

# Involutions on $S^6$ with 3–dimensional fixed point set

MARTIN OLBERMANN

In this article, we classify all involutions on  $S^6$  with 3–dimensional fixed point set. In particular, we discuss the relation between the classification of involutions with fixed point set a knotted 3–sphere and the classification of free involutions on homotopy  $\mathbb{C}P^3$ 's.

[57S17](#); [55M35](#), [57R65](#)

## 1 Introduction

As a general assumption, we are interested in smooth involutions on connected, closed, smooth manifolds.

The study of group actions on very simple manifolds such as disks, spheres or Euclidean spaces has been a very active subject. In his MathSciNet review [17] of Pawalowski [21], Masuda notes: “Representations of groups provide examples of group actions on Euclidean spaces, disks or spheres, and an important natural question in transformation groups is to what extent arbitrary actions on those spaces resemble actions provided by representations.” The first highlight theorems are due to P A Smith. For an involution on a sphere, Smith proved that the fixed point set is a  $\mathbb{Z}_2$ –homology sphere. There are various converses to this theorem which can be found in the literature, eg Davis and Weinberger [4] and Pawalowski [21] and references therein. However, the following theorem seems to be new:

**Theorem 1.1** (Olbermann [19]) *Every  $\mathbb{Z}_2$ –homology 3–sphere is the fixed point set of an involution on  $S^6$ .*

Moreover, the method used by the author in [19] generalizes to a classification of these involutions, which is the subject of the present paper. (A shorter proof of the above existence theorem can be given using Dovermann’s equivariant surgery approach along the lines of Schultz [24] and Dovermann, Masuda and Schultz [6].)

Recently, a class of involutions called *conjugations* was defined by Hausmann, Holm and Puppe [11] and various aspects of conjugations were studied by Franz and Puppe [7].

the author [20], Hausmann and Holm [10] and Hambleton and Hausmann [9]. Conjugations  $\tau$  on topological spaces  $X$  have the property that the fixed point set has  $\mathbb{Z}_2$ -cohomology ring isomorphic to the  $\mathbb{Z}_2$ -cohomology ring of  $X$ , with the slight difference that all degrees are divided by two. In the case of smooth involutions on (positive-dimensional) spheres, the conjugations are exactly the involutions on even-dimensional spheres with half-dimensional fixed point set. In dimension 2, the Schönflies theorem gives a classification: every conjugation is conjugate to the reflection of  $S^2$  at the equator. In dimension 4, work of Gordon and Sumners shows that there are infinitely many nonequivalent conjugations on  $S^4$ . Hambleton and Hausmann recently reduced the study of such involutions to a nonequivariant four-dimensional knot theory question [9]. Knot theory of  $n$ -spheres in  $S^{2n}$  is easier for  $n > 2$ , so that the study of conjugations gets a different flavor for  $n > 2$ .

The study of free involutions on simply connected spin manifolds with the same homology groups as  $\mathbb{C}P^3$  was motivated by the question whether it is possible to define an “equivariant Montgomery–Yang correspondence”. After Haefliger [8] proved that the group  $C_3^3$  of knotted  $S^3$ 's in  $S^6$  (isotopy classes of smooth embeddings, or equivalently diffeomorphism classes relative to  $S^3$ ) is isomorphic to  $\mathbb{Z}$ , Montgomery and Yang [18] showed that there is a natural bijection between  $C_3^3$  and the set of diffeomorphism classes of homotopy  $\mathbb{C}P^3$ 's. Wall's classification of all simply connected spin manifolds with the same homology groups as  $\mathbb{C}P^3$  [28] also uses the bijection between diffeomorphism classes of such manifolds, together with a basis of  $H^2$ , and isotopy classes of framed  $S^3$ -knots in  $S^6$ . In the equivariant setting, Li and Lü [15] show that the existence of a free involution on a homotopy  $\mathbb{C}P^3$  implies the existence of an involution on  $S^6$  with fixed point set the corresponding knotted  $S^3$ . Similarly, in our approach, the same surgery arguments apply in both cases. However, it is not possible to produce a nice bijection on the set of equivariant diffeomorphism classes of these, as claimed by Su [26]. Our classification results are:

**Theorem 1.2** *Let  $M$  be a smooth closed simply connected spin manifold with  $H_2(M) = \mathbb{Z}$  and  $H_3(M) = 0$ . Let  $x \in H^2(M; \mathbb{Z})$  be a generator. Assume that  $M \not\cong S^2 \times S^4$ .*

- *If  $\langle (p_1(M)x - 4x^3)/24, [M] \rangle$  is odd, there exists no free involution on  $M$ .*
- *If  $\langle x^3, [M] \rangle$  is odd, and  $\langle (p_1(M)x - 4x^3)/24, [M] \rangle$  is even, there exist up to diffeomorphism exactly two free involutions on  $M$ .*
- *If  $\langle x^3, [M] \rangle$  is even, and  $\langle (p_1(M)x - 4x^3)/24, [M] \rangle$  is even, there exist up to diffeomorphism exactly five free involutions on  $M$ .*

*If  $M \cong S^2 \times S^4$ , then the same classification holds for orientation-reversing involutions which act by  $-1$  on  $H^2(M)$ .*

In the case of homotopy  $\mathbb{C}P^3$ 's the classification of free involutions was given by Petrie [22] (whose result contains a mistake, corrected by Dovermann, Masuda and Schultz [6]), and Su [26]. Li and Su (unpublished) give the answer to the existence question in the case of odd  $\langle x^3, [M] \rangle$ . Our method reproves all these results in a different way, extends to a larger class of manifolds and gives classification results in all cases. A different generalization of Petrie's result is the classification of free involutions on homotopy complex projective spaces of arbitrary dimension given by Sady [23].

**Theorem 1.3** *For every even element of  $C_3^3$  there are (up to equivariant diffeomorphism relative  $S^3$ ) exactly four involutions (conjugations) on  $S^6$  with the knot as fixed point set. For every odd element of  $C_3^3$ , there is no involution on  $S^6$  with the knot as fixed point set.*

The new part of the theorem is the classification of these involutions. Li and Lü proved that the existence of an involution with fixed point set a given knot is equivalent to the existence of a free involution on the corresponding homotopy  $\mathbb{C}P^3$  under the Montgomery–Yang correspondence, so that together with Su's work mentioned above, the existence question was answered.

The case of involutions on  $S^6$  with fixed point set  $S^3$  is especially interesting since, given another 6–manifold with an involution that has a 3–dimensional fixed point set, (equivariant) connected sum gives a possibly different involution on the same 6–manifold with same fixed point set. (This is in analogy with the fact that connected sum with a homotopy sphere gives a possibly new smooth structure on the same underlying topological manifold.) Similarly, connected sum with a conjugation on  $S^6$  with fixed point set different from  $S^3$  gives a new conjugation on the same 6–manifold, with different fixed point space.

Our main result is a classification of smooth involutions on  $S^6$  with arbitrary three–dimensional fixed point set, using a recent classification of embeddings of closed oriented connected 3–manifolds into  $S^6$  by Skopenkov [25]. Skopenkov proves that for a 3–dimensional  $\mathbb{Z}_2$ –homology sphere  $M$  the set of isotopy classes of embeddings  $i: M \rightarrow S^6$  has a free action by  $C_3^3$  and the orbits are in canonical bijection with  $H_1(M)$ .

**Theorem 1.4** *Let  $M$  be a  $\mathbb{Z}_2$ –homology sphere of dimension 3. The set of isotopy classes of embeddings  $i: M \rightarrow S^6$  which are the fixed point sets of involutions (conjugations) is contained in the orbit corresponding to  $0 \in H_1(M)$ . Moreover, it is acted upon freely and transitively by  $2\mathbb{Z} \subseteq \mathbb{Z} \cong C_3^3$ . There are up to equivariant diffeomorphism relative to  $i$  exactly four such conjugations for every such  $i$ .*

In our case, we can also classify involutions without additional identification of the fixed point set with a given 3–manifold. However, we consider involutions together with an orientation of their fixed point sets, and the equivalence relation is equivariant diffeomorphism which respects the orientations of both  $S^6$  and the fixed point set. Equipping the involution with an orientation of the fixed point set is necessary in order to perform connected sums. Thus it is more natural to determine this more structured set of equivariant diffeomorphism classes  $\text{Inv}_M(S^6)$ .

Since the action of the mapping class group of  $M$  on the above set of equivariant diffeomorphism classes of involutions relative to the fixed point set  $M$  is trivial, we get the same classification as in [Theorem 1.4](#):

**Theorem 1.5** *Conjugations up to conjugation (preserving orientations) with fixed point set of a fixed oriented diffeomorphism type of  $\mathbb{Z}_2$ –homology 3–spheres are in bijection with  $\mathbb{Z} \oplus \mathbb{Z}_4$ . Under connected sum,  $\text{Inv}_{S^3}(S^6) \cong \mathbb{Z} \oplus \mathbb{Z}_4$  is an isomorphism of groups, and  $\text{Inv}_{S^3}(S^6)$  acts freely and transitively on  $\text{Inv}_M(S^6)$  for every  $\mathbb{Z}_2$ –homology 3–sphere  $M$ .*

**Remark 1.6** In principle, using the machinery described by Kreck [\[13\]](#), it is also possible to prove classification results for conjugations on  $X^6$  with fixed point set  $M^3$  in other cases as  $X = S^6$ . (However, the argument we use to show the surgery obstruction is trivial does not extend to the case of free involutions on other manifolds.)

One would compute the set of equivariant diffeomorphism classes of the involution together with an identification of the fixed point set with a prescribed 3–manifold  $M$ , ie the set

$$\text{Emb}_{\mathbb{Z}_2}(M \rightarrow X) = \{f: (M, \text{id}) \rightarrow (X, \tau)\} / \sim$$

where  $f$  is an inclusion onto the fixed point set of  $\tau$ , and  $f: (M, \text{id}) \rightarrow (X, \tau)$  is identified with  $f': (M, \text{id}) \rightarrow (X', \tau')$  if there is an equivariant diffeomorphism  $\phi: (X, \tau) \rightarrow (X', \tau')$  making the obvious triangle commute. One of the difficulties to overcome is “due” to Wall’s classification: in order to determine which of the resulting 6–manifolds are diffeomorphic to  $X$ , one would need a good understanding of the isomorphism classes of the algebraic invariants (trilinear forms), and this problem seems to be very hard in general.

**Acknowledgements** I would like to thank Diarmuid Crowley, Jean-Claude Hausmann, Matthias Kreck and Arturo Prat-Waldron for many helpful discussions and remarks. Special thanks to Yang Su, whose article [\[26\]](#) was the origin of this paper, and who shared with me the so far unpublished notes on a generalization of [\[26\]](#) due to himself and Banghi Li.

## 2 Preliminaries

### 2.1 Modified surgery

We will use Kreck’s modified surgery theory [14; 13] which is also suited to give classification results. By the equivariant tubular neighbourhood theorem, we can write  $S^6 = M \times D^3 \cup_{\partial} V$ , where  $V$  is a manifold with boundary  $M \times S^2$  equipped with a free involution, which restricts to  $(\text{id}, -\text{id})$  on the boundary. (It is not hard to see that the normal bundle of  $M$  in  $S^6$  is trivial. See [20] for a proof.)

A classification of manifolds  $V$  with free involutions  $\tau$  up to equivariant diffeomorphism is the same as a classification of the quotient manifolds  $W = V/\tau$  up to diffeomorphism. This is what modified surgery theory will give us.

We first determine the normal 2–type of the manifolds  $W$  under consideration. The normal 2–type  $B$  of a 6–manifold (see the precise Definition 3.2) is a fibration  $B \rightarrow BO$ . It carries roughly the information of a 3–skeleton of the manifold together with the restriction of the normal bundle to this 3–skeleton. After computing the bordism group of manifolds with normal  $B$ –structure, we show that in every bordism class there exists a manifold (together with a map to  $B$ ) which qualifies as the  $W$  above. Moreover, we show that given two normally  $B$ –bordant manifolds  $W$  as above, the obstruction, which a priori lies in the complicated monoid  $l_7(\mathbb{Z}_2, -1)$ , for the existence of an  $s$ –cobordism (ie a diffeomorphism) is zero. The diffeomorphism classification of the manifolds  $W$  under consideration is given by the set of orbits of the action of the group of fiber homotopy self–equivalences  $B \rightarrow BO$ .

### 2.2 Conjugations on manifolds

This section explains what conjugation spaces are and shows that the smooth involutions on  $S^{2n}$  with  $n$ –dimensional fixed point set are exactly the smooth conjugations. The rest of the paper does not depend on the material in this section.

A conjugation on a topological space  $X$  is an involution  $\tau: X \rightarrow X$ , which we consider as an action of the group  $\mathbb{Z}_2 \cong C = \{\text{id}, \tau\}$  on  $X$ , and which satisfies the following cohomological pattern: We denote the Borel equivariant cohomology of  $X$  by  $H_C^*(X; \mathbb{Z}_2)$ . It is a module over  $H_C^*(\text{pt}; \mathbb{Z}_2) = \mathbb{Z}_2[u]$ . The restriction maps in equivariant cohomology are denoted by

$$\rho: H_C^*(X; \mathbb{Z}_2) \rightarrow H^*(X; \mathbb{Z}_2) \quad \text{and} \quad r: H_C^*(X; \mathbb{Z}_2) \rightarrow H_C^*(X^\tau; \mathbb{Z}_2) \cong H^*(X^\tau; \mathbb{Z}_2)[u].$$

**Definition 2.1** [11]  $X$  is a conjugation space if

- $H^{\text{odd}}(X; \mathbb{Z}_2) = 0$ ,
- there exists a (ring) isomorphism  $\kappa: H^{2*}(X; \mathbb{Z}_2) \rightarrow H^*(X^\tau; \mathbb{Z}_2)$  and a (multiplicative) section  $\sigma: H^*(X; \mathbb{Z}_2) \rightarrow H_C^*(X; \mathbb{Z}_2)$  of  $\rho$  such that the so-called conjugation equation holds:

$$r\sigma(x) = \kappa(x)u^k + \text{terms of lower degree in } u.$$

One does not need to require that  $\kappa$  and  $\sigma$  be ring homomorphisms, it is a consequence of the definition. Moreover, the “structure maps”  $\kappa$  and  $\sigma$  are unique, and natural with respect to equivariant maps between conjugation spaces.

There are many examples of such conjugations: complex conjugation on the projective space  $\mathbb{C}P^n$  and on complex Grassmannians, natural involutions on smooth toric manifolds [5] and on polygon spaces [12]. Every cell complex with the property that each cell is a unit disk in  $\mathbb{C}^n$  with complex conjugation, and with equivariant attaching maps, is a conjugation space. Coadjoint orbits of semisimple Lie groups with the Chevalley involution are conjugation spaces. Moreover, there are various constructions of new conjugation spaces out of other conjugation spaces.

A conjugation manifold is a conjugation space consisting of a smooth manifold  $X$  with a smooth involution  $\tau$ . As a consequence, a closed conjugation manifold  $X$  must be even-dimensional, say of dimension  $2n$ , and  $M$  is of dimension  $n$ .

In [20] we proved that it is possible to give a definition of conjugation spaces without the nongeometric maps  $\kappa$  and  $\sigma$ , which is moreover well-adapted to the case of conjugation manifolds, where the fixed point set has an equivariant tubular neighbourhood.

**Proposition 2.2** Every smooth involution on  $S^{2n}$  with (nonempty)  $n$ -dimensional fixed point set is a conjugation.

**Proof** Let  $\text{pt} \in S^{2n}$  be a fixed point of the involution. Then  $\text{pt}$  is a conjugation space and  $(S^{2n}, \text{pt})$  is a conjugation pair, by the same proof as in Example 3.5 of [11]. Then the extension property for triples, Proposition 4.1 in [11], shows that  $S^{2n}$  is a conjugation space. (A slightly different proof is given in [9].)  $\square$

### 3 Free involutions on certain 6-manifolds

We are considering smooth involutions on smooth closed simply connected spin manifolds  $M$  with  $H_2(M) = \mathbb{Z}$ , and  $H_3(M) = 0$ . The classification by Wall in [28]

also uses a generator  $x \in H^2(M)$  and an orientation of  $M$ . Then pairs  $(M, x)$  up to diffeomorphism are classified by the bordism class of

$$M \xrightarrow{x} \mathbb{C}P^\infty \in \Omega_6^{\text{Spin}}(\mathbb{C}P^\infty) \cong \mathbb{Z}^2,$$

and  $(M, x)$  is mapped under the isomorphism to

$$\left( \left\langle \frac{p_1(M)x - 4x^3}{24}, [M] \right\rangle, \langle x^3, [M] \rangle \right).$$

Both switching the sign of the generator of  $H^2(M)$  and the orientation of  $M$  induce multiplication with  $-1$ , so that diffeomorphism classes are in bijection with  $\mathbb{Z}^2 / -1$ .

As observed in [15], the Lefschetz fixed point theorem implies:

**Lemma 3.1** *If  $\langle x^3, [M] \rangle$  is nonzero, then a free involution on  $M$  must be orientation reversing, and act by  $-1$  on  $H^2(M)$ .*

If  $\langle x^3, [M] \rangle = 0$ , we consider only orientation reversing free involutions which are  $-1$  on  $H^2(M)$ . From Li and Su we learned that except for the case  $M = S^2 \times S^4$ , these are all free involutions: If the involution acts by  $-1$  on  $H^4(M)$ , the first Pontryagin class must be 0, and we use the classification. The remaining case is handled as above by the Lefschetz fixed point theorem.

Obviously, for  $M = S^2 \times S^4$ , our classification of orientation reversing free involutions which are  $-1$  on  $H^2(M)$  does not include all free involutions.

### 3.1 The normal 2-type

**Definition 3.2** The normal 2-type of a compact manifold  $N$  is a fibration  $B_2(N) \rightarrow BO$  which is obtained as a Postnikov factorization of the stable normal bundle map  $N \rightarrow BO$ : There is a 3-connected map  $N \rightarrow B_2(N)$ , the fibration  $B_2(N) \rightarrow BO$  is 3-coconnected (ie  $\pi_i(BO, B_2(N)) = 0$  for  $i > 3$ ), and the composition is the stable normal bundle map. This determines  $B_2(N) \rightarrow BO$  up to fiber homotopy equivalence.

**Lemma 3.3** *Let  $\tau$  be an involution on  $M$  as above, and let  $N = M/\tau$  be the quotient space of the involution. The second space in a Postnikov tower for  $N$  is either  $P = (\mathbb{C}P^\infty \times S^\infty)/(c, -1)$ , where  $c$  is complex conjugation, or  $Q = (\mathbb{C}P^\infty \times S^\infty)/(\tau, -1)$ , where  $\tau$  is fiberwise the antipodal involution on  $S^2 \rightarrow \mathbb{C}P^\infty \rightarrow \mathbb{H}P^\infty$ .*

**Proof** The first space in the Postnikov tower is a  $K(\mathbb{Z}_2, 1)$ , and the second space is a  $K(\mathbb{Z}, 2)$  fibration over it, with  $\pi_1$  acting nontrivially on  $\pi_2$ . Such fibrations are

classified by their  $k$ -invariant in  $H^3(K(\mathbb{Z}_2, 1); \mathbb{Z}_-) \cong \mathbb{Z}_2$ . The spaces  $P$  and  $Q$  have the required properties, and they are not homotopy equivalent, as eg  $H^2(P; \mathbb{Z}_2) \cong \mathbb{Z}_2^2$  and  $H^2(Q; \mathbb{Z}_2) \cong \mathbb{Z}_2$ . So they represent all isomorphism classes of fibrations. ( $P$  has  $k$ -invariant 0, and  $Q$  has nonzero  $k$ -invariant.)  $\square$

**Lemma 3.4** *The  $\mathbb{Z}_2$ -cohomology ring of  $N$  is  $\mathbb{Z}_2[q, t]/\langle t^3, q^2 \rangle$ , where  $\deg(q) = 4$ ,  $\deg(t) = 1$ .*

**Proof** (This was proved in [26] in the case of homotopy  $\mathbb{C}P^3$ 's, and we generalize his proof.) We consider the Serre spectral sequence of the fibration  $M \rightarrow N \rightarrow \mathbb{R}P^\infty$ , with  $\mathbb{Z}_2$  (and also sometimes with integral) coefficients. The first case is that  $d_3: E_3^{0,2} \rightarrow E_3^{3,0}$  is nontrivial. Then by multiplicativity the  $E_4$ -term has exactly one  $\mathbb{Z}_2$  in  $E_4^{p,0}$  and  $E_4^{p,4}$  for each  $p = 0, 1, 2$ . There are no further differentials, and we get the above cohomology ring. The second case is that  $d_3: E_3^{0,2} \rightarrow E_3^{3,0}$  is trivial. We will show that this leads to a contradiction. By multiplicativity, also  $d_3: E_3^{0,6} \rightarrow E_3^{3,4}$  is trivial. If we remember that we need the limit to have no cohomology in degrees  $> 6$ , then we see in the sequence with integral coefficients that there must be a nontrivial  $d_3$ -differential between the fourth and second line. Then the same must hold for the  $\mathbb{Z}_2$  coefficient sequence. And we get a  $d_7$ -differential from the sixth to the zeroth line. The  $E_\infty$ -term has exactly one  $\mathbb{Z}_2$  in  $E_\infty^{p,0}$  for  $p = 0, \dots, 6$  and  $E_\infty^{p,2}$  for  $p = 0, 1, 2$ .

This gives a cohomology ring with a generator  $t \in H^1(N; \mathbb{Z}_2)$ , and another generator  $x \in H^2(N; \mathbb{Z}_2)$ . Since in this case  $H^2(N; \mathbb{Z}_2) \cong \mathbb{Z}_2^2$ , the second Postnikov space must be  $P$ , and we can choose  $x \in H^2(N; \mathbb{Z}_2)$  coming from  $P$ ; we choose the class in  $H^2(P; \mathbb{Z}_2)$  which maps to 0 under a section of  $P \rightarrow \mathbb{R}P^\infty$ . Note that  $x$  maps nontrivially to  $H^2(M; \mathbb{Z}_2)$ .

We have the relations  $t^7 = t^3x = x^2 + at^2x + t^4 = 0$ , where  $a \in \mathbb{Z}_2$ . (We have  $Sq^1 x = tx$  as this is true in  $P$ , thus  $t^3x = Sq^1(t^2x) = 0$  since  $H^5(N; \mathbb{Z}) = 0$ . By Poincaré duality  $t^2x^2$  can't be zero. This implies that in  $x^2 + at^2x + bt^4 = 0$  we have  $b = 1$ .)

We see that  $Sq^1(t^5) = t^6, Sq^2(t^4) = 0, Sq^2(t^2x) = t^2x^2 \neq 0$ . It follows that the first Wu class is  $t$  and the second Wu class is  $x$ . But that implies that the second Stiefel–Whitney class of  $N$  maps to a nontrivial class in  $H^2(M; \mathbb{Z}_2)$ . But since  $M$  is spin, this image must be zero, as it is the second Stiefel–Whitney class of  $M$ . Contradiction.  $\square$

**Corollary 3.5** *The second space in a Postnikov tower for  $N$  is  $Q$ .*

**Proof** This follows from the fact that  $H^2(N; \mathbb{Z}_2) \cong \mathbb{Z}_2$ .  $\square$

**Proposition 3.6** *If such a manifold  $M$  with  $\langle x^3, [M] \rangle$  odd has a free involution  $\tau$ , then the quotient space  $N$  has normal 2-type  $B = BSpin \times Q \rightarrow BO \times BO(1) \xrightarrow{\oplus} BO$ .*

**Proof** If  $\langle x^3, [M] \rangle$  is odd, then the map  $H^4(Q; \mathbb{Z}_2) \rightarrow H^4(N; \mathbb{Z}_2)$  is a bijection, since then we have isomorphisms  $H^4(Q; \mathbb{Z}_2) \rightarrow H^4(\mathbb{C}P^\infty; \mathbb{Z}_2) \rightarrow H^4(M; \mathbb{Z}_2) \leftarrow H^4(N; \mathbb{Z}_2)$ . Then  $Sq^2: H^4(N; \mathbb{Z}_2) \rightarrow H^6(N; \mathbb{Z}_2)$  is zero, so is the second Wu class of  $M/\tau$ , and as a consequence the quotient space has a spin structure twisted by the nontrivial real line bundle  $L \rightarrow Q$ .  $\square$

**Proposition 3.7** *If such a manifold  $M$  has a free involution  $\tau$  (satisfying the extra conditions if  $M = S^2 \times S^4$ ), then the quotient space  $N$  has one of the following normal 2-types:*

$$A = Q \times BSpin \rightarrow BO(1) \times BO(1) \times BO(1) \times BO \xrightarrow{\oplus} BO,$$

$$B = Q \times BSpin \rightarrow BO(1) \times BO \xrightarrow{\oplus} BO,$$

where the maps to all  $BO(1)$ 's are the projections  $p: Q \rightarrow \mathbb{R}P^\infty = BO(1)$ .

**Proof** If  $\langle x^3, [M] \rangle$  is even, then there is a second possibility for the normal 2-type. If we fix the second space in a Postnikov tower to be  $Q$ , the second Stiefel–Whitney class of  $N$  can be  $t^2$  or 0. Thus  $N$  either admits spin structures twisted by  $L$  or spin structures twisted by  $L \oplus L \oplus L = 3L$ .  $\square$

A normal  $B$ -structure on a manifold  $N$  can be defined in three equivalent ways.

- It is a vertical homotopy class of lifts of the normal bundle map  $N \rightarrow BO$  to  $B$  (this is independent of the particular map  $N \rightarrow BO$  coming from an embedding of  $N$  into some Euclidean space).
- It is a spin structure on  $N$  twisted by  $L$ , ie a map  $f: N \rightarrow Q$  (up to homotopy) together with a spin structure on the bundle  $\nu_N - f^*(L)$ , where  $L$  is the nontrivial real line bundle on  $Q$ .
- It is a map  $f: N \rightarrow Q$  together with a homotopy  $\eta$  (and this up to homotopy) in the following square:

$$\begin{array}{ccc}
 N & \xrightarrow{\nu} & BO \\
 \downarrow f & \nearrow \eta & \downarrow w_1 \times w_2 \\
 Q & \xrightarrow[t \times 0]{} & K(\mathbb{Z}_2, 1) \times K(\mathbb{Z}_2, 2)
 \end{array}$$

Here we fix maps corresponding to the classes  $w_1 \in H^1(BO; \mathbb{Z}_2)$ ,  $w_2 \in H^2(BO; \mathbb{Z}_2)$ , the generator  $t \in H^1(Q; \mathbb{Z}_2)$ ,  $0 \in H^2(Q; \mathbb{Z}_2)$ .

(For fixed  $f$ , the homotopy classes of homotopies  $\eta$  have a free and transitive action by  $\pi_1((K(\mathbb{Z}_2, 1) \times K(\mathbb{Z}_2, 2))^N) \cong H^1(N; \mathbb{Z}_2) \times H^0(N; \mathbb{Z}_2)$ .)

Similarly, normal  $A$ -structures are defined. (In the first definition, replace  $B$  by  $A$ . In the second definition, replace  $L$  by  $3L$ . In the third definition, replace  $0$  by  $t^2$ .)

**Remark 3.8** As a converse to Proposition 3.7, a manifold with normal  $B$ -structure  $N^6 \rightarrow B$  which is a 3-connected map is the quotient of an involution on a closed simply connected spin manifold  $M$  with  $H_2(M) = \mathbb{Z}$ , and  $H_3(M) = 0$  if and only if  $H_3(N; \Lambda) = 0$ . Here  $\Lambda = \mathbb{Z}[\mathbb{Z}_2]$  is the group ring of the fundamental group. The same holds for normal  $A$ -structures.

### 3.2 Computation of the bordism groups

We compute bordism groups of manifolds with normal  $A$ -structures (respectively  $B$ -structures). The (co)homology of  $Q$  is described in [20]. To compute the bordism groups  $\Omega_6^A$  and  $\Omega_6^B$  we use the Atiyah–Hirzebruch spectral sequence (which computes the group up to an extension problem) and an Adams spectral sequence (which can help solve the extension problem). We get:

**Theorem 3.9** We have isomorphisms  $\Omega_6^A \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2$  and  $\Omega_6^B \cong \mathbb{Z}^2 \oplus \mathbb{Z}_4$ .

**Proof** The case of  $\Omega_6^B$  was proven in [20]. For  $\Omega_6^A$ , the Atiyah–Hirzebruch spectral sequence is

$$H_p(Q; \underline{\Omega}_q^{\text{Spin}}) \Rightarrow \Omega_{p+q}^A,$$

the  $d^2$ -differential

$$E_{p,1}^2 \cong H_p(Q; \mathbb{Z}_2) \rightarrow H_{p-2}(Q; \mathbb{Z}_2) \cong E_{p-2,2}^2$$

is the dual of  $\text{Sq}^2 + t \text{Sq}^1 + t^2 \text{Sq}^0$ , and the  $d^2$ -differential

$$E_{p,0}^2 \cong H_p(Q; \mathbb{Z}_-) \rightarrow H_{p-2}(Q; \mathbb{Z}_2) \cong E_{p-2,1}^2$$

is reduction modulo 2 composed with the dual of  $\text{Sq}^2 + t \text{Sq}^1 + t^2 \text{Sq}^0$ . (We obtain this using the Thom isomorphism

$$\Omega_6^{\text{Spin}}(Q; 3L) \cong \Omega_9^{\text{Spin}}(D(3L), S(3L))$$

and the Atiyah–Hirzebruch spectral sequence for the latter. See also [27].) From the calculations in [20] it immediately follows that the differentials  $d^2: E_{6,1}^2 \rightarrow E_{4,2}^2$

and  $d^2: E_{6,0}^2 \rightarrow E_{4,1}^2$  are nontrivial. This implies that the nonzero terms on the sixth diagonal in the  $E^\infty$ -term are

$$E_{2,4}^\infty \cong \mathbb{Z}, \quad E_{5,1}^\infty \cong \mathbb{Z}_2, \quad E_{6,0}^\infty \cong 2\mathbb{Z}.$$

Thus  $\Omega_6^A \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2$  or  $\Omega_6^A \cong \mathbb{Z}^2$ .

Now we consider the Adams spectral sequence

$$\text{Ext}_{\mathcal{A}}^{s,t}(H^*(M \text{ Spin} \wedge T(3L); \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s-3}^A,$$

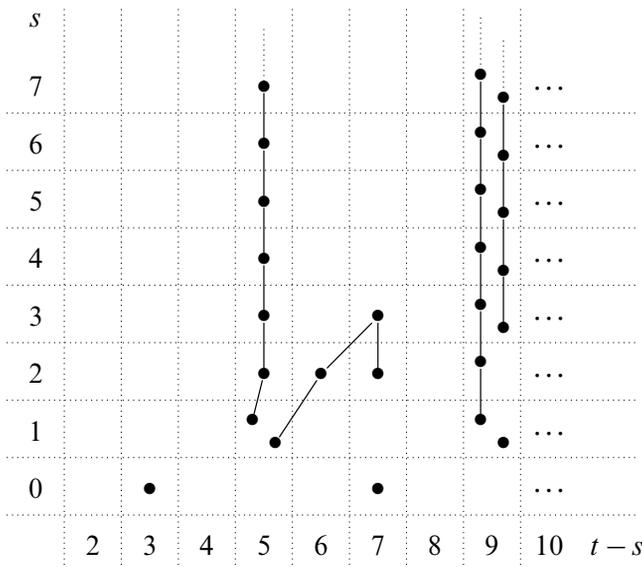
where  $\mathcal{A}$  is the mod 2 Steenrod algebra. We compute the left hand side for  $t-s-3 \leq 6$ . We have

$$H^*(M \text{ Spin} \wedge T(3L); \mathbb{Z}_2) \cong H^*(M \text{ Spin}; \mathbb{Z}_2) \otimes \tilde{H}^*(T(3L); \mathbb{Z}_2),$$

and  $\tilde{H}^*(T(3L); \mathbb{Z}_2)$  is a free  $H^*(Q; \mathbb{Z}_2)$ -module on one generator  $u_3$  of degree 3 (the Thom class). We have

$$\text{Sq}(u_3) = w(3L)u_3 = u_3 + tu_3 + t^2u_3 + t^3u_3.$$

All of this allows to write down the  $\mathcal{A}$ -module structure of  $H^*(M \text{ Spin} \wedge T(3L); \mathbb{Z}_2)$  in degrees  $\leq 10$ . Then we compute a free  $\mathcal{A}$ -resolution (in low degrees). From this we get the  $E^2$ -term of the spectral sequence, which is displayed in the following diagram:



Since there are no differentials starting or ending at  $(t-s, s) = (9, 1)$ , we obtain  $\Omega_6^A \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2$ . □

### 3.3 Construction and classification up to normal $B$ -bordism

By Wall’s classification, every bordism class in  $\Omega_6^{\text{Spin}}(K(\mathbb{Z}, 2)) \cong \mathbb{Z}^2$  contains a unique normal 2-smoothing with  $H_3 = 0$  (up to diffeomorphism relative  $B\text{Spin} \times K(\mathbb{Z}, 2)$ ).

**Theorem 3.10** *In every bordism class in  $\Omega_6^A$  there is a unique manifold  $W \rightarrow A$  (up to diffeomorphism relative to  $A$ ) such that  $W \rightarrow A$  is 3-connected and  $H_3(W; \Lambda) = 0$ . The same is true if one replaces  $A$  by  $B$ .*

**Proof** For the construction and classification of free involutions on these manifolds, we use surgery theory. The existence proof is a simplified version of the proof of the main theorem in [19].

We start with any 6-dimensional closed manifold with normal  $A$ -structure, and we do surgery to get manifolds  $W$  such that the map  $W \rightarrow A$  is 3-connected and  $H_3(W; \Lambda) = 0$ .

Surgery below the middle dimension is always possible [14]. It allows to modify any closed 6-manifold  $W$  with normal  $A$ -structure into one with normal 2-type  $A$ . Let us denote this new manifold again by  $W$ . It remains to kill  $H_3(W; \Lambda)$ .

By the Hurewicz theorem (its extended version) we have a surjection  $\pi_3(W) \rightarrow H_3(W; \Lambda)$ , where  $\Lambda = \mathbb{Z}[\mathbb{Z}_2]$  is the group ring of the fundamental group, and  $H_3(W; \Lambda)$  can be identified with the homology of the universal cover of  $W$ .

The map  $H_3(W; \Lambda) \rightarrow H_3(W; \mathbb{Z}_2)$  factors through  $H_3(W; \Lambda) \otimes_{\Lambda} \mathbb{Z}_2$ , more precisely the relation is given by a universal coefficient spectral sequence

$$\text{Tor}_p^{\Lambda}(H_q(W; \Lambda), \mathbb{Z}_2) \Rightarrow H_{p+q}(W; \mathbb{Z}_2).$$

(This can also be interpreted as the Serre spectral sequence for the fibration  $\widetilde{W} \rightarrow W \rightarrow \mathbb{R}P^{\infty}$ .) Here the zeroth and the second row are related by nontrivial differentials: we compare with the corresponding situation for the space  $Q$ . As a result,  $H_3(W; \mathbb{Z}_2) \cong H_3(W; \Lambda) \otimes_{\Lambda} \mathbb{Z}_2$ .

By Poincaré duality,  $H_3(W; \Lambda) \cong H^3(W; \Lambda)$ , and this is free over  $\mathbb{Z}$ , as there is no  $\mathbb{Z}$ -torsion in  $H_2(W; \Lambda)$ . Since  $H_3(W; \Lambda)$  is free over  $\mathbb{Z}$ , it is a sum of summands of the form  $\Lambda, \mathbb{Z}_+$  and  $\mathbb{Z}_-$  [3]. We also get that  $H^3(W; \Lambda) \cong \text{Hom}_{\Lambda}(H_3(W; \Lambda), \Lambda)$ , for example again from a universal coefficient spectral sequence. The map  $H_3(W; \Lambda) \rightarrow H^3(W; \Lambda) \rightarrow \text{Hom}_{\Lambda}(H_3(W; \Lambda), \Lambda)$  describes the  $\Lambda$ -valued intersection form on  $H_3(W; \Lambda)$ . The  $\mathbb{Z}_2$ -valued intersection form on  $H_3(W; \mathbb{Z}_2)$  is given by tensoring with  $\mathbb{Z}_2$ .

But this implies that for a generator  $x \in H_3(W; \Lambda)$  of a  $\mathbb{Z}_\pm$  summand, its image in  $H_3(W; \mathbb{Z}_2)$  has intersection 0 with all other elements, hence it must be 0: if  $Tx = \pm x$ , then  $T\lambda(x, y) = \lambda(Tx, y) = \lambda(\pm x, y) = \pm\lambda(x, y)$ , so  $\lambda(x, y)$  is a multiple of  $(1 \pm T)$  and its reduction in  $\mathbb{Z}_2$  is zero.

Hence  $H_3(W; \Lambda)$  is a free  $\Lambda$ –module with nondegenerate intersection form. Since we are in dimension 3, there is a quadratic refinement in  $\Lambda/\langle x + \bar{x}, 1 \rangle$  which is uniquely determined by the intersection form. Hence we obtain an element in  $\tilde{L}_6(\Lambda, w = -) = 0$ , as  $L_6 \cong \mathbb{Z}_2$  is given by the Arf invariant. Thus it is possible (after stabilization) to do surgery which makes  $H_3(W; \Lambda) = 0$ .

The argument shows that every class of  $\Omega_6^A$  contains a manifold  $W$  with normal 2–type  $A$  and  $H_3(W; \Lambda) = 0$ .

For the uniqueness result we take two such manifolds  $W_0, W_1$ , and assume there is a normal  $A$ –bordism between them. By Kreck’s theory [14, page 734], there is a surgery obstruction in  $\tilde{L}_7(\Lambda, w = -1)$  for turning the normal  $A$ –bordism into an  $s$ –cobordism. By surgery below the middle dimension we may assume that the bordism  $Y$  is equipped with a 3–connected map to  $A$ . Now Kreck defines the surgery obstruction using a construction of a certain disjoint union  $U$  of submanifolds of  $Y$  diffeomorphic to  $S^3 \times D^4$ , and defines the surgery obstruction to be the kernel of  $H_3(\partial U; \Lambda) \rightarrow H_3(Y \setminus \text{int}(U), W_0; \Lambda)$ . He also notes that the orthogonal complement of this kernel is the kernel of  $H_3(\partial U; \Lambda) \rightarrow H_3(Y \setminus \text{int}(U), W_1; \Lambda)$ . But in our case  $H_3(W_i; \Lambda) = 0$  so that both these kernels are equal to the kernel of  $H_3(\partial U; \Lambda) \rightarrow H_3(Y \setminus \text{int}(U); \Lambda)$ .

This implies that the surgery obstruction just defined lies in the group  $\tilde{L}_7(\Lambda, w = -1)$  which is trivial as computed by Wall. Hence we get as a result that  $A$ –bordant manifolds  $W_i$  with normal 2–type  $A$  and  $H_3(W; \Lambda) = 0$  are diffeomorphic (relative to  $A$ ).

The proof for normal 2–type  $B$  is obtained by replacing all occurrences of  $A$  by  $B$ .  $\square$

### 3.4 The transfer

For the transfer (double cover) map  $\Omega_6^A \rightarrow \Omega_6^{\tilde{A}} \cong \Omega_6^{\text{Spin}}(\mathbb{C}P^\infty)$  we compare the Atiyah–Hirzebruch spectral sequences. We fix the isomorphism

$$\Omega_6^{\text{Spin}}(\mathbb{C}P^\infty) \cong \mathbb{Z}^2$$

$$[f: M \rightarrow \mathbb{C}P^\infty] \mapsto \left( \left\langle \frac{p_1(M)f^*x - 4f^*x^3}{24}, [M] \right\rangle, \langle f^*x^3, [M] \rangle \right).$$

One computes the homology transfers using the long exact sequences coming from short exact coefficient sequences:

$$\begin{aligned} H_6(Q; \mathbb{Z}_-) &\xrightarrow{\cong} H_6(\mathbb{C}P^\infty), \\ H_4(Q; \mathbb{Z}_2) &\xrightarrow{0} H_4(\mathbb{C}P^\infty; \mathbb{Z}_2), \\ H_2(Q; \mathbb{Z}_-) &\xrightarrow{\cong} H_2(\mathbb{C}P^\infty). \end{aligned}$$

We see that the transfer gives a map of short exact sequences

$$\begin{array}{ccccc} F_5 \cong \mathbb{Z}_2 \oplus \mathbb{Z} & \xrightarrow{(a,b) \mapsto (a,b,0)} & \Omega_6^A \cong \mathbb{Z}_2 \oplus \mathbb{Z}^2 & \xrightarrow{(a,b,c) \mapsto c} & E_{6,0}^\infty \cong \mathbb{Z} \\ (a,b) \mapsto 2b \downarrow & & \downarrow \text{tr} & & \downarrow c \mapsto 2c \\ \tilde{F}_5 \cong \mathbb{Z} & \xrightarrow{b \mapsto (b,0)} & \Omega_6^{\tilde{A}} \cong \mathbb{Z}^2 & \xrightarrow{(b,c) \mapsto c} & \tilde{E}_{6,0}^\infty \cong \mathbb{Z} \end{array}$$

which shows (using the snake lemma) that  $\Omega_6^A \rightarrow \Omega_6^{\tilde{A}}$  has a cokernel of order 4. But the composition of projection and transfer:  $\Omega_6^{\tilde{A}} \rightarrow \Omega_6^A \rightarrow \Omega_6^{\tilde{A}}$  is multiplication by 2. So the image of the transfer consists exactly of all classes divisible by 2.

It also follows that one can find generators for the free summands in  $\Omega_6^A$  as images of the generators of  $\Omega_6^{\tilde{A}}$ .

**Theorem 3.11** [19] *The image of the transfer map  $\Omega_6^B \rightarrow \Omega_6^{\tilde{B}} \cong \Omega_6^{\text{Spin}}(\mathbb{C}P^\infty)$  is  $2\mathbb{Z} \oplus \mathbb{Z}$ .*

### 3.5 Generators for the bordism group $\Omega_6^B$

We use the Thom isomorphism for twisted spin bordism:

$$\Omega_6^B \cong \Omega_6^{\text{Spin}}(Q; L) \cong \Omega_7^{\text{Spin}}(DL, SL).$$

Under this isomorphism, the boundary map

$$\Omega_7^{\text{Spin}}(DL, SL) \rightarrow \Omega_6^{\text{Spin}}(SL) \cong \Omega_6^{\text{Spin}}(\mathbb{C}P^\infty)$$

corresponds to the transfer map  $\Omega_6^B \rightarrow \Omega_6^{\tilde{B}}$ . It follows that the torsion elements in

$$\Omega_6^B \cong \Omega_7^{\text{Spin}}(DL, SL)$$

come from

$$\Omega_7^{\text{Spin}}(DL) \cong \Omega_7^{\text{Spin}}(Q).$$

Moreover,  $\Omega_7^{\text{Spin}}(\mathbb{C}P^\infty) = 0$ , as one sees easily from the Atiyah–Hirzebruch spectral sequence. Thus  $\Omega_7^{\text{Spin}}(Q) \cong \mathbb{Z}_4$  must be responsible for the torsion. Moreover, a study

of the Atiyah–Hirzebruch spectral sequence shows that the map

$$\Omega_7^{\text{Spin}}(\mathbb{C}\mathbb{P}^3/\tau) \rightarrow \Omega_7^{\text{Spin}}(Q)$$

is an isomorphism. Now there is a bundle  $\mathbb{R}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^3/\tau \rightarrow S^4$ . So  $\mathbb{C}\mathbb{P}^3/\tau - \mathbb{R}\mathbb{P}^2$  is an  $\mathbb{R}\mathbb{P}^2$ -bundle over  $D^4$ , so homotopy equivalent to  $\mathbb{R}\mathbb{P}^2$ . Again the Atiyah–Hirzebruch spectral sequence shows that

$$\Omega_7^{\text{Spin}}(\mathbb{C}\mathbb{P}^3/\tau - \mathbb{R}\mathbb{P}^2) = \Omega_6^{\text{Spin}}(\mathbb{C}\mathbb{P}^3/\tau - \mathbb{R}\mathbb{P}^2) = 0.$$

Thus  $\Omega_7^{\text{Spin}}(\mathbb{C}\mathbb{P}^3/\tau) \cong \Omega_7^{\text{Spin}}(\mathbb{C}\mathbb{P}^3/\tau, \mathbb{C}\mathbb{P}^3/\tau - \mathbb{R}\mathbb{P}^2)$ ,

and the latter is isomorphic to  $\Omega_3^{\text{Spin}}(\mathbb{R}\mathbb{P}^2) \cong \mathbb{Z}_4$  by the Thom isomorphism. Again the Atiyah–Hirzebruch spectral sequence shows that the transfer

$$\Omega_3^{\text{Spin}}(\mathbb{R}\mathbb{P}^2) \rightarrow \Omega_3^{\text{Spin}}(S^2) \cong \mathbb{Z}_2$$

is surjective. Thus one can detect a generator of  $\Omega_7^{\text{Spin}}(\mathbb{C}\mathbb{P}^3/\tau)$  by the composition

$$\Omega_7^{\text{Spin}}(\mathbb{C}\mathbb{P}^3/\tau) \rightarrow \Omega_7^{\text{Spin}}(\mathbb{C}\mathbb{P}^3/\tau, \mathbb{C}\mathbb{P}^3/\tau - \mathbb{R}\mathbb{P}^2) \rightarrow \Omega_3^{\text{Spin}}(\mathbb{R}\mathbb{P}^2) \rightarrow \Omega_3^{\text{Spin}}(S^2).$$

Now take the seven-dimensional manifold  $(\mathbb{C}\mathbb{P}^3 \times S^1)/(\tau, c)$ , where  $c$  is complex conjugation. The spin structure on  $\mathbb{C}\mathbb{P}^3 \times S^1$  which restricts to the nonbounding one on  $S^1$  is preserved by the involution  $(\tau, c)$ , so that we obtain a spin structure on the quotient. The map to  $\mathbb{C}\mathbb{P}^3/\tau$  is projection on the first coordinate. This intersects  $\mathbb{R}\mathbb{P}^2 = \mathbb{C}\mathbb{P}^1/\tau$  transversely, so that the Thom isomorphism maps it to the pullback  $(\mathbb{C}\mathbb{P}^1 \times S^1)/(\tau, c) \rightarrow \mathbb{C}\mathbb{P}^1/\tau$  whose double cover is the projection  $\mathbb{C}\mathbb{P}^1 \times S^1 \rightarrow \mathbb{C}\mathbb{P}^1$ , which is a generator for  $\Omega_3^{\text{Spin}}(S^2)$  since the Spin structure restricts to the nonboundant one on  $S^1$ .

We have to apply the Thom isomorphism

$$\Omega_7^{\text{Spin}}(DL|_{\mathbb{C}\mathbb{P}^3/\tau}, SL|_{\mathbb{C}\mathbb{P}^3/\tau}) \rightarrow \Omega_6^{\text{Spin}}(\mathbb{C}\mathbb{P}^3/\tau; L)$$

to the map  $(\mathbb{C}\mathbb{P}^3 \times S^1)/(\tau, c) \rightarrow \mathbb{C}\mathbb{P}^3/\tau \rightarrow (\mathbb{C}\mathbb{P}^3 \times D^1)/(\tau, -1)$ . For this, we homotope the map to make it transversal to the zero section: take

$$\begin{aligned} (\mathbb{C}\mathbb{P}^3 \times S^1)/(\tau, c) &\rightarrow (\mathbb{C}\mathbb{P}^3 \times D^1)/(\tau, -1) \\ [x, y] &\mapsto [x, \text{Im}(y)] \end{aligned}$$

and intersect it with the zero section: we obtain two copies of  $\mathbb{C}\mathbb{P}^3/\tau$  with different  $B$ -structures.

It follows that the torsion  $\mathbb{Z}_4$  in  $\Omega_6^B$  is generated by the sum (or the difference) of two copies of  $\mathbb{C}P^3/\tau$  with different  $B$ -structures. Let us denote one of them by  $\sigma$  and the other by  $\sigma'$ .

Thus we obtain as generators for  $\Omega_6^B \cong \Omega_7^{\text{Spin}}(DL, SL)$ :

- The map  $f \times \text{id}: X \times D^1 \rightarrow \mathbb{C}P^\infty \times D^1$ , where  $X$  is a simply connected spin 6-manifold with  $H^3(X) = 0$ , trivial cup product  $H^2(X) \times H^2(X) \rightarrow H^4(X)$ , and  $f: X \rightarrow \mathbb{C}P^\infty$  defines a generator of  $H^2(X)$  such that the first Pontrjagin class of  $X$  is equal to 24 times the generator of  $H^4(X)$  which is dual to  $f \in H^2(X)$ .
- $((\mathbb{C}P^3 \times D^1)/(\tau, -1), \sigma)$ .
- $((\mathbb{C}P^3 \times D^1)/(\tau, -1), \sigma) - ((\mathbb{C}P^3 \times D^1)/(\tau, -1), \sigma')$ .

The two former generators generate free summands, the latter generates the torsion summand  $\mathbb{Z}_4$ . In this basis, the map  $\Omega_6^B \cong \mathbb{Z}^2 \oplus \mathbb{Z}_4 \rightarrow \mathbb{Z}^2 \cong \Omega_6^{\text{Spin}}(\mathbb{C}P^\infty)$  is given as  $(a, b, c) \mapsto (2a, b)$ .

### 3.6 The automorphism groups of $A$ and $B$ and their action on the bordism groups

The automorphism group  $\text{Aut}(B)_{BO}$  of fiber homotopy classes of fiber homotopy self-equivalences of  $B$  acts on  $\Omega_6^B$ . We saw that  $\Omega_6^B \cong \Omega_7^{\text{Spin}}(DL, SL)$ , where  $L$  is the nontrivial real line bundle  $(\mathbb{C}P^\infty \times \mathbb{R})/(\tau, -1)$ .

**Lemma 3.12** *The set of equivariant oriented diffeomorphism classes of free involutions on 6-manifolds with  $H_3 = 0$  and whose quotient spaces have normal 2-type  $B$  are given as the orbits. Again,  $B$  can be replaced by  $A$  in the theorem.*

**Proof** Equivariant diffeomorphism classes of free involutions are the same as diffeomorphism classes of the quotients. Now the lemma follows from the uniqueness of the Postnikov decomposition, ie for a given manifold  $W$  with normal 2-type  $B$ , the map  $W \rightarrow B$  is uniquely determined up to fiber homotopy self equivalences of  $B$  over  $BO$ . See also [13]. □

The restriction of the first component of a fiber homotopy self-equivalence of  $Q \times B\text{Spin}$  to  $Q$  is a self-homotopy equivalence of  $Q$ . There is a unique free homotopy class of maps  $Q \rightarrow Q$  which is an isomorphism on  $\pi_1$ , and thus (using obstruction theory to extend homotopies from  $Q$  to  $Q \times B\text{Spin}$ ) a unique free homotopy class of maps  $Q \times B\text{Spin} \rightarrow Q$  which are an isomorphism on  $\pi_1$ . The vertical homotopy classes of fiber homotopy equivalences thus correspond to the different choices (up to homotopy)

of a homotopy  $\eta$  from  $Q \times BSpin \rightarrow Q \rightarrow K(\mathbb{Z}_2, 1) \times K(\mathbb{Z}_2, 2)$  to  $Q \times BSpin \rightarrow BO \rightarrow K(\mathbb{Z}_2, 1) \times K(\mathbb{Z}_2, 2)$ . So the group has four elements, and the action of the group on the set of normal  $B$ -structures on a manifold  $f: M \rightarrow B$  (ie spin structures on  $\nu_M - f^*L$ ) is given by just changing the spin structure  $\sigma$  into  $\sigma, -\sigma, \sigma + f^*t, -\sigma + f^*t$ , where  $t \in H^1(B; \mathbb{Z}_2)$  is the generator.

On  $\Omega_7^{Spin}(DL, SL)$ , the group  $Aut(B)_{BO}$  acts in the following way: the negative spin generator acts by  $-1$ , and the spin flip generator acts by

$$((\mathbb{C}P^3 \times D^1)/(\tau, -1), \sigma) \mapsto ((\mathbb{C}P^3 \times D^1)/(\tau, -1), \sigma')$$

and is the identity on  $X$ . So in the decomposition  $\Omega_7^{Spin}(DL, SL) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_4$  given by the above generators, the negative spin generator acts by  $(a, b, c) \mapsto (-a, -b, -c)$ , and the spin flip generator acts by  $(a, b, c) \mapsto (a, b, b - c)$ . We obtain orbits of the form

$$\{(a, b, c), (a, b, b - c), (-a, -b, -c), (-a, -b, -b + c)\}.$$

The group  $Aut(A)_{BO}$  also has four elements which act on the set of normal  $A$ -structures on a manifold  $f: M \rightarrow A$  (ie spin structures on  $\nu_M - f^*(3L)$ ) by just changing the spin structure  $\sigma$  into  $\sigma, -\sigma, \sigma + f^*t, -\sigma + f^*t$ , where  $t \in H^1(A; \mathbb{Z}_2)$  is the generator.

This is either the identity or minus the identity on the generators for the free summands in  $\Omega_6^A$  as they are images of the generators of  $\Omega_6^A$  under projection. All group elements must act by the identity on the torsion generator. We obtain orbits

$$\{(a, b, c), (-a, -b, -c)\}.$$

The group  $Aut(K(\mathbb{Z}, 2) \times BSpin)_{BO}$  also has four elements, which act on the bordism group by reversing the spin structure and/or the class in  $H^2$ . It follows that here the orbits are of the form

$$\{(a, b), (-a, -b)\}.$$

For the proof of [Theorem 1.2](#), it is now sufficient to count preimages and orbits: An element of the form  $(2a + 1, b) \in \Omega_6^{Spin}(\mathbb{C}P^\infty)$  has no preimages in  $\Omega_6^A$  or  $\Omega_6^B$ . An element  $(2a, 2b + 1) \in \Omega_6^{Spin}(\mathbb{C}P^\infty)$  has four preimages  $(a, 2b + 1, c)$  in  $\Omega_6^B$  and no preimages in  $\Omega_6^A$ . These four preimages, together with the four preimages of  $(-2a, -2b - 1)$ , decompose into two orbits. For  $(2a, 2b) \in \Omega_6^{Spin}(\mathbb{C}P^\infty)$  we obtain four preimages  $(a, 2b, c)$  in  $\Omega_6^B$ . These four preimages, together with the four preimages of  $(-2a, -2b)$ , decompose into three orbits. The element  $(2a, 2b) \in \Omega_6^{Spin}(\mathbb{C}P^\infty)$  has two preimages  $(a, b, c)$  in  $\Omega_6^A$ . These two preimages, together with the two preimages of  $(-2a, -2b)$ , decompose into two orbits.

## 4 Involutions on $S^6$ with three-dimensional fixed point set

### 4.1 Nonequivariant classification of embeddings in $S^6$

Before we classify involutions on  $S^6$ , let us recall the nonequivariant results on embeddings of three-manifolds into  $S^6$ .

Naturally the most interesting case is the one of knotted 3–spheres in the six-sphere. Here the results are due to Haefliger. One could consider various equivalence relations on knotted  $S^3$ 's in  $S^6$ . In all cases there is an addition defined using connected sums:

First we can look at the group  $C_3^3$  of isotopy classes of embeddings of  $S^3$  into  $S^6$ . Second, the group  $\Theta$  of diffeomorphism classes of embeddings of  $S^3$  into oriented manifolds  $X$  diffeomorphic to  $S^6$ , relative to  $S^3$ . (We require a diffeomorphism to be the identity on  $S^3$  and to preserve orientations.) Third, the group  $\Sigma$  of orientation-preserving diffeomorphism classes of pairs  $(X, M)$ , where the oriented manifold  $X$  is diffeomorphic to  $S^6$  and the oriented submanifold  $M$  is diffeomorphic to  $S^3$ .

There are obvious surjective group homomorphisms  $C_3^3 \rightarrow \Theta \rightarrow \Sigma$ . Haefliger showed that  $C_3^3$  and  $\Sigma$  are both isomorphic to  $\mathbb{Z}$ , so that both of the above maps are isomorphisms.

To prove that  $C_3^3 \rightarrow \Theta$  is an isomorphism, one needs to show that a diffeomorphism  $S^6 \rightarrow S^6$  relative the embedded  $S^3$  can be replaced by an isotopy. This is true since  $\pi_0(\text{Diff}(D^n, \partial)) \rightarrow \pi_0(\text{Diff}(S^n))$  is surjective. This means it is always possible to modify the original diffeomorphism on a disk such that the resulting diffeomorphism is isotopic to the identity.

One explanation of the isomorphism  $\Theta \rightarrow \Sigma$  is Cerf's result that  $\text{Diff}^+(S^3)$  is connected [2]. Thus every orientation-preserving diffeomorphism of  $S^3$  is isotopic to the identity, and this isotopy extends to an ambient equivariant isotopy of  $S^6$ .

The negative of an isotopy class is given by precomposing the embedding  $S^3 \rightarrow S^6$  with an orientation-reversing self-diffeomorphism of  $S^3$ .

Isotopy classes of framed embeddings  $S^3 \times D^3 \rightarrow S^6$  are in bijection with  $\mathbb{Z}^2$ , the framing giving an additional integer invariant (the isomorphism to  $\mathbb{Z}^2$  depends on a choice). Note also that in the PL category, all nonframed knots are trivial, but the isotopy classes of framed knots are in bijection with smooth isotopy classes of smooth framed knots.

For embeddings of general closed oriented connected 3–manifolds  $M^3$  into  $S^6$ , the argument that diffeomorphism relative to the submanifold implies isotopy of the embeddings still holds. Isotopy classes of embeddings  $\text{Emb}(M, S^6)$  have been classified

by Skopenkov in [25]. To an isotopy class of embeddings  $i: M \rightarrow S^6$  he associates its Whitney invariant  $W(i) \in H_1(N)$ . (For the precise definition we refer to [25], we give a description in special cases in Remark 4.5.) The map  $W: \text{Emb}(M, S^6) \rightarrow H_1(N)$  is surjective, and  $C_3^3 \cong \mathbb{Z}$  acts transitively on the fibers by connected sum. This action has nontrivial stabilizers in general: There is a bijective map (the Kreck invariant) from the fiber over  $u \in H_1(N)$  to  $\mathbb{Z}_{d(u)}$ , where  $d(u)$  is the divisibility of  $\bar{u} \in H_1(M)/\{\text{torsion}\}$ . In general, both the Whitney and the Kreck invariant depend on choices. For  $\mathbb{Z}_2$ –homology spheres  $M$ , the Whitney invariant does not depend on choices, and  $C_3^3$  acts freely on the fibers, so that  $\text{Emb}(M, S^6)$  is in noncanonical bijection with  $\mathbb{Z} \times H_1(M)$ . Instead of a map to  $\mathbb{Z}$ , the Kreck invariant describes an action of  $\mathbb{Z}$  on  $\text{Emb}(M, S^6)$  which leaves the Whitney invariant fixed.

## 4.2 What should we classify in the equivariant case?

In the equivariant case, there are again various equivalence relations one can put on the set of embeddings respectively involutions. However, in order to get a well-defined connected sum operation, it is necessary to orient both the 6–manifold and the fixed point set.

**Remark 4.1** The proof of the uniqueness of the nonequivariant connected sum construction can be generalized to show that the equivariant diffeomorphism type of the equivariant connected sum of two conjugation manifolds of dimension  $2n$  only depends on the connected component of the set of equivariant isomorphisms of a tangent space at a fixed point with  $(\mathbb{R}^{2n}, (1^n, -1^n))$ . More generally, varying the chosen fixed point, we get a bundle of such isomorphisms over the fixed point set, and the equivariant diffeomorphism type depends only on the connected component in the total space of this bundle. (The total space has two components if the fixed point set is orientable, and one component if it is not.) See also Definition 1.1 and Lemma 1.2 of [16]. In particular, the connected sum is unique up to equivariant diffeomorphism if we are provided with orientations of the conjugation manifolds and their fixed point sets, but in general depends also on an orientation of the fixed point sets. This answers a question in [11], and it also explains why we are less interested in the classification of conjugations without an orientation of the fixed point set.

We fix  $M^3$ , allow various involutions  $\tau$  on  $S^6$ , and consider equivariant embeddings  $(M, \text{id}) \rightarrow (S^6, \tau)$  such that the image of  $i$  is the fixed point set of  $\tau$ . Again there are several equivalence relations which we can put on this set.

- Two embeddings  $i_0: (M, \text{id}) \rightarrow (S^6, \tau_0)$  and  $i_1: (M, \text{id}) \rightarrow (S^6, \tau_1)$  are equivalent if there is a diffeomorphism  $\phi: (S^6, \tau_0) \rightarrow (S^6, \tau_1)$  relative  $M$  which is

equivariant and orientation-preserving, ie there is a commutative triangle

$$\begin{array}{ccc}
 (M, \text{id}) & \xrightarrow{i_0} & (S^6, \tau_0) \\
 & \searrow i_1 & \downarrow \phi \\
 & & (S^6, \tau_1).
 \end{array}$$

This classifies involutions together with an identification of the fixed point set with the given 3–manifold  $M$ . We get a set  $\text{Emb}_{\mathbb{Z}_2}(M, S^6)$ .

- Two embeddings  $i_0: (M, \text{id}) \rightarrow (S^6, \tau_0)$  and  $i_1: (M, \text{id}) \rightarrow (S^6, \tau_1)$  are equivalent if there is an equivariant diffeomorphism  $\phi: (S^6, \tau_0) \rightarrow (S^6, \tau_1)$  which restricts to some self-diffeomorphism  $\phi_M$  of  $M$ . We require that both  $\phi$  and  $\phi_M$  are orientation-preserving. There is a commutative square

$$\begin{array}{ccc}
 (M, \text{id}) & \xrightarrow{i_0} & (S^6, \tau_0) \\
 \downarrow \phi_M & & \downarrow \phi \\
 (M, \text{id}) & \xrightarrow{i_1} & (S^6, \tau_1).
 \end{array}$$

This classifies (up to orientation-preserving diffeomorphism) involutions plus an orientation of the fixed point set. One might call this the classification of conjugations up to conjugation. We get a set  $\text{Inv}_M(S^6)$ .

**Remark 4.2** One could also define an equivariant version of isotopy classes of embeddings: we say two equivariant embeddings  $i_0: (M, \text{id}) \rightarrow (S^6, \tau_0)$  and  $i_1: (M, \text{id}) \rightarrow (S^6, \tau_1)$  are equivalent if there is an equivariant embedding  $i: (M \times I, \text{id}) \rightarrow (S^6 \times I, \tau)$  such that  $i(x, t) = (i_t(x), t)$  and  $\tau(y, t) = (\tau_t(y), t)$ , hence in particular it restricts on both ends to  $i_0$  respectively  $i_1$ . (We also require that the image of all the embeddings involved is the whole fixed point set.)

### 4.3 Analysis of involutions on $S^6$ with three-dimensional fixed point set

We recall the classical result:

**Theorem 4.3** (P A Smith [1]) *If an involution on  $S^n$  has fixed points, the fixed point set is a  $\mathbb{Z}_2$ –homology sphere.*

Comparing with Skopenkov’s classification of embeddings, our first result is the following.

**Proposition 4.4** *If  $i: M^3 \rightarrow S^6$  is the embedding of a fixed point set of an involution, then the Whitney invariant of the embedding vanishes:  $W(i) = 0$ .*

**Proof** Let  $\tau$  be such an involution, and let  $\sigma$  be the reflection of  $S^6$  at the equator. In [25] it is proved that  $W(\sigma \circ i) = -W(i)$ . But we have a commutative square

$$\begin{array}{ccc} M & \xrightarrow{i} & S^6 \\ \downarrow = & & \downarrow \sigma \circ \tau \\ M & \xrightarrow{\sigma \circ i} & S^6 \end{array}$$

which shows that  $i$  and  $\sigma \circ i$  are isotopic. Thus  $W(i) = -W(i)$ , and since  $H_1(M)$  consists of odd torsion only, we have  $W(i) = 0$ . □

**Remark 4.5** Up to a multiplicative factor, which is invertible in the case of  $\mathbb{Z}_2$ -homology spheres, the Whitney invariant can also be defined in the following way: Let  $C_i = S^6 \setminus i(M^3)$ . By Alexander duality,  $H^2(C_i) \cong \mathbb{Z}$  and  $H^4(C_i) \cong H^2(M) \cong H_1(M)$ . Then the invariant is given by the square of a generator of  $H^2(C_i)$ . In the case of a fixed point set of a conjugation, the involution acts by multiplication with  $-1$  on both  $H^2(C_i)$  and  $H^4(C_i)$ . It follows that the square of a generator of  $H^2(C_i)$  must be 0.

Recall that by the equivariant tubular neighborhood theorem, for every involution on  $S^6$  with fixed point set  $M$  we can write  $S^6 = M \times D^3 \cup V$ , such that the involution is  $\text{id} \times -\text{id}$  on  $M \times D^3$ , and free on  $V$ . The quotient  $W = V/\tau$  is a manifold with fundamental group  $\mathbb{Z}_2$  and boundary  $M \times \mathbb{R}P^2$ . The normal 2-type of  $W$  is  $B$ . The inclusion of the boundary needs to induce an isomorphism  $\pi_2(\mathbb{R}P^2) \rightarrow \pi_2(B)$ .

In order to define the bordism set  $\Omega_6^{(B, M \times \mathbb{R}P^2)}$ , we have to fix a normal  $B$ -structure on  $M \times \mathbb{R}P^2$ . Since we are interested in a classification, we first consider all relevant normal  $B$ -structures.

For any such structure, the homotopy class of maps  $M \times \mathbb{R}P^2 \rightarrow Q \rightarrow \mathbb{R}P^\infty$  is the nontrivial class in  $H^1(M \times \mathbb{R}P^2; \mathbb{Z}_2)$ . We have to lift this nontrivial map  $M \times \mathbb{R}P^2 \rightarrow \mathbb{R}P^\infty$  to  $Q$ . Any lift, together with a choice of a spin structure on  $\nu_{M \times \mathbb{R}P^2} - f^*(L)$ , will then be a normal  $B$ -structure on  $M \times \mathbb{R}P^2$ . It is easy to find a lift  $f$  on  $\mathbb{R}P^2$  (see [20, page 47]), and composing any lift with the projection  $M \times \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$  gives a lift  $M \times \mathbb{R}P^2 \rightarrow Q$ . Obstruction theory shows that pointed homotopy classes of lifts on  $\mathbb{R}P^2$  are classified by  $H^2(\mathbb{R}P^2, \mathbb{Z}_-) \cong \mathbb{Z}$ , and that every lift on  $\mathbb{R}P^2$  extends uniquely up to homotopy to a lift on  $M \times \mathbb{R}P^2$  (see [20, pages 50–52]).

But there is a further condition. Only two pointed homotopy classes of lifts induce an isomorphism  $\pi_2(\mathbb{R}P^2) \rightarrow \pi_2(Q)$ , and one obtains one from the other by precomposing with the nontrivial pointed homotopy class of maps  $\mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ . However, this map is freely homotopic to the identity of  $\mathbb{R}P^2$ . Thus we get up to (free) homotopy a unique

map  $f: \mathbb{R}P^2 \rightarrow Q$ . There are four spin structures on  $\nu_{M \times \mathbb{R}P^2} - f^*(L)$ , since spin structures on unoriented (but orientable and spin) bundles over  $N$  are in bijection with  $H^0(N; \mathbb{Z}_2) \times H^1(N; \mathbb{Z}_2)$ . Thus there are four distinguished normal  $B$ -structures on  $M \times \mathbb{R}P^2$  which can be used in the construction.

**Theorem 4.6** *For every distinguished normal  $B$ -structure on  $M \times \mathbb{R}P^2$ , every element of the bordism set  $\Omega_6^{(B, M \times \mathbb{R}P^2)}$  contains (up to diffeomorphism relative to  $B$  and the boundary) a unique manifold  $W$  which produces a conjugation on  $S^6$ .*

**Remark 4.7** Actually we also see that  $H_3(W; \Lambda)$  consisting just of odd torsion is a necessary and sufficient condition for this.

**Proof** The proof of existence is a slight modification of the proof of [Theorem 3.10](#). (For full details, see the proof of Theorem 1.3 in [19].) Also the uniqueness extends from the proof of [Theorem 3.10](#): If we take two manifolds  $W_0, W_1$  with the same normal  $B$ -structure on the boundary, and which both produce conjugations on  $S^6$ , and assume that there is a normal  $B$ -bordism between them, we have to modify the argument from [Section 3.3](#) slightly to show that Kreck’s surgery obstruction is 0: For both  $i = 0, 1$  the kernel of  $H_3(\partial U; \Lambda) \rightarrow H_3(Y \setminus \text{int}(U), W_i; \Lambda)$  is equal to the kernel of  $H_3(\partial U; \Lambda) \rightarrow H_3(Y \setminus \text{int}(U); \Lambda) / \{\text{torsion}\}$ . So again the surgery obstruction lies in  $\tilde{L}_7(\Lambda, w = -1) = 0$ . □

**Corollary 4.8** *Diffeomorphism classes of  $W$  relative to  $M \times \mathbb{R}P^2$  and to  $B$  (fixing  $M \times \mathbb{R}P^2 \rightarrow B$ ) producing conjugations on  $S^6$  are in bijection with  $\Omega_6^{(B, M \times \mathbb{R}P^2)} \cong \mathbb{Z}^2 \oplus \mathbb{Z}_4$ .*

Diffeomorphism classes of  $W$  relative to  $M \times \mathbb{R}P^2$  and to  $B$  (where  $M \times \mathbb{R}P^2 \rightarrow B$  is not fixed) producing conjugations on  $S^6$  are in bijection with the disjoint union of these four bordism sets.

Equivariant connected sum of two conjugations with fixed point sets  $M_1$  respectively  $M_2$  corresponds to a map of bordism sets defined by gluing along part of the boundary (or parametrized boundary connected sum): Choose disks  $D^3$  in  $M_1, M_2$  centered at the points where the connected sum is performed. Then there is a map

$$\begin{aligned} \Omega_6^{(B, M_1 \times \mathbb{R}P^2)} \times \Omega_6^{(B, M_2 \times \mathbb{R}P^2)} &\rightarrow \Omega_6^{(B, (M_1 \# M_2) \times \mathbb{R}P^2)} \\ (W_1, W_2) &\mapsto W_1 \cup_{D^3 \times \mathbb{R}P^2} W_2 \end{aligned}$$

This is equivariant with respect to the action of the bordism group  $\Omega_6^B$ . Hence it equips  $\Omega_6^{(B, S^3 \times \mathbb{R}P^2)}$  with a group structure, and we obtain an action of  $\Omega_6^{(B, S^3 \times \mathbb{R}P^2)}$  on  $\Omega_6^{(B, M \times \mathbb{R}P^2)}$  for any  $M$ .

To compare with the nonequivariant embedding results of Skopenkov, it suffices to consider the image under the transfer map: Comparing with [25], we see that for the embeddings with Whitney invariant 0, the Kreck invariant describes an action of  $\mathbb{Z}$  on  $\text{Emb}(M, S^6)$  which corresponds precisely to the action of

$$\Omega_6^{\text{Spin}}(K(\mathbb{Z}, 2), \partial = S^3 \times S^2)/(0 \oplus \mathbb{Z} \subset \Omega_6^{\text{Spin}}(K(\mathbb{Z}, 2)))$$

on the set  $\Omega_6^{\text{Spin}}(K(\mathbb{Z}, 2), \partial = M \times S^2)/(0 \oplus \mathbb{Z} \subset \Omega_6^{\text{Spin}}(K(\mathbb{Z}, 2)))$ .

We will see in the next section how the quotients arise also in the equivariant setting.

#### 4.4 Equivariant diffeomorphism classes as a quotient by group actions

Now we have to relate the set of relative diffeomorphism classes of the previous section to the set  $\text{Emb}_{\mathbb{Z}_2}(M, S^6)$  of equivariant diffeomorphism classes of six-spheres with involution whose fixed point set is identified with  $M$ .

Basically, we forget the  $B$ -structure, we construct the equivariant inclusion  $M \times D^3 \rightarrow X$  from  $M \times \mathbb{R}P^2 \rightarrow W$ , and we forget the tubular neighbourhood and the framing of the normal bundle.

More precisely, we have the following sets of equivalence classes:

- (1) The set  $T_1$  of diffeomorphism classes of manifolds  $W$  relative to  $M \times \mathbb{R}P^2$  and  $BO$ . An element is represented by  $M \times \mathbb{R}P^2 \rightarrow W \rightarrow B$ , such that the first map is the inclusion of the boundary. Two representatives  $W$  and  $W'$  are equivalent if there is a diffeomorphism  $W \rightarrow W'$  which commutes with the maps from  $M \times \mathbb{R}P^2$  and to  $BO$ .
- (2) The set  $T_2$  of diffeomorphism classes of manifolds  $W$  relative to  $M \times \mathbb{R}P^2$ . An element is represented by  $M \times \mathbb{R}P^2 \rightarrow W$ , which is the inclusion of the boundary. Two representatives  $W$  and  $W'$  are equivalent if there is a diffeomorphism  $W \rightarrow W'$  which commutes with the maps from  $M \times \mathbb{R}P^2$ .
- (3) The set  $T_3$  of equivariant diffeomorphism classes of manifolds  $V$  relative to  $M \times S^2$ . An element is represented by  $(M \times S^2, (\text{id}, -\text{id})) \rightarrow (V, \tau)$ , which is the inclusion of the boundary, and where  $\tau$  is a free involution. Two representatives  $V$  and  $V'$  are equivalent if there is a diffeomorphism  $(V, \tau) \rightarrow (V', \tau')$  which commutes with the maps from  $M \times S^2$ .
- (4) The set  $T_4$  of equivariant diffeomorphism classes of closed manifolds  $X$  relative to  $M \times D^3$ . An element is represented by an equivariant embedding  $(M \times D^3, (\text{id}, -\text{id})) \rightarrow (X, \tau)$ , where  $\tau$  is free on the complement of the image. Two representatives  $X$  and  $X'$  are equivalent if there is a diffeomorphism  $(X, \tau) \rightarrow (X', \tau')$  which commutes with the maps from  $M \times D^3$ .

(5) The set  $\text{Emb}_{\mathbb{Z}_2}(M, S^6)$  of equivariant diffeomorphism classes of manifolds  $X$  relative to  $M$ . An element is represented by an equivariant embedding  $(M, \text{id}) \rightarrow (X, \tau)$ , where  $\tau$  is free on the complement of the image. Two representatives  $X$  and  $X'$  are equivalent if there is a diffeomorphism  $(X, \tau) \rightarrow (X', \tau')$  which commutes with the maps from  $M$ .

On the set  $T_1$ , the automorphism group  $\text{Aut}(B)_{BO}$  of fiber homotopy classes of fiber homotopy self-equivalences of  $B$  acts by postcomposing  $M \times \mathbb{R}P^2 \rightarrow B$  with  $B \rightarrow B$ .

We saw that the group  $\text{Aut}(B)_{BO}$  has four elements, and the action of the group on the set of normal  $B$ -structures on a manifold  $f: M \rightarrow B$  (ie spin structures on  $\nu_M - f^*L$ ) is given by just changing the spin structure  $\sigma$  into  $\sigma, -\sigma, \sigma + f^*t, -\sigma + f^*t$ , where  $t \in H^1(B; \mathbb{Z}_2)$  is the generator.

Thus the group  $\text{Aut}(B)_{BO}$  acts freely and transitively on the set of distinguished normal  $B$ -structures on  $M \times \mathbb{R}P^2$ . In particular each orbit of the action on the set in (1) contains a unique element from each of the four bordism sets.

By uniqueness of the Postnikov decomposition, forgetting the map to  $B$  is a bijection from the quotient of  $T_1$  by  $\text{Aut}(B)_{BO}$  to the set  $T_2$ . See also [13]. Thus the set  $T_2$  is in bijection with  $\Omega_6^{(B, M \times \mathbb{R}P^2)} \cong \mathbb{Z}^2 \oplus \mathbb{Z}_4$ .

The sets  $T_2, T_3$  and  $T_4$  are in bijective correspondence: One gets from  $M \times D^3 \rightarrow X$  by restriction to  $M \times S^2 \rightarrow V = X \setminus \text{int}(M \times D^3)$  and from  $M \times S^2 \rightarrow V$  one takes the quotient by the involution to obtain  $M \times \mathbb{R}P^2 \rightarrow W = V/\tau$ . Vice-versa, from  $M \times \mathbb{R}P^2 \rightarrow W$ , take any nontrivial double covering  $V$  of  $W$  and any map  $M \times S^2 \rightarrow V$  inducing the given  $M \times \mathbb{R}P^2 \rightarrow W$ . The involution on  $V$  is the nontrivial deck transformation. Up to equivalence, this does not depend on the choices made. And from  $\phi: M \times S^2 \rightarrow V$ , obtain  $M \times D^3 \rightarrow X = M \times D^3 \cup_{\phi} V$ . The smooth structure on the latter depends on choices, but only up to equivalence.

On the sets  $T_2, T_3$  and  $T_4$ , we have an action of the group of bundle automorphisms  $\text{Map}(M, O(3))$  of  $M \times D^3$ : every bundle automorphism induces a self-diffeomorphism of the boundary  $M \times S^2$  which induces a self-diffeomorphism of  $M \times \mathbb{R}P^2$ , and we precompose with these diffeomorphisms. A bundle automorphism  $f: M \rightarrow O(3)$  corresponds to the diffeomorphism

$$\begin{aligned} \phi_f: M \times \mathbb{R}P^2 &\rightarrow M \times \mathbb{R}P^2 \\ (x, \pm y) &\mapsto (x, \pm f(x) \cdot y) \end{aligned}$$

for  $y \in S^2$ . So the bundle automorphism which is minus the identity on each fiber acts trivially on  $M \times \mathbb{R}P^2$ , hence the action is trivial also on the other sets. (This

corresponds in  $T_4$  to the fact that the embeddings

$$M \times D^3 \xrightarrow{i} X \quad \text{and} \quad M \times D^3 \xrightarrow{\text{id} \times -\text{id}} M \times D^3 \xrightarrow{i} X$$

are related by the equivariant diffeomorphism  $\tau: (X, \tau) \rightarrow (X, \tau)$ . Thus we may restrict to orientation preserving bundle automorphisms.

The equivariant tubular neighbourhood of the fixed point set is unique up to isotopy and bundle automorphisms. But isotopies of embeddings can be enlarged to isotopies of the ambient space, so that they act trivially on the set of equivariant diffeomorphism classes. It follows that the action of the group of bundle automorphisms descends to an action of  $[M, \text{SO}(3)]$ , and that dividing out this action, we get the set  $\text{Emb}_{\mathbb{Z}_2}(M, S^6)$ .

The part which is a little more complicated is to see the action of  $\mathbb{Z} \cong \pi_3(\text{SO}(3)) \cong [M, \text{SO}(3)] \ni f$  on a bordism set  $\Omega_6^{(B, M \times \mathbb{R}P^2)}$ . We can also precompose with the corresponding self-diffeomorphism  $\phi_f$  of  $M \times \mathbb{R}P^2$ , but this changes the precise map  $M \times \mathbb{R}P^2 \rightarrow B$ . Still the normal  $B$ -structure is preserved: Up to homotopy, we may assume that  $f: M \rightarrow \text{SO}(3)$  is trivial on a whole disk  $D^3$ , so that the normal  $B$ -structure on  $D^3 \times \mathbb{R}P^2$  does not change. Spin structures on vector bundles over  $M \times \mathbb{R}P^2$  are in bijection with spin structures on their restrictions to  $\mathbb{R}P^2$  (using the natural framing of the normal bundle), and the latter are invariant. Hence these bundle automorphisms preserve the normal  $B$ -structure on  $M \times \mathbb{R}P^2$ , and we may consider the action of  $f$  on the bordism group as given by gluing a mapping cylinder of  $f$ .

**Lemma 4.9** *Let  $B \rightarrow BO$  be a fibration, let  $N$  be an  $(n - 1)$ -manifold with normal  $B$ -structure,  $W_1, W_2, W_3$  be normal  $B$ -nullbordisms of  $N$ , let  $\phi: N \rightarrow N$  be a diffeomorphism preserving the normal  $B$ -structure, let  $C_\phi$  be the mapping cylinder of  $\phi$ , let  $i_0, i_1: N \rightarrow C_\phi$  be the two natural inclusions, and let  $T_\phi$  be the mapping torus of  $\phi$ . Then, in the bordism group  $\Omega_n^B$ , we have*

$$(W_1 \cup_{i_0} C_\phi \cup_{i_1} -W_2) - (W_1 \cup_{\text{id}_N} -W_2) = T_\phi = W_3 \cup_{i_0} C_\phi \cup_{i_1} -W_3.$$

**Proof**  $W \times I$  can be considered as a bordism between the manifolds with boundary  $\partial W \times I$  and  $-W \cup W$ . The second equality in the statement follows from applying this to  $W = W_3$ . Similarly, the first equality is obtained by applying this to  $W = W_1$  and  $W = W_2$ . □

It follows that the action of  $f$  on the bordism set  $\Omega_6^{(B, S^3 \times \mathbb{R}P^2)}$  is the same as taking the disjoint sum with  $D^4 \times \mathbb{R}P^2 \cup_{\phi_f} D^4 \times \mathbb{R}P^2$ . The latter is an  $\mathbb{R}P^2$ -bundle on  $S^4$ , and the corresponding double cover is a  $S^2$ -bundle over  $S^4$  which can be identified with  $S^2 \rightarrow \mathbb{C}P^3 \rightarrow \mathbb{H}P^1$ . Thus the induced action on  $\Omega_6^{(\tilde{B}, S^3 \times S^2)} \cong \mathbb{Z}^2$  is by a generator for the first summand of  $\Omega_6^{\tilde{B}} \cong \mathbb{Z}^2$ .

It follows that the set  $\text{Emb}_{\mathbb{Z}_2}(M, S^6)$  of orbits of the action on  $T_4$  is in bijection with  $\mathbb{Z} \oplus \mathbb{Z}_4$ . In particular  $\text{Emb}_{\mathbb{Z}_2}(M, S^6)$  is a group which acts freely and transitively on  $\text{Emb}_{\mathbb{Z}_2}(M, S^6)$  for all  $M$ .

Comparing with the nonequivariant classification of embeddings in  $S^6$ , forgetting the involution defines a map  $\text{Emb}_{\mathbb{Z}_2}(M, S^6) \cong \mathbb{Z} \oplus \mathbb{Z}_4 \rightarrow \text{Emb}(M, S^6) \cong \mathbb{Z} \oplus H_1(M)$  which is equivariant with respect to the group homomorphism  $\text{Emb}_{\mathbb{Z}_2}(S^3, S^6) \cong \mathbb{Z} \oplus \mathbb{Z}_4 \rightarrow \text{Emb}(S^3, S^6) \cong \mathbb{Z}$  given by  $(a, b) \mapsto 2a$ . In particular the image of  $\text{Emb}_{\mathbb{Z}_2}(M, S^6)$  is acted upon freely by  $2\mathbb{Z} \subseteq \mathbb{Z}$ , and the map is 4-to-1.

For embeddings of  $\mathbb{Z}_2$ -homology spheres we saw that the elements  $[i: M \rightarrow S^6] \in \text{Emb}(M, S^6)$  with vanishing Whitney invariant are acted freely and transitively upon by  $C_3^3 \cong \mathbb{Z}$ . The subset of isotopy classes of embeddings which are the fixed point sets of conjugations are acted freely and transitively upon by  $2\mathbb{Z} \subseteq \mathbb{Z}$ . There are up to equivariant diffeomorphism relative to  $i$  exactly four such conjugations for every  $i$ . This proves [Theorem 1.3](#) and [Theorem 1.4](#).

#### 4.5 From embeddings to submanifolds – proof of [Theorem 1.5](#)

The more natural thing is to classify involutions without the additional identification of the fixed point set with a fixed 3-manifold  $M$ . The invariant of the involution should be its fixed point set, ie a submanifold of  $S^6$ . In order to get from embeddings up to diffeomorphism to submanifolds up to diffeomorphism, it suffices to divide out the action of the group of self-diffeomorphisms  $\text{Diff}(M)$ . Since isotopies extend to ambient isotopies (this also holds in this equivariant case, since it suffices to extend a vector field on the fixed point set to an equivariant vector field on the whole space), and these give equivariant diffeomorphisms, the action of  $\text{Diff}(M)$  factors through the mapping class group of  $M$ .

Since in our case, the map  $M \times \mathbb{R}P^2 \rightarrow Q$  factors through  $\mathbb{R}P^2$ , this map does not change, so that we get the same normal  $B$ -structure. Then the action of a self-diffeomorphism  $f: M \rightarrow M$  on the bordism set  $\Omega_6^{(B, M \times \mathbb{R}P^2)}$  is by disjoint union with the mapping torus  $T_{f \times \text{id}}$  of  $f \times \text{id}: M \times \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ . Now  $T_{f \times \text{id}} = T_f \times \mathbb{R}P^2$ , and the map to  $Q$  factors again through  $\mathbb{R}P^2$ . Spin structures twisted by  $L$  on  $T_f \times \mathbb{R}P^2$  are products of a spin structure on  $M$  and a spin-structure twisted by  $L$  on  $\mathbb{R}P^2$ . Thus a spin-nullbordism of  $T_f$  gives a normal  $B$ -nullbordism of  $T_{f \times \text{id}}$ . Recall that a four-dimensional spin manifold is zero bordant if and only if its signature is zero, and that the signature of a mapping torus is always zero. Hence the action of the mapping class group of  $M$  on the bordism set is trivial. As a consequence, the mapping class group of  $M$  acts trivially on  $\text{Emb}_{\mathbb{Z}_2}(M, S^6)$ . This proves [Theorem 1.5](#).

## References

- [1] **G E Bredon**, *Introduction to compact transformation groups*, Pure and Applied Math. 46, Academic Press, New York (1972) [MR0413144](#)
- [2] **J Cerf**, *Sur les difféomorphismes de la sphère de dimension trois* ( $\Gamma_4 = 0$ ), Lecture Notes in Math. 53, Springer, Berlin (1968) [MR0229250](#)
- [3] **C W Curtis, I Reiner**, *Representation theory of finite groups and associative algebras*, Pure and Applied Math. XI, Wiley-Interscience, New York-London (1962) [MR0144979](#)
- [4] **J F Davis, S Weinberger**, *Characteristic fixed-point sets of semifree actions on spheres*, Comm. Pure Appl. Math. 52 (1999) 935–947 [MR1686973](#)
- [5] **M W Davis, T Januszkiewicz**, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. 62 (1991) 417–451 [MR1104531](#)
- [6] **K H Dovermann, M Masuda, R Schultz**, *Conjugation involutions on homotopy complex projective spaces*, Japan. J. Math. (N.S.) 12 (1986) 1–35 [MR914304](#)
- [7] **M Franz, V Puppe**, *Steenrod squares on conjugation spaces*, C. R. Math. Acad. Sci. Paris 342 (2006) 187–190 [MR2198191](#)
- [8] **A Haefliger**, *Knotted  $(4k-1)$ -spheres in  $6k$ -space*, Ann. of Math. (2) 75 (1962) 452–466 [MR0145539](#)
- [9] **I Hambleton, J-C Hausmann**, *Conjugation spaces and 4-manifolds* [arXiv:0906.5057](#)
- [10] **J-C Hausmann, T Holm**, *Conjugation spaces and edges of compatible torus actions* [arXiv:0807.3289](#)
- [11] **J-C Hausmann, T Holm, V Puppe**, *Conjugation spaces*, Algebr. Geom. Topol. 5 (2005) 923–964 [MR2171799](#)
- [12] **J-C Hausmann, A Knutson**, *The cohomology ring of polygon spaces*, Ann. Inst. Fourier (Grenoble) 48 (1998) 281–321 [MR1614965](#)
- [13] **M Kreck**, *An extension of results of Browder, Novikov and Wall about surgery on compact manifolds*, Preprint (1985)
- [14] **M Kreck**, *Surgery and duality*, Ann. of Math. (2) 149 (1999) 707–754 [MR1709301](#)
- [15] **B-h Li, Z Lü**, *Smooth free involution of  $H\mathbb{C}P^3$  and Smith conjecture for imbeddings of  $S^3$  in  $S^6$* , Math. Ann. 339 (2007) 879–889 [MR2341905](#)
- [16] **P Löffler**, *Homotopielineare  $Z_p$ -Operationen auf Sphären*, Topology 20 (1981) 291–312 [MR608602](#)
- [17] **M Masuda**, *Review of [21]*, Math Reviews (2002) [MR1943326](#)
- [18] **D Montgomery, C T Yang**, *Differentiable actions on homotopy seven spheres*, Trans. Amer. Math. Soc. 122 (1966) 480–498 [MR0200934](#)

- [19] **M Olbermann**, *Conjugations on 6–manifolds with free integral cohomology* [arXiv:1001.0911](#)
- [20] **M Olbermann**, *Conjugations on 6–manifolds*, PhD thesis, Universität Heidelberg (2007) Available at <http://www.ub.uni-heidelberg.de/archiv/7450/>
- [21] **K Pawalowski**, *Manifolds as fixed point sets of smooth compact Lie group actions*, from: “Current trends in transformation groups”, (A Bak, M Morimoto, F Ushitaki, editors), *K–Monogr. Math.* 7, Kluwer Acad. Publ., Dordrecht (2002) 79–104 [MR1943326](#)
- [22] **T Petrie**, *Involutions on homotopy complex projective spaces and related topics*, from: “Proceedings of the Second Conference on Compact Transformation Groups (Univ. Massachusetts, Amherst, 1971), Part I”, (H T Ku, L N Mann, J L Sicks, J C Su, editors), *Lecture Notes in Math.* 298, Springer, Berlin (1972) 234–259 [MR0418133](#)
- [23] **R H Sady**, *Free involutions of complex projective spaces*, *Michigan Math. J.* 24 (1977) 51–64 [MR0474350](#)
- [24] **R Schultz**, *Exotic spheres as stationary sets of homotopy sphere involutions*, *Michigan Math. J.* 29 (1982) 121–122 [MR646378](#)
- [25] **A Skopenkov**, *A classification of smooth embeddings of 3–manifolds in 6–space*, *Math. Z.* 260 (2008) 647–672 [MR2434474](#)
- [26] **Y Su**, *A note on the  $\mathbb{Z}_2$ –equivariant Montgomery–Yang correspondence* [arXiv:0908.3053](#)
- [27] **P Teichner**, *On the signature of four-manifolds with universal covering spin*, *Math. Ann.* 295 (1993) 745–759 [MR1214960](#)
- [28] **C T C Wall**, *Classification problems in differential topology. V. On certain 6–manifolds*, *Invent. Math.* 1 (1966), 355–374; corrigendum, *ibid* 2 (1966) 306 [MR0215313](#)

Max-Planck-Institut für Mathematik  
Vivatsgasse 7, 53111 Bonn, Germany

[olber@mpim-bonn.mpg.de](mailto:olber@mpim-bonn.mpg.de)

Received: 5 May 2010      Revised: 22 July 2010