Meridional destabilizing number of knots

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We define the meridional destabilizing number of a knot. This together with Heegaard genus (or tunnel number) gives a binary complexity of knots. We study its behavior under connected sum of tunnel number one knots.

57M25; 57N10

1 Introduction

1.1 Backgrounds

From a viewpoint of Heegaard theory, we have two types of natural positions of knots in connected closed orientable 3–manifolds: (i) a bridge position with respect to a Heegaard surface, and (ii) a core position of a handlebody bounded by a Heegaard surface. A Heegaard surface of type (ii) corresponds to that of a knot exterior. Hence it has a close connection to Heegaard genus and tunnel number of knots defined below.

Let M be a connected closed orientable 3–manifold and $(V_1, V_2; S)$ a (genus g) *Heegaard splitting* of M, that is, (1) V_1 and V_2 are (genus g) handlebodies, (2) $V_1 \cup V_2 = M$ and (3) $V_1 \cap V_2 = \partial V_1 = \partial V_2 = S$. Such a surface S is called a *Heegaard surface* of M. A knot K, that is, a connected closed 1–manifold in M is in a (g, b)*–bridge position* if K is in a b–bridge position with respect to a Heegaard surface of genus g (see Section 2.1 for the precise definition). Set $\mathcal{M} = (M, K)$, $\mathcal{V}_i = (V_i, V_i \cap K)$ $(i = 1, 2)$ and $\mathcal{S} = (S, S \cap K)$. If a Heegaard splitting $(V_1, V_2; S)$ of M gives a (g, b)–bridge position of K, then $(\mathcal{V}_1, \mathcal{V}_2; \mathcal{S})$ is called a (g, b)*–bridge splitting* of $\mathcal{M}$, and $\mathcal{S}$ is called a (g, b)*–bridge surface*. This is introduced by Doll [2] and is a natural generalization of classical bridge decompositions of knots in the 3–sphere S^3.

A Heegaard splitting $(V_1, V_2; S)$ of M is also called a *Heegaard splitting* of $\mathcal{M} = (M, K)$ if $K \subset V_i$, say $i = 1$, and the exterior of K in V_1 is a compression body. Such a surface S is also called a *Heegaard surface* of $\mathcal{M}$. The *Heegaard genus* of $K \subset M$, denoted by $\mathrm{hg}(K)$, is the minimal value g such that $\mathcal{M}$ admits a Heegaard surface of genus g. We notice that $t(K) := \mathrm{hg}(K) - 1$ is called the *tunnel number* of $K \subset M$.

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Let $(V_1, V_2; S)$ be a genus g Heegaard splitting of $\mathcal{M} = (M, K)$ with $K \subset V_1$. Suppose that there are compressing disks D_i $(i = 1, 2)$ of V_i such that D_1 intersects K transversely in a single point and that ∂D_1 intersects ∂D_2 transversely in a single point. Then S is said to be *meridionally stabilized*. Under this condition we obtain a $(g-1, 1)$–bridge splitting of $\mathcal{M}$ as follows. Let V_1' be a 3–manifold obtained by cutting V_1 along D_1. Since D_1 is non-separating in V_1, we see that V_1' is a handlebody of genus $g-1$. Moreover, $V_1' \cap K$ is a trivial arc in V_1', that is, $V_1' \cap K$ is a simple arc which is isotopic into $\partial V_1'$ relative to boundary. Attaching a (closed) regular neighborhood of D_1 in V_1 to V_2, we obtain a 3–manifold V_2' which is also a handlebody of genus $g-1$. We also see that $V_2' \cap K$ is a trivial arc in V_2'. Therefore $S' := \partial V_1' = \partial V_2'$ gives a $(g-1, 1)$–bridge position of K, that is, $(\mathcal{V}_1', \mathcal{V}_2'; \mathcal{S}')$ is a $(g-1, 1)$–bridge splitting of $\mathcal{M}$, where $\mathcal{V}_i' = (V_i', V_i' \cap K)$ $(i = 1, 2)$ and $\mathcal{S}' = (S', S' \cap K)$. We call this operation *meridional destabilization*. We can similarly define a *meridionally stabilized* (g, b)–bridge surface and obtain a $(g-1, b+1)$–bridge surface from such a surface by meridional destabilization.

It could not be said that there is a close relationship between a bridge number $b_g(K)$ and Heegaard genus $\mathrm{hg}(K)$, where $b_g(K)$ is the minimal bridge number of K with respect to a genus g Heegaard surface of M. If, of course, $\mathcal{M}$ admits a (g, b)–bridge position, then we obtain a genus $g + b$ Heegaard splitting of $\mathcal{M}$ by repeating the converse operation of meridional destabilization and hence we see $\mathrm{hg}(K) \le g + b$. However, Minsky, Moriah and Schleimer [10, Theorem 4.2] showed that for any integers $g \ge 2$ and $b \ge 1$, there is a knot $K \subset S^3$ with $\mathrm{hg}(K) = g$ such that K does not admit a (g, b)–bridge position (see also Johnson and Thompson [5] for the case of $g = 2$). In this paper, we define *meridional destabilizing number* of a knot $K \subset M$ as follows:

Definition 1.1 Let K be a knot in a connected closed orientable 3–manifold M. *Meridional destabilizing number* of K, denoted by $\mathrm{md}(K)$, is defined by the maximal number of m such that $\mathcal{M} = (M, K)$ admits a $(\mathrm{hg}(K)-m, m)$–bridge position. In particular, $\mathrm{md}(K) = 0$ if none of the minimal genus Heegaard splittings of $\mathcal{M}$ are meridionally stabilized.

By the definition above, we see that $\mathrm{md}(K) \le \mathrm{hg}(K)$ for any knot K.

Notation 1.2 Let K be a knot in S^3. We describe $K \in \mathcal{K}_g^m$ if $\mathrm{hg}(K) = g$ and $\mathrm{md}(K) = m$.

For example, $K \in \mathcal{K}_1^1$ if and only if K is a trivial knot. We can divide tunnel number one knots into three families, $\mathcal{K}_2^2$, $\mathcal{K}_2^1$ and $\mathcal{K}_2^0$. Knots in $\mathcal{K}_2^2$ are non-trivial 2–bridge knots, those in $\mathcal{K}_2^1$ are $(1, 1)$–knots which are not 2–bridge knots, and those in $\mathcal{K}_2^0$ are the other tunnel number one knots.

1.2 Results

We study behavior of meridional destabilizing number under connected sum of knots. Let K be a knot in S^3. We denote by nK the connected sum of n copies of K. Then we have: $n \leq t(nK) \leq n \cdot t(K) + (n-1)$ or equivalently

$$n + 1 \leq \mathrm{hg}(nK) \leq n \cdot \mathrm{hg}(K).$$

The upper bound is well-known and is easy to understand. However, the lower bound is highly non-trivial and is obtained by Scharlemann and Schultens [15, Theorem 14]. If $\mathrm{md}(K) \neq 0$, then we also have $\mathrm{hg}(nK) \leq n \cdot \mathrm{hg}(K) - n + 1$ and $\mathrm{md}(nK) \neq 0$ (see Proposition 2.14). Hence if $\mathrm{hg}(K) = 2$ and $\mathrm{md}(K) \neq 0$, then $\mathrm{hg}(nK) = n + 1$. It follows from Schubert's formula on bridge number [16] that K is a 2–bridge knot if and only if nK is an $(n+1)$–bridge knot. Similarly we have:

Observation 1.3 $K_1, \ldots, K_n \in \mathcal{K}_2^2$ if and only if $K_1 \# \cdots \# K_n \in \mathcal{K}_{n+1}^{n+1}$.

In this paper, we show:

Theorem 1.4 *Let K be a knot in S^3.*

(1) *If $K_i \in \mathcal{K}_2^1$ $(i = 1, 2, 3)$, then $K_1 \# K_2 \in \mathcal{K}_3^1$ and $K_1 \# K_2 \# K_3 \in \mathcal{K}_4^1$.*

(2) *If $K_j \in \mathcal{K}_2^0$ $(j = 1, 2)$, then $K_1 \# K_2 \in \mathcal{K}_4^0$ or $\mathcal{K}_4^1$.*

The most famous examples of knots in $\mathcal{K}_2^0$ would be so-called MSY knots K_{MSY} introduced by Morimoto, Sakuma and Yokota [14]. It follows from Morimoto [11, Corollary 2] that $\mathrm{hg}(2K_{\mathrm{MSY}}) = 4$. Since K_{MSY} admits a $(1,2)$–bridge position, we see that $\mathrm{md}(2K_{\mathrm{MSY}}) \geq 1$ (see Kobayashi and Rieck [6, Theorem A.1], see also Proposition 2.14). Therefore we have the following as a corollary of Theorem 1.4.

Corollary 1.5 $K_{\mathrm{MSY}} \in \mathcal{K}_2^0$ *and* $2K_{\mathrm{MSY}} \in \mathcal{K}_4^1$.

On the other hand, Kobayashi and Rieck [8] showed that there is a knot $K \in \mathcal{K}_2^0$ with $2K \in \mathcal{K}_4^0$. This implies that (2) of Theorem 1.4 is best possible. As a summary, we have Figure 1.

Remark 1.6 More generally, Kobayashi and Rieck proved the following: given an integer $m \geq 1$, there are infinitely many knots K in S^3 such that $\mathrm{hg}(m'K) = m' \cdot \mathrm{hg}(K)$ for any positive integer $m' \leq m$ (see Kobayashi and Rieck [7, Corollary 1.6]). This implies that given an integer $n \geq 1$, there are infinitely many knots $K \in \mathcal{K}_2^0$ such that $\mathrm{md}(n'K) = 0$ for any positive integer $n' \leq n$.

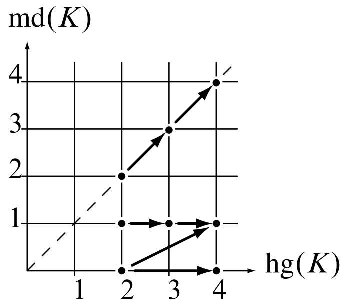

Figure 1: Relation between $\mathrm{hg}(\cdot)$ and $\mathrm{md}(\cdot)$ under connected sum

Based on the results above, we would like to ask some questions on tunnel number one knots.

Question 1.7 (1) Is there a knot $K \in \mathcal{K}_2^1$ with $\mathrm{md}(nK) > 1$ for some integer n?

(2) Is there a knot $K \in \mathcal{K}_2^0$ with $\mathrm{md}(nK) = 0$ for any integer n?

It would be much interesting and challenging to take the connected sum of knots with tunnel number greater than one, because there is a possibility of sub-additivity of tunnel number under connected sum (see Kobayashi and Saito [9, Assertion 6.4]).

2 Preliminaries

Throughout this paper, we work in the piecewise linear category. Let B be a submanifold of a manifold A. The notation $\eta(B; A)$ denotes a (closed) regular neighborhood of B in A. By $\mathrm{Ext}(B; A)$, we mean the *exterior* of B in A, that is, $\mathrm{Ext}(B; A) = \mathrm{cl}(A \setminus \eta(B; A))$, where $\mathrm{cl}(\cdot)$ means the closure. The notation $|\cdot|$ indicates the number of connected components. Let M be a connected compact orientable 3–manifold. A *link* in M is a closed 1–manifold in M and a *knot* in M is a connected closed 1–manifold in M. Let J be a 1–manifold properly embedded in M and F a surface properly embedded in M. Here, a *surface* means a connected compact 2–manifold. We always assume that J is not *split*, that is, there is no 2–sphere in $M \setminus J$ which separates the components of J, and also assume that F intersects J transversely. Set $\mathcal{M} = (M, J)$ and $\mathcal{F} = (F, F \cap J)$. For convenience, we also call $\mathcal{F}$ a *surface*, and we say that $\mathcal{F}$ is *closed* if F is closed. Whenever we use such calligraphic symbols, we consider not only a 2– or 3–manifold but intersections with a 1–manifold.

2.1 Fundamental definitions

A simple closed curve or a simple arc properly embedded in $F \setminus J$ is said to be *trivial* in $\mathcal{F}$ if it cuts off a disk from F which is disjoint from J. A simple closed curve properly embedded in $F \setminus J$ is said to be *inessential* in $\mathcal{F}$ if it bounds a disk in F intersecting J in at most one point. A simple closed curve properly embedded in $F \setminus J$ is said to be *essential* in $\mathcal{F}$ if it is not inessential in $\mathcal{F}$. A surface $\mathcal{F}$ is *compressible* in $\mathcal{M}$ if there is a disk $D \subset M \setminus J$ such that $D \cap F = \partial D$ and ∂D is essential in $\mathcal{F}$. Such a disk D is called a *compressing disk* of $\mathcal{F}$. We say that $\mathcal{F}$ is *incompressible* in $\mathcal{M}$ if $\mathcal{F}$ is not compressible in $\mathcal{M}$.

Suppose that $\partial M \neq \varnothing$ and $\partial F \neq \varnothing$. We say that $\mathcal{F}$ is *∂–compressible* in $\mathcal{M}$ if there is a disk $D \subset M$ such that $D \cap F = \partial D \cap F =: \gamma$ is a non-trivial arc in $\mathcal{F}$, and $\mathrm{cl}(\partial D \setminus \gamma)$ is an arc in ∂M. The disk D is called a *∂–compressing disk* of $\mathcal{F}$. We say that $\mathcal{F}$ is *∂–incompressible* in $\mathcal{M}$ if $\mathcal{F}$ is not ∂–compressible in $\mathcal{M}$. A surface $\mathcal{F}$ is *∂–parallel* in $\mathcal{M}$ if F cuts off $\mathcal{M}'$ from $\mathcal{M}$ with $\mathcal{M}' \cong \mathcal{F} \times [0,1] (= (F \times [0,1], (F \cap J) \times [0,1]))$.

We say that $\mathcal{M}$ is *reducible* if there is a 2–sphere disjoint from J which does not bound a 3–ball B^3. We say that $\mathcal{M}$ is *∂–reducible* if there is a disk $\bar{D} \subset M \setminus J$ such that $\bar{D} \cap \partial M = \partial \bar{D}$ and $\partial \bar{D}$ is essential in $\partial \mathcal{M} = (\partial M, \partial M \cap J)$. We say that $\mathcal{M}$ is *∂–irreducible* if $\mathcal{M}$ is not ∂–reducible.

A 3–manifold C is called a (genus g) *compression body* if there exists a closed surface F of genus g such that C is obtained from $F \times [0,1]$ by attaching 2–handles along mutually disjoint loops in $F \times \{0\}$ and filling in some resulting 2–sphere boundary components with 3–handles. We denote $F \times \{1\}$ by $\partial_+ C$ and $\partial C \setminus \partial_+ C$ by $\partial_- C$. A compression body C is called a *handlebody* if $\partial_- C = \varnothing$. The triplet $(C_1, C_2; S)$ is called a (genus g) *Heegaard splitting* of M if C_1 and C_2 are (genus g) compression bodies with $C_1 \cup C_2 = M$ and $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = S$.

A simple arc γ properly embedded in a compression body C is said to be *vertical* if γ is isotopic to an arc with $\{\text{a point}\} \times [0,1] \subset \partial_- C \times [0,1]$. A simple arc γ properly embedded in C is said to be *trivial* if there is a disk δ in C with $\gamma \subset \partial\delta$ and $\partial\delta \setminus \gamma \subset \partial_+ C$. Such a disk δ is called a *bridge disk* of γ. A disjoint union of trivial arcs is said to be *mutually trivial* if they admits a disjoint union of bridge disks.

Let L be a link in a connected compact orientable 3–manifold M. We say that L admits a $(g, 0)$*–bridge position* if there is a genus g Heegaard splitting $(C_1, C_2; S)$ of M with $L \cap S = \varnothing$ such that $\mathrm{cl}(C_i \setminus \eta(L; C_i))$ $(i = 1, 2)$ are compression bodies. We say that L admits a (g, b)*–bridge position* $(b > 0)$ if there is a genus g Heegaard splitting $(C_1, C_2; S)$ of M such that $C_i \cap L$ consists of b arcs which are mutually trivial for each $i = 1, 2$. Set $\mathcal{C}_i = (C_i, C_i \cap L)$ and $\mathcal{S} = (S, S \cap L)$. We call the triplet

$(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a (g, b)*–bridge splitting* of $\mathcal{M} = (M, L)$ and $\mathcal{S}$ is called a (g, b)*–bridge surface*, or a *bridge surface* for short. We notice that a $(g, 0)$–bridge splitting of $\mathcal{M} = (M, L)$ is also called a *Heegaard splitting* of $\mathcal{M}$ and a $(g, 0)$–bridge surface of $\mathcal{M}$ is called a *Heegaard surface* of $\mathcal{M}$.

Definition 2.1 Let K be a knot in a connected compact orientable 3–manifold M. The *Heegaard genus* of $K \subset M$, denoted by $\mathrm{hg}(K)$, is the minimal value g such that (M, K) admits a Heegaard surface of genus g.

2.2 C–bodies and cH–splittings

We recall definitions of a *c–compression body* and a *c–Heegaard splitting* given by Tomova [18]. In this paper, they are abbreviated as *a c–body* and *a cH–splitting* respectively.

Definition 2.2 Let J be a 1–manifold properly embedded in a connected compact orientable 3–manifold M. A surface $\mathcal{F} = (F, F \cap J)$ is *c–compressible in* $\mathcal{M} = (M, J)$ if there is a disk $D \subset M$ such that $D \cap F = \partial D$, ∂D is essential in $\mathcal{F}$ and D intersects J in at most one point. If $|D \cap J| = 1$, then D is called a *cut disk of* $\mathcal{F}$. We say that $\mathcal{F}$ is *c–incompressible in* $\mathcal{M}$ if $\mathcal{F}$ is not c–compressible in $\mathcal{M}$. A *c–disk* is a compressing disk or a cut disk.

Definition 2.3 Let $\mathcal{C}$ be a pair of a genus g compression body C and a 1–manifold J properly embedded in C. Then $\mathcal{C}$ is called a (genus g) *c–body* if there is a disjoint union $\mathbb{D}$ of c–disks and bridge disks which cuts $\mathcal{C}$ into some 3–balls and a 3–manifold homeomorphic to $\partial_- C \times [0, 1]$ with vertical arcs. Then $\mathbb{D}$ is called a *complete c–disk system* of $\mathcal{C}$. We set $\partial_+\mathcal{C} = (\partial_+ C, \partial_+ C \cap J)$ and $\partial_-\mathcal{C} = (\partial_- C, \partial_- C \cap J)$.

The next two lemmas are obtained by standard innermost/outermost disk arguments.

Lemma 2.4 *Let $\mathcal{C}$ be a c–body. Suppose that there is a compressing disk D of $\mathcal{C}$ which cuts $\mathcal{C}$ into $(\partial_- C \times [0, 1]$, vertical arcs$)$ and $(B^3$, a trivial arc$)$. Then any compressing disk of $\mathcal{C}$ is isotopic to D.*

Lemma 2.5 *Let $\mathcal{C}$ be a c–body. Suppose that there is a non-separating compressing disk (resp. a cut disk) D of $\mathcal{C}$ which cuts $\mathcal{C}$ into $(\partial_- C \times [0, 1]$, vertical arcs$)$. Then any non-separating compressing disk (resp. a cut disk) of $\mathcal{C}$ is isotopic to D.*

We now recall the following obtained by Hayashi and Shimokawa [4] and by Tomova [18].

Lemma 2.6 (Hayashimo [4, Lemma 2.4]) *Let $\mathcal{C} = (C, J)$ be a c–body such that each component of J is trivial or vertical in C. If $\mathcal{F} = (F, F \cap J)$ is an incompressible, ∂–incompressible surface in $\mathcal{C}$, then F is*

1. *a 2–sphere intersecting J in 0 or 2 points,*
2. *a disk intersecting J in 0 or 1 points,*
3. *a vertical annulus disjoint from J, or*
4. *a closed surface parallel to a component of $\partial_- \mathcal{C}$.*

Lemma 2.7 (Tomova [18, Corollary 3.7]) *If $\mathcal{F} = (F, F \cap J)$ is a c–incompressible, ∂–incompressible surface in a c–body $\mathcal{C} = (C, J)$, then F is*

1. *a 2–sphere intersecting J in 0 or 2 points,*
2. *a disk intersecting J in 0 or 1 points,*
3. *a vertical annulus disjoint from J, or*
4. *a closed surface parallel to a component of $\partial_- \mathcal{C}$.*

Definition 2.8 Let J be a 1–manifold properly embedded in a connected compact orientable 3–manifold M. The triplet $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ is a (genus g) *cH–splitting* of $\mathcal{M} = (M, J)$ if $\mathcal{C}_i$ ($i = 1$ and 2) are (genus g) c–bodies with $\mathcal{C}_1 \cup \mathcal{C}_2 = \mathcal{M}$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \partial_+ \mathcal{C}_1 = \partial_+ \mathcal{C}_2 = \mathcal{S}$. The surface $\mathcal{S}$ is called a *cH–surface* of $\mathcal{M}$.

We notice that a (g, b)–bridge splitting of $\mathcal{M}$ is a genus g cH–splitting and that if $\mathcal{M} = (M, J)$ with $J = \varnothing$, then a cH–splitting is a Heegaard splitting of M.

Definition 2.9 Let J be a 1–manifold properly embedded in a connected compact orientable 3–manifold M, and let $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ be a cH–splitting of $\mathcal{M} = (M, J)$.

1. The cH–surface $\mathcal{S}$ is said to be *∂–reducible* if there is a ∂–reducing disk $\bar{D}$ of $\mathcal{M}$ such that $\bar{D} \cap S$ is a single curve.
2. The cH–surface $\mathcal{S}$ is said to be *reducible* if there are compressing disks D_i ($i = 1, 2$) of $\mathcal{C}_i$ with $\partial D_1 = \partial D_2$. The cH–surface $\mathcal{S}$ is said to be *irreducible* if it is not reducible.
3. The cH–surface $\mathcal{S}$ is said to be *stabilized* if there are compressing disks D_i ($i = 1, 2$) of $\mathcal{C}_i$ such that ∂D_1 and ∂D_2 intersect transversely in a single point.
4. The cH–surface $\mathcal{S}$ is said to be *meridionally stabilized* if there are a compressing disk D_i of $\mathcal{C}_i$ and a cut disk D_j of $\mathcal{C}_j$ ($(i, j) = (1, 2)$ or $(2, 1)$) such that ∂D_1 and ∂D_2 intersect transversely in a single point.

(5) The cH–surface $\mathcal{S}$ is said to be *weakly reducible* if there are compressing disks D_i $(i = 1, 2)$ of $\mathcal{C}_i$ with $\partial D_1 \cap \partial D_2 = \varnothing$. The cH–surface $\mathcal{S}$ is said to be *strongly irreducible* if it is not weakly reducible.

(6) The cH–surface $\mathcal{S}$ is said to be *c–weakly reducible* if there are c–disks D_i $(i = 1, 2)$ of $\mathcal{C}_i$ with $\partial D_1 \cap \partial D_2 = \varnothing$. The cH–surface $\mathcal{S}$ is said to be *c–strongly irreducible* if it is not c–weakly reducible.

The next lemma is proved by Tomova [18].

Lemma 2.10 (Tomova [18, Theorem 6.2]) *Let J be a 1–manifold properly embedded in a connected compact orientable irreducible 3–manifold M, and let $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ be a cH–splitting of $\mathcal{M} = (M, J)$. If $\mathcal{M}$ is ∂–reducible, then $\mathcal{S}$ is ∂–reducible.*

Corollary 2.11 *Let J be a 1–manifold properly embedded in a connected compact orientable irreducible 3–manifold M, and let $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ be a cH–splitting of $\mathcal{M} = (M, J)$. Let $\bar{D}$ be a ∂–reducing disk of $\mathcal{M}$ with $|\bar{D} \cap S| = 1$ and $\partial \bar{D} \subset \partial_- C_i$ $(i = 1$ or $2)$. Suppose that $\mathcal{C}_i$ admits a compressing disk. Then there is a compressing disk D of $\mathcal{C}_i$ such that $D \cap \bar{D} = \varnothing$ and hence $\mathcal{S}$ is weakly reducible.*

2.3 C–weak reduction

We briefly recall the operation called a *c–weak reduction* treated in [18]. Though we here recall c–weak reductions only for bridge splittings, such operations are applied to those for cH–splittings as in [18].

Let L be a link in a connected compact orientable 3–manifold M, and let $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ be a bridge splitting of $\mathcal{M} = (M, L)$. Suppose that $\mathcal{S}$ is c–weakly reducible. Then there are disjoint unions of c–disks of $\mathcal{C}_i$ $(i = 1, 2)$, say $\mathbb{D}_i$, such that $\partial\mathbb{D}_1 \cap \partial\mathbb{D}_2 = \varnothing$. Since each c–disk cuts a c–body into c–bodies, we obtain a collection of c–bodies $\mathcal{C}_{11}$ by cutting $\mathcal{C}_1$ along $\mathbb{D}_1$. Let $\mathcal{C}_{12}$ be a 3–manifold obtained from $\partial_+\mathcal{C}_{11} \times [0, 1]$ by attaching $\eta(\mathbb{D}_2; \mathcal{C}_2)$. We notice that $\mathcal{C}_{12}$ is also a collection of c–bodies with $\partial_+\mathcal{C}_{12} = \partial_+\mathcal{C}_{11}$. Let $\mathcal{C}_{21}$ be a 3–manifold obtained from $\partial_-\mathcal{C}_{12} \times [0, 1]$ by attaching $\eta(\mathbb{D}_1; \mathcal{C}_1)$. We also notice that $\mathcal{C}_{21}$ is a collection of c–bodies with $\partial_-\mathcal{C}_{21} = \partial_-\mathcal{C}_{12} =: \mathcal{F}$. Finally, we let $\mathcal{C}_{22}$ be a collection of c–bodies by cutting $\mathcal{C}_2$ along $\mathbb{D}_2$. Set $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$ for each $i = 1, 2$. Then we see that $\{\mathcal{C}_{i1}, \mathcal{C}_{i2}\}$ gives a collection of cH–splittings of $\mathcal{M}_i$ for each $i = 1, 2$ (see Figure 2). We say that a collection of surfaces $\mathcal{F}$ is obtained by the *c–weak reduction* with respect to $(\mathbb{D}_1, \mathbb{D}_2)$. If $\mathbb{D}_i$ $(i = 1, 2)$ are disjoint unions of compressing disks of $\mathcal{C}_i$ such that $\partial\mathbb{D}_1 \cap \partial\mathbb{D}_2 = \varnothing$, then such an operation is originally introduced by Casson and Gordon [1] and is called a *weak reduction*.

In this paper, we slightly modify the operation as in the following remark.

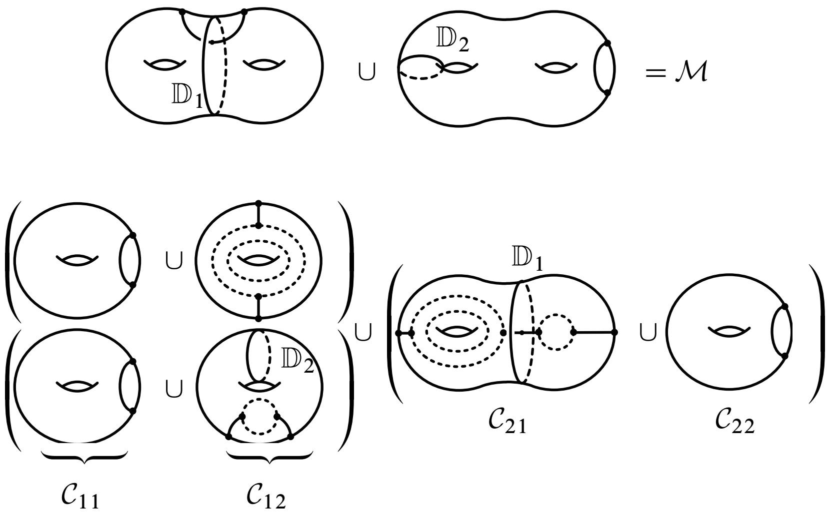

Figure 2: An example of c–weak reduction

Remark 2.12 Suppose that $\mathbb{D}_i$, say $i = 2$, consists of a compressing disk D_2 which cuts off a 3–ball $\mathcal{B} = (B, B \cap L)$ from $\mathcal{C}_2$ such that $B \cap L$ is a collection of mutually trivial arcs and that $\partial B \cap \partial \mathbb{D}_1 = \varnothing$. Then we slightly modify the operation as follows: let $\mathcal{C}_{12}$ be a 3–manifold obtained from $\partial_+ \mathcal{C}_{11} \times [0,1]$ by attaching $\mathcal{B}$ along $\mathrm{cl}(B \setminus D_2)$, and let $\mathcal{C}_{22}$ be a 3–manifold obtained from $\mathcal{C}_2$ by cutting $\mathcal{B}$ off. Then we see that $\mathcal{C}_{12}$ and $\mathcal{C}_{22}$ are c–bodies and that $\{\mathcal{C}_{i1}, \mathcal{C}_{i2}\}$ similarly gives a collection of cH–splittings of $\mathcal{M}_i$ for each $i = 1, 2$. We notice that D_2 is naturally extended to be a compressing disk $\widehat{D}_2$ of $\mathcal{C}_{12}$ (see, for example, Figure 3).

2.4 Meridional destabilizing number

Let L be a link in a connected compact orientable 3–manifold M, and let $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ be a (g, b)–bridge splitting of $\mathcal{M} = (M, L)$. Suppose that $\mathcal{S}$ is meridionally stabilized. Then there are a compressing disk D_i of $\mathcal{C}_i$ and a cut disk D_j of $\mathcal{C}_j$ ($(i, j) = (1, 2)$ or $(2, 1)$, say the latter) such that ∂D_1 intersects ∂D_2 transversely in a single point. We notice that D_1 and D_2 are non-separating in C_1 and C_2 respectively. Cutting $\mathcal{C}_1$ along D_1, we obtain a pair $\mathcal{C}_1'$ of a genus $g-1$ compression body and $b+1$ mutually trivial arcs. Gluing $\mathcal{C}_2$ together with $\eta(D_1; C_1)$ containing a trivial arc, we also obtain a pair $\mathcal{C}_2'$ of a genus $g-1$ compression body and $b+1$ mutually trivial arcs. Hence $\{\mathcal{C}_1', \mathcal{C}_2'\}$ gives a $(g-1, b+1)$–bridge splitting of $\mathcal{M}$. Such an operation is called a *meridional destabilization*.

Definition 2.13 Let K be a knot in M. *Meridional destabilizing number* of K, denoted by $\mathrm{md}(K)$, is defined by the maximal number of m such that $\mathcal{M} = (M, K)$ admits a $(\mathrm{hg}(K) - m, m)$–bridge $\mathrm{md}(K) = 0$ if none of the minimal genus Heegaard splittings of $\mathcal{M}$ are meridionally stabilized.

2.5 Connected sum

For each $i = 1, 2$, let J_i be a 1–manifold properly embedded in a connected compact orientable 3–manifold M_i, and take a point p_i in the interior of J_i. We notice that $\eta(p_i; M_i) \cong B^3$ and $\eta(p_i; M_i) \cap J_i$ is a trivial arc in $\eta(p_i; M_i)$. Set $M_i' = \mathrm{cl}(M_i \setminus \eta(p_i; M_i))$, $\partial_0 M_i' = \partial M_i' \setminus \partial M_i$, and let $h : \partial_0 M_1' \to \partial_0 M_2'$ be a gluing map with $h(\partial_0 M_1' \cap J_1) = \partial_0 M_2' \cap J_2$ and $h_*([\partial_0 M_1' \cap J_1]) = -[\partial_0 M_2' \cap J_2]$, where $[\cdot]$ is a homology class and h_* is the homomorphism induced by h. The 3–manifold $M_1' \cup_h M_2'$ is denoted by $M_1 \# M_2$, and the 1–manifold $(M_1' \cap J_1) \cup_h (M_2' \cap J_2)$ is denoted by $J_1 \# J_2$. We call this operation the *connected sum*. If $(M_i', M_i' \cap J_i)$ is neither (a 3–manifold, a trivial arc) nor $(S^2 \times [0, 1]$, two vertical arcs) for each $i = 1, 2$, then the image Σ of $\partial_0 M_i'$ in $M_1 \# M_2$ is called a *decomposing sphere* of $J_1 \# J_2$.

Proposition 2.14 *For each $i = 1, 2$, let K_i be a knot in a connected compact orientable 3–manifold M_i which admits a (g_i, b_i)–bridge position. If $b_1 \neq 0$ or $b_2 \neq 0$, then $K_1 \# K_2$ admits a $(g_1 + g_2, b_1 + b_2 - 1)$–bridge position.*

Proof For each $i = 1, 2$, let $(\mathcal{V}_{i1}, \mathcal{V}_{i2}; \mathcal{S}_i)$ be a (g_i, b_i)–bridge splitting of (M_i, K_i), where $\mathcal{V}_{ij} = (V_{ij}, V_{ij} \cap K_i)$ $(j = 1, 2)$. We first assume that each b_i is not equal to zero. Let γ_{i2} be a component of $V_{i2} \cap K_i$. We notice that γ_{i2} is trivial in $\mathcal{V}_{i2}$ and hence γ_{i2} admits a bridge disk, say δ_{i2}. To take the connected sum of K_1 and K_2, we let V_{i2}' be a 3–manifold obtained from V_{i2} by removing $\eta(\gamma_{i2}; V_{i2})$. We notice that $V_{i2}' \cap \eta(\gamma_{i2}; V_{i2})$ consists of an annulus, say A_{i2} which admits a ∂–compressing disk $\delta_{i2}' = \delta_{i2} \cap V_{i2}'$. Set $\mathcal{V}_{i2}' = (V_{i2}', V_{i2} \cap K_i)$, $M_i' = V_{i1} \cup V_{i2}'$ and $K_i' = M_i' \cap K$. Then we obtain the connected sum $K_1 \# K_2$ by gluing (M_1', K_1') to (M_2', K_2') along a map h. In particular, A_{12} is identified with A_{22} by a map h. This implies that $V_{12}' \cup_h V_{22}'$ is a compression body of genus $g_1 + g_2 + 1$ which intersects $K_1 \# K_2$ in $b_1 + b_2 - 2$ mutually trivial arcs. We also see that $V_{11} \cup_h V_{21}$ is a compression body of genus $g_1 + g_2 + 1$ which intersects $K_1 \# K_2$ in $b_1 + b_2 - 2$ mutually trivial arcs. We hereafter take such a map h that $A_{12} \cap \delta_{12}'$ is identified with $A_{22} \cap \delta_{22}'$. Set $W_1 = V_{11} \cup_h V_{21}$ and $W_2 = V_{12}' \cup_h V_{22}'$. Then $\{W_1, W_2\}$ gives a $(g_1 + g_2 + 1, b_1 + b_2 - 2)$–bridge splitting of $K_1 \# K_2$. Since $\mathrm{cl}(\partial \eta(\gamma_{i2}; V_{i2}) \setminus A_{i2})$ consists of two disks in ∂V_{i1}, we let $D_1 \subset W_1$ be a copy of such a component. Then

D_1 is a cut disk of W_1. We notice that $D_2 = \delta'_{12} \cup_h \delta'_{22}$ is a compressing disk of W_2 such that ∂D_1 and ∂D_2 intersect transversely in a single point. Hence the bridge splitting given by $\{W_1, W_2\}$ is meridionally stabilized and therefore $K_1 \# K_2$ admits a (g_1+g_2, b_1+b_2-1)–bridge position.

Suppose next that b_1 or b_2, say the latter, is equal to zero. We may assume that $K_2 \subset V_{21}$. Let V'_{12} be as above and set $V'_{21} = \mathrm{cl}(V_{21} \setminus E_{21})$, where E_{21} is a cut disk of V_{21}. We reset $W_1 = V_{11} \cup V'_{21}$ and $W_2 = V'_{12} \cup_h V_{22}$. Then we also see that $\{W_1, W_2\}$ gives a (g_1+g_2, b_1+b_2-1)–bridge splitting of $K_1 \# K_2$. □

3 Incompressible surfaces and cH–splittings

Let M be a connected compact orientable irreducible 3–manifold, J a 1–manifold properly embedded in M. Recall that we assume that J is not split. We obtain the following by using a standard innermost disk argument.

Lemma 3.1 *Let J be a 1–manifold properly embedded in a connected compact orientable irreducible 3–manifold M. Let $\mathcal{F} = (F, F \cap J)$ and $\mathcal{F}' = (F', F' \cap J)$ be closed surfaces incompressible in $\mathcal{M} = (M, J)$. Then there is a closed incompressible surface $\mathcal{F}'' = (F'', F'' \cap J)$ with $F'' \cong F'$ and $|F'' \cap J| = |F' \cap J|$ such that $F'' \cap F = \emptyset$ or that each component of $F'' \cap F$ is non-trivial both in $\mathcal{F}''$ and in $\mathcal{F}$. Moreover, $\mathcal{F}''$ is ambient isotopic to $\mathcal{F}'$ in $\mathcal{M}$ if $M = S^3$.*

Remark 3.2 To prove Lemma 3.1 by using a standard innermost disk argument, we suppose that $F \cap F' \neq \emptyset$ and there is a component α of $F \cap F'$ which is trivial in $\mathcal{F}$ or $\mathcal{F}'$. Then we notice that α must be trivial both in $\mathcal{F}$ *and* in $\mathcal{F}'$ because M is irreducible and J is properly embedded in M.

The following is essentially obtained by Schultens [17] (see also Morimoto [12]).

Lemma 3.3 (Schultens [17, Lemma 6]) *Let J be a 1–manifold properly embedded in a connected compact orientable irreducible 3–manifold M, and let $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ be a cH–splitting of $\mathcal{M} = (M, J)$. Suppose that $\mathcal{S}$ is strongly irreducible and that J admits a decomposing sphere. Then there is a decomposing sphere Σ of J such that each component of $\Sigma \cap S$ is non-trivial in $(\Sigma, J \cap \Sigma)$ and is essential in $\mathcal{S}$.*

Proof Morimoto's argument in the proof of [12, Lemma 2.3] will work here. Recall that $\mathcal{C}_i = (C_i, C_i \cap J)$ $(i = 1, 2)$ and $\mathcal{S} = (S, S \cap J)$. We assume that C_1 is small enough to intersect a decomposing sphere Σ only in c–disks of $\mathcal{C}_1$. Set $\Sigma_i = \Sigma \cap \mathcal{C}_i$

$(i=1,2)$. We take a decomposing sphere Σ so that $(|\Sigma_1|, |\Sigma_2 \cap J|)$ is lexicographically minimal among all such decomposing spheres. Then each component of $\Sigma_2^{(0)} := \Sigma_2$ is c–incompressible in $\mathcal{C}_2$. Hence we have a sequence of ∂–compressions for Σ_2 which realizes a *hierarchy* $\{(\Sigma_2^{(j)}, a_j)\}_{0 \le j \le n}$, that is, a_j is a non-trivial arc in $\Sigma_2^{(j)}$, $\Sigma_2^{(j+1)}$ is obtained by cutting $\Sigma_2^{(j)}$ along a_j, and $\Sigma_2^{(n)}$ consists of c–disks of $\mathcal{C}_2$. Set $\Sigma_1^{(j)} = \mathrm{cl}\left(\Sigma \setminus \Sigma_2^{(j)}\right)$. By the minimality, we may also assume that each component of $\Sigma_1^{(j)} \cap S$ is essential in $\mathcal{S}$ for any integer j with $0 \le j \le n$.

If $\Sigma_1^{(0)}$ or $\Sigma_2^{(n)}$ consists of cut disks of $\mathcal{C}_1$ or $\mathcal{C}_2$ respectively, then we are done. Hence we assume that both $\Sigma_1^{(0)}$ and $\Sigma_2^{(n)}$ contain compressing disks. Let k be the maximal integer such that $\Sigma_1^{(k)}$ contains a compressing disk, say D_1, of $\mathcal{C}_1$. Suppose that $\Sigma_2^{(k+1)}$ contains a compressing disk, say D_2, of $\mathcal{C}_2$. We notice that D_2 is obtained by cutting an annulus component, say A_2, of $\Sigma_2^{(k)}$ along a_k. It follows from strong irreducibility of $\mathcal{S}$ that ∂D_1 is a component of ∂A_2. Taking a parallel copy, say D_2', of D_2 in $\mathcal{C}_2$ with $A_2 \cap D_2' = \varnothing$, we see that $\partial D_1 \cap \partial D_2' = \varnothing$, contradicting strong irreducibility of $\mathcal{S}$. Therefore $\Sigma_2^{(k+1)}$ contains no compressing disks of $\mathcal{C}_2$ and hence we have the desired result because $\Sigma_1^{(k+1)}$ also contains no compressing disks of $\mathcal{C}_1$. □

Corollary 3.4 *Let J be a 1–manifold properly embedded in a connected compact orientable irreducible 3–manifold M, and let $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ be a cH–splitting of $\mathcal{M} = (M, J)$. Suppose that $\mathcal{S}$ is strongly irreducible and that J admits a decomposing sphere. Then either*

(1) *there are separating cut disks E_1 and E_2 of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively with $\partial E_i = \partial E_j$, or*

(2) *there is a cut disk E_i of $\mathcal{C}_i$ and a compressing disk E_j of $\mathcal{C}_j$ such that $\partial E_i \cap \partial E_j = \varnothing$ for $(i,j) = (1,2)$ or $(2,1)$.*

Proof It follows from Lemma 3.3 that there is a decomposing sphere Σ of J such that each component of $\Sigma \cap S$ is essential in $\mathcal{S}$, and the components of Σ cut along $\Sigma \cap S$ consist of two disks Δ and Δ' with $|\Delta \cap J| = |\Delta' \cap J| = 1$ and possibly annuli disjoint from J. Without loss of generality, we assume that Δ is a cut disk of $\mathcal{C}_1$. If $\Sigma \cap C_2$ contains no annulus components ,then $\Sigma \cap C_1$ consists of the disk Δ and $\Sigma \cap C_2$ similarly consists of the disk Δ'. Hence we have the conclusion (1) of Corollary 3.4. Otherwise, $\Sigma \cap C_2$ contains an annulus component which is ∂–compressible in $\mathcal{C}_2$. Let A be an annulus component of $\Sigma \cap C_2$ such that a ∂–compressing disk δ of A is disjoint from the other components of $\Sigma \cap C_2$. After the ∂–compression along δ, we have a compressing disk D_2 of $\mathcal{C}_2$. A parallel copy of Δ and D_2 satisfy the conclusion (2) of Corollary 3.4. □

Corollary 3.5 *Let L be a link in a connected compact orientable irreducible 3–manifold M, and let $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ be a bridge splitting of $\mathcal{M} = (M, L)$. Suppose that $\mathcal{S}$ is strongly irreducible and that L admits a decomposing sphere. Then there are a non-separating cut disk E_i of $\mathcal{C}_i$ and a compressing disk E_j of $\mathcal{C}_j$ such that $\partial E_i \cap \partial E_j = \varnothing$ for $(i, j) = (1, 2)$ or $(2, 1)$.*

Proof By assumption, we have one of the conclusions of Corollary 3.4. If the conclusion (2) of Corollary 3.4 holds, then we may assume that there are a cut disk E_1 of $\mathcal{C}_1$ and a compressing disk E_2 of $\mathcal{C}_2$ such that $\partial E_1 \cap \partial E_2 = \varnothing$. Suppose, towards a contradiction, E_1 is separating in C_1. Then E_1 cuts C_1 into two compression bodies C_{11} and C_{12}. Let β_1 be the component of $C_1 \cap L$ with $\beta_1 \cap E_1 \neq \varnothing$. Then C_{11} (resp. C_{12}) contains $\beta_1 \cap C_{11}$ (resp. $\beta_1 \cap C_{12}$) as a trivial arc. Since ∂E_1 is essential in $\mathcal{S}$, we see that there are compressing disks D_{11} and D_{12} of $\mathcal{C}_{11} = (C_{11}, C_{11} \cap L)$ and $\mathcal{C}_{12} = (C_{12}, C_{12} \cap L)$ respectively. We notice that D_{11} and D_{12} are also compressing disks of $\mathcal{C}_1$.

We now suppose that the conclusion (1) of Corollary 3.4 holds. Then we similarly see that E_2 cuts C_2 into two compression bodies C_{21} and C_{22} and that there are compressing disks D_{21} and D_{22} of $\mathcal{C}_{21} = (C_{21}, C_{21} \cap L)$ and $\mathcal{C}_{22} = (C_{22}, C_{22} \cap L)$. Since $\partial E_1 = \partial E_2$, we have either $\partial D_{11} \cap \partial D_{21} = \varnothing$ or $\partial D_{12} \cap \partial D_{21} = \varnothing$, contradicting strong irreducibility of $\mathcal{S}$. If the conclusion (2) of Corollary 3.4 holds, then we also see that $\partial D_{11} \cap \partial E_2 = \varnothing$ or $\partial D_{12} \cap \partial E_2 = \varnothing$. This again contradicts strong irreducibility of $\mathcal{S}$. Therefore E_1 is non-separating in C_1 and we have the desired conclusion. □

4 $(1, 2)$–bridge splittings

Theorem 4.1 *Let K be a knot in S^3 and $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a $(1, 2)$–bridge splitting of (S^3, K). Suppose that $\mathcal{S}$ is strongly irreducible and that K admits a decomposing sphere. Then one of the following holds.*

(1) *$\mathcal{S}$ is meridionally stabilized.*

(2) *There is a c–weak reduction yielding a 2–sphere which intersects K in four points and is incompressible in (S^3, K).*

Proof Since $\mathcal{S}$ is strongly irreducible and K admits a decomposing sphere, we have the conclusion of Corollary 3.5. Without loss of generality, we assume that there are a non-separating cut disk E_1 of $\mathcal{C}_1$ and a compressing disk E_2 of $\mathcal{C}_2$ such that $\partial E_1 \cap \partial E_2 = \varnothing$. Then we have:

(i) E_2 is a non-separating compressing disk of $\mathcal{C}_2$,

(ii) E_2 cuts off a 3–ball with two mutually trivial arcs from $\mathcal{C}_2$, or

(iii) E_2 cuts off a 3–ball with a single trivial arc from $\mathcal{C}_2$.

If we have the condition (i), then $C_1 \cup C_2$ contains a non-separating 2–sphere, a contradiction. The condition (ii) implies that $\mathcal{S}$ is meridionally stabilized. Hence we consider the condition (iii). We now do the c–weak reduction with respect to (E_1, E_2) (see Figure 3). We notice that $\mathcal{C}_{11}$ is a c–body obtained by cutting $\mathcal{C}_1$ along E_1, $\mathcal{C}_{12}$ is a c–body obtained from $\partial_+\mathcal{C}_{11} \times [0,1]$ by attaching a 3–ball $\mathcal{B}$ with a trivial arc which is obtained by cutting $\mathcal{C}_2$ along E_2, $\mathcal{C}_{21}$ is a c–body obtained from $\partial_-\mathcal{C}_{12} \times [0,1]$ by attaching $\eta(E_1; C_1)$ with a trivial arc, and $\mathcal{C}_{22}$ is a c–body obtained from $\mathcal{C}_2$ by cutting $\mathcal{B}$ off. We notice that E_2 is naturally extended to a compressing disk $\widehat{E}_2$ of $\mathcal{C}_{12}$. Set $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$, $\mathcal{S}_i = \mathcal{C}_{i1} \cap \mathcal{C}_{i2}$ for each $i = 1, 2$ and $\mathcal{F} = \partial_-\mathcal{C}_{12} = \partial_-\mathcal{C}_{21}$. If both $\mathcal{M}_1$ and $\mathcal{M}_2$ are ∂–irreducible, then $\mathcal{F}$ is incompressible in (S^3, K) and hence we have the conclusion (2) of Theorem 4.1. Hence we assume that $\mathcal{M}_1$ or $\mathcal{M}_2$ is ∂–reducible.

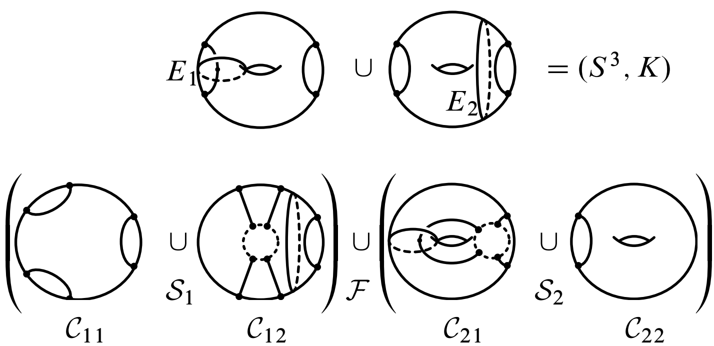

Figure 3: The c–weak reduction with respect to (E_1, E_2)

If $\mathcal{M}_1$ is ∂–reducible, then there is a ∂–reducing disk $\overline{D}_1$ with $|\overline{D}_1 \cap S_1| = 1$ by Lemma 2.10, and there is a compressing disk D_{12} of $\mathcal{C}_{12}$ with $D_{12} \cap \overline{D}_1 = \varnothing$ by Corollary 2.11. It follows from Lemma 2.4 that D_{12} is isotopic to $\widehat{E}_2$ in $\mathcal{C}_{12}$. Hence we see that $\overline{D}_1 \cap \widehat{E}_2 = \varnothing$. Since $\overline{D}_1$ can be regarded as a compressing disk of $\mathcal{C}_1$ and E_2 is contained in $\widehat{E}_2$, we see that $\mathcal{S}$ is weakly reducible, a contradiction.

If $\mathcal{M}_2$ is ∂–reducible, then it follows from Lemma 2.10 that there is a ∂–reducing disk $\overline{D}_2$ with $|\overline{D}_2 \cap S_2| = 1$, that is, $\overline{D}_2$ intersects C_{21} in a vertical annulus in C_{21}. Since $\mathcal{C}_{21}$ is ambient isotopic to a regular neighborhood of $\partial_- C_{21} \cup (C_{21} \cap K)$, we see that $\overline{D}_2$ is isotoped to be disjoint from E_1. The disk $\overline{D}_2$ can be regarded as a compressing disk of $\mathcal{C}_2$ which is disjoint from E_2 and is not parallel to E_2. We notice that $\partial E_1 \cap \partial(E_2 \cup \bar{D}_2) = \varnothing$. Since $\bar{D}_2$ is separating in C_2, we see that $\mathcal{S}$ is meridionally stabilized. □

Theorem 4.2 *Let K be a knot in S^3 and $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a $(1,2)$–bridge splitting of (S^3, K). Suppose that $\mathcal{S}$ is weakly reducible. Then $K = K_1 \# K_2$ such that K_1 admits a $(0,2)$–bridge position and K_2 admits a $(1,1)$–bridge position.*

Proof Let D_1 and D_2 be compressing disks of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively such that $\partial D_1 \cap \partial D_2 = \varnothing$. For each $i = 1, 2$, we have:

(i) D_i is a non-separating compressing disk of $\mathcal{C}_i$,

(ii) D_i cuts off a 3–ball with two mutually trivial arcs from $\mathcal{C}_i$, or

(iii) D_i cuts off a 3–ball with a single trivial arc from $\mathcal{C}_i$.

Suppose first that D_1 satisfies the condition (i). If D_2 also satisfies the condition (i), then $C_1 \cup C_2$ contains a non-separating 2–sphere, a contradiction. If D_2 satisfies the condition (ii), then we see that K admits a $(0,2)$–bridge position and hence we have the desired conclusion. Suppose that D_2 satisfies the condition (iii). Then we have the desired conclusion by *extraction operation* as follows. We first notice that D_2 cuts $\mathcal{C}_2$ into a solid torus $\mathcal{C}_2'$ with a trivial arc and a 3–ball $\mathcal{C}_2''$ with a trivial arc. Attaching $\eta(D_1; C_1)$ to $\mathcal{C}_2'$, we have a 3–ball B with a properly embedded arc J. We notice that (B, J) forms S^3 with a knot, say K', which admits a $(1,1)$–bridge position after gluing a 3–ball with a trivial arc along their boundaries. Let K'' be a knot obtained from K by replacing J with a trivial arc in B. Then we see that K'' admits a $(0,2)$–bridge position. Hence we see that $K = K' \# K''$ such that K' admits a $(1,1)$–bridge position and K'' admits a $(0,2)$–bridge position.

Suppose next that D_1 satisfies the condition (ii). If D_2 also satisfies the condition (ii), then we see that K admits a $(0,2)$–bridge position and hence we have the desired conclusion. If D_2 satisfies the condition (iii), then there is a non-separating compressing disk of $\mathcal{C}_1$ which is disjoint from D_2 and hence we are done.

The other case is that both D_1 and D_2 satisfy the condition (iii). However, this implies that K consists of two components, a contradiction. □

5 $(1,3)$–bridge splittings

Theorem 5.1 *Let K be a knot in S^3 and $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a $(1,3)$–bridge splitting of (S^3, K). Suppose that $\mathcal{S}$ is strongly irreducible and that K admits a decomposing sphere. Then one of the following holds.*

(1) *$\mathcal{S}$ is meridionally stabilized.*

(2) *There is a c–weak reduction yielding a 2–sphere which intersects K in four or six points and is incompressible in (S^3, K).*

Proof Since $\mathcal{S}$ is strongly irreducible and K admits a decomposing sphere, we have the conclusion of Corollary 3.5. Without loss of generality, we assume that there are a non-separating cut disk E_1 of $\mathcal{C}_1$ and a compressing disk E_2 of $\mathcal{C}_2$ such that $\partial E_1 \cap \partial E_2 = \varnothing$. Then we have:

(i) E_2 is a non-separating compressing disk of $\mathcal{C}_2$,

(ii) E_2 cuts off a 3–ball with three mutually trivial arcs from $\mathcal{C}_2$,

(iii) E_2 cuts off a 3–ball with two mutually trivial arcs from $\mathcal{C}_2$, or

(iv) E_2 cuts off a 3–ball with a single trivial arc from $\mathcal{C}_2$.

If we have the condition (i), then $C_1 \cup C_2$ contains a non-separating 2–sphere, a contradiction. The condition (ii) implies that $\mathcal{S}$ is meridionally stabilized. The conditions (iii) and (iv) are very similar to the condition (iii) in the proof of Theorem 4.1. As in the proof of Theorem 4.1, we can do the c–weak reduction with respect to (E_1, E_2) to obtain $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$, $\mathcal{S}_i = \mathcal{C}_{i1} \cap \mathcal{C}_{i2}$ for each $i = 1, 2$ and $\mathcal{F} = \partial_- \mathcal{C}_{12} = \partial_- \mathcal{C}_{21}$. Then $\mathcal{F}$ is a 2–sphere which intersects K in four or six points depending on the conditions. If both $\mathcal{M}_1$ and $\mathcal{M}_2$ are ∂–irreducible, then $\mathcal{F}$ is incompressible in (S^3, K) and hence we have the conclusion (2) of Theorem 5.1. Hence we assume that $\mathcal{M}_1$ or $\mathcal{M}_2$ is ∂–reducible.

If $\mathcal{M}_1$ is ∂–reducible, then it follows from Corollary 2.11 that $\mathcal{S}_1$ is weakly reducible. This implies that $\mathcal{S}$ is weakly reducible, a contradiction.

If $\mathcal{M}_2$ is ∂–reducible, then it follows from Lemma 2.10 that there is a ∂–reducing disk $\bar{D}_2$ with $|\bar{D}_2 \cap S_2| = 1$, that is, $\bar{D}_2$ intersects C_{21} in a vertical annulus in C_{21}. Since $\mathcal{C}_{21}$ is ambient isotopic to a regular neighborhood of $\partial_- C_{21} \cup (C_{21} \cap K)$, we see that $\bar{D}_2$ is isotoped to be disjoint from E_1. The disk $\bar{D}_2$ can be regarded as a compressing disk of $\mathcal{C}_2$ which is disjoint from E_2 and is not parallel to E_2. We notice that $\partial E_1 \cap \partial(\bar{D}_2 \cup E_2) = \varnothing$. If $\bar{D}_2$ is non-separating in C_2, then $C_1 \cup C_2$ contains a non-separating 2–sphere, a contradiction. Hence $\bar{D}_2$ is separating in C_2. If we have the condition (iii), then we obtain the conclusion (1) of Theorem 5.1. Suppose that we have the condition (iv). If $\bar{D}_2$ cuts off a 3–ball with a single trivial arc, then we can find a compressing disk of $\mathcal{C}_2$ which satisfies the condition (iii) and is disjoint from E_1 by taking, if necessary, band-sum of E_2 and $\bar{D}_2$ disjoint from ∂E_1. If $\bar{D}_2$ cuts off a 3–ball with two mutually trivial arcs, then we obtain the conclusion (1) of Theorem 5.1.

This completes the proof of Theorem 5.1. □

Theorem 5.2 *Let K be a knot in S^3 and $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a $(1,3)$–bridge splitting of (S^3, K). Suppose that $\mathcal{S}$ is weakly reducible. Then one of the following holds.*

(1) *$K = K_1 \# K_2$ such that K_1 admits a $(0,2)$–bridge position and K_2 admits a $(1,2)$–bridge position.*

(2) *$K = K_1 \# K_2$ such that K_1 admits a $(0,3)$–bridge position and K_2 admits a $(1,1)$–bridge position.*

(3) *There is a weak reduction yielding a 2–sphere which intersects K in four points and is incompressible in (S^3, K).*

(4) *There is a weak reduction yielding a torus which intersects K in two points and is incompressible in (S^3, K).*

Proof Let D_1 and D_2 be compressing disks of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively such that $\partial D_1 \cap \partial D_2 = \varnothing$. For each $i = 1, 2$, we have:

(i) D_i is a non-separating compressing disk of $\mathcal{C}_i$,

(ii) D_i cuts off a 3–ball with three mutually trivial arcs from $\mathcal{C}_i$,

(iii) D_i cuts off a 3–ball with two mutually trivial arcs from $\mathcal{C}_i$, or

(iv) D_i cuts off a 3–ball with a single trivial arc from $\mathcal{C}_i$.

Case 1 The disk D_1 satisfies the condition (i).

If D_2 also satisfies the condition (i), then $C_1 \cup C_2$ contains a non-separating 2–sphere, a contradiction. If D_2 satisfies the condition (ii), then we see that K admits a $(0,3)$–bridge position and hence we have the conclusion (2) of Theorem 5.2. If D_2 satisfies the condition (iii), then we also have the conclusion (2) of Theorem 5.2 by extraction operation (see the proof of Theorem 4.2). We therefore suppose that D_2 satisfies the condition (iv). We obtain $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$, $\mathcal{S}_i = \mathcal{C}_{i1} \cap \mathcal{C}_{i2}$ for each $i = 1, 2$ and $\mathcal{F} = \partial_- \mathcal{C}_{12} = \partial_- \mathcal{C}_{21}$ by the weak reduction with respect to (D_1, D_2). We notice that $\mathcal{F}$ is a 2–sphere intersecting K in four points. If both $\mathcal{M}_1$ and $\mathcal{M}_2$ are ∂–irreducible, then we have the conclusion (3) of Theorem 5.2. Hence we assume that $\mathcal{M}_1$ or $\mathcal{M}_2$ is ∂–reducible.

If $\mathcal{M}_1$ is ∂–reducible, then there is a ∂–reducing disk $\bar{D}_1$ with $|\bar{D}_1 \cap S_1| = 1$ by Lemma 2.10, and there is a compressing disk D_{12} of $\mathcal{C}_{12}$ with $D_{12} \cap \bar{D}_1 = \varnothing$ by Corollary 2.11. It follows from Lemma 2.4 that D_{12} is isotopic to $\hat{D}_2$ in $\mathcal{C}_{12}$, where $\hat{D}_2$ is a compressing disk of $\mathcal{C}_{12}$ which is obtained by extending D_2 naturally. Hence we see that $\bar{D}_1 \cap \hat{D}_2 = \varnothing$. The disk $\bar{D}_1$ can be regarded as a compressing disk of $\mathcal{C}_1$ which is disjoint from D_1 and is not parallel to D_1. We notice that $\partial(D_1 \cup \bar{D}_1) \cap \partial D_2 = \varnothing$.

Whether $\bar{D}_1$ is separating or non-separating in C_1, we have the conclusion (1) of Theorem 5.2 by extraction operation.

If $\mathcal{M}_2$ is ∂–reducible, then there is a ∂–reducing disk $\bar{D}_2$ with $|\bar{D}_2 \cap S_2| = 1$ by Lemma 2.10, and there is a compressing disk D_{21} of $\mathcal{C}_{21}$ with $D_{21} \cap \bar{D}_2 = \varnothing$ by Corollary 2.11. We may assume that D_{21} is non-separating in C_{21}. It follows from Lemma 2.5 that D_{21} is isotopic to D_1 in $\mathcal{C}_{21}$. Hence we see that $D_1 \cap \bar{D}_2 = \varnothing$. The disk $\bar{D}_2$ can be regarded as a compressing disk of $\mathcal{C}_2$ which is disjoint from D_2 and is not parallel to D_2. We notice that $\partial D_1 \cap \partial(D_2 \cup \bar{D}_2) = \varnothing$. Since $\bar{D}_2$ is separating in C_2, $\bar{D}_2$ separates two arcs $C_{22} \cap K$ (Figure 4 (a) or (b)) or not (Figure 4 (c) or (d)). In each case, we have the conclusion (2) of Theorem 5.2.

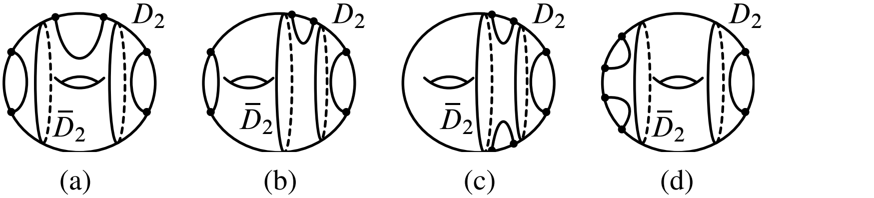

Figure 4: Possible positions of $D_2 \cup \bar{D}_2$ in $\mathcal{C}_{22}$

Case 2 The disk D_1 satisfies the condition (ii).

If D_2 also satisfies the condition (ii), then we see that K admits a $(0,3)$–bridge position. If D_2 satisfies the condition (iii) or (iv), then there is a non-separating compressing disk of $\mathcal{C}_1$ disjoint from D_2 and hence we are done in Case 1.

Case 3 The disk D_1 satisfies the condition (iii).

If D_2 also satisfies the condition (iii), then we see that K is not connected, a contradiction. Hence D_2 satisfies the condition (iv) and therefore we have the conclusion (2) of Theorem 5.2 by extraction operation (see the proof of Theorem 4.2).

Case 4 The disk D_1 satisfies the condition (iv).

Then it suffices to consider the case that D_2 also satisfies the condition (iv). We obtain $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$, $\mathcal{S}_i = \mathcal{C}_{i1} \cap \mathcal{C}_{i2}$ for each $i = 1, 2$ and $\mathcal{F} = \partial_- \mathcal{C}_{12} = \partial_- \mathcal{C}_{21}$ by the weak reduction with respect to (D_1, D_2) (see Figure 5). We notice that $\mathcal{F}$ is a torus intersecting K in two points. If both $\mathcal{M}_1$ and $\mathcal{M}_2$ are ∂–irreducible, then we have the conclusion (4) of Theorem 5.2. Hence we may assume that $\mathcal{M}_2$ is ∂–reducible

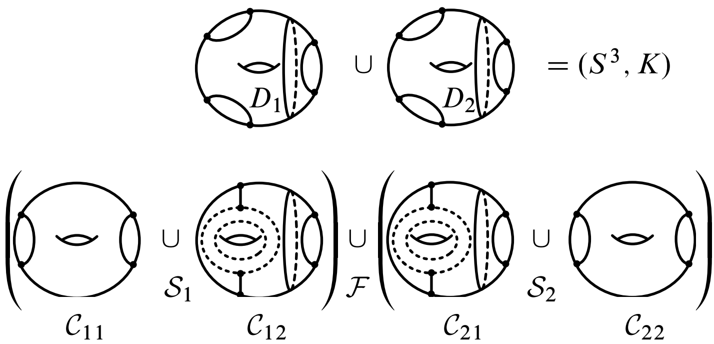

Figure 5: The weak reduction with respect to (D_1, D_2)

without loss of generality. Then there is a ∂–reducing disk $\bar{D}_2$ with $|\bar{D}_2 \cap S_2| = 1$ by Lemma 2.10, and there is a compressing disk D_{21} of $\mathcal{C}_{21}$ with $D_{21} \cap \bar{D}_2 = \varnothing$ by Corollary 2.11. It follows from Lemma 2.4 that D_{21} is isotopic to $\hat{D}_1$ in $\mathcal{C}_{21}$, where $\hat{D}_1$ is a compressing disk of $\mathcal{C}_{21}$ which is obtained by extending D_1 naturally. Hence we see that $\hat{D}_1 \cap \bar{D}_2 = \varnothing$. The disk $\bar{D}_2$ can be regarded as a compressing disk of $\mathcal{C}_2$ which is disjoint from D_2 and is not parallel to D_2. We notice that $\partial D_1 \cap \partial(\bar{D}_2 \cup D_2) = \varnothing$. If $\bar{D}_2$ is non-separating in C_2, then we have the conclusion (1) of Theorem 5.2. Hence we assume that $\bar{D}_2$ is separating in C_2. If $\bar{D}_2$ separates two arcs $C_{22} \cap K$, then this implies that K is not connected, a contradiction. Hence $D_2 \cup \bar{D}_2$ is as illustrated in Figure 4 (c) or (d), and therefore we also have the conclusion (1) of Theorem 5.2.

This completes the proof of Theorem 5.2. □

6 (2, 2)–bridge splittings

Theorem 6.1 *Let K be a knot in S^3 and $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a $(2,2)$–bridge splitting of (S^3, K). Suppose that $\mathcal{S}$ is strongly irreducible and that K admits a decomposing sphere. Then one of the following holds.*

(1) *$\mathcal{S}$ is meridionally stabilized.*

(2) *There is a c–weak reduction yielding a 2–sphere which intersects K in four or six points and is incompressible in (S^3, K).*

(3) *There is a c–weak reduction yielding a torus which intersects K in two or four points and is incompressible in (S^3, K).*

Proof Since $\mathcal{S}$ is strongly irreducible and K admits a decomposing sphere, we have the conclusion of Corollary 3.5. Without loss of generality, we assume that there are a non-separating cut disk E_1 of $\mathcal{C}_1$ and a compressing disk E_2 of $\mathcal{C}_2$ such that $\partial E_1 \cap \partial E_2 = \emptyset$. Then we have:

(i) E_2 is a non-separating compressing disk of $\mathcal{C}_2$,

(ii) E_2 cuts off a solid torus with two mutually trivial arcs from $\mathcal{C}_2$,

(iii) E_2 cuts off a solid torus with a trivial arc from $\mathcal{C}_2$,

(iv) E_2 cuts off a 3–ball with two mutually trivial arcs from $\mathcal{C}_2$, or

(v) E_2 cuts off a 3–ball with a single trivial arc from $\mathcal{C}_2$.

We do the c–weak reduction with respect to (E_1, E_2) to obtain $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$, $\mathcal{S}_i = \mathcal{C}_{i1} \cap \mathcal{C}_{i2}$ for each $i = 1, 2$ and $\mathcal{F} = \partial_- \mathcal{C}_{12} = \partial_- \mathcal{C}_{21}$.

Case 1 We have the condition (i), (ii) or (iii).

Then there is a non-separating compressing disk of $\mathcal{C}_2$ such that its boundary is disjoint from ∂E_1. Hence it suffices to consider the condition (i) and therefore $\mathcal{F}$ is a 2–sphere intersecting K in six points. As in the proof of Theorem 5.1, we see that $\mathcal{M}_1$ is ∂–irreducible. If $\mathcal{M}_2$ is also ∂–irreducible, then $\mathcal{F}$ is incompressible in (S^3, K) and hence we have the conclusion (2) of Theorem 6.1. Therefore we suppose that $\mathcal{M}_2$ is ∂–reducible. Then Lemma 2.10 and Corollary 2.11 imply that there is a ∂–reducing disk $\bar{D}_2$ of $\mathcal{M}_2$ with $\bar{D}_2 \cap E_1 = \emptyset$. The disk $\bar{D}_2$ can be regarded as a compressing disk of $\mathcal{C}_2$ which is disjoint from E_2 and is not parallel to E_2. Hence $\bar{D}_2$ is non-separating in C_2 (Figure 6 (a), (b) or (c)), $\bar{D}_2$ cuts C_2 into two solid tori (Figure 6 (d) or (e)), or $\bar{D}_2$ cuts off a 3–ball from C_2 (Figure 6 (f) or (g)).

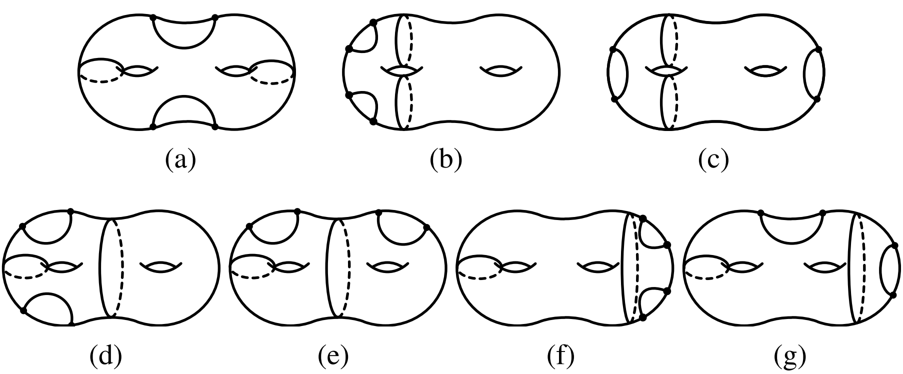

Figure 6: Possible positions of $\bar{D}_2 \cup E_2$ in $\mathcal{C}_2$

If $\bar{D}_2 \cup E_2$ is as illustrated in Figure 6 (a), then $C_1 \cup C_2$ contains a non-separating 2–sphere, a contradiction. If $\bar{D}_2 \cup E_2$ is as illustrated in Figure 6 (b), (d) or (f), then we can take E_2 so that E_2 satisfies the condition (iv). We will consider this condition in Case 2. If $\overline{D}_2 \cup E_2$ is as illustrated in Figure 6 (c), (e) or (g), then we can take E_2 so that E_2 satisfies the condition (v). We will consider this condition in Case 3.

Case 2 We have the condition (iv).

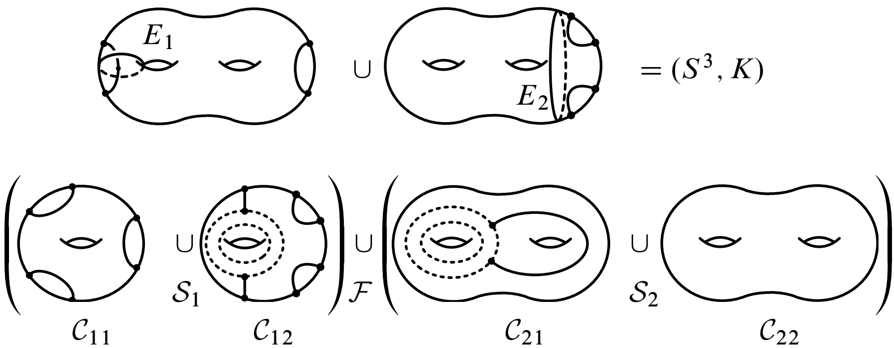

Figure 7: The c–weak reduction with respect to (E_1, E_2)

Then $\mathcal{F}$ is a torus intersecting K in two points (see Figure 7). If $\mathcal{M}_1$ is ∂–reducible, then it follows from Corollary 2.11 that $\mathcal{S}_1$ is weakly reducible and hence $\mathcal{S}$ is weakly reducible, a contradiction. Hence we see that $\mathcal{M}_1$ is ∂–irreducible. If $\mathcal{M}_2$ is also ∂–irreducible, then $\mathcal{F}$ is incompressible in (S^3, K) and hence we have the conclusion (3) of Theorem 6.1. Therefore we suppose that $\mathcal{M}_2$ is ∂–reducible. Then Lemma 2.10 and Corollary 2.11 imply that there is a ∂–reducing disk $\overline{D}_2$ of $\mathcal{M}_2$ with $|\overline{D}_2 \cap S_2| = 1$ and $\overline{D}_2 \cap E_1 = \varnothing$. The disk $\overline{D}_2$ can be regarded as a compressing disk of $\mathcal{C}_2$ which is disjoint from E_2 and is not parallel to E_2. Whether $\overline{D}_2$ is separating or non-separating in C_2, we have the conclusion (1) of Theorem 6.1.

Case 3 We have the condition (v).

Then $\mathcal{F}$ is a torus intersecting K in four points (see Figure 8). As in Case 1, we see that $\mathcal{M}_1$ is ∂–irreducible. If $\mathcal{M}_2$ is also ∂–irreducible, then $\mathcal{F}$ is incompressible in (S^3, K) and hence we have the conclusion (3) of Theorem 6.1. Therefore we suppose that $\mathcal{M}_2$ is ∂–reducible. Then Lemma 2.10 and Corollary 2.11 imply that there is a ∂–reducing disk $\overline{D}_2$ of $\mathcal{M}_2$ with $|\overline{D}_2 \cap S_2| = 1$ and $\overline{D}_2 \cap E_1 = \varnothing$. The disk $\overline{D}_2$ can be regarded as a compressing disk of $\mathcal{C}_2$ which is disjoint from E_2 and is not parallel to E_2. Hence there is a non-separating compressing disk D_2 of $\mathcal{C}_2$ with $D_2 \cap E_2 = \varnothing$ and $\partial E_1 \cap \partial(D_2 \cup E_2) = \varnothing$, or we can retake E_2 so that E_2 satisfies the condition

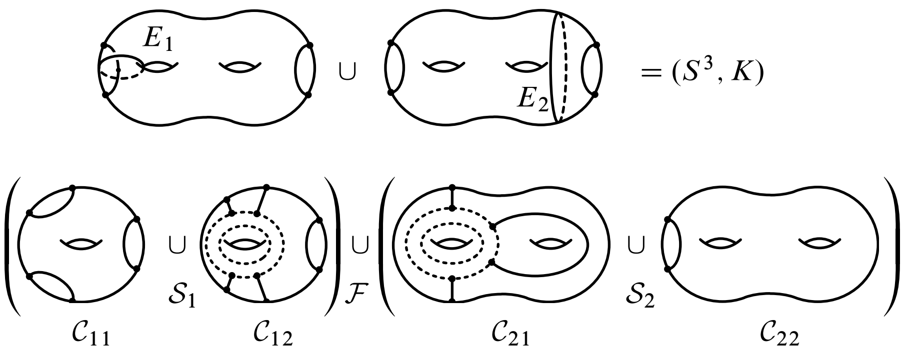

Figure 8: The c–weak reduction with respect to (E_1, E_2)

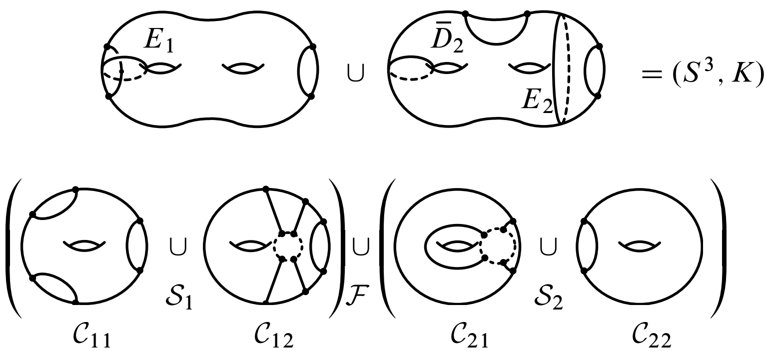

Figure 9: The c–weak reduction with respect to (E_1, E_2)

(iv). We are done in Case 1 if the latter occurs. Therefore we suppose that the former occurs.

By the c–weak reduction with respect to $(E_1, D_2 \cup E_2)$, we reset $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$, $\mathcal{S}_i = \mathcal{C}_{i1} \cap \mathcal{C}_{i2}$ for each $i = 1, 2$ and $\mathcal{F} = \partial_- \mathcal{C}_{12} = \partial_- \mathcal{C}_{21}$. We notice that $\mathcal{F}$ is a 2–sphere intersecting K in four points (see Figure 9). Since $\mathcal{S}$ is strongly irreducible, we see that $\mathcal{M}_1$ is ∂–irreducible. If $\mathcal{M}_2$ is also ∂–irreducible, then $\mathcal{F}$ is incompressible in (S^3, K) and hence we have the conclusion (2) of Theorem 6.1. Therefore we suppose that $\mathcal{M}_2$ is ∂–reducible. Then Lemma 2.10 and Corollary 2.11 imply that there is a ∂–reducing disk $\bar{D}'_2$ of $\mathcal{M}_2$ with $|\bar{D}'_2 \cap S_2| = 1$ and $\bar{D}'_2 \cap E_1 = \varnothing$. The disk $\bar{D}'_2$ can be regarded as a compressing disk of $\mathcal{C}_2$ which is disjoint from $D_2 \cup E_2$ and is parallel neither to D_2 nor to E_2. We notice that $\partial E_1 \cap \partial(D_2 \cup \bar{D}'_2 \cup E_2) = \varnothing$. Hence we have the conclusion (1) of Theorem 6.1.

This completes the proof of Theorem 6.1. □

Theorem 6.2 *Let K be a knot in S^3 and $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a $(2,2)$–bridge splitting of (S^3, K). Suppose that $\mathcal{S}$ is weakly reducible. Then one of the following holds.*

(1) *$K = K_1 \# K_2$ such that K_1 admits a $(0,2)$–bridge position and K_2 admits a $(2,1)$–bridge position.*

(2) *$K = K_1 \# K_2$ such that K_1 admits a $(1,1)$–bridge position and K_2 admits a $(1,2)$–bridge position.*

(3) *There is a weak reduction yielding a 2–sphere which intersects K in four points and is incompressible in (S^3, K).*

(4) *There is a weak reduction yielding a torus which intersects K in two points and is incompressible in (S^3, K).*

(5) *There is a weak reduction yielding a torus disjoint from K which is incompressible in (S^3, K) and cuts (S^3, K) into the exterior of a tunnel number one knot and a solid torus V with K. Moreover, K can be put in a $(1,2)$–bridge position with respect to a genus one Heegaard surface of V.*

Proof Let D_1 and D_2 be compressing disks of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively such that $\partial D_1 \cap \partial D_2 = \varnothing$. For each $i = 1, 2$, we have:

(i) D_i is a non-separating compressing disk of $\mathcal{C}_i$,

(ii) D_i cuts off a solid torus with two mutually trivial arcs from $\mathcal{C}_i$,

(iii) D_i cuts off a solid torus with a trivial arc from $\mathcal{C}_i$,

(iv) D_i cuts off a 3–ball with two mutually trivial arcs from $\mathcal{C}_i$, or

(v) D_i cuts off a 3–ball with a single trivial arc from $\mathcal{C}_i$.

We do the weak reduction with respect to (D_1, D_2) to obtain $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$, $\mathcal{S}_i = \mathcal{C}_{i1} \cap \mathcal{C}_{i2}$ for each $i = 1, 2$ and $\mathcal{F} = \partial_- \mathcal{C}_{12} = \partial_- \mathcal{C}_{21}$.

Case 1 The disk D_1 satisfies the condition (i), (ii) or (iii).

Suppose first that D_2 satisfies the condition (i), (ii) or (iii). Then it suffices to consider the case that both D_1 and D_2 satisfy the condition (i). Hence $\mathcal{F}$ is a 2–sphere intersecting K in four points. If both $\mathcal{M}_1$ and $\mathcal{M}_2$ are ∂–irreducible, then we have the conclusion (3) of Theorem 6.2. Hence we may assume that $\mathcal{M}_2$ is ∂–reducible without loss of generality. Then Lemma 2.10 and Corollary 2.11 imply that there is a ∂–reducing disk $\bar{D}_2$ of $\mathcal{M}_2$ with $|\bar{D}_2 \cap S_2| = 1$ and $\bar{D}_2 \cap D_1 = \varnothing$. The disk $\bar{D}_2$ can be regarded as a compressing disk of $\mathcal{C}_2$ which is disjoint from D_2 and is not parallel to D_2. We notice that $\partial D_1 \cap \partial(D_2 \cup \bar{D}_2) = \varnothing$. Since $C_1 \cup C_2$ does not contain a

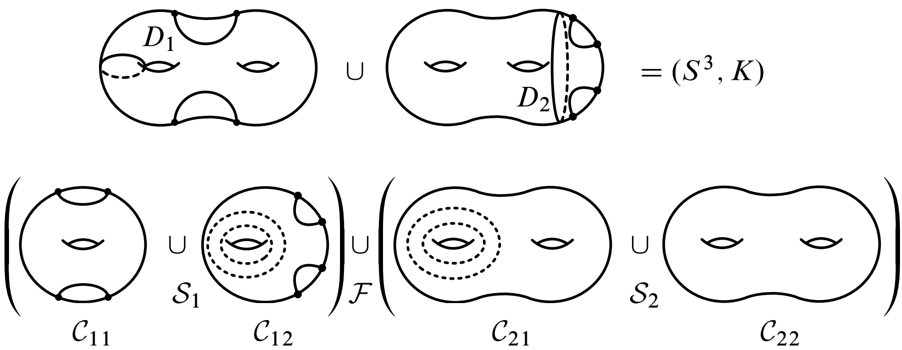

Figure 10: The weak reduction with respect to (D_1, D_2)

non-separating 2–sphere, we see that $D_2 \cup \bar{D}_2$ is as illustrated in Figure 6 except (a). In each case, we have the conclusion (2) of Theorem 6.2.

Suppose next that D_2 satisfies the condition (iv). Then $\mathcal{F}$ is a torus disjoint from K (see Figure 10). If both $\mathcal{M}_1$ and $\mathcal{M}_2$ are ∂–irreducible, then we have the conclusion (5) of Theorem 6.2. Hence we assume that $\mathcal{M}_1$ or $\mathcal{M}_2$ is ∂–reducible. If $\mathcal{M}_1$ is ∂–reducible, then Lemma 2.10 implies that there is a ∂–reducing disk $\bar{D}_1$ of $\mathcal{M}_1$ with $|\bar{D}_1 \cap S_1| = 1$. The disk $\bar{D}_1$ can be regarded as a compressing disk of $\mathcal{C}_1$ which is disjoint from D_1 and is not parallel to D_1. Hence we see that $D_1 \cup \bar{D}_1$ is as illustrated in Figure 6. It follows from Corollary 2.11 that there is a compressing disk D_2' of $\mathcal{C}_{12}$ with $\bar{D}_1 \cap D_2' = \varnothing$. We may assume that D_2' cuts off a 3–ball with a trivial arc from $\mathcal{C}_{12}$. Hence the disk D_2' can be regarded as a compressing disk of $\mathcal{C}_2$ with $\partial(D_1 \cup \bar{D}_1) \cap \partial D_2' = \varnothing$. Therefore we have the conclusion (1) of Theorem 6.2. If $\mathcal{M}_2$ is ∂–reducible, then $\mathcal{M}_2$ is a solid torus. This implies that K admits a $(1,2)$–bridge position and therefore we have the conclusion (2) of Theorem 6.2.

Suppose finally that D_2 satisfies the condition (v) (see Figure 11). If both $\mathcal{M}_1$ and $\mathcal{M}_2$ are ∂–irreducible, then we have the conclusion (4) of Theorem 6.2. If $\mathcal{M}_1$ is ∂–reducible, then we have the conclusion (1) of Theorem 6.2 by an argument similar to the above. If $\mathcal{M}_2$ is ∂–reducible, then we have the conclusion (2) of Theorem 6.2, or we can retake D_2 so that $\partial D_1 \cap \partial D_2 = \varnothing$ and D_2 satisfies the condition (iv) and hence we are done.

Case 2 The disk D_1 satisfies the condition (iv) or (v).

Suppose that D_1 satisfies the condition (iv). Then we have the conclusion (1) of Theorem 6.2, whether D_2 satisfies the condition (iv) or (v). It is impossible to have the condition (v), because K is connected.

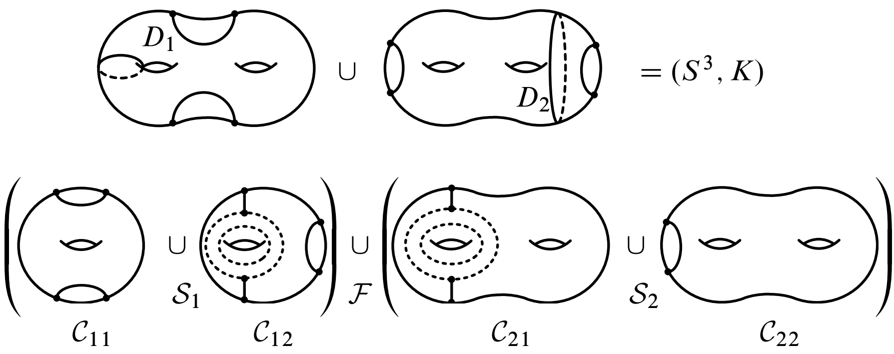

Figure 11: The weak reduction with respect to (D_1, D_2)

This completes the proof of Theorem 6.2. □

7 The connected sum of n–string prime knots

Proposition 7.1 *For each $i = 1, 2$, let K_i be a non-trivial knot in S^3 and set $K = K_1 \# K_2$. Suppose that (S^3, K) admits a closed incompressible surface $\mathcal{F} = (F, F \cap K)$ with $F \cap K \neq \varnothing$ and $\chi(F \cap \mathrm{Ext}(K; S^3)) < 0$, where $\chi(\cdot)$ is the Euler characteristic. Then for $i = 1$ or 2, (S^3, K_i) admits a closed surface $\mathcal{F}' = (F', F' \cap K_i)$, which is obtained from a subsurface of F by filling its boundary with disjoint disks, such that $\mathcal{F}'$ is incompressible in (S^3, K_i), $F' \cap K_i \neq \varnothing$ and $\chi(F' \cap \mathrm{Ext}(K_i; S^3)) < 0$.*

Proof Let Σ be a decomposing sphere of K with $K = K_1 \#_\Sigma K_2$. Then Σ divides (S^3, K) into $\mathcal{B}_1 = (B_1, K_1')$ and $\mathcal{B}_2 = (B_2, K_2')$, where B_i is a 3–ball and $K_i' = K \cap B_i$ $(i = 1, 2)$. We notice that (B_1, K_1') forms (S^3, K_1) after gluing a 3–ball B with a trivial arc γ. We set $\mathcal{B} = (B, \gamma)$. If $F \cap \Sigma = \varnothing$, then we are done. Hence we assume that $F \cap \Sigma \neq \varnothing$. It follows from Lemma 3.1 that F and Σ are isotoped so that each component of $F \cap \Sigma$ is non-trivial both in $\mathcal{F}$ and in Σ. Since $\chi(F \cap \mathrm{Ext}(K; S^3)) < 0$, there is a component F_0 of F cut along $F \cap \Sigma$ such that $\chi(F_0 \cap \mathrm{Ext}(K; S^3)) < 0$. Without loss of generality, we may assume that $F_0 \subset B_1$. Recall that $(S^3, K_1) = \mathcal{B}_1 \cup \mathcal{B}$. Let F' be a closed surface obtained from F_0 by filling its boundary with disjoint disks in B such that each disk intersects the trivial arc γ in a single point. Then F' is a closed surface in $S^3 = B_1 \cup B$. Set $\mathcal{F}' = (F', F' \cap K_1)$. Suppose that $\mathcal{F}'$ is compressible in (S^3, K_1), and let δ be its compressing disk. We may assume that $\partial\delta$ is contained in $F_0 \subset F'$ and moreover $\delta \subset B_1$. By an innermost disk argument, if necessary, we see that $\mathcal{F}$ is compressible in (S^3, K), a contradiction. Hence $\mathcal{F}'$ is incompressible in (S^3, K_1). □

A knot in S^3 is said to be *n–string prime* ($n>0$) if there is no incompressible 2–sphere intersecting the knot in $2n$ points.

Corollary 7.2 *Let K be the connected sum of non-trivial knots of n–string prime for all n. Then (S^3, K) admits no incompressible 2–spheres intersecting K in more than two points.*

Proposition 7.3 *Let K be the connected sum of non-trivial knots of n–string prime for all n. Suppose that (S^3, K) admits an incompressible torus $\mathcal{T} = (T, T\cap K)$ which is not isotopic to $\partial\eta(K; S^3)$. Then there is a decomposing sphere of K disjoint from T.*

Proof Let Σ be a decomposing sphere of K. If $\Sigma\cap T = \varnothing$, then we are done. Hence we assume that $\Sigma\cap T \neq \varnothing$. Then it follows from Lemma 3.1 that Σ and T are isotoped so that each component of $\Sigma\cap T$ is non-trivial both in Σ and in $\mathcal{T}$. We take Σ so that $|\Sigma\cap T|$ is minimal among such all decomposing spheres of K.

We first suppose that $T\cap K = \varnothing$. Then T cuts off a pair of a solid torus V and the knot K from (S^3, K). Let Δ be a disk component of Σ cut along $\Sigma\cap T$. We notice that Δ intersects K in a single point and hence Δ is a cut disk of (V, K). Let Σ' be a 2–sphere obtained by cutting T along $\partial\Delta$ and attaching copies of Δ to the resulting boundaries. Since K is not a core loop of V, we see that Σ' bounds a 3–ball B in V which contains a non-trivial arc. Since T is incompressible in (S^3, K), we see that $\mathrm{Ext}(V; S^3)$ is not a solid torus and therefore $\mathrm{Ext}(B; S^3)$ is a 3–ball which contains a non-trivial arc. Hence Σ' is a decomposing sphere of K disjoint from T. This contradicts the minimality of $|\Sigma\cap T|$.

We now suppose that $T\cap K\neq\varnothing$. If a component of $\Sigma\cap T$ is essential in $\mathcal{T}$, then there is a component T_0 of T cut along $\Sigma\cap T$ such that $T_0\cap E(K; S^3)$ is a planar surface with $\chi(T_0\cap \mathrm{Ext}(K; S^3)) < 0$. This together with Proposition 7.1 implies that for a factor knot K' of K, (S^3, K') admits an incompressible 2–sphere $\mathcal{P} = (P, P\cap K')$ with $P\cap K'\neq\varnothing$ and $\chi(P\cap \mathrm{Ext}(K'; S^3)) < 0$. This contradicts that K is the connected sum of non-trivial knots of n–string prime for all n. Hence each component of $\Sigma\cap T$ bounds a disk in T which intersects K in a single point. Let α be a component of $\Sigma\cap T$ which is innermost in T and δ_α its innermost disk. Since each component of $\Sigma\cap T$ is non-trivial in Σ, α bounds a disk δ'_α in Σ intersecting K in a single point. If $\Sigma' = \delta_\alpha\cup\delta'_\alpha$ bounds a 3–ball with a trivial arc, then we can isotope Σ and T so that $|\Sigma\cap T|$ is reduced, a contradiction. Therefore Σ' is a decomposing sphere of K. A slight isotopy implies that $|\Sigma'\cap T| < |\Sigma\cap T|$. This also contradicts the minimality of $|\Sigma\cap T|$. □

Let K be the connected sum of non-trivial knots of n–string prime for all n, and let $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ be a $(1,3)$– or $(2,2)$–bridge splitting of (S^3, K). Then it follows from Corollary 7.2 that (S^3, K) admits no incompressible 2–spheres intersecting K in more than two points. Toward Theorem 1.4, we need to study up on all the cases such that an incompressible torus is obtained by a weak or c–weak reduction in Sections 5 and 6. Namely, $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ is one of the following:

(I) a $(1,3)$–bridge splitting as illustrated in Figure 5,

(II) a $(2,2)$–bridge splitting as illustrated in Figure 7,

(III) a $(2,2)$–bridge splitting as illustrated in Figure 8,

(IV) a $(2,2)$–bridge splitting as illustrated in Figure 10, and

(V) a $(2,2)$–bridge splitting as illustrated in Figure 11.

Theorem 7.4 *Let K be the connected sum of non-trivial knots of n–string prime for all n and $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a $(1,3)$–bridge splitting of (S^3, K). Suppose that there are compressing disks D_1 and D_2 of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively such that $\partial D_1 \cap \partial D_2 = \varnothing$ and that each D_i cuts off a 3–ball with a single trivial arc from $\mathcal{C}_i$. Then a torus obtained by the weak reduction with respect to (D_1, D_2) is compressible in (S^3, K).*

Proof As in Figure 5, we obtain $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$, $\mathcal{S}_i = \mathcal{C}_{i1} \cap \mathcal{C}_{i2}$ for each $i = 1, 2$ and $\mathcal{F} = \partial_- \mathcal{C}_{12} = \partial_- \mathcal{C}_{21}$ by the weak reduction with respect to (D_1, D_2). We notice that $\mathcal{F} = (F, F \cap K)$ is a torus. Suppose, towards a contradiction, that $\mathcal{F} = (F, F \cap K)$ is incompressible in (S^3, K).

Claim The surface $\mathcal{S}_i = (S_i, S_i \cap K)$ is strongly irreducible for each $i = 1, 2$.

Proof Suppose that $\mathcal{S}_i$, say $i = 1$, is weakly reducible. We notice that a compressing disk of $\mathcal{C}_{12}$ is isotopic to $\widehat{D}_2$, where $\widehat{D}_2$ is a compressing disk of $\mathcal{C}_{12}$ which is obtained by extending D_2 naturally. Hence there is a compressing disk D_{11} of $\mathcal{C}_{11}$ with $\partial D_{11} \cap \partial \widehat{D}_2 = \varnothing$. We notice that $\mathcal{S}_1$ is irreducible because K is connected. This implies that there is a vertical annulus A_{12} in C_{12} with $A_{12} \cap K = \varnothing$, $A_{12} \cap \widehat{D}_2 = \varnothing$ and $\partial A_{12} \supset \partial D_{11}$. Hence $D_{11} \cup A_{12}$ is a ∂–reducing disk of $\mathcal{M}_1$ and therefore $\mathcal{F}$ is compressible, contrary to the hypothesis. □

It follows from Proposition 7.3 that there is a decomposing sphere of K disjoint from F. Without loss of generality, we may assume that a decomposing sphere is contained in $\mathcal{M}_1$. Then the cH–splitting $(\mathcal{C}_{11}, \mathcal{C}_{12}; \mathcal{S}_1)$ satisfies one of the conclusions of Corollary 3.4. Moreover, we see that the conclusion (1) of Corollary 3.4 does not hold by an argument similar to that in the proof of Corollary 3.5. Hence we have only the conclusion (2) of Corollary 3.4.

Case 1 There are a cut disk E_{11} of $\mathcal{C}_{11}$ and a compressing disk E_{12} of $\mathcal{C}_{12}$ such that $\partial E_{11} \cap \partial E_{12} = \varnothing$.

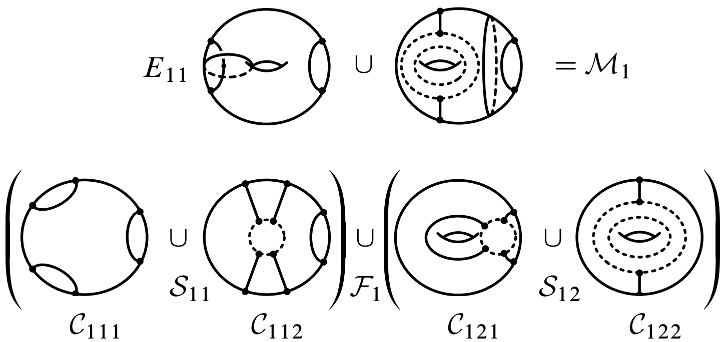

Figure 12: The c–weak reduction with respect to $(E_{11}, \widehat{D}_2)$

Then we see that E_{11} is non-separating C_{11} by an argument similar to that in the proof of Corollary 3.5. We notice that E_{12} is isotopic to $\widehat{D}_2$ by Lemma 2.4. We now do the c–weak reduction with respect to $(E_{11}, \widehat{D}_2)$. As usual, we set $\mathcal{M}_{1i} = \mathcal{C}_{1i1} \cup \mathcal{C}_{1i2}$, $\mathcal{S}_{1i} = \mathcal{C}_{1i1} \cap \mathcal{C}_{1i2}$ for each $i = 1, 2$ and $\mathcal{F}_1 = \partial_- \mathcal{C}_{112} = \partial_- \mathcal{C}_{121}$ (see Figure 12). We notice that $\mathcal{F}_1$ is a 2–sphere intersecting K in four points. Since we assume that K is the connected sum of non-trivial knots of n–string prime for all n, it follows from Corollary 7.2 that $\mathcal{F}_1$ is compressible in (S^3, K). Let D be a compressing disk of $\mathcal{F}_1$. Since $\mathcal{F}$ is incompressible in (S^3, K), we may assume that D is disjoint from F. This implies that $\mathcal{F}_1$ is compressible in $\mathcal{M}_1$ and hence either $\mathcal{M}_{11}$ or $\mathcal{M}_{12}$ is ∂–reducible. If $\mathcal{M}_{11}$ is ∂–reducible, then we see that $\mathcal{S}_1$ is weakly reducible by Corollary 2.11. This contradicts the claim above. We also see that $\mathcal{M}_{12}$ is ∂–irreducible because C_{12j} is ambient isotopic to a regular neighborhood of $\partial_- C_{12j} \cup (C_{12j} \cup K)$ for each $j = 1, 2$. Therefore Case 1 does not hold.

Case 2 There are a compressing disk E_{11} of $\mathcal{C}_{11}$ and a cut disk E_{12} of $\mathcal{C}_{12}$ such that $\partial E_{11} \cap \partial E_{12} = \varnothing$.

Since C_{12} is homeomorphic to $\partial_- C_{12} \times [0, 1]$, we see that E_{12} cuts off a 3–ball B from C_{12}. Since ∂E_{12} is essential in $\mathcal{S}_1$, we see that E_{12} intersects a vertical arc of $C_{12} \cap K$ and hence B intersects K in two mutually trivial arcs one of which is the trivial arc in $\mathcal{C}_{12}$. Namely, E_{12} cuts $\mathcal{C}_{12}$ into a 3–manifold $\partial_- C_{12} \times [0, 1]$ with two vertical arcs and the 3–ball B with two trivial arcs. If E_{11} is non-separating in C_{11}, then ∂E_{11} is disjoint from B. This implies that $\mathcal{S}_1$ is weakly reducible, a contradiction.

Hence E_{11} cuts off a 3–ball with one or two trivial arcs from $\mathcal{C}_{11}$. However, this also implies that $\mathcal{S}_1$ is weakly reducible, a contradiction.

This completes the proof of Theorem 7.4. □

Theorem 7.5 *Let K be the connected sum of non-trivial knots of n–string prime for all n and $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a strongly irreducible $(2,2)$–bridge splitting of (S^3, K). Suppose that there are a non-separating cut disk E_1 of $\mathcal{C}_1$ and a compressing disk E_2 of $\mathcal{C}_2$ such that $\partial E_1 \cap \partial E_2 = \varnothing$ and that E_2 cuts off a 3–ball with two mutually trivial arcs from $\mathcal{C}_2$. Then a torus obtained by the c–weak reduction with respect to (E_1, E_2) is compressible in (S^3, K).*

Proof Suppose, towards a contradiction, that a torus obtained by the c–weak reduction with respect to (E_1, E_2) is incompressible in (S^3, K). As in the proof of Theorem 6.1, we obtain $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$, $\mathcal{S}_i = \mathcal{C}_{i1} \cap \mathcal{C}_{i2}$ for each $i = 1, 2$ and $\mathcal{F} = \partial_- \mathcal{C}_{12} = \partial_- \mathcal{C}_{21}$ by the c–weak reduction with respect to (E_1, E_2) (see Figure 7). Recall that $\mathcal{F} = (F, F \cap K)$ is a torus of which we suppose incompressibility. It follows from Proposition 7.3 that there is a decomposing sphere of K disjoint from F. Suppose that a decomposing sphere is contained in $\mathcal{M}_2$. We notice that $\mathcal{S}_2$ is strongly irreducible because C_{21} is ambient isotopic to a regular neighborhood of $\partial_- C_{21} \cup (C_{21} \cap K)$. Hence the cH–splitting $(\mathcal{C}_{21}, \mathcal{C}_{22}; \mathcal{S}_2)$ satisfies one of the conclusions of Corollary 3.4. Moreover, we see that the conclusion (1) of Corollary 3.4 does not hold because $\mathcal{C}_{21}$ admits no separating cut disks. Hence we have only the conclusion (2) of Corollary 3.4. This implies that there are a cut disk E_{21} of $\mathcal{C}_{21}$ and a compressing disk E_{22} of $\mathcal{C}_{22}$ such that $\partial E_{21} \cap \partial E_{22} = \varnothing$. Then we can extend E_{22} into C_1 so that the extended disk is a compressing disk of $\mathcal{F}$, a contradiction. Therefore a decomposing sphere is contained in $\mathcal{M}_1$.

Claim There are a non-separating cut disk E_{11} of $\mathcal{C}_{11}$ and a compressing disk E_{12} of $\mathcal{C}_{12}$ with $\partial E_{11} \cap \partial E_{12} = \varnothing$.

Proof If $\mathcal{S}_1$ is weakly reducible, then $\mathcal{S}$ is also weakly reducible. Hence $\mathcal{S}_1$ is strongly irreducible. Hence the cH–splitting $(\mathcal{C}_{11}, \mathcal{C}_{12}; \mathcal{S}_1)$ satisfies one of the conclusions of Corollary 3.4. Moreover, we have only the conclusion (2) of Corollary 3.4. If there are a compressing disk E_{11} of $\mathcal{C}_{11}$ and a cut disk E_{12} of $\mathcal{C}_{12}$ such that $\partial E_{11} \cap \partial E_{12} = \varnothing$, then either $\mathcal{S}_1$ is weakly reducible, or $\mathcal{F}$ is compressible. Hence there are a cut disk E_{11} of $\mathcal{C}_{11}$ and a compressing disk E_{12} of $\mathcal{C}_{12}$ such that $\partial E_{11} \cap \partial E_{12} = \varnothing$. By an argument similar to that in the proof of Corollary 3.5, we see that E_{11} must be non-separating in C_{11} because $\mathcal{S}_1$ is strongly irreducible. □

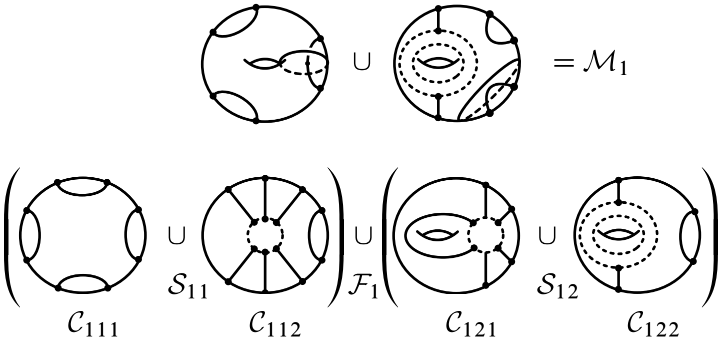

Figure 13: The c–weak reduction with respect to (E_{11}, E_{12})

Case 1 The disk E_{12} cuts off a 3–ball with a trivial arc from $\mathcal{C}_{12}$.

We now do the c–weak reduction with respect to (E_{11}, E_{12}). As usual, we set $\mathcal{M}_{1i} = \mathcal{C}_{1i1} \cup \mathcal{C}_{1i2}$, $\mathcal{S}_{1i} = \mathcal{C}_{1i1} \cap \mathcal{C}_{1i2}$ for each $i = 1, 2$ and $\mathcal{F}_1 = \partial_- \mathcal{C}_{112} = \partial_- \mathcal{C}_{121}$ (see Figure 13). We notice that $\mathcal{F}_1$ is a 2–sphere intersecting K in six points. Since we assume that K is the connected sum of non-trivial knots of n–string prime for all n, it follows from Proposition 7.1 that $\mathcal{F}_1$ is compressible in (S^3, K). Let D be a compressing disk of $\mathcal{F}_1$. Since $\mathcal{F}$ is incompressible in (S^3, K), we may assume that D is disjoint from F. This implies that $\mathcal{F}_1$ is compressible in $\mathcal{M}_1$ and hence either $\mathcal{M}_{11}$ or $\mathcal{M}_{12}$ is ∂–reducible. If $\mathcal{M}_{11}$ is ∂–reducible, then $\mathcal{S}_{11}$ is weakly reducible and hence $\mathcal{S}_1$ is weakly reducible, a contradiction. Hence $\mathcal{M}_{12}$ is ∂–reducible and therefore there is a compressing disk D_{122} of $\mathcal{C}_{122}$ such that its boundary is disjoint from ∂E_{11}. We can regard ∂D_{122} as a compressing disk of $\mathcal{C}_{12}$ which is disjoint from E_{12} and is not parallel to E_{12}. We notice that $\partial E_{11} \cap \partial(D_{122} \cup E_{12}) = \varnothing$. This implies that there is a compressing disk of $\mathcal{C}_{12}$, which is obtained by joining D_{122} to E_{12} with a band, such that its boundary is disjoint from ∂E_{11} and that it cuts off a 3–ball with two mutually trivial arcs from $\mathcal{C}_{12}$. We consider such a case in the following.

Case 2 The disk E_{12} cuts off a 3–ball with two mutually trivial arcs from $\mathcal{C}_{12}$.

We now do the c–weak reduction with respect to (E_{11}, E_{12}). As usual, we set $\mathcal{M}_{1i} = \mathcal{C}_{1i1} \cup \mathcal{C}_{1i2}$, $\mathcal{S}_{1i} = \mathcal{C}_{1i1} \cap \mathcal{C}_{1i2}$ for each $i = 1, 2$ and $\mathcal{F}_1 = \partial_- \mathcal{C}_{112} = \partial_- \mathcal{C}_{121}$. We notice that $\mathcal{F}_1$ is a 2–sphere intersecting K in four points. If $\mathcal{M}_{11}$ is ∂–reducible, then we see that $\mathcal{S}_1$ is weakly reducible, a contradiction. Since C_{12i} is ambient isotopic to a regular neighborhood of $\partial_- C_{12i} \cup (C_{12i} \cup K)$ for each $i = 1, 2$, we see that $\mathcal{M}_{12}$ is also ∂–irreducible. This implies that $\mathcal{F}_1$ is incompressible in (S^3, K), contradicting that K is the connected sum of non-trivial knots of n–string prime for all n.

This completes the proof of Theorem 7.5. □

Theorem 7.6 *Let K be the connected sum of non-trivial knots of n–string prime for all n and $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a strongly irreducible $(2,2)$–bridge splitting of (S^3, K). Suppose that there are a non-separating cut disk E_1 and a compressing disk E_2 of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively such that $\partial E_1 \cap \partial E_2 = \varnothing$ and that E_2 cuts off a 3–ball with a trivial arc from $\mathcal{C}_2$. Then a torus obtained by the c–weak reduction with respect to (E_1, E_2) is compressible in (S^3, K).*

Proof Suppose, towards a contradiction, that a torus obtained by the c–weak reduction with respect to (E_1, E_2) is incompressible in (S^3, K). As in the proof of Theorem 6.1, we obtain $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$, $\mathcal{S}_i = \mathcal{C}_{i1} \cap \mathcal{C}_{i2}$ for each $i = 1, 2$ and $\mathcal{F} = \partial_- \mathcal{C}_{12} = \partial_- \mathcal{C}_{21}$ by the c–weak reduction with respect to (E_1, E_2) (see Figure 8). Recall that $\mathcal{F} = (F, F \cap K)$ is a torus of which we suppose incompressibility. It follows from Proposition 7.3 that there is a decomposing sphere of K disjoint from F. By an argument similar to that in the first half of the proof of Theorem 7.5, we see that a decomposing sphere is contained in $\mathcal{M}_1$. The argument to obtain the desired result is almost the same as that in the proof of Theorem 7.4. □

Theorem 7.7 *Let K be the connected sum of non-trivial knots of n–string prime for all n and $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a $(2,2)$–bridge splitting of (S^3, K). Suppose that there are compressing disks D_1 and D_2 of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively such that $\partial D_1 \cap \partial D_2 = \varnothing$, D_1 is non-separating, and that D_2 cuts off a 3–ball with two mutually trivial arcs from $\mathcal{C}_2$. Suppose also that a torus obtained by the weak reduction with respect to (D_1, D_2) is incompressible in (S^3, K). Then one of the following holds.*

(1) *K contains a non-trivial 2–bridge knot as a connected summand.*

(2) *$K = K_1 \# K_2$ such that each K_i admits a $(1,1)$–bridge position.*

(3) *$K = K_1 \# K_2$ such that K_1 admits a $(0,3)$–bridge position and K_2 admits a $(2,0)$–bridge position.*

Proof As in the proof of Theorem 6.2, we obtain $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$, $\mathcal{S}_i = \mathcal{C}_{i1} \cap \mathcal{C}_{i2}$ for each $i = 1, 2$ and $\mathcal{F} = \partial_- \mathcal{C}_{12} = \partial_- \mathcal{C}_{21}$ by the weak reduction with respect to (D_1, D_2) (see Figure 10). Recall that $\mathcal{F} = (F, F \cap K)$ is a torus of which we suppose incompressibility. It follows from Proposition 7.3 that there is a decomposing sphere of K disjoint from F. Hence $\mathcal{M}_1$ contains a decomposing sphere of K. Suppose that $\mathcal{S}_1$ is weakly reducible. Then there are compressing disks D_{11} and D_{12} of $\mathcal{C}_{11}$

and $\mathcal{C}_{12}$ respectively such that $\partial D_{11} \cap \partial D_{12} = \varnothing$. We may assume that D_{11} is non-separating in C_{11} and that D_{12} cuts off a 3–ball with a trivial arc from $\mathcal{C}_{12}$. Then K contains a non-trivial 2–bridge knot as a connected summand, or $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ is simplified so that (S^3, K) admits a $(2,1)$–bridge decomposition. If the latter occurs, then it follows from Morimoto [13, Theorem 1.6] that we have the conclusion (1) or (2) of Theorem 7.7. Therefore we assume that $\mathcal{S}_1$ is strongly irreducible. This implies that there are a non-separating cut disk E_{11} of $\mathcal{C}_{11}$ and a compressing disk E_{12} of $\mathcal{C}_{12}$ with $\partial E_{11} \cap \partial E_{12} = \varnothing$ (see Corollary 3.5).

The following argument is quite similar to that in the proof of Theorem 7.5. We now do the c–weak reduction with respect to (E_{11}, E_{12}). As usual we set $\mathcal{M}_{1i} = \mathcal{C}_{1i1} \cup \mathcal{C}_{1i2}$, $\mathcal{S}_{1i} = \mathcal{C}_{1i1} \cap \mathcal{C}_{1i2}$ for each $i = 1, 2$ and $\mathcal{F}_1 = \partial_- \mathcal{C}_{112} = \partial_- \mathcal{C}_{121}$. We suppose that E_{12} cuts off a 3–ball with a trivial arc from $\mathcal{C}_{12}$. Then $\mathcal{F}_1$ is a 2–sphere intersecting K in four points. Since we assume that K is the connected sum of non-trivial knots of n–string prime for all n, we see that $\mathcal{F}_1$ is compressible in $\mathcal{M}_1$ and hence either $\mathcal{M}_{11}$ or $\mathcal{M}_{12}$ is ∂–reducible. If $\mathcal{M}_{11}$ is ∂–reducible, then $\mathcal{S}_1$ is weakly reducible, a contradiction. Hence $\mathcal{M}_{12}$ is ∂–reducible. Since C_{121} is isotopic to a regular neighborhood of $\partial_- C_{121} \cup (C_{121} \cap K)$, there is a ∂–reducing disk $\bar{D}_{12}$ of $\mathcal{M}_{12}$ with $|\bar{D}_{12} \cap \mathcal{S}_{12}| = 1$ and $\bar{D}_{12} \cap E_{11} = \varnothing$. We can regard $\bar{D}_{12}$ as a compressing disk of $\mathcal{C}_{12}$ which is disjoint from E_{12} and is not parallel to E_{12}. This implies that we can obtain a ∂–compressing disk of $\mathcal{C}_{12}$ such that its boundary is disjoint from ∂E_{11} and that it cuts off a 3–ball with two mutually trivial arcs from $\mathcal{C}_{12}$. Thus we suppose that E_{12} cuts off a 3–ball with two mutually trivial arcs from $\mathcal{C}_{12}$. Then $\mathcal{F}_1$ is a 2–sphere intersecting K in two points and hence we have the conclusion (3) of Theorem 7.7 by extraction operation. □

Theorem 7.8 *Let K be the connected sum of non-trivial knots of n–string prime for all n and $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a $(2,2)$–bridge splitting of (S^3, K). Suppose that there are compressing disks D_1 and D_2 of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively such that $\partial D_1 \cap \partial D_2 = \varnothing$, D_1 is non-separating, and that D_2 cuts off a 3–ball with a trivial arc from $\mathcal{C}_2$. Suppose also that a torus obtained by the weak reduction with respect to (D_1, D_2) is incompressible in (S^3, K). Then one of the following holds.*

1. *$\mathcal{S}$ is meridionally stabilized.*
2. *K contains a non-trivial 2–bridge knot as a connected summand.*
3. *$K = K_1 \# K_2$ such that K_1 admits a $(1,1)$–bridge position and K_2 admits a $(1,2)$–bridge position.*

Proof As in Figure 11, we obtain $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$, $\mathcal{S}_i = \mathcal{C}_{i1} \cap \mathcal{C}_{i2}$ for each $i = 1, 2$ and $\mathcal{F} = \partial_- \mathcal{C}_{12} = \partial_- \mathcal{C}_{21}$ by the weak reduction with respect to (D_1, D_2). Recall that $\mathcal{F} = (F, F \cap K)$ is a torus of which we suppose incompressibility. It follows from Proposition 7.3 that there is a decomposing sphere of K disjoint from F. If a decomposing sphere is contained in $\mathcal{M}_1$, then we have a contradiction by the same argument as in the proof of Theorem 7.4. Hence we assume that a decomposing sphere is contained in $\mathcal{M}_2$. By an argument similar to the proof of the claim in the proof of Theorem 7.4, we also see that $\mathcal{S}_2$ is strongly irreducible. Hence the cH–splitting $(\mathcal{C}_{21}, \mathcal{C}_{22}; \mathcal{S}_2)$ satisfies one of the conclusions of Corollary 3.4. Moreover, we see that the conclusion (1) of Corollary 3.4 does not hold because $\mathcal{S}_2$ is strongly irreducible. Hence we have only the conclusion (2) of Corollary 3.4.

Case 1 There are a compressing disk E_{21} of $\mathcal{C}_{21}$ and a cut disk E_{22} of $\mathcal{C}_{22}$ such that $\partial E_{21} \cap \partial E_{22} = \varnothing$.

If E_{22} is separating in C_{22}, then $\mathcal{S}_2$ is weakly reducible, a contradiction. Hence E_{22} is a non-separating cut disk of $\mathcal{C}_{22}$. This implies that $\mathcal{S}$ is meridionally stabilized or that there is a non-separating compressing disk of $\mathcal{C}_{21}$ such that its boundary is disjoint from $\partial D_2 \cup \partial E_{22}$. If the latter occurs, then we have the conclusion (2) or (3) of Theorem 7.8 by Lemma 7.9 which we prove below.

Case 2 There are a separating cut disk E_{21} of $\mathcal{C}_{21}$ and a compressing disk E_{22} of $\mathcal{C}_{22}$ such that $\partial E_{21} \cap \partial E_{22} = \varnothing$.

Then E_{21} cuts $\mathcal{C}_{21}$ into ({a solid torus} $\times [0, 1]$, two vertical arcs) and (a solid torus V, a trivial arc). We may assume that E_{22} is a non-separating compressing disk of $\mathcal{C}_{22}$. Since $\mathcal{F}$ is incompressible in (S^3, K), we see that ∂E_{22} is contained in ∂V. This implies that we have the conclusion (3) of Theorem 7.8 by extraction operation (see Figure 14).

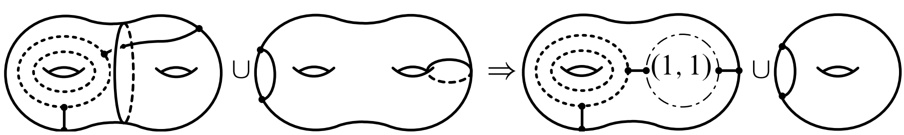

Figure 14: Removing $\eta(E_{22}, C_{22})$ from C_{22} and attaching it to C_{21}

Case 3 There are a non-separating cut disk E_{21} of $\mathcal{C}_{21}$ and a compressing disk E_{22} of $\mathcal{C}_{22}$ such that $\partial E_{21} \cap \partial E_{22} = \varnothing$.

It follow from Lemma 3.3 that there is a decomposing sphere Σ of $K_2 = K \cap M_2$ such that each component of $\Sigma \cap S_2$ is essential in $\mathcal{S}_2$, and the components of Σ cut along $\Sigma \cap S_2$ consist of two disks Δ and Δ' with $|\Delta \cap K_2| = |\Delta' \cap K_2| = 1$ and possibly annuli disjoint from K_2. We take Σ so that $|\Sigma \cap S_2|$ is minimal among all such decomposing spheres. If Δ or Δ' is contained in C_{22}, then we have the conclusion (1) of Corollary 3.4 or the condition of Case 1 by an argument in the proof of Corollary 3.4. Hence we assume that both Δ and Δ' are contained in C_{21}. Moreover, if Δ or Δ' is separating in C_{21}, then we have the condition of Case 2. Therefore we also assume that each of Δ and Δ' is non-separating in C_{21}. Then we have either (i) Δ and Δ' are mutually parallel in $\mathcal{C}_{21}$, or (ii) Δ and Δ' are not mutually parallel in $\mathcal{C}_{21}$ (see Figure 15).

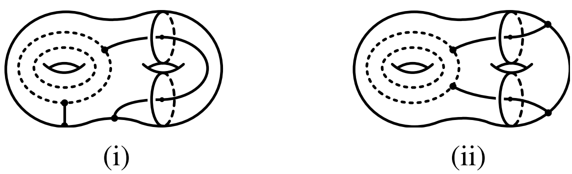

Figure 15: Possible positions of $\Delta \cup \Delta'$ in $\mathcal{C}_{21}$

We first suppose that Δ and Δ' satisfy the condition (i). If $\Sigma \cap C_{21}$ contains no annulus component, then $\Sigma \cap C_{22}$ consists of an annulus A_{22}. We notice that A_{22} is obtained by joining a compressing disk, which cuts C_{22} into two solid tori, to itself with a band. Hence A_{22} is ∂–parallel in $\mathcal{C}_{22}$, because otherwise M_2 contains a lens space as a connected summand. This implies that Σ is isotoped to be contained in C_{21}, a contradiction. Therefore $\Sigma \cap C_{21}$ contains an annulus component. We notice that such an annulus component is obtained by joining a non-separating compressing disk to itself with a band. Let A_{21} be the annulus component of $\Sigma \cap C_{21}$ such that $A_{21} \cup \Delta$ or $A_{21} \cup \Delta'$, say the former, cuts off a solid torus V with a trivial arc from $\mathcal{C}_{21}$ and that the interior of V is disjoint from Σ (see Figure 16). Since $\Sigma \cap C_{22}$ also contains an annulus component, we can obtain a compressing disk D_{22} of $\mathcal{C}_{22}$ by an appropriate ∂–compression for a component of $\Sigma \cap C_{22}$. If V is not affected by the ∂–compression, then this implies that $\mathcal{S}_2$ is weakly reducible, a contradiction. Hence after the ∂–compression, A_{21} is joined to Δ with a band. Let V' be the solid torus obtained by cutting C_{21} along A_{21} joined to Δ with a band. We notice that V' is a submanifold of V. Attaching $\eta(D_{22}; C_{22})$ to V', we obtain a 3–ball B with a single arc. If the arc is trivial in B, then we can isotope Σ to delete A_{21} and Δ as components of $\Sigma \cap C_{21}$, contradicting the minimality of $|\Sigma \cap S_2|$. Hence the arc contained in B is non-trivial and therefore K contains a non-trivial 2–bridge knot as a connected summand. Thus we have the conclusion (2) of Theorem 7.8.

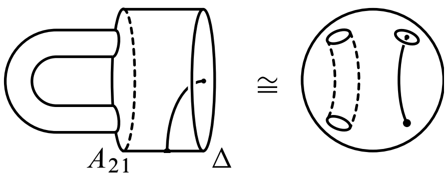

Figure 16: $A_{21} \cup \Delta$ cuts off a solid torus with a trivial arc from V

We next suppose that Δ and Δ' satisfy the condition (ii). Then $\Sigma \cap C_{21}$ contains no annulus component and hence $\Sigma \cap C_{22}$ consists of an annulus A'_{22}. We notice that $\Delta \cup \Delta'$ cuts off a 3–ball with two trivial arcs from $\mathcal{C}_{21}$ and that A'_{22} is obtained by joining a separating compressing disk D_{22} of $\mathcal{C}_{22}$ to itself with a band. If D_{22} separates C_{22} into two solid tori, then we see that $\mathcal{S}_2$ is weakly reducible, a contradiction. Hence D_{22} separates $\mathcal{C}_{22}$ into a genus two handlebody and a 3–ball with a trivial arc. This implies that A'_{22} cuts off a solid torus with a trivial arc as illustrated at the right side of Figure 16. Therefore K contains a non-trivial 2–bridge knot as a connected summand and hence we have the conclusion (2) of Theorem 7.8.

This completes the proof of Theorem 7.8. □

Lemma 7.9 *Let K be a knot in S^3 and $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{S})$ a $(2,2)$–bridge splitting of (S^3, K). Suppose that there are compressing disks D_1 and D_2 of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively and a non-separating cut disk E_2 of $\mathcal{C}_2$ such that $\partial D_1 \cap \partial(D_2 \cup E_2) = \varnothing$, D_1 is non-separating, D_2 cuts off a 3–ball with a trivial arc from $\mathcal{C}_2$ and E_2 is disjoint from D_2. Then one of the following holds.*

1. *$\mathcal{S}$ is meridionally stabilized.*
2. *$K = K_1 \# K_2$ such that K_1 admits a $(0,2)$–bridge position and K_2 admits a $(2,1)$–bridge position.*
3. *$K = K_1 \# K_2$ such that K_1 admits a $(1,1)$–bridge position and K_2 admits a $(1,2)$–bridge position.*
4. *There is a c–weak reduction yielding a 2–sphere which intersects K in four points and is incompressible in (S^3, K).*

Proof By the c–weak reduction with respect to $(D_1, D_2 \cup E_2)$, we obtain $\mathcal{M}_i = \mathcal{C}_{i1} \cup \mathcal{C}_{i2}$, $\mathcal{S}_i = \mathcal{C}_{i1} \cap \mathcal{C}_{i2}$ for each $i = 1, 2$ and $\mathcal{F} = \partial_- \mathcal{C}_{12} = \partial_- \mathcal{C}_{21}$. We notice that $\mathcal{F}$ is a 2–sphere intersecting K in four points (see Figure 17). If both $\mathcal{M}_1$ and $\mathcal{M}_2$ are ∂–irreducible, then we have the conclusion (4) of Lemma 7.9. Hence we assume that $\mathcal{M}_1$ or $\mathcal{M}_2$ is ∂–reducible.

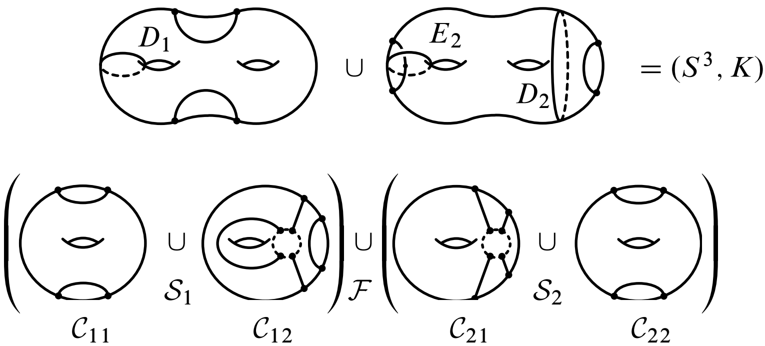

Figure 17: The c–weak reduction with respect to $(D_1, D_2 \cup E_2)$

Suppose that $\mathcal{M}_1$ is ∂–reducible. Then there is a ∂–reducing disk $\bar{D}_1$ with $|\bar{D}_1 \cap S_1| = 1$ by Lemma 2.10, and there is a compressing disk D_{12} of $\mathcal{C}_{12}$ with $D_{12} \cap \bar{D}_1 = \varnothing$ by Corollary 2.11. It follows from Lemma 2.4 that D_{12} is isotopic to $\hat{D}_2$ in $\mathcal{C}_{12}$, where $\hat{D}_2$ is a compressing disk of $\mathcal{C}_{12}$ which is obtained by extending D_2 naturally. Hence we see that $\bar{D}_1 \cap \hat{D}_2 = \varnothing$. The disk $\bar{D}_1$ can be regarded as a compressing disk of $\mathcal{C}_1$ which is disjoint from D_1 and is not parallel to D_1. We notice that $\partial(D_1 \cup \bar{D}_1) \cap \partial D_2 = \varnothing$. Hence we have the conclusion (1) or (2) of Lemma 7.9 (see Figure 6).

Suppose that $\mathcal{M}_2$ is ∂–reducible. Then there is a ∂–reducing disk $\bar{D}_2$ with $|\bar{D}_2 \cap S_2| = 1$ by Lemma 2.10, and there is a compressing disk D_{21} of $\mathcal{C}_{21}$ with $D_{21} \cap \bar{D}_2 = \varnothing$ by Corollary 2.11. Since we may assume that D_{21} is non-separating in C_{21}, it follows from Lemma 2.4 that D_{21} is isotopic to D_1 in $\mathcal{C}_{21}$. Hence we see that $D_1 \cap \bar{D}_2 = \varnothing$. The disk $\bar{D}_2$ can be regarded as a compressing disk of $\mathcal{C}_2$ which is disjoint from $D_2 \cup E_2$ and is parallel neither to D_2 nor to E_2. We notice that $\partial D_1 \cap \partial(D_2 \cup E_2 \cup \bar{D}_2) = \varnothing$. Hence we have the conclusion (3) of Lemma 7.9. □

Proof of Theorem 1.4 The proof for (2) of Theorem 1.4 is quite similar to that for (1). Hence we give a proof only for (1) of Theorem 1.4. Let K_i $(i = 1, 2, 3)$ be knots in S^3 with $K_i \in \mathcal{K}_2^1$. We notice that $\mathrm{hg}(K_1 \# K_2) = 3$ and $K_1 \# K_2 \notin \mathcal{K}_3^3$ (see Observation 1.3). It follows from Observation 1.3, Theorems 4.1 and 4.2 that $K_1 \# K_2$ cannot admit a $(1,2)$–bridge position, that is, $K_1 \# K_2 \notin \mathcal{K}_3^2$. On the other hand, we see that $K_1 \# K_2$ admits a $(2,1)$–bridge position by Proposition 2.14. Hence we see that $K_1 \# K_2 \in \mathcal{K}_3^1$.

We now consider meridional destabilizing number of $K_1 \# K_2 \# K_3$. We notice that $\mathrm{hg}(K_1 \# K_2 \# K_3) = 4$ and $K_1 \# K_2 \# K_3 \notin \mathcal{K}_4^4$ (see Observation 1.3). Suppose first that $K_1 \# K_2 \# K_3$ admits a $(1,3)$–bridge position. Since each K_i is n–string prime for all n (see Gordon and Reid [3, Corollary 1.2]), we have the conclusion (2) or (4) of

Theorem 5.2 by Section 5. If the conclusion (2) of Theorem 5.2 holds, then $K_i \# K_j$, say $(i,j)=(1,2)$, must admit a $(0,3)$–bridge position because $K_3 \in \mathcal{K}_2^1$. This, however, implies that K_1 admits a $(0,2)$–bridge position (see Observation 1.3), contradicting $K_1 \in \mathcal{K}_2^1$. It follows from Theorem 7.4 that the conclusion (4) of Theorem 5.2 is impossible, because each K_i is n–string prime for all n. Thus we see that $K_1 \# K_2 \# K_3$ does not admit a $(1,3)$–bridge position, that is, $K_1 \# K_2 \# K_3 \notin \mathcal{K}_4^3$. Suppose next that $K_1 \# K_2 \# K_3$ admits a $(2,2)$–bridge position. Then by Section 6, we have the conclusion (3) of Theorem 6.1, the conclusion (2), (4) or (5) of Theorem 6.2. If the conclusion (2) of Theorem 6.2 holds, then $K_i \# K_j$, say $(i,j)=(1,2)$, must admit a $(1,2)$–bridge position, contradicting $K_1, K_2 \in \mathcal{K}_2^1$ by Section 4. The conclusion (3) of Theorem 6.1 is impossible by Theorem 7.5. If the conclusion (4) or (5) of Theorem 6.2 holds, then we see that $K_1 \# K_2 \# K_3$ contains a non-trivial 2–bridge knot as a connected summand by Theorems 7.6–7.8, a contradiction. Hence we see that $K_1 \# K_2 \# K_3$ does not admit a $(2,2)$–bridge position, that is, $K_1 \# K_2 \# K_3 \notin \mathcal{K}_4^2$. On the other hand, we see that $K_1 \# K_2 \# K_3$ admits a $(3,1)$–bridge position by Proposition 2.14. Therefore we see that $K_1 \# K_2 \# K_3 \in \mathcal{K}_4^1$. □

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