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A Thomason model structure on the category of small n–fold categories

THOMAS M FIORE
SIMONA PAOLI

We construct a cofibrantly generated Quillen model structure on the category of small n–fold categories and prove that it is Quillen equivalent to the standard model structure on the category of simplicial sets. An n–fold functor is a weak equivalence if and only if the diagonal of its n–fold nerve is a weak equivalence of simplicial sets. This is an n–fold analogue to Thomason's Quillen model structure on **Cat**. We introduce an n–fold Grothendieck construction for multisimplicial sets, and prove that it is a homotopy inverse to the n–fold nerve. As a consequence, we completely prove that the unit and counit of the adjunction between simplicial sets and n–fold categories are natural weak equivalences.

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1 Introduction

An n–fold category is a higher and wider categorical structure obtained by n applications of the internal category construction. In this paper we study the homotopy theory of n–fold categories. Our main result is Theorem 9.28. Namely, we have constructed a cofibrantly generated model structure on the category of small n–fold categories in which an n–fold functor is a weak equivalence if and only if its nerve is a diagonal weak equivalence, this model structure is Quillen equivalent to the standard one on the category of simplicial sets, and the unit and counit of the Quillen equivalence are natural weak equivalences. As a consequence, topological spaces have combinatorial models in terms of n–fold categories. Our main tools are model category theory, the n–fold nerve and an n–fold Grothendieck construction for multisimplicial sets. Notions of nerve and versions of the Grothendieck construction are very prominent in homotopy theory and higher category theory, as we now explain. The Thomason model structure on **Cat** is also often present, at least implicitly.

The Grothendieck nerve of a category and the Grothendieck construction for functors are fundamental tools in homotopy theory. Theorems A and B of Quillen [75], and Thomason's theorem [84] on Grothendieck constructions as models for certain

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homotopy colimits, are still regularly applied decades after their creation. Functors with nerves that are weak equivalences of simplicial sets feature prominently in these theorems. Such functors form the weak equivalences of Thomason's model structure on **Cat** [85], which is Quillen equivalent to **SSet**. Earlier, Illusie [46] proved that the nerve and the Grothendieck construction are homotopy inverses. Although the nerve and the Grothendieck construction are not adjoints[1], the equivalence of homotopy categories can be realized by adjoint functors; see Fritsch–Latch [26; 27] and Thomason [85]. Related results on homotopy inverses are those of Latch [61], Lee [62] and Waldhausen [87]. More recently, Cisinski [13] has proved two conjectures of Grothendieck concerning this circle of ideas; see also Jardine [48].

On the other hand, notions of nerve play an important role in various definitions of n–category (see Leinster's survey [63]), namely in the definitions of Simpson [80], Street [82] and Tamsamani [83], as well as in the theory of quasi-categories developed by Joyal [49; 50; 51] and also Lurie [67; 68]. For notions of nerve for bicategories, see for example Duskin [16; 17] and Lack–Paoli [60], and for left adjoints to singular functors in general see also Gabriel–Zisman [29] and Kelly [56]. Fully faithful cellular nerves have been developed for higher categories by Berger [3], together with characterizations of their essential images. Nerve theorems can be established in a very general context, as proved by Leinster [64] and Weber [88] and discussed in the n–Category Café by Leinster and others [65]. As an example, Kock proves in [57] a nerve theorem for polynomial endofunctors in terms of trees.

Model category techniques are only becoming more important in the theory of *higher* categories. They have been used to prove that, in a precise sense, simplicial categories, Segal categories, complete Segal spaces and quasi-categories are all equivalent models for $(\infty,1)$–categories; see Bergner [6; 4; 5], Joyal–Tierney [54], Rezk [77] and Toën [86]. In other directions, although the cellular nerve of C Berger [3] does not transfer a model structure from cellular sets to ω–categories, Berger proves in [3] that the homotopy category of cellular sets is equivalent to the homotopy category of ω–categories. For this, a Quillen equivalence between *cellular spaces* and *simplicial ω–categories* is constructed. There is also the work of Pellisier [74] and Simpson [80; 81], developing model structures on n–categories for the purpose of n–stacks, and also a model structure for (∞,n)–categories.

[1] In fact, the Grothendieck construction is not even homotopy equivalent to c, the left adjoint to the nerve, as follows. For any simplicial set X, let Δ/X denote the Grothendieck construction on X. Then $N(\Delta/\partial\Delta[3])$ is homotopy equivalent to $\partial\Delta[3]$ by Illusie's result. On the other hand, $Nc\partial\Delta[3]=Nc\Delta[3]=\Delta[3]$, since cX only depends on 0–, 1– and 2–simplices. Clearly, $\partial\Delta[3]$ and $\Delta[3]$ are not homotopy equivalent, so the Grothendieck construction is not naturally homotopy equivalent to c.

In low dimensions several model structures have already been investigated. On **Cat**, there is the categorical structure of Joyal–Tierney [53], Rezk [76], as well as the topological structure of Thomason [85], corrected by Cisinksi in [12]. A model structure on pro-objects in **Cat** was proved by Golasinski [31; 32; 33]. The articles of Heggie [40; 41; 42] are closely related to the Thomason structure and the Thomason homotopy colimit theorem. More recently, the Thomason structure on **Cat** was proved by Cisinski [13, Theorem 5.2.12] in the context of Grothendieck test categories and fundamental localizers. The homotopy categories of spaces and categories are proved equivalent by del Hoyo [45] without using model categories.

On **2-Cat** there is the categorical structure of Lack [58; 59], as well as the Thomason structure of Worytkiewicz–Hess–Parent–Tonks [89]. Model structures on **2FoldCat** have been studied by Fiore, Paoli and Pronk [25] in great detail. The homotopy theory of 2–fold categories is very rich, since there are numerous ways to view 2–fold categories: as internal categories in **Cat**, as certain simplicial objects in **Cat** or as algebras over a 2–monad. Fiore, Paoli and Pronk associate in [25] a model structure to each point of view, and compare these model structures.

However, there is another way to view 2–fold categories not treated by Fiore–Paoli–Pronk [25], namely as certain bisimplicial sets. There is a natural notion of fully faithful double nerve, which associates to a 2–fold category a bisimplicial set. An obvious question is: does there exist a Thomason-like model structure on **2FoldCat** that is Quillen equivalent to some model structure on bisimplicial sets via the double nerve? Unfortunately, the left adjoint to double nerve is homotopically poorly behaved as it extends the left adjoint c to ordinary nerve, which is itself poorly behaved. So any attempt at a model structure must address this issue.

Fritsch and Latch [26; 27] and Thomason [85] noticed that the composite of c with second barycentric subdivision Sd^2 is much better behaved than c alone. In fact, Thomason used the adjunction $c\mathrm{Sd}^2 \dashv \mathrm{Ex}^2 N$ to construct his model structure on **Cat**. This adjunction is a Quillen equivalence, as the right adjoint preserves weak equivalences and fibrations by definition, and the unit and counit are natural weak equivalences.

Following this lead, we move from bisimplicial sets to simplicial sets via δ^* (restriction to the diagonal) in order to correct the homotopy type of double categorification using Sd^2. Moreover, our method of proof works for n–fold categories as well, so we shift our focus from 2–fold categories to general n–fold categories. In this paper, we construct a cofibrantly generated model structure on **nFoldCat** using the fully faithful

n–fold nerve and the adjunction below,

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "\mathrm{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathrm{Ex}^2"] \arrow[r, bend left, "\delta_!"] \arrow[r, phantom, "\perp"] & \mathbf{SSet^n} \arrow[l, bend left, "\delta^*"] \arrow[r, bend left, "c^n"] \arrow[r, phantom, "\perp"] & \mathbf{nFoldCat} \arrow[l, bend left, "N^n"] \end{tikzcd} \tag{1}$$

prove that this Quillen adjunction is a Quillen equivalence, and show that the unit and counit are natural weak equivalences. Our method for the Quillen adjunction is to apply Kan's Lemma on Transfer of Structure. First we prove Thomason's classical theorem in Theorem 6.3, and then use this proof as a basis for the general n–fold case in Theorem 8.2. We introduce the n–fold Grothendieck construction in Definition 9.1, prove that it is homotopy inverse to the n–fold nerve in Theorems 9.21 and 9.22, and conclude in Proposition 9.27 that the Quillen adjunction (1) is a Quillen equivalence and the unit and counit are natural weak equivalences. In a different way, Fritsch and Latch proved that the unit and counit of the classical Thomason adjunction $\mathbf{SSet} \dashv \mathbf{Cat}$ are natural weak equivalences in [26; 27].

Recent interest in n–fold categories has focused on the $n = 2$ case. In many cases, this interest stems from the fact that 2–fold categories provide a good context for incorporating two types of morphisms, and this is useful for applications. For example, between rings there are ring homomorphisms and bimodules, between topological spaces there are continuous maps and parametrized spectra as in May–Sigurdsson [70], between manifolds there are smooth maps and cobordisms, and so on. In this direction, see for example Grandis–Paré [36], Fiore [23; 24], Morton [72] and Shulman [78; 79]. C Ehresmann originally introduced 2–fold categories under the name *double categories* [21; 22], and his pioneering works with A Ehresmann include [2; 18; 19; 20]. The theory of double categories is now flourishing, with many contributions by Brown and Mosa [11], Grandis and Paré [36; 37; 38; 39], Dawson, Paré and Pronk [15], Dawson and Paré [14], Fiore, Paoli and Pronk [25] and Shulman [78; 79] and many others – these are only a few examples.

There has also been interest in general n–fold categories from various points of view. Connected homotopy $(n+1)$–types are modelled by n–fold categories internal to the category of groups in the work of Loday [66], as summarized in the survey paper of Paoli [73]. Edge symmetric n–fold categories have been studied by Brown and Higgins [7; 8; 9; 10] and others for many years now. There are also the more recent *symmetric weak cubical categories* of Grandis [35; 34]. The homotopy theory of cubical sets has been studied by Jardine [47].

The present article is the first to consider a Thomason structure on the category of n–fold categories. Our paper is organized as follows. Section 2 recalls n–fold categories, introduces the n–fold nerve N^n and its left adjoint n–fold categorification c^n, and describes how c^n interacts with $\delta_!$, the left adjoint to precomposition with the diagonal. In Section 3 we recall barycentric subdivision, including explicit descriptions of $\mathrm{Sd}^2\,\Lambda^k[m]$, $\mathrm{Sd}^2\,\partial\Delta[m]$ and $\mathrm{Sd}^2\,\Delta[m]$. More importantly, we present a decomposition of the poset $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ into the union of three posets **Comp**, **Center** and **Outer** in Proposition 3.10, as pictured in Figure 1 for $m = 2$ and $k = 1$. Though Section 3 may appear technical, the statements become clear after a brief look at the example in Figure 1. This section is the basis for the verification of the pushout axiom (iv) of Corollary 6.1, completed in the proofs of Theorem 6.3 and Theorem 8.2.

Section 4 and Section 5 make further preparations for the verification of the pushout axiom. Proposition 4.3 gives a deformation retraction of $|N(\mathbf{Comp} \cup \mathbf{Center})|$ to part of its boundary; see Figure 1. This deformation retraction finds application in Equation (17). The highlights of Section 5 are Proposition 5.1 and Corollary 5.5 on the commutation of nerve with certain colimits of posets. Proposition 5.1 on commutation of nerve with certain pushouts finds application in Equation (17). Other highlights of Section 5 are Proposition 5.3, Proposition 5.4 and Corollary 5.9 on the expression of certain posets (respectively their nerves) as a colimit of two ordinals (respectively two standard simplices). Section 6 pulls these results together and quickly proves the classical Thomason theorem.

Section 7 proves the n–fold versions of the results in Sections 3, 4 and 5. The n–fold version of Proposition 5.3 on colimit decompositions of certain posets is Proposition 7.4. The n–fold version of Corollary 5.5 on the commutation of nerve with certain colimits of posets is Proposition 7.13. The n–fold version of the deformation retraction in Proposition 4.3 is Corollary 7.14. The n–fold version of Proposition 5.1 on commutation of nerve with certain pushouts is Proposition 7.18. Proposition 7.15 displays a calculation of a pushout of double categories, and the diagonal of its nerve is characterized in Proposition 7.16.

Section 8 pulls together the results of Section 7 to prove the Thomason structure on **nFoldCat** in Theorem 8.2. In Section 9, we introduce a Grothendieck construction for multisimplicial sets and prove that it is a homotopy inverse for n–fold nerve in Theorems 9.21 and 9.22. As a consequence, we have in Proposition 9.27 that the Quillen adjunction in (1) is a Quillen equivalence, and the unit and counit are natural weak equivalences.

Section 10 is an appendix on the Multidimensional Eilenberg–Zilber Lemma.

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2 n–Fold categories

In this section we quickly recall the inductive definition of n–fold category, present an equivalent combinatorial definition of n–fold category, discuss completeness and cocompleteness of **nFoldCat**, introduce the n–fold nerve N^n, prove the existence of its left adjoint c^n, and recall the adjunction $\delta_! \dashv \delta^*$.

Definition 2.1 A *small n–fold category* $\mathbb{D} = (\mathbb{D}_0, \mathbb{D}_1)$ is a category object in the category of small $(n-1)$–fold categories. In detail, $\mathbb{D}_0$ and $\mathbb{D}_1$ are $(n-1)$–fold categories equipped with $(n-1)$–fold functors

$$\begin{tikzcd} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\circ"] & \mathbb{D}_1 \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \mathbb{D}_0 \arrow[l, "u"'] \end{tikzcd}$$

that satisfy the usual axioms of a category. We denote the category of n–fold categories by **nFoldCat**.

Since we will always deal with small n–fold categories, we leave off the adjective "small". Also, all of our n–fold categories are strict. The following equivalent combinatorial definition of n–fold category is more explicit than the inductive definition. The combinatorial definition will only be needed in a few places, so the reader may skip the combinatorial definition if it appears more technical than one's taste.

Definition 2.2 The data for an *n–fold category* $\mathbb{D}$ are

(i) sets $\mathbb{D}_\epsilon$, one for each $\epsilon \in \{0,1\}^n$,

(ii) for every $1 \leq i \leq n$ and $\epsilon' \in \{0,1\}^n$ with $\epsilon'_i = 1$ we have *source* and *target* functions
$$s^i, t^i \colon \mathbb{D}_{\epsilon'} \longrightarrow \mathbb{D}_\epsilon$$
where $\epsilon \in \{0,1\}^n$ satisfies $\epsilon_i = 0$ and $\epsilon_j = \epsilon'_j$ for all $j \neq i$ (for ease of notation we do not include ϵ' in the notation for s^i and t^i, despite the ambiguity),

(iii) for every $1 \leq i \leq n$ and $\epsilon, \epsilon' \in \{0,1\}^n$ with $\epsilon_i = 0$, $\epsilon'_i = 1$, and $\epsilon_j = \epsilon'_j$ for all $j \neq i$, we have a *unit* $u^i \colon \mathbb{D}_\epsilon \longrightarrow \mathbb{D}_{\epsilon'}$,

(iv) for every $1 \leq i \leq n$ and $\epsilon, \epsilon' \in \{0,1\}^n$ with $\epsilon_i = 0$, $\epsilon'_i = 1$, and $\epsilon_j = \epsilon'_j$ for all $j \neq i$, we have a *composition*
$$\mathbb{D}_{\epsilon'} \times_{\mathbb{D}_\epsilon} \mathbb{D}_{\epsilon'} \xrightarrow{\circ^i} \mathbb{D}_{\epsilon'}.$$

To form an *n–fold category*, these data are required to satisfy the following axioms.

(i) *Compatibility of source and target*: For all $1 \leq i \leq n$ and all $1 \leq j \leq n$,
$$s^i s^j = s^j s^i$$
$$t^i t^j = t^j t^i$$
$$s^i t^j = t^j s^i$$
whenever these composites are defined.

(ii) *Compatibility of units with units*: For all $1 \leq i \leq n$ and all $1 \leq j \leq n$,
$$u^i u^j = u^j u^i$$
whenever these composites are defined.

(iii) *Compatibility of units with source and target*: For all $1 \leq i \leq n$ and all $1 \leq j \leq n$,
$$s^i u^j = u^j s^i$$
$$t^i u^j = u^j t^i$$
whenever these composites are defined.

(iv) *Categorical structure*: For every $1 \leq i \leq n$ and $\epsilon, \epsilon' \in \{0,1\}^n$ with $\epsilon_i = 0$, $\epsilon'_i = 1$, and $\epsilon_j = \epsilon'_j$ for all $j \neq i$, the diagram in **Set**

$$\begin{tikzcd} \mathbb{D}_{\epsilon'} \times_{\mathbb{D}_\epsilon} \mathbb{D}_{\epsilon'} \arrow[r, "\circ^i"] & \mathbb{D}_{\epsilon'} \arrow[r, bend left, "s^i"] \arrow[r, bend right, "t^i"'] & \mathbb{D}_\epsilon \arrow[l, "u^i"'] \end{tikzcd}$$

is a category.

(v) *Interchange law*: For every $i \neq j$ and every $\epsilon \in \{0,1\}^n$ with $\epsilon_i = 1 = \epsilon_j$, the compositions $\circ^i$ and $\circ^j$ can be interchanged, that is, if $w, x, y, z \in \mathbb{D}_\epsilon$, and

$$t^i(w) = s^i(x), \quad t^i(y) = s^i(z),$$
$$t^j(w) = s^j(y), \quad t^j(x) = s^j(z),$$

$$\begin{array}{cc} & \xrightarrow{i} \\ j \downarrow & \end{array} \qquad \begin{array}{|c|c|} \hline w & x \\ \hline y & z \\ \hline \end{array}$$

then $(z \circ^j y) \circ^i (x \circ^j w) = (z \circ^i x) \circ^j (y \circ^i w)$.

We define $|\epsilon|$ to be the number of 1's in ϵ, that is

$$|\epsilon| := |\{1 \leq i \leq n \mid \epsilon_i = 1\}| = \sum_{i=1}^n \epsilon_i.$$

If $k = |\epsilon|$, an element of $\mathbb{D}_\epsilon$ is called a k–*cube*.

Remark 2.3 If $\mathbb{D}_\epsilon = \mathbb{D}_{\epsilon'}$ for all $\epsilon, \epsilon' \in \{0,1\}^n$ with $|\epsilon| = |\epsilon'|$, then the data (i), (ii), (iii) satisfying axioms (i), (ii), (iii) are an n–truncated *cubical complex* in the sense of Section 1 of [9]. Compositions and the interchange law are also similar. The situation of [9] is *edge symmetric* in the sense that $\mathbb{D}_\epsilon = \mathbb{D}_{\epsilon'}$ for all $\epsilon, \epsilon' \in \{0,1\}^n$ with $|\epsilon| = |\epsilon'|$, and the $|\epsilon|$ compositions on $\mathbb{D}_\epsilon$ coincide with the $|\epsilon'|$ compositions on $\mathbb{D}_{\epsilon'}$. In the present article we study the non-edge-symmetric case, in the sense that we do *not* require $\mathbb{D}_\epsilon$ and $\mathbb{D}_{\epsilon'}$ to coincide when $|\epsilon| = |\epsilon'|$, and hence, the $|\epsilon|$ compositions on $\mathbb{D}_\epsilon$ are not required to be the same as the $|\epsilon'|$ compositions on $\mathbb{D}_{\epsilon'}$.

Remark 2.4 The generalized interchange law follows from the interchange law in (v). For example, if we have eight compatible 3–dimensional cubes arranged as a 3–dimensional cube, then all possible ways of composing these eight cubes down to one cube are the same.

Proposition 2.5 *The inductive notion of n–fold category in Definition 2.1 is equivalent to the combinatorial notion of n–fold category in Definition 2.2 in the strongest possible sense: the categories of such are equivalent.*

Proof For $n = 1$ the categories are clearly the same. Suppose the proposition holds for $n - 1$ and call the categories $\mathbf{(n{-}1)FoldCat(ind)}$ and $\mathbf{(n{-}1)FoldCat(comb)}$. Then internal categories in $\mathbf{(n{-}1)FoldCat(ind)}$ are equivalent to internal categories in $\mathbf{(n{-}1)FoldCat(comb)}$, while internal categories in $\mathbf{(n{-}1)FoldCat(comb)}$ are the same as $\mathbf{nFoldCat(comb)}$. □

A 2–fold category, that is, a category object in **Cat**, is precisely a *double category* in the sense of Ehresmann. A double category consists of a set $\mathbb{D}_{00}$ of *objects*, a set $\mathbb{D}_{01}$ of *horizontal morphisms*, a set $\mathbb{D}_{10}$ of *vertical morphisms*, and a set $\mathbb{D}_{11}$ of *squares* equipped with various sources, targets, and associative and unital compositions satisfying the interchange law. Several homotopy theories for double categories were considered by Fiore, Paoli and Pronk [25].

Example 2.6 There are various standard examples of double categories. To any category, one can associate the double category of commutative squares. Any 2–category can be viewed as a double category with trivial vertical morphisms or as a double category with trivial horizontal morphisms. To any 2–category, one can also associate the double category of *quintets*: a square is a square of morphisms inscribed with a 2–cell in a given direction.

Example 2.7 In nature, one often finds *pseudo double categories*. These are like double categories, except one direction is a bicategory rather than a 2–category (see Grandis–Paré [36] for a more precise definition). For example, one may consider 1–manifolds, 2–cobordisms, smooth maps, and appropriate squares. Another example is rings, bimodules, ring maps, and twisted equivariant maps. For these examples and more, see Grandis–Paré [36], Fiore [24] and other articles on double categories listed in the introduction.

Example 2.8 Any n–category is an n–fold category in numerous ways, just like a 2–category can be considered as a double category in several ways.

An important method of constructing n–fold categories from n ordinary categories is the external product, which is compatible with the external product of simplicial sets. This was called the *square product* on page 251 of [2].

Definition 2.9 If $\mathbf{C}_1, \dots, \mathbf{C}_n$ are small categories, the *external product* $\mathbf{C}_1 \boxtimes \cdots \boxtimes \mathbf{C}_n$ is an n–fold category with object set $\operatorname{Obj} \mathbf{C}_1 \times \cdots \times \operatorname{Obj} \mathbf{C}_n$. Morphisms in the i–th direction are n–tuples $(f_1, \dots, f_n)$ of morphisms in $\mathbf{C}_1 \times \cdots \times \mathbf{C}_n$ where all but the i–th entry are identities. Squares in the ij–plane are n–tuples where all entries are identities except the i–th and j–th entries, and so on. An n–cube is an n–tuple of morphisms, possibly all nonidentity morphisms.

Proposition 2.10 *The category* **nFoldCat** *is locally finitely presentable.*

Proof We prove this by induction. The category **Cat** of small categories is known to be locally finitely presentable (see for example Gabriel–Ulmer [28]). Assume **(n−1)FoldCat** is locally finitely presentable. The category **nFoldCat** is the category of models in **(n−1)FoldCat** for a sketch with finite diagrams. Since **(n−1)FoldCat** is locally finitely presentable, we conclude from Adámek–Rosický [1, Proposition 1.53] that **nFoldCat** is also locally finitely presentable. □

Proposition 2.11 *The category* **nFoldCat** *is complete and cocomplete.*

Proof Completeness follows quickly, because **nFoldCat** is a category of algebras. For example, the adjunction between n–fold graphs and n–fold categories is monadic by the Beck Monadicity Theorem. This means that the algebras for the induced monad are precisely the n–fold categories.

The category **nFoldCat** is cocomplete because it is locally finitely presentable. □

The colimits of certain k–fold subcategories are the k–fold subcategories of the colimit. To prove this, we introduce some notation.

Notation 2.12 Let $\leq$ denote the lexicographic order on $\{0,1\}^n$, and let $\overline{k} \in \{0,1\}^n$ with $k = |\overline{k}|$. The forgetful functor

$$U_{\overline{k}} \colon \mathbf{nFoldCat} \longrightarrow \mathbf{kFoldCat}$$

assigns to an n–fold category $\mathbb{D}$ the k–fold category consisting of those sets $\mathbb{D}_\epsilon$ with $\epsilon \leq \overline{k}$ and all the source, target, and identity maps of $\mathbb{D}$ between them. If we picture $\mathbb{D}$ as an n–cube with $\mathbb{D}_\epsilon$'s at the vertices and source, target, identity maps on the edges, then the k–fold subcategory $U_{\overline{k}}(\mathbb{D})$ is a k–face of this n–cube. For example, if $n = 2$ and $k = 1$, then $U_{\overline{k}}(\mathbb{D})$ is either the horizontal or vertical subcategory of the double category $\mathbb{D}$.

Proposition 2.13 *The forgetful functor* $U_{\overline{k}}$: **nFoldCat**$\longrightarrow$**kFoldCat** *admits a right adjoint* $R_{\overline{k}}$, *and thus preserves colimits: for any functor* F *into* **nFoldCat** *we have*

$$U_{\overline{k}}\,(\operatorname{colim} F) = \operatorname{colim} U_{\overline{k}} F.$$

Proof For a k–fold category $\mathbb{E}$, the n–fold category $R_{\overline{k}}\mathbb{E}$ has $U_{\overline{k}}R_{\overline{k}}\mathbb{E} = \mathbb{E}$, in particular the objects of $R_{\overline{k}}\mathbb{E}$ are the same as the objects of $\mathbb{E}$. The other cubes are defined inductively. If $k_i = 0$, then $R_{\overline{k}}\mathbb{E}$ has a unique morphism (1–cube) in direction i between any two objects. Suppose the j–cubes of $R_{\overline{k}}\mathbb{E}$ have already been defined, that is $\left(R_{\overline{k}}\mathbb{E}\right)_\epsilon$ has been defined for all ϵ with $|\epsilon| = j$. For any ϵ with $|\epsilon| = j+1$ and $\epsilon \not\leq \overline{k}$, there is a unique element of $\left(R_{\overline{k}}\mathbb{E}\right)_\epsilon$ for each boundary of j–cubes.

The natural bijection

$$\mathbf{kFoldCat}(U_{\overline{k}}\mathbb{D}, \mathbb{E}) \cong \mathbf{nFoldCat}(\mathbb{D}, R_{\overline{k}}\mathbb{E})$$

is given by uniquely extending k–fold functors defined on $U_{\overline{k}}\mathbb{D}$ to n–fold functors into $R_{\overline{k}}\mathbb{E}$. □

We next introduce the n–fold nerve functor, prove that it admits a left adjoint, and also prove that an n–fold natural transformation gives rise to a simplicial homotopy after pulling back along the diagonal.

Definition 2.14 The *n–fold nerve* of an n–fold category $\mathbb{D}$ is the multisimplicial set $N^n\mathbb{D}$ with $\overline{p}$–simplices

$$(N^n\mathbb{D})_{\overline{p}} := \mathrm{Hom}_{\mathbf{nFoldCat}}([p_1] \boxtimes \cdots \boxtimes [p_n], \mathbb{D}).$$

A $\overline{p}$–simplex is a $\overline{p}$–array of composable n–cubes.

Remark 2.15 The n–fold nerve is the same as iterating the nerve construction for internal categories n times.

Example 2.16 The n–fold nerve is compatible with external products:

$$N^n(\mathbf{C}_1 \boxtimes \cdots \boxtimes \mathbf{C}_n) = N\mathbf{C}_1 \boxtimes \cdots \boxtimes N\mathbf{C}_n.$$

In particular,

$$N^n([m_1] \boxtimes \cdots \boxtimes [m_n]) = \Delta[m_1] \boxtimes \cdots \boxtimes \Delta[m_n] = \Delta[m_1, \ldots, m_n].$$

Proposition 2.17 *The functor* N^n: **nFoldCat**$\longrightarrow$$\mathbf{SSet^n}$ *is fully faithful.*

Proof We proceed by induction. For $n = 1$, the usual nerve functor is fully faithful.

Consider now $n > 1$, and suppose

$$N^{n-1}\colon \mathbf{(n{-}1)FoldCat} \longrightarrow \mathbf{SSet^{n-1}}$$

is fully faithful. We have a factorization

$$\begin{tikzcd} \mathsf{Cat}(\mathbf{(n{-}1)FoldCat}) \arrow[r, "N"'] \arrow[rr, bend left, "N^n"] & {[\Delta^{\mathrm{op}}, \mathbf{(n{-}1)FoldCat}]} \arrow[r, "N_*^{n-1}"'] & \left[\Delta^{\mathrm{op}}, \mathbf{SSet^{n-1}}\right], \end{tikzcd}$$

where the brackets mean functor category. The functor N is faithful, as $(NF)_0$ and $(NF)_1$ are F_0 and F_1. It is also full, for if $F'\colon N\mathbb{D} \longrightarrow N\mathbb{E}$, then F_0' and F_1' form an n–fold functor with nerve F' (compatibility of F' with the inclusions $e_{i,i+1}\colon [1] \longrightarrow [m]$ determines F_m' from F_0' and F_1').

The functor N_*^{n-1} is faithful, since it is faithful at every degree by hypothesis. If $(G_m')_m\colon (N^{n-1}\mathbb{D}_m)_m \longrightarrow (N^{n-1}\mathbb{E}_m)_m$ is a morphism in $\left[\Delta^{\mathrm{op}}, \mathbf{SSet^{n-1}}\right]$, there exist $(n{-}1)$–fold functors G_m such that $N^{n-1}G_m = G_m'$, and these are compatible with the structure maps for $(\mathbb{D}_m)_m$ and $(\mathbb{E}_m)_m$ by the faithfulness of N^{n-1}. So N_*^{n-1} is also full.

Finally, $N^n = N^{n-1} \circ N$ is a composite of fully faithful functors.

This proposition can also be proved using the Nerve Theorem of Weber [88, Nerve Theorem 4.10]. We will present a direct proof in the case $n = 2$ in a future paper. □

Proposition 2.18 *The n–fold nerve functor N^n admits a left adjoint c^n called fundamental n–fold category or n–fold categorification.*

Proof The functor N^n is defined as the singular functor associated to an inclusion. Since **nFoldCat** is cocomplete, a left adjoint to N^n is obtained by left Kan extending along the Yoneda embedding. This is the Lemma from Kan about singular-realization adjunctions. □

Example 2.19 If $X_1, \dots, X_n$ are simplicial sets, then

$$c^n(X_1 \boxtimes \cdots \boxtimes X_n) = cX_1 \boxtimes \cdots \boxtimes cX_n$$

where c is ordinary categorification. The symbol $\boxtimes$ on the left means external product of simplicial sets, and the symbol $\boxtimes$ on the right means external product of categories as in Definition 2.9. We will present a direct proof in the case $n = 2$ in a future paper.

Since the nerve of a natural transformation is a simplicial homotopy, we expect the diagonal of the n–fold nerve of an n–fold natural transformation to be a simplicial homotopy.

Definition 2.20 An *n–fold natural transformation* $\alpha\colon F \Longrightarrow G$ between n–fold functors $F, G\colon \mathbb{D} \longrightarrow \mathbb{E}$ is an n–fold functor

$$\alpha\colon \mathbb{D} \times [1]^{\boxtimes n} \longrightarrow \mathbb{E}$$

such that $\alpha|_{\mathbb{D}\times\{0\}}$ is F and $\alpha|_{\mathbb{D}\times\{1\}}$ is G.

Essentially, an n–fold natural transformation associates to an object an n–cube with source corner that object, to a morphism in direction i a square in direction ij for all $j \neq i$ in $1 \leq j \leq n$, to an ij–square a 3–cube in direction ijk for all $k \neq i, j$ in $1 \leq k \leq n$ etc, and these are appropriately functorial, natural, and compatible.

Example 2.21 If $\alpha_i\colon \mathbf{C}_i \times [1] \longrightarrow \mathbf{C}_i'$ are ordinary natural transformations between ordinary functors for $1 \leq i \leq n$, then $\alpha_1 \boxtimes \cdots \boxtimes \alpha_n$ is an n–fold natural transformation because of the isomorphism

$$(\mathbf{C}_1 \times [1]) \boxtimes \cdots \boxtimes (\mathbf{C}_n \times [1]) \cong (\mathbf{C}_1 \boxtimes \cdots \boxtimes \mathbf{C}_n) \times ([1] \boxtimes \cdots \boxtimes [1]).$$

Proposition 2.22 *Suppose $\alpha\colon \mathbb{D} \times [1]^{\boxtimes n} \longrightarrow \mathbb{E}$ is an n–fold natural transformation. Then $(\delta^* N^n \alpha) \circ (1_{\delta^* N^n \mathbb{D}} \times d)$ is a simplicial homotopy from $\delta^*(N^n \alpha|_{\mathbb{D}\times\{0\}})$ to $\delta^*(N^n \alpha|_{\mathbb{D}\times\{1\}})$.*

Proof We have the diagonal of the n–fold nerve of α

$$\delta^*(N^n \mathbb{D}) \times \delta^*(N^n [1]^{\boxtimes n}) \xrightarrow{\delta^* N^n \alpha} \delta^* N^n \mathbb{E},$$

which we then precompose with $1_{\delta^* N^n \mathbb{D}} \times d$ to get

$$(\delta^* N^n \mathbb{D}) \times \Delta[1] \xrightarrow{1_{\delta^* N^n \mathbb{D}} \times d} \delta^*(N^n \mathbb{D}) \times \Delta[1]^{\times n} \xrightarrow{\delta^* N^n \alpha} \delta^* N^n \mathbb{E}\ . \qquad \square$$

Lastly, we consider the behavior of c^n on the image of the left adjoint $\delta_!$. The diagonal functor

$$\begin{aligned} \delta\colon \Delta &\longrightarrow \Delta^n \\ [m] &\longmapsto ([m], \dots, [m]) \end{aligned}$$

induces $\delta^*\colon \mathbf{SSet^n} \longrightarrow \mathbf{SSet}$ by precomposition. The functor δ^* admits both a left and right adjoint by Kan extension. The left adjoint $\delta_!$ is uniquely characterized by two properties:

(i) $\delta_!(\Delta[m]) = \Delta[m,\dots,m]$.

(ii) $\delta_!$ preserves colimits.

Thus $$\delta_! X = \delta_!\left(\operatorname*{colim}_{\Delta[m]\to X} \Delta[m]\right) = \operatorname*{colim}_{\Delta[m]\to X} \delta_!\Delta[m] = \operatorname*{colim}_{\Delta[m]\to X} \Delta[m,\dots,m]$$

where the colimit is over the simplex category of the simplicial set X. Further, since c^n preserves colimits, we have

$$c^n\delta_! X = \operatorname*{colim}_{\Delta[m]\to X} c^n\Delta[m,\dots,m] = \operatorname*{colim}_{\Delta[m]\to X} [m]\boxtimes\cdots\boxtimes[m].$$

Clearly, $c^n\delta_!\Delta[m]=[m]\boxtimes\cdots\boxtimes[m]$. The calculation of $c^n\delta_!\operatorname{Sd}^2\Delta[m]$ and $c^n\delta_!\operatorname{Sd}^2\Lambda^k[m]$ is not as simple, because external product does not commute with colimits. We will give a general procedure of calculating the n–fold categorification of nerves of certain posets in Section 7.

3 Barycentric subdivision and decomposition of $\mathbf{P}\operatorname{Sd}\Delta[m]$

The adjunction

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "\mathrm{Sd}"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathrm{Ex}"] \end{tikzcd} \tag{2}$$

between barycentric subdivision Sd and Kan's functor Ex is crucial to Thomason's transfer from **Cat** to **SSet**. We will need a good understanding of subdivision for the Thomason structure on **nFoldCat** as well, so we recall it in this section. Explicit descriptions of certain subsimplices of the double subdivisions $\operatorname{Sd}^2\Lambda^k[m]$, $\operatorname{Sd}^2\partial\Delta[m]$, and $\operatorname{Sd}^2\Delta[m]$ will be especially useful later. In Proposition 3.10, we present a decomposition of the poset $\mathbf{P}\operatorname{Sd}\Delta[m]$, which is pictured in Figure 1 for the case $m=2$ and $k=1$. The nerve of the poset $\mathbf{P}\operatorname{Sd}\Delta[m]$ is of course $\operatorname{Sd}^2\Delta[m]$. This decomposition allows us to describe a deformation retraction of part of $|\operatorname{Sd}^2\Delta[m]|$ in a very controlled way (Proposition 4.3). In particular, each m–subsimplex is deformation retracted onto one of its faces. This allows us to do a deformation retraction of the n–fold categorifications as well in Corollary 7.14. These preparations are essential for verifying the pushout-axiom in Kan's Lemma on Transfer of Model Structures.

We begin now with our recollection of barycentric subdivision. The simplicial set $\operatorname{Sd}\Delta[m]$ is the nerve of the poset $\mathbf{P}\Delta[m]$ of nondegenerate simplices of $\Delta[m]$. The ordering is the face relation. Recall that the poset $\mathbf{P}\Delta[m]$ is isomorphic to the poset of nonempty subsets of $[m]$ ordered by inclusion. Thus a q–simplex v of $\operatorname{Sd}\Delta[m]$ is a tuple $(v_0, \ldots, v_q)$ of nonempty subsets of $[m]$ such that v_i is a subset of v_{i+1} for all $0 \le i \le q-1$. For example, the tuple

$$(\{0\}, \{0,2\}, \{0,1,2,3\}) \tag{3}$$

is a 2–simplex of $\operatorname{Sd}\Delta[3]$. A p–simplex u is a *face* of a q–simplex v in $\operatorname{Sd}\Delta[m]$ if and only if

$$\{u_0, \ldots, u_p\} \subseteq \{v_0, \ldots, v_q\}. \tag{4}$$

For example the 1–simplex

$$(\{0\}, \{0,1,2,3\}) \tag{5}$$

is a face of the 2–simplex in Equation (3). A face that is a 0–simplex is called a *vertex*. The vertices of v are written simply as $v_0, \ldots, v_q$. A q–simplex v of $\operatorname{Sd}\Delta[m]$ is nondegenerate if and only if all v_i are distinct. The simplices in Equations (3) and (5) are both nondegenerate.

The barycentric subdivision of a general simplicial set K is defined in terms of the barycentric subdivisions $\operatorname{Sd}\Delta[m]$ that we have just recalled.

Definition 3.1 The *barycentric subdivision* of a simplicial set K is

$$\operatorname*{colim}_{\Delta[n]\to K} \operatorname{Sd}\Delta[n]$$

where the colimit is indexed over the category of simplices of K.

The right adjoint to Sd is the Ex functor of Kan, and is defined in level m by

$$(\operatorname{Ex} X)_m = \mathbf{SSet}(\operatorname{Sd}\Delta[m], X).$$

As pointed out on page 311 of Thomason's article [85], there is a particularly simple description of $\operatorname{Sd} K$ whenever K is a classical simplicial complex each of whose simplices has a linearly ordered vertex set compatible with face inclusion. In this case, $\operatorname{Sd} K$ is the nerve of the poset $\mathbf{P}K$ of nondegenerate simplices of K. The cases $K = \operatorname{Sd}\Delta[m], \Lambda^k[m], \operatorname{Sd}\Lambda^k[m], \partial\Delta[m]$, and $\operatorname{Sd}\partial\Delta[m]$ are of particular interest to us.

We first consider the case $K = \operatorname{Sd}\Delta[m]$ in order to describe the simplicial set $\operatorname{Sd}^2\Delta[m]$. This is the nerve of the poset $\mathbf{P}\operatorname{Sd}\Delta[m]$ of nondegenerate simplices of $\operatorname{Sd}\Delta[m]$. A q–simplex of $\operatorname{Sd}^2\Delta[m]$ is a sequence $V = (V_0, \ldots, V_q)$ where each $V_i = (v^i_0, \ldots, v^i_{r_i})$ is a nondegenerate simplex of $\operatorname{Sd}\Delta[m]$ and $V_{i-1} \subseteq V_i$. For example,

$$\big((\{01\}),\ (\{0\},\{01\}),\ (\{0\},\{01\},\{012\})\big) \tag{6}$$

is a 2–simplex in $\mathrm{Sd}^2\,\Delta[2]$. A p–simplex U is a face of a q–simplex V in $\mathrm{Sd}^2\,\Delta[m]$ if and only if

$$\{U_0, \dots, U_p\} \subseteq \{V_0, \dots, V_q\}. \tag{7}$$

For example, the 1–simplex

$$\big((\{01\}),\ (\{0\}, \{01\}, \{012\})\big) \tag{8}$$

is a subsimplex of the 2–simplex in Equation (6). The vertices of V are $V_0, \dots, V_q$. A q–simplex V of $\mathrm{Sd}^2\,\Delta[m]$ is nondegenerate if and only if all V_i are distinct. The simplices in Equations (6) and (8) are both nondegenerate. Figure 1 displays the poset $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$, the nerve of which is $\mathrm{Sd}^2\,\Delta[m]$.

Next we consider $K = \Lambda^k[m]$ in order to describe $\mathrm{Sd}\,\Lambda^k[m]$ as the nerve of the poset $\mathbf{P}\Lambda^k[m]$ of nondegenerate simplices of $\Lambda^k[m]$. The simplicial set $\Lambda^k[m]$ is the smallest simplicial subset of $\Delta[m]$ which contains all nondegenerate simplices of $\Delta[m]$ except the sole m–simplex $1_{[m]}$ and the $(m-1)$–face opposite the vertex $\{k\}$. The n–simplices of $\Lambda^k[m]$ are

$$(\Lambda^k[m])_n = \{f\colon [n] \longrightarrow [m] \mid \operatorname{im} f \not\supseteq [m]\backslash\{k\}\}. \tag{9}$$

A q–simplex $(v_0, \dots, v_q)$ of $\mathrm{Sd}\,\Delta[m]$ is in $\mathrm{Sd}\,\Lambda^k[m]$ if and only if each v_i is a face of $\Lambda^k[m]$. More explicitly, $(v_0, \dots, v_q)$ is in $\mathrm{Sd}\,\Lambda^k[m]$ if and only if $|v_q| \leq m$ and in case of equality $k \in v_q$. This follows from Equation (9). Similarly, a q–simplex V in $\mathrm{Sd}^2\,\Delta[m]$ is in $\mathrm{Sd}^2\,\Lambda^k[m]$ if and only if all v^i_j are faces of $\Lambda^k[m]$. This is the case if and only if for all $0 \leq i \leq q$, $|v^i_{r_i}| \leq m$ and in case of equality $k \in v^i_{r_i}$. This, in turn, is the case if and only if $|v^q_{r_q}| \leq m$ and in case of equality $k \in v^q_{r_q}$. See again Figure 1.

Lastly, we similarly describe $\mathrm{Sd}\,\partial\Delta[m]$ and $\mathrm{Sd}^2\,\partial\Delta[m]$. The simplicial set $\partial\Delta[m]$ is the simplicial subset of $\Delta[m]$ obtained by removing the sole m–simplex $1_{[m]}$. A q–simplex $(v_0, \dots, v_q)$ of $\mathrm{Sd}\,\Delta[m]$ is in $\mathrm{Sd}\,\partial\Delta[m]$ if and only if $v_q \neq \{0, 1, \dots, m\}$. A q–simplex V of $\mathrm{Sd}^2\,\Delta[m]$ is in $\mathrm{Sd}^2\,\partial\Delta[m]$ if and only if $v^i_{r_i} \neq \{0, 1, \dots, m\}$ for all $0 \leq i \leq q$, which is the case if and only if $v^q_{r_q} \neq \{0, 1, \dots, m\}$. See again Figure 1.

Remark 3.2 Also of interest to us is the way that the nondegenerate m–simplices of $\mathrm{Sd}^2\,\Delta[m]$ are glued together along their $(m-1)$–subsimplices. In the following, let $V = (V_0, \dots, V_m)$ be a nondegenerate m–simplex of $\mathrm{Sd}^2\,\Delta[m]$. Each $V_i = (v^i_0, \dots, v^i_{r_i})$ is then a distinct nondegenerate simplex of $\mathrm{Sd}\,\Delta[m]$. See Figure 1 for intuition.

(i) Then $r_i = i$, $|V_i| = i + 1$, and hence also $v^m_m = \{0, 1, \dots, m\}$.

(ii) If $v^{m-1}_{m-1} \neq \{0, 1, \dots, m\}$, then the m–th face $(V_0, \dots, V_{m-1})$ of V is not shared with any other nondegenerate m–simplex V' of $\mathrm{Sd}^2\,\Delta[m]$.

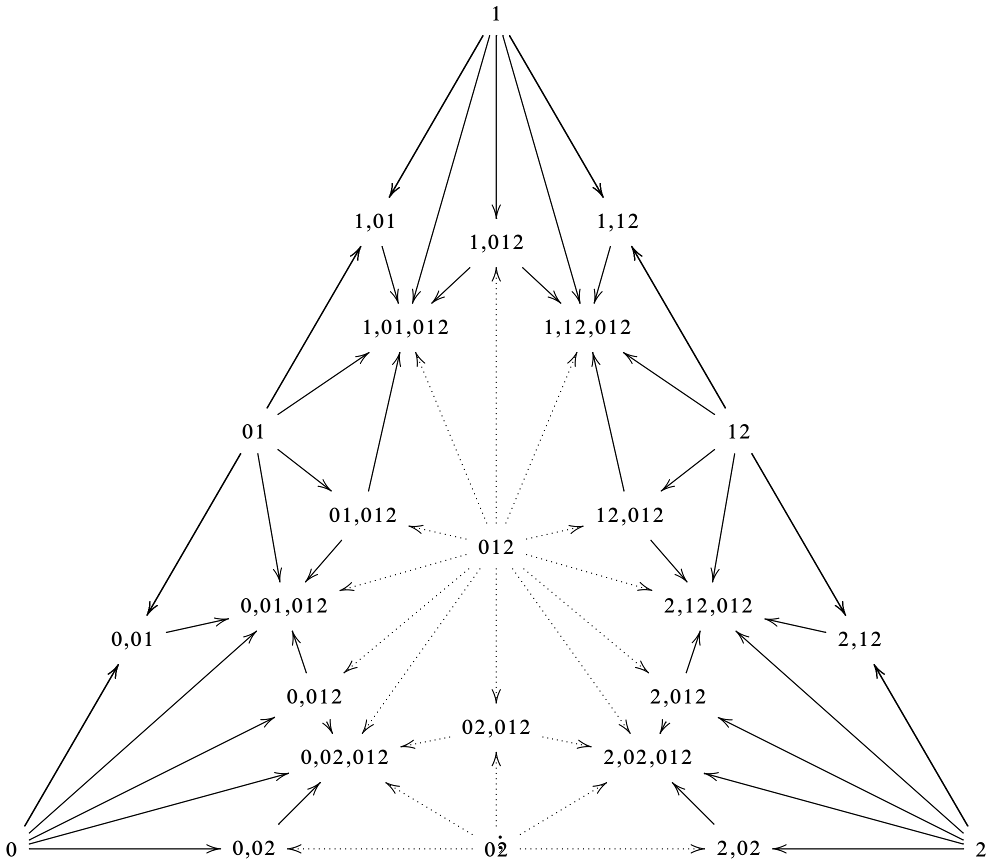

Figure 1: Decomposition of the poset $\mathbf{P}\,\mathrm{Sd}\,\Delta[2]$. The dark arrows form the poset $\mathbf{P}\,\mathrm{Sd}\,\Lambda^1[2]$, while its up-closure **Outer** consists of all solid arrows. The poset **Center** consists of all the triangles emanating from 012; these triangles all have two dotted sides emanating from 012. The poset **Comp** consists of the four triangles at the bottom emanating from 02; these four triangles each have two dotted sides emanating from 02. The geometric realization of all triangles with at least two dotted edges, namely $|N(\mathbf{Comp} \cup \mathbf{Center})|$, is topologically deformation retracted onto the solid part of its boundary.

Proof If $v_{m-1}^{m-1} \neq \{0, 1, \ldots, m\}$, then the $(m-1)$–simplex $(V_0, \ldots, V_{m-1})$ lies in $\mathrm{Sd}^2\,\partial\Delta[m]$ by the description of $\mathrm{Sd}^2\,\partial\Delta[m]$ above, and hence does not lie in any other nondegenerate m–simplex V' of $\mathrm{Sd}^2\,\Delta[m]$. □

(iii) If $v_{m-1}^{m-1} = \{0, 1, \ldots, m\}$, then the m–th face $(V_0, \ldots, V_{m-1})$ of V is shared with one other nondegenerate m–simplex V' of $\mathrm{Sd}^2\,\Delta[m]$.

Proof If $v^{m-1}_{m-1} = \{0, 1, \dots, m\}$, then there exists a unique $0 \leq i \leq m-1$ with $v^{m-1}_i \backslash v^{m-1}_{i-1} = \{a, a'\}$ with $a \neq a'$ (since the sequence $v^{m-1}_0, v^{m-1}_1, \dots, v^{m-1}_{m-1} = \{0, 1, \dots, m\}$ is strictly ascending). Here we define $v^{m-1}_{i-1} = \varnothing$ whenever $i = 0$. Thus, the $(m-1)$–simplex $(V_0, \dots, V_{m-1})$ is also a face of the nondegenerate m–simplex V' where

$$
\begin{aligned}
V'_\ell &= V_\ell \quad \text{for } 0 \leq \ell \leq m-1 \\
V'_m &= (v^{m-1}_0, \dots, v^{m-1}_{i-1}, v^{m-1}_{i-1} \cup \{a'\}, v^{m-1}_i, \dots, v^{m-1}_{m-1})
\end{aligned}
$$

where we also have

$$V_m = (v^{m-1}_0, \dots, v^{m-1}_{i-1}, v^{m-1}_{i-1} \cup \{a\}, v^{m-1}_i, \dots, v^{m-1}_{m-1}). \qquad \square$$

(iv) If $0 \leq j \leq m-1$, then V shares its j–th face $(\dots, \widehat{V_j}, \dots, V_m)$ with one other nondegenerate m–simplex V' of $\mathrm{Sd}^2\, \Delta[m]$.

Proof Since $|V_i| = i+1$, we have $V_{j+1} \backslash V_{j-1} = \{v, v'\}$ with $v \neq v'$ (we define $V_{j-1} = \varnothing$ whenever $j = 0$). Then $(\dots, \widehat{V_j}, \dots, V_m)$ is shared by the two nondegenerate m–simplices

$$
\begin{aligned}
V &= (V_0, \dots, V_{j-1}, V_{j-1} \cup \{v\}, V_{j+1}, \dots, V_m) \\
V' &= (V_0, \dots, V_{j-1}, V_{j-1} \cup \{v'\}, V_{j+1}, \dots, V_m)
\end{aligned}
$$

and no others. $\square$

After this brief discussion of how the nondegenerate m–simplices of $\mathrm{Sd}^2\, \Delta[m]$ are glued together, we turn to some comments about the relationships between the second subdivisions of $\Lambda^k[m]$, $\partial\Delta[m]$, and $\Delta[m]$. Since the counit $cN \Longrightarrow 1_{\mathbf{Cat}}$ is a natural isomorphism[2], the categories $c\,\mathrm{Sd}^2\, \Lambda^k[m]$, $c\,\mathrm{Sd}^2\, \partial\Delta[m]$ and $c\,\mathrm{Sd}^2\, \Delta[m]$ are respectively the posets $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$, $\mathbf{P}\,\mathrm{Sd}\,\partial\Delta[m]$ and $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ of nondegenerate simplices. Moreover, the induced functors

$$c\,\mathrm{Sd}^2\, \Lambda^k[m] \longrightarrow c\,\mathrm{Sd}^2\, \Delta[m] \qquad c\,\mathrm{Sd}^2\, \partial\Delta[m] \longrightarrow c\,\mathrm{Sd}^2\, \Delta[m]$$

are simply the poset inclusions

$$\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m] \longrightarrow \mathbf{P}\,\mathrm{Sd}\,\Delta[m] \qquad \mathbf{P}\,\mathrm{Sd}\,\partial\Delta[m] \longrightarrow \mathbf{P}\,\mathrm{Sd}\,\Delta[m]\ .$$

The down-closure of $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ in $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ is easily described.

Proposition 3.3 *The subposet $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ of $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ is down-closed.*

[2] The nerve functor is fully faithful, so the counit is a natural isomorphism by IV.3.1 of Mac Lane [69].

Proof A q–simplex $(v_0, \dots, v_q)$ of $\mathrm{Sd}\,\Delta[m]$ is in $\mathrm{Sd}\,\Lambda^k[m]$ if and only if $|v_q| \leq m$ and in case of equality $k \in v_q$. If $(v_0, \dots, v_q)$ has this property, then so do all of its subsimplices. $\square$

The rest of this section is dedicated to a decomposition of $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ into the union of three up-closed subposets: **Comp**, **Center**, and **Outer**. This culminates in Proposition 3.10, and will be used in the construction of the retraction in Section 4 as well as the transfer proofs in Section 6 and Section 8. The reader is encouraged to compare with Figure 1 throughout. We begin by describing these posets. The poset **Outer** is the up-closure of $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ in $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$. Although **Outer** depends on k and m, we omit these letters from the notation for readability.

Proposition 3.4 *Let* **Outer** *denote the smallest up-closed subposet of* $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ *which contains* $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$.

(i) *The subposet* **Outer** *consists of those* $(v_0, \dots, v_q) \in \mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ *such that there exists a* $(u_0, \dots, u_p) \in \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ *with*

$$\{u_0, \dots, u_p\} \subseteq \{v_0, \dots, v_q\}.$$

In particular, $(v_0, \dots, v_q) \in \mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ *is in* **Outer** *if and only if some* v_i *satisfies* $|v_i| \leq m$ *and in case of equality* $k \in v_i$.

(ii) *Define a functor* $r\colon \mathbf{Outer} \longrightarrow \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ *by* $r(v_0, \dots, v_q) := (u_0, \dots, u_p)$ *where* $(u_0, \dots, u_p)$ *is the maximal subset*

$$\{u_0, \dots, u_p\} \subseteq \{v_0, \dots, v_q\}$$

that is in $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$. *Let* $\mathrm{inc}\colon \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m] \longrightarrow \mathbf{Outer}$ *be the inclusion. Then* $r \circ \mathrm{inc} = 1_{\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]}$ *and there is a natural transformation* $\alpha\colon \mathrm{inc} \circ r \Longrightarrow 1_{\mathbf{Outer}}$ *which is the identity morphism on objects of* $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$. *Hence,* $|\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]|$ *is a deformation retract of* $|\mathbf{Outer}|$. *See Figure 1 for a geometric picture.*

Proof (i) An element of $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ is in the up-closure of $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ if and only if it lies above some element of $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$, and the order is the face relation as in Equation (4). For the last part, we use the observation that $(u_0, \dots, u_p) \in \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ if and only if $|u_p| \leq m$ and in the case of equality $k \in u_p$, as in the discussion after (9), and also the fact that $(u_j) \leq (u_0, \dots, u_p)$.

(ii) For $(v_0, \dots, v_q) \in \mathbf{Outer}$, we define $\alpha(v_0, \dots, v_q)$ to be the unique arrow in **Outer** from $r(v_0, \dots, v_q)$ to $(v_0, \dots, v_q)$. Naturality diagrams must commute, since **Outer** is a poset. The rest is clear. $\square$

The following trivial remark will be of use later.

Remark 3.5 Since $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ is down-closed by Proposition 3.3, any morphism of $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ that ends in $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ must also be contained in $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$. Since **Outer** is the up-closure of the poset $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ in $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$, any morphism that begins in $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ ends in **Outer**.

We can similarly characterize the up-closure **Center** of $(\{0,1,\dots,m\})$ in $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$. We call a nondegenerate m–simplex of $\mathrm{Sd}^2\,\Delta[m]$ a *central m–simplex* if it has $(\{0,1,\dots,m\})$ as its 0–th vertex.

Proposition 3.6 *The smallest up-closed subposet* **Center** *of* $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ *containing* $(\{0,1,\dots,m\})$ *consists of those* $(v_0,\dots,v_q)\in\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ *such that* $v_q=\{0,1,\dots,m\}$. *The nerve* N**Center** *consists of all central m–simplices of* $\mathrm{Sd}^2\,\Delta[m]$ *and all their faces. A q–simplex* $(V_0,\dots,V_q)$ *of* $\mathrm{Sd}^2\,\Delta[m]$ *is in* N**Center** *if and only if* $v^i_{r_i}=\{0,1,\dots,m\}$ *for all* $0\le i\le q$.

For example, the 2–simplex

$$\big((\{012\}),\ (\{01\},\{012\}),\ (\{0\},\{01\},\{012\})\big) \tag{10}$$

is a central 2–simplex of $\mathrm{Sd}^2\,\Delta[2]$ and the 1–simplex

$$\big((\{01\},\{012\}),\ (\{0\},\{01\},\{012\})\big) \tag{11}$$

is in N**Center**, as it is a face of the 2–simplex in Equation (10). A glance at Figure 1 makes all of this apparent.

Remark 3.7 We need to understand more thoroughly the way the central m–simplices are glued together in N**Center**. Suppose V is a central m–simplex, so that $v^i_i=\{0,1,\dots,m\}$ for all $0\le i\le m$ by Proposition 3.6. From the description of V' in Remark 3.2 (iii)–(iv), and again Proposition 3.6, we see for $j=1,\dots,m$ that the neighboring nondegenerate m–simplex V' containing the $(m-1)$–face $(V_0,\dots,\widehat{V}_j,\dots)$ of V is also central. The face $(V_1,\dots,V_m)$ of V opposite $V_0=(\{0,1,\dots,m\})$, is not shared with any other central m–simplex as every central m–simplex has $\{0,\dots,m\}$ as its 0–th vertex. Thus, each central m–simplex V shares exactly m of its $(m-1)$–faces with other central m–simplices. A glance at Figure 1 shows that the central simplices fit together to form a 2–ball. More generally, the central m–simplices of $\mathrm{Sd}^2\,\Delta[m]$ fit together to form an m–ball with center vertex $\{0,\dots,m\}$.

There is still one last piece of $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ that we discuss, namely **Comp**.

Proposition 3.8 *Let $0 \leq k \leq m$. The smallest up-closed subposet* **Comp** *of* $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ *containing the object* $(\{0,1,\ldots,\widehat{k},\ldots,m\})$ *consists of those* $(v_0,\ldots,v_q) \in \mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ *with*

$$\{0,1,\ldots,\widehat{k},\ldots,m\} \in \{v_0,\ldots,v_q\}.$$

We describe how the nondegenerate m–simplices of $N\mathbf{Comp}$ are glued together in terms of collections C^ℓ of nondegenerate m–simplices. A nondegenerate m–simplex $V \in N_m\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ is in $N_m\mathbf{Comp}$ if and only if each $V_0,\ldots,V_m$ is in **Comp**, and this is the case if and only if $V_0 = (\{0,\ldots,\widehat{k},\ldots,m\})$ (recall $|V_i| = i+1$ and Proposition 3.8). For $1 \leq \ell \leq m$, we let C^ℓ denote the set of those nondegenerate m–simplices V in $N_m\mathbf{Comp}$ which have their first ℓ vertices $V_0,\ldots,V_{\ell-1}$ on the k–th face of $|\Delta[m]|$. A nondegenerate m–simplex $V \in N_m\mathbf{Comp}$ is in C^ℓ if and only if $v_i^i = \{0,\ldots,\widehat{k},\ldots,m\}$ for all $0 \leq i \leq \ell-1$ and $v_i^i = \{0,\ldots,m\}$ for all $\ell \leq i \leq m$.

Proposition 3.9 *Let $V \in C^\ell$. Then the j–th face of V is shared with some other $V' \in C^\ell$ if and only if $j \neq 0, \ell-1, \ell$.*

Proof By Remark 3.2 we know exactly which other nondegenerate m–simplex V' shares the j–th face of V. So, for each ℓ and j we only need to check whether or not V' is in C^ℓ. Let $V \in C^\ell$.

Cases $1 \leq \ell \leq m$ and $j = 0$. For all $U \in C^\ell$, we have $U_0 = (\{0,\ldots,\widehat{k},\ldots,m\}) = V_0$, so we conclude from the description of V' in Remark 3.2 (iv) that V' is not in C^ℓ.

Case $\ell = m$ and $j = m-1$. In this case, $v_{m-1}^{m-1} = \{0,\ldots,\widehat{k},\ldots,m\}$ and $v_m^m = \{0,1,\ldots,m\}$. By Remark 3.2 (iv), the $(m-1)$–st face of V is shared with the V' which agrees with V everywhere except in V_{m-1}, where we have $(v')_{m-1}^{m-1} = \{0,\ldots,m\}$ instead of $v_{m-1}^{m-1} = \{0,\ldots,\widehat{k},\ldots,m\}$. But this V' is not an element of C^m.

Case $\ell = m$ and $j = m$. In this case, $v_{m-1}^{m-1} = \{0,\ldots,\widehat{k},\ldots,m\} \neq \{0,1,\ldots,m\}$, so we are in the situation of Remark 3.2 (ii). The m–th face $(V_0,\ldots,V_{m-1})$ does not lie in any other nondegenerate m–simplex V', let alone in a V' in C^m.

Case $\ell = m$ and $j \neq 0, m-1, m$. By Remark 3.2 (iv), the j–th face is shared with the V' that agrees with V in V_0, V_{m-1}, and V_m, so that $V' \in C^m$.

At this point we conclude from the above cases that if $\ell = m$, the j–th face of $V \in C^m$ is shared with another $V' \in C^m$ if and only if $j \neq 0, m-1, m$.

Cases $1 \leq \ell \leq m-1$ and $j = \ell-1$. The $(\ell-1)$–st face of V is shared with that V' which agrees with V everywhere except in $V_{\ell-1}$, where we have $(v')_{\ell-1}^{\ell-1} = \{0,\ldots,m\}$ instead of $v_{\ell-1}^{\ell-1} = \{0,\ldots,\widehat{k},\ldots,m\}$. Hence V' is not in C^ℓ.

Cases $1 \le \ell \le m-1$ and $j = \ell$. Similarly, the ℓ–th face of V is shared with that V' which agrees with V everywhere except in V_ℓ, where we have $(v')^\ell_\ell = \{0, \dots, \widehat{k}, \dots, m\}$ instead of $v^\ell_\ell = \{0, \dots, m\}$. Hence V' is not in C^ℓ.

Cases $1 \le \ell \le m-1$ and $j \neq 0, \ell-1, \ell$. Here the j–th face is shared with a V' that agrees with V in V_0, $V_{\ell-1}$, and V_ℓ, so that $V' \in C^\ell$.

We conclude that the j–th face of $V \in C^\ell$ is shared with some other $V' \in C^\ell$ if and only if $j \neq 0, \ell-1, \ell$. □

Proposition 3.10 *Let $0 \le k \le m$. Recall that* **Comp**, **Center** *and* **Outer** *denote the up-closure in* $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ *of* $(\{0, 1, \dots, \widehat{k}, \dots, m\})$, $(\{0, 1, \dots, m\})$, *and* $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ *respectively.*

Then the poset $\mathbf{P}\,\mathrm{Sd}\,\Delta[m] = c\,\mathrm{Sd}^2\,\Delta[m]$ *is the union of these three up-closed subposets:*

$$\mathbf{P}\,\mathrm{Sd}\,\Delta[m] = \mathbf{Comp} \cup \mathbf{Center} \cup \mathbf{Outer}.$$

The partial order on $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ *is given in (7).*

4 Deformation retraction of $|N(\mathbf{Comp} \cup \mathbf{Center})|$

In this section we construct a retraction of $|N(\mathbf{Comp} \cup \mathbf{Center})|$ to that part of its boundary which lies in **Outer**. As stated in Proposition 4.3, each stage of the retraction is part of a deformation retraction, and is thus a homotopy equivalence. The retraction is done in such a way that we can adapt it later to the n–fold case. We first treat the retraction of $|N\,\mathbf{Comp}|$ in detail.

Proposition 4.1 *Let $C^m, C^{m-1}, \dots, C^1$ be the collections of nondegenerate m–simplices of $N\,\mathbf{Comp}$ defined in Section 3. Then there is an m stage retraction of $|N\,\mathbf{Comp}|$ onto $|N(\mathbf{Comp} \cap (\mathbf{Center} \cup \mathbf{Outer}))|$ which retracts the individual simplices of $C^m, C^{m-1}, \dots, C^1$ to subcomplexes of their boundaries. Further, each retraction of each simplex is part of a deformation retraction.*

Proof To illustrate, we first prove the case $m = 1$ and $k = 0$. The poset $\mathbf{P}\,\mathrm{Sd}\,\Delta[1]$ is

$$(\{\mathbf{0}\}) \longrightarrow (\{0\}, \{01\}) \leftarrow (\{01\}) \rightarrow (\{1\}, \{01\}) \xleftarrow{f} (\{1\})$$

and $\mathbf{P}\,\mathrm{Sd}\,\Lambda^0[1]$ consists only of the object $(\{0\})$. Of the nontrivial morphisms in $\mathbf{P}\,\mathrm{Sd}\,\Delta[1]$, the only one in **Outer** is the solid one on the far left. The poset **Center** consists of the two middle morphisms, emanating from $(\{01\})$. The only morphism in **Comp** is the one labelled f. The intersection $\mathbf{Comp} \cap (\mathbf{Center} \cup \mathbf{Outer})$ is the vertex $(\{1\}, \{01\})$, which is the target of f.

Clearly, after geometrically realizing, the interval $|f|$ can be deformation retracted to the vertex $(\{1\}, \{01\})$. The case $m = 1$ with $k = 1$ is exactly the same. In fact, k does not matter, since the simplices no longer have a direction after geometric realization.

The case $m = 2$ and $k = 1$ can be similarly observed in Figure 1.

For general $m \in \mathbb{N}$, we construct a *topological* retraction in m steps, starting with Step 0. In Step 0 we retract those nondegenerate m–simplices of $N_m\mathbf{Comp}$ which have an entire $(m-1)$–face on the k–th face of $\Delta[m]$, ie, in Step 0 we retract the elements of C^m. Generally, in Step ℓ we retract those nondegenerate m–simplices of $N_m\mathbf{Comp}$ which have exactly ℓ vertices on the k–th face of $\Delta[m]$, ie, in Step ℓ we retract the elements of $C^{m-\ell}$.

We describe Step $m-\ell$ in detail for $2 \leq \ell \leq m$. We retract each $V \in C^\ell$ to

$$(V_0, \dots, \widehat{V}_{\ell-1}, V_\ell, \dots) \cup (V_1, \dots, V_m)$$

in such a way that for each $j \neq 0, \ell-1, \ell$ the j–th face

$$(V_0, \dots, \widehat{V}_j, \dots, V_{\ell-1}, V_\ell, \dots)$$

is retracted *within itself* to its subcomplex

$$(V_0, \dots, \widehat{V}_j, \dots, \widehat{V}_{\ell-1}, V_\ell, \dots) \cup (\widehat{V}_0, \dots, \widehat{V}_j, \dots, V_{\ell-1}, V_\ell, \dots).$$

We can do this to all $V \in C^\ell$ *simultaneously* because the prescription agrees on the overlaps: V shares the face $(V_0, \dots, \widehat{V}_j, \dots, V_{\ell-1}, V_\ell, \dots)$ with only one other nondegenerate m–simplex $V' \in C^\ell$, and V' differs from V only in V'_j by Proposition 3.9.

This procedure is done for Step 0 up to and including Step $m-2$. After Step $m-2$, the only remaining nondegenerate m–simplices in $N_m\mathbf{Comp}$ are those which have only the first vertex (ie, only V_0) on the k–th face of $\Delta[m]$. This is the set C^1.

Every $V \in C^1$ has

$$V_0 = (\{0, \dots, \widehat{k}, \dots, m\})$$
$$V_1 = (\{0, \dots, \widehat{k}, \dots, m\}, \{0, \dots, m\}),$$

so all $V \in C^1$ intersect in this edge. In Step $m-1$, we retract each $V \in C^1$ to $(V_1, \dots, V_m)$ in such a way that for $j \neq 0, 1$ we retract the j–th face V to $(V_1, \dots, \widehat{V}_j, \dots)$, and further we retract the 1–simplex (V_0, V_1) to the vertex V_1. We can do this simultaneously to all $V \in C^1$, as the procedure agrees in overlaps by

Proposition 3.9, and the observation about (V_0, V_1) we made above. For each $V \in C^1$, the 0–th face $(V_1, \dots, V_m)$ is also the 0–th face of a nondegenerate m–simplex U not in $N_m\mathbf{Comp}$, namely

$$U_0 = (\{0, \dots, m\})$$
$$U_j = V_j \quad \text{for } j \geq 1$$

by Remark 3.2 (iv). The simplex U is even central. Thus, $(V_1, \dots, V_m)$ is in the intersection $|N(\mathbf{Comp} \cap (\mathbf{Center} \cup \mathbf{Outer}))|$ and we have succeeded in retracting $|N\mathbf{Comp}|$ to $|N(\mathbf{Comp} \cap (\mathbf{Center} \cup \mathbf{Outer}))|$ in such a way that each nondegenerate m–simplex is retracted within itself. Further, each retraction is part of a deformation retraction. □

Proposition 4.2 *There is a multistage retraction of the space* $|N\mathbf{Center}|$ *onto the space* $|N(\mathbf{Center} \cap \mathbf{Outer})|$ *which retracts each nondegenerate* m*–simplex to a subcomplex of its boundary. Further, this retraction is part of a deformation retraction.*

Proof We describe how this works for the case $m = 2$ pictured in Figure 1. The poset **Center** consists of all the central triangles emanating from 012. These have two dotted sides emanating from 012. The intersection $\mathbf{Center} \cap \mathbf{Outer}$ consists of the indicated solid lines on those triangles and their vertices (the two triangles at the bottom have no solid lines). To topologically deformation retract $|N\mathbf{Center}|$ onto $|N(\mathbf{Center} \cap \mathbf{Outer})|$, we first deformation retract the vertical, downward pointing edge $012 - 02, 012$ by pulling the vertex $02, 012$ up to 012 while at the same time deforming the left bottom triangle to the edge $012 - 0, 02, 012$ and the right bottom triangle to the edge $012 - 2, 02, 012$.

Then we consecutively deform each of the left triangles emanating from 012 to the its solid edge and the edge of the next one, holding the vertex 012 fixed. We deform the left triangles in this manner all the way until we reach the vertically pointing edge $012 - 1, 012$.

Similarly, we consecutively deform each of the right triangles emanating from 012 to the its solid edge and the edge of the next one, holding the vertex 012 fixed. We deform the right triangles in this manner all the way until we reach the vertically pointing edge $012 - 1, 012$.

Finally, we deformation retract the last remaining edge $012 - 1, 012$ up to the vertex $1, 012$, and we are finished.

It is possible to describe this in arbitrary dimensions, although it gets rather technical, as we already have seen in Proposition 4.1. □

Proposition 4.3 *There is a multistage retraction of the space $|N(\mathbf{Comp} \cup \mathbf{Center})|$ to the space $|N((\mathbf{Comp} \cup \mathbf{Center}) \cap \mathbf{Outer})|$ which retracts each nondegenerate m–simplex to a subcomplex of its boundary. Further, each retraction of each simplex is part of a deformation retraction. See Figure 1.*

Proof This follows from Proposition 4.1 and Proposition 4.2. □

5 Nerve, pushouts and colimit decompositions of subposets of $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$

In this section we prove that the nerve is compatible with certain colimits and express posets satisfying a chain condition as a colimit of two finite ordinals, in a way compatible with nerve. The somewhat technical results of this section are crucial for the verification of the pushout axiom in the proof of the Thomason model structure on **Cat** and **nFoldCat** in Section 6 and Section 8. The results of this section will have n–fold versions in Section 7.

We begin by proving that the nerve preserves certain pushouts in Proposition 5.1. The question of commutation of nerve with certain pushouts is an old one, and has been studied by Fritsch and Latch [27, Section 5].

The next task is to express posets satisfying a chain condition as a colimit of two finite ordinals $[m-1]$ and $[m]$ in Proposition 5.3, and similarly express their nerves as a colimit of $\Delta[m-1]$ and $\Delta[m]$ in Proposition 5.4. As a consequence, the nerve functor preserves these colimits in Corollary 5.5. The combinatorial proof that our posets of interest, namely $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$, $\mathbf{Center}$, $\mathbf{Outer}$, $\mathbf{Comp}$, $\mathbf{Comp} \cup \mathbf{Center}$, $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$, and $\mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center})$, satisfy the chain conditions, is found in Remark 5.6 and Proposition 5.7. Corollary 5.9 summarizes the nerve commutation for the decompositions of the posets of interest.

Proposition 5.1 *Suppose $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ are categories, and $\mathbf{S}$ is a full subcategory of $\mathbf{Q}$ and $\mathbf{R}$ such that:*

(i) *If $f\colon x \longrightarrow y$ is a morphism in $\mathbf{Q}$ and $x \in \mathbf{S}$, then $y \in \mathbf{S}$.*

(ii) *If $f\colon x \longrightarrow y$ is a morphism in $\mathbf{R}$ and $x \in \mathbf{S}$, then $y \in \mathbf{S}$.*

Then the nerve of the pushout is the pushout of the nerves, that is,

$$N\Big(\mathbf{Q} \coprod_{\mathbf{S}} \mathbf{R}\Big) \cong N\mathbf{Q} \coprod_{N\mathbf{S}} N\mathbf{R}. \tag{12}$$

Proof First we claim that there are no free composites in $\mathbf{Q} \coprod_{\mathbf{S}} \mathbf{R}$. Suppose f is a morphism in $\mathbf{Q}$ and g is a morphism in $\mathbf{R}$ and that these are composable in the pushout $\mathbf{Q} \coprod_{\mathbf{S}} \mathbf{R}$.

$$w \xrightarrow{f} x \xrightarrow{g} y$$

Then $x \in \operatorname{Obj} \mathbf{Q} \cap \operatorname{Obj} \mathbf{R} = \mathbf{S}$, so $y \in \mathbf{S}$ by hypothesis (ii). Since $\mathbf{S}$ is full, g is a morphism of $\mathbf{S}$. Then $g \circ f$ is a morphism in $\mathbf{Q}$ and is not free. The other case f in $\mathbf{R}$ and g in $\mathbf{Q}$ is exactly the same. Thus the pushout $\mathbf{Q} \coprod_{\mathbf{S}} \mathbf{R}$ has no free composites.

Let $(f_1, \dots, f_p)$ be a p–simplex in $N(\mathbf{Q} \coprod_{\mathbf{S}} \mathbf{R})$. Then each f_j is a morphism in $\mathbf{Q}$ or $\mathbf{R}$, as there are no free composites. Further, by repeated application of the argument above, if f_1 is in $\mathbf{Q}$ then every f_j is in $\mathbf{Q}$. Similarly, if f_1 is in $\mathbf{R}$ then every f_j is in $\mathbf{R}$. Thus we have a morphism $N(\mathbf{Q} \coprod_{\mathbf{S}} \mathbf{R}) \longrightarrow N\mathbf{Q} \coprod_{N\mathbf{S}} N\mathbf{R}$. Its inverse is the canonical morphism $N\mathbf{Q} \coprod_{N\mathbf{S}} N\mathbf{R} \longrightarrow N(\mathbf{Q} \coprod_{\mathbf{S}} \mathbf{R})$. □

Proposition 5.2 *The full subcategory* $(\mathbf{Comp} \cup \mathbf{Center}) \cap \mathbf{Outer}$ *of the categories* $\mathbf{Comp} \cup \mathbf{Center}$ *and* $\mathbf{Outer}$ *satisfies (i) and (ii) of Proposition 5.1.*

Proof Since $\mathbf{Comp}$ and $\mathbf{Center}$ are up-closed, the union $\mathbf{Comp} \cup \mathbf{Center}$ is up-closed, as is its intersection with up-closed poset $\mathbf{Outer}$. Hence conditions (i) and (ii) of Proposition 5.1 follow. □

Proposition 5.3 *Let* $\mathbf{T}$ *be a poset and* $m \geq 1$ *a positive integer such that the following hold.*

(i) *Any linearly ordered subposet* $U = \{U_0 < U_1 < \cdots < U_p\}$ *of* $\mathbf{T}$ *with* $|U| \leq m+1$ *is contained in a linearly ordered subposet* V *of* $\mathbf{T}$ *with* $m+1$ *distinct elements.*

(ii) *Suppose* x *and* y *are in* $\mathbf{T}$ *and* $x \leq y$. *If* V *and* V' *are linearly ordered subposets of* $\mathbf{T}$ *with exactly* $m+1$ *elements, and both* V *and* V' *contain* x *and* y, *then there exist linearly ordered subposets* $W^0, W^1, \dots, W^k$ *of* $\mathbf{T}$ *such that:*
   (a) $W^0 = V$.
   (b) $W^k = V'$.
   (c) *For all* $0 \leq j \leq k$, *the linearly ordered poset* W^j *has exactly* $m+1$ *elements.*
   (d) *For all* $0 \leq j \leq k$, *we have* $x \in W^j$ *and* $y \in W^j$.
   (e) *For all* $0 \leq j \leq k-1$, *the poset* $W^j \cap W^{j+1}$ *has* m *elements.*

(iii) *If* $m = 1$, *we further assume that there are no linearly ordered subposets with 3 or more elements, that is, there are no nontrivial composites* $x < y < z$. *Whenever* $m = 1$, *hypothesis (ii) is vacuous.*

Let $\mathbf{J}$ *denote the poset of linearly ordered subposets* U *of* $\mathbf{T}$ *with exactly* m *or* $m+1$ *elements. Then* $\mathbf{T}$ *is the colimit of the functor*

$$\begin{aligned} F\colon \mathbf{J} &\longrightarrow \mathbf{Cat} \\ U &\longmapsto U. \end{aligned}$$

The components of the universal cocone $\pi\colon F \Longrightarrow \Delta_{\mathbf{T}}$ *are the inclusions* $F(U) \longrightarrow \mathbf{T}$.

Proof Suppose $\mathbf{S} \in \mathbf{Cat}$ and $\alpha\colon F \Longrightarrow \Delta_{\mathbf{S}}$ is a natural transformation. We define a functor $G\colon \mathbf{T} \longrightarrow \mathbf{S}$ as follows. Let x and y be elements of $\mathbf{T}$ and suppose $x \leq y$. By hypothesis (i), there is a linearly ordered subposet V of $\mathbf{T}$ which contains x and y and has exactly $m+1$ elements. We define $G(x \leq y) := \alpha_V(x \leq y)$.

We claim G is well defined. If V' is another linearly ordered subposet of $\mathbf{T}$ which contains x and y and has exactly $m+1$ elements, then we have a sequence $W^0, \ldots, W^k$ as in hypothesis (ii), and the naturality diagrams below.

$$\begin{tikzcd}
W^i \arrow[rr, "\alpha_{W^i}"] & & \mathbf{S} \arrow[d, equal] \\
W^i \cap W^{i+1} \arrow[u] \arrow[d] \arrow[rr, "\alpha_{W^i \cap W^{i+1}}"] & & \mathbf{S} \arrow[d, equal] \\
W^{i+1} \arrow[rr, "\alpha_{W^{i+1}}"'] & & \mathbf{S}
\end{tikzcd}$$

Thus we have a string of equalities

$$\alpha_{W^0}(x \leq y) = \alpha_{W^1}(x \leq y) = \cdots = \alpha_{W^k}(x \leq y),$$

and we conclude $\alpha_V(x \leq y) = \alpha_{V'}(x \leq y)$ so that $G(x \leq y)$ is well defined.

The assignment G is a functor, as follows. It preserves identities because each α_V does. If $m = 1$, then there are no nontrivial composites by hypothesis (iii), so G vacuously preserves all compositions. If $m \geq 2$, and the elements $x < y < z$ are in $\mathbf{T}$, then there exists a V containing all three of x, y, and z. The functor α_V preserves this composition, so G does also.

By construction, for each linearly ordered subposet V of $\mathbf{T}$ with $m+1$ elements we have $\alpha_V = G \circ \pi_V$. Further, G is the unique such functor, since such posets V cover $\mathbf{T}$ by hypothesis (i).

Lastly we claim that $\alpha_U = G \circ \pi_U$ for any linearly ordered subposet U of $\mathbf{T}$ with m elements. By hypothesis (i) there exists a linearly ordered subposet V of $\mathbf{T}$ with $m+1$

elements such that $U \subseteq V$. If i denotes the inclusion of U into V, by naturality of α and π we have

$$\alpha_U = \alpha_V \circ i = G \circ \pi_V \circ i = G \circ \pi_U. \qquad \square$$

Proposition 5.4 *Let $\mathbf{T}$ be a poset and $m \geq 1$ a positive integer such that the following hold.*

(i) *Any linearly ordered subposet $U = \{U_0 < U_1 < \cdots < U_p\}$ of $\mathbf{T}$ is contained in a linearly ordered subposet V of $\mathbf{T}$ with $m+1$ distinct elements, in particular, any linearly ordered subposet of $\mathbf{T}$ has at most $m+1$ elements.*

(ii) *Suppose $x_0 < x_1 < \cdots < x_\ell$ are in $\mathbf{T}$ and $\ell \leq m$. If V and V' are linearly ordered subposets of $\mathbf{T}$ with exactly $m+1$ elements, and both V and V' contain $x_0 < x_1 < \cdots < x_\ell$, then there exist linearly ordered subposets $W^0, W^1, \ldots, W^k$ of $\mathbf{T}$ such that:*
   (a) $W^0 = V$.
   (b) $W^k = V'$.
   (c) *For all $0 \leq j \leq k$, the linearly ordered poset W^j has exactly $m+1$ elements.*
   (d) *For all $0 \leq j \leq k$, the elements $x_0 < x_1 < \cdots < x_\ell$ are all in W^j.*
   (e) *For all $0 \leq j \leq k-1$, the poset $W^j \cap W^{j+1}$ has exactly m distinct elements.*

As in Proposition 5.3, let $\mathbf{J}$ denote the poset of linearly ordered subposets U of $\mathbf{T}$ with exactly m or $m+1$ elements, let F be the functor

$$\begin{aligned} F\colon \mathbf{J} &\longrightarrow \mathbf{Cat} \\ U &\longmapsto U, \end{aligned}$$

and π the universal cocone $\pi\colon F \Longrightarrow \Delta_{\mathbf{T}}$. The components of π are the inclusions $F(U) \longrightarrow \mathbf{T}$. Then $N\mathbf{T}$ is the colimit of the functor

$$\begin{aligned} NF\colon \mathbf{J} &\longrightarrow \mathbf{SSet} \\ U &\longmapsto NF(U) \end{aligned}$$

and $N\pi\colon NF \Longrightarrow \Delta_{N\mathbf{T}}$ is its universal cocone.

Proof The principle of the proof is similar to the direct proof of Proposition 5.3. Suppose $S \in \mathbf{SSet}$ and $\alpha\colon NF \Longrightarrow \Delta_S$ is a natural transformation. We induce a morphism of simplicial sets $G\colon N\mathbf{T} \longrightarrow S$ by defining G on the m–skeleton as follows.

Let Δ_m denote the full subcategory of Δ on the objects $[0], [1], \ldots, [m]$ and let $\mathrm{tr}_m\colon \mathbf{SSet} \longrightarrow \mathbf{Set}^{\Delta_m^{\mathrm{op}}}$ denote the m–th truncation functor. The truncation $\mathrm{tr}_m N\mathbf{T}$

is a union of the truncated simplicial subsets $\mathrm{tr}_m NV$ for $V \in \mathbf{J}$ with $|V| = m+1$, since $\mathbf{T}$ is a union of such V. We define

$$G_m|_{\mathrm{tr}_m NV} \colon \mathrm{tr}_m NV \longrightarrow \mathrm{tr}_m S$$

simply as $\mathrm{tr}_m \alpha_V$.

The morphism G_m is well-defined, for if $0 \leq \ell \leq m$ and $x \in (\mathrm{tr}_m NV)_\ell$ and $x \in (\mathrm{tr}_m NV')_\ell$ with $|V| = m+1 = |V|$, then V and V' can be connected by a sequence $W^0, W^1, \dots, W^k$ of $(m{+}1)$–element linearly ordered subsets of $\mathbf{T}$ that all contain the linearly ordered subposet x and satisfy the properties in hypothesis (ii). By a naturality argument as in the proof of Proposition 5.3, we have a string of equalities

$$\alpha_{W^0}(x) = \alpha_{W^1}(x) = \cdots = \alpha_{W^k}(x),$$

and we conclude $\alpha_V(x) = \alpha_{V'}(x)$ so that $G_m(x)$ is well defined.

By definition $\Delta_{G_m} \circ \mathrm{tr}_m N\pi = \mathrm{tr}_m \alpha$. We may extend this to nontruncated simplicial sets using the following observation: if $\mathbf{C}$ is a category in which composable chains of morphisms have at most m–morphisms, and sk_m is the left adjoint to tr_m, then the counit inclusion

$$\mathrm{sk}_m \mathrm{tr}_m(N\mathbf{C}) \longrightarrow N\mathbf{C}$$

is the identity. Thus G_m extends to $G \colon N\mathbf{T} \longrightarrow S$ and $\Delta_G \circ N\pi = \alpha$.

Lastly, the morphism G is unique, since the simplicial subsets NV for $|V| = m+1$ cover $N\mathbf{T}$ by hypothesis (i). □

Corollary 5.5 *Under the hypotheses of Proposition 5.4, the nerve functor commutes with the colimit of F.*

Since $\mathrm{Sd}^2\,\Delta[m]$ geometrically realizes to a *connected* simplicial complex that is a union of nondegenerate m–simplices, it is clear that we can move from any nondegenerate m–simplex V of $\mathrm{Sd}^2\,\Delta[m]$ to any other V' by a chain of nondegenerate m–simplices in which consecutive ones share an $(m{-}1)$–subsimplex. However, if x and y are two vertices contained in both V and V', it is not clear that a chain can be chosen from V to V' in which all nondegenerate m–simplices contain both x and y. The following extended remark explains how to choose such a chain.

Remark 5.6 Our next task is to prepare for the proof of Proposition 5.7, which says that the posets $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$, $\mathbf{Center}$, $\mathbf{Outer}$, $\mathbf{Comp}$, and $\mathbf{Comp} \cup \mathbf{Center}$ satisfy the hypotheses of Proposition 5.4 for m, and the posets $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ and $\mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center})$ satisfy the hypotheses of Proposition 5.4 for $m-1$. Building on Remark 3.2, we describe a way of moving from a nondegenerate m–simplex V of $\mathrm{Sd}^2\,\Delta[m]$ to another nondegenerate m–simplex V' of $\mathrm{Sd}^2\,\Delta[m]$ via a chain of nondegenerate m–simplices, in which consecutive m–simplices overlap in an $(m-1)$–simplex, and each nondegenerate m–simplex in the chain contains specified vertices $x_0 < x_1 < \cdots < x_\ell$ contained in both V and V'. Observe that the respective elements $x_0, x_1, \ldots, x_\ell$ are in the same respective positions in V and V', for if they were in different respective positions, we would arrive at a linearly ordered subposet of length greater than $m+1$, a contradiction.

We first prove the analogous statement about moving from V to V' for $\mathrm{Sd}\,\Delta[m]$. The nondegenerate m–simplices of $\mathrm{Sd}\,\Delta[m]$ are in bijective correspondence with the permutations of $\{0,1,\ldots,m\}$. Namely, the simplex $v=(v_0,\ldots,v_m)$ corresponds to $a_0,\ldots,a_m$ where $a_i = v_i \backslash v_{i-1}$. For example, $(\{1\},\{1,2\},\{0,1,2\})$ corresponds to $1,2,0$. Swapping a_i and a_{i+1} gives rise to a nondegenerate m–simplex w which shares an $(m-1)$–subsimplex with v, that is, v and w differ only in the i–th spot: $v_i \neq w_i$. Since transpositions generate the symmetric group, we can move from any nondegenerate m–simplex of $\mathrm{Sd}\,\Delta[m]$ to any other by a sequence of moves in which we only change one vertex at a time. Suppose v and v' are the same at spots $s_0 < s_1 < \cdots < s_\ell$, that is $v_{s_i} = v'_{s_i}$ for $0 \leq i \leq \ell$. Then, using transpositions, we can traverse from v to v' through a chain $w^1,\ldots,w^k$ of nondegenerate m–simplices of $\mathrm{Sd}\,\Delta[m]$, each of which is equal to $v_{s_1}, v_{s_2}, \ldots, v_{s_\ell}$ in spots $s_1, s_2, \ldots, s_\ell$. Indeed, this corresponds to the embedding of symmetric groups

$$\mathrm{Sym}(v_{s_1}) \times \left(\prod_{i=2}^{\ell} \mathrm{Sym}(v_{s_i} \backslash v_{s_{i-1}}) \right) \times \mathrm{Sym}(\{0,\ldots,n\} \backslash v_{s_\ell}) \longrightarrow \mathrm{Sym}(\{0,\ldots,n\})$$

and generation by the relevant transpositions.

Similar, but more involved, arguments allow us to navigate the nondegenerate m–simplices of $\mathrm{Sd}^2\,\Delta[m]$. For a *fixed* nondegenerate m–simplex $V_m = (v_0^m, \ldots, v_m^m)$ of $\mathrm{Sd}\,\Delta[m]$, the nondegenerate m–simplices $V=(V_0,\ldots,V_m)$ of $\mathrm{Sd}^2\,\Delta[m]$ ending in the *fixed* V_m correspond to permutations $A_0,\ldots,A_m$ of the vertices of V_m. For example, the 2–simplex in (6) corresponds to the permutation

$$\{01\},\ \{0\},\ \{012\}.$$

Again, arguing by transpositions, we can move from any nondegenerate m–simplex of $\mathrm{Sd}^2\,\Delta[m]$ ending in V_m to any other ending in V_m by a sequence of moves in which we only change one vertex at a time, and at every step, we preserve the specified vertices $x_0 < x_1 < \cdots < x_\ell$. Holding V_m fixed corresponds to moving (in $\mathrm{Sd}^2\,\Delta[m]$) within the subdivision of one of the nondegenerate m–simplices of $\mathrm{Sd}\,\Delta[m]$ (the subdivision is isomorphic to $\operatorname{Sd}\Delta[m]$, the case treated above). See for example Figure 1 for a convincing picture.

But how do we move between nondegenerate m–simplices that do not agree in the m–th spot, in other words, how do we move from nondegenerate m–simplices of one subdivided nondegenerate m–simplex of $\operatorname{Sd}\Delta[m]$ to nondegenerate m–simplices in another subdivided nondegenerate m–simplex of $\operatorname{Sd}\Delta[m]$? First, we say how to move without requiring containment of the specified vertices $x_0 < x_1 < \cdots < x_\ell$. Note that if V and W in $\operatorname{Sd}^2\Delta[m]$ only differ in the last spot m, then V_m and W_m agree in all but one spot, say $v_i^m \neq w_i^m$, and the permutations corresponding to V and W are respectively

$$A_0, \dots, A_{m-1}, v_i^m$$
$$A_0, \dots, A_{m-1}, w_i^m.$$

Given arbitrary nondegenerate m–simplices V and V' of $\operatorname{Sd}^2\Delta[m]$, we construct a chain connecting V and V' as follows. First we choose a chain of m–simplices $\{\overline{W}^p\}_{p=0}^q$ in $\operatorname{Sd}\Delta[m]$

$$\overline{W}_m^p = (w_0^p, \dots, w_m^p)$$

$0 \leq p \leq q$ from V_m to V_m' which corresponds to transpositions. This we can do by the first paragraph of this Remark. We define an m–simplex $\overline{W}^p$ in $\operatorname{Sd}^2\Delta[m]$ by

$$\overline{W}^p := (\dots, \overline{W}_m^p \backslash w_{i_p}^p, \overline{W}_m^p)$$

where $w_{i_p}^p$ is the vertex of $\overline{W}_m^p$ which distinguishes it from $\overline{W}_m^{p-1}$ for $1 \leq p \leq q$. The last letter in the permutation corresponding to $\overline{W}^p$ is $w_{i_p}^p$. The other vertices of $\overline{W}^p$ indicated by $\dots$ are any subsimplices of $\overline{W}_m^p$ written in increasing order. Now, our chain $\{W^j\}_j$ in $\operatorname{Sd}^2\Delta[m]$ from V to V' begins at V and traverses to $\overline{W}^1$: starting from V, we pairwise transpose $v_{i_1}^m$ to the end of the permutation corresponding to V, then we replace $v_{i_1}^m$ by $w_{i_1}^1$, and then we pairwise transpose the first m letters of the resulting permutation to arrive at the permutation corresponding to $\overline{W}^1$. Similarly, starting from $\overline{W}^1$ we move $w_{i_2}^1$ to the end, replace it by $w_{i_2}^2$, and then pairwise transpose the first m letters to arrive at $\overline{W}^2$. Continuing in this fashion, we arrive at V' through a chain $\{W^j\}_j$ of nondegenerate m–simplices W^j in $\operatorname{Sd}^2\Delta[m]$ in which W^j and W^{j+1} share an $(m-1)$–subsimplex.

Lastly, we must prove that if V and V' both contain specified vertices $x_0 < x_1 < \cdots < x_\ell$, then the chain $\{W^j\}_j$ of nondegenerate m–simplices can be chosen so that each W^j contains all of the specified vertices $x_0 < x_1 < \cdots < x_\ell$. Suppose

$$V_{s_i} = x_i = V_{s_i}'$$

for all $0 \le i \le \ell$ and $s_0 < s_1 < \cdots < s_\ell$. Then V_m and V'_m both contain all of the vertices of $x_0, x_1, \dots, x_\ell$ since

$$V_m \supseteq V_{s_\ell} = x_\ell \supseteq x_{\ell-1} \supseteq \cdots \supseteq x_0 = V_{s_0}$$
$$V'_m \supseteq V'_{s_\ell} = x_\ell \supseteq x_{\ell-1} \supseteq \cdots \supseteq x_0 = V'_{s_0}.$$

We first choose the chain $\{\overline{W}_m^p\}_p$ in $\operatorname{Sd}\Delta[m]$ so that each $\overline{W}_m^p$ contains all of the vertices of $x_0, x_1, \dots, x_\ell$ (this can be done by the discussion of $\operatorname{Sd}\Delta[m]$ above). Since we have $\overline{W}_m^p \supseteq x_\ell$, all $w_{i_p}^p$ must satisfy $i_p > s_\ell$. The first vertices of the nondegenerate m–simplex $\overline{W}^p$ in $\operatorname{Sd}^2\Delta[m]$ indicated by $\dots$ are chosen so that in spots $s_0, s_1, \dots, s_\ell$ we have $x_0, x_1, \dots, x_\ell$. For fixed W_m^p we can transpose as we wish, without perturbing $x_0, x_1, \dots, x_\ell$ (again by the discussion of $\operatorname{Sd}\Delta[m]$ above, but this time applied to the $\operatorname{Sd}\Delta[m]$ isomorphic to the collection of m–simplices of $\operatorname{Sd}\Delta[m]$ ending in $\overline{W}_m^p$.) On the other hand, the part of $\{W^j\}_j$ in which we move $w_{i_p}^{p-1}$ to the right does not perturb any of $x_0, x_1, \dots, x_\ell$ because $i_p > s_\ell$. Thus, each W^j has $x_0, x_1, \dots, x_\ell$ in spots $s_0, s_1, \dots, s_\ell$ respectively.

Proposition 5.7 *Let $m \ge 1$ be a positive integer. The posets $\mathbf{P}\operatorname{Sd}\Delta[m]$, **Center**, **Outer**, **Comp**, and **Comp** $\cup$ **Center** satisfy (i) and (ii) of Proposition 5.4 for m. Similarly, $\mathbf{P}\operatorname{Sd}\Lambda^k[m]$ and **Outer** $\cap$ (**Comp** $\cup$ **Center**) satisfy (i) and (ii) of Proposition 5.4 for $m-1$. The hypotheses of Proposition 5.4 imply those of Proposition 5.3, so Proposition 5.3 also applies to these posets.*

Proof We first consider $m = 1$ and the various subposets of $\mathbf{P}\operatorname{Sd}\Delta[1]$. Let $k = 0$ (the case $k = 1$ is symmetric). The poset $\mathbf{P}\operatorname{Sd}\Delta[1]$ is

$$(\{\mathbf{0}\}) \longrightarrow (\{0\},\{01\}) \longleftarrow (\{01\}) \longrightarrow (\{1\},\{01\}) \xleftarrow{f} (\{1\})$$

and $\mathbf{P}\operatorname{Sd}\Lambda^0[1]$ consists only of the object $(\{0\})$ (the typography is chosen to match with Figure 1). Of the nontrivial morphisms in $\mathbf{P}\operatorname{Sd}\Delta[1]$, the only one in **Outer** is the solid one on the far left. The poset **Center** consists of the two middle morphisms, emanating from $(\{01\})$. The only morphism in **Comp** is the one labelled f. The union **Comp** $\cup$ **Center** consist of all the dotted arrows and their sources and targets. The intersection **Outer** $\cap$ (**Comp** $\cup$ **Center**) consists only of the vertex $(\{0\},\{0,1\})$. The hypotheses (i) and (ii) of Proposition 5.4 are clearly true by inspection for $\mathbf{P}\operatorname{Sd}\Delta[1]$, **Center**, **Outer**, **Comp**, and **Comp** $\cup$ **Center** and also $\mathbf{P}\operatorname{Sd}\Lambda^k[1]$ and **Outer** $\cap$ (**Comp** $\cup$ **Center**).

We next prove that $\mathbf{P}\operatorname{Sd}\Delta[m]$ satisfies hypothesis (i) of Proposition 5.4 for $m \ge 2$, and also its various subposets satisfy hypothesis (i). Suppose $U = \{U_0 < U_1 < \cdots < U_p\}$ is a linearly ordered subposet of $\mathbf{P}\operatorname{Sd}\Delta[m]$. As before, we write $U_i = (u_0^i, \dots, u_{r_i}^i)$.

We extend U to a linearly ordered subposet V with $m+1$ elements so that U_i occupies the r_i–th place (the lowest element is in the 0–th place). For $j \leq r_0$, let $V_j = (u_0^0, \ldots, u_j^0)$. For $j = r_i$, $V_j := U_i$. For $r_i \leq j < r_{i+1}-1$, we define V_{j+1} as V_j with one additional element of $U_{i+1} \backslash U_i$. If $|U_p| = m+1$, then we are now finished. If $|U_p| = r_p + 1 < m+1$, then extend U_p to a strictly increasing chain of subsets of $\{0, \ldots, m\}$ of length $m+1$, where the new subsets are $v_1, \ldots, v_{m+1-(r_p+1)}$ and define for $j = 1, \ldots, m - r_p$

$$V_{r_p+j} := V_{r_p} \cup \{v_1, \ldots, v_j\}.$$

Then we have U contained in $V = \{V_0 < \cdots < V_m\}$.

Easy adjustments show that the poset **Center** satisfies hypothesis (i) for $m \geq 2$. If U is a linearly ordered subposet of **Center**, then each $u^i_{r_i}$ is $\{0, 1, \ldots, m\}$ by Proposition 3.6. We take $V_0 = (\{0, 1, \ldots, m\})$ and then successively throw in $u_0^0, \ldots, u_{r_0-1}^0$ to obtain $V_1, \ldots, V_{r_0}$. The higher V_j's are as above. By Proposition 3.6, the extension V lies in **Center**. A similar argument works for **Comp**, since it is also the up-closure of a single point, namely $(\{0, 1, \ldots, \widehat{k}, \ldots, m\})$. The union **Comp** $\cup$ **Center** also satisfies hypothesis (i) for $m \geq 2$: if U is a subposet of the union, then U_0 is in at least one of **Comp** or **Center**, and all the other U_i's are also contained in that one, so the proof for **Comp** or **Center** then finishes the job.

The poset **Outer** satisfies hypothesis (i) for $m \geq 2$, for if U is a subposet of **Outer**, then U_0 must contain some u_i^0 in $\Lambda^k[m]$ by Proposition 3.4. We extend to the left of U_0 by taking $V_0 = (u_i^0)$ and then successively throwing in the remaining elements of U_0. The rest of the extension proceeds as above, since everything above U_0 also contains $u_i^0 \in \Lambda^k[m]$. The poset **Outer** $\cap$ **Comp** satisfies hypothesis (i) for $m-1$ rather than m because any element in the intersection must have at least 2 vertices, namely a vertex in $\Lambda^k[m]$ and $\{0, \ldots, \widehat{k}, \ldots, m\}$. Similarly, the poset **Outer** $\cap$ **Center** satisfies hypothesis (i) for $m-1$ rather than m because any element in the intersection must have at least 2 vertices, namely a vertex in $\Lambda^k[m]$ and $\{0, \ldots, m\}$. The proofs that **Outer** $\cap$ **Comp** and **Outer** $\cap$ **Center** satisfy hypothesis (i) are similar to the above. Since unions of subposets of $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ that satisfy hypothesis (i) for $m-1$ also satisfy hypothesis (i) for $m-1$, we see that

$$(\mathbf{Outer} \cap \mathbf{Comp}) \cup (\mathbf{Outer} \cap \mathbf{Center}) = \mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center}) \tag{13}$$

also satisfies hypothesis (i) for $m-1$.

Lastly $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ satisfies hypothesis (i) for $m-1$. It is down closed by Proposition 3.3, so for a subposet U, the extension of U to the left in $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ described above is also in $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$. Any extension to the right which includes k in the final m–element set is also in $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ by the discussion after Equation (9).

Next we turn to hypothesis (ii) of Proposition 5.4 for the subposets of $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ in question, where $m \geq 2$. The poset $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ satisfies hypothesis (ii) by Remark 5.6.

The poset **Center** is the up-closure of $(\{0,1,\dots,m\})$ in $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$. Every linearly ordered subposet of **Center** with $m+1$ elements must begin with $(\{0,1,\dots,m\})$. Given $(m+1)$–element, linearly ordered subposets V and V' of **Center** with specified elements $x_0 < x_1 < \cdots < x_\ell$ in common, we can select the chain $\{W^j\}_j$ in Remark 5.6 so that each W^j has $(\{0,1,\dots,m\})$ as its 0–vertex. Thus **Center** satisfies hypothesis (ii). The poset **Comp** similarly satisfies hypothesis (ii), as it is also the up-closure of an element in $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$.

The union $\mathbf{Comp} \cup \mathbf{Center}$ satisfies hypothesis (ii) as follows. If V and V' (of cardinality $m+1$) are both linearly ordered subposets of **Comp** or are both linearly ordered subposets of **Center** respectively with the specified elements in common, then we may simply take the chain in **Comp** or **Center** respectively. If V is in **Center** and V' is in **Comp**, then $V_0 = (\{0,1,\dots,m\})$ and $V'_0 = (\{0,\dots,\widehat{k},\dots,m\})$. Suppose

$$V_{s_i} = x_i = V'_{s_i}$$

for all $0 \leq i \leq \ell$ and $s_0 < s_1 < \cdots < s_\ell$. Then x_0 contains both $\{0,1,\dots,m\}$ and $\{0,\dots,\widehat{k},\dots,m\}$. Then we move from V' to V'' by transposing $\{0,1,\dots,m\}$ down to vertex 0, leaving everything else unchanged. This chain from V' to V'' is in **Comp** until it finally reaches V'', which is in **Center**. From V we can reach V'' via a chain in **Center** as above. Putting these two chains together, we move from V to V' as desired.

To show **Outer** satisfies hypothesis (ii), suppose V and V' are linearly ordered subposets of cardinality $m+1$ with $V_{s_i} = x_i = V'_{s_i}$ for all $0 \leq i \leq \ell$ and $s_0 < s_1 < \cdots < s_\ell$. If $V_0 = V'_0$, then we can make certain that the chain $\{W^j\}_j$ in Remark 5.6 satisfies $W^j_0 = V_0 = V'_0 \in \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$. Then each W^j lies in **Outer**, and we are finished. If $V_0 \neq V'_0$, then we move from V' to V'' with $V''_0 = V_0$ as follows. The elements V_0 and V'_0 are both in $V_{s_0} = x_0 = V'_{s_0}$, so we can transpose V_0 in V' down to the 0–vertex and interchange V_0 and V'_0. Each step of the way is in **Outer**. The result is V'', to which we can move from V on a chain in **Outer**.

We claim that the subposet $\mathbf{Outer} \cap \mathbf{Comp}$ of $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ satisfies hypothesis (ii) for $m-1$. Suppose V and V' are linearly ordered subposets of cardinality m with $V_{s_i} = x_i = V'_{s_i}$ for all $0 \leq i \leq \ell$ and $s_0 < s_1 < \cdots < s_\ell$, where $1 \leq \ell \leq m-1$. Then $V_0 = (v, \{0,\dots,\widehat{k},\dots,m\})$ and $V'_0 = (v', \{0,\dots,\widehat{k},\dots,m\})$ where v and v' are elements of $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$. We extend the m–element linearly ordered posets V and V' to $(m+1)$–element linearly ordered posets $\overline{V}$ and $\overline{V}'$ in **Comp** by putting $(\{0,\dots,\widehat{k},\dots,m\})$ in the 0–th spot of $\overline{V}$ and $\overline{V}'$. If $v = v'$, then we can find a chain

$\{W^j\}_j$ from $\bar{V}$ to $\bar{V}'$ in **Comp** which preserves $x_0, x_1, \ldots, x_\ell$, and v using the above result that **Comp** satisfies (ii) for m. Truncating the 0–th spot of each W^j, we obtain the desired chain in **Outer** $\cap$ **Comp**. If $v \neq v'$, then we find a chain in **Comp** from $\bar{V}'$ to a $\bar{V}''$ with $v'' = v$, like above, and then find a chain in **Comp** from $\bar{V}$ to $\bar{V}''$. Combining chains, and truncating the 0–th spot again gives us the desired path from V to V'.

By a similar argument, with the role of $\{0, \ldots, \widehat{k}, \ldots, m\}$ played by $\{0, 1, \ldots, m\}$, the poset **Outer** $\cap$ **Center** satisfies hypothesis (ii) for $m-1$. Next we claim that the union of **Outer** $\cap$ **Comp** with **Outer** $\cap$ **Center** also satisfies hypothesis (ii) for $m-1$. Suppose that $V \subseteq$ **Outer** $\cap$ **Comp** and $V' \subseteq$ **Outer** $\cap$ **Center** are m–element linearly ordered subposets with $V_{s_i} = x_i = V'_{s_i}$ for all $0 \leq i \leq \ell$ and $s_0 < s_1 < \cdots < s_\ell$, where $1 \leq \ell \leq m-1$. Then v, $v', \{0, \ldots, \widehat{k}, \ldots, m\}$ and $\{0, 1, \ldots, m\}$ are in x_0, so we can transpose v and $\{0, \ldots, \widehat{k}, \ldots, m\}$ down in V' to take the place of v' and $\{0, 1, \ldots, m\}$, without perturbing $x_0, x_1, \ldots, x_\ell$. The resulting poset V'' is in **Outer** $\cap$ **Comp**, and was reached from V' by a chain in **Outer** $\cap$ **Center**. By the above, we can reach V'' from V by a chain in **Outer** $\cap$ **Comp**. Thus we have connected V and V' by a chain in (13), always preserving $x_0, x_1, \ldots, x_\ell$, and therefore **Outer** $\cap$ (**Comp** $\cup$ **Center**) satisfies hypothesis (ii) for $m-1$. $\square$

Remark 5.8 The posets $\mathbf{C}^\ell$ do not satisfy the hypotheses of Proposition 5.4, nor those of Proposition 5.3.

Corollary 5.9 *Let $m \geq 1$ be a positive integer.*

(i) *The posets* $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$, **Center**, **Outer**, **Comp**, *and* **Comp** $\cup$ **Center** *are each a colimit of finite ordinals* $[m-1]$ *and* $[m]$. *Similarly, the posets* $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ *and* **Outer** $\cap$ (**Comp** $\cup$ **Center**) *are each a colimit of finite ordinals* $[m-2]$ *and* $[m-1]$. *(By definition* $[-1] = \varnothing$*.)*

(ii) *The simplicial sets* $N(\mathbf{P}\,\mathrm{Sd}\,\Delta[m])$, N(**Center**), N(**Outer**), N(**Comp**) *and* N(**Comp** $\cup$ **Center**) *are each a colimit of simplicial sets of the form* $\Delta[m-1]$ *and* $\Delta[m]$. *Similarly, the two simplicial sets* $N(\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m])$ *and* N(**Outer** $\cap$ (**Comp** $\cup$ **Center**)) *are each a colimit of simplicial sets of the form* $\Delta[m-2]$ *and* $\Delta[m-1]$. *(By definition* $[-1] = \varnothing$*.)*

(iii) *The nerve of the colimit decomposition in* **Cat** *in (i) is the colimit decomposition in* **SSet** *in (ii).*

Proof (i) By Proposition 5.7, the posets $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$, **Center**, **Outer**, **Comp**, and **Comp** $\cup$ **Center** satisfy hypotheses (i) and (ii) of Proposition 5.4 for m, as do the posets $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ and **Outer** $\cap$ (**Comp** $\cup$ **Center**) for $m-1$. The hypotheses of

Proposition 5.4 imply the hypotheses of Proposition 5.3, so part (i) of the current corollary follows from Proposition 5.3.

(ii) By Proposition 5.7, the posets $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$, $\mathbf{Center}$, $\mathbf{Outer}$, $\mathbf{Comp}$, and $\mathbf{Comp} \cup \mathbf{Center}$ satisfy hypotheses (i) and (ii) of Proposition 5.4 for m, as do the posets $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ and $\mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center})$ for $m-1$. So Proposition 5.4 applies and we immediately obtain part (ii) of the current corollary.

(iii) This follows from Corollary 5.5 and Proposition 5.7. □

6 Thomason structure on Cat

The Thomason structure on **Cat** [85] is transferred from the standard model structure on **SSet** by transferring across the adjunction

(14)
$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "\mathrm{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathrm{Ex}^2"] \arrow[r, bend left, "c"] \arrow[r, phantom, "\perp"] & \mathbf{Cat}. \arrow[l, bend left, "N"] \end{tikzcd}$$

In other words, a functor F in **Cat** is a weak equivalence or fibration if and only if $\mathrm{Ex}^2 NF$ is. We present a quick proof that this defines a model structure using a corollary to Kan's Lemma on Transfer. Although Thomason did not do it exactly this way, it is practically the same, in spirit. Our proof relies on the results in the previous sections: the decomposition of $\mathrm{Sd}^2\,\Delta[m]$, the commutation of nerve with certain colimits, and the deformation retraction.

This proof of the Thomason structure on **Cat** will be the basis for our proof of the Thomason structure on **nFoldCat**. The key corollary to Kan's Lemma on Transfer is the following Corollary, inspired by Worytkiewicz–Hess–Parent–Tonks [89, Proposition 3.4.1].[3]

Corollary 6.1 *Let* $\mathbf{C}$ *be a cofibrantly generated model category with generating cofibrations* I *and generating acyclic cofibrations* J*. Suppose* $\mathbf{D}$ *is complete and cocomplete, and that* $F \dashv G$ *is an adjunction as in (15).*

(15)
$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

[3]The difference between [89, Proposition 3.4.1] and Corollary 6.1 of the present paper is that in hypothesis (i) we require Fi and Fj to be small with respect to the entire category $\mathbf{D}$, rather than merely small with respect to FI and FJ.

Assume the following.

(i) *For every* $i \in I$ *and* $j \in J$, *the objects* $\operatorname{dom} Fi$ *and* $\operatorname{dom} Fj$ *are small with respect to the entire category* $\mathbf{D}$.

(ii) *For any ordinal* λ *and any colimit preserving functor* $X\colon \lambda \longrightarrow \mathbf{C}$ *such that* $X_\beta \longrightarrow X_{\beta+1}$ *is a weak equivalence, the transfinite composition*

$$X_0 \longrightarrow \operatorname*{colim}_{\lambda} X$$

is a weak equivalence.

(iii) *For any ordinal* λ *and any colimit preserving functor* $Y\colon \lambda \longrightarrow \mathbf{D}$, *the functor* G *preserves the colimit of* Y.

(iv) *If* j' *is a pushout of* $F(j)$ *in* $\mathbf{D}$ *for* $j \in J$, *then* $G(j')$ *is a weak equivalence in* $\mathbf{C}$.

Then there exists a cofibrantly generated model structure on $\mathbf{D}$ *with generating cofibrations* FI *and generating acyclic cofibrations* FJ. *Further,* f *is a weak equivalence in* $\mathbf{D}$ *if and only* $G(f)$ *is a weak equivalence in* $\mathbf{C}$, *and* f *is a fibration in* $\mathbf{D}$ *if and only* $G(f)$ *is a fibration in* $\mathbf{C}$.

Proof For a proof of a similar statement, see Fiore–Paoli–Pronk [25]. The only difference between the statement here and the one proved in [25] is that here we only require in hypothesis (iii) that G preserves colimits indexed by any ordinal λ, rather than more general filtered colimits. The proof of the statement here is the same as in [25]: it is a straightforward application of Kan's Lemma on Transfer. □

Lemma 6.2 *The functor* Ex *preserves and reflects weak equivalences. That is, a morphism* f *of simplicial sets is a weak equivalence if and only if* $\operatorname{Ex} f$ *is a weak equivalence.*

Proof There is a natural weak equivalence $1_{\mathbf{SSet}} \Longrightarrow \operatorname{Ex}$ by Kan [55, Lemma 3.7], or more recently Joyal–Tierney [52, Theorem 6.2.4] or Goerss–Jardine [30, Theorem 4.6]. The naturality diagram

$$\begin{tikzcd} X \arrow[r, "\text{w. e.}"] \arrow[d, "f"'] & \operatorname{Ex} X \arrow[d, "\operatorname{Ex} f"] \\ Y \arrow[r, "\text{w. e.}"'] & \operatorname{Ex} Y \end{tikzcd}$$

then implies the Proposition. □

We may now prove Thomason's Theorem.

Theorem 6.3 *There is a model structure on* **Cat** *in which a functor F is a weak equivalence respectively fibration if and only if $\mathrm{Ex}^2 NF$ is a weak equivalence respectively fibration in* **SSet**. *This model structure is cofibrantly generated with generating cofibrations*

$$\{ c\,\mathrm{Sd}^2\,\partial\Delta[m] \longrightarrow c\,\mathrm{Sd}^2\,\Delta[m] \mid m \geq 0 \}$$

and generating acyclic cofibrations

$$\{ c\,\mathrm{Sd}^2\,\Lambda^k[m] \longrightarrow c\,\mathrm{Sd}^2\,\Delta[m] \mid 0 \leq k \leq m \text{ and } m \geq 1 \}.$$

These functors were explicitly described in Section 3.

Proof (i) The categories $c\,\mathrm{Sd}^2\,\partial\Delta[m]$ and $c\,\mathrm{Sd}^2\,\Lambda^k[m]$ each have a finite number of morphisms, hence they are finite, and are small with respect to **Cat**. For a proof, see Fiore–Paoli–Pronk [25, Proposition 7.6].

(ii) The model category **SSet** is cofibrantly generated, and the domains and codomains of the generating cofibrations and generating acyclic cofibrations are finite. As in Hovey's book [44, Corollary 7.4.2], this implies that transfinite compositions of weak equivalences in **SSet** are weak equivalences.

(iii) The nerve functor preserves filtered colimits. Every ordinal is filtered, so the nerve functor preserves λ–sequences.

The Ex functor preserves colimits of λ–sequences as well. We use the idea in the proof by Worytkiewicz–Hess–Parent–Tonks [89, Theorem 4.5.1]. First recall that for each m, the simplicial set $\mathrm{Sd}\,\Delta[m]$ is finite, so that $\mathbf{SSet}(\mathrm{Sd}\,\Delta[m], -)$ preserve colimits of all λ–sequences. If $Y\colon \lambda \longrightarrow \mathbf{SSet}$ is a λ–sequence, then

$$\begin{aligned}\Big(\mathrm{Ex}\,\operatorname*{colim}_{\lambda} Y\Big)_m &= \mathbf{SSet}\Big(\mathrm{Sd}\,\Delta[m], \operatorname*{colim}_{\lambda} Y\Big)\\ &\cong \operatorname*{colim}_{\lambda} \mathbf{SSet}(\mathrm{Sd}\,\Delta[m], Y)\\ &\cong \Big(\operatorname*{colim}_{\lambda} \mathrm{Ex}\,Y\Big)_m.\end{aligned}$$

Colimits in **SSet** are formed pointwise, we see that Ex preserves λ–sequences.

Thus $\mathrm{Ex}^2 N$ preserves λ–sequences.

(iv) Let $j\colon \Lambda^k[m] \longrightarrow \Delta[m]$ be a generating acyclic cofibration for **SSet**. Let the functor j' be the pushout along L as in the following diagram with $m \geq 1$.

$$\begin{tikzcd} c\,\mathrm{Sd}^2\,\Lambda^k[m] \arrow[r, "L"] \arrow[d, "c\,\mathrm{Sd}^2\,j"'] & \mathbf{B} \arrow[d, "j'"] \\ c\,\mathrm{Sd}^2\,\Delta[m] \arrow[r] & \mathbf{P} \end{tikzcd}$$

We factor j' into two inclusions

$$\mathbf{B} \xrightarrow{i} \mathbf{Q} \longrightarrow \mathbf{P}$$

and show that the nerve of each is a weak equivalence.

By Remark 3.5 the only free composites that occur in the pushout $\mathbf{P}$ are of the form (f_1, f_2)

$$\xrightarrow{f_1} \ \xrightarrow{f_2}$$

where f_1 is a morphism in $\mathbf{B}$ and f_2 is a morphism of **Outer** with source in $c\,\mathrm{Sd}^2\,\Lambda^k[m]$ and target outside of $c\,\mathrm{Sd}^2\,\Lambda^k[m]$ (see for example the drawing of $c\,\mathrm{Sd}^2\,\Delta[m]$ in Figure 1). Hence, $\mathbf{P}$ is the union

$$\mathbf{P} = \overbrace{(\mathbf{B} \coprod_{c\,\mathrm{Sd}^2\,\Lambda^k[m]} \mathbf{Outer})}^{\mathbf{Q}} \cup \overbrace{(\mathbf{Comp} \cup \mathbf{Center})}^{\mathbf{R}} \tag{16}$$

by Proposition 3.10, all free composites are in $\mathbf{Q}$, and they have the form (f_1, f_2).

We claim that the nerve of the inclusion $i\colon \mathbf{B} \longrightarrow \mathbf{Q}$ is a weak equivalence. Let $\overline{r}\colon \mathbf{Q} \longrightarrow \mathbf{B}$ be the identity on $\mathbf{B}$, and for any $(v_0, \dots, v_q) \in \mathbf{Outer}$ we define $\overline{r}(v_0, \dots, v_q) = (u_0, \dots, u_p)$ where $(u_0, \dots, u_p)$ is the maximal subset

$$\{u_0, \dots, u_p\} \subseteq \{v_0, \dots, v_q\}$$

that is in $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ (recall Proposition 3.4 (ii)). On free composites in $\mathbf{Q}$ we then have $\overline{r}(f_1, f_2) = (f_1, \overline{r}(f_2))$. More conceptually, we define $\overline{r}\colon \mathbf{Q} \longrightarrow \mathbf{B}$ using the universal property of the pushout $\mathbf{Q}$ and the maps $1_{\mathbf{B}}$ and Lr (the functor r is as in Proposition 3.4 (ii)).

Then $\overline{r}i = 1_{\mathbf{B}}$, and there is a unique natural transformation $i\overline{r} \Longrightarrow 1_{\mathbf{Q}}$ which is the identity morphism on the objects of $\mathbf{B}$. Thus $|Ni|\colon |N\mathbf{B}| \longrightarrow |N\mathbf{Q}|$ includes $|N\mathbf{B}|$ as a deformation retract of $|N\mathbf{Q}|$.

Next we show that the nerve of the inclusion $\mathbf{Q} \longrightarrow \mathbf{P}$ is also a weak equivalence. The intersection of $\mathbf{Q}$ and $\mathbf{R}$ in (16) is equal to

$$\mathbf{S} = \mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center}).$$

Proposition 5.2 then implies that $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ satisfy the hypotheses of Proposition 5.1. Then

$$
\begin{aligned}
|N\mathbf{Q}| &\cong |N\mathbf{Q}| \coprod_{|N\mathbf{S}|} |N\mathbf{S}| && \text{(pushout along identity)}\\
&\simeq |N\mathbf{Q}| \coprod_{|N\mathbf{S}|} |N\mathbf{R}| && \text{(Proposition 4.3 and Gluing Lemma)}\\
&\cong |N\mathbf{Q} \coprod_{N\mathbf{S}} N\mathbf{R}| && \text{(realization is a left adjoint)}\\
&\cong |N(\mathbf{Q} \coprod_{\mathbf{S}} \mathbf{R})| && \text{(Propositions 5.1 and 5.2)}\\
&= |N\mathbf{P}|.
\end{aligned}
\tag{17}
$$

In the second line, for the application of the Gluing Lemma (see Goerss–Jardine [30, Lemma 8.12] or Hirschhorn [43, Proposition 13.5.4]), we use two identities and the inclusion $|N\mathbf{S}| \longrightarrow |N\mathbf{R}|$. It is a homotopy equivalence whose inverse is the retraction in Proposition 4.3. We conclude that the inclusion $|N\mathbf{Q}| \longrightarrow |N\mathbf{P}|$ is a weak equivalence, as it is the composite of the morphisms in Equation (17). It is even a homotopy equivalence by Whitehead's Theorem.

We conclude that $|Nj'|$ is the composite of two weak equivalences

$$|N\mathbf{B}| \xrightarrow{|Ni|} |N\mathbf{Q}| \longrightarrow |N\mathbf{P}|$$

and is therefore a weak equivalence. By Lemma 6.2, the functor Ex preserves weak equivalences, so that $\mathrm{Ex}^2 Nj'$ is also a weak equivalence of simplicial sets. Part (iv) of Corollary 6.1 then holds, and we have the Thomason model structure on **Cat**. □

7 Pushouts and colimit decompositions of $c^n \delta_! \operatorname{Sd}^2 \Delta[m]$

Next we enhance the proof of the **Cat**–case to obtain the **nFoldCat**–case. The preparations of Section 3, Section 4, and Section 5 are adapted in this section to n–fold categorification.

Proposition 7.1 *Let $d^i \colon [m-1] \longrightarrow [m]$ be the injective order preserving map which skips i. Then the pushout in* **nFoldCat**

$$
\begin{tikzcd}
{[m-1] \boxtimes \cdots \boxtimes [m-1]} \arrow[r, "d^i \boxtimes \cdots \boxtimes d^i"] \arrow[d, "d^i \boxtimes \cdots \boxtimes d^i"'] & {[m] \boxtimes \cdots \boxtimes [m]} \arrow[d] \\
{[m] \boxtimes \cdots \boxtimes [m]} \arrow[r] & \mathbb{P}
\end{tikzcd}
\tag{18}
$$

does not have any free composites, and is an n–fold poset.

Proof We do the proof for $n = 2$.

We consider horizontal morphisms, the proof for vertical morphisms and more generally squares is similar. We denote the two copies of $[m]\boxtimes[m]$ by $\mathbb{N}_1$ and $\mathbb{N}_2$ for convenience. A free composite occurs whenever there are

$$f_1\colon A_1 \longrightarrow B_1$$

$$g_2\colon B_2 \longrightarrow C_2$$

in $\mathbb{N}_1$ and $\mathbb{N}_2$ respectively such that B_1 and B_2 are identified in the pushout, and further, the images of $[m-1]\boxtimes[m-1]$ contain neither f_1 nor g_2. Inspection of $d^i \boxtimes d^i$ shows that this does not occur. $\square$

Remark 7.2 The gluings of Proposition 7.1 are the only kinds of gluings that occur in $c^n\delta_!\operatorname{Sd}^2\Delta[m]$ and $c^n\delta_!\operatorname{Sd}^2\Lambda^k[m]$ because of the description of glued simplices in Remark 3.2 and the fact that $c^n\delta_!$ is a left adjoint.

Corollary 7.3 *Consider the pushout $\mathbb{P}$ in Proposition 7.1. The application of δ^*N^n to Diagram (18) is a pushout and is drawn in Diagram (19).*

(19)
$$\begin{tikzcd}
\Delta[m-1]\times\cdots\times\Delta[m-1] \arrow[r, "\delta^*N^n(d^i\boxtimes\cdots\boxtimes d^i)"] \arrow[d, "\delta^*N^n(d^i\boxtimes\cdots\boxtimes d^i)"'] & \Delta[m]\times\cdots\times\Delta[m] \arrow[d] \\
\Delta[m]\times\cdots\times\Delta[m] \arrow[r] & \delta^*N^n\mathbb{P}
\end{tikzcd}$$

Proof The functor N^n preserves a pushout whenever there are no free composites in that pushout, which is the case here by Proposition 7.1. Also, δ^* is a left adjoint (it admits a right adjoint by Kan extension), so δ^* preserves any pushout. $\square$

The n–fold version of Proposition 5.3 is as follows.

Proposition 7.4 *Let $\mathbf{T}$ and F be as in Proposition 5.3. In particular, $\mathbf{T}$ could be $\mathbf{P}\operatorname{Sd}\Delta[m]$, $\mathbf{Center}$, $\mathbf{Outer}$, $\mathbf{Comp}$ or $\mathbf{Comp}\cup\mathbf{Center}$ by Proposition 5.7. Then $c^n\delta_!N\mathbf{T}$ is the union inside of $\mathbf{T}\boxtimes\mathbf{T}\boxtimes\cdots\boxtimes\mathbf{T}$ given by*

$$c^n\delta_!N\mathbf{T} = \bigcup_{\substack{U\subseteq\mathbf{T}\ \text{lin. ord.}\\ |U|=m+1}} U\boxtimes U\boxtimes\cdots\boxtimes U. \tag{20}$$

Similarly, if $\mathbf{S} = \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ *or* $\mathbf{S} = \mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center})$, *then by Proposition 5.7,* $c^n\delta_! N\mathbf{S}$ *is the union inside of* $\mathbf{S} \boxtimes \mathbf{S} \boxtimes \cdots \boxtimes \mathbf{S}$ *given by*

$$c^n\delta_! N\mathbf{S} = \bigcup_{\substack{U \subseteq \mathbf{S} \text{ lin. ord.} \\ |U|=m}} U \boxtimes U \boxtimes \cdots \boxtimes U. \tag{21}$$

If $\mathbf{T}$ *or* $\mathbf{S}$ *is any of the respective posets above, then*

$$\begin{aligned} c^n\delta_! N\mathbf{T} &\subseteq \mathbf{P}\,\mathrm{Sd}\,\Delta[m] \boxtimes \mathbf{P}\,\mathrm{Sd}\,\Delta[m] \boxtimes \cdots \boxtimes \mathbf{P}\,\mathrm{Sd}\,\Delta[m] \\ c^n\delta_! N\mathbf{S} &\subseteq \mathbf{P}\,\mathrm{Sd}\,\Delta[m] \boxtimes \mathbf{P}\,\mathrm{Sd}\,\Delta[m] \boxtimes \cdots \boxtimes \mathbf{P}\,\mathrm{Sd}\,\Delta[m]. \end{aligned}$$

Proof For any linearly ordered subposet U of $\mathbf{T}$ we have

$$\begin{aligned} c^n\delta_! NU &= c^n(NU \boxtimes NU \boxtimes \cdots \boxtimes NU) \\ &= cNU \boxtimes cNU \boxtimes \cdots \boxtimes cNU \\ &= U \boxtimes U \boxtimes \cdots \boxtimes U. \end{aligned}$$

Thus we have

$$\begin{aligned} c^n\delta_! N\mathbf{T} &= c^n\delta_! N\Big(\operatorname*{colim}_{\mathbf{J}} F\Big) && \text{by Proposition 5.3} \\ &= c^n\delta_!\Big(\operatorname*{colim}_{\mathbf{J}} NF\Big) && \text{by Corollary 5.5} \\ &= \operatorname*{colim}_{\mathbf{J}} c^n\delta_! NF && \text{because } c^n\delta_! \text{ is a left adjoint} \\ &= \operatorname*{colim}_{U \in \mathbf{J}} U \boxtimes U \boxtimes \cdots \boxtimes U \\ &= \bigcup_{\substack{U \subseteq \mathbf{T} \text{ lin. ord.} \\ |U|=m+1}} U \boxtimes U \boxtimes \cdots \boxtimes U. \end{aligned}$$

This last equality follows for the same reason that $\mathbf{T}$ (=colimit of F) is the union of the linearly ordered subposets U of $\mathbf{T}$ with exactly $m+1$ elements. See also Proposition 7.1. □

Remark 7.5 Note that

$$\mathbf{T} \boxtimes \mathbf{T} \boxtimes \cdots \boxtimes \mathbf{T} \supsetneq \bigcup_{\substack{U \subseteq \mathbf{T} \text{ lin. ord.} \\ |U|=m+1}} U \boxtimes U \boxtimes \cdots \boxtimes U.$$

Definition 7.6 An n–fold category is an *n–fold preorder* if for any two objects A and B, there is at most one n–cube with A in the $(0,0,\dots,0)$–corner and B in the $(1,1,\dots,1)$–corner. If $\mathbb{D}$ is an n–fold preorder, we define an ordinary preorder on

$\operatorname{Obj}\mathbb{D}$ by $A \leq B$ if and only if there exists an n–cube with A in the $(0,0,\dots,0)$–corner and B in the $(1,1,\dots,1)$–corner. We call an n–fold preorder an *n–fold poset* if $\leq$ is additionally antisymmetric as a preorder on $\operatorname{Obj}\mathbb{D}$, that is, $(\operatorname{Obj}\mathbb{D},\leq)$ is an n–fold poset. If $\mathbb{T}$ is an n–fold preorder and $\mathbb{S}$ is a sub–n–fold preorder, then $\mathbb{S}$ is *down-closed in* $\mathbb{T}$ if $A \leq B$ and $B \in \mathbb{S}$ implies $A \in \mathbb{S}$. If $\mathbb{T}$ is an n–fold preorder and $\mathbb{S}$ is a sub–n–fold preorder, then the *up-closure* of $\mathbb{S}$ in $\mathbb{T}$ is the full sub–n–category of $\mathbb{T}$ on the objects B in $\mathbb{T}$ such that $B \geq A$ for some object $A \in \mathbb{S}$.

Example 7.7 If $\mathbf{T}$ is a poset, $\mathbf{T}\boxtimes\mathbf{T}\boxtimes\cdots\boxtimes\mathbf{T}$ is an n–fold poset, and $(a_1,\dots,a_n) \leq (b_1,\dots,b_n)$ if and only if $a_i \leq b_i$ in $\mathbf{T}$ for all $1 \leq i \leq n$. If $\mathbf{T}$ is as in Proposition 5.3, then the n–fold category $c^n\delta_! N\mathbf{T}$ is also an n–fold poset, as it is contained in the n–fold poset $\mathbf{T}\boxtimes\mathbf{T}\boxtimes\cdots\boxtimes\mathbf{T}$ by Equation (20).

Proposition 7.8 *The n–fold poset $c^n\delta_! N\mathbf{P}\operatorname{Sd}\Lambda^k[m]$ is down-closed in the n–fold poset $c^n\delta_! N\mathbf{P}\operatorname{Sd}\Delta[m]$.*

Proof Suppose $(a_1,\dots,a_n) \leq (b_1,\dots,b_n)$ in $c^n\delta_! N\mathbf{P}\operatorname{Sd}\Delta[m]$ and $(b_1,\dots,b_n) \in c^n\delta_! N\mathbf{P}\operatorname{Sd}\Lambda^k[m]$. We make use of Equations (20) and (21) in Proposition 7.4. There is a linearly ordered subposet V of $\mathbf{P}\operatorname{Sd}\Lambda^k[m]$ such that $|V| = m$ and $b_1,\dots,b_n \in V$. There also exists a linearly ordered subposet U of $\mathbf{P}\operatorname{Sd}\Delta[m]$ such that $|U| = m+1$ and $a_1,\dots,a_n \in U$. In particular, $\{a_1,\dots,a_n\}$ is linearly ordered.

The preorder on $\operatorname{Obj} c^n\delta_! N\mathbf{P}\operatorname{Sd}\Delta[m]$ then implies that $a_i \leq b_i$ in $\mathbf{P}\operatorname{Sd}\Delta[m]$ for all i, so that $a_i \in \mathbf{P}\operatorname{Sd}\Lambda^k[m]$ by Proposition 3.3. Since the length of a maximal chain in $\mathbf{P}\operatorname{Sd}\Lambda^k[m]$ is m, the linearly ordered poset $\{a_1,\dots,a_n\}$ has at most m elements. By Proposition 5.7, there exists a linearly ordered subposet U' of $\mathbf{P}\operatorname{Sd}\Lambda^k[m]$ such that $|U'| = m$ and $a_1,\dots,a_n \in U'$. Consequently, $(a_1,a_2,\dots,a_n) \in c^n\delta_! N\mathbf{P}\operatorname{Sd}\Lambda^k[m]$, again by Equation (21). □

Proposition 7.9 *The up-closure of $c^n\delta_! N\mathbf{P}\operatorname{Sd}\Lambda^k[m]$ in $c^n\delta_! N\mathbf{P}\operatorname{Sd}\Delta[m]$ is contained in $c^n\delta_! N\,\mathbf{Outer}$.*

Proof An explicit description of all three n–fold posets is given in Equations (20) and (21) of Proposition 7.4. Recall that $\mathbf{P}\operatorname{Sd}\Lambda^k[m]$ and $\mathbf{Outer}$ satisfy hypothesis (i) of Proposition 5.3 for m and $m+1$ respectively (by Proposition 5.7).

Suppose $$A = (a_1,a_2,\dots,a_n) \leq (b_1,b_2,\dots,b_n) = B$$

in $c^n\delta_! N\mathbf{P}\operatorname{Sd}\Delta[m]$, $A \in c^n\delta_! N\mathbf{P}\operatorname{Sd}\Lambda^k[m]$, and $B \in c^n\delta_! N\mathbf{P}\operatorname{Sd}\Delta[m]$. Then

$$\{a_1,a_2,\dots,a_n\} \subseteq U$$

for some linearly ordered subposet $U \subseteq \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ with $|U| = m$, and

$$\{b_1, b_2, \ldots, b_n\} \subseteq V$$

for some linearly ordered subposet $V \subseteq \mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ with $|V| = m+1$. We also have $a_i \leq b_i$ in $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ for all i, so that each b_i is in the up-closure of $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ in $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$, namely in $\mathbf{Outer}$. Since Equation (20) holds for $\mathbf{Outer}$, we see $B \in c^n \delta_! N\mathbf{Outer}$, and therefore the up-closure of $c^n \delta_! N \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$ is contained in $c^n \delta_! N\mathbf{Outer}$. □

Remark 7.10 (i) If α is an n–cube in $c^n \delta_! N \mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ whose i–th target is in $c^n \delta_! N \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$, then α is in $c^n \delta_! N \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$.

(ii) If α is an n–cube in $c^n \delta_! N \mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ whose i–th source is in $c^n \delta_! N \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$, then α is in $c^n \delta_! N\mathbf{Outer}$.

Proof (i) If α is an n–cube in $c^n \delta_! N \mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ whose i–th target is in the n–fold poset $c^n \delta_! N \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$, then its $(1,1,\ldots,1)$–corner is in $c^n \delta_! N \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$, as this corner lies on the i–th target. By Proposition 7.8, we then have α is in $c^n \delta_! N \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$.

(ii) If α is an n–cube in $c^n \delta_! N \mathbf{P}\,\mathrm{Sd}\,\Delta[m]$ whose i–th source is in $c^n \delta_! N \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$, then the $(0,0,\ldots,0)$–corner is in $c^n \delta_! N \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$, as this corner lies on the i–th source. By Proposition 7.9, we then have α is in $c^n \delta_! N\mathbf{Outer}$. □

Next we describe the diagonal of the nerve of certain n–fold categories as a union of n–fold products of standard simplices in Proposition 7.13. This proposition is also an analogue of Corollary 5.5 since it says the composite functor $\delta^* N^n c^n \delta_! N$ preserves colimits of certain posets.

Lemma 7.11 *For any finite, linearly ordered poset U we have*

$$\delta^* N^n c^n \delta_! NU = NU \times NU \times \cdots \times NU.$$

Proof Since U is a finite, linearly ordered poset, NU is isomorphic to $\Delta[m]$ for some nonnegative integer m, and we have

$$\begin{aligned}
\delta^* N^n c^n \delta_! NU &= \delta^* N^n c^n (NU \boxtimes NU \boxtimes \cdots \boxtimes NU) \\
&= \delta^* N^n (cNU \boxtimes cNU \boxtimes \cdots \boxtimes cNU) \\
&= \delta^* N^n (U \boxtimes U \boxtimes \cdots \boxtimes U) \\
&= \delta^* (NU \boxtimes NU \boxtimes \cdots \boxtimes NU) \\
&= NU \times NU \times \cdots \times NU.
\end{aligned}$$
□

Lemma 7.12 *For any finite, linearly ordered poset U, the simplicial set*

$$\delta^* N^n c^n \delta_! NU = NU \times NU \times \cdots \times NU$$

is M–skeletal for a large enough M depending on n and the cardinality of U.

Proof We prove that there is an M such that all simplices in degrees greater than M are degenerate.

Without loss of generality, we may assume U is $[m]$. We have

$$\begin{aligned} c^n \delta_! N[m] &= c^n \delta_! \Delta[m] \\ &= c^n \left(\Delta[m] \boxtimes \Delta[m] \boxtimes \cdots \boxtimes \Delta[m]\right) \\ &= (c\Delta[m]) \boxtimes (c\Delta[m]) \boxtimes \cdots \boxtimes (c\Delta[m]) \\ &= [m] \boxtimes [m] \boxtimes \cdots \boxtimes [m] \end{aligned}$$

by Example 2.19. An ℓ–simplex in $\delta^* N^n \left([m] \boxtimes [m] \boxtimes \cdots \boxtimes [m]\right)$ is an $\ell \times \ell \times \cdots \times \ell$ array of composable n–cubes in $[m] \boxtimes [m] \boxtimes \cdots \boxtimes [m]$, that is, a collection of n sequences of ℓ composable morphisms in $[m]$, namely $\left((f_j^1)_j, (f_j^2)_j, \ldots, (f_j^n)_j\right)$ where $1 \leq j \leq \ell$ and $f_{j+1}^i \circ f_j^i$ is defined for $j+1 \leq \ell$. An ℓ–simplex is degenerate if and only if there is a j_0 such that $f_{j_0}^1, f_{j_0}^2, \ldots, f_{j_0}^n$ are all identities. An ℓ–simplex has ℓ–many n–cubes along its diagonal, namely

$$(f_j^1, f_j^2, \ldots, f_j^n)$$

for $1 \leq j \leq \ell$. Since $[m]$ is finite, there is an integer M such that for any $\ell \geq 0$ and any ℓ–simplex y, there are at most M–many nontrivial n–cubes in y, that is, there are at most M–many tuples

$$(f_{j_1}^1, f_{j_2}^2, \ldots, f_{j_n}^n)$$

which have at least one $f_{j_i}^i$ nontrivial.

If $\ell > M$ then at least one of the ℓ–many n–cubes on the diagonal must be trivial, by the pigeon-hole principle. Hence, for $\ell > M$, every ℓ–simplex of $\delta^* N^n c^n \delta_! N[m]$ is degenerate. Finally, $\delta^* N^n c^n \delta_! N[m]$ is M–skeletal. □

Proposition 7.13 *Let $m \geq 1$ be a positive integer and $\mathbf{T}$ a poset satisfying the hypotheses (i) and (ii) of Proposition 5.4. In particular, $\mathbf{T}$ could be $\mathbf{P}\,\mathrm{Sd}\,\Delta[m]$, **Center**, **Outer**, **Comp** or **Comp** $\cup$ **Center** by Proposition 5.7. Let the functor $F\colon \mathbf{J} \longrightarrow \mathbf{Cat}$ and the universal cocone $\pi\colon F \Longrightarrow \Delta_{\mathbf{T}}$ be as indicated in Proposition 5.3. Then*

$$\begin{aligned} \delta^* N^n c^n \delta_! N\mathbf{T} &= \operatorname*{colim}_{\mathbf{J}} \delta^* N^n c^n \delta_! NF \\ &= \operatorname*{colim}_{\mathbf{J}} (NF \times \cdots \times NF) \end{aligned}$$

where $NF(U)$ is isomorphic to $\Delta[m-1]$ or $\Delta[m]$ for all $U \in \mathbf{J}$. Similarly, the simplicial sets $\delta^* N^n c^n \delta_! N(\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m])$ and

$$\delta^* N^n c^n \delta_! N(\mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center}))$$

are each a colimit of simplicial sets of the form $\Delta[m-2]\times\cdots\times\Delta[m-2]$ and $\Delta[m-1]\times \cdots\times\Delta[m-1]$. (By definition $[-1]=\varnothing$.)

Proof We first directly prove $\delta^* N^n c^n \delta_! N\mathbf{T}$ is a colimit of $\delta^* N^n c^n \delta_! NF\colon \mathbf{J}\longrightarrow\mathbf{SSet}$ along the lines of the proof of Proposition 5.4.

Let $M > m$ be a large enough integer such that the simplicial set $\delta^* N^n c^n \delta_! N[m]$ is M–skeletal. Such an M is guaranteed by Lemma 7.12.

Suppose $S \in \mathbf{SSet}$ and $\alpha\colon \delta^* N^n c^n \delta_! NF \Longrightarrow \Delta_S$ is a natural transformation. We induce a morphism of simplicial sets

$$G\colon \delta^* N^n c^n \delta_! N\mathbf{T} \longrightarrow S$$

by defining G on the M–skeleton as follows.

As in the proof of Proposition 5.4, Δ_M denotes the full subcategory of Δ on the objects $[0],[1],\dots,[M]$ and $\mathrm{tr}_M\colon \mathbf{SSet}\longrightarrow\mathbf{Set}^{\Delta_M^{\mathrm{op}}}$ denotes the M–th truncation functor. The truncation $\mathrm{tr}_M(\delta^* N^n(c^n\delta_! N\mathbf{T}))$ is a union of the truncated simplicial subsets $\mathrm{tr}_M(\delta^* N^n(c^n\delta_! N\mathbf{V}))$ for $V\in\mathbf{J}$ with $|V|=m+1$, since

- $c^n\delta_! N\mathbf{T}$ is a union of such $c^n\delta_! N\mathbf{V}$ by Proposition 7.4,
- any maximal linearly ordered subset of $\mathbf{T}$ has $m+1$ elements, and
- $\delta^* N^n$ preserves unions.

We define

$$G_M|_{\mathrm{tr}_M(\delta^* N^n(c^n\delta_! N\mathbf{V}))}\colon \mathrm{tr}_M(\delta^* N^n(c^n\delta_! N\mathbf{V})) \longrightarrow \mathrm{tr}_M S$$

simply as $\mathrm{tr}_M\alpha_V$.

The morphism G_M is well-defined, for if $0\le \ell\le M$ and $x\in(\mathrm{tr}_M(\delta^* N^n c^n \delta_! N\mathbf{V}))_\ell$ and $x\in(\mathrm{tr}_M(\delta^* N^n c^n \delta_! N\mathbf{V}))_\ell$ with $|V|=m+1=|V'|$, then V and V' can be connected by a sequence $W^0, W^1,\dots,W^k$ of $(m{+}1)$–element linearly ordered subsets of $\mathbf{T}$ that all contain the linearly ordered subposet x and satisfy the properties in hypothesis (ii). By a naturality argument as in the proof of Proposition 5.3, we have a string of equalities

$$\alpha_{W^0}(x)=\alpha_{W^1}(x)=\cdots=\alpha_{W^k}(x),$$

and we conclude $\alpha_V(x)=\alpha_{V'}(x)$ so that $G_M(x)$ is well defined.

By definition $\Delta_{G_M} \circ \mathrm{tr}_M N\pi = \mathrm{tr}_M \alpha$. We may extend this to nontruncated simplicial sets by recalling from above that the simplicial set $\delta^* N^n c^n \delta_! N\mathbf{T}$ is M*–skeletal*, that is, the counit inclusion

$$\mathrm{sk}_M \mathrm{tr}_M(\delta^* N^n c^n \delta_! N\mathbf{T}) \longrightarrow \delta^* N^n c^n \delta_! N\mathbf{T}$$

is the identity.

Thus G_M extends to $G\colon N\mathbf{T} \longrightarrow S$ and $\Delta_G \circ N\pi = \alpha$.

Lastly, the morphism G is unique, since the simplicial subsets $\delta^* N^n c^n \delta_! N\mathbf{V}$ for $|V| = m+1$ in $\mathbf{J}$ cover $\delta^* N^n c^n \delta_! N\mathbf{T}$ by hypothesis (i).

So far we have proved $\delta^* N^n c^n \delta_! N\mathbf{T} = \mathrm{colim}_{\mathbf{J}}\, \delta^* N^n c^n \delta_! NF$. It only remains to show $\mathrm{colim}_{\mathbf{J}}\, \delta^* N^n c^n \delta_! NF = \mathrm{colim}_{\mathbf{J}}(NF \times \cdots \times NF)$. But this follows from Lemma 7.11 and that fact that $FV = V$ for all $V \in \mathbf{J}$. □

The n–fold version of Proposition 4.3 is the following.

Corollary 7.14 *The space $|\delta^* N^n c^n \delta_! N(\mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center}))|$ includes into the space $|\delta^* N^n c^n \delta_! N(\mathbf{Comp} \cup \mathbf{Center})|$ as a deformation retract.*

Proof Recall realization $|\cdot|$ commutes with colimits since it is a left adjoint, and $|\cdot|$ also commutes with products. We do the multistage deformation retraction of Proposition 4.3 to each factor $|\Delta[m]|$ of $|\Delta[m]| \times \cdots \times |\Delta[m]|$ in the colimit of Proposition 7.13. This is the desired deformation retraction of $|\delta^* N^n c^n \delta_! N(\mathbf{Comp} \cup \mathbf{Center})|$ to $|\delta^* N^n c^n \delta_! N(\mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center}))|$. □

Proposition 7.15 *Consider $n = 2$. Let $j\colon \Lambda^k[m] \longrightarrow \Delta[m]$ be a generating acyclic cofibration for* **SSet**, *$\mathbb{B}$ a double category, and L a double functor as below. Then the pushout $\mathbb{Q}$ in the diagram*

$$\begin{tikzcd} c^2\delta_!\,\mathrm{Sd}^2\,\Lambda^k[m] \arrow[r, "L"] \arrow[d, "c^2\delta_!\,\mathrm{Sd}^2\, j"'] & \mathbb{B} \arrow[d] \\ c^2\delta_! N\,\mathbf{Outer} \arrow[r] & \mathbb{Q} \end{tikzcd} \tag{22}$$

has the following form.

(i) *The object set of $\mathbb{Q}$ is the pushout of the object sets.*

(ii) *The set of horizontal morphisms of* $\mathbb{Q}$ *consists of the set of horizontal morphisms of* $\mathbb{B}$*, the set of horizontal morphisms of* $c^2\delta_! N\,\mathbf{Outer}$*, and the set of formal composites of the form*

$$\xrightarrow{f_1}\ \xrightarrow{(1,f_2)}$$

where f_1 *is a horizontal morphism in* $\mathbb{B}$*,* f_2 *is a morphism in* $\mathbf{Outer}$*, and the target of* f_1 *is the source of* $(1, f_2)$ *in* $\operatorname{Obj}\mathbb{Q}$*.*

(iii) *The set of vertical morphisms of* $\mathbb{Q}$ *consists of the set of vertical morphisms of* $\mathbb{B}$*, the set of vertical morphisms of* $c^2\delta_! N\,\mathbf{Outer}$*, and the set of formal composites of the form*

$$\begin{tikzcd} {} \arrow[d, "g_1"] \\ {} \\ {} \arrow[d, "{(g_2,1)}"] \\ {} \end{tikzcd}$$

where g_1 *is a vertical morphism in* $\mathbb{B}$*,* g_2 *is a morphism in* $\mathbf{Outer}$*, and the target of* g_1 *is the source of* $(g_2, 1)$ *in* $\operatorname{Obj}\mathbb{Q}$*.*

(iv) *The set of squares of* $\mathbb{Q}$ *consists of the set of squares of* $\mathbb{B}$*, the set of squares of* $c^2\delta_! N\,\mathbf{Outer}$*, and the set of formal composites of the following three forms.*

(a)
$$\begin{tikzcd} {} \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\alpha_1"] & (W, A') \arrow[r, "{(1_W,f_2)}"] \arrow[d, "{(g,1_{A'})}"] & (W, B') \arrow[d, "{(g,1_{B'})}"] \\ {} \arrow[r, "p_1"'] & (A, A') \arrow[r, "{(1_A,f_2)}"'] & (A, B') \end{tikzcd}$$

(b)
$$\begin{tikzcd} {} \arrow[rr, "f_1"] \arrow[d, "g_1"'] \arrow[drr, phantom, "\beta_1"] & & {} \arrow[d, "q_1"] \\ (A, W') \arrow[rr, "{(1_A,f)}"] \arrow[d, "{(g_2,1_{W'})}"'] & & (A, A') \arrow[d, "{(g_2,1_{A'})}"] \\ (B, W') \arrow[rr, "{(1_B,f)}"'] & & (B, A') \end{tikzcd}$$

(c)
$$\begin{tikzcd}
{} \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\gamma_1"] & (W,A') \arrow[r, "(1_W,f_2)"] \arrow[d, "(g,1_{A'})"] & (W,B') \arrow[d, "(g,1_{B'})"] \\
(A,W') \arrow[r, dashed, "(1_A,f)"] \arrow[d, "(g_2,1_{W'})"'] & (A,A') \arrow[r, dashed, "(1_A,f_2)"] \arrow[d, "(g_2,1_{A'})"] & (A,B') \arrow[d, "(g_2,1_{B'})"] \\
(B,W') \arrow[r, "(1_B,f)"'] & (B,A') \arrow[r, "(1_B,f_2)"'] & (B,B')
\end{tikzcd}$$

where $\alpha_1, \beta_1, \gamma_1$ are squares in $\mathbb{B}$, the horizontal morphisms f_1, p_1 are in $\mathbb{B}$, the vertical morphisms g_1, q_1 are in $\mathbb{B}$, and the morphisms f, f_2, g, g_2 are in **Outer**. *Further, each boundary of each square in $c^2\delta_! N$* **Outer** *must belong to a linearly ordered subset of* **Outer** *of cardinality $m+1$ (see Proposition 7.4). So for example, f and g_2 must belong to a linearly ordered subset of* **Outer** *of cardinality $m+1$, and f_2 and g must belong to another linearly ordered subset of* **Outer** *of cardinality $m+1$. Of course, the sources and targets in each of (a), (b) and (c) must match appropriately.*

Proof All of this follows from the colimit formula in **DblCat**, which is Theorem 4.6 of [25], and is also a special case of Proposition 2.13 in the present paper. The horizontal and vertical 1–categories of $\mathbb{Q}$ are the pushouts of the horizontal and vertical 1–categories, so (i) follows, and then (ii) and (iii) follow from Remark 3.5. To see (iv), one observes that the only free composite pairs of squares that can occur are of the first two forms, again from Remark 3.5. Certain of these can be composed with a square in $c^2\delta_! N$ **Outer** to obtain the third form. No further free composites can be obtained from these ones because of Remark 3.5 and the special form of $c^2\delta_! N$ **Outer**. □

Proposition 7.16 *Consider $n=2$ and the pushout $\mathbb{Q}$ in diagram (22). Then any q–simplex in $\delta^* N^2\mathbb{Q}$ is a $q\times q$–matrix of composable squares of $\mathbb{Q}$ which has the form in Figure 2. The submatrix labelled $\mathbb{B}$ is a matrix of squares in $\mathbb{B}$. The submatrix labelled a is a single column of squares of the form (a) in Proposition 7.15 (iv) (the α_1's may be trivial). The submatrix labelled b is a single row of squares of the form (b) in Proposition 7.15 (iv) (the β_1's may be trivial). The submatrix labelled c is a single square of the form (c) in Proposition 7.15 (iv) (part of the square may be trivial). The remaining squares in the q–simplex are squares of $c^2\delta_! N$* **Outer**.

Proof These are the only composable $q\times q$–matrices of squares because of the special form of the horizontal and vertical 1–categories. □

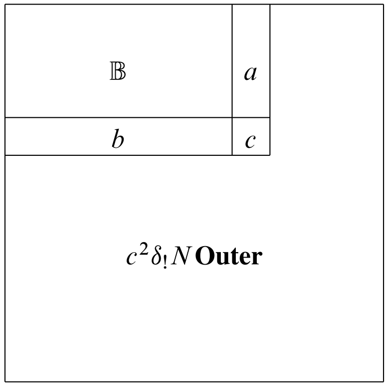

Figure 2: A q–simplex in $\delta^* N^2\mathbb{Q}$

Remark 7.17 The analogues of Proposition 7.15 and Proposition 7.16 clearly hold in higher dimensions as well, only the notation gets more complicated. Proposition 2.13 provides the key to proving the higher dimensional versions, namely, it allows us to calculate the pushout in **nFoldCat** in steps: first the object set of the pushout, then sub–1–categories of the pushout in all n–directions, then the squares in the sub-double-categories of the pushout in each direction ij, then the cubes in the sub–3–fold-categories of the pushout in each direction ijk, and so on. Since we do not need the explicit formulations of Proposition 7.15 and Proposition 7.16 for $n > 2$ in this paper, we refrain from stating and proving them. In fact, we do not even need the case $n = 2$ for this paper; we only presented Proposition 7.15 and Proposition 7.16 as an illustration of how the pushout in **nFoldCat** works in a specific case.

The n–fold version of Proposition 5.1 is the following.

Proposition 7.18 *Suppose $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{S}$ are n–fold categories, and $\mathbb{S}$ is an n–foldly full n–fold subcategory of $\mathbb{Q}$ and $\mathbb{R}$ such that:*

(i) *If $f\colon x \longrightarrow y$ is a 1–morphism in $\mathbb{Q}$ (in any direction) and $x \in \mathbb{S}$, then $y \in \mathbb{S}$.*

(ii) *If $f\colon x \longrightarrow y$ is a 1–morphism in $\mathbb{R}$ (in any direction) and $x \in \mathbb{S}$, then $y \in \mathbb{S}$.*

Then the nerve of the pushout of n–fold categories is the pushout of the nerves, that is,

$$N^n(\mathbb{Q} \coprod_{\mathbb{S}} \mathbb{R}) \cong N^n\mathbb{Q} \coprod_{N^n\mathbb{S}} N^n\mathbb{R}. \tag{23}$$

Proof We claim that there are no free composite n–cubes in the pushout $\mathbb{Q} \coprod_{\mathbb{S}} \mathbb{R}$. Suppose that α is an n–cube in $\mathbb{Q}$ and β is an n–cube in $\mathbb{R}$ and that these are composable in the i–th direction. In other words, the i–th target of α is the i–th source of β, which we will denote by γ. Then γ must be an $(n-1)$–cube in $\mathbb{S}$, as it lies in both $\mathbb{Q}$ and $\mathbb{R}$. Since the corners of γ are in $\mathbb{S}$, we can use hypothesis (ii) to see that all corners of β are in $\mathbb{S}$ by travelling along edges that emanate from γ. By the fullness of $\mathbb{S}$, the cube β is in $\mathbb{S}$, and also $\mathbb{Q}$. Then $\beta \circ_i \alpha$ is in $\mathbb{Q}$ and is not free.

If α is in $\mathbb{R}$ and β is in $\mathbb{Q}$, we can similarly conclude that β is in $\mathbb{S}$, $\beta \circ_i \alpha$ is in $\mathbb{R}$, and $\beta \circ_i \alpha$ is not a free composite.

Thus, the pushout $\mathbb{Q} \coprod_{\mathbb{S}} \mathbb{R}$ has no free composite n–cubes, and hence no free composites of any cells at all.

Let $(\alpha_{\bar{j}})_{\bar{j}}$ be a p–simplex in $N^n(\mathbb{Q} \coprod_{\mathbb{S}} \mathbb{R})$. Then each $\alpha_{\bar{j}}$ is an n–cube in $\mathbb{Q}$ or $\mathbb{R}$, since there are no free composites. By repeated application of the argument above, if $\alpha_{(0,\dots,0)}$ is in $\mathbb{Q}$ then every $\alpha_{\bar{j}}$ is in $\mathbb{Q}$. Similarly, if $\alpha_{(0,\dots,0)}$ is in $\mathbb{R}$ then every $\alpha_{\bar{j}}$ is in $\mathbb{R}$. Thus we have a morphism $N^n(\mathbb{Q} \coprod_{\mathbb{S}} \mathbb{R}) \longrightarrow N^n\mathbb{Q} \coprod_{N^n\mathbb{S}} N^n\mathbb{R}$. Its inverse is the canonical morphism $N^n\mathbb{Q} \coprod_{N^n\mathbb{S}} N^n\mathbb{R} \longrightarrow N^n(\mathbb{Q} \coprod_{\mathbb{S}} \mathbb{R})$.

Note that we have not used the higher dimensional version of Proposition 7.15 nor Proposition 7.16 anywhere in this proof. □

8 Thomason structure on nFoldCat

We apply Corollary 6.1 to transfer across the adjunction below.

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "\mathrm{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathrm{Ex}^2"] \arrow[r, bend left, "\delta_!"] \arrow[r, phantom, "\perp"] & \mathbf{SSet^n} \arrow[l, bend left, "\delta^*"] \arrow[r, bend left, "c^n"] \arrow[r, phantom, "\perp"] & \mathbf{nFoldCat} \arrow[l, bend left, "N^n"] \end{tikzcd} \tag{24}$$

Proposition 8.1 *Let F be an n–fold functor. Then the morphism of simplicial sets $\delta^* N^n F$ is a weak equivalence if and only if $\mathrm{Ex}^2\, \delta^* N^n F$ is a weak equivalence.*

Proof This follows from two applications of Lemma 6.2. □

Theorem 8.2 *There is a model structure on* **nFoldCat** *in which an n–fold functor F a weak equivalence respectively fibration if and only if $\mathrm{Ex}^2 \delta^* N^n F$ is a weak equivalence respectively fibration in* **SSet**. *Moreover, this model structure on* **nFoldCat** *is cofibrantly generated with generating cofibrations*

$$\{ c^n \delta_!\, \mathrm{Sd}^2\, \partial\Delta[m] \longrightarrow c^n \delta_!\, \mathrm{Sd}^2\, \Delta[m] \mid m \geq 0 \}$$

and generating acyclic cofibrations

$$\{ c^n \delta_!\, \mathrm{Sd}^2\, \Lambda^k[m] \longrightarrow c^n \delta_!\, \mathrm{Sd}^2\, \Delta[m] \mid 0 \leq k \leq m \text{ and } m \geq 1 \}.$$

Proof We apply Corollary 6.1.

(i) The n–fold categories $c^n\delta_!\operatorname{Sd}^2\partial\Delta[m]$ and $c^n\delta_!\operatorname{Sd}^2\Lambda^k[m]$ each have a finite number of n–cubes, hence they are finite, and are small with respect to **nFoldCat**. For a proof, see Fiore–Paoli–Pronk [25, Proposition 7.7] and the remark immediately afterwards.

(ii) This holds as in the proof of (ii) in Theorem 6.3.

(iii) The n–fold nerve functor N^n preserves filtered colimits. Every ordinal is filtered, so N^n preserves λ–sequences. The functor δ^* preserves all colimits, as it is a left adjoint. The functor Ex preserves λ–sequences as in the proof of (iii) in Theorem 6.3.

(iv) Let $j\colon \Lambda^k[m] \longrightarrow \Delta[m]$ be a generating acyclic cofibration for **SSet**. Let the functor j' be the pushout along L as in the following diagram with $m \geq 1$.

$$\begin{tikzcd} c^n\delta_!\operatorname{Sd}^2\Lambda^k[m] \arrow[r, "L"] \arrow[d, "c^n\delta_!\operatorname{Sd}^2 j"'] & \mathbb{B} \arrow[d, "j'"] \\ c^n\delta_!\operatorname{Sd}^2\Delta[m] \arrow[r] & \mathbb{P} \end{tikzcd} \tag{25}$$

We factor j' into two inclusions

$$\mathbb{B} \xrightarrow{i} \mathbb{Q} \longrightarrow \mathbb{P} \tag{26}$$

and show that δ^*N^n applied to each yields a weak equivalence. For the first inclusion i, we will see in Lemma 8.3 that $\delta^*N^n i$ is a weak equivalence of simplicial sets.

By Remark 7.10, the only free composites of an n–cube in $c^n\delta_!\operatorname{Sd}^2\Delta[m]$ with an n–cube in $\mathbb{B}$ that can occur in $\mathbb{P}$ are of the form $\beta\circ_i\alpha$ where α is an n–cube in $\mathbb{B}$ and β is an n–cube in $c^n\delta_!N\,\mathbf{Outer}$ with i–th source in $c^n\delta_!N\mathbf{P}\operatorname{Sd}\Lambda^k[m]$ and i–th target outside of $c^n\delta_!N\mathbf{P}\operatorname{Sd}\Lambda^k[m]$. Of course, there are other free composites in $\mathbb{P}$, most generally of a form analogous to Proposition 7.15 (c), but these are obtained by composing the free composites of the form $\beta\circ_i\alpha$ above. Hence $\mathbb{P}$ is the union

$$\mathbb{P} = \overbrace{(\mathbb{B} \coprod_{c^n\delta_!N\mathbf{P}\operatorname{Sd}\Lambda^k[n]} c^n\delta_!N\,\mathbf{Outer})}^{\mathbb{Q}} \cup \overbrace{(c^n\delta_!N(\mathbf{Comp}\cup\mathbf{Center}))}^{\mathbb{R}}. \tag{27}$$

Note that we have not used the higher dimensional versions of Proposition 7.15 and Proposition 7.16 to draw this conclusion.

We show that $\delta^* N^n$ applied to the second inclusion $\mathbb{Q} \longrightarrow \mathbb{P}$ in Equation (26) is a weak equivalence. The intersection of $\mathbb{Q}$ and $\mathbb{R}$ in (27) is equal to

$$\begin{aligned}\mathbb{S} &= c^n \delta_! N(\mathbf{Outer}) \cap c^n \delta_! N(\mathbf{Comp} \cup \mathbf{Center})\\ &= c^n \delta_! N(\mathbf{Outer} \cap (\mathbf{Comp} \cup \mathbf{Center})).\end{aligned}$$

Proposition 5.2 and Proposition 7.4 then imply that $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{S}$ satisfy the hypotheses of Proposition 7.18. Then

$$\begin{aligned}
|\delta^* N^n \mathbb{Q}| &\cong |\delta^* N^n \mathbb{Q}| \coprod_{|\delta^* N^n \mathbb{S}|} |\delta^* N^n \mathbb{S}| && \text{(pushout along identity)}\\
&\simeq |\delta^* N^n \mathbb{Q}| \coprod_{|\delta^* N^n \mathbb{S}|} |\delta^* N^n \mathbb{R}| && \text{(Corollary 7.14 and Gluing Lemma)}\\
&\cong \left| \delta^* \left(N^n \mathbb{Q} \coprod_{N^n \mathbb{S}} N^n \mathbb{R} \right) \right| && \text{(the functors } |\cdot| \text{ and } \delta^* \text{ are left adjoints)}\\
&\cong \left| \delta^* N^n \left(\mathbb{Q} \coprod_{\mathbb{S}} \mathbb{R} \right) \right| && \text{(Proposition 7.18)}\\
&= |\delta^* N^n \mathbb{P}|.
\end{aligned}$$

In the second line, for the application of the Gluing Lemma, we use two identities and the inclusion $|\delta^* N^n \mathbb{S}| \longrightarrow |\delta^* N^n \mathbb{R}|$. It is a homotopy equivalence whose inverse is the retraction in Corollary 7.14. We conclude that the inclusion $|\delta^* N^n \mathbb{Q}| \longrightarrow |\delta^* N^n \mathbb{P}|$ is a weak equivalence, as it is the composite of the morphisms above. It is even a homotopy equivalence by Whitehead's Theorem.

We conclude that $|\delta^* N^n j'|$ is the composite of two weak equivalences

$$|\delta^* N^n \mathbb{B}| \xrightarrow{|\delta^* N^n i|} |\delta^* N^n \mathbb{Q}| \longrightarrow |\delta^* N^n \mathbb{P}|$$

and is therefore a weak equivalence. Thus $\delta^* N^n j'$ is a weak equivalence of simplicial sets. By Lemma 6.2, the functor Ex preserves weak equivalences, so that $\mathrm{Ex}^2 \delta^* N^n j'$ is also a weak equivalence of simplicial sets. Part (iv) of Corollary 6.1 then holds, and we have the Thomason model structure on **nFoldCat**. □

Lemma 8.3 *The inclusion $\delta^* N^n i\colon \delta^* N^n \mathbb{B} \longrightarrow \delta^* N^n \mathbb{Q}$ embeds the simplicial set $\delta^* N^n \mathbb{B}$ into $\delta^* N^n \mathbb{Q}$ as a simplicial deformation retract.*

Proof Recall $i\colon \mathbb{B} \longrightarrow \mathbb{Q}$ is the inclusion in Equation (26) and $\mathbb{Q}$ is defined as in Equation (27). We define an n–fold functor $\overline{r}\colon \mathbb{Q} \longrightarrow \mathbb{B}$ using the universal property of the pushout $\mathbb{Q}$ and the functor from Proposition 3.4 (ii) called $r\colon \mathbf{Outer} \longrightarrow \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$.

If $(v_0, \ldots, v_q) \in \mathbf{Outer}$ then $r(v_0, \ldots, v_q) := (u_0, \ldots, u_p)$ where $(u_0, \ldots, u_p)$ is the maximal subset

$$\{u_0, \ldots, u_p\} \subseteq \{v_0, \ldots, v_q\}$$

that is in $\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$. We have

$$\begin{aligned} c^n\delta_! N\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m] &= \bigcup_{\substack{U \subseteq \mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m] \text{ lin. ord.} \\ |U|=m}} U \boxtimes U \boxtimes \cdots \boxtimes U \\ &\subseteq \bigcup_{\substack{U \subseteq \mathbf{Outer} \text{ lin. ord.} \\ |U|=m+1}} U \boxtimes U \boxtimes \cdots \boxtimes U \\ &= c^n\delta_! N\mathbf{Outer}. \end{aligned}$$

Recall L is the n–fold functor in Diagram (25). We define $\overline{r}$ on $c^n\delta_! N\mathbf{Outer}$ to be

$$L \circ (r \boxtimes r \boxtimes \cdots \boxtimes r)\colon c^n\delta_! N\mathbf{Outer} \longrightarrow \mathbb{B}$$

and we define $\overline{r}$ to be the identity on $\mathbb{B}$. This induces the desired n–fold functor $\overline{r}\colon \mathbb{Q} \longrightarrow \mathbb{B}$ by the universal property of the pushout $\mathbb{Q}$.

By definition we have $\overline{r}i = 1_{\mathbb{B}}$. We next define an n–fold natural transformation $\overline{\alpha}\colon i\overline{r} \Longrightarrow 1_{\mathbb{Q}}$ (see Definition 2.20), which will induce a simplicial homotopy from $\delta^* N^n(i\overline{r})$ to $1_{\delta^* N^n\mathbb{Q}}$ as in Proposition 2.22. Let

$$\begin{aligned} f_1\colon \mathbb{B} &\longrightarrow \mathbb{B}^{[1]\boxtimes\cdots\boxtimes[1]} \\ f_2\colon c^n\delta_! N\mathbf{Outer} &\longrightarrow \mathbb{B}^{[1]\boxtimes\cdots\boxtimes[1]} \end{aligned}$$

be the n–fold functors corresponding to the n–fold natural transformations

$$\begin{aligned} \mathrm{pr}_{\mathbb{B}}\colon \mathbb{B} \times ([1] \boxtimes \cdots \boxtimes [1]) &\longrightarrow \mathbb{B} \\ L \circ (\alpha \boxtimes \cdots \boxtimes \alpha)\colon c^n\delta_! N\mathbf{Outer} \times ([1] \boxtimes \cdots \boxtimes [1]) &\longrightarrow \mathbb{B} \end{aligned}$$

(recall **nFoldCat** is Cartesian closed by Ehresmann–Ehresmann [19], the definition of α in Proposition 3.4 (ii), and Example 2.21). Then the necessary square involving f_1, f_2, L and the inclusion

$$c^n\delta_! N\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m] \longrightarrow c^n\delta_! N\mathbf{Outer}$$

commutes ($\alpha \boxtimes \cdots \boxtimes \alpha$ is trivial on $c^n\delta_! N\mathbf{P}\,\mathrm{Sd}\,\Lambda^k[m]$), so we have an n–fold functor $f\colon \mathbb{Q} \longrightarrow \mathbb{B}^{[1]\boxtimes\cdots\boxtimes[1]}$, which corresponds to an n–fold natural transformation

$$\overline{\alpha}\colon i\overline{r} \Longrightarrow 1_{\mathbb{Q}}\ .$$

Thus $\overline{\alpha}$ induces a simplicial homotopy from $\delta^* N^n(i) \circ \delta^* N^n(\overline{r})$ to $1_{\delta^* N^n\mathbb{Q}}$ and from above we have $\delta^* N^n(\overline{r}) \circ \delta^* N^n(i) = 1_{\delta^* N^n\mathbb{B}}$. This completes the proof that the inclusion $\delta^* N^n i\colon \delta^* N^n \mathbb{B} \longrightarrow \delta^* N^n \mathbb{Q}$ embeds the simplicial set $\delta^* N^n \mathbb{B}$ into $\delta^* N^n \mathbb{Q}$ as a simplicial deformation retract.

We next write out what this simplicial homotopy is in the case $n = 2$. We denote by σ this simplicial homotopy from $\delta^* N^n (i\overline{r})$ to $1_{\delta^* N^n \mathbb{Q}}$. For each q, we need to define $q+1$ maps $\sigma_\ell\colon (\delta^* N^n \mathbb{Q})_q \longrightarrow (\delta^* N^n \mathbb{Q})_{q+1}$ compatible with the face and degeneracy maps, $\delta^* N^n (i\overline{r})$, and $1_{\delta^* N^n \mathbb{Q}}$. We define σ_ℓ on a q–simplex α of the form in Proposition 7.16. *This q–simplex α has nothing to do with the n–fold natural transformation α above.* Suppose that the unique square of type (c) of Proposition 7.15 is in entry (u, v) and $u \leq v$.

If $\ell < u$, then $\sigma_\ell(\alpha)$ is obtained from α by inserting a row of vertical identities between rows ℓ and $\ell+1$ of α, as well as a column of horizontal identity squares between columns ℓ and $\ell+1$ of α. Thus $\sigma_\ell(\alpha)$ is vertically trivial in row $\ell+1$ and horizontally trivial in column $\ell+1$ of α.

If $\ell = u$ and $u < v$, then to obtain $\sigma_\ell(\alpha)$ from α, we replace row u by the two rows that make row u into a row of formal vertical composites, and we insert a column of horizontal identity squares between column u and column $u+1$ of α.

If $\ell = u$ and $u = v$, then to obtain $\sigma_\ell(\alpha)$ from α, we replace row u by the two rows that make row u into a row of formal vertical composites, and we replace column u by the two columns that make column u into a column of formal horizontal composites.

If $u < \ell < v$, then to obtain $\sigma_\ell(\alpha)$ from α, we replace row u by the row of squares β_1 in $\mathbb{B}$ that make up the first part of the formal vertical composite row u (consisting partly of region b of Proposition 7.16), then rows $u+1, u+2, \ldots, \ell$ of $\sigma_\ell(\alpha)$ are identity rows, row $\ell+1$ of $\sigma_\ell(\alpha)$ is the composite of the bottom half of row u of α with rows $u+1, u+2, \ldots, \ell$ of α, and the remaining rows of $\sigma_\ell(\alpha)$ are the remaining rows of α (shifted down by 1). We also insert a column of horizontal identity squares between column ℓ and column $\ell+1$ of α.

If $u < \ell = v$, then to obtain $\sigma_\ell(\alpha)$ from α, we do the row construction as in the case $u < \ell < v$, and we also replace column v by the two columns that make column v into a column of formal horizontal composites.

If $u \leq v < \ell$, then to obtain $\sigma_\ell(\alpha)$ from α, we do the row construction as in the case $u < \ell < v$, and we also do the analogous column construction.

The maps σ_ℓ for $0 \leq \ell \leq q$ are compatible with the boundary operators, $\delta^* N^n (i\overline{r})$, and $1_{\delta^* N^n \mathbb{Q}}$ for the same reason that the analogous maps associated to a natural transformation of functors are compatible with the face and degeneracy maps and the functors. Indeed, the σ_ℓ's are defined precisely as those for a natural transformation, we merely take into account the horizontal and vertical aspects.

In conclusion, we have morphisms of simplicial sets

$$\delta^* N^n(i)\colon \delta^* N^n \mathbb{B} \hookrightarrow \delta^* N^n \mathbb{Q}$$

$$\delta^* N^n(\overline{r})\colon \delta^* N^n \mathbb{Q} \longrightarrow \delta^* N^n \mathbb{B}$$

such that $(\delta^* N^n(\overline{r})) \circ (\delta^* N^n(i)) = 1_{\delta^* N^n \mathbb{B}}$ and $(\delta^* N^n(i)) \circ (\delta^* N^n(\overline{r}))$ is simplicially homotopic to $1_{\delta^* N^n \mathbb{Q}}$ via the simplicial homotopy σ. □

9 Unit and counit are weak equivalences

In this section we prove that the unit and counit of the adjunction in (24) are weak equivalences. Our main tool is the n–fold Grothendieck construction and the theorem that, in certain situations, a natural weak equivalence between functors induces a weak equivalence between the colimits of the functors. We prove that N^n and the n–fold Grothendieck construction are "homotopy inverses". From this, we conclude that our Quillen adjunction (24) is actually a Quillen equivalence. The left and right adjoints of (24) preserve weak equivalences, so the unit and counit are weak equivalences.

Definition 9.1 Let $Y\colon (\Delta^{\times n})^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a multisimplicial set. We define the *n–fold Grothendieck construction* $\Delta^{\boxtimes n}/Y \in \mathbf{nFoldCat}$ as follows. The *objects* of the n–fold category $\Delta^{\boxtimes n}/Y$ are

$$\mathrm{Obj}\, \Delta^{\boxtimes n}/Y = \{(y, \overline{k}) \mid \overline{k} = ([k_1], \dots, [k_n]) \in \Delta^{\times n},\, y \in Y_{\overline{k}}\}.$$

An *n–cube* in $\Delta^{\boxtimes n}/Y$ with $(0,0,\dots,0)$–vertex $(y, \overline{k})$ and $(1,1,\dots,1)$–vertex $(z, \overline{\ell})$ is a morphism $\overline{f} = (f_1, \dots, f_n)\colon \overline{k} \longrightarrow \overline{\ell}$ in $\Delta^{\times n}$ such that

$$\overline{f}^*(z) = y. \tag{28}$$

For $\epsilon_\ell \in \{0,1\}$, the $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$–vertex of such an n–cube is

$$(f_1^{1-\epsilon_1}, f_2^{1-\epsilon_2}, \dots, f_n^{1-\epsilon_n})^*(z).$$

For $1 \leq i \leq n$, a *morphism in direction i* is an n–cube $\overline{f}$ that has f_j the identity except at $j = i$. A *square in direction $i\,i'$* is an n–cube $\overline{f}$ such that f_j is the identity except at $j = i$ and $j = i'$, etc. In this way, the edges, subsquares, subcubes, etc. of an n–cube $\overline{f}$ are determined.

Example 9.2 If $n = 1$, then the Grothendieck construction of Definition 9.1 is the usual Grothendieck construction of a simplicial set.

Example 9.3 The Grothendieck construction $\Delta/\Delta[m]$ of the simplicial set $\Delta[m]$ is the comma category $\Delta/[m]$.

Example 9.4 The Grothendieck construction commutes with external products, that is, for simplicial sets $X_1, X_2, \dots, X_n$ we have

$$\Delta^{\boxtimes n}/(X_1 \boxtimes X_2 \boxtimes \cdots \boxtimes X_n) = (\Delta/X_1) \boxtimes (\Delta/X_2) \boxtimes \cdots \boxtimes (\Delta/X_n).$$

Remark 9.5 We describe the n–fold nerve of the n–fold Grothendieck construction. We learned the $n=1$ case from Joyal–Tierney [52, Chapter 6]. Let $Y\colon (\Delta^{\times n})^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a multisimplicial set and $\overline{p} = ([p_1], \dots, [p_n]) \in \Delta^{\times n}$. Then a $\overline{p}$–multisimplex of $N^n(\Delta^{\boxtimes n}/Y)$ consists of n composable paths of morphisms in Δ of lengths $p_1, p_2, \dots, p_n$

$$\begin{gathered}
\langle f^1_1, \dots, f^1_{p_1}\rangle\colon [k^1_0] \xrightarrow{f^1_1} [k^1_1] \xrightarrow{f^1_2} \cdots \xrightarrow{f^1_{p_1}} [k^1_{p_1}] \\
\langle f^2_1, \dots, f^2_{p_2}\rangle\colon [k^2_0] \xrightarrow{f^2_1} [k^2_1] \xrightarrow{f^2_2} \cdots \xrightarrow{f^2_{p_2}} [k^2_{p_2}] \\
\vdots \\
\langle f^n_1, \dots, f^n_{p_n}\rangle\colon [k^n_0] \xrightarrow{f^n_1} [k^n_1] \xrightarrow{f^n_2} \cdots \xrightarrow{f^n_{p_n}} [k^n_{p_n}]
\end{gathered}$$

and a multisimplex z of Y in degree

$$\overline{k_{\overline{p}}} := (k^1_{p_1}, k^2_{p_2}, \dots, k^n_{p_n}).$$

The last vertex in this $\overline{p}$–array of n–cubes in $\Delta^{\boxtimes n}/Y$ is

$$(z, ([k^1_{p_1}], [k^2_{p_2}], \dots, [k^n_{p_n}])).$$

The other vertices of this array are determined from z by applying the f's and their composites as in Equation (28). Thus, the set of $\overline{p}$–multisimplices of $N^n(\Delta^{\boxtimes n}/Y)$ is

$$\coprod_{\substack{\langle f^1_1,\dots,f^1_{p_1}\rangle \\ \langle f^2_1,\dots,f^2_{p_2}\rangle \\ \vdots \\ \langle f^n_1,\dots,f^n_{p_n}\rangle}} Y_{\overline{k_{\overline{p}}}}. \tag{29}$$

Proposition 9.6 *The functor $Y \mapsto N^n(\Delta^{\boxtimes n}/Y)$ preserves colimits.*

Proof The set of $\overline{p}$–multisimplices of $N^n(\Delta^{\boxtimes n}/Y)$ is (29). The assignment of Y to the expression in (29) preserves colimits. □

Remark 9.7 We can also describe the p–simplices of $\delta^* N^n(\Delta^{\boxtimes n}/Y)$. We learned the $n=1$ case from Joyal–Tierney [52, Chapter 6]. A p–simplex of $\delta^* N^n(\Delta^{\boxtimes n}/Y)$ is a composable path of p n–cubes

$$\overline{f^i}\colon (y^{i-1}, \overline{k^{i-1}}) \longrightarrow (y^i, \overline{k^i})$$

$(i=1,\dots,p)$. Each y^i is determined from y^p by the $\overline{f^i}$'s, as in Equation (28). The last target, namely $(y^p, \overline{k^p})$, is the same as a morphism of multisimplicial sets $\Delta^{\times n}[\overline{k^p}] \longrightarrow Y$. So by Yoneda, a p–simplex of $\delta^* N^n(\Delta^{\boxtimes n}/Y)$ is the same as a composable path of morphisms of multisimplicial sets

$$\Delta^{\times n}[\overline{k^0}] \longrightarrow \Delta^{\times n}[\overline{k^1}] \longrightarrow \cdots \longrightarrow \Delta^{\times n}[\overline{k^p}] \longrightarrow Y\ .$$

The set of p–simplices of $\delta^* N^n(\Delta^{\boxtimes n}/Y)$ is

$$\coprod_{\Delta^{\times n}[\overline{k^0}]\to\Delta^{\times n}[\overline{k^1}]\to\cdots\to\Delta^{\times n}[\overline{k^p}]} Y_{\overline{k^p}}. \tag{30}$$

Let us recall the natural morphism of simplicial sets $N(\Delta/X)\longrightarrow X$ described in Joyal–Tierney [52, Section 6.1], and which we shall call ρ_X as in Appendix A of Moerdijk–Svensson [71]. First note that any path of morphisms in Δ

$$[k_0] \longrightarrow [k_1] \longrightarrow \cdots \longrightarrow [k_p] \tag{31}$$

determines a morphism in Δ

$$\begin{aligned} [p] &\longrightarrow [k_p] \\ i &\longmapsto \operatorname{im} k_i \end{aligned} \tag{32}$$

where $\operatorname{im} k_i$ refers to the image of k_i under the composite of the last $p-i$ morphisms in (31). Note also that paths of the form (31) are in bijective correspondence with paths of the form

$$\Delta[k_0] \longrightarrow \Delta[k_1] \longrightarrow \cdots \longrightarrow \Delta[k_p] \tag{33}$$

by the Yoneda Lemma. The morphism $\rho_X\colon N(\Delta/X)\longrightarrow X$ sends a p–simplex

$$\Delta[k_0] \longrightarrow \Delta[k_1] \longrightarrow \cdots \longrightarrow \Delta[k_p] \longrightarrow X \tag{34}$$

to the composite

$$\Delta[p] \longrightarrow \Delta[k_p] \longrightarrow X \tag{35}$$

where the first morphism in (35) is the image of (32) under the Yoneda embedding (p–simplices of $N(\Delta/X)$ have the form (34) by the $n = 1$ case of Remark 9.7 with $k_i := \overline{k^i}$). As is well known, the morphism $\rho_X\colon N(\Delta/X) \longrightarrow X$ is a natural weak equivalence (see Joyal–Tierney [52, Theorem 6.2.2], Illusie [46, page 21] or Waldhausen [87, page 359]).

We analogously define a morphism of multisimplicial sets

$$\rho_Y\colon N^n(\Delta^{\boxtimes n}/Y) \longrightarrow Y$$

natural in Y. Consider a $\overline{p}$–multisimplex of $N^n(\Delta^{\boxtimes n}/Y)$ as in Remark 9.5. For each $1 \leq j \leq n$, the path $\langle f_1^j, \ldots, f_{p_j}^j \rangle$ gives rise to a morphism in Δ

$$[p_j] \longrightarrow [k^j_{p_j}]$$

as in (31) and (32). Together these form a morphism in $\Delta^{\times n}$, which induces a morphism of multisimplicial sets

$$\Delta^{\times n}[\overline{p}] \longrightarrow \Delta^{\times n}[\overline{k_{\overline{p}}}]\ .$$

The morphism ρ_Y assigns to the $\overline{p}$–multisimplex that we are considering the $\overline{p}$–multisimplex

$$\Delta^{\times n}[\overline{p}] \longrightarrow \Delta^{\times n}[\overline{k_{\overline{p}}}] \xrightarrow{z} Y\ .$$

This completes the definition of the natural transformation ρ.

Remark 9.8 The natural transformation ρ is compatible with external products. If $X_1, X_2, \ldots, X_n$ are simplicial sets and $Y = X_1 \boxtimes X_2 \boxtimes \cdots \boxtimes X_n$, then

$$\rho_Y\colon N^n(\Delta^{\boxtimes n}/Y) \longrightarrow Y$$

is equal to

$$\rho_{X_1} \boxtimes \rho_{X_2} \boxtimes \cdots \boxtimes \rho_{X_n}\colon$$
$$N(\Delta/X_1) \boxtimes N(\Delta/X_2) \boxtimes \cdots \boxtimes N(\Delta/X_n) \longrightarrow X_1 \boxtimes X_2 \boxtimes \cdots \boxtimes X_n.$$

Thus $\delta^*\rho_Y = \rho_{X_1} \times \rho_{X_2} \times \cdots \times \rho_{X_n}$ is a weak equivalence, since in **SSet** any finite product of weak equivalences is a weak equivalence. We conclude that ρ_Y is a weak equivalence of multisimplicial sets whenever Y is an external product. (For us, a morphism f of multisimplicial sets is a *weak equivalence* if and only if $\delta^* f$ is a weak equivalence of simplicial sets.) As we shall soon see, ρ_Y is a weak equivalence for all Y.

We quickly recall what we will need regarding Reedy model structures. The following definition and proposition are part of Definitions 5.1.2, 5.2.2, and Theorem 5.2.5 of Hovey [44], or Definitions 15.2.3, 15.2.5, and Theorem 15.3.4 of Hirschhorn [43].

Definition 9.9 Let $(\mathcal{B}, \mathcal{B}_+, \mathcal{B}_-)$ be a Reedy category and $\mathcal{C}$ a category with all small colimits and limits. For $i \in \mathcal{B}$, the *latching category* $\mathcal{B}_i$ is the full subcategory of $\mathcal{B}_+/i$ on the *nonidentity* morphisms $b \longrightarrow i$. For $F \in \mathcal{C}^{\mathcal{B}}$ the *latching object of F at i* is the colimit $L_i F$ of the composite functor

$$\mathcal{B}_i \longrightarrow \mathcal{B} \xrightarrow{F} \mathcal{C} \,. \tag{36}$$

For $i \in \mathcal{B}$, the *matching category* $\mathcal{B}^i$ is the full subcategory of $i/\mathcal{B}_-$ on the *nonidentity* morphisms $i \longrightarrow b$. For $F \in \mathcal{C}^{\mathcal{B}}$ the *matching object of F at i* is the limit $M_i F$ of the composite functor

$$\mathcal{B}^i \longrightarrow \mathcal{B} \xrightarrow{F} \mathcal{C} \,. \tag{37}$$

Theorem 9.10 (Kan) *Let $(\mathcal{B}, \mathcal{B}_+, \mathcal{B}_-)$ be a Reedy category and $\mathcal{C}$ a model category. Then the levelwise weak equivalences, Reedy fibrations, and Reedy cofibrations form a model structure on the category $\mathcal{C}^{\mathcal{B}}$ of functors $\mathcal{B} \longrightarrow \mathcal{C}$.*

Remark 9.11 A consequence of the definitions is that a functor $\mathcal{B} \longrightarrow \mathcal{C}$ is *Reedy cofibrant* if and only if the induced morphism $L_i F \longrightarrow Fi$ is a cofibration in $\mathcal{C}$ for all objects i of $\mathcal{B}$.

Proposition 9.12 (Compare with Example 15.1.19 of Hirschhorn [43].) *The category of multisimplices*

$$\Delta^{\times n} Y := \Delta^{\times n}/Y$$

of a multisimplicial set $Y \colon (\Delta^{\times n})^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a Reedy category. The degree of a $\overline{p}$–multisimplex is $p_1 + p_2 \cdots + p_n$. The direct subcategory $(\Delta^{\times n} Y)_+$ consists of those morphisms $(f_1, \ldots, f_n)$ that are iterated coface maps in each coordinate, ie, injective maps in each coordinate. The inverse subcategory $(\Delta^{\times n} Y)_-$ consists of those morphisms $(f_1, \ldots, f_n)$ that are iterated codegeneracy maps in each coordinate, ie, surjective maps in each coordinate.

Proposition 9.13 (Compare with Proposition 15.10.4(1) of Hirschhorn [43].) *If $\mathcal{B}$ is the category of multisimplices of a multisimplicial set, then for every $i \in \mathcal{B}$, the matching category $\mathcal{B}^i$ is either connected or empty.*

Proof This follows from the multidimensional Eilenberg–Zilber Lemma, recalled in Proposition 10.3. Let $Y\colon (\Delta^{\times n})^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a multisimplicial set and $\mathcal{B} = \Delta^{\times n} Y$ its category of multisimplices.

Let $i\colon \Delta^{\times n}[\overline{p}] \longrightarrow Y$ be a degenerate multisimplex. Then there exists a nontrivial, componentwise surjective map $\overline{\tau}$ and a totally nondegenerate multisimplex t with $i = (\overline{\tau})^* t$. The pair $(\overline{\tau}, t)$ is an object of the matching category $\mathcal{B}^i$. If $(\overline{\eta}, b)$ is another object of $\mathcal{B}^i$, there exists a componentwise surjective map $\overline{g}$ and a totally nondegenerate $b' \in \mathcal{B}$ such that $b = (\overline{g})^* b'$. But $i = (\overline{\eta})^* b = (\overline{\eta})^* (\overline{g})^* b'$ implies that $b' = t$, $\overline{g} \circ \overline{\eta} = \overline{\tau}$, and $\overline{g}$ is a morphism in $\mathcal{B}^i$ from $(\overline{\eta}, b)$ to $(\overline{\tau}, t)$. Thus, whenever i is degenerate, there is a morphism from any object of $\mathcal{B}^i$ to $(\overline{\tau}, t)$ and $\mathcal{B}^i$ is connected. One can also show $(\overline{\tau}, t)$ is a terminal object of $\mathcal{B}^i$, but we do not need this.

Let $i\colon \Delta^{\times n}[\overline{p}] \longrightarrow Y$ be a totally nondegenerate multisimplex. An object of the matching category $\mathcal{B}^i$ is a nontrivial, componentwise surjective map $\overline{\eta}$ and a multisimplex b with $i = (\overline{\eta})^* b$. Such $\overline{\eta}$ and b cannot exist because i is totally nondegenerate. Thus, whenever i is totally nondegenerate, the matching category $\mathcal{B}^i$ is empty. □

Theorem 9.14 *Suppose $\mathcal{C}$ is a model category and $\mathcal{B}$ is a Reedy category such that for all $i \in \mathcal{B}$, the matching category $\mathcal{B}^i$ is either connected or empty. Then the colimit functor*

$$\mathrm{colim}\colon \mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}$$

takes levelwise weak equivalences between Reedy cofibrant functors to weak equivalences between cofibrant objects of $\mathcal{C}$.

Proof This is merely a summary of Definition 15.10.1(2), Proposition 15.10.2(2) and Theorem 15.10.9(2) of Hirschhorn [43]. □

Notation 9.15 Let $Y\colon (\Delta^{\times n})^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a multisimplicial set, $\mathcal{B} = \Delta^{\times n} Y$, $\mathcal{C} = \mathbf{SSet}$, and $i\colon \Delta^{\times n}[\overline{m}] \longrightarrow Y$ an object of $\mathcal{B}$. Then the set of nonidentity morphisms in $\mathcal{B}_+$ with target i is the set of morphisms $(f_1, \ldots, f_n)$ in $\Delta^{\times n}$ with target $[\overline{m}]$ such that each f_j is injective and not all f_j's are the identity.

Notation 9.16 Let F and G be the following two functors.

$$\begin{aligned} F\colon \Delta^{\times n} Y &\longrightarrow \mathbf{SSet^n} \\ \left[\Delta^{\times n}[\overline{m}] \to Y\right] &\longmapsto N^n(\Delta^{\boxtimes n} / \Delta^{\times n}[\overline{m}]) \\ G\colon \Delta^{\times n} Y &\longrightarrow \mathbf{SSet^n} \\ \left[\Delta^{\times n}[\overline{m}] \to Y\right] &\longmapsto \Delta^{\times n}[\overline{m}] \end{aligned}$$

Note that $\delta^* \circ F$ and $\delta^* \circ G$ are in $\mathcal{C}^{\mathcal{B}}$. The natural transformation ρ induces a natural transformation we denote by

$$\rho^Y \colon F \Longrightarrow G \ .$$

Remark 9.17 The natural transformation ρ^Y is levelwise a weak equivalence by Remark 9.8.

Lemma 9.18 *The morphism in* $\mathbf{SSet^n}$

$$\operatorname*{colim}_{\Delta^{\times n} Y} \rho^Y \colon \operatorname*{colim}_{\Delta^{\times n} Y} F \longrightarrow \operatorname*{colim}_{\Delta^{\times n} Y} G$$

is equal to

$$\rho_Y \colon N^n(\Delta^{\boxtimes n}/Y) \longrightarrow Y.$$

Proof By Proposition 9.6, we have

$$\begin{aligned}\operatorname*{colim}_{\Delta^{\times n} Y} F &= \operatorname*{colim}_{\Delta^{\times n}[\overline{m}] \to Y} N^n(\Delta^{\boxtimes n}/\Delta^{\times n}[\overline{m}]) \\ &= N^n\Big(\Delta^{\boxtimes n}\Big/\Big(\operatorname*{colim}_{\Delta^{\times n}[\overline{m}] \to Y} \Delta^{\times n}[\overline{m}]\Big)\Big) = N^n(\Delta^{\boxtimes n}/Y).\end{aligned}$$

□

Lemma 9.19 *The functor*

$$\begin{aligned}\delta^* \circ F \colon \Delta^{\times n} Y &\longrightarrow \mathbf{SSet} \\ \big[\Delta^{\times n}[\overline{m}] \to Y\big] &\longmapsto N(\Delta/\Delta[m_1]) \times N(\Delta/\Delta[m_2]) \times \cdots \times N(\Delta/\Delta[m_n])\end{aligned}$$

is Reedy cofibrant.

Proof We use Notations 9.15 and 9.16. The colimit of Equation (36) is

$$L_i(\delta^* \circ F) = \bigcup_{1 \leq j \leq n} N(\Delta/\Delta[m_1]) \times \cdots \times N(\Delta/\partial\Delta[m_j]) \times \cdots \times N(\Delta/\Delta[m_n])$$

and $\delta^* \circ F(i) = N(\Delta/\Delta[m_2]) \times \cdots \times N(\Delta/\Delta[m_n])$. The map

$$L_i(\delta^* \circ F) \longrightarrow \delta^* \circ F(i)$$

is injective, or equivalently, a cofibration. Remark 9.11 now implies that $\delta^* \circ F$ is Reedy cofibrant. □

Lemma 9.20 *The functor*

$$\begin{aligned}\delta^* \circ G \colon \Delta^{\times n} Y &\longrightarrow \mathbf{SSet} \\ \big[\Delta^{\times n}[\overline{m}] \to Y\big] &\longmapsto \Delta[m_1] \times \Delta[m_2] \times \cdots \times \Delta[m_n]\end{aligned}$$

is Reedy cofibrant.

Proof We use Notations 9.15 and 9.16. The colimit of Equation (36) is

$$L_i(\delta^* \circ G) = \bigcup_{1 \leq j \leq n} \Delta[m_1] \times \cdots \times \partial\Delta[m_j] \times \cdots \times \Delta[m_n]$$

and $\delta^* \circ G(i) = \Delta[m_1] \times \Delta[m_2] \times \cdots \times \Delta[m_n]$. The morphism

$$L_i(\delta^* \circ G) \longrightarrow \delta^* \circ G(i)$$

is injective, or equivalently, a cofibration. Remark 9.11 now implies that $\delta^* \circ G$ is Reedy cofibrant. $\square$

Theorem 9.21 *For every multisimplicial set $Y\colon (\Delta^{\times n})^{\mathrm{op}} \longrightarrow \mathbf{Set}$, the morphism*

$$\rho_Y\colon N^n(\Delta^{\boxtimes n}/Y) \longrightarrow Y$$

is a weak equivalence of multisimplicial sets.

Proof Fix a multisimplicial set Y, and let F, G, and ρ^Y be as in Notation 9.16. The natural transformation $\delta^*\rho^Y\colon \delta^*F \Longrightarrow \delta^*G$ is levelwise a weak equivalence of simplicial sets by Remark 9.17, and is a natural transformation between Reedy cofibrant functors by Lemma 9.19 and Lemma 9.20. By Proposition 9.13, each matching category of the Reedy category $\Delta^{\times n}Y$ is connected or empty. Theorem 9.14 then guarantees that the morphism

$$\operatorname*{colim}_{\Delta^{\times n}Y} \delta^*\rho^Y\colon \operatorname*{colim}_{\Delta^{\times n}Y} \delta^* \circ F \longrightarrow \operatorname*{colim}_{\Delta^{\times n}Y} \delta^* \circ G$$

is a weak equivalence of simplicial sets. Since δ^* is a left adjoint, it commutes with colimits, and we have

$$\operatorname*{colim}_{\Delta^{\times n}Y} \delta^*\rho^Y = \delta^* \operatorname*{colim}_{\Delta^{\times n}Y} \rho^Y = \delta^*\rho_Y$$

by Lemma 9.18. We conclude $\delta^*\rho_Y$ is a weak equivalence, and that ρ_Y is a weak equivalence of multisimplicial sets. $\square$

We also define an n–fold functor

$$\lambda_{\mathbb{D}}\colon \Delta^{\boxtimes n}/N^n(\mathbb{D}) \longrightarrow \mathbb{D}$$

natural in $\mathbb{D}$, by analogy to Appendix A of Moerdijk–Svensson [71], and many others. If $(y, \overline{k})$ is an object of $\Delta^{\boxtimes n}/N^n(\mathbb{D})$, then $\lambda(y, \overline{k})$ is the n–fold category in the last vertex of the array of n–cubes y, namely

$$\lambda_{\mathbb{D}}(y, \overline{k}) = y_{\overline{k}}.$$

Theorem 9.22 *For any n–fold category $\mathbb{D}$, we have $N^n(\lambda_{\mathbb{D}}) = \rho_{N^n(\mathbb{D})}$. In particular, $\lambda_{\mathbb{D}}$ is a weak equivalence of n–fold categories.*

Corollary 9.23 *The functor $N^n\colon \mathbf{nFoldCat} \longrightarrow \mathbf{SSet^n}$ induces an equivalence of categories*

$$\mathrm{Ho}\,\mathbf{nFoldCat} \simeq \mathrm{Ho}\,\mathbf{SSet^n}.$$

Here Ho *refers to the category obtained by formally inverting weak equivalences. There is no reference to any model structure.*

Proof An "inverse" to N^n is the n–fold Grothendieck construction, since ρ and λ induce natural isomorphisms after passing to homotopy categories by Theorem 9.21 and Theorem 9.22. □

The following simple proposition, pointed out to us by Denis-Charles Cisinski, will be of use.

Proposition 9.24 *Let*

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

be a Quillen equivalence. If both F and G preserve weak equivalences, then:

(i) *Both F and G detect weak equivalences.*

(ii) *The unit and counit of the adjunction $F \dashv G$ are weak equivalences.*

Proof (i) We prove F detects weak equivalences; the proof that G detects weak equivalences is similar. Let $Q\colon \mathbf{C} \longrightarrow \mathbf{C}$ be a cofibrant replacement functor on $\mathbf{C}$, that is, QC is cofibrant for all objects C in $\mathbf{C}$ and there is a natural acyclic fibration $q\colon QC \longrightarrow C$. Suppose Ff is a weak equivalence. Then FQf is a weak equivalence (apply F to the naturality diagram for f and Q and use the 3-for-2 property). The total left derived functor $\mathbf{L}F$ is the composite

$$\begin{tikzcd} \mathrm{Ho}\,\mathbf{C} \arrow[r, "\mathrm{Ho}\,Q"'] \arrow[rr, bend left, "\mathbf{L}F"] & \mathrm{Ho}\,\mathbf{C}_c \arrow[r, "\mathrm{Ho}\,F|_{\mathbf{C}_c}"'] & \mathrm{Ho}\,\mathbf{D} \end{tikzcd},$$

where $\mathbf{C}_c$ is the full subcategory of $\mathbf{C}$ on the cofibrant objects of $\mathbf{C}$. Then $\mathbf{L}F[f]$ is an isomorphism in $\mathrm{Ho}\,\mathbf{D}$, as FQf is a weak equivalence in $\mathbf{D}$. The functor $\mathbf{L}F$ detects isomorphisms, as it is an equivalence of categories, so $[f]$ is an isomorphism in $\mathrm{Ho}\,\mathbf{C}$. Finally, a morphism in $\mathbf{C}$ is a weak equivalence if and only if its image in $\mathrm{Ho}\,\mathbf{C}$ is an isomorphism, so f is a weak equivalence in $\mathbf{C}$, and F detects weak equivalences.

(ii) We prove that the unit of the adjunction $F \dashv G$ is a natural weak equivalence; the proof that the counit is a natural weak equivalence is similar. Let $Q\colon \mathbf{C} \longrightarrow \mathbf{C}$ be a cofibrant replacement functor on $\mathbf{C}$, that is, QC is cofibrant for every object C in $\mathbf{C}$ and there is a natural acyclic fibration $q_C\colon QC \longrightarrow C$. Let $R\colon \mathbf{D} \longrightarrow \mathbf{D}$ be a fibrant replacement functor on $\mathbf{D}$, that is, RD is fibrant for every object D in $\mathbf{D}$ and there is a natural acyclic cofibration $r_D\colon D \longrightarrow RD$. Since $F \dashv G$ is a Quillen equivalence, the composite

$$QC \xrightarrow{\eta_{QC}} GFQX \xrightarrow{Gr_{FQX}} GRFQX$$

is a weak equivalence by Hovey [44, Proposition 1.3.13]. Then η_{QC} is a weak equivalence by the 3-for-2 property and the hypothesis that G preserves weak equivalences. An application of 3-for-2 to the naturality diagram for η

$$\begin{tikzcd} QC \arrow[r, "\eta_{QC}"] \arrow[d, "q_C"'] & GFQC \arrow[d, "GFq_C"] \\ C \arrow[r, "\eta_C"'] & GFC \end{tikzcd}$$

shows that η_C is a weak equivalence (recall GF preserves weak equivalences). $\square$

Lemma 9.25 *Let $G\colon \mathbf{D} \longrightarrow \mathbf{C}$ be a right Quillen functor. Suppose $\operatorname{Ho} G\colon \operatorname{Ho}\mathbf{D} \longrightarrow \operatorname{Ho}\mathbf{C}$ is an equivalence of categories. Then the total right derived functor*

$$\begin{tikzcd} \operatorname{Ho}\mathbf{D} \arrow[r, "\operatorname{Ho} R"'] \arrow[rr, bend left, "\mathbf{R}G"] & \operatorname{Ho}\mathbf{D}_f \arrow[r, "\operatorname{Ho} G|_{\mathbf{D}_f}"'] & \operatorname{Ho}\mathbf{C} \end{tikzcd}$$

is an equivalence of categories. Here R is a fibrant replacement functor on $\mathbf{D}$, and $\mathbf{D}_f$ is the full subcategory of $\mathbf{D}$ on the fibrant objects.

Proof The functors

$$\begin{tikzcd} \operatorname{Ho}\mathbf{D} \arrow[r, shift left, "\operatorname{Ho} R"] & \operatorname{Ho}\mathbf{D}_f \arrow[l, shift left, "\operatorname{Ho} i"] \end{tikzcd}$$

are equivalences of categories, "inverse" to one another. Then $\operatorname{Ho} G|_{\mathbf{D}_f} = (\operatorname{Ho} G) \circ (\operatorname{Ho} i)$ is a composite of equivalences. $\square$

Lemma 9.26 *Suppose $L \dashv R$ is an adjunction and R is an equivalence of categories. Then the unit η and counit ε of this adjunction are natural isomorphisms.*

Proof By Mac Lane [69, Theorem 1, page 93], R is part of an adjoint equivalence $L' \dashv R$ with unit η' and counit ε'. By the universality of η and η' there exists an isomorphism $\theta_X \colon LX \longrightarrow L'X$ such that $(R\theta_X) \circ \eta_X = \eta'_X$. Since η'_X is also an isomorphism, we see that η_X is an isomorphism. A similar argument shows that the counit ε is a natural isomorphism. □

Proposition 9.27 *The Quillen adjunction of (24)*

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "\mathrm{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathrm{Ex}^2"] \arrow[r, bend left, "\delta_!"] \arrow[r, phantom, "\perp"] & \mathbf{SSet^n} \arrow[l, bend left, "\delta^*"] \arrow[r, bend left, "c^n"] \arrow[r, phantom, "\perp"] & \mathbf{nFoldCat} \arrow[l, bend left, "N^n"] \end{tikzcd}$$

is a Quillen equivalence and the unit and counit are weak equivalences.

Proof Let $F \dashv G$ denote the adjunction in (24). This is a Quillen adjunction by Theorem 8.2. We first prove it is even a Quillen equivalence. The functor $\mathrm{Ex}^2\delta^*$ is known to induce an equivalence of homotopy categories, and N^n induces an equivalence of homotopy categories by Corollary 9.23, so $G = \mathrm{Ex}^2\delta^* N^n$ induces an equivalence of homotopy categories $\mathrm{Ho}\, G$. Lemma 9.25 then says that the total right derived functor $\mathbf{R}G$ is an equivalence of categories. The derived adjunction $\mathbf{L}F \dashv \mathbf{R}G$ is then an adjoint equivalence by Lemma 9.26, so $F \dashv G$ is a Quillen equivalence.

By Ken Brown's Lemma, the left Quillen functor F preserves weak equivalences (every simplicial set is cofibrant). The right Quillen functor G preserves weak equivalences by definition. Proposition 9.24 now guarantees that the unit and counit are weak equivalences. □

We summarize our main results of Theorem 8.2, Corollary 9.23 and Proposition 9.27.

Theorem 9.28 (i) *There is a cofibrantly generated model structure on* **nFoldCat** *such that an n–fold functor F is a weak equivalence (respectively fibration) if and only if $\mathrm{Ex}^2\delta^* N^n(F)$ is a weak equivalence (respectively fibration). In particular, an n–fold functor is a weak equivalence if and only if the diagonal of its nerve is a weak equivalence of simplicial sets.*

(ii) *The adjunction*

$$\begin{tikzcd} \mathbf{SSet} \arrow[r, bend left, "\mathrm{Sd}^2"] \arrow[r, phantom, "\perp"] & \mathbf{SSet} \arrow[l, bend left, "\mathrm{Ex}^2"] \arrow[r, bend left, "\delta_!"] \arrow[r, phantom, "\perp"] & \mathbf{SSet^n} \arrow[l, bend left, "\delta^*"] \arrow[r, bend left, "c^n"] \arrow[r, phantom, "\perp"] & \mathbf{nFoldCat} \arrow[l, bend left, "N^n"] \end{tikzcd}$$

is a Quillen equivalence.

(iii) *The unit and counit of this Quillen equivalence are weak equivalences.*

Corollary 9.29 *The homotopy category of n–fold categories is equivalent to the homotopy category of topological spaces.*

Another approach to proving that N^n and the n–fold Grothendieck construction are homotopy inverse would be to apply a multisimplicial version of the following Weak Equivalence Extension Theorem of Joyal–Tierney. We apply the present Weak Equivalence Extension Theorem to prove that there is a natural isomorphism

$$\delta^* N^n(\Delta^{\boxtimes n}/\delta_!\text{-}) \Longrightarrow 1_{\mathrm{Ho}\,\mathbf{SSet}}\ .$$

Theorem 9.30 (Weak Equivalence Extension Theorem 6.2.1 of Joyal–Tierney [52]) *Let $\phi\colon F \Longrightarrow G$ be a natural transformation between functors $F, G\colon \Delta \longrightarrow \mathbf{SSet}$. We denote by $\phi^+\colon F^+ \Longrightarrow G^+$ the left Kan extension along the Yoneda embedding $Y\colon \Delta \longrightarrow \mathbf{SSet}$.*

$$\begin{tikzcd} \mathbf{SSet} \arrow[dr, "F^+{,}G^+"] & \\ \Delta \arrow[u, "Y"] \arrow[r, "F{,}G"'] & \mathbf{SSet} \end{tikzcd}$$

Suppose that G satisfies the following condition:

$$\operatorname{im} G\epsilon^0 \cap \operatorname{im} G\epsilon^1 = \varnothing, \text{ where } \epsilon^i\colon [0] \longrightarrow [1] \text{ is the injection which misses } i.$$

If $\phi[m]\colon F[m] \longrightarrow G[m]$ is a weak equivalence for all $m \geq 0$, then

$$\phi^+ X\colon F^+X \longrightarrow G^+X$$

is a weak equivalence for every simplicial set X.

Lemma 9.31 *The functor*

$$\begin{aligned} \mathbf{SSet^n} &\longrightarrow \mathbf{SSet} \\ Y &\longmapsto \delta^* N^n(\Delta^{\boxtimes n}/Y) \end{aligned}$$

preserves colimits.

Proof The functor which assigns to Y the expression in (30) is colimit preserving. □

Proposition 9.32 *For every simplicial set X, the canonical morphism*

$$\delta^* N^n(\Delta^{\boxtimes n}/\delta_! X) \longrightarrow \delta^*\delta_! X$$

is a weak equivalence.

Proof We apply the Weak Equivalence Extension Theorem 9.30. Let $F, G\colon \Delta \longrightarrow$ **SSet** be defined by

$$F[m] = \delta^* N^n(\Delta^{\boxtimes n}/\delta_! \Delta[m])$$
$$G[m] = \delta^* \delta_! \Delta[m].$$

The functor
$$\delta^* N^n(\Delta^{\boxtimes n}/\delta_! \text{-})\colon \mathbf{SSet} \longrightarrow \mathbf{SSet}$$

preserves colimits by Lemma 9.31 and the fact that $\delta_!$ is a left adjoint. The functor

$$\delta^* \delta_!\colon \mathbf{SSet} \longrightarrow \mathbf{SSet}$$

preserves colimits since both δ^* and $\delta_!$ are both left adjoints. Thus the canonical comparison morphisms

$$F^+ X \longrightarrow \delta^* N^n(\Delta^{\boxtimes n}/\delta_! X)$$
$$G^+ X \longrightarrow \delta^* \delta_! X$$

are isomorphisms.

The condition on G listed in Theorem 9.30 is easy to verify, since

$$G\epsilon^0 = \epsilon^0 \times \epsilon^0\colon \Delta[0] \times \Delta[0] \longrightarrow \Delta[1] \times \Delta[1]$$
$$G\epsilon^1 = \epsilon^1 \times \epsilon^1\colon \Delta[0] \times \Delta[0] \longrightarrow \Delta[1] \times \Delta[1].$$

All that remains is to define natural morphisms

$$\phi[m]\colon \delta^* N^n(\Delta^{\boxtimes n}/\Delta[m, \dots, m]) \longrightarrow \Delta[m] \times \cdots \times \Delta[m]$$

and to show that each is a weak equivalence of simplicial sets. By the description in Definition 9.1, an object of $\Delta^{\boxtimes n}/\Delta[m, \dots, m]$ is a morphism

$$y = (y_1, \dots, y_n)\colon \overline{k} \longrightarrow ([m], \dots, [m])$$

in $\Delta^{\times n}$. An n–cube $\overline{f}$ is a morphism in $\Delta^{\times n}$ making the diagram

$$\begin{tikzcd} \overline{k} \arrow[rr, "\overline{f}"] \arrow[dr, "y"'] & & \overline{k'} \arrow[dl, "y'"] \\ & ([m], \dots, [m]) & \end{tikzcd}$$

commute. A p–simplex in $\delta^* N^n(\Delta^{\boxtimes n}/\Delta[m, \dots, m])$ is a path $\overline{f^1}, \dots, \overline{f^p}$ of composable morphisms in $\Delta^{\times n}$ making the appropriate triangles commute. We see that

$$\delta^* N^n(\Delta^{\boxtimes n}/\Delta[m, \dots, m]) \cong N(\Delta/\Delta[m]) \times \cdots N(\Delta/\Delta[m]).$$

We define $\phi[m]$ to be the product of n–copies of the weak equivalence

$$\rho_{\Delta[m]}\colon N(\Delta/\Delta[m]) \longrightarrow \Delta[m]$$

defined in Equations (34) and (35). Since $\phi[m]$ is a weak equivalence for all m, we conclude from Theorem 9.30 that the canonical morphism

$$\phi^+ X\colon \delta^* N^n(\Delta^{\boxtimes n}/\delta_! X) \longrightarrow \delta^*\delta_! X$$

is a weak equivalence for every simplicial set X. □

Lemma 9.33 *There is a natural weak equivalence $\delta^*\delta_! X \longleftarrow X$.*

Proof In Theorem 9.30, let F be the Yoneda embedding and G once again $\delta^*\delta_!$. The diagonal morphism

$$\Delta[m] \longrightarrow \Delta[m] \times \cdots \times \Delta[m]$$

is a weak equivalence, as both the source and target are contractible. □

Proposition 9.34 *There is a zigzag of natural weak equivalences between the functor $\delta^* N^n(\Delta^{\boxtimes n}/\delta_!\text{-})$ and the identity functor on* **SSet**. *Consequently, there is a natural isomorphism*

$$\delta^* N^n(\Delta^{\boxtimes n}/\delta_!\text{-}) \Longrightarrow 1_{\mathrm{Ho}\,\mathbf{SSet}} .$$

Proof This follows from Proposition 9.32 and Lemma 9.33. □

10 Appendix: Multidimensional Eilenberg–Zilber Lemma

In Proposition 9.13 we made use of the multidimensional Eilenberg–Zilber Lemma to prove that the matching category $\mathcal{B}^i$ is either connected or empty whenever $\mathcal{B}$ is a category of multisimplices $\Delta^{\times n} Y$. In this Appendix, we prove the multidimensional Eilenberg–Zilber Lemma. We merely paraphrase Joyal–Tierney's proof of the two-dimensional case in [52, Chapter 5: Bisimplicial Sets] in order to make the present paper more self-contained.

Proposition 10.1 (Eilenberg–Zilber Lemma) *Let Y be simplicial set and $y \in Y_p$. Then there exists a unique surjection $\eta\colon [p] \longrightarrow [q]$ and a unique nondegenerate simplex $y' \in Y_p$ such that $y = \eta^*(y')$.*

Proof Proofs can be found in many books on simplicial homotopy theory; for example see Hirschhorn [43, Lemma 15.8.4]. □

Definition 10.2 Let $Y\colon (\Delta^{\times n})^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a multisimplicial set. A multisimplex $y \in Y_{\overline{p}}$ is *degenerate in direction i* if there exists a surjection $\eta_i\colon [p_i] \longrightarrow [q_i]$ and a multisimplex $y' \in Y_{p_1,\dots,p_{i-1},q_i,p_{i+1},\dots,p_n}$ such that

$$y = (\mathrm{id}_{p_1}, \dots, \mathrm{id}_{p_{i-1}}, \eta, \mathrm{id}_{p_i}, \dots, \mathrm{id}_{p_n})^*(y').$$

A multisimplex $y \in Y_{\overline{p}}$ is *nondegenerate in direction i* if it is not degenerate in direction i. A multisimplex $y \in Y_{\overline{p}}$ is *totally nondegenerate* if is it not degenerate in any direction.

Proposition 10.3 (Multidimensional Eilenberg–Zilber Lemma) *If $Y\colon (\Delta^{\times n})^{\mathrm{op}} \longrightarrow$ $\mathbf{Set}$ is a multisimplicial set and $y \in Y_{\overline{p}}$, then there exist unique surjections $\eta_i\colon [p_i] \longrightarrow$ $[q_i]$ and a unique totally nondegenerate multisimplex $y_n \in Y_{\overline{q}}$ such that $y = (\overline{\eta})^* y_n$.*

Proof We simply reproduce Joyal–Tierney's bisimplicial proof in [52, Chapter 5: Bisimplicial Sets] for multisimplicial sets.

Let $y = y_0$ for the proof of existence. The Eilenberg–Zilber Lemma for $\mathbf{SSet}$, recalled in Proposition 10.1, guarantees surjections $\eta_i\colon [p_i] \longrightarrow [q_i]$ and multisimplices $y_i \in Y_{q_1,\dots,q_{i-1},q_i,p_{i+1},\dots,p_n}$ such that

$$y_{i-1} = (\mathrm{id}_{q_1}, \dots, \mathrm{id}_{q_{i-1}}, \eta_i, \mathrm{id}_{p_{i+1}}, \dots, \mathrm{id}_{p_n})^*(y_i)$$

and each y_i is nondegenerate in direction i for all $i = 1, 2, \dots, n$. Then $y = (\eta_1, \dots, \eta_n)^*(y_n)$. The multisimplex y_n is totally nondegenerate, for if it were degenerate in direction i, so that

$$y_n = (\mathrm{id}_{q_1}, \dots, \mathrm{id}_{q_{i-1}}, \eta'_i, \mathrm{id}_{q_{i+1}}, \dots, \mathrm{id}_{q_n})^*(y'_i),$$

we would have y_i degenerate in direction i:

$$\begin{aligned} y_i &= (\mathrm{id}_{q_1}, \dots, \mathrm{id}_{q_i}, \eta_{i+1}, \dots, \eta_n)^*(y_n) \\ &= (\mathrm{id}_{q_1}, \dots, \mathrm{id}_{q_i}, \eta_{i+1}, \dots, \eta_n)^*(\mathrm{id}_{q_1}, \dots, \mathrm{id}_{q_{i-1}}, \eta'_i, \mathrm{id}_{q_{i+1}}, \dots, \mathrm{id}_{q_n})^*(y'_i) \\ &= (\mathrm{id}_{q_1}, \dots, \mathrm{id}_{q_{i-1}}, \eta'_i, \mathrm{id}_{p_{i+1}}, \dots, \mathrm{id}_{p_n})^*(\mathrm{id}_{q_1}, \dots, \mathrm{id}_{q_i}, \eta_{i+1}, \dots, \eta_n)^*(y'_i). \end{aligned}$$

But y_i is nondegenerate in direction i.

For the uniqueness, suppose $\eta'_i\colon [p_i] \longrightarrow [q'_i]$ and $y'_n \in Y_{\overline{q'}}$ is another totally nondegenerate multisimplex such that $y = (\overline{\eta'})^* y'_n$. The diagram in $\Delta^{\times n}$ associated to the n pushouts in Δ

$$\begin{tikzcd} {[p_i]} \arrow[r, "\eta_i"] \arrow[d, "\eta'_i"'] & {[q_i]} \arrow[d, "\mu_i"] \\ {[q'_i]} \arrow[r, "\mu'_i"'] & {[r_i]} \end{tikzcd}$$

is a pushout in $\Delta^{\times n}$ (η_i and η'_i are all surjective). The Yoneda embedding then gives us a pushout in $\mathbf{SSet^n}$.

$$\begin{tikzcd}
\Delta^{\times n}[\overline{p}] \arrow[r, "\Delta^{\times n}[\overline{\eta}]"] \arrow[d, "\Delta^{\times n}[\overline{\eta'}]"'] & \Delta^{\times n}[\overline{q}] \arrow[d, "\Delta^{\times n}[\overline{\mu}]"] \\
\Delta^{\times n}\left[\overline{q'}\right] \arrow[r, "\Delta^{\times n}[\overline{\mu'}]"'] & \Delta^{\times n}[\overline{r}]
\end{tikzcd}$$

Since
$$(\overline{\eta'})^* y'_n = y = (\overline{\eta})^* y_n,$$
the universal property of this pushout produces a unique multisimplex $z \in Y_{\overline{r}}$ such that
$$y'_n = (\overline{\mu'})^*(z), \quad y_n = (\overline{\mu})^*(z).$$
The multisimplices y_n and y'_n are totally nondegenerate, so $\overline{\mu} = \overline{\mathrm{id}}$ and $\overline{\mu'} = \overline{\mathrm{id}}$, and consequently $\overline{\eta'} = \overline{\eta}$ and $y'_n = y_n$. □

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Department of Mathematics and Statistics, University of Michigan-Dearborn
4901 Evergreen Road, Dearborn, MI 48128

Department of Mathematics and Statistics, Penn State Altoona
3000 Ivyside Park, Altoona, PA 16601-3760

tmfiore@umd.umich.edu, sup24@psu.edu

http://www-personal.umd.umich.edu/~tmfiore/,
http://math.aa.psu.edu/~simona/

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