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A Khovanov stable homotopy type for colored links

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We extend Lipshitz and Sarkar's definition of a stable homotopy type associated to a link L whose cohomology recovers the Khovanov cohomology of L. Given an assignment c (called a *coloring*) of a positive integer to each component of a link L, we define a stable homotopy type $\mathcal{X}_{\mathrm{col}}(L_c)$ whose cohomology recovers the c–colored Khovanov cohomology of L. This goes via Rozansky's definition of a categorified Jones–Wenzl projector P_n as an infinite torus braid on n strands.

We then observe that Cooper and Krushkal's explicit definition of P_2 also gives rise to stable homotopy types of colored links (using the restricted palette $\{1, 2\}$), and we show that these coincide with $\mathcal{X}_{\mathrm{col}}$. We use this equivalence to compute the stable homotopy type of the $(2, 1)$–colored Hopf link and the 2–colored trefoil. Finally, we discuss the Cooper–Krushkal projector P_3 and make a conjecture of $\mathcal{X}_{\mathrm{col}}(U_3)$ for U the unknot.

1 Introduction

1.1 Categorification

Given a semisimple Lie algebra $\mathfrak{g}$ and a link $L \subset S^3$ in which each component of L is decorated by an irreducible representation of $\mathfrak{g}$, the Reshetikhin–Turaev construction returns an invariant of that link that can, in principle, be computed combinatorially from any diagram of L. The standard example is the Jones polynomial, which arises from decorating all components with the fundamental representation $V = V^1$ of $\mathfrak{sl}_2$ (here the superscript 1 on the representation refers to the highest weight of V being 1). There are then two obvious first directions in which one can generalize.

On the one hand, one might vary the Lie algebra and consider instead $\mathfrak{sl}_n$, but still with the fundamental representation of $\mathfrak{sl}_n$. Each invariant obtained this way is a 1–variable specialization of the 2–variable HOMFLYPT polynomial, and satisfies an oriented skein relation, which yields the benefit of easy computability.

On the other hand, one might stick with $\mathfrak{sl}_2$, but vary the irreducible representation. There is one irreducible $(n+1)$–dimensional representation V^n (of highest weight n) for each $n \geq 1$. Decorating with V^n gives rise to the so-called n–colored Jones polynomial. The colored Jones polynomials no longer satisfy such pleasant skein relations, but they are powerful — for example giving rise to 3–manifold invariants (also called Reshetikhin–Turaev invariants or, in another form, Turaev–Viro invariants).

Both the $\mathfrak{sl}_n$ polynomials and the colored Jones polynomials admit categorifications — that is, they can be exhibited as the graded Euler characteristic of bigraded cohomology theories. In the case of $\mathfrak{sl}_n$, this is Khovanov–Rozansky cohomology [6]. In the case of the colored Jones polynomial there are constructions due to many authors, some inequivalent, although the two we shall be considering in fact give isomorphic cohomologies. The first is due to Rozansky [10] and the second due to Cooper and Krushkal [3]. In both cases, the fundamental representation of $\mathfrak{sl}_2$ gives Khovanov cohomology [5].

1.2 Spacification

Recently it has been shown that Khovanov cohomology admits a *spacification*, that is, for any link there is a stable homotopy type $\mathcal{X}(L)$ whose cohomology gives Khovanov cohomology (the *bi*grading of Khovanov cohomology is recovered from a splitting of $\mathcal{X}(L)$ into wedge of spaces indexed by the integers). This is work due to Lipshitz and Sarkar [8]. We note that the term "spacification" is not yet well-defined, since it is unclear exactly what properties one should require of it. (For example: should just taking a wedge of the Moore spaces determined by the cohomology count as a spacification?) Nevertheless, we find it a convenient shorthand for now.

It is a natural question if other Reshetikhin–Turaev invariants admitting categorifications can further be spacified. In the $\mathfrak{sl}_n$ case, work by two of the authors with Dan Jones [4] has constructed an $\mathfrak{sl}_n$ stable homotopy type given the input of a *matched* knot diagram. There is good evidence that this stable homotopy type should be diagram-independent. For $n = 2$ it agrees with the stable homotopy type due to Lipshitz and Sarkar.

The case of the colored Jones invariants is, in a sense, a little easier. In particular, Rozansky's categorification admits spacification. In the case of the c–colored unknot whose categorification is, in Rozansky's construction, the stable limit of the Khovanov cohomology of c–stranded torus links as the number of twists goes to infinity, this has been observed by Willis [12], whose paper appeared on the arXiv while this one was being written. The case of a c–colored link in general is no harder, and in fact Rozansky has already taken care of the difficult work.

Since the Cooper–Krushkal and the Rozansky categorifications are equivalent, the natural expectation is that one can lift the Cooper–Krushkal categorification to a spacification equivalent to the Rozansky spacification. This turns out to be straightforward in the 2–colored case, but at least the more obvious attempt fails in the 3–colored case, as we discuss later.

1.3 Computational results

We shall define a stable homotopy type $\mathcal{X}_{\mathrm{col}}(L_c)$, where L_c is a framed link with a coloring c of its components by positive integers. Picking the coloring 1 for each component returns the stable homotopy type $\mathcal{X}_{\mathrm{col}}(L_1)$, a grading-shifted version of Lipshitz and Sarkar's stable homotopy type $\mathcal{X}(L)$.

We make some computations for certain links and colorings in Section 4. Already in the simplest case these show interesting behavior: the link with the lowest positive crossing number is the Hopf link and the first coloring which has not yet been considered by Lipshitz and Sarkar is where one component is colored with 2 and the other with 1. The tail of the colored Khovanov cohomology of the $(2, 1)$–colored Hopf link agrees with the tail of the colored Khovanov cohomology of the $(2, 1)$–colored 2–component unlink. Nevertheless, we observe that even these tails can be distinguished by the stable homotopy type.

Although we are not yet able to compute fully the stable homotopy type of the 3–colored unknot, we make a conjecture based on some partial computations. This conjecture is interesting because its truth would imply that the periodicity of the tail of the stable homotopy type of a colored link (even in the case of the 3–colored unknot) can be longer than the periodicity of the tail of its cohomology.

1.4 Plan of the paper

In Section 2 we first observe that we can combine Rozansky's insight with the work of Lipshitz and Sarkar. This combination is straightforward and yields a stable homotopy type of a framed colored link whose cohomology recovers colored Khovanov cohomology. Secondly, we give ourselves a framework in which to make computations. For this it makes more sense to use the Cooper–Krushkal categorification, which, at least in the case of colors 2 and 3, is entirely explicit. We define what we mean by a lift of the Cooper–Krushkal categorification to a spacification and show that any such lift gives the same stable homotopy type as that arising from Rozansky's construction.

In Section 3, we construct such a lift of the Cooper–Krushkal categorification for colorings taken from the restricted palette $\{1, 2\}$. The case of 3–colored cannot be

made to work in the way that one might expect (there is an explicit obstruction to this). Finally, in Section 4 we make computations as already discussed in Section 1.3. At the end of this section we give a discussion of the Cooper–Krushkal 3–colored case.

2 Two approaches to a colored stable homotopy type

The colored Jones polynomial is an invariant of framed links L in which each component of L has been assigned a *color*, or in other words a positive integer weight. We write the color of a component k of L as $c(k)$, and often keep track of the coloring as a subscript L_c.

To compute the polynomial one takes a diagram of L_c in which the self-writhe of each component is equal to its framing. Then one replaces each component k by $c(k)$ parallel copies following the blackboard framing. Finally, one places on each component a *Jones–Wenzl projector*. This projector is an element of the relevant Temperley–Lieb algebra, with coefficients in rational functions of q. Finally, one applies the Kauffman bracket, and obtains an element of $\mathbb{Z}[\![q, q^{-1}]$ by expanding in powers of q (or an element of $\mathbb{Z}[q, q^{-1}]\!]$ by expanding in powers of q^{-1}).

The Jones–Wenzl projector is idempotent and satisfies turnback-triviality. It turns out that these two universal properties are enough to determine it completely. The Jones–Wenzl projector should in principle lift, in a categorification of the colored Jones polynomial, to a complex in Bar-Natan's tangles-and-cobordisms category [1], satisfying idempotence and turnback-triviality up to chain homotopy equivalence. Cooper and Krushkal [3] and Rozansky [10] give ways of achieving such a lift. Cooper and Krushkal proceed explicitly and give a categorified projector that they define inductively, while Rozansky realizes the categorified projector as a limit of the complexes associated to torus braids. It is surprising that the latter approach had apparently not been considered even at the decategorified level until Rozansky's insight! As observed by Cooper and Krushkal, categorified universal properties imply that the two competing categorifications give identical cohomological groups.

2.1 Grading and other conventions

We note that there is a discrepancy in the grading conventions between the original paper of Khovanov's [5], Rozansky's torus braids paper [10] and Cooper and Krushkal's
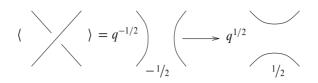
Figure 1: We follow the grading conventions as depicted in the complex that we associate to a single crossing. The complex is supported in cohomological degrees $\pm\frac{1}{2}$, and a quantum grading shift is applied. The differential increases the cohomological degree by 1 and preserves the quantum grading. A crossingless circle has complex supported in cohomological degree 0 and quantum degrees $+1$ and -1.

paper [3]. We apologize for possibly adding to the confusion. We shall essentially work with the bigrading conventions used by Bar-Natan [1] up to an overall shift. The overall shift makes it easier to treat the colored Khovanov cohomology as an invariant of a colored framed link, with no choice of orientation. The convention is depicted in Figure 1.

With these conventions, the Khovanov complex $\langle D \rangle$ of a diagram D is invariant up to bigraded homotopy equivalence under the second and third Reidemeister moves, but it is only invariant up to an overall shift under the first Reidemeister move. Hence it becomes a chain homotopy invariant of framed links (where the framing is given by the blackboard-framing of a diagram). If, on the other hand, D and D' differ by the first Reidemeister move with the writhes satisfying $w(D) = w(D') + 1$, then there is a bigraded homotopy equivalence between

$$\langle D \rangle \quad \text{and} \quad h^{-1/2} q^{-3/2} \langle D' \rangle,$$

where the powers of h and q represent cohomological and quantum degree shifts in the usual way.

2.2 Rozansky spacification

Rozansky [10] has given an approach to colored Khovanov cohomology that expresses the c–colored cohomology of a link L as the limit of the Khovanov cohomologies of a c–strand cable of L in which one puts an increasing number of twists. The stabilization of the cohomology was observed earlier by Stošić in the case of L being the unknot, which amounts to the stabilization of the cohomology of the (p, c)–torus link as $p \to \infty$.

We now summarize the construction. We shall be sticking with the convention of right-handed full twists, although there is an analogous story for left-handed twists. In
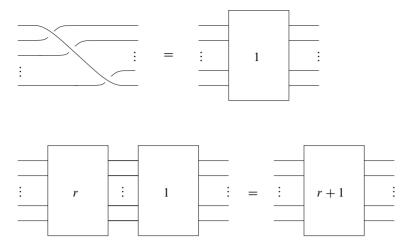
Figure 2: This shows inductively what is meant by twisting r times positively on an n–stranded braid.

Figure 2, we describe what is meant by twisting r times on an n–stranded braid. We write this braid as $B_{r,n}$. To each such braid, Bar-Natan's construction [1] associates a complex, which we shall denote by $\langle B_{r,n} \rangle$. In this complex, each cochain group is a vector of tangle smoothings, each such smoothing coming with a quantum degree. We shall apply a bigrading shift to this complex so that the resolution which is the identity braid group element is in cohomological degree 0 and comes with quantum degree shift 0. We write the shifted complex as $h^{r(n-1)/2} q^{r(n-1)/2} \langle B_{r,n} \rangle$, where the exponents of h and q denote cohomological and quantum degree shifts, respectively. Note that all other resolutions of the braid now lie in positive cohomological degrees.

For each $r \geq 1$ there is a map of complexes

$$F_r \colon (hq)^{rn(n-1)/2} \langle B_{rn,n} \rangle \to (hq)^{(r-1)n(n-1)/2} \langle B_{(r-1)n,n} \rangle,$$

given by taking F_1 to be the identity in cohomological degree 0, and then defining F_r to be the tensor product of F_1 with the identity on $(hq)^{(r-1)n(n-1)/2} \langle B_{(r-1)n,n} \rangle$.

Rozansky shows that for large r the cone complex $\mathrm{Cone}(F_r)$ is homotopy equivalent to a complex in which each smoothing that appears has high cohomological and quantum degrees. For our purposes, we are mainly interested in the quantum degree; we have:

Proposition 2.1 [10, Theorem 4.4] *The cone* $\mathrm{Cone}(F_r)$ *is homotopy equivalent to a complex made up of circleless smoothings, where each such smoothing is shifted in quantum degree by at least* $2n(r-1) + 1$.

The precise form of the quantum degree shift is unimportant for us; rather we note that it increases at least linearly with r.

Definition 2.2 Let D_c be an unoriented link diagram in which each component is colored by a positive integer weight (we write the coloring by weights as c), and each component k carries a basepoint. Let the diagram D_c^r be given by the blackboard-framed c–stranded cable of D_c in which each component k receives $c(k)r$ positive twists at the basepoint. In other words, the diagram is cut open at each basepoint and $B_{rc(k),c(k)}$ is inserted.

Definition 2.3 Let

$$G_r\colon (hq)^{\sum_k rc(k)(c(k)-1)/2}\langle D_c^r\rangle \to (hq)^{\sum_k (r-1)c(k)(c(k)-1)/2}\langle D_c^{r-1}\rangle$$

be induced by the tensor product of the maps F_r at each basepoint.

Lemma 2.4 *It follows from [Proposition 2.1](#) that, for fixed j and for all large enough r, the map of cohomologies*

$$H^{i,j}\big((hq)^{\sum_k rc(k)(c(k)-1)/2}\langle D_c^r\rangle\big) \to H^{i,j}\big((hq)^{\sum_k (r-1)c(k)(c(k)-1)/2}\langle D_c^{r-1}\rangle\big)$$

induced by G_r is an isomorphism.

Proof There is more than one way to see this. For example, label the components $k_1, \ldots, k_s$ and write

$$e_\alpha = \sum_{\beta=1}^{\beta=\alpha-1} \tfrac{1}{2}(r-1)c(k_\beta)(c(k_\beta)-1) + \sum_{\beta=\alpha}^{\beta=s} \tfrac{1}{2}rc(k_\beta)(c(k_\beta)-1),$$

and denote by $D_c^{k_\alpha}$ the result of taking the c–cable of D and adding rc twists at the basepoints of $k_\alpha, \ldots, k_s$ and $(r-1)c$ twists at the basepoints of $k_1, \ldots, k_{\alpha-1}$. Then we can write

$$G_r = \overline{F_r^{k_s}} \circ \cdots \circ \overline{F_r^{k_1}},$$

where

$$\overline{F_r^{k_\alpha}}\colon (hq)^{e_\alpha}\langle D_c^{k_\alpha}\rangle \to (hq)^{e_{\alpha+1}}\langle D_c^{k_{\alpha+1}}\rangle$$

is induced by F_r at a chosen basepoint. The cone $\mathrm{Cone}(\overline{F_r^{k_\alpha}})$ is homotopy equivalent to a complex made up of the tensor product of three Bar-Natan complexes of tangles. Namely:

- A complex of circleless smoothings at the chosen basepoint whose quantum degree increases linearly with r.

- At the other basepoints, the complexes $(hq)^{rn(n-1)/2}\langle B_{rn,n}\rangle$. After circle removal, these consist of circleless smoothings each in a nonnegative quantum degree. This can be seen by observing that the identity braid is in cohomological and quantum degree 0. Smoothings in cohomological degree d differ from the identity braid by exactly d surgeries and so contain at most $d-1$ circles.

- A complex independent of r arising from the Bar-Natan complex of the diagram away from the basepoints.

Finally we recall the homological algebra fact that

$$\text{Cone}(k \circ l) = \text{Cone}(\Sigma^{-1}\text{Cone}(k) \to \text{Cone}(l))$$

for maps of complexes $k\colon C \to C'$ and $l\colon C'' \to C$. This implies that $\text{Cone}(G_r)$ can be represented by circleless smoothings such that the minimal quantum degree among them increases at least linearly with r. Since j was fixed, we can choose r large enough that cohomology of $\text{Cone}(G_r)$ is 0 in quantum degree j, which means G_r induces an isomorphism in quantum degree j. $\qquad\square$

Hence we can make the following definition:

Definition 2.5 For fixed j, the c–colored Khovanov cohomology of the diagram D framed by the componentwise writhe is defined to be the group

$$\text{Kh}^{i,j}_{\text{col}}(D_c) = H^{i,j}\big((hq)^{\sum_k rc(k)(c(k)-1)/2}\langle D_c^r\rangle\big)$$

for sufficiently large r.

Independence of the cohomology under Reidemeister moves II and III and under choice of basepoints follows immediately from the independence under Reidemeister moves II and III of standard Khovanov cohomology. The fact that a suitable Euler characteristic of the cohomology agrees with the c–colored Jones polynomial of D is due to Rozansky.

Since $H^{i,j}\big((hq)^{\sum_k rc(k)(c(k)-1)/2}\langle D_c^r\rangle\big)$ is simply a grading-shifted version of the usual Khovanov cohomology of D_c^r, the construction of Lipshitz and Sarkar gives rise to a stable homotopy type $\overline{\mathcal{X}}^j(D_c^r)$ which recovers the Khovanov cohomology as its (suitably shifted) singular cohomology groups.

Furthermore, observe that the map G_r is induced by quotienting out a subcomplex generated by standard generators of the Khovanov complex. This subcomplex corresponds to an upward-closed subcategory of the framed flow category associated by Lipshitz and Sarkar to D_c^r. It follows that G_r is induced by a map

$$g_r\colon \overline{\mathcal{X}}^j(D_c^{r-1}) \to \overline{\mathcal{X}}^j(D_c^r).$$

Since g_r gives an isomorphism on cohomology for all sufficiently large r, Whitehead's theorem implies that g_r is a stable homotopy equivalence for sufficiently large r.

Definition 2.6 We can now define the colored stable homotopy type for fixed j to be

$$\mathcal{X}^j_{\mathrm{col}}(D_c) = \overline{\mathcal{X}}^j(D^r_c)$$

for sufficiently large r. In other words, this is the homotopy colimit of the directed system of maps g_r.

The invariance of this stable homotopy type under choice of basepoints and under Reidemeister moves II and III follows from the invariance of the Lipshitz–Sarkar homotopy type under Reidemeister moves II and III.

Remark 2.7 Willis [12] gave Definition 2.6 in the case that D is the unknot and gave an independent argument that the limit of the system g_r exists. Using his own estimates of quantum degree rather than Rozansky's, Willis has independently defined the Rozansky spacification, in a paper appearing on the arXiv shortly after ours [13]. He further showed a stabilization of the spacification of the c–colored unknot as $c \to \infty$.

Remark 2.8 Definition 2.6 implies that the framing of the link components only affects the colored stable homotopy type up to an overall shift in bigrading, as is the case for the colored Khovanov cohomology. This is because the blackboard-framed c–cable of a 1–crossing Reidemeister 1–tangle is equivalent to a full twist in a c–stranded braid by a sequence of Reidemeister moves involving c Reidemeister I moves. Reidemeister moves preserve the stable homotopy type according to Lipshitz and Sarkar, but Reidemeister I moves introduce a shift (with our grading conventions).

2.3 Cooper–Krushkal spacification

In this subsection we give the properties that one might expect of a spacification based on the Cooper–Krushkal categorification. These properties are enough to imply that any such spacification is stably homotopy equivalent to the Rozansky spacification, as is verified in Section 2.4. The construction of such spacifications is, however, not straightforward, and we leave discussion of these to Section 3.

Suppose for each $n \geq 1$ that P_n is a complex of (n, n)–tangle smoothings in the sense of Bar-Natan [1], so that each P_n is a *universal projector* by [3, Definition 3.1]. Cooper and Krushkal have given a way of constructing such universal projectors. We note that a part of their definition of P_n is that the identity n–braid smoothing appears only

once and in degree $(0,0)$, and that the quantum and cohomological degrees of every smoothing in the complex are nonnegative.

Suppose that T is a tangle diagram in the plane punctured by k discs with $2n_i$ ordered boundary points on the ith disc. Then we may define the Khovanov cochain complex (of free abelian groups) $\langle T_P \rangle$ by taking the tensor product of the Bar-Natan complex $\langle T \rangle$ and P_{n_i} for $i = 1, \ldots, k$ in the obvious way.

Definition 2.9 A *Cooper–Krushkal framed flow category* (CKffc) is a choice of finite-object framed flow category (see [8] for the definition and references) $\mathcal{C}(T_P)$ refining the Khovanov cochain complex $\langle T_P \rangle$ for each such T. Choosing a particular crossing of the tangle T we write T^0 and T^1 for the 0– and 1–resolutions of that crossing. We require that the standard generators corresponding to the subcomplex $\langle T_P^1 \rangle$ (resp. the quotient complex $\langle T_P^0 \rangle$) correspond to upward-closed (resp. downward-closed) framed flow subcategories of $\mathcal{C}(T_P)$ such that the associated CW–complex is stably homotopy equivalent to $|\mathcal{C}(T_P^1)|$ (resp. $|\mathcal{C}(T_P^0)|$).

Furthermore, if we denote by T^{id} the tangle diagram produced by filling the kth boundary disc of T with the identity n_k–braid, then $\langle T_P^{\text{id}} \rangle$ is naturally a quotient complex of $\langle T_P \rangle$ generated by standard generators of $\langle T_P \rangle$. We require this quotient complex to correspond to a downward-closed subcategory of $\mathcal{C}(T_P)$ with associated CW–complex stably homotopy equivalent to $|\mathcal{C}(T_P^{\text{id}})|$.

Remark 2.10 We can restrict this definition, if we like, to certain values of n. In particular in this paper we give a genuine CKffc only for the color $n = 2$. For the color $n = 3$ we may slightly alter the definition of a CKffc, to arrive at a framed flow category spacifying a cohomology theory that has its graded Euler characteristic a nonstandard normalization of the 3–colored Jones polynomial. If we insist on the standard normalization we run into difficulties. We discuss this in Section 3.

Remark 2.11 The condition that a CKffc assigns a *finite-object* framed flow category is equivalent to the condition that the minimal quantum degree of the circleless smoothings in the ith cochain group of P_n tends to infinity as $i \to \infty$. Although this is true for the explicit examples of universal projectors constructed by Cooper and Krushkal, it is not required by them axiomatically.

2.4 The equivalence

CKffcs are nice since, *if they exist*, they give an honest framed flow category whose associated stable homotopy type recovers colored Khovanov cohomology as its singular
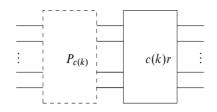
Figure 3: We describe how to form the complex $C_{P,r}^{i,j}(D)$ from a based $c-$ colored diagram D. We take the blackboard-framed $c-$cable of D and at the basepoint of each component k of D we tensor in $P_{c(k)}$ and add $c(k)r$ twists as shown in the diagram. We then take the corresponding cochain complex and shift by $hq^{\sum_k rc(k)(c(k)-1)/2}$.

cohomology. Going via Rozansky's construction we are producing instead a stable homotopy type as a homotopy colimit of spaces arising from framed flow categories.

Nevertheless, we shall next see that CKffcs, if they exist, would give rise to the same stable homotopy types as does $\mathcal{X}_{\mathrm{col}}^j$. More precisely, let D be a link diagram framed by the componentwise writhe with each component k having a basepoint, and each being colored by a positive integer weight $c(k)$. We write D^{cab} for the tangle formed by cutting D open at each basepoint and then taking the blackboard-framed $c-$cable. Then we can consider the Bar-Natan cochain complex of free abelian groups formed by tensoring in each $P_{c(k)}$ corresponding to component k in the obvious way. This cochain complex is the Cooper–Krushkal complex that categorifies the colored Jones polynomial of D, and it is refined by the framed flow category $\mathcal{C}(D_P^{\mathrm{cab}})$ provided by the CKffc. Writing $\mathcal{C}^j(D_P^{\mathrm{cab}})$ for the part of this framed flow category in quantum degree j, we have the following result:

Proposition 2.12 *With the diagram D as above we have*

$$\mathcal{X}_{\mathrm{col}}^j(D_c) \simeq |\mathcal{C}^j(D_P^{cab})|.$$

Proof We fix j. We write $C_{P,r}^{i,j}(D)$ to be the cochain complex of free abelian groups formed by following the procedure as outlined in Figure 3. By the definition of a CKffc, there is a framed flow category $\mathcal{A}_{P,r}^j(D)$ that refines $C_{P,r}^{i,j}(D)$.

Consider the quotient complex C_1 of $C_{P,r}^{i,j}(D)$ consisting of all generators corresponding to taking the 0–resolution at each of the crossings of the twist regions at the basepoints. This corresponds to a downward-closed subcategory $\mathcal{A}_1$ of $\mathcal{A}_{P,r}^j(D)$. We observe firstly that $|\mathcal{A}_1|$ is stably homotopy equivalent to $|\mathcal{A}_{P,0}^j(D)|$, which is exactly $|\mathcal{C}^j(D_P^{\mathrm{cab}})|$, and secondly that the corresponding upward-closed subcategory has trivial

cohomology by the turnback-triviality condition on the projectors $P_{c(k)}$. Hence we have that

$$|\mathcal{A}^j_{P,r}(D)| \simeq |\mathcal{A}_1| \simeq |\mathcal{C}^j(D^{\mathrm{cab}}_P)|.$$

On the other hand, for any value of r, the complex $C^{i,j}_{P,r}(D)$ can be written as the total complex

$$(hq)^{\sum_k rc(k)(c(k)-1)/2}\langle D^r_c \rangle \to \Gamma^r_1 \to \cdots \to \Gamma^r_s \to \cdots,$$

where each Γ^r_s carries an internal differential arising from all crossings of D^r_c, while the part of the differential from Γ^r_s to Γ^r_{s+1} is induced by the differentials of the $P_{c(k)}$. This is because the identity braid smoothing is the only smoothing appearing in cohomological degree zero of each complex $P_{c(k)}$.

Now the minimal quantum degree of a generator in $\bigoplus_r \Gamma^r_s$ tends to $+\infty$ as s tends to $+\infty$ (see Remark 2.11). On the other hand, each Γ^r_s is chain-homotopy equivalent by Gauss-elimination to a complex in which the minimal quantum degree is bounded below by $b(r)$, a function independent of s and tending to $+\infty$ as r tends to $+\infty$. This follows from [10, Formula (4.9)] (taking into account our different grading conventions) and the observation that the cohomological Reidemeister I and II relations can be proved by Gauss-elimination.

Hence the lowest quantum degree of the support of the cohomology of the subcomplex

$$\Gamma^r_1 \to \cdots \to \Gamma^r_s \to \cdots$$

tends to $+\infty$ as r tends to $+\infty$. The quotient complex $(hq)^{\sum_k rc(k)(c(k)-1)/2}\langle D^r_c \rangle$ corresponds to a downward-closed subcategory of $\mathcal{A}^j_{P,r}(D)$ with associated stable homotopy type $\overline{\mathcal{X}}^j(D^r_c)$. So for large enough r we have

$$\mathcal{X}^j_{\mathrm{col}}(D_c) \simeq \overline{\mathcal{X}}^j(D^r_c) \simeq |\mathcal{A}^j_{P,r}(D)| \simeq |\mathcal{A}_1| \simeq |\mathcal{C}^j(D^{\mathrm{cab}}_P)|. \qquad \square$$

Remark 2.13 We have worked here with colored links, but all of what we have done applies, *mutatis mutandis*, to more general (in other words, not just diagrams obtained by cabling) closed diagrams containing Jones–Wenzl projectors.

3 Lifting the Cooper–Krushkal projectors

In this section we give a CKffc associated to link diagrams colored with colors drawn from the palette $\{1, 2\}$. It would seem a priori very likely that the methods used in this construction should extend to the color 3, since for this color we have (due to Cooper and Krushkal [3]) an explicit and fairly simple cohomological projector. However, it turns out that there is an unexpected nontrivial obstruction to this extension. The

Figure 4: We show here the Cooper–Krushkal projector. We suppress the degree shifts for ease of visualization. The degree shifts can be determined by noting that the identity-braid or horizontal smoothing on the far left is in cohomological degree 0 and quantum degree 0, and all differentials raise the cohomological degree by 1 and preserve the quantum degree.

obstruction can be obviated by renormalizing the 3–colored Jones invariant of the 0–framed unknot to be

$$(q^{-2} + 1 + q^2)(1 - q^2 + q^4 - q^6 + \cdots) \quad \text{rather than} \quad q^{-2} + 1 + q^2.$$

We briefly discuss the obstruction and renormalization at the end of Section 4, but we do not give in this paper the full construction of the renormalized spacification.

3.1 A 2–colored Cooper–Krushkal projector

Two of the authors and Dan Jones [4] considered the 2–stranded braid of k crossings, each of the same sign. The Bar-Natan complex of this tangle has a particularly simple form: it is homotopy equivalent to a complex which has one circleless smoothing in each cohomological degree from $-\frac{1}{2}k$ to $\frac{1}{2}k$ (with the grading conventions used in this paper). Indeed, in Figure 4, we give the Cooper–Krushkal projector for the color 2; the Bar-Natan complex for the positively twisted k–crossing 2–braid is, up to an overall shift, the quotient complex of this projector consisting of all tangles of cohomological degree less than $k + 1$.

Decomposing a closed link diagram D into a tensor product of such tangles, one can consider the tensor product of their simplified chain homotopy class representatives. This gives a cochain complex $\langle D \rangle^{\text{simp}}$ (depending on the decomposition of D) of free abelian groups, and $\langle D \rangle^{\text{simp}}$ is refined by a framed flow category given in [4]. The associated stable homotopy type was shown to be independent of the choice of decomposition, and it was observed that the decomposition in which each tangle has a single crossing returns the Lipshitz–Sarkar framed flow category.

Taking a suitably normalized version of this construction for $k = \infty$ gives a construction of a CKffc. In particular, this construction enables us to make nontrivial calculations of the colored stable homotopy types of the $(2, 1)$–colored Hopf link as well as of the 2–colored trefoil.

Suppose that T is a tangle diagram in the plane punctured by k discs each with 4 ordered boundary points. Let the closed diagram T^r be given by filling in each disc with r positive full twists.

We consider a particular decomposition of T^r into a tensor product of tangles — specifically, we take one tangle (of $2r$ crossings) at each filled disc, one tangle for every other crossing of T^r, and finally the rest of the diagram which is crossingless.

Such a decomposition into tangles is exactly the input into the construction of [4]. So, incorporating now an overall shift and fixing a quantum degree j, there is a framed flow category $\mathcal{A}^j(T^r)$ refining the quantum degree j part of the simplified cochain complex $(hq)^{kr} \langle T^r \rangle^{\mathrm{simp}}$.

Finally we note that for fixed j and large enough r, the quantum degree j part of $(hq)^{kr} \langle T^r \rangle^{\mathrm{simp}}$ agrees with the quantum degree j part of the Cooper–Krushkal complex $\langle T_P \rangle$. This is because, in the $\mathfrak{sl}_2$ case, the construction of [4] gives a framed flow category refining the simplified Bar-Natan complex of link diagrams decomposed into tangles, each of which is a 2–braid. So, taking r to be large, the framed flow category $\mathcal{A}^j(T^r)$ provides our candidate for a CKffc. The remaining properties required of a CKffc are now straightforward to verify.

4 Examples

4.1 The 2–colored unknot

Consider a diagram of the blackboard framed 2–cable of the 0–crossing unknot containing a Cooper–Krushkal projector P_2. The generators in the resulting cochain complex come from smoothings with two circles in homological degree 0, and one circle in homological degree bigger than 0; compare Figure 4. The minimal quantum degree in which we get a generator is therefore $q = -2$ with one generator in homological degree 0. For $q = 0$ we get two generators in homological degree 0 and one in homological degree 1. For $q = 2$ there is one generator in homological degrees 0, 1 and 2 each.

For $q = 2j$ with $j \geq 2$ we get two generators, one in homological degree $j - 1$ and one in degree j. The coboundary map alternates between multiplication by 0 and 2. The cohomology is therefore easily calculated, and determines the stable homotopy types because of thinness. We thus get

$$\mathcal{X}^{-2}_{\mathrm{col}}(U_2) = S^0, \quad \mathcal{X}^0_{\mathrm{col}}(U_2) = S^0, \quad \mathcal{X}^2_{\mathrm{col}}(U_2) = S^2,$$
$$\mathcal{X}^{4j}_{\mathrm{col}}(U_2) = M(\mathbb{Z}/2, 2j) \qquad \text{for } j \geq 1,$$
$$\mathcal{X}^{4j+2}_{\mathrm{col}}(U_2) = S^{2j+1} \vee S^{2j+2} \quad \text{for } j \geq 1.$$
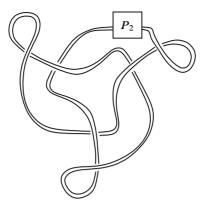
Figure 5: The 0–framed 2–cable of the right-handed trefoil with a Cooper–Krushkal projector placed on it

Note that the notation $M(G, n)$ stands for a Moore space, a space whose only nontrivial integral homology group is G in degree n.

4.2 The 2–colored trefoil

In Figure 5 we give a diagram of a 2–cable of the right-handed trefoil T containing a Cooper–Krushkal projector P_2. The extra loops ensure that we get the 0–framed 2–cable, and we denote it by T_2^0. For each quantum degree j this diagram gives rise to a framed flow category $\mathcal{A}$ as described in Section 3.1.

For calculational purposes, we want to remove the three double loops. Performing two Reidemeister I moves and one Reidemeister III move turns each double loop into $B_{-2,2}$, which can be absorbed by the projector P_2. However, because of the Reidemeister I moves, we get a shift in homological and quantum degrees. More precisely, we get $\langle D_2^r \rangle = h^3 q^9 \langle D_2'^{r-3} \rangle$, where D' is the standard 3–crossing diagram of the right-handed trefoil. Denoting the 2–colored right-hand trefoil with framing 3 by T_2^3, we get $\mathrm{Kh}_{\mathrm{col}}^{i,j}(T_2^0) = \mathrm{Kh}_{\mathrm{col}}^{i-6, j-12}(T_2^3)$.

Taking these shifts into account and working with the diagram for T_2^3, we see that the least quantum degree in $\mathcal{A} = \mathcal{A}_0$ which admits an object is given by $q = 2$ with homological degree $h = 0$, coming from a smoothened diagram with 4 circles. This is indeed the only object in this quantum degree.

The projector P_2 gives rise to upward-closed subcategories $\mathcal{A}_k$ for $k \geq 0$ generated by objects that arise from a tangle in P_2 of cohomological degree at least k. The highest quantum degree of an object in $\mathcal{A}_0 - \mathcal{A}_1$ is $q = 24$ coming from 6 circles in

the smoothened diagram. It follows that for quantum degree $q \geq 26$ the relevant flow category $\mathcal{A}^q$ is a full subcategory of $\mathcal{A}_1$.

The quotient category $\mathcal{A}_k/\mathcal{A}_{k+1}$ for $k \geq 1$ is, up to degree shifts, the Lipshitz–Sarkar flow category of a diagram of the unknot with 12 crossings. Furthermore, this diagram can be transformed into the standard unknot diagram by performing six Reidemeister II moves. The category $\mathcal{A}_k/\mathcal{A}_{k+1}$ for $k \geq 1$ is therefore stably equivalent to a flow category containing two objects of homological degree $k + 6$, one of quantum degree $2k + 12$, the other of quantum degree $2k + 10$.

Also notice that the associated cochain complexes to the flow categories $\mathcal{A}^q$ and $\mathcal{A}^{q+4}$ for $q \geq 26$ only differ in a cohomological shift by 2. If the tail turns out to be cohomologically thin (as it does), it follows that the stable homotopy types for q up to 28 determine all the stable homotopy types. The stable homotopy types for q up to 28 may be determined using the diagram $D_2'^{\,r}$ for large r. It turns out that $r = 8$ is sufficient, and the following calculations have been done using the program KnotJob [11].

We can identify all stable homotopy types from cohomology and Steenrod square calculations using the classification result of Baues and Hennes [2] with the exception of $q = 10$, where $\mathcal{X}^{10}_{\mathrm{col}(2)}(T)$ is either $S^3 \vee S^4 \vee S^6$ or $X(\varepsilon, 3) \vee S^4$. Recall that $X(\varepsilon, n)$ is the space obtained by attaching an $(n+3)$–cell to S^n using the nontrivial element of $\pi_2^{\mathrm{st}} \cong \mathbb{Z}/2$. Excluding this, we get

$$\mathcal{X}^2_{\mathrm{col}}(T_2^0) = S^0, \qquad\qquad \mathcal{X}^4_{\mathrm{col}}(T_2^0) = S^0,$$
$$\mathcal{X}^6_{\mathrm{col}}(T_2^0) = S^2, \qquad\qquad \mathcal{X}^8_{\mathrm{col}}(T_2^0) = X(_2\eta, 2),$$
$$\mathcal{X}^{12}_{\mathrm{col}}(T_2^0) = X(\eta 2, 5) \vee S^6, \qquad \mathcal{X}^{14}_{\mathrm{col}}(T_2^0) = X(\eta 2, 5) \vee S^7 \vee S^8 \vee S^8,$$
$$\mathcal{X}^{16}_{\mathrm{col}}(T_2^0) = S^7 \vee M(\mathbb{Z}/4, 8) \vee M(\mathbb{Z}/2, 8), \qquad \mathcal{X}^{18}_{\mathrm{col}}(T_2^0) = S^9 \vee S^9 \vee M(\mathbb{Z}/2, 9) \vee S^{10},$$
$$\mathcal{X}^{20}_{\mathrm{col}}(T_2^0) = M(\mathbb{Z}/2, 9) \vee M(\mathbb{Z}/2, 10) \vee S^{11}, \qquad \mathcal{X}^{22}_{\mathrm{col}}(T_2^0) = S^{11} \vee M(\mathbb{Z}/2, 11) \vee S^{12},$$
$$\mathcal{X}^{24}_{\mathrm{col}}(T_2^0) = S^{12} \vee M(\mathbb{Z}/2, 12)$$

The tail is given by

$$\mathcal{X}^{4j+2}_{\mathrm{col}}(T_2^0) = S^{2j+1} \vee S^{2j+2} \quad \text{for } j \geq 6,$$
$$\mathcal{X}^{4j}_{\mathrm{col}}(T_2^0) = M(\mathbb{Z}/2, 2j) \qquad \text{for } j \geq 7.$$

Notice that for $j \geq 26$ we have $\mathcal{X}^j_{\mathrm{col}}(T_2^0) = \mathcal{X}^j_{\mathrm{col}}(U_2)$.

The notation $X(\eta 2, n)$ is taken from [2], and stands for an elementary Chang complex. It is an appropriate suspension of $\mathbb{RP}^4/\mathbb{RP}^1$ such that the first nontrivial homology
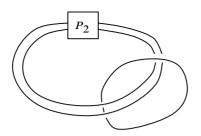
Figure 6: This is the 0–framed $(2, 1)$–cable of the Hopf link, in which the 2–cabled component receives a Cooper–Krushkal projector.

group is in degree n. Similarly, $X(_2\eta, m)$ is a suspension of $\mathbb{RP}^5/\mathbb{RP}^2$ such that the first nontrivial homology group is in degree m. Both spaces have nontrivial Sq^2 and are therefore not wedges of Moore spaces.

4.3 The $(2, 1)$–colored Hopf link

We denote the $(2, 1)$–colored 0–framed Hopf link by $H_{2,1}$. In Figure 6 we give a diagram of the Hopf link, in which one of the components has been replaced by a 0–framed 2–cable containing a Cooper–Krushkal projector P_2. For each quantum degree j this diagram gives rise to a framed flow category, as described in Section 3.1. The associated stable homotopy type is $\mathcal{X}_{\mathrm{col}}(H_{2,1})$.

Note that the diagram consists of the tensor product of three parts: the projector P_2 and then two tangles, each of which is a 2–crossing 2–braid. As before, we can filter the flow category via the projector, leading to categories $\mathcal{A}_j$ for $j \geq 0$.

For actual calculations, we replace the projector with a $(2r)$–tangle, so the resulting diagram is that of the $P(-2, 2, 2r)$ pretzel link. For a given quantum degree we can then use the method of [4] to get a flow category built from three tangles. The lowest quantum degree for which we can get an object is $q = -5$, for which there is exactly one object of homological degree -2.

For $q \geq 7$, all objects are contained in $\mathcal{A}_1$, and the categories $\mathcal{A}^{2j-1}$ and $\mathcal{A}^{2j+3}$ for $j \geq 4$ have the following similarity. If α is an object of $\mathcal{A}^{2j-1}$ which also sits in $\mathcal{A}_k$ for $k \geq 1$, there is a corresponding object $\bar{\alpha}$ in $\mathcal{A}^{2j+3}$ also in $\mathcal{A}_{k+2}$ with $|\bar{\alpha}| = |\alpha| + 2$. It is clear from the framing formulas in [4] that $\mathcal{M}(\alpha, \beta) \cong \mathcal{M}(\bar{\alpha}, \bar{\beta})$ as framed manifolds, provided these are at most 1–dimensional.

Therefore the colored Khovanov cohomology of the tail is periodic, and since we only get nontrivial cohomology groups in three adjacent degrees, we also get periodicity of the stable homotopy type in the tail. This uses that the 1–dimensional moduli spaces

agree with framing for $\mathcal{A}^{2j-1}$ and $\mathcal{A}^{2j+3}$. Calculation of Khovanov cohomology and the second Steenrod Square shows that

$$\mathcal{X}_{\mathrm{col}}^{-5}(H_{2,1}) = S^{-2}, \qquad\qquad \mathcal{X}_{\mathrm{col}}^{-3}(H_{2,1}) = S^{-2},$$

$$\mathcal{X}_{\mathrm{col}}^{-1}(H_{2,1}) = S^{0}, \qquad\qquad \mathcal{X}_{\mathrm{col}}^{1}(H_{2,1}) = X(_2\eta, 0),$$

$$\mathcal{X}_{\mathrm{col}}^{3}(H_{2,1}) = S^{1} \vee S^{2} \vee S^{2}, \quad \mathcal{X}_{\mathrm{col}}^{5}(H_{2,1}) = X(_2\eta, 2).$$

The tail is given by

$$\mathcal{X}_{\mathrm{col}}^{4j-1}(H_{2,1}) = X(\eta 2, 2j-1) \vee S^{2j} \quad \text{for } j \geq 2,$$

$$\mathcal{X}_{\mathrm{col}}^{4j+1}(H_{2,1}) = X(_2\eta, 2j) \vee S^{2j+1} \quad \text{for } j \geq 2.$$

The $(2,1)$–colored unlink $U_{2,1}$ is the disjoint union of the 1–colored 0–framed unknot U_1 and the 2–colored 0–framed unknot U_2. The stable homotopy type can therefore be derived using [7, Theorem 1]. More precisely, we get

$$\mathcal{X}_{\mathrm{col}}^{j}(U_{2,1}) = (\mathcal{X}^{1}(U) \wedge \mathcal{X}_{\mathrm{col}}^{j-1}(U_2)) \vee (\mathcal{X}^{-1}(U) \wedge \mathcal{X}_{\mathrm{col}}^{j+1}(U_2)).$$

Since both $\mathcal{X}^{1}(U) = S^{0} = \mathcal{X}^{-1}(U)$, we have that $\mathcal{X}_{\mathrm{col}}^{j}(U_{2,1})$ is a wedge of Moore spaces for all j. In the tail we have

$$\mathcal{X}_{\mathrm{col}}^{4j-1}(U_{2,1}) = S^{2j-1} \vee S^{2j} \vee M(\mathbb{Z}/2, 2j) \qquad \text{for } j \geq 2,$$

$$\mathcal{X}_{\mathrm{col}}^{4j+1}(U_{2,1}) = M(\mathbb{Z}/2, 2j) \vee S^{2j+1} \vee S^{2j+2} \quad \text{for } j \geq 2.$$

In particular, we have

$$\mathrm{Kh}_{\mathrm{col}}^{i,j}(U_{2,1}) = \mathrm{Kh}_{\mathrm{col}}^{i,j}(H_{2,1})$$

for all $j \geq 7$ (a result that for high enough j is not unexpected, and that can be derived in ways other than brute calculation), but

$$\mathcal{X}_{\mathrm{col}}^{j}(U_{2,1}) \not\simeq \mathcal{X}_{\mathrm{col}}^{j}(H_{2,1}).$$

4.4 A conjecture on the 3–colored unknot

The stable homotopy type of the 0–framed 3–colored unknot $\mathcal{X}_{\mathrm{col}}^{j}(U_3)$ was partially computed by Willis [12], who showed that it was not a wedge of Moore spaces and so, in some sense, more interesting than just the colored Khovanov cohomology.

The 3–colored Khovanov cohomology can easily be calculated from [3, Section 4.4]. We summarize this in Table 1.

We observe that the tail is 3–periodic in quantum degrees $q = 2j + 1$ starting from $j \geq 2$ with a homological shift by 4. Also, by simply looking at these groups we see

	i			
	1	2	3	4
$Kh_{col}^{i-4,-3}(U_3)$				$\mathbb{Z}$
$Kh_{col}^{i-4,-1}(U_3)$				$\mathbb{Z}$
$Kh_{col}^{i,1}(U_3)$		$\mathbb{Z}$		
$Kh_{col}^{i,3}(U_3)$			$\mathbb{Z}/2$	$\mathbb{Z}$
$Kh_{col}^{i+4j,6j+5}(U_3)$, $j \geq 0$			$\mathbb{Z}$	$\mathbb{Z}$
$Kh_{col}^{i+4j,6j+1}(U_3)$, $j \geq 1$	$\mathbb{Z}$	$\mathbb{Z}$		
$Kh_{col}^{i+4j,6j+3}(U_3)$, $j \geq 1$	$\mathbb{Z}$		$\mathbb{Z}/2$	$\mathbb{Z}$

Table 1: The 3–colored Khovanov cohomology of the unknot

that except for quantum degrees $q = 6j + 3$ with $j \geq 0$ the stable homotopy types are wedges of spheres. In quantum degree $q = 3$ we have the nontrivial Steenrod Square coming from the torus knot $T_{4,3}$ first observed in [9], and which stably survives by [12].

The quantum degree $q = 9$ can be realized by the torus knot $T_{7,3}$, and computer calculations show a nontrivial Sq^2 in degree 5, with Sq^2 trivial in degree 6. The triviality in degree 6 indicates that the tail of the stable homotopy types is not 3–periodic, as the difference in 3–colored Khovanov cohomology in quantum degrees $q = 3$ and $q = 9$ comes from an extra generator in homological degree 0 killing the cocycle in degree 1, which survives in degree 5 for $q = 9$.

Computer calculations on $T_{8,3}$ show a trivial Sq^2 in degree 9, although this is not yet in the stable range for $q = 15$. Using a suitable diagram with a low number of tangles we have made computer calculations for $T_{13,3}$ which give evidence for the conjecture below:

Conjecture 4.1 *For $j \geq 1$ we have*

$$\mathcal{X}_{col}^{12j-3}(U_3) = X(\eta 2, 8j - 3) \vee S^{8j},$$
$$\mathcal{X}_{col}^{12j+3}(U_3) = S^{8j+1} \vee X(_2\eta, 8j + 2).$$

Note that these two spaces are not stably homotopy equivalent, although they are Spanier–Whitehead dual when appropriately shifted. Following consideration of the Cooper–Krushkal projector P_3 explicitly described in [3] this conjecture is somewhat

surprising. From P_3 the 3–periodicity follows immediately, so one may expect the same periodicity in the tail of the stable homotopy type.

Indeed, if one attempts a spacification based on lifting the Cooper–Krushkal projector P_3 to a framed flow category, one finds that the natural first attempt gives rise to 1–dimensional moduli spaces the framings of which also follow 3–periodicity. However, if one then pushes a little further to determine if one can genuinely lift P_3 to a CKffc, one runs into "ladybug matching" type problems which cannot all be solved simultaneously in a natural way, at least to the authors' eyes.

On the other hand, suppose that D is a tangle diagram in a disc with 6 ordered boundary points. This gives a cochain complex of free abelian groups $\langle D_P \rangle$. Now consider the "reduced" subcomplex $\langle D_P \rangle^{\mathrm{red}}$ of $\langle D_P \rangle$ obtained by restricting to half the generators of $\langle D_P \rangle$. Specifically, restrict to only those generators arising from a decoration by v_- of a chosen boundary point of D. In such a situation one can lift the cochain complex $\langle D_P \rangle^{\mathrm{red}}$ to a framed flow category refining it. The ladybug matching problems no longer occur since we have thrown out enough objects of the flow category to kill them.

Unfortunately, this subcomplex is not really a very natural object to consider. The graded Euler characteristic is a renormalized version of the 3–colored Reshetikhin–Turaev invariant as discussed at the start of Section 3, but it is hard to motivate why one should consider this renormalization. Therefore we do not pursue this further here.

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