},$$

and a number

$$d_i(m_1,\dots,m_r)=\gcd\Big\{\prod_{\substack{1\le j\le r\\ j\neq i}}(k_j^{m_i}-k_j^{m_j})\;\Big|\;k_j\in\mathbb{N}^+\text{ for }1\le j\le r,\ j\neq i\Big\}.$$

Notice that π_i induces a linear transformation $\bar{\pi}_i$ on $\bigoplus_{i=1}^r V_i$ which is the natural projection onto the i^{th} component V_i. For any $x\in\widetilde{\mathcal{H}}$, we have

$$\pi_i(x)\cdot\prod_{\substack{1\le j\le r\\ j\neq i}}(k_j^{m_i}-k_j^{m_j})=\prod_{\substack{1\le j\le r\\ j\neq i}}(\psi^{k_j}-k_j^{m_j})(x)\in\widetilde{\mathcal{H}}.$$

Accordingly,

$$\pi_i(x)d_i(m_1,\dots,m_r)\in\widetilde{\mathcal{H}}.$$

If we write $x=\sum_i\bar{\pi}_i(x-v)+v$ for some $v\in B$, then we also have

$$\bar{\pi}_i(x-v)d_i(m_1,\dots,m_r)\in\widetilde{\mathcal{H}}.$$

Now we make a crucial assumption that for each i

$$p^{m_i}\nmid d_i(m_1,\dots,m_r). \tag{2-1}$$

Since B is a $\{\psi^p\}$–module, we have

$$\begin{aligned}\psi^p(x)&=\sum_i\psi^p(\bar{\pi}_i(x-v))+\psi^p(v)\\&=\sum_i p^{m_i}\frac{\bar{\pi}_i(x-v)d_i(m_1,\dots,m_r)}{d_i(m_1,\dots,m_r)}+\psi^p(v)\\&=py+\psi^p(v)\in p\widetilde{\mathcal{H}}+B,\end{aligned}$$

ie $x^p\equiv\psi^p(x)\equiv 0 \bmod (p,B)$. Again, as in [4], $\bar{x}^p\equiv 0 \bmod (\bar{B})$ on the cohomology level, where $\bar{x}$ denotes the corresponding element of x in $\bar{\mathcal{H}}\subset H^*(X,\mathbb{Z}/p\mathbb{Z})$.

Remark 2.1 Notice that when $\bar{\mathcal{H}}=H^*(X,\mathbb{Z}/p\mathbb{Z})$ and $\bar{B}=0$, the above result is exactly [4, Corollary].

3 Proof of Theorem 1.1 and the finiteness theorem

3.1 Proof of Theorem 1.1

We prove the theorem by contradiction. The main task is to prove the condition (2-1) holds. We have to do some number theory first.

Definition 3.1 Let n be a positive integer.

(1) Define $e(n)=f$ if $n=p^f\cdot x$ and $p\nmid x$.

(2) Define ν by

$$\nu(n)=\begin{cases}f+1 & \text{if } n=p^f(p-1)x \text{ and } p\nmid x,\\ 0 & \text{if } p-1\nmid n.\end{cases}$$

Suppose k is a primitive root modulo p^2. Then k is also a primitive root modulo p^f for all $f \in \mathbb{N}^+$. Then for any positive integer n, we have

$$k^n \equiv 1 \bmod p^f \iff n \equiv 0 \bmod p^{f-1}(p-1). \tag{3-1}$$

So $\nu(n)$ is the exact exponent of p in the prime factorization of $k^n - 1$ if $p-1 \mid n$.

The following lemma is well known and basic in number theory:

Lemma 3.2 (Legendre 1808) *We have*

$$e(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor = \frac{n - s_p(n)}{p-1},$$

where $s_p(n) = a_k + a_{k-1} + \cdots + a_1 + a_0$ is the sum of all the digits in the expansion of n in base p.

From above, we easily get:

Corollary 3.3 (1) *$e(a!) + e(b!) \le e((a+b)!)$;*

(2) *$e((ab)!) \le a + e(a!)$ if $b \le p$.*

Now we are ready to prove our main lemma, which is a generalization of [4, Lemma 3.5]:

Lemma 3.4 *Let p be an odd prime, k be a primitive root modulo p^2, $m, t \in \mathbb{N}^+$ be such that $m \nmid p-1$, and set*

$$\Pi := \prod_{\substack{j \le t \le tp \\ j \ne i}} (k^{mi} - k^{mj}).$$

Then we have

$$e(\Pi) < mt. \tag{3-2}$$

Proof We set $\gcd(m, p-1) = h$, $m = ah$, and $p-1 = bh$. Then $a > 1$ since $m \nmid p-1$. Then we have

$$\prod_{\substack{j \le t \le tp \\ j \ne i}} (k^{mi} - k^{mj}) = \prod_{t \le j < i} k^{mj}(k^{m(i-j)} - 1) \cdot \prod_{i < j \le tp} k^{mi}(1 - k^{m(j-i)}).$$

By (3-1), we only need to consider values of j satisfying $p-1 \mid m(i-j)$, ie $b \mid i-j$. Then we have

$$
\begin{aligned}
e(\Pi) &= \prod_{t\le j<i} e(k^{m(i-j)}-1)\cdot \prod_{i<j\le tp} e(1-k^{m(j-i)}) \\
&= \prod_{1\le \frac{i-j}{b}\le \lfloor \frac{i-t}{b}\rfloor} e(k^{mb\frac{i-j}{b}}-1)\cdot \prod_{1\le \frac{j-i}{b}\le \lfloor \frac{tp-i}{b}\rfloor} e(k^{mb\frac{j-i}{b}}-1) \\
&= \prod_{1\le j\le \lfloor \frac{i-t}{b}\rfloor} e(k^{mbj}-1)\cdot \prod_{1\le l\le \lfloor \frac{tp-i}{b}\rfloor} e(k^{mbl}-1) \\
&= \sum_{1\le j\le \lfloor \frac{i-t}{b}\rfloor} \nu(mbj)+ \sum_{1\le l\le \lfloor \frac{tp-i}{b}\rfloor} \nu(mbl) \\
&= (e(m)+1)\Big(\Big\lfloor \frac{i-t}{b}\Big\rfloor+\Big\lfloor \frac{tp-i}{b}\Big\rfloor\Big)+e\Big(\Big\lfloor \frac{i-t}{b}\Big\rfloor !\Big)+e\Big(\Big\lfloor \frac{tp-i}{b}\Big\rfloor !\Big) \\
&\le (e(m)+1)\frac{tp-t}{b}+e\Big(\Big(\Big\lfloor \frac{i-t}{b}\Big\rfloor+\Big\lfloor \frac{tp-i}{b}\Big\rfloor\Big)!\Big) \\
&\le (e(m)+1)th+e((th)!).
\end{aligned}
$$

Now if $h=1$, then

$$
\begin{aligned}
e(\Pi) &\le (e(m)+1)t+e(t!) \\
&= (e(m)+1)t+\frac{t-s_p(t)}{p-1} \\
&< t\Big(e(m)+1+\frac{1}{p-1}\Big).
\end{aligned}
$$

If $h\ge 2$, then

$$
\begin{aligned}
e(\Pi) &\le (e(m)+1)th+t+e(t!) \\
&= (e(m)+1)th+t+\frac{t-s_p(t)}{p-1} \\
&< t\Big((e(m)+1)h+1+\frac{1}{p-1}\Big).
\end{aligned}
$$

On the other hand, the inequality $a-e(m)-1\ge 1$ always holds, for otherwise $e(a)+1=e(m)+1=a$ implies $a=1$ (we use $p\ge 3$ here). Now combining all above, it is easy to see $e(\Pi)<mt$ in both cases. □

Now we are going to prove Theorem 1.1. First we recall some background on A_n–spaces, for which Stasheff's original papers [21; 22] are the standard reference. Stasheff's A_n–spaces can be defined inductively with the help of Stasheff polytopes, which are also called associahedra. Explicitly, an associahedron K_n is an $(n-2)$–dimensional convex polytope whose vertices are in one to one correspondence with the parenthesizings of the word $x_1x_2\dots x_n$ and whose edges correspond to single application of the associativity rule. In particular, K_2 is a point, K_3 is a interval and K_4 is the convex hull of a pentagon. There are canonical maps between the K_n. Indeed, the family $\mathcal{K}=\{K_n\}$ can be endowed with an operadic structure such that any $\mathcal{K}$–space is the so-called A_∞–space ($\mathcal{K}$ is called A_∞–operad). Then an A_n–space is just an space with the action of $\mathcal{K}$ only up to the n–stage (the corresponding operad is called the A_n–operad). Stasheff also gave another equivalent description of A_n–spaces, which he used as definition:

Definition 3.5 [21, Definition 1] An A_n–structure on a space X consists of an n–tuple of maps

$$\begin{tikzcd}
X \arrow[r, equal] & E_1 \arrow[r, hook] \arrow[d, "p_1"] & E_2 \arrow[r, hook] \arrow[d, "p_2"] & \cdots \arrow[r, hook] & E_n \arrow[d, "p_n"] \\
* \arrow[r, equal] & B_1 \arrow[r, hook] & B_2 \arrow[r, hook] & \cdots \arrow[r, hook] & B_n
\end{tikzcd}$$

such that each p_i is a quasifibration and there is a contracting homotopy $h\colon CE_{n-1}\to E_n$ such that $h(CE_{i-1})\subset E_i$.

Note that if A_n–structure is given by the operadic action, the above diagram can be constructed such that B_i is the i^{th} "projective space" $P_n(X)$ over X (as in Milnor's construction). The reverse process was done by Stasheff. The projective space is crucial for there are nontrivial n^{th} powers in its cohomology ring.

Here, the key construction for our proof of Theorem 1.1 is the so-called modified projective space of Hemmi [13] which is an analogy of Stasheff's n–projective space [21]. Since we will not use the explicit construction of this concept, we only recall some properties stated in the following lemma.

Lemma 3.6 (see [13, Theorem 1.1]) *Let $n\geq 3$ and let X be a finite A_n–space with cohomology ring*

$$H^*(X,\mathbb{Z}/p\mathbb{Z})\cong \bigwedge(x_{2m_1-1},\dots,x_{2m_r-1}),\quad \deg(x_{2m_i-1})=2m_i-1.$$

Then there exists a modified projective space $R_n(X)$ with a map $\varepsilon\colon \Sigma X\to R_n(X)$ such that

$$\overline{\mathcal{H}}\cong \overline{A}\oplus\overline{B}=\mathbb{Z}/p\mathbb{Z}[y_{2m_1},\dots,y_{2m_r}]/(\text{height } n+1)\oplus\overline{B}$$

as rings, for some subalgebra $\overline{\mathcal{H}}$ of $H^*(R_n(X),\mathbb{Z}/p\mathbb{Z})$ and $\varepsilon^*(y_{2m_i})=\sigma^*(x_{2m_i-1})$, where the ideal under quotient in the first factor is generated by monomials of length greater than or equal to $n+1$. Further, $\overline{\mathcal{H}}$ and $\overline{B}$ are closed under the action of the mod p Steenrod algebra $\mathscr{A}_p$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 We prove the theorem by contradiction, and assume $m \nmid p-1$. By Lemma 3.6, $H^*(R_p(X))$ contains a truncated polynomial algebra

$$\mathbb{Z}/p\mathbb{Z}[y_{2m_1}, \dots, y_{2m_r}]/(\text{height } p+1) \hookrightarrow H^*(R_p(X)).$$

Let us define $Y(X) = R_p^{2pm_1+1}(X)$ to be the $(2pm_1+1)$–skeleton of $R_p(X)$. We then have a ring decomposition

$$i^*(\bar{\mathcal{H}}) \cong i^*(\bar{A}) \oplus i^*(\bar{B}),$$

where $i\colon Y(X) \hookrightarrow R_p(X)$ is the canonical inclusion. Then $y_{2m_1}^p \not\equiv 0 \bmod (i^*(\bar{B}))$. We then set $m_i = ms_i$, and apply Lemma 3.4 for $t = s_1$ and $m = m$ since $m \nmid p-1$ by assumption. Then we get $e(\Pi) < ms_1 = m_1$, which implies the condition (2-1) holds for $Y(X)$ since m_1 is the lowest degree. Further, $i^*(\bar{\mathcal{H}})$ and $i^*(\bar{B})$ are closed under the action of $\mathscr{A}_p$, hence by the argument in Section 2, $\bar{x}^p \equiv 0 \bmod (i^*(\bar{B}))$ for any $\bar{x} \in i^*(\bar{\mathcal{H}})$, which contradicts the fact that $y_{2m_1}^p \not\equiv 0 \bmod (i^*(\bar{B}))$. The proof of Theorem 1.1 is completed. □

3.2 The finiteness theorem for finite A_p–spaces

As another application, we prove the following theorem of Hubbuck and Mimura:

Theorem 3.7 [16] *Let X be a connected finite* mod p *A_p–space of rank r. Then there are only finitely many possible homotopy types for the space X.*

Proof Suppose X has the type $(m_1, m_2, \dots, m_r)$ with $m_1 \le m_2 \le \cdots \le m_r$, and form the space

$$Y(X) = \frac{R_p^{2pm_r+1}(X)}{R_p^{2m_r-1}(X)},$$

which is the $(2pm_r+1)$–skeleton of $R_p(X)$ with the $(2m_r-1)$–skeleton pinched to a point. As in the proof of Theorem 1.1, we can get a ring decomposition

$$p^{*-1}i^*(\bar{\mathcal{H}}) \cong p^{*-1}i^*(\bar{A}) \oplus p^{*-1}i^*(\bar{B})$$

using the canonical inclusion and projection, such that $p^{*-1}i^*(\bar{\mathcal{H}})$ and $p^{*-1}i^*(\bar{B})$ are closed under the action of $\mathscr{A}_p$, and $y_{2m_r}^p$ is the nontrivial module $p^{*-1}i^*(\bar{B})$. We may also fix a number $N(p,r)$ only depending on p and r such that $N(p,r) \ge \dim\ p^{*-1}i^*(\bar{\mathcal{H}})$, and notice that the largest difference of the degrees of any two elements in $p^{*-1}i^*(\bar{\mathcal{H}})$ is bounded by $2(p-1)m_r$. Suppose the even part of $p^{*-1}i^*(\bar{\mathcal{H}})$

concentrates in dimension $2t_1, 2t_2, \ldots$. Then for sufficiently large m_r we have

$$\begin{aligned} e\Big(\prod_{j\neq i}(k^{t_i}-k^{t_j})\Big) &\leq \sum_{j\neq i}(e(t_i-t_j)+1) \\ &\leq N(p,r)\lfloor \log_p(2(p-1)m_r)\rfloor + N(p,r) \\ &< m_r \end{aligned}$$

for any i, ie the condition (2-1) holds, which contradicts the existence of the nontrivial p^{th} power in $p^{*-1}i^*(\bar{\mathcal{H}})$. Accordingly the largest dimension of the generators is bounded and there are only finitely many possible types for X. Also by [6, Corollary 4.2], there are only finitely many homotopy types for each certain type. Then in all there are finitely many homotopy types for fixed rank. □

4 Rank 3 mod 3 homotopy associative H–spaces

For rank 3 mod 3 homotopy associative H–spaces, we will consider Stasheff's 3–projective space instead of Hemmi's modified projective space used in the proof of Theorem 1.1. The key lemma analogous to Lemma 3.6 for projective spaces is the following well-known result.

Lemma 4.1 (see eg [17]) *Let $n \geq 3$ and X be a finite A_n–space with cohomology ring*

$$H^*(X, \mathbb{Z}/p\mathbb{Z}) \cong \bigwedge(x_{2m_1-1}, \ldots, x_{2m_r-1}), \quad \deg(x_{2m_i-1}) = 2m_i - 1,$$

such that each x_{2m_i-1} is A_n–primitive, ie x_{2m_i-1} lies in the image of a series of natural morphisms

$$H^*(P_n(X)) \to H^*(P_{n-1}(X)) \to \cdots \to H^*(P_1(X) = \Sigma X) \overset{\cong}{\leftarrow} H^{*-1}(X).$$

Then we have ring isomorphism

$$H^*(P_n(X), \mathbb{Z}/p\mathbb{Z}) \cong A \oplus B = \mathbb{Z}/p\mathbb{Z}[y_{2m_1}, \ldots, y_{2m_r}]/(\text{height } n+1) \oplus B$$

as $\mathscr{A}_p$–modules and $A^+ \cdot B = 0$, where $\deg(y_{2m_i}) = 2m_i$.

Notice that the corresponding result in the context of K–theory can be easily deduced, and for rank 3 mod 3 homotopy associative H–spaces, the primitivity assumption is automatically satisfied. To prove Theorem 1.2, we will also use the following theorem of Wilkerson.

Theorem 4.2 [24, Theorems 6.1 and 6.2] *Let X be a finite* mod p *A_p–space with cohomology ring $H^*(X,\mathbb{Z}/p\mathbb{Z}) \cong \bigwedge(x_{2m_1-1},\dots,x_{2m_r-1})$, with $m_1 \le m_2 \le \cdots \le m_r$ and $m_r > p$. Then:*

(1) *There is an x_{2m_k-1} with $m_r - m_k = s(p-1)$ for some $1 \le s \le e(m_r)+1$.*

(2) *If $p \nmid m_i$ for some i, there is an x_{2m_j-1} such that $m_j = k_j m_i - p + 1$ for some $1 \le k_j \le p$.*

Combining Theorem 1.1 and Theorem 4.2, we are left to consider the following four cases for the possible types of the mod 3 A_3–space X in Theorem 1.2:

Case 1 $3 \mid m$, $3 \mid n$ and $m-n = 2s$ with $1 \le s \le e(m)+1$,

Case 2 $3 \mid m$, $3 \nmid n$ and $m-n = 2s$ with $1 \le s \le e(m)+1$,

Case 3 $3 \nmid m$ and $m-n = 2s$ with $1 \le s \le e(m)+1$,

Case 4 $m-r = 2t$ with $1 \le t \le e(m)+1$, and $m-n \ne 2s$ for any s such that $1 \le s \le e(m)+1$.

For Case 1, we need the following lemma:

Lemma 4.3 *Under the condition of Theorem 1.2 and Case 1, we have:*

(1) *If $r=2$, $m>n>6$ and $e(m) \ge e(n)+2$, then*

$$8e(n)+23 \ge n.$$

(2) *If $r=2$, $m>n>6$ and $e(m) = e(n)+1$, then*

$$8\max\{e(3n-m), e(3n-2m)\}+15 \ge n.$$

(3) *If $m \le 3r$, $e(m) \ge e(n)+2$, then*

$$7e(n)+\lfloor \log_3(m-r)\rfloor + 24 \ge m \quad \text{or} \quad 8\lfloor \log_3(m-r)\rfloor + 24 \ge 3r.$$

(4) *If $m \le 3r$, $e(m) = e(n)+1$, then*

$$7\max\{e(3n-m), e(3n-2m)\} + \lfloor \log_3(m-r)\rfloor + 17 \ge m$$

or

$$8\lfloor \log_3(m-r)\rfloor + 24 \ge 3r.$$

Proof By the condition, we have a $\{\psi^k\}$–module $K = \mathbb{Z}_{(3)}[x_r, x_n, x_m]/(\text{height } 4)$, where the subscripts refer to the filtration degree. For (1) and (2) we have $r = 2$, and we only need to consider $K' = K - \{x_r^i \mid i = 1, 2, 3\}$. We can set

$$S = \big\{2i + jn + km \mid (i, j, k) \neq (i, 0, 0),\ 0 \leq |j|, |k| \leq 3,\ 0 \leq |i| \leq 2\big\},$$

and define $\Phi(i, j, k) = |2i + jn + km|$. For (1) we have $e(\Phi(0, j, k)) \leq e(n) + 1$ and $e(\Phi(i, j, k)) = 0$ if $|i| = 1$, or 2. And we notice that there are nine elements of the form $x_n^* x_m^*$, five elements of the form $x_r^1 x_n^* x_m^*$, and two elements of the form $x_r^2 x_n^* x_m^*$ in K'. Then

$$\begin{aligned} e\Big(\prod_{(\tilde{i},\tilde{j},\tilde{k})\neq(0,j,k)} (2^{jn+km} - 2^{2\tilde{i}+\tilde{j}n+\tilde{k}m})\Big) &\leq \sum e\big(\Phi(-\tilde{i}, j-\tilde{j}, k-\tilde{k})\big) + 15 \\ &\leq 8(e(n)+1) + 15 \\ &= 8e(n) + 23. \end{aligned}$$

Similarly, we have

$$e\Big(\prod_{(\tilde{i},\tilde{j},\tilde{k})\neq(1,j,k)}\Big) \leq 4e(n) + 19 \quad \text{and} \quad e\Big(\prod_{(\tilde{i},\tilde{j},\tilde{k})\neq(2,j,k)}\Big) \leq e(n) + 16.$$

Since condition (2-1) should fail for X, we must have $8e(n) + 23 \geq n$.

The remaining three claims can be proved similarly, and notice that for (3) and (4), we work with $K' = K - \{x_r, x_n\}$ if $m \leq 2r$ and with $K' = K - \{x_r, x_n, x_r^2\}$ if $m > 2r$. $\square$

Now we are ready to deal with Case 1:

Proposition 4.4 *Under the condition of Theorem 1.2 and Case 1, the only possible types of X are*

$$\begin{gathered}(2,3,9),\ (2,12,18),\ (2,21,27),\ (2,30,36),\ (2,39,45),\\ (7,12,18),\ (10,12,18),\ (16,30,36),\ (19,30,36).\end{gathered}$$

Proof By Theorem 1.1, we have $\gcd(r, n, m) \leq 2$, so $3 \nmid r$. Hence by Theorem 4.2, we have $x = \lambda r - 2$ with $\lambda \in \{1, 2, 3\}$ and $x \in \{r, n, m\}$. Then $r = 2$ or $n = 2r - 2$ or $m = 2r - 2$.

We prove the proposition under the condition $e(m) > e(n)$ first:

(1) If $r=2$, $n>6$ and $e(m)\geq e(n)+2$, by Lemma 4.3 we have $8e(n)+23\geq n$. Then

$$\begin{aligned} 3^{e(m)}\cdot f &= m = n+2s \\ &\leq 8e(n)+23+2(e(n)+1) \\ &= 10e(n)+25 \\ &\leq 10e(m)+5. \end{aligned}$$

Since $e(m)\geq 3$, we have $m=27$ and $e(m)=3$. Then $e(n)=e(s)=1$ and n is odd. Now it is not hard to check that $(2,21,27)$ is the only possible type satisfying all the conditions.

(2) If $r=2$, $n>6$ and $e(m)=e(n)+1$, by Lemma 4.3,

$$8\max\{e(3n-m),e(3n-2m)\}+15\geq n.$$

If $8e(3n-m)+15\geq n$, then

$$8e(n-s)+12\geq 8e(n-s)+15-s\geq n-s$$

for

$$e(n-s)=e(2(n-s)=3n-m)\geq e(m)\geq 2 \quad \text{and} \quad s\geq 3.$$

Then it is easy to show that $n-s=9,18$ or 27. In any case, $s\leq e(m)+1\leq 4$, which implies $s=3$. And then $m-n=6$ and $n=12,21$ or 30. But since $e(m)=e(n)+1=2$, only $(2,12,18)$ or $(2,30,36)$ is possible for our X.

If $8e(3n-2m)+15\geq n$, then

$$8e(n-4s)+3\geq 8e(n-4s)+15-4s\geq n-4s$$

for

$$n-4s=3n-2m \quad \text{and} \quad s\geq 3.$$

Then we get $n-4s=9,18$ or 27. Again since $e(n-4s)\geq e(m)\geq 2$ and $s\leq e(m)+1$, we have $s=3$. Then $m-n=6$ and $n=21,30$ or 39 and only $(2,30,36)$ and $(2,39,45)$ survive.

(3) If $m\leq 3r$, $e(m)\geq e(n)+2$, by Lemma 4.3 we have

$$7e(n)+\lfloor\log_3(m-r)\rfloor+24\geq m \quad \text{or} \quad 8\lfloor\log_3(m-r)\rfloor+24\geq 3r.$$

We also notice that $r \neq 2$, which by our earlier discussion implies $n = 2r-2$ or $n = 2m-2$. If the first inequality and $n = 2r-2$ hold, then

$$\begin{aligned} 2r-2 = n < m &\leq 7e(n) + \lfloor \log_3(m-r) \rfloor + 24 \\ &\leq 7e(r-1) + \lfloor \log_3(2r) \rfloor + 24 \\ &\leq 8\lfloor \log_3 r \rfloor + 25, \end{aligned}$$

which implies $r \leq 21$. Then $m \leq 3r \leq 63$ implies $m = 27$ or 54 for $e(m) \geq 3$. So $e(m) = 3$ and $e(n) = 1$. Since $3 \mid s$ and $s \leq e(m)+1$, we have $s = 3$ and $m-n = 6$. Then we see $m = 27$ is impossible for n is even, while $m = 54$ leads to $r = 25$, which contradicts our previous calculation. Similar arguments can be applied to the other three cases, which will show there are no types left.

(4) If $m \leq 3r$, $e(m) = e(n)+1$, by Lemma 4.3 and similar calculations as in part (3), we get $(r,n,m) = (7,12,18)$, $(10,12,18)$, $(16,30,36)$ or $(19,30,36)$.

(5) By Theorem 1.1, the only remaining case under condition $e(m) > e(n)$ is $n \leq 3r$ but $m > 3r$. If $r = 2$, then $n = 3$ or 6, which gives $(r,n,m) = (2,3,9)$. When $n = 2r-2$, we have $\frac{1}{3}m + 2 < m-n = 2s \leq 2e(m)+2$, which is impossible. Further, $m = 2r-2$ can not hold by our assumption.

We have proved the proposition when $e(m) > e(n)$. If $e(n) \geq e(m)$, then $e(s) = e(m-n) \geq e(m) \geq s-1 \geq 0$, which implies $s = 1$ and $m-n = 2$. However, since $3 \mid m$ and $3 \mid n$, this is impossible. □

For the remaining cases, we will also use a theorem of Hemmi:

Theorem 4.5 ([12, Theorem 1.2]; also see [13, Section 8]) *Let X be a homotopy H–space with $H^*(X;\mathbb{Z}/3\mathbb{Z})$ being finite. Then for any $n \in \mathbb{Z}$ with $n \not\equiv 0 \bmod 3$ and $n > 3$, if*

$$QH^{2(3^a \cdot 2t)-1}(X,\mathbb{Z}/3\mathbb{Z}) = 0 \quad \text{for } t \geq n-1, \tag{4-1}$$

then

$$\mathscr{P}^{3^a}\colon QH^{2(3^a(n-2))-1}(X,\mathbb{Z}/3\mathbb{Z}) \to QH^{2(3^a n)-1}(X,\mathbb{Z}/3\mathbb{Z}) \tag{4-2}$$

is an epimorphism, where $QH^*(X,\mathbb{Z}/3\mathbb{Z}) = H^*(X,\mathbb{Z}/3\mathbb{Z})/DH^*(X,\mathbb{Z}/3\mathbb{Z})$ and $DH^*(X,\mathbb{Z}/3\mathbb{Z})$ is the submodule consisting of decomposable elements.

Proposition 4.6 *Under the conditions of Theorem 1.2 and Case 2, the only possible types of X are*

$$(2,4,6),\ (3,4,6),\ (3,5,9),\ (6,8,12).$$

Proof Since $3 \nmid n$, by Theorem 4.2, we have $x = \lambda n - 2$ with x and λ as before. Then either $r = n-2$ or $m = 2n-2$.

(1) $r = n-2$. If $m > 3r$, then $m - n > m - \left(\frac{1}{3}m + 2\right) = \frac{2}{3}m - 2$. So we have $\frac{2}{3}m - 2 = 2s < 2e(m) + 2$, which implies $m = 9$. Then $(r,n,m) = (2,4,9)$ contradicts the fact that $m-n$ is even.

If $2n - 2 = 2r + 2 \le m \le 3r$, then $\frac{1}{2}m - 1 \le m - n = 2s \le 2e(m) + 2$, which implies $(r,n,m) = (2,4,6)$ or $(3,5,9)$.

If $m < 2n-2$ and $n = 3k+2$ for some k, then in the $\mathscr{A}_p$–module

$$\bar{K} = \mathbb{Z}/3\mathbb{Z}[x_r, x_n, x_m]/(\text{height } 4),$$

$\mathscr{P}^1(x_r) = c x_n$ with $c \not\equiv 0 \bmod 3$ by Theorem 4.5. By the Adem relation

$$\mathscr{P}^1 \mathscr{P}^3 \mathscr{P}^{3k-1} = \epsilon \mathscr{P}^1 \mathscr{P}^{3k+2} + 2\mathscr{P}^{3k+2} \mathscr{P}^1, \tag{4-3}$$

we have $\mathscr{P}^{3k-1}(x_r) \neq 0$, which implies $9k - 2 = 3n - 8$ has to be the degree of some monomial in K. Then by direct computation, we get $n = 8$ and $r = 6$, which implies $m < 14$. Since $3 \mid m$, we have $m = 9$ or 12. When $m = 9$, $m - n = 1$ is odd, which is impossible. So we have $(r,n,m) = (6,8,12)$.

If $m < 2n-2$ and $n = 3k+1$, then $r = 3k-1$ which by Theorem 4.2 implies $x = \lambda r - 2$ with $x \in \{r,n,m\}$ and $\lambda \in \{1,2,3\}$. Then we have $r = 2$ or $n = 2r-2$, both of which are impossible.

(2) $m = 2n-2$. We have $\frac{1}{2}m - 1 = m - n = 2s \le 2e(m) + 2$, which implies $(r,n,m) = (2,4,6)$ or $(3,4,6)$. □

Proposition 4.7 *Under the conditions of Theorem 1.2 and Case 3, the only possible types of X are*

$$(2,3,5),\ (2,6,8),\ (3,5,7),\ (3,6,8),\ (4,6,8),\ (5,6,8), (6,8,10),$$
$$(8,12,14),\ (12,18,20),\ (18,24,26),\ (21,27,29),\ (30,36,38).$$

Proof Since $3 \nmid m$, we have $m - n = 2$. Then by Theorem 4.5, we have $\mathscr{P}^1(x_n) \neq 0$.

(1) If $m = 3k+1$, we have $n = 3k-1$, which by Theorem 4.2 implies $x = \lambda n - 2$ as before. Then $r = n-2$, or $m = 2n-2$, or $m = 3n-2$; the latter two cases are easy to check and are impossible. For $r = n-2$, we apply Theorem 4.5 to get $\mathscr{P}^1(x_r) \neq 0$, and again by Adem relation (4-3), we get $\mathscr{P}^{r-1}(x_r) \neq 0$, which implies $(r,n,m) = (3,5,7)$ or $(6,8,10)$.

(2) If $m = 3k+2$, again by Adem relation (4-3) we have $\mathscr{P}^{n-1}(x_n) \neq 0$. By comparing the degree and applying Theorem 4.2, we get a list of possible types:

$(2,3,5)$, $(2,6,8)$, $(3,6,8)$, $(4,6,8)$, $(5,6,8)$, $(8,12,14)$ and also a special type $(r,r+6,r+8)$ with $3\mid r$. For this remaining case, if $r=3l$ with $l\not\equiv 1 \bmod 3$, Theorem 4.5 implies $\mathscr{P}^3(x_r)\neq 0$. By the Adem relation

$$\mathscr{P}^9\mathscr{P}^{3l-1}=\epsilon_1\mathscr{P}^{3l+8}+\epsilon_2\mathscr{P}^{3l+7}\mathscr{P}^1+\epsilon_3\mathscr{P}^{3l+6}\mathscr{P}^2+\mathscr{P}^{3l+5}\mathscr{P}^3, \tag{4-4}$$

we have $\mathscr{P}^{3l-1}(x_r)\neq 0$, which gives $(r,n,m)=(18,24,26)$.

For $l\equiv 1 \bmod 3$, we argue similarly as in Lemma 4.3 to get the condition $m\leq 44$. Then the possible types are $(12,18,20)$, $(21,27,29)$ and $(30,36,38)$. □

Proposition 4.8 *Under the condition of Theorem 1.2 and Case 4, the only possible types of X are*

$$(2,3,4),\ (2,3,6).$$

Proof If $m>3r$, then $2t=m-r>2r$, ie $r<t$. Then we have

$$m=r+2t<3t\leq 3e(m)+3,$$

which is impossible. So we have $m\leq 3r$.

If $3\nmid m$, then $m-r=2$ and $(r,n,m)=(r,r+1,r+2)$. Further, if $3\mid r$, then $3\nmid n$, which implies $x=\lambda n-2$ as usual. However, it is easy to check the latter is impossible. Then we get $3\nmid r$, which implies $x=\lambda r-2$. In this case, the only possible type is $(r,n,m)=(2,3,4)$.

Now suppose $3\mid m$. If $3\nmid r$, we have $r=2$, $n=2r-2$, $n=3r-2$ or $m=2r-2$ by Theorem 4.2. When $r=2$, we get $(r,n,m)=(2,3,6)$, while $(2,5,6)$ is impossible since $\lambda 5-2\in\{3,8,13\}$. When $n=2r-2$, we have $r=\frac{1}{2}n+1<\frac{1}{2}m+1$. Then $\frac{1}{2}m-1<m-r=2t\leq 2e(m)+2$, which implies $m=6$ or 9. When n is even, $n=4$ when $m=6$, which implies $r=3$. But $3\nmid r$, so $m=6$ is impossible. If $m=9$, then we have $(r,n,m)=(4,6,9)$ or $(5,8,9)$, both of which are impossible since $9-4\neq 2t$ and $\lambda 8-2\in\{6,14,22\}$. The other two cases can be treated similarly and lead to no possible types.

If $3\mid r$, then $3\nmid n$, which implies $r=n-2$ or $m=2n-2$. When $r=n-2$, we argue exactly as in the proof of the first case in Proposition 4.6 and get no possible types in this case. When $m=2n-2$, we see $r<n=\frac{1}{2}m+1$, which implies $\frac{1}{2}m-1<m-r=2t\leq 2e(m)+2$. Again, no types survive. □

We recall the following theorem of Wilkerson and Zabrodsky [26], which was also reproved by McCleary [18], and later strengthened by Hemmi in [14] where the assumption of the primitivity of the generators was removed:

Theorem 4.9 *Let X be a simply connected* mod p *H–space with cohomology ring $H^*(X,\mathbb{Z}/p\mathbb{Z}) = \bigwedge(x_{2m_1-1},\dots,x_{2m_r-1})$, with $m_1 \le m_2 \le \cdots \le m_r$. If $m_r - m_1 < 2(p-1)$, then X is p–quasiregular, ie X is p–equivalent to a product of odd spheres and copies of $B_n(p)$, where $B_n(p)$ is the S^{2n+1}–fibration over $S^{2n+1+2(p-1)}$ characterized by α_p.*

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 We collect all the types obtained from Propositions 4.4, 4.6, 4.7 and 4.8, and prove the theorem case by case.

First, we notice that $(2,3,4)$, $(2,3,5)$, $(3,4,6)$ and $(5,6,8)$ are quasiregular by Theorem 4.9.

If $(r,n,m)=(4,6,8)$, we already know $\mathscr{P}^1(x_n)=x_m$ in $\bar{K}=\mathbb{Z}/3\mathbb{Z}[x_r,x_n,x_m]/(\text{height } 4)$. Then for degree reasons we have

$$\mathscr{P}^4(x_r)=\mathscr{P}^1\mathscr{P}^3(x_r)=\mathscr{P}^1(\lambda x_r x_n)=\lambda\mathscr{P}^1(x_r)x_n+\lambda x_r x_m,$$

which contradicts that $\mathscr{P}^4(x_r)=x_r^3$. So $(4,6,8)$ cannot be the type of X.

If $(r,n,m)=(3,5,9)$, we still have $\mathscr{P}^1(x_r)=x_n$ by Theorem 4.5. Then by Adem relation (4-3), we have $\mathscr{P}^2(x_r)\neq 0$, which is impossible since $K_7=0$.

If $(r,n,m)=(8,12,14)$, we know $\mathscr{P}^1(x_n)=x_m$ in $\bar{K}$. Then for degree reasons we have

$$2\mathscr{P}^8(x_r)=\mathscr{P}^1\mathscr{P}^1\mathscr{P}^6(x_r)=\mathscr{P}^1\mathscr{P}^1(\lambda x_r x_n)=\lambda x_r\mathscr{P}^1(x_m),$$

which implies $\mathscr{P}^1(x_m)=\mu x_r^2$ with $3\nmid\mu$. On the other hand, we have $\mathscr{P}^{11}(x_n)\neq 0$ from the proof of Proposition 4.7, which implies that $\mathscr{P}^1\colon \bar{K}_{30}=\mathbb{Z}/p\mathbb{Z}(x_r^2x_m)\to\bar{K}_{32}$ is not the zero map. But $\mathscr{P}^1(x_r^2x_m)=x_r^2\mathscr{P}^1(x_m)=0$ and then $(r,n,m)=(8,12,14)$ is impossible.

If $(r,n,m)=(10,12,18)$, we have $\mathscr{P}^1\mathscr{P}^9(x_r)=\mathscr{P}^{10}(x_r)=x_r^3$, which implies $\mathscr{P}^1(x_r x_m)=x_r\mathscr{P}^1(x_m)+\mathscr{P}^1(x_r)x_m=\lambda x_r^3$ with $3\nmid\lambda$. Then we have $\mathscr{P}^1(x_r)=0$ and $\mathscr{P}^1(x_m)=\lambda x_r^2$. Then by the Adem relation

$$\mathscr{P}^3\mathscr{P}^7=-\mathscr{P}^{10}+\mathscr{P}^9\mathscr{P}^1, \tag{4-5}$$

we have $\mathscr{P}^3(x_n^2)=\mu x_r^3$ with $3\nmid\mu$. However, $\mathscr{P}^3(x_n^2)=2x_n\mathscr{P}^3(x_n)$ is not equal to μx_r^3, so $(10,12,18)$ cannot be the type of X.

If $(r,n,m)=(12,18,20)$, we have $\mathscr{P}^1(x_n)=x_m$. Again, by Adem relation (4-3), we have $\mathscr{P}^{17}(x_n)\neq 0$ and $\mathscr{P}^3\colon \bar{K}_{52}=\mathbb{Z}/p\mathbb{Z}(x_r x_m^2)\to\bar{K}_{58}=\mathbb{Z}/p\mathbb{Z}(x_n x_m^2)$ is not the zero map, which implies $\mathscr{P}^3(x_r)=x_n$. However, the Adem relation

$$\mathscr{P}^3\mathscr{P}^9=\mathscr{P}^{12}+\mathscr{P}^{11}\mathscr{P}^1 \tag{4-6}$$

implies $\mathscr{P}^3(x_r x_n) = \pm x_r^3$, which contradicts the equality

$$\mathscr{P}^3(x_r x_n) = x_r \mathscr{P}^3(x_n) + \mathscr{P}^3(x_r) x_n = x_r \mathscr{P}^3(x_n) + x_n^2.$$

So $(r, n, m) = (12, 18, 20)$ is impossible.

For $(r, n, m) = (2, 12, 18)$, or $(7, 12, 18)$, we first prove the following lemma:

Lemma 4.10 *Let X be a p–local A_p–space with cohomology ring $H^*(X, \mathbb{Z}/p\mathbb{Z}) \cong \bigwedge(x_{2m_1-1}, \dots, x_{2m_r-1})$, such that each x_{2m_i-1} is A_p–primitive, $m_1 \le m_j$ for all j, and $p < m_r$. Then there is an x_{2m_k-1} such that $\mathscr{P}^i(x_{2m_k-1}) = x_{2m_r-1}$ for some suitable nonzero i.*

Proof This is essentially [24, Lemma 4.4], which claims that in the $\{\psi^p\}$–submodule $K = \mathbb{Z}_{(p)}[x_{m_1}, \dots, x_{m_r}]/(\text{height } p+1)$ of $K(P_p(X)) \otimes \mathbb{Z}_{(p)}$, there is an x_{m_k} such that

$$\psi^p(x_{m_k}) = \lambda x_{m_r} + \text{other terms}$$

with $\lambda \ne 0$, for in [5, Theorem 6.5], Atiyah has shown that if $\psi^p(x_q) = \sum_i p^{q-i} x_i$, then $\mathscr{P}^i(\bar{x}_q) = \bar{x}_i$ holds on the cohomology level. □

Now we return to the proof Theorem 1.2. Using Lemma 4.10, we see $\mathscr{P}^3(x_{12}) = x_{18}$ holds in $\bar{K} \subset H^*(P_3(X))$ for both mentioned cases. Then we apply Adem relation (4-6) to x_{12}. Since in both cases $\mathscr{P}^{11}\mathscr{P}^1(x_{12}) = 0$, we have $\mathscr{P}^3\mathscr{P}^9(x_{12}) = \pm x_{12}^3$. However, $\bar{K}_{30} = \mathbb{Z}/p\mathbb{Z}(x_{12}x_{18})$, and since $\bar{K}$ is truncated,

$$\mathscr{P}^3(x_{12}x_{18}) = x_{12}\mathscr{P}^3(x_{18}) + \mathscr{P}^3(x_{12})x_{18} = x_{12}\mathscr{P}^3(x_{18}) + x_{18}^2,$$

which is not equal to $\pm x_{12}^3$. Accordingly, neither case can be the type of X.

We notice that $(r, n, m) = (2, 3, 6)$ is impossible directly by the above lemma.

For the remaining cases which do not appear in the final list, we can check whether the condition (2-1) fails or not in an appropriate $\{\psi^k\}$–module K' constructed from K (with the help of a computer), and find that (2-1) holds when (r, n, m) is one of $(2, 3, 9)$, $(2, 21, 27)$, $(2, 30, 36)$, $(2, 39, 45)$, $(18, 24, 26)$, $(16, 30, 36)$, $(19, 30, 36)$, $(21, 27, 29)$ or $(30, 36, 38)$, which implies X cannot be a mod 3 A_3–space. □

Acknowledgements

The authors would like to thank Professor Stephen D Theriault and Professor Mamoru Mimura for helpful discussions and comments, and are also indebted to Professor John R Harper for suggesting the reference [12] and valuable knowledge about H–spaces of rank p and higher associativity. We wish to thank the referee most warmly for suggestions and comments on using the modified projective space of Hemmi [13] which has greatly improved the article, and also the careful reading of our manuscript. We are also indebted to Professor Jérôme Scherer and Professor Fred Cohen for careful reading of the manuscript and many valuable suggestions which have improved the paper.

The authors are partially supported by the Singapore Ministry of Education research grant (AcRF Tier 1 WBS No. R-146-000-222-112). The second author is also supported by a grant (No. 11329101) of NSFC of China.

References

[1] **J F Adams**, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. 72 (1960) 20–104 MR

[2] **J F Adams**, *The sphere, considered as an H–space* mod p, Quart. J. Math. Oxford. Ser. 12 (1961) 52–60 MR

[3] **J F Adams**, *H–spaces with few cells*, Topology 1 (1962) 67–72 MR

[4] **J F Adams**, **M F Atiyah**, *K–theory and the Hopf invariant*, Quart. J. Math. Oxford Ser. 17 (1966) 31–38 MR

[5] **M F Atiyah**, *Power operations in K–theory*, Quart. J. Math. Oxford Ser. 17 (1966) 165–193 MR

[6] **R A Body**, **R R Douglas**, *Homotopy types within a rational homotopy type*, Topology 13 (1974) 209–214 MR

[7] **R R Douglas**, **F Sigrist**, *Sphere bundles over spheres and H–spaces*, Topology 8 (1969) 115–118 MR

[8] **J Grbić**, **J Harper**, **M Mimura**, **S Theriault**, **J Wu**, *Rank $p-1$ mod-p H–spaces*, Israel J. Math. 194 (2013) 641–688 MR

[9] **N L Hagelgans**, *Local spaces with three cells as H–spaces*, Canad. J. Math. 31 (1979) 1293–1306 MR

[10] **J R Harper**, *The* mod 3 *homotopy type of F_4*, from "Localization in group theory and homotopy theory, and related topics" (P Hilton, editor), Lecture Notes in Math. 418, Springer (1974) 58–67 MR

[11] **J Harper**, **A Zabrodsky**, *Evaluating a p^{th} order cohomology operation*, Publ. Mat. 32 (1988) 61–78 MR

[12] **Y Hemmi**, *Homotopy associative finite H–spaces and the* mod 3 *reduced power operations*, Publ. Res. Inst. Math. Sci. 23 (1987) 1071–1084 MR

[13] **Y Hemmi**, *On exterior A_n–spaces and modified projective spaces*, Hiroshima Math. J. 24 (1994) 583–605 MR

[14] **Y Hemmi**, *Mod p decompositions of mod p finite H–spaces*, Mem. Fac. Sci. Kochi Univ. Ser. A Math. 22 (2001) 59–65 MR

[15] **J R Hubbuck**, *Generalized cohomology operations and H–spaces of low rank*, Trans. Amer. Math. Soc. 141 (1969) 335–360 MR

[16] **J R Hubbuck**, **M Mimura**, *The number of mod p $A(p)$–spaces*, Illinois J. Math. 33 (1989) 162–169 MR

[17] **N Iwase**, *On the K–ring structure of X–projective n–space*, Mem. Fac. Sci. Kyushu Univ. Ser. A 38 (1984) 285–297 MR

[18] **J McCleary**, *Mod p decompositions of H–spaces; another approach*, Pacific J. Math. 87 (1980) 373–388 MR

[19] **M Mimura**, **G Nishida**, **H Toda**, *On the classification of H–spaces of rank 2*, J. Math. Kyoto Univ. 13 (1973) 611–627 MR

[20] **M Mimura**, **G Nishida**, **H Toda**, *Mod p decomposition of compact Lie groups*, Publ. Res. Inst. Math. Sci. 13 (1977/78) 627–680 MR

[21] **J D Stasheff**, *Homotopy associativity of H–spaces, I*, Trans. Amer. Math. Soc. 108 (1963) 275–292 MR

[22] **J D Stasheff**, *Homotopy associativity of H–spaces, II*, Trans. Amer. Math. Soc. 108 (1963) 293–312 MR

[23] **J D Stasheff**, *The mod p decomposition of Lie groups*, from "Localization in group theory and homotopy theory, and related topics" (P Hilton, editor), Lecture Notes in Math. 418, Springer (1974) 142–149 MR

[24] **C Wilkerson**, *K–theory operations in mod p loop spaces*, Math. Z. 132 (1973) 29–44 MR

[25] **C Wilkerson**, *Spheres which are loop spaces mod p*, Proc. Amer. Math. Soc. 39 (1973) 616–618 MR

[26] **C Wilkerson**, *Mod p decomposition of mod p H–spaces*, from "Algebraic and geometrical methods in topology" (L F McAuley, editor), Lecture Notes in Math. 428, Springer (1974) 52–57 MR

[27] **A Zabrodsky**, *Homotopy associativity and finite CW complexes*, Topology 9 (1970) 121–128 MR

[28] **A Zabrodsky**, *The classification of simply connected H–spaces with three cells, I*, Math. Scand. 30 (1972) 193–210 MR

[29] **A Zabrodsky**, *The classification of simply connected H–spaces with three cells, II*, Math. Scand. 30 (1972) 211–222 MR

[30] **A Zabrodsky**, *On the realization of invariant subgroups of $\pi_*(X)$*, Trans. Amer. Math. Soc. 285 (1984) 467–496 MR

Department of Mathematics, National University of Singapore
10 Lower Kent Ridge Road, Singapore 119076, Singapore

a0123769@u.nus.edu, matwuj@nus.edu.sg

http://www.math.nus.edu.sg/~matwujie

Received: 11 February 2016 Revised: 9 December 2016