Homotopy normal maps

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A group property made homotopical is a property of the corresponding classifying space. This train of thought can lead to a homotopical definition of normal maps between topological groups (or loop spaces).

In this paper we deal with such maps, called *homotopy normal maps*, which are topological group maps $N \to G$ being "normal" in that they induce a compatible topological group structure on the homotopy quotient $G/\!\!/N := EN \times_N G$. We develop the notion of homotopy normality and its basic properties and show it is invariant under homotopy monoidal endofunctors of topological spaces, eg localizations and completions. In the course of characterizing normality, we define a notion of a *homotopy action* of a loop space on a space phrased in terms of Segal's 1–fold delooping machine. Homotopy actions are "flexible" in the sense they are invariant under homotopy monoidal functors, but can also rigidify to (strict) group actions.

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1 Introduction

Homotopy normality is an attempt to derive a homotopical analogue for the inclusion of a normal subgroup via classifying spaces. An inclusion of topological groups $N \hookrightarrow G$ is the inclusion of a normal subgroup if and only if it is the kernel inclusion of some group map $G \to H$. Since any map is, up to homotopy, an inclusion, one needs to consider all group maps $N \to G$. Such a map should then be "homotopy normal" if $BN \to BG$ is the map from the homotopy fiber to the total space for some map $BG \dashrightarrow W$. There is another angle from which this notion makes sense. To every group map $N \to G$, one can associate the Borel construction $EN \times_N G =: G/\!\!/N$, which is the "correct" quotient in the homotopical world. We note that such an extension $BG \dashrightarrow W$ induces a loop space structure on $G/\!\!/N$, and a loop map structure (up to map equivalence) on $G \to G/\!\!/N$, providing a second analogy to the group theoretic notion: a group inclusion $N \hookrightarrow G$ is the inclusion of a normal subgroup if and only if G/N admits a group structure for which the natural quotient map $G \to G/N$ is a group map.

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Let $f\colon X \to Y$ be a pointed map of connected spaces. Consider the Puppe–Nomura sequence [16]

$$\Omega X \to \Omega Y \to \Omega Y /\!\!/ \Omega X \to X \to Y,$$

where we denote $\Omega Y /\!\!/ \Omega X := \mathrm{hfib}(f)$.

The following is essentially taken from Farjoun and Hess [9, Section 5].

Definition 1.1 A loop map $\Omega f\colon \Omega X \to \Omega Y$ is *homotopy normal* if there exist a connected space W with a map $\pi\colon Y \to W$, so that

$$X \xrightarrow{f} Y \xrightarrow{\pi} W$$

is a homotopy fibration sequence. The map $\pi\colon Y \to W$ is called a *normal structure*.

Remarks 1.2 (a) We see that a loop map $\Omega f\colon \Omega X \to \Omega Y$ is homotopy normal if and only if $f\colon X \to Y$ admits a structure of a homotopy principal fibration, ie equivalent to a principal fibration. In particular, the homotopy fiber of such a loop map has the structure of a double loop space.

(b) If $\Omega f\colon \Omega X \to \Omega Y$ is homotopy normal, the group map $\pi_0(\Omega f)\colon \pi_0(\Omega X) \to \pi_0(\Omega Y)$ is normal in the sense of [9], ie underlies a crossed module structure on the corresponding groups. Whitehead showed [23] that crossed modules correspond to connected 2–types. We note that if a discrete group map $N \to G$ is normal (in the sense of [9]) and $BG \to W$ its normal structure then W is the corresponding connected 2–type.

Example 1 If $F \to E \to B$ is a fibration sequence, the map $\pi_1 F \to \pi_1 E$ is a homotopy normal map of discrete groups. It is also true that any homotopy normal map of discrete groups is of this form (see Brown, Higgins and Sivera [3, Section 2.6] and Loday [13, Corollary 1.5]).

Example 2 Any double loop map $\Omega^2 f\colon \Omega^2 X \to \Omega^2 Y$ where X, Y are simply connected spaces is homotopy normal: take $W = \mathrm{hfib}(X \to Y)$; W is then a connected space which extends the Puppe–Nomura sequence.

Example 3 Let F be a pointed connected space. Then the universal fibration in Gottlieb [11], $F \to Baut_*(F) \to Baut(F)$ induces a homotopy normal map $\Omega F \to \Omega Baut_*(F)$. This map may be viewed as a universally initial homotopy normal map in the following sense: for every homotopy normal map $\Omega F \to \Omega X$ there exist a loop map $\Omega X \to \Omega\, Baut_*(F)$ and a homotopy commutative triangle

$$\begin{array}{ccc} \Omega F & \longrightarrow & \Omega\, Baut_*(F) \\ & \searrow & \uparrow \\ & & \Omega X. \end{array}$$

The dashed arrow is obtained as follows. Assume $F \to X \to W$ is a homotopy fibration sequence giving a normal structure on $\Omega F \to \Omega X$. By [11], there exists a "classifying map" $c\colon W \to Baut(F)$ such that $X \to W$ is obtained as a homotopy pullback

$$\begin{array}{ccc} X & \longrightarrow & W \\ \downarrow & & \downarrow c \\ Baut_*(F) & \longrightarrow & Baut(F). \end{array}$$

This can be extended to a homotopy commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & W \\ \downarrow \simeq & & \downarrow & & \downarrow c \\ F & \longrightarrow & Baut_*(F) & \longrightarrow & Baut(F) \end{array}$$

and looping down $X \to Baut_*(F)$ gives the desired map.

Main results

Given a group map $N \to G$, each level of the bar construction $Bar_\bullet(G, N) = \{G \times N^k\}_{k \geq 0}$ (see May [14, Section 7]) admits an action of G, namely the one induced from the group inclusions $s_0\colon G \to G \times N$, $s_1 s_0\colon G \to G \times N^2$, etc. Similarly, in any simplicial group $\Gamma_\bullet$, Γ_0 acts on each level via degeneracies (as above) and endows $\Gamma_\bullet$ with a structure of Γ_0–simplicial set.

The following is the main theorem in [9, Section 4], rephrased.

Theorem 1.3 *A map of discrete groups $f\colon N \to G$ is homotopy normal if and only if there exists a simplicial group $\Gamma_\bullet$, with an isomorphism $\Gamma_0 \cong G$ which extends to a G–equivariant isomorphism of simplicial sets*

$$Bar_\bullet(G, N) \to \Gamma_\bullet.$$

The main goal of this work is to describe a generalization of Theorem 1.3 that characterizes all normal maps $\Omega X \to \Omega Y$. Our strategy is as follows.

In Section 3 we define a homotopical analogue to the bar construction $Bar_\bullet(\Omega Y, \Omega X)$ in the case of loop maps $\Omega X \to \Omega Y$. In the degenerate case of $\Omega Y \simeq *$, $Bar_\bullet(*, \Omega X) = Bar_\bullet(\Omega X)$, and one recovers Segal's 1–fold delooping machine (Definition 2.2) for ΩX.

Next, in Section 5 we define the notion of a homotopy action of a loop space on a space. We study its basic properties and establish a weak equivalence between the category of homotopy actions of a fixed loop space and the category of spaces with an action of a fixed topological group. The simplicial space $Bar_\bullet(\Omega Y, \Omega X)$ admits a canonical homotopy action of ΩY. A homotopy action of ΩY is also defined for any simplicial loop space $\Gamma_\bullet$ satisfying $\Gamma_0 \simeq \Omega Y$. Using this setup we can state a homotopical analogue of Theorem 1.3.

Theorem A *A loop map $\Omega f\colon \Omega X \to \Omega Y$ is homotopy normal if and only if there exists a simplicial loop space $\Gamma_\bullet$ with $\Gamma_0 \simeq \Omega Y$ (as loop spaces), and such that the canonical homotopy actions of ΩY on $\Gamma_\bullet$ and on $Bar_\bullet(\Omega Y, \Omega X)$ are weakly equivalent.*

As often happens, Theorem 1.3 is a special case of Theorem A in that it is precisely its π_0 statement. One consequence of Theorem A is the fact that homotopy normal maps are invariant under homotopy monoidal functors.

Definition 1.4 A functor $L\colon Top \to Top$ is called a *homotopy monoidal* (HM) functor if it preserves homotopy equivalences, contractible spaces and finite products up to homotopy. The last condition can also be formulated as follows: for every pair of spaces X, Y, the canonical map $L(X \times Y) \xrightarrow{\simeq} LX \times LY$ is a homotopy equivalence.

Let L be an HM functor and $\Omega f\colon \Omega X \to \Omega Y$ a loop map. It is implicit in Bousfield [2] and Farjoun [7] and can be proved also by using the delooping theorem of Segal [20] that $L(\Omega X)$ always has the homotopy type of a loop space and $L(\Omega f)$ is always equivalent to a loop map.

Remark 1.5 Although HM functors preserve the property of having (the homotopy type of) a loop space, they do not commute with the functor $\Omega\colon Top_* \to Top_*$.

Using the fact that homotopy actions of loop spaces can be described in terms of maps between finite products of spaces we show that HM functors preserve homotopy normality.

Theorem B *Let $\Omega f\colon \Omega X \to \Omega Y$ be a homotopy normal map. If $L\colon Top \to Top$ is an HM functor, then $L(\Omega f)\colon L\Omega X \to L\Omega Y$ is a homotopy normal map.*

This, in turn, gives an immediate proof of a theorem due to Dwyer and Farjoun [5, Section 3] which we restate.

Theorem C *Let $f\colon X \to Y$ be a map of pointed connected spaces and $p\colon E \to B$ be a homotopy principal fibration of connected spaces. If $L_{\Sigma f}$ is the localization functor by $\Sigma f\colon \Sigma X \to \Sigma Y$, then $L_{\Sigma f}(p)\colon L_{\Sigma f}E \to L_{\Sigma f}B$ is a homotopy principal fibration.*

Remark 1.6 In what follows, we use L to denote an arbitrary HM functor. The notation L reflects the special case of localization by a map.

Refer to related work of Farjoun and Hess [8] on homotopy (co)normal structures in a category with a class of weak equivalences and some additional structure, called a twisted homotopical category.

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2 Preliminaries

Throughout this paper, *topological spaces* or *spaces* will mean topological spaces of the homotopy type of CW complexes. We denote the corresponding category by *Top*. Thus, by Whitehead's theorem, every weak equivalence is in fact a homotopy equivalence. All mapping spaces will be taken with the compact-open topology. The *path space* PX of a pointed space X is the space of maps $\{\alpha\colon I \to X \mid \alpha(0) = *\}$; a *loop space* is understood to be a space of the form $\Omega X := \{\alpha\colon I \to X \mid \alpha(0) = * = \alpha(1)\}$, where X is a pointed connected space and a *loop map* is a map of the form $\Omega f\colon \Omega X \to \Omega Y$ where $f\colon X \to Y$ is a pointed map. The following is a well-known fact, essentially contained in Kan [12] and Milnor [15].

Theorem 2.1 *If X is a (pointed) connected space, then there exists a topological group G, with $X \overset{\simeq}{\rightarrow} BG$. Moreover, one can construct G functorially in X, ie if $\Omega f\colon \Omega X \to \Omega Y$ is a loop map, there is a commutative diagram*

$$\begin{array}{ccc} \Omega X & \longrightarrow & \Omega Y \\ {\scriptstyle\simeq}\downarrow & & {\scriptstyle\simeq}\downarrow \\ G & \longrightarrow & H \end{array}$$

with the vertical arrows being homotopy equivalences, and the bottom arrow being a topological group map.

A map $E \to B$ is a (Serre) *fibration* if it has the right lifting property with respect to all inclusions of the form $D^n \hookrightarrow D^n \times I$ that include the n–disc D^n as $D^n \times \{0\}$. A *fibration sequence* is a sequence of the form $F \to E \overset{p}{\to} B$, where $p\colon E \to B$ is a fibration and either (B, b_0) is pointed and $F = p^{-1}(b_0)$ or $F = p^{-1}(b)$ for some $b \in B$ and B is connected. A sequence $X \to Y \to Z$ is called a *homotopy fibration sequence* if there is a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ {\scriptstyle\simeq}\downarrow & & {\scriptstyle\simeq}\downarrow & & {\scriptstyle\simeq}\downarrow \\ F & \longrightarrow & E & \longrightarrow & B \end{array}$$

with vertical arrows being homotopy equivalences and the bottom being a fibration sequence. A homotopy fibration sequence $X \to Y \to Z$ is called a *homotopy principal fibration sequence* if there is a connected space B and a map $Z \to B$, called the *classifying map* such that $Y \to Z \to B$ is a homotopy fibration sequence. In that case, $X \simeq \Omega B$ and there is a principal fibration sequence $G \to E \to E/G$, and a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ {\scriptstyle\simeq}\downarrow & & {\scriptstyle\simeq}\downarrow & & {\scriptstyle\simeq}\downarrow \\ G & \longrightarrow & E & \longrightarrow & E/G \end{array}$$

with all vertical maps being homotopy equivalences and the left vertical map being equivalent to a loop map $\Omega B \to \Omega BG$.

As usual, we denote by Δ the category of finite ordinals $[n] = (0, \dots, n)$ with ordinal maps between them. Given a category $\mathcal{C}$, a *simplicial object* in $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$, and we denote it by $X_\bullet$ with X_n for its value on $[n]$.

Of special importance to this paper are simplicial objects in *Top*, namely *simplicial spaces*. If X is a space, we shall denote the constant simplicial space on it by X

when there is no risk of confusion. An *equivalence of simplicial spaces* (or: simplicial equivalence) is a simplicial map $f\colon X_\bullet \to Y_\bullet$ such that, $f_n\colon X_n \to Y_n$ is a homotopy equivalence for each n. Similarly, a *(homotopy) fibration sequence of simplicial spaces* is a diagram of simplicial spaces $F_\bullet \to E_\bullet \to B_\bullet$ which is a level-wise (homotopy) fibration sequence.

We will often use a particular class of simplicial spaces introduced in a preprint of Segal [20] and originally called "group-like special Δ–spaces". Influenced by the Rezk's terminology [19], we call them *reduced Segal spaces*; these are defined as follows.

Definition 2.2 (cf [20]) (a) A *reduced Segal space* is a simplicial space $B_\bullet$ such that

(i) $B_0 \simeq *$;

(ii) for each $n \geq 1$, the maps $p_n\colon B_n \to B_1 \times \cdots \times B_1$ (called Segal maps) induced by the maps

$$i_k\colon [1] \to [n] \quad (1 \leq k \leq n)$$
$$0 \mapsto k-1, \quad 1 \mapsto k,$$

are homotopy equivalences;

(iii) the monoid structure on $\pi_0(B_1)$ admits inverses (ie is a group).

(b) We say that $B_\bullet$ is a *reduced Segal space for* ΩX if it comes equipped with a homotopy equivalence $|B_\bullet| \xrightarrow{\simeq} X$; if $B_\bullet$ and $B'_\bullet$ are reduced Segal spaces for ΩX, a map (respectively equivalence) between them is a simplicial map (respectively equivalence) $B_\bullet \to B'_\bullet$ which makes the triangle of loop maps below commutative.

$$\begin{array}{ccc} \Omega|B_\bullet| & \longrightarrow & \Omega|B'_\bullet| \\ & {\scriptstyle\simeq}\searrow \quad \swarrow{\scriptstyle\simeq} & \\ & \Omega X & \end{array}$$

Remark 2.3 By [20, 1.5], it follows that if $B_\bullet$ is a reduced Segal space for ΩX there is a natural homotopy equivalence $B_1 \xrightarrow{\simeq} \Omega|B_\bullet|$. Thus, a reduced Segal space for ΩY can equivalently be defined as a reduced Segal space $B_\bullet$ equipped with a loop equivalence $B_1 \xrightarrow{\simeq} \Omega X$. The diagram of Definition 2.2(b) should then be changed accordingly.

For a topological group G and $a\colon X \times G \to X$ a right action of G on a space X which we denote by $x \mapsto xg$ for $x \in X$ and $g \in G$, the *bar construction* (cf [14, Section 7]) is the simplicial space $Bar_\bullet(X,G)$, consisting of

(1) for every $n \geq 0$, $Bar_n(X,G) := X \times G^n$

together with

(2) face maps $d_i^{(n)} \equiv d_i\colon Bar_n(X,G) \to Bar_{n-1}(X,G)$ for every $n \geq 1$ and every $0 \leq i \leq n$ given by

$$d_i\colon (x, g_1, \dots, g_n) \mapsto \begin{cases} (x \cdot g_1, g_2, \dots, g_n) & \text{if } i = 0, \\ (x, g_1, .., g_{i-1}, g_i \cdot g_{i+1}, g_{i+2}, \dots, g_n) & \text{if } 1 \leq i < n, \\ (x, g_1, \dots, g_{n-1}) & \text{if } i = n, \end{cases}$$

(3) degeneracy maps $s_i\colon Bar_n(X,G) \to Bar_{n+1}(X,G)$ for every $n \geq 1$ and every $0 \leq i \leq n$ given by

$$s_i\colon (x, g_1, \dots, g_n) \mapsto (x, g_1, \dots, g_i, e, g_{i+1}, \dots, g_n).$$

3 The homotopy power of a map

Given a fibration $p\colon E \to B$, one can define a simplicial space $Pow_\bullet(E \to B)$, called the *power of p*, by $Pow_n(E \to B) = E \times_B E \cdots \times_B E$ $(n+1$ times) with face and degeneracies being the obvious projections and diagonals. In [13], it is shown that for (E nonempty and) B connected, $|Pow_\bullet(E \to B)| \simeq B$. We note that for a nonconnected space B, $|Pow_\bullet(E \to B)|$ is homotopy equivalent to the disjoint union of connected components of B intersecting the image of p.

Here, we wish to construct such a power space for an arbitrary map $f\colon X \to B$ by means of homotopy pullbacks, thus turning it to a homotopically invariant construction.

We define the *n–th homotopy power* of $f\colon X \to B$ to be

$$hPow_n(X \to B) = \operatorname{map}\begin{pmatrix} \Delta[n]_0 & & X \\ \iota\downarrow & , & \downarrow f \\ \Delta[n] & & B \end{pmatrix} = \operatorname{holim}\begin{pmatrix} X & X & \cdots & X \\ & \searrow & \searrow\ \swarrow & \\ & & B & \end{pmatrix},$$

with $\iota\colon \Delta[n]_0 \to \Delta[n]$ being the inclusion of the 0–skeleton into the topological n–simplex.

This clearly yields a functorial construction over Δ^{op}, and we define:

Definition 3.1 The *homotopy power* of a map $f\colon X \to B$, denoted $hPow_\bullet(X \to B)$, is the simplicial space with $hPow_n(X \to B)$ on level n, and face and degeneracies given by the functorial construction above.

Note that for a fibration $p\colon E \to B$ one gets an equivalence of simplicial spaces $hPow_\bullet(E \to B) \simeq Pow_\bullet(E \to B)$.

Remark 3.2 When calculating the homotopy power of a map $f\colon X \to B$ we will often use a slightly different but equivalent construction. We first replace f by an equivalent fibration $p\colon E_f \to B$, ie one for which there is a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & E_f \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle p} & \\ B & & \end{array}$$

and then take the power of p, as in [13]. This construction is functorial as well. We also note that if $X \to B$ is a pointed map, $hPow_\bullet(X \to B)$ naturally becomes a pointed simplicial space.

4 The homotopy bar construction

Consider a topological group G acting on a space X and the corresponding (homotopy) principal fibration $G \to X \to X/\!\!/G$. One has the "usual" bar construction $Bar_\bullet(X, G) = \{X \times G^k\}_{k\geq 0}$ with $|Bar_\bullet(X, G)| = X/\!\!/G$. On the other hand, we can resolve $X/\!\!/G$ by taking homotopy powers of the map $q\colon X \to X/\!\!/G$.

Proposition 4.1 *Let G act on X as above. Then there are simplicial equivalences*

$$Bar_\bullet(X, G) \rightleftarrows hPow_\bullet(X \to X/\!\!/G).$$

Proof Replacing $q\colon X \to EG \times_G X$ by the fibration $p\colon EG \times X \to EG \times_G X$ and taking the pullback, we get $hPow_1(X \to X/\!\!/G) = (EG \times X) \times_{X/\!\!/G} (EG \times X) \cong EG \times G \times X$, since $EG \times X$ is a free G–space. In general,

$$hPow_n(X \to X/\!\!/G) = (EG \times X) \times_{X/\!\!/G} \cdots \times_{X/\!\!/G} (EG \times X) \cong EG \times X \times G^n,$$

and the obvious map $EG \times X \times G^n \to X \times G^n$ defines a simplicial equivalence $hPow_\bullet(X \to X/\!\!/G) \to Bar_\bullet(X, G)$. Taking (for example) Milnor's join construction, we have a natural base point for EG and hence a canonical map $X \times G^n \to EG \times X \times G^n$, which in turn defines another simplicial equivalence. □

In light of the last proposition, we define:

Definition 4.2 Given a (homotopy) principal fibration sequence $\Omega Y \to X \xrightarrow{q} Q$, the *homotopy bar construction* $Bar_\bullet(X, \Omega Y)$ is the homotopy power $hPow_\bullet(X \to Q)$.

Remark 4.3 In the case of a loop map $\Omega f\colon \Omega Y \to \Omega Z$, $Bar_\bullet(\Omega Z, \Omega Y)$ is the homotopy power of the map $q\colon \Omega Z \to \Omega Z /\!\!/ \Omega Y := \mathrm{hfib}(f)$. If $\Omega Z \simeq *$, $Bar_\bullet(*, \Omega Y)$ becomes the power of the map $PY \to Y$ which is a reduced Segal space for ΩY. Put differently, one can recover Segal's delooping machine by using homotopy powers.

It is useful to have the following property.

Proposition 4.4 *Let $f\colon X \to B$ be any pointed map. The canonical map induces an equivalence of simplicial spaces $\Omega(hPow_\bullet(X \to B)) \simeq hPow_\bullet(\Omega X \to \Omega B)$.*

The proof is essentially the fact that given a pointed diagram $A \to X \leftarrow Y$, we have a weak equivalence $\Omega\,\mathrm{holim}(A \to X \leftarrow Y) \simeq \mathrm{holim}(\Omega A \to \Omega X \leftarrow \Omega Y)$.

4.1 From homotopy normality to a simplicial loop space structure on the homotopy bar construction

Let $\Omega f\colon \Omega X \to \Omega Y$ be a homotopy normal map. We form the Puppe–Nomura sequence

$$\Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{q} \Omega Y /\!\!/ \Omega X \longrightarrow X \longrightarrow Y \xrightarrow{\pi} W.$$

Then by [16] there is a commutative triangle in which the vertical arrow is a homotopy equivalence

$$\begin{array}{ccc} \Omega Y & \xrightarrow{q} & \Omega Y /\!\!/ \Omega X \\ & \searrow^{\Omega\pi} & \downarrow \\ & & \Omega W. \end{array}$$

Passing to (homotopy) powers, we get an equivalence of simplicial spaces

$$hPow_\bullet(\Omega Y \to \Omega W) \simeq hPow_\bullet(\Omega Y \to \Omega Y /\!\!/ \Omega X)$$

and, by Proposition 4.4, an equivalence of simplicial spaces

$$\Omega(hPow_\bullet(Y \to W)) \simeq hPow_\bullet(\Omega Y \to \Omega Y /\!\!/ \Omega X).$$

Using the argument above and Definition 4.2 we have just proved the following result.

Theorem 4.5 *If $\Omega f\colon \Omega X \to \Omega Y$ is homotopy normal, there are natural simplicial equivalences $Bar_\bullet(\Omega Y, \Omega X) \rightleftarrows \Omega(hPow_\bullet(Y \to W))$.*

Notation 4.6 (cf Theorem 4.5) (1) For a homotopy normal map $\Omega f\colon \Omega X \to \Omega Y$ and a given normal structure $\pi\colon Y \to W$, we denote by $Q_\bullet$ the simplicial loop space $\Omega(hPow_\bullet(Y \to W))$.

(2) The equivalences given in Theorem 4.5 will be denoted

$$\epsilon\colon Bar_\bullet(\Omega Y, \Omega X) \rightleftarrows Q_\bullet : \eta.$$

Remark 4.7 Notice that the maps

$$\epsilon_0\colon \Omega Y \rightleftarrows Q_0 : \eta_0$$

are loop maps by construction, but for $n \geq 1$, the maps

$$\epsilon_n\colon Bar_n(\Omega Y, \Omega X) \rightleftarrows Q_n : \eta_n$$

need not be loop maps. This means that we have, in general, two different loop space structures on $\Omega Y \times (\Omega X)^n$. The nontrivial one is given by the equivalence $Bar_n(\Omega Y, \Omega X) \simeq Q_n$.

5 Homotopy actions

By Remarks 1.2(a) a homotopy normal map is a loop map with its underlying map having the structure of a principal fibration (of connected spaces). Furthermore, Theorem 1.3 involves (strict) group actions. Hence, characterization and invariance of homotopy normal maps under HM functors should include characterization and invariance of group actions "up to homotopy" to some extent. Given an action of a topological group G on a space X and an HM functor $L\colon \mathit{Top} \to \mathit{Top}$, we would like to construct a canonical "action" of LG (not a group, not a loop space) on LX. In other words, we would like to have a homotopical notion of an action of (a space of the homotopy type of) a loop space on a space, invariant under HM functors. One approach we wish to refer the reader to is that of A_∞–actions introduced by Nowlan [17] and recently used by Stasheff [21]. For our purpose, we could not use A_∞–actions since it is not clear they are invariant under HM functors. As demonstrated in Section 5.2, homotopy actions can be rigidified into (strict) group actions. This rigidification gives in fact a "proxy action" on X in the sense of Dwyer and Wilkinson [6] so all the homotopically-invariant information (eg homotopy fixed points) is preserved. Homotopy actions have more flexibility than proxy actions since the object which "acts" need not be a topological group but rather a loop space.

5.1 Definition and basic properties

If a topological group G acts on a space X, one has a simplicial fibration sequence of the form $X \to Bar_\bullet(X, G) \to B_\bullet G$, where the maps $X \to Bar_n(X, G)$ and $Bar_n(X, G) \to B_n G$ are given by $s_n \cdots s_0$ and projection respectively.

Under realization, this becomes a (homotopy) fibration sequence $X \to X /\!\!/ G \to BG$ with a connected base space, ie an "action up to homotopy" in the sense of Dror, Dwyer and Kan [4]. The above simplicial fibration sequence is trivial in each level $X \to X \times G^n \to G^n$, and hence constitutes a useful resolution. We note also that for all n, the map $d_1 d_2 \cdots d_n \colon Bar_n(X, G) \to Bar_0(X, G)$ is the projection on X and the map $d_0 d_0 \cdots d_0 \colon Bar_n(X, G) \to Bar_0(X, G)$ is given by $(x, g_1, \dots, g_n) \mapsto x \cdot (g_1 \cdots g_n)$.

As we saw, the simplicial spaces $Bar_\bullet(X, G)$ and $B_\bullet G$ can be relaxed to their "homotopy versions", namely $Bar_\bullet(X, \Omega Y)$ and $Bar_\bullet(*, \Omega Y)$ (which is a reduced Segal space for ΩY when $BG \simeq Y$).

Definition 5.1 We say that a space S of the homotopy type of a loop space, *homotopy acts* on a space X, if there exist a simplicial map

$$A_\bullet \xrightarrow{\pi} B_\bullet$$

such that

1. (1) $A_0 \simeq X$;
2. (2) $B_\bullet$ is a reduced Segal space for S;
3. (3) for every n, the maps

$$A_n \underset{d_0 \cdots d_0 \times \pi_n}{\overset{d_1 \cdots d_n \times \pi_n}{\rightrightarrows}} A_0 \times B_n$$

are homotopy equivalences.

Maps are defined as follows.

Definition 5.2 Given two homotopy actions of S on X and on X', represented by $A_\bullet \to B_\bullet$ and $A'_\bullet \to B'_\bullet$ respectively, a map between them is a commutative square

$$\begin{array}{ccc} A_\bullet & \longrightarrow & B_\bullet \\ \downarrow & & \downarrow \simeq \\ A'_\bullet & \longrightarrow & B'_\bullet \end{array}$$

such that the map $B_\bullet \to B'_\bullet$ is an equivalence of reduced Segal spaces (see Definition 2.2).

Notation 5.3 We denote by $Top^{h\Omega Y}$ the category of homotopy actions of (spaces of the homotopy type of) ΩY on spaces.

Remark 5.4 If $S \to S'$ is a loop equivalence and S homotopy acts on X, then S' homotopy acts on X, since a reduced Segal space $B_\bullet$ for S induces a reduced Segal space for S' simply by composing the map $B_1 \overset{\simeq}{\twoheadrightarrow} S$ with $S \overset{\simeq}{\twoheadrightarrow} S'$ (see Definition 2.2).

We will need a generalization of Definition 5.1 as follows.

Definition 5.5 A homotopy action of ΩY on a simplicial space $X_\bullet$ is a map of bisimplicial spaces $A_{\bullet\bullet} \to B_{\bullet\bullet}$ such that for each n, $A_{\bullet n} \to B_{\bullet n}$ is a homotopy action of ΩY on X_n and for every map $\theta\colon [n] \to [m]$ in Δ, $\theta^*\colon B_{\bullet m} \to B_{\bullet n}$ is an equivalence of reduced Segal spaces for ΩY; maps and equivalences are defined in the obvious way.

Observation 5.6 If a topological group G acts on a space X, the simplicial map $p\colon Bar_\bullet(X,G) \to B_\bullet(G)$ is a homotopy action of G on X. To see this, note that $B_\bullet(G)$ is a reduced Segal space for G and the maps $(d_1 \cdots d_n) \times p_n\colon Bar_n(X,G) \to Bar_0(X,G) \times B_n(G)$ are the identity maps $X \times G^n \to X \times G^n$. One can verify that the maps $(d_0 \cdots d_0) \times p_n\colon Bar_n(X,G) \to Bar_0(X,G) \times B_n(G)$, ie the action of G^n on X (arising from multiplying n elements in G and then act on X) multiplied by the projection p_n, are homeomorphisms.

Nowlan [17] defined an action of an A_∞–space on a topological space. The difference between this approach and ours is essentially the difference between the approaches of Stasheff [22] and Segal [20] to the characterization of loop spaces.

It is commonly said that in every fibration sequence, the loop space of the base "acts" on the fiber. We wish to demonstrate how a homotopy action interprets this statement.

Theorem 5.7 *Given a fibration sequence $F \xrightarrow{i} E \xrightarrow{p} B$ with B pointed connected, there is a homotopy action of ΩB on F, represented by $\pi\colon A_\bullet \to B_\bullet$, such that the map $|\pi|\colon |A_\bullet| \to |B_\bullet|$ is equivalent to $p\colon E \to B$.*

Proof Consider the commutative square

$$\begin{array}{ccc} F & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & B. \end{array}$$

Taking homotopy powers in each row produces a simplicial map

$$\pi\colon A_\bullet := hPow_\bullet(F \to E) \to hPow_\bullet(* \to B) =: B_\bullet.$$

By Remark 4.3, $B_\bullet$ is a reduced Segal space and thus $|B_\bullet| \simeq B$. Since B is connected, it follows from Section 3 that $|A_\bullet| \simeq E$. To see that $\pi\colon A_\bullet \to B_\bullet$ is a homotopy action, we first replace $i\colon F \to E$ and $* \to B$ by equivalent fibrations $\mathrm{ev}_1\colon F_i \to E$ and $\mathrm{ev}_1\colon PB \to B$, where PB is the path space and $F_i \subseteq F \times E^I$ is the space $\{(f,\alpha) | \alpha(0) = i(f)\}$. Taking $\pi_0\colon F_i \to PB$ to be $\pi_0(f,\alpha) = p \circ \alpha$ we obtain the commutative square

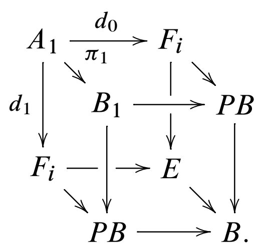
$$(*)\qquad \begin{array}{ccc} F_i & \xrightarrow{\mathrm{ev}_1} & E \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle p} \\ PB & \xrightarrow{\mathrm{ev}_1} & B, \end{array}$$

and taking powers (ie fiber products) of the rows, we obtain a simplicial map we denote as $\pi\colon A_\bullet \to B_\bullet$.

Let us show that the maps

$$A_1 \overset{d_1 \cdots d_n \times \pi_1}{\underset{d_0 \cdots d_0 \times \pi_1}{\rightrightarrows}} A_0 \times B_1$$

are homotopy equivalences. We have a commutative cube



We want to show that the left-hand and upper faces are homotopy cartesian squares, which follows directly from the cartesian-ness of the lower, right-hand and outer faces using the fact that a square is cartesian if and only if the comparison map between homotopy fibers of rows/columns is a homotopy equivalence; see Goodwillie [10, 1.18].

One proceeds similarly to show that the maps $(d_0 \cdots d_0) \times \pi_n$ and $(d_1 \cdots d_n) \times \pi_n$ $(n > 1)$ are homotopy equivalence. Thus, $\pi\colon A_\bullet \to B_\bullet$ is a homotopy action.

Lastly, since the equivalences $|Pow_\bullet(F_i \to E)| \simeq E$ and $|Pow_\bullet(PB \to B)| \simeq B$ are natural, and in light of $(*)$ the map $|\pi|\colon |A_\bullet| \to |B_\bullet|$ is equivalent to $p\colon E \to B$. $\square$

The importance of Theorem 5.7 can be seen, for example, from the fact that it allows one to classify fibrations using homotopy actions.

Homotopy actions arise in our context in the following form.

Corollary 5.8 *If $\Omega f\colon \Omega X \to \Omega Y$ is a loop map, then Ωf induces a homotopy action of ΩX on ΩY, natural in f.*

Proof This follows from Theorem 5.7 if we consider the homotopy fibration sequence $\Omega Y \to \Omega Y /\!\!/ \Omega X \to X$. Alternatively, if we (functorially) rigidify $\Omega f\colon \Omega X \to \Omega Y$ to a topological group map $G \to H$ as in Theorem 2.1, then as we saw, $Bar_\bullet(H,G) \to B_\bullet G$ is a homotopy action. □

Finally, let us see that homotopy actions are invariant under HM functors.

Proposition 5.9 *If $A_\bullet \to B_\bullet$ is a homotopy action of ΩY on X, and $L\colon Top \to Top$ is an HM functor, then $LA_\bullet \to LB_\bullet$ is a homotopy action of $L\Omega Y$ on LX.*

Proof $LB_\bullet$ is a reduced Segal space for LB_1. In particular, LB_1 is of the homotopy type of a loop space. Applying L to the structure maps of the homotopy action yields the structure maps for $LA_\bullet \to LB_\bullet$, and L preserves homotopy equivalences. □

For the sake of completeness, we wish to define a map between homotopy actions of two non–homotopy equivalent loop spaces. The simplicity of the definition demonstrates the "flexibility" of homotopy actions. For example, it allows one to talk about the category of *all* homotopy actions.

Definition 5.10 Given two homotopy actions of ΩY on X and of $\Omega(Y')$ on X', represented by $A_\bullet \longrightarrow B_\bullet$ and $A'_\bullet \longrightarrow B'_\bullet$, a map between them is a commutative square of simplicial spaces

$$\begin{array}{ccc} A_\bullet & \longrightarrow & B_\bullet \\ \downarrow & & \downarrow \\ A'_\bullet & \longrightarrow & B'_\bullet. \end{array}$$

Such a map will be called an *equivalence* if both vertical maps are simplicial equivalences.

5.2 A weakly inverse correspondence with group actions

Our goal here is to establish a weakly inverse correspondence between the category Top_{BG} of spaces over BG and the category $Top^{h\Omega Y}$ of homotopy actions of ΩY where $Y \simeq BG$. Since Top_{BG} is Quillen equivalent to the category of G–spaces, we obtain a correspondence between homotopy actions and group actions which may be referred to as a "rigidification" of the homotopy action. Our functors will be weak inverses in the following sense.

Definition 5.11 Maps $f\colon X \to Y$ and $f'\colon X' \to Y'$ are called *weakly equivalent* if there is a zigzag of commutative squares with all horizontal arrows being homotopy equivalences

$$\begin{array}{ccccccccc} X & \xrightarrow{\simeq} & X_1 & \xleftarrow{\simeq} & \cdots & \xrightarrow{\simeq} & X_n & \xleftarrow{\simeq} & X' \\ \downarrow f & & \downarrow & & & & \downarrow & & \downarrow f' \\ Y & \xrightarrow{\simeq} & Y_1 & \xleftarrow{\simeq} & \cdots & \xrightarrow{\simeq} & Y_n & \xleftarrow{\simeq} & Y'. \end{array}$$

Similarly, simplicial maps $f\colon X_\bullet \to Y_\bullet$ and $f'\colon X'_\bullet \to Y'_\bullet$ are called *weakly equivalent* if there is a zigzag of commutative squares as above, but with objects being simplicial spaces and maps being simplicial maps. The number of squares involved in such a zigzag is said to be its *length*. In particular, maps are called *equivalent* if they are weakly equivalent via a zigzag of length 1.

Definition 5.12 Let G be a topological group, ΩY a loop space and $Y \to BG$ a fixed homotopy equivalence.

(1) The functor $\mathcal{P}\colon \mathit{Top}_{BG} \to \mathit{Top}^{h\Omega Y}$ is defined as follows. Given a map $E \to BG$, let X be its homotopy fiber. Thus, there is a commutative square

$$\begin{array}{ccc} X & \longrightarrow & E \\ \downarrow & & \downarrow \\ PBG & \xrightarrow{\mathrm{ev}_1} & BG. \end{array}$$

Then $\mathcal{P}(E \to BG)$ is the map $\mathit{hPow}_\bullet(X \to E) \to \mathit{hPow}_\bullet(PBG \to BG)$, which is a homotopy action of ΩY by Theorem 5.7.

(2) The functor $\mathcal{R}\colon \mathit{Top}^{h\Omega Y} \to \mathit{Top}_{BG}$ is defined as follows. Given a homotopy action $\pi\colon A_\bullet \to B_\bullet$ of ΩY on X, $\mathcal{R}(A_\bullet \to B_\bullet)$ is the composition

$$|A_\bullet| \overset{|\pi|}{\to} |B_\bullet| \overset{\simeq}{\to} Y \overset{\simeq}{\to} BG,$$

where the second map comes from the fact that $B_\bullet$ is a reduced Segal space for ΩY (see Definition 2.2).

Proposition 5.13 *The functors above satisfy the following properties.*

(a) *If $E \to BG$ is in Top_{BG}, then $\mathcal{P}(E \to BG)$ is a homotopy action of ΩY on $X := \mathrm{hfib}(E \to BG)$.*

(b) *If $\pi\colon A_\bullet \to B_\bullet$ is a homotopy action of ΩY on X, then $\mathcal{R}(A_\bullet \to B_\bullet)$ is a space over BG with X as its homotopy fiber.*

Proof (a) This follows from Theorem 5.7.

(b) Given a homotopy action $\pi\colon A_\bullet \to B_\bullet$ of ΩY on X, define a simplicial map $i\colon A_0 \to A_\bullet$ by $i_n = s_{n-1}\cdots s_0$. Choose $b_0 \in B_0$ and endow B_n with a basepoint $s_{n-1}\cdots s_0(b_0)$. By definition, the map $(d_1\cdots d_n)\times\pi_n\colon A_n \to A_0\times B_n$ is a homotopy equivalence and hence the map $\pi_n\colon A_n \to B_n$ is equivalent to the trivial fibration $A_0\times B_n \to B_n$. We now claim that

$$A_0 \xrightarrow{i_n} A_n \xrightarrow{\pi_n} B_n$$

is a homotopy fibration sequence. To see this, note that by simplicial identities, the composite

$$A_0 \xrightarrow{(d_1\cdots d_n\times\pi_n)\circ i_n} A_0\times B_n$$

equals $1_{A_0}\times(\pi_n\circ i_n)$ and, since B_0 is contractible, $\pi_n\circ i_n = s_{n-1}\cdots s_0\circ\pi_0$ is null-homotopic. Hence, i_n is equivalent to the fiber inclusion $A_0 \to A_0\times B_n$. It follows that the sequence $A_0 \to A_\bullet \to B_\bullet$ is a homotopy fibration sequence in each level and so $A_0 \to |A_\bullet| \to |B_\bullet|$ is a homotopy fibration sequence by Puppe [18]. By definition, $A_0 \simeq X$, and we are done. □

Theorem 5.14 *The functors* $\mathcal{R}\colon Top^{h\Omega Y} \rightleftarrows Top_{BG} :t\mathcal{P}$ *of Definition 5.12 constitute a weakly inverse correspondence in the sense that*

(i) $\mathcal{R}\mathcal{P}(E \to BG)$ *is weakly equivalent to* $E \to BG$*;*

(ii) $\mathcal{P}\mathcal{R}(A_\bullet \to B_\bullet)$ *is weakly equivalent to* $A_\bullet \to B_\bullet$*.*

Theorem 5.14 establishes a "rigidification theorem", which we wish to state separately.

Theorem 5.15 *Given a homotopy action of* ΩY *on* X*, represented by* $\pi\colon A_\bullet \to B_\bullet$*, there is a topological group* G *with* $BG \simeq Y$ *and a space* $X' \simeq X$ *together with a (strict) action of* G *on* X' *such that the simplicial map* π *is weakly equivalent to the simplicial map* $Bar_\bullet(X', G) \to B_\bullet(G)$*.*

The proof of Theorem 5.14 will require some technical preparation.

Definition 5.16 If $A_\bullet$ is a simplicial space, the *simplicial path space on* $A_\bullet$, denoted $PA_\bullet$, is the simplicial space defined by $PA_n = A_{n+1}$ with face maps $d_i := d_{i+1}$ and degeneracy maps $s_i := s_{i+1}$.

Observation 5.17 Let $A_\bullet$ be a simplicial space and let A_0 denote the constant simplicial space. There are simplicial maps $\iota\colon A_0 \to A_\bullet$ and $\rho\colon PA_\bullet \to A_0$ defined on level n via the maps $[n+1] \to [0]$ and $[0] \hookrightarrow [n] (0 \mapsto 0)$, respectively. $PA_\bullet$ is simplicially homotopy equivalent to the constant simplicial space A_0; in particular, $|PA_\bullet| \simeq A_0$. In addition, the face map $d_0\colon A_{n+1} \to A_n$ defines a simplicial map $PA_\bullet \to A_\bullet$.

In addition, we will need the following result.

Lemma 5.18 *Let $\pi\colon A_\bullet \to B_\bullet$ be a homotopy action. Then for each $n \geq 0$, the square*

$$\begin{array}{ccc} A_{n+1} & \longrightarrow & |PA_\bullet| \\ \downarrow & & \downarrow \\ A_n & \longrightarrow & |A_\bullet| \end{array}$$

is homotopy cartesian.

Proof From the axioms of a homotopy action, there is a commutative square with horizontal maps homotopy equivalences

$$\begin{array}{ccc} A_{n+1} & \xrightarrow{(d_0 \cdots d_0) \times \pi_{n+1}} & A_0 \times B_{n+1} \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle 1 \times d_0} \\ A_n & \xrightarrow{(d_0 \cdots d_0) \times \pi_n} & A_0 \times B_n. \end{array}$$

Since $B_\bullet$ is a reduced Segal space, by [20, 1.6], for each $k \geq 0$, the square

$$\begin{array}{ccc} B_{k+1} & \longrightarrow & |PB_\bullet| \\ {\scriptstyle d_0}\downarrow & & \downarrow \\ B_k & \longrightarrow & |B_\bullet| \end{array} \tag{1}$$

is homotopy cartesian.

Thus, the homotopy fiber of $d_0\colon B_{n+1} \to B_n$ is (canonically) equivalent to B_1. The homotopy fiber of $d_0\colon A_{n+1} \to A_n$ is therefore homotopy equivalent to B_1, which is also the homotopy fiber of $|PA_\bullet| \to |A_\bullet|$. It follows that the square

$$\begin{array}{ccc} A_{n+1} & \longrightarrow & |PA_\bullet| \\ {\scriptstyle d_0}\downarrow & & \downarrow \\ A_n & \longrightarrow & |A_\bullet| \end{array}$$

is homotopy cartesian. □

Proof of Theorem 5.14 (i) Given, without loss of generality, a fibration sequence $X \to X/G \to BG$, the map $hPow_\bullet(X \to X/G) \to hPow_\bullet(* \to BG)$ obtained just as in Theorem 5.7 has X as a homotopy fiber in each level. Since $|hPow_\bullet(X \to X/G)| \simeq X/G$ and $|hPow_\bullet(* \to BG)| \simeq BG$, the map $|hPow_\bullet(X \to X/G)| \to |hPow_\bullet(* \to BG)|$ is equivalent to $X/G \to BG$.

(ii) Given a homotopy action $\pi\colon A_\bullet \to B_\bullet$, $B_\bullet$ is a reduced Segal space, and thus by [20, Proposition 1.6], for each $k \geq 0$, the following square is homotopy cartesian:

$$\begin{array}{ccc} B_{k+1} & \longrightarrow & |PB_\bullet| \\ {\scriptstyle d_0}\downarrow & & \downarrow \\ B_k & \longrightarrow & |B_\bullet| \end{array} \tag{2}$$

By Lemma 5.18, the same holds for $A_\bullet$, ie for each $k \geq 0$, the square

$$\begin{array}{ccc} A_{k+1} & \longrightarrow & |PA_\bullet| \\ {\scriptstyle d_0}\downarrow & & \downarrow \\ A_k & \longrightarrow & |A_\bullet| \end{array} \tag{3}$$

is homotopy cartesian. We construct a map $A_\bullet \to hPow_\bullet(|PA_\bullet| \to |A_\bullet|)$ by induction on n. For $n = 0$, the map $A_0 \to |PA_\bullet|$ is the realization of $\iota\colon A_0 \to PA_\bullet$ defined in Observation 5.17. For $n = 1$, consider the commutative square

$$\begin{array}{ccc} A_0 & \longrightarrow & |A_\bullet| \\ \downarrow & & \downarrow \\ |PA_\bullet| & \longrightarrow & |A_\bullet|. \end{array} \tag{4}$$

Since (2) is homotopy cartesian for $k = 0$, the map $A_1 \to A_0 \times^h_{|A_\bullet|} |PA_\bullet|$ is a homotopy equivalence, and the map $A_1 \to hPow_1(|PA_\bullet| \to |A_\bullet|)$ is obtained by composing the last map with $A_0 \times^h_{|A_\bullet|} |PA_\bullet| \to |PA_\bullet| \times^h_{|A_\bullet|} |PA_\bullet|$ induced by (3). Let us define the map for $n+1$: the square (2) with index n is homotopy cartesian, and thus there is a homotopy equivalence $A_{n+1} \to A_n \times^h_{|A_\bullet|} |PA_\bullet|$. Using the map $A_n \to hPow_n(|PA_\bullet| \to |A_\bullet|)$ that was defined, we get a natural homotopy equivalence $A_{n+1} \to hPow_\bullet(|PA_\bullet| \to |A_\bullet|)$. It is clear from the construction that one gets a simplicial map $A_\bullet \to hPow_\bullet(|PA_\bullet| \to |A_\bullet|)$. Similarly, there is a simplicial map $B_\bullet \to hPow_\bullet(|PB_\bullet| \to |B_\bullet|)$. The zigzag of commutative squares

$$\begin{array}{ccccccc} A_0 & \xrightarrow{\simeq} & |PA_\bullet| & \longrightarrow & |PB_\bullet| & \xleftarrow{\simeq} & B_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ |A_\bullet| & \xrightarrow{\simeq} & |A_\bullet| & \longrightarrow & |B_\bullet| & \xleftarrow{\simeq} & |B_\bullet| \end{array}$$

induces a zigzag of commutative simplicial squares

$$\begin{array}{ccccc} A_\bullet & \xrightarrow{\simeq} & hPow_\bullet(|PA_\bullet| \to |A_\bullet|) & \xleftarrow{\simeq} & hPow_\bullet(A_0 \to |A_\bullet|) \\ \downarrow & & \downarrow & & \downarrow \\ B_\bullet & \xrightarrow{\simeq} & hPow_\bullet(|PB_\bullet| \to |B_\bullet|) & \xleftarrow{\simeq} & hPow_\bullet(B_0 \to |B_\bullet|). \end{array}$$

Note that by Proposition 4.1, there is also a square

$$\begin{array}{ccc} hPow_\bullet(A_0 \to |A_\bullet|) & \longleftarrow & Bar_\bullet(X, G) \\ \downarrow & & \downarrow \\ hPow_\bullet(B_0 \to |B_\bullet|) & \longleftarrow & B_\bullet(G) \end{array}$$

for a topological group G with $BG \simeq |B_\bullet|$. □

6 An invariant characterization of normality

Theorem 1.3 characterizes homotopy normal maps of discrete groups in terms of a simplicial group, equivariantly equivalent to the bar construction. By analogy, the mere fact that the homotopy bar construction $Bar_\bullet(\Omega Y, \Omega X)$ is simplicially equivalent to a simplicial loop space $\Gamma_\bullet$ with $\Gamma_0 \simeq \Omega Y$, is a necessary but not sufficient condition for a loop map $\Omega f\colon \Omega X \to \Omega Y$ to be homotopy normal.

In both simplicial spaces $Bar_\bullet(\Omega Y, \Omega X)$ and $Q_\bullet$ (see Notation 4.6), the map $s_{n-1}\cdots s_0$ is a loop map, therefore it induces a homotopy action of ΩY on Q_n and $Bar_n(\Omega Y, \Omega X)$ (see Corollary 5.8).

We begin with the following.

Proposition 6.1 *Let $\Omega f\colon \Omega X \to \Omega Y$ be a homotopy normal map and $Q_\bullet$ its corresponding simplicial loop space. For each n, the homotopy actions induced by the loop maps $Q_0 \to Q_n$ and $\Omega Y \to Bar_n(\Omega Y, \Omega X)$ are equivalent via the map $\eta\colon Q_\bullet \to Bar_\bullet(\Omega Y, \Omega X)$, defined in Notation 4.6.*

Proof We do only the case $n = 1$ since other cases are similar. Write $\sigma := s_0\colon Q_0 \to Q_1$ and $s := s_0\colon Bar_0(\Omega Y, \Omega X) \to Bar_1(\Omega Y, \Omega X)$. The simplicial equivalence $\eta\colon Q_\bullet \to Bar_\bullet(\Omega Y, \Omega X)$ induces a commutative square with vertical arrows being homotopy equivalences, and with the left vertical arrow being a loop map

$$\begin{array}{ccc} Q_0 & \xrightarrow{\sigma} & Q_1 \\ {\scriptstyle \eta_0}\downarrow & & \downarrow{\scriptstyle \eta_1} \\ \Omega Y & \xrightarrow{s} & \Omega Y \times \Omega X. \end{array}$$

Finding the dashed arrow

$$\begin{array}{ccc} Q_1 & \xrightarrow{\gamma} & Q_1 /\!\!/ Q_0 \\ {\scriptstyle \eta_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ \Omega Y \times \Omega X & \xrightarrow[c]{} & \Omega X \end{array}$$

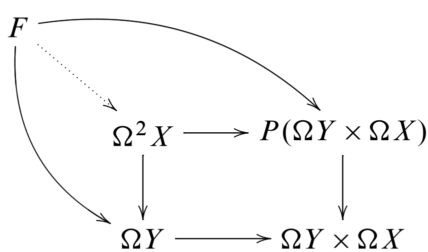
will end the proof since the first and second homotopy actions are built out of homotopy powers of γ and c, respectively. Both σ and s have (spaces of the homotopy type of) loop spaces as their homotopy fiber, and the Puppe–Nomura sequence will provide the dashed arrow, once we show that the equivalence between the homotopy fibers $F := \mathrm{hfib}(\sigma) \to \mathrm{hfib}(s) \simeq \Omega^2 X$ is a loop map. To prove the last statement we use the path-space to model the homotopy fiber. On the one hand, we have the pullback square

$$\begin{array}{ccc} \Omega^2 X & \longrightarrow & P(\Omega Y \times \Omega X) \\ \downarrow & & \downarrow \\ \Omega Y & \xrightarrow{s} & \Omega Y \times \Omega X, \end{array}$$

and on the other hand, in the pullback square

$$\begin{array}{ccc} F & \longrightarrow & P(Q_1) \\ \downarrow & & \downarrow \\ Q_0 & \xrightarrow{\sigma} & Q_1, \end{array}$$

all maps are of the homotopy type of loop maps. The map $F \to \Omega^2 X$ is the universal map to the pullback $\Omega^2 X$, obtained from the diagram



where the curved maps are $F \to Q_0 \to \Omega Y$ and $F \to P(Q_1) \to P(\Omega Y \times \Omega X)$; these maps are (of the homotopy type of) loop maps, and thus the map they induce $F \to \Omega^2 X$ is itself (of the homotopy type of) a loop map. □

As we have just seen, the loop maps $s_{n-1} \cdots s_0 \colon Q_0 \to Q_n$ ($n = 0$ understood as the identity map) induce homotopy actions of Q_0 on Q_n. We can pack all the maps into one simplicial map $Q_0 \to Q_\bullet$, which will then induce a simplicial object in the category of homotopy actions. Recalling Definition 5.5, this is a homotopy action of Q_0 on $Q_\bullet$. Similarly, one has a homotopy action of ΩY on $Bar_\bullet(\Omega Y, \Omega X)$ and the loop space equivalence $Q_0 \simeq \Omega Y$ makes the first homotopy action into one of ΩY on $Q_\bullet$ (see Corollary 5.8). Note that any simplicial loop space $\Gamma_\bullet$ with $\Gamma_0 \simeq \Omega Y$ could play the role of $Q_\bullet$ in defining these homotopy actions.

Given a loop map $\Omega f \colon \Omega X \to \Omega Y$ and a simplicial loop space $\Gamma_\bullet$ with $\Gamma_0 \simeq \Omega Y$, we call the actions above the *canonical homotopy actions* of ΩY on $\Gamma_\bullet$ and $Bar_\bullet(\Omega Y, \Omega X)$. The additional condition for a characterization of normality is that the two are equivalent.

We can now restate and prove Theorem A.

Theorem A *A loop map $\Omega f \colon \Omega X \to \Omega Y$ is homotopy normal if and only if there exist a simplicial loop space $\Gamma_\bullet$ with $\Gamma_0 \simeq \Omega Y$ (as loop spaces), and such that the canonical homotopy actions of ΩY on $\Gamma_\bullet$ and on $Bar_\bullet(\Omega Y, \Omega X)$ (as above) are weakly equivalent.*

Remark 6.2 The weak equivalence of homotopy actions above implies, in particular, the equivalence of simplicial spaces $Bar_\bullet(\Omega Y, \Omega X)$ and $\Gamma_\bullet$.

Proof Assume Ωf is homotopy normal. We have a commutative square of simplicial spaces

$$\begin{array}{ccccc}
\Omega Y & \xrightarrow{\sigma} & Q_\bullet & \longrightarrow & Q_\bullet /\!\!/ Q_0 \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle d} \\
\Omega Y & \xrightarrow{s} & Bar_\bullet(\Omega Y, \Omega X) & \longrightarrow & Bar_\bullet(\Omega Y, \Omega X) /\!\!/ \Omega Y
\end{array}$$

with φ the simplicial equivalence of Theorem 4.5; the dashed arrow d with d_1 (of Proposition 6.1) as its first component, and the analogous d_n as its n–th component. This gives the desired equivalence of the canonical actions.

Conversely, if we have a zigzag of equivalent homotopy actions (see Definition 5.5), then taking the homotopy quotient of each homotopy action, we get a zigzag of simplicial spaces

$$\begin{array}{ccccc}
\Gamma_0 & \longrightarrow & \Gamma_\bullet & \longrightarrow & \Gamma_\bullet /\!\!/ \Gamma_0 \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\
\uparrow{\simeq} & & \uparrow{\simeq} & & \uparrow{\simeq} \\
\Omega Y & \longrightarrow & Bar_\bullet(\Omega Y, \Omega X) & \xrightarrow{q} & Bar_\bullet(\Omega Y, \Omega X) /\!\!/ \Omega Y.
\end{array}$$

The map q in the bottom row is in fact $\pi\colon Bar_\bullet(\Omega Y, \Omega X) \to Bar_\bullet(*, \Omega X)$, and upon realization we have a zigzag of equivalent principal fibrations

$$\begin{array}{ccccc} \Gamma_0 & \longrightarrow & |\Gamma_\bullet| & \longrightarrow & |\Gamma_\bullet /\!\!/ \Gamma_0| \\ \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\ \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\ \uparrow{\simeq} & & \uparrow{\simeq} & & \uparrow{\simeq} \\ \Omega Y & \longrightarrow & \Omega Y /\!\!/ \Omega X & \longrightarrow & X. \end{array}$$

The operation of taking loops commutes with that of realization, and hence $|\Gamma_\bullet| \simeq \Omega W$ for some connected space W. The map $\Gamma_0 \to |\Gamma_\bullet|$ is the realization of a simplicial loop map $\Gamma_0 \to \Gamma_\bullet$, hence a loop map itself, and delooping it gives the desired extension $Y \dashrightarrow W$. □

As an application of Theorem A we will show that homotopy normal maps are preserved by HM functors.

Let $A_\bullet \to B_\bullet$ be a homotopy action. From Proposition 5.13 (b), it follows that there is a homotopy fibration sequence $A_0 \xrightarrow{\sigma} |A_\bullet| \to |B_\bullet|$, where σ is the realization of the simplicial map $A_0 \to A_\bullet$ that has as n–th component the map $s_{n-1} \cdots s_0$. Since $B_\bullet$ is a reduced Segal space, $\Omega|B_\bullet| \simeq B_1$. We denote by $\psi\colon B_1 \to A_0$ the canonical map from the homotopy fiber of $\sigma\colon A_0 \to |A_\bullet|$ to A_0 and endow A_0 with a basepoint via ψ. Denote by $i\colon B_1 \to A_0 \times B_1$ the natural inclusion. We shall need the following technical lemma.

Lemma 6.3 *For any choice of homotopy inverse $e\colon A_0 \times B_1 \to A_1$ for the map $d_1 \times \pi_1\colon A_1 \to A_0 \times B_1$, the composite*

$$B_1 \xrightarrow{i} A_0 \times B_1 \xrightarrow{e} A_1 \xrightarrow{d_0} A_0$$

is homotopic to ψ.

Proof The following square is homotopy commutative:

$$\begin{array}{ccc} A_1 & \xrightarrow{d_1} & A_0 \\ {\scriptstyle d_0}\downarrow & & \downarrow \\ A_0 & \longrightarrow & |A_\bullet|. \end{array}$$

We thus obtain a homotopy commutative diagram of solid arrows

$$\begin{array}{ccccc}
B_1 & \xrightarrow{i} & A_0 \times B_1 & \xrightarrow{\mathrm{pr}} & A_0 \\
{\scriptstyle c_2}\uparrow{\scriptstyle \simeq} & & {\scriptstyle d_1 \times \pi_1}\uparrow \downarrow{\scriptstyle e} & & {\scriptstyle 1}\uparrow \\
B_1 & \longrightarrow & A_1 & \xrightarrow{d_1} & A_0 \\
{\scriptstyle c_1}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle \sigma} \\
B_1 & \xrightarrow{\psi} & A_0 & \xrightarrow{\sigma} & |A_\bullet|,
\end{array}$$

where the map $B_1 \to A_1$ is the canonical map from the homotopy fiber, the map c_1 is the comparison map between the homotopy fibers of d_1 and σ, which is a homotopy equivalence, and the map c_2 is the comparison map between the homotopy fibers of d_1 and pr, which is again a homotopy equivalence. The lemma now follows from inverting c_2. □

Theorem 6.4 *Let $\Omega f\colon \Omega X \to \Omega Y$ be a loop map and $L\colon Top \to Top$ an HM functor. Then the map $L\,Bar_\bullet(\Omega Y, \Omega X) \to L\,Bar_\bullet(*, \Omega X)$ is weakly equivalent to $Bar_\bullet(L\Omega Y, L\Omega X) \to Bar_\bullet(*, L\Omega X)$ where the latter is induced from $L\Omega f$.*

Proof Since $L\Omega Y \to |L\,Bar_\bullet(\Omega Y, \Omega X)| \to |L\,Bar_\bullet(*, \Omega X)|$ is the realization of a simplicial fibration sequence, it is a homotopy fibration sequence, and since $|L\,Bar_\bullet(*, \Omega X)| \simeq B(L\Omega X)$ ($L\,Bar_\bullet(*, \Omega X)$ is a reduced Segal space for $L\Omega X$), there is a map $\varphi\colon L\Omega X \to L\Omega Y$, which is the map from the homotopy fiber of $L\Omega Y \to |L\,Bar_\bullet(\Omega Y, \Omega X)|$ to $L\Omega Y$.

Abbreviate $A_\bullet := Bar_\bullet(\Omega Y, \Omega X)$ and $B_\bullet := Bar_\bullet(*, \Omega X)$. If $e\colon A_0 \times B_1 \to A_1$ is a homotopy inverse to $d_1 \times \pi_1$, then Le is a homotopy inverse for $L(d_1 \times \pi_1)$, which is equivalent to $L(d_1) \times L(\pi_1)$. By Lemma 6.3, Ωf is homotopic to the composite

$$B_1 \xrightarrow{i} A_0 \times B_1 \xrightarrow{e} A_1 \xrightarrow{d_0} A_0,$$

and so $L\Omega f$ is homotopic to the composition $Ld_0 \circ Le \circ Li$. The last composite is homotopic to the composite

$$LB_1 \hookrightarrow LA_0 \times LB_1 \xrightarrow{Le \circ w} LA_1 \xrightarrow{Ld_0} A_0$$

(where w is some homotopy inverse for $L(A_0 \times B_1) \to LA_0 \times LB_1$), which is in turn homotopic to φ by Lemma 6.3 ($Le \circ w$ is a homotopy inverse for $Ld_1 \times L\pi_1$). It follows that $L\Omega f$ is equivalent to φ. Thus, the map $L\Omega Y \to |L\,Bar_\bullet(\Omega Y, \Omega X)|$ is equivalent to $L\Omega Y \to L\Omega Y /\!\!/ L\Omega X$ and using Proposition 4.1 and Theorem 5.14, we deduce that $L\,Bar_\bullet(\Omega Y, \Omega X) \to L\,Bar(*, \Omega X)$ is weakly equivalent to $Bar_\bullet(L\Omega Y, L\Omega X) \to Bar_\bullet(*, L\Omega X)$. □

Let us rephrase Theorem 6.4. Given a loop map Ωf and an HM functor L, there are two homotopy actions: the first is given by applying L to the homotopy action induced by Ωf, and the second is the homotopy action induced from $L\Omega f$. The theorem then says that the two are weakly equivalent. We note that if we are given a homotopy action of a loop space on a simplicial space, in which the homotopy actions in each level are induced by loop maps, an analogous statement holds.

Using the machinery of reduced Segal spaces, one can easily see that applying an HM functor to a simplicial loop space in every level yields a simplicial space *simplicially equivalent* to a simplicial loop space.

Thus, we now know all the ingredients used in Theorem A are invariant under HM functors and we deduce Theorem B (which we restate for convenience).

Theorem B *Let $\Omega f\colon \Omega X \to \Omega Y$ be a homotopy normal map. If $L\colon Top \to Top$ is an HM functor, then $L(\Omega f) : L\Omega X \to L\Omega Y$ is a homotopy normal map.*

Let us demonstrate a use of Theorem B by applying it to prove Theorem C (which we restate).

Theorem C *Let $p\colon E \to B$ be a principal fibration with B connected, $f\colon X \to Y$ a map of pointed connected spaces and $L_{\Sigma f}$ the localization with respect to its suspension. Then $L_{\Sigma f} E \to L_{\Sigma f} B$ is equivalent to a principal fibration.*

Remark 6.5 Note that if G is the structure group of $E \to B$, $L_{\Sigma f} G$ need not be the structure group of $L_{\Sigma f} E \to L_{\Sigma f} B$.

Proof of Theorem C Note that $\Omega E \to \Omega B$ is homotopy normal. So $L_f \Omega E \to L_f \Omega B$ is homotopy normal. Since for any pointed space A there is a natural equivalence $L_f \Omega A \simeq \Omega L_{\Sigma f} A$, we get that $\Omega L_{\Sigma f} E \to \Omega L_{\Sigma f} B$ is homotopy normal and thus $L_{\Sigma f} E \to L_{\Sigma f} B$ is a homotopy principal fibration. □

7 Higher normality

As mentioned in Example 2, any double loop map with simply connected underlying spaces is automatically homotopy normal. However, in the case of a double loop map, it is more natural to ask when the homotopy quotient admits a natural double loop space structure.

Definition 7.1 A 0–homotopy normal map is a pointed map which admits a structure of a (homotopy) principal fibration of connected spaces. For $k \geq 1$, call a k–fold loop map $\Omega^k f\colon \Omega^k X \to \Omega^k Y$ *k–homotopy normal* if f is 0–homotopy normal.

Thus, if a k–fold loop map $\Omega^k f$ is k–homotopy normal, the homotopy quotient $\Omega^k Y /\!\!/ \Omega^k X$ (which is always a $(k-1)$–fold loop space) admits a structure of a k–fold loop space in a natural way.

Remark 7.2 One may wonder about the definition of "∞–homotopy normality". However, any infinite loop map $X \to Y$ induces a principal fibration sequence of infinite loop spaces $X \to Y \to Y /\!\!/ X$. Thus any infinite loop map is "∞–normal" in the naive sense. This is a reflection of the fact that any inclusion map of abelian (topological) groups is the inclusion of a normal subgroup.

We begin with an extension of Theorem A.

Theorem 7.3 *A k–fold loop map $\Omega^k f\colon \Omega^k X \to \Omega^k Y$ is k–homotopy normal if and only if there exists a k–fold simplicial loop space $\Gamma_\bullet$ with $\Gamma_0 \simeq \Omega^k Y$, and such that the canonical homotopy actions of $\Omega^k Y$ on $Bar_\bullet(\Omega^k Y, \Omega^k X)$ and $\Gamma_\bullet$ are naturally equivalent.*

Proof This is analogous to the proof of Theorem C. If $\Omega^k f$ is k–homotopy normal, then Ωf is homotopy normal, and looping down its extension $Y \to W$ k times gives a k–fold loop map equivalent to $\Omega^k Y \to \Omega^k Y /\!\!/ \Omega^k X$. Taking the (homotopy) power of that map gives the desired k–fold loop space. Conversely, such a k–fold loop space gives a (homotopy) principal fibration sequence of k–fold loop spaces $\Omega^k X \to \Omega^k Y \to |\Gamma_\bullet|$, equivalent to the Borel construction, providing the k–homotopy normality required. □

We wish to use the same methods as before to prove invariance of k–homotopy normal maps under HM functors. For that, we need that k–fold loop spaces are invariant under these functors. A slight generalization of reduced Segal spaces is the tool needed.

Definition 7.4 Let k be a positive integer. A *k–simplicial space* is a functor

$$\Delta^{\mathrm{op}} \times \cdots \times \Delta^{\mathrm{op}} \to \mathit{Top} \quad (k \text{ times}).$$

The following is taken from Balteanu et al [1].

Definition 7.5 A k–simplicial space $X_{\bullet \dots \bullet}$ is called a *reduced Segal k–space* if

(1) $X_{0,\dots,0} \simeq *$;

(2) the Segal maps induce homotopy equivalences $X_{p_1,\dots,p_k} \xrightarrow{\simeq} (X_{1,\dots,1})^{p_1 \cdots p_k}$;

(3) the monoid $\pi_0(X_{1,\dots,1})$ admits inverses (ie is a group).

Building on Segal's delooping machine, the characterization of k–fold loop spaces takes the following form.

Theorem 7.6 *A space X is of the homotopy type of a k–fold loop space if and only if there exist a reduced Segal k–space $X_{\bullet,\dots,\bullet}$ with $X_{1,\dots,1} \simeq X$.*

Corollary 7.7 *Homotopy monoidal endofunctors of spaces preserve k–fold loop spaces.*

Using exactly the same arguments of Theorem C, Theorem 7.3 implies that L preserves higher homotopy normality.

Theorem 7.8 *If $\Omega^k f\colon \Omega^k X \to \Omega^k Y$ is k–homotopy normal and $L\colon Top \to Top$ an HM functor, then $L(\Omega^k f)$ is k–homotopy normal.*

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