A stable range description of the space of link maps

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We study the space $\mathrm{Link}(P,Q;N)$ of link maps: maps from $P \sqcup Q$ to N such that the images of P and Q are disjoint. We identify the homotopy fiber of the inclusion $\mathrm{Link}(P,Q;N) \to \mathrm{Map}(P,N) \times \mathrm{Map}(Q,N)$ in a stable range, showing that it has a $(2(n-p-q)-3)$–connected map to the infinite loopspace of a certain Thom spectrum.

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1 Introduction

Let N be a smooth manifold and let P and Q be smooth compact manifolds. A (smooth) *link map* of P and Q in N is a pair $(f\colon P \to N, g\colon Q \to N)$ of smooth maps such that $f(P)$ is disjoint from $g(Q)$. The set of link maps, denoted by $\mathrm{Link}(P,Q;N)$, is an open subspace of $\mathrm{Map}(P,N) \times \mathrm{Map}(Q,N) = \mathrm{Map}(P \sqcup Q, N)$.

For brevity we will write $\mathcal{M}$ for $\mathrm{Map}(P,N)\times\mathrm{Map}(Q,N)$ and denote the complement of the set of link maps in $\mathcal{M}$ by $\mathcal{B}$. We prove that a certain "linking number" map

$$\ell\colon \mathrm{hofiber}_{(f_1,g_1)}(\mathcal{M}-\mathcal{B} \to \mathcal{M}) \to \Omega Q_+^{TN-(TP\oplus TQ)}\,\mathrm{holim}(P \xrightarrow{f_1} N \xleftarrow{g_1} Q)$$

is $(2(n-p-q)-3)$–connected, where p, q and n are the dimensions of the manifolds. The map was defined by the second author in [5], although the version we reference below is of a more homotopy-theoretic flavor, and is given by Klein and Williams [3]. Its domain is the homotopy fiber of the inclusion $\mathcal{M}-\mathcal{B} \to \mathcal{M}$ with respect to any point $(f_1,g_1) \in \mathcal{M}-\mathcal{B}$. Its codomain is the infinite loopspace associated to the Thom spectrum of a virtual vector bundle. Both of these spaces are $(n-p-q-2)$–connected. In the case when $p+q = n-1$ it was shown in [5] that the effect of the map ℓ on π_0 can be interpreted as a generalized linking number.

Functor calculus (the manifold version developed by Weiss [6] and the first author and Weiss [2]) offers one point of view on link maps. Consider the functor $(U,V) \mapsto \mathrm{Link}(U,V;N)$ whose domain is the poset $\mathcal{O}(P\sqcup Q) = \mathcal{O}(P)\times\mathcal{O}(Q)$ of open subsets of $P \sqcup Q$. Its best linear approximation is $\mathrm{Map}(U,N)\times\mathrm{Map}(V,N)$. Our result can

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be interpreted as a statement about a quadratic approximation to the same functor, but we will not pursue this here. This work overlaps the recent work of Klein and Williams; in particular, some of the material in Section 3 also appears in [3].

Our main result is:

Theorem 1.1 *The map*

$$\Lambda\colon \Sigma\,\mathrm{hofiber}(\mathcal{M}-\mathcal{B}\to\mathcal{M})\to Q_+^{TN-(TP\oplus TQ)}\,\mathrm{holim}(P\to N\leftarrow Q)$$

adjoint to ℓ is $(2(n-p-q)-1)$–connected.

The fact that ℓ is $(2(n-p-q)-3)$–connected then follows immediately by the Freudenthal Theorem, since the domain of ℓ is $(n-p-q-2)$–connected. Note that the connectivity claimed for Λ is negative if $p+q\geq n$, so it is no loss to assume $p+q<n$.

1.1 Conventions

A space X is k–connected if for every j with $-1\leq j\leq k$ every map $S^j\to X$ can be extended to a map $D^{j+1}\to X$. In other words, (-1)–connected means nonempty and if $k\geq 0$ then k–connected means that there is exactly one path-component and that the homotopy groups vanish through dimension k. A map is k–connected if each of its homotopy fibers is $(k-1)$–connected. A (weak) *equivalence* is an ∞–connected map.

We write $QX=\Omega^\infty\Sigma^\infty X$ if X is a based space. If X is unbased, then X_+ means X with a disjoint basepoint added and Q_+X means $Q(X_+)$. For a vector bundle ξ over a space X, the unit disk bundle and the unit sphere bundle are $D(X;\xi)$ and $S(X;\xi)$. The Thom space X^ξ is the quotient $D(X;\xi)/S(X;\xi)$, or equivalently the homotopy cofiber of the projection $S(X;\xi)\to X$. If ξ and η are two vector bundles on X, then by choosing a vector bundle monomorphism $\eta\to\epsilon^i$ to a trivial bundle we can define $Q_+^{\xi-\eta}X=\Omega^iQX^{\xi\oplus\epsilon^i/\eta}$. This is essentially independent of the choice of $i\geq 0$ and vector bundle monomorphism, in the sense that for large i the weak homotopy type of this space is independent of those choices.

2 Sketch of the proof of Theorem 1.1

To prove Theorem 1.1 we will use the diagram (1) below and obtain the connectivity of Λ from the connectivities of all the other maps. For this we must introduce another closed set $\mathcal{V}\subset\mathcal{B}$. Recall that a point $(f,g)\in\mathcal{M}$ belongs to $\mathcal{B}$ if the statement $f(x)=z=g(y)$ holds for some pair $(x,y)\in P\times Q$ and some point $z\in N$. The closed set $\mathcal{B}$ has codimension $n-p-q$ in $\mathcal{M}$ in some sense. Inside this space $\mathcal{B}$ of "bad" maps is a set $\mathcal{V}$ of "very bad" maps, having codimension $2(n-p-q)$ in $\mathcal{M}$. A point (f,g) is in $\mathcal{V}$ if either the statement $f(x)=z=g(y)$ holds for more than one choice of (x,y,z) or else it holds for one such choice in such a way that the associated map of tangent spaces $T_xP\oplus T_yQ\to T_zN$ is not injective. The set $\mathcal{B}-\mathcal{V}$ may be regarded as a submanifold of $\mathcal{M}$. It has maps to P, Q and N given by x, y and z. Pulling back tangent bundles via these maps, we obtain vector bundles on $\mathcal{B}-\mathcal{V}$, which we will denote simply by TP, TQ and TN. There is also a monomorphism $TP\oplus TQ\to TN$, and its cokernel $TN/(TP\oplus TQ)$ may be thought of as the normal bundle of $\mathcal{B}-\mathcal{V}$ in $\mathcal{M}$.

The next result immediately implies Theorem 1.1.

Theorem 2.1 *In the homotopy commutative diagram below, the maps F and H are equivalences, the maps G, C and D are $(2(n-p-q)-1)$–connected, and the map E is $(3(n-p-q)-2)$–connected.*

$$
\begin{array}{ccc}
\Sigma\,\mathrm{hofiber}(\mathcal{M}-\mathcal{B}\to\mathcal{M}) & \xrightarrow{\ \Lambda\ } & Q_+^{TN-(TP\oplus TQ)}\,\mathrm{holim}(P\to N\leftarrow Q) \\
 & & \uparrow{\scriptstyle D} \\
\uparrow{\scriptstyle G} & & Q_+^{TN-(TP\oplus TQ)}\,\mathrm{hofiber}(\mathcal{B}-\mathcal{V}\to\mathcal{M}) \\
 & & \uparrow{\scriptstyle H} \\
\Sigma\,\mathrm{hofiber}(\mathcal{M}-\mathcal{B}\to\mathcal{M}-\mathcal{V}) & & Q\,\mathrm{hofiber}(\mathcal{B}-\mathcal{V}\to\mathcal{M})^{TN/(TP\oplus TQ)} \\
{\scriptstyle F}\uparrow & & \uparrow{\scriptstyle C} \\
\mathrm{hofiber}(\mathcal{B}-\mathcal{V}\to\mathcal{M}-\mathcal{V})^{TN/(TP\oplus TQ)} & \xrightarrow{\ E\ } & \mathrm{hofiber}(\mathcal{B}-\mathcal{V}\to\mathcal{M})^{TN/(TP\oplus TQ)}
\end{array}
\tag{1}
$$

We now briefly define the maps in the diagram and explain about their connectivities. Steps that are sketchy here will be filled in the following sections. Let $c=n-p-q$.

The equivalence F is essentially an instance of the following general fact. If Y is a smooth submanifold of X and also a closed subset, then the suspension of the homotopy fiber of the inclusion $X-Y\to X$ is equivalent to the Thom space, over the homotopy fiber of $Y\to X$, of the normal bundle of Y in X. This general fact will be proved, and adapted to the present function-space setting, in Section 4.

The map G is an inclusion map. Since $\mathcal{V}$ has codimension $2c$ in $\mathcal{M}$, the inclusion $\mathcal{M}-\mathcal{V}\to\mathcal{M}$ is $(2c-1)$–connected. (This will be worked out in detail in Section 3.) Therefore the map of homotopy fibers is $(2c-2)$–connected and the map G of suspensions is $(2c-1)$–connected.

The map E is a map of Thom spaces. For a k–connected map $Z \to W$ of spaces and a vector bundle ξ on W with fiber dimension d, the associated map $Z^\xi \to W^\xi$ is $(k+d)$–connected. In our case $d=c$ and $k=2c-2$; the inclusion of hofiber$(\mathcal{B}-\mathcal{V}\to\mathcal{M}-\mathcal{V})$ into hofiber$(\mathcal{B}-\mathcal{V}\to\mathcal{M})$ is $(2c-2)$–connected, again because the inclusion of $\mathcal{M}-\mathcal{V}$ into $\mathcal{M}$ is $(2c-1)$–connected.

The map C is the canonical map $Z \to QZ$, where the space Z is $(c-1)$–connected, being the Thom space of a vector bundle of rank c. By the Freudenthal Theorem, the map is $(2c-1)$–connected.

The equivalence H is simply a matter of rewriting the Thom spectrum of a virtual vector bundle $\xi-\eta$ as the suspension spectrum of the Thom space of ξ/η when η is a subbundle of ξ.

The map D arises from a $(c-1)$–connected map from hofiber$(\mathcal{B}-\mathcal{V}\to\mathcal{M})$ to holim$(P \to N \leftarrow Q)$. To explain further, we need the space $\widetilde{\mathcal{B}}$ of all $((f,g),x,y,z)\in \mathcal{M}\times P\times Q\times N$ such that $f(x)=z=g(y)$. Projection to $\mathcal{M}$ gives a map from $\widetilde{\mathcal{B}}$ onto $\mathcal{B}$. Let $\widetilde{\mathcal{V}}\subset\widetilde{\mathcal{B}}$ be the preimage of $\mathcal{V}$. The projection $\widetilde{\mathcal{B}}-\widetilde{\mathcal{V}}\to\mathcal{B}-\mathcal{V}$ is an isomorphism. The inclusion $\widetilde{\mathcal{B}}-\widetilde{\mathcal{V}}\to\widetilde{\mathcal{B}}$ is $(c-1)$–connected for reasons of codimension (again, the details are in Section 3), and therefore the induced map hofiber$(\mathcal{B}-\mathcal{V}\to\mathcal{M})\to$ hofiber$(\widetilde{\mathcal{B}}\to\mathcal{M})\simeq$ holim$(P\to N\leftarrow Q)$ is also $(c-1)$–connected. There are vector bundles TP, TQ and TN on $\widetilde{\mathcal{B}}$ pulling back to their namesakes on $\mathcal{B}-\mathcal{V}$. (The monomorphism $df\oplus dg\colon TP\oplus TQ\to TN$ is not available on the holim$(P\to N\leftarrow Q)$ side, which is why we switched from Thom spaces to Thom spectra).

We end this section with a brief account of the commutativity of diagram (1). First we need to define the map Λ. As mentioned in Section 1, Λ is adjoint to a map ℓ: hofiber$_{(f_1,g_1)}(\mathcal{M}-\mathcal{B}\to\mathcal{M})\to\Omega Q_+^{TN-(TP\oplus TQ)}$ holim$(P\to N\leftarrow Q)$, which is a composite described below (also see Klein and Williams [3, Section 9]). Let $(f_t,g_t)\in$ hofiber$_{(f_1,g_1)}(\mathcal{M}-\mathcal{B}\to\mathcal{M})$. The map $\mathcal{M}\to\mathrm{Map}(P\times Q,N\times N)$ given by $(f,g)\mapsto f\times g$ induces a map

$$\mathrm{hofiber}_{(f_1,g_1)}(\mathcal{M}-\mathcal{B}\to\mathcal{M})\to\mathrm{hofiber}_{f_1\times g_1}(\mathrm{Map}(P\times Q,N\times N-\Delta_N)\to\mathrm{Map}(P\times Q,N\times N)).$$

We can identify the latter homotopy fiber as a space of sections as follows.

Let $$E=\mathrm{holim}(P\times Q\xrightarrow{f_1\times g_1}N\times N\longleftarrow N\times N-\Delta_N).$$

The projection map $E\to P\times Q$ is a fibration with fiber over (p,q) the space $\Phi_2(N)=$ hofiber$_{(f_1(p),g_1(q))}(N\times N-\Delta_N\to N\times N)$. Let $\Gamma(P\times Q,E)$ be its space of sections. This space of sections has a preferred basepoint given by (f_1,g_1). It is equivalent to $\text{hofiber}_{f_1\times g_1}(\text{Map}(P\times Q, N\times N-\Delta_N)\to \text{Map}(P\times Q, N\times N))$ by inspection. Let $Q_f S_f E\to P\times Q$ be the fibration whose fibers are $QS\Phi_2(N)$, where S stands for the unreduced suspension. The canonical map $\Gamma(P\times Q, E)\to\Omega\Gamma(P\times Q, Q_f S_f E)$ is easily shown to be $(2n-p-q-1)$–connected, and there is an equivalence

$$\Omega\Gamma(P\times Q, Q_f S_f E)\simeq \Omega Q_+^{TN-(TP\oplus TQ)}\,\text{holim}(P\times Q\xrightarrow{f_1\times g_1} N\times N\leftarrow \Delta_N)$$

which is the identity on the loop coordinate. Moreover, there is a homeomorphism

$$\text{holim}(P\times Q\xrightarrow{f_1\times g_1} N\times N\leftarrow\Delta_N)\cong \text{holim}(P\xrightarrow{f_1} N\xleftarrow{g_1} Q).$$

The composite map

$$\text{hofiber}_{(f_1,g_1)}(\mathcal{M}-\mathcal{B}\to\mathcal{M})\to\Omega Q_+^{TN-(TP\oplus TQ)}\,\text{holim}(P\xrightarrow{f_1} N\xleftarrow{g_1} Q)$$

is the map ℓ, and Λ is its adjoint.

Now let $(f_t, g_t, v)\in\text{hofiber}(\mathcal{B}-\mathcal{V}\to\mathcal{M}-\mathcal{V})^{TN/TP\oplus TQ}$. Here v is a vector of length $0\le |v|\le 1$, and (f_t, g_t, v) is identified to a point when $|v|=1$. After applying the maps E, C, H and D in diagram (1), it is clear that (f_t, g_t, v) is sent to $((x_0,\beta,y_0), v)\in Q_+^{TN-(TP\oplus TQ)}\,\text{holim}(P\to N\leftarrow Q)$, where $(x_0,y_0)\in P\times Q$ is the unique pair such that $f_0(x_0)=g_0(y_0)$ and $\beta\colon I\to N$ is the path defined by $\beta(s)=f_{1-2s}(x_0)$ for $0\le s\le 1/2$ and $\beta(s)=g_{2s-1}(y_0)$ for $1/2\le s\le 1$.

Now we must apply F, G and Λ to (f_t, g_t, v). A careful examination of the material in Section 4 reveals that F sends (f_t, g_t, v) to the point $s\wedge(\tilde f_t,\tilde g_t)$, where $s=1-|v|$ and $(\tilde f_t,\tilde g_t)\in\text{hofiber}(\mathcal{M}-\mathcal{B}\to\mathcal{M})$ is defined as follows. For $s\le t\le 1$, we have $(\tilde f_t,\tilde g_t)=(f_{(t-s)/(1-s)}, g_{(t-s)/(1-s)})$. For $0\le t\le s$, $(\tilde f_t,\tilde g_t)$ has the following properties: $(\tilde f_t,\tilde g_t)\in\mathcal{M}-\mathcal{B}$ for $t<s$, $(\tilde f_s,\tilde g_s)=(f_0,g_0)$ has a unique pair $(x_0,y_0)\in P\times Q$ such that $f_0(x_0)=g_0(y_0)=z_0\in N$ and such that $f_0'(x_0)-g_0'(y_0)\in T_{z_0}N$, when projected to $T_{z_0}/T_{x_0}P\oplus T_{y_0}Q$, is equal to v (here f_0' and g_0' are the derivatives with respect to t). From this description of F and the description of Λ above, the diagram commutes.

3 Codimension and connectivity

The proof outlined above uses that the pair $(\mathcal{M},\mathcal{M}-\mathcal{V})$ is $(2n-2p-2q-1)$–connected and that the pair $(\widetilde{\mathcal{B}},\widetilde{\mathcal{B}}-\widetilde{\mathcal{V}})$ is $(n-p-q-1)$–connected. We now justify these statements more carefully.

For the first, it suffices if for every smooth manifold W of dimension $k<2n-2p-2q$, and for every map of pairs $\phi\colon(W,\partial W)\to(\mathcal{M},\mathcal{M}-\mathcal{V})$, there is a homotopy of pairs to a map that is disjoint from $\mathcal{V}$.

Consider the adjoint map $\Phi\colon W \times (P \sqcup Q) \to N$. By a preliminary homotopy we can assume that Φ is smooth, and we can make the homotopy small enough in the C^0 sense so that it corresponds to a homotopy of pairs. If we can show that the condition $\phi^{-1}(\mathcal{V}) = \varnothing$ holds for a dense set of all such smooth maps Φ, then another small homotopy will complete the job. For the density statement we will use the multijet transversality theorem of Mather [4, Proposition 3.3] (which appears in [1] as Theorem 4.13).

Recall the setup: Two smooth maps $\Phi, \Psi\colon X \to Y$ have the same m–jet at $x \in X$ if $\Phi(x) = \Psi(x)$ and Φ and Ψ have the same derivatives through order m. Let $X^{(r)} \subset X^r$ be the space of configurations of r distinct points in X. The maps Φ and Ψ have the same m–multijet at $(x_1, \dots, x_r) \in X^{(r)}$ if for every $i \in \{1, \dots r\}$ they have the same m–jet at x_i. The manifold $J^{(r)}_m(X, Y)$ of multijets has a point for each r–tuple $(x_1, \dots, x_r)$ and each equivalence class of maps as above. A smooth map $\Phi\colon X \to Y$ determines a smooth map $j^{(r)}_m(\Phi)\colon X^{(r)} \to J^{(r)}_m(X, Y)$. The multijet transversality theorem asserts that, for every submanifold Z of $J^{(r)}_m(X, Y)$, the set of all Φ such that $j^{(r)}_m(\Phi)$ is transverse to Z is a countable intersection of dense open sets in the function space $\mathrm{Map}(X, Y)$. It follows that such a set, or even the intersection of countably many such sets, is dense.

We now introduce various submanifolds Z of $J^{(r)}_m(W \times (P \sqcup Q), N)$, for various values of r and m. The point is that the condition $\phi^{-1}(\mathcal{V}) = \varnothing$ will hold if and only if for each of these the set $j^{(r)}_m(\Phi)$ is disjoint from Z. The codimension of Z will always be big enough so that in order for $j^{(r)}_m(\Phi)$ to be transverse to Z it must be disjoint. Therefore the theorem will guarantee that there are maps $W \to \mathcal{M} - \mathcal{V}$ arbitrarily close to a given map $\Phi\colon W \to \mathcal{M}$.

Let k be the dimension of W. We consider the various ways in which ϕ could hit $\mathcal{V}$.

(1) There might exist distinct x_1 and x_2 in P and distinct y_1 and y_2 in Q such that for some $w \in W$ we have $\Phi(w, x_1) = \Phi(w, y_1)$ and $\Phi(w, x_2) = \Phi(w, y_2)$. Then the point

$$((w, x_1), (w, x_2), (w, y_1), (w, y_2)) \in (W \times (P \sqcup Q))^{(4)}$$

maps into a certain submanifold of $J^{(4)}_0(W \times (P \sqcup Q), N)$ whose codimension is $3k + 2n$. (That is $3k$ to make four points of W equal to each other and $2n$ for two coincidences in N.) This codimension is greater than the dimension $4k + 2p + 2q$ of (the relevant open and closed part of) $(W \times (P \sqcup Q))^{(4)}$, so that transverse means disjoint.

(2) There might exist distinct x_1 and x_2 in P and y in Q such that $\Phi(w, x_1) = \Phi(w, y) = \Phi(w, x_2)$. This leads to a submanifold of $J^{(3)}_0(W \times (P \sqcup Q), N)$

whose codimension is $2k+2n$, greater than the dimension $3k+2p+q$ of (part of) $(W\times(P\sqcup Q))^{(3)}$.

(3) There might exist x in P and distinct y_1 and y_2 in Q such that $\Phi(w,x)=\Phi(w,y_1)=\Phi(w,y_2)$. The relevant manifold has codimension $2k+2n$ in $J^{(3)}_0(W\times(P\sqcup Q),N)$, greater than $3k+p+2q$.

(4) There might exist $x\in P$ and $y\in Q$ such that $\Phi(w,x)=z=\Phi(w,y)$ and such that the linear map $T_xP\oplus T_yQ\to T_zN$ given by differentiation of $\phi(w)$ at x and y has rank less than $p+q$. For each fixed rank $r<p+q$ this leads to a submanifold of $J^{(2)}_1(W\times(P\sqcup Q),N)$ whose codimension $k+(n-r)(p+q-r)$ is greater than $2k+p+q$.

This completes the proof that the pair $(\mathcal{M},\mathcal{M}-\mathcal{V})$ is $(2n{-}2p{-}2q{-}1)$–connected.

To prove that the pair $(\widetilde{\mathcal{B}},\widetilde{\mathcal{B}}-\widetilde{\mathcal{V}})$ is $(n{-}p{-}q{-}1)$–connected, essentially the same kind of standard dimension-counting will succeed, but a simple reference as before to the multijet transversality theorem will not suffice because $\widetilde{\mathcal{B}}$ is not simply the space of maps from one manifold to another.

First observe that both the projection $\widetilde{\mathcal{B}}\to P\times Q\times N$ and its restriction $\widetilde{\mathcal{B}}-\widetilde{\mathcal{V}}\to P\times Q\times N$ are fibrations. It therefore suffices if, for a point $(x_0,y_0,z_0)\in P\times Q\times N$, the pair $(\widetilde{\mathcal{B}}_0,\widetilde{\mathcal{B}}_0-\widetilde{\mathcal{V}}_0)$ of fibers is $(n{-}p{-}q{-}1)$–connected. Here $\widetilde{\mathcal{B}}_0\subset\mathcal{M}$ is the set of all ϕ such that $\phi(x_0)=z_0=\phi(y_0)$, and $\widetilde{\mathcal{V}}_0\subset\widetilde{\mathcal{B}}_0$ is the set of all ϕ such that in addition at least one of the following is true:

(1) $\phi(x)=\phi(y)$ for some $x\in P-x_0$ and some $y\in Q-y_0$.

(2) $\phi(x)=z_0$ for some $x\in P-x_0$.

(3) $\phi(y)=z_0$ for some $y\in Q-y_0$.

(4) The linear map $T_{x_0}P\oplus T_{y_0}Q\to T_{z_0}N$ has rank less than $p+q$.

To deal first with (4), note that $\widetilde{\mathcal{B}}_0$ is fibered over the space $\mathcal{L}$ of all linear maps $T_{x_0}P\oplus T_{y_0}Q\to T_{z_0}N$. Let $\mathcal{L}^{\max}\subset\mathcal{L}$ be the open set of maps of rank $p+q$ and let $\widetilde{\mathcal{B}}^{\max}_0\subset\widetilde{\mathcal{B}}_0$ be its preimage. The pair $(\widetilde{\mathcal{B}}_0,\widetilde{\mathcal{B}}^{\max}_0)$ is $(n{-}p{-}q)$–connected (one better than needed), because the pair $(\mathcal{L},\mathcal{L}^{\max})$ is $(n{-}p{-}q)$–connected, because the closed set $\mathcal{L}-\mathcal{L}^{\max}$ is the union of finitely many submanifolds having codimension at least $n-p-q+1$.

It remains to show that the pair $(\widetilde{\mathcal{B}}^{\max}_0,\widetilde{\mathcal{B}}_0-\widetilde{\mathcal{V}}_0)$ is $(n{-}p{-}q{-}1)$–connected. Both $\widetilde{\mathcal{B}}^{\max}_0$ and $\widetilde{\mathcal{B}}_0-\widetilde{\mathcal{V}}_0$ fiber over $L^{\max}$, so we can replace the two spaces by their fibers, say $\widetilde{\mathcal{B}}_L$ and $\widetilde{\mathcal{B}}_L-\widetilde{\mathcal{V}}_L$, over a given $L\in\mathcal{L}$.

Now given a map $\phi\colon W \to \widetilde{\mathcal{B}}_L$, we want to perturb it slightly so as to eliminate behaviors (1), (2) and (3). None of these can occur for x near x_0 or y near y_0 anyway, given the choice of L, so we look for perturbations that are fixed near x_0 and y_0. In other words, we look for a small compactly supported change in the map $\Phi\colon W \times ((P-x_0) \sqcup (Q-y_0)) \to N$. This goes as before: case (1) leads to a submanifold of $J_0^{(2)}(W \times ((P-x_0) \sqcup (Q-y_0)), N)$ with codimension $k+n$, greater than $2k+p+q$; case (2) leads to a submanifold of $J_0^{(1)}(W \times ((P-x_0) \sqcup (Q-y_0)), N)$ with codimension n, greater than $k+p$; and case (3) leads to a submanifold of $J_0^{(1)}(W \times ((P-x_0) \sqcup (Q-y_0)), N)$ with codimension n, greater than $k+q$.

4 Normal bundles and homotopy cofibers

Suppose that X is a smooth manifold, and that the closed subset $Y \subset X$ is a smooth submanifold with normal bundle ν.

Of course, the Thom space Y^ν is equivalent to the homotopy cofiber of the inclusion map $X - Y \to X$. This follows from the fact that there is a homotopy pushout square

$$\begin{array}{ccc} S(Y;\nu) & \longrightarrow & D(Y;\nu) \\ \downarrow & & \downarrow \\ X-Y & \longrightarrow & X. \end{array} \tag{2}$$

The homotopy fibers over X of the four spaces above form another homotopy pushout square

$$\begin{array}{ccc} \mathrm{hofiber}(S(Y;\nu)\to X) & \longrightarrow & \mathrm{hofiber}(D(Y;\nu)\to X) \\ \downarrow & & \downarrow \\ \mathrm{hofiber}(X-Y\to X) & \longrightarrow & \mathrm{hofiber}(X\to X)\simeq *. \end{array}$$

Comparing homotopy cofibers of the rows in this square, we obtain an equivalence

$$\mathrm{hofiber}(Y \to X)^\nu \to \Sigma\, \mathrm{hofiber}(X - Y \to X).$$

Here we have written ν for the pullback of ν to $\mathrm{hofiber}(Y \to X)$.

We need statements like those above in which the manifolds X and Y are replaced by the function spaces $\mathcal{M} - \mathcal{V}$ and $\mathcal{B} - \mathcal{V}$ and the role of the normal bundle is played by the vector bundle $TN/(TP \oplus TQ)$ on $\mathcal{B} - \mathcal{V}$. The only little difficulty is that the square (2) depended on having a tubular neighborhood. We will write down a substitute for (2) that avoids this dependence.

Let $P(Y,X)$ be the space of all smooth paths $\gamma\colon [0,1]\to X$ such that $\gamma^{-1}(Y)=0$ and $\gamma'(0)$ is not tangent to Y. We have the homotopy-commutative square

$$\begin{array}{ccc} P(Y,X) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X-Y & \longrightarrow & X \end{array} \tag{3}$$

in which the top and left maps are evaluation at 0 and at 1 respectively.

There are equivalences

$$\text{hocofiber}(P(Y,X)\to Y)\to \text{hocofiber}(S(Y;\nu)\to Y)=Y^{\nu}, \tag{4}$$

$$\text{hocofiber}(P(Y,X)\to Y)\to \text{hocofiber}(X-Y\to X). \tag{5}$$

The logic is as follows:

For (4) we use the map $P(Y,X)\to S(Y;\nu)$ that sends γ to the projection of $\gamma'(0)$ in the direction perpendicular to Y, normalized to have unit length. It is a map over Y between two spaces fibered over Y, and it is an equivalence because for each point in Y the map of fibers is an equivalence.

For (5) we need to see that the homotopy-commutative square (3) is a homotopy pushout, in the sense that the associated map from the homotopy colimit of

$$X-Y\leftarrow P(Y,X)\to Y$$

to X is an equivalence. After choosing a tubular neighborhood of Y in X, one can map $S(Y;\nu)$ to $P(Y,X)$ by using radial paths perpendicular to Y. This map is an equivalence because it is a one-sided inverse to an equivalence. It follows that in showing that the square is a homotopy pushout we may consider instead the square

$$\begin{array}{ccc} S(Y;\nu) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X-Y & \longrightarrow & X. \end{array}$$

But this comes down to considering the same strictly commutative square (2) that we began with.

Note that although a tubular neighborhood was used in proving (5) to be an equivalence, the definitions of (4) and (5) did not use it. This is the point of introducing $P(Y,X)$.

Now for the function spaces: Again we will obtain equivalences

$$\text{hocofiber}(P(\mathcal{B}-\mathcal{V},\mathcal{M}-\mathcal{V})\to \mathcal{M}-\mathcal{V})\to(\mathcal{B}-\mathcal{V})^{\nu}$$

(where ν now means the bundle $TN/(TP\oplus TQ)$ on $\mathcal{B}-\mathcal{V}$) and

$$\text{hocofiber}(P(\mathcal{B}-\mathcal{V},\mathcal{M}-\mathcal{V})\to \mathcal{M}-\mathcal{V})\to \text{hocofiber}(\mathcal{M}-\mathcal{B}\to\mathcal{M}-\mathcal{V}).$$

We define the space $P(\mathcal{B}-\mathcal{V},\mathcal{M}-\mathcal{V})$. A point in it is a map $\gamma\colon [0,1]\to\mathcal{M}$ meeting the following conditions. Write $\gamma(t)=(f_t,g_t)$. The conditions are:

(1) γ is smooth in the sense that the adjoint maps $(t,x)\mapsto f_t(x)$ and $(t,x)\mapsto g_t(x)$ from $[0,1]\times P$ and $[0,1]\times Q$ to N are smooth.

(2) For every $t>0$, γ_t is in $\mathcal{M}-\mathcal{B}$, that is, $f_t(P)\cap g_t(Q)=\varnothing$.

(3) $\gamma_0\in\mathcal{B}-\mathcal{V}$, that is, (a) there is exactly one point $(x_0,z_0,y_0)\in P\times N\times Q$ such that $f_0(x)=z_0=g_0(y)$ and (b) $df_0\oplus dg_0\colon T_{x_0}P\oplus T_{y_0}Q\to T_{z_0}N$ is injective.

(4) $\gamma'(0)$ is not tangent to $\mathcal{B}-\mathcal{V}$, that is, the vector $f_0'(x_0)-g_0'(y_0)\in T_{z_0}(N)$ does not belong to the subspace $(D_{x_0}f_0)(T_{x_0}P)\oplus(D_{y_0}g_0)(T_{y_0}Q)$. Here f' and g' are derivatives with respect to t.

Consider the homotopy-commutative square

$$\begin{array}{ccc} P(\mathcal{B}-\mathcal{V},\mathcal{M}-\mathcal{V}) & \longrightarrow & \mathcal{B}-\mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{M}-\mathcal{B} & \longrightarrow & \mathcal{M}-\mathcal{V}, \end{array}$$

where the upper map and the left map take $\gamma=(f,g)$ to (f_0,g_0) and (f_1,g_1) respectively. We argue much as in the finite-dimensional case.

First, there is an equivalence $P(\mathcal{B}-\mathcal{V},\mathcal{M}-\mathcal{V})\to S(\mathcal{B}-\mathcal{V};\nu)$ that respects the projection to $\mathcal{B}-\mathcal{V}$, namely the map that takes $\gamma=(f,g)$ to the unit vector in $T_{z_0}N/(T_{x_0}P\oplus T_{y_0}Q)$ determined by the element $f_0'(x_0)-g_0'(y_0)$ of $T_{x_0}P\oplus T_{y_0}Q$. It is an equivalence because it is a map between spaces fibered over $\mathcal{B}-\mathcal{V}$ and it induces equivalences fiber by fiber.

Second, the square is a homotopy pushout. For this step, instead of trying to come up with a tubular neighborhood we reduce to the finite-dimensional case.

To show that the map from the homotopy colimit of

$$\mathcal{M}-\mathcal{B}\leftarrow P(\mathcal{B}-\mathcal{V},\mathcal{M}-\mathcal{V})\to\mathcal{B}-\mathcal{V}$$

to $\mathcal{M}-\mathcal{V}$ is surjective on homotopy groups, let $X=S^k$ and take any map $\phi\colon X\to\mathcal{M}-\mathcal{V}$, with adjoint $\Phi=(F,G)$, $F\colon X\times P\to N$, $G\colon X\times Q\to N$. Deforming by a homotopy that stays within $\mathcal{M}-\mathcal{V}$, make Φ “transverse to $\mathcal{B}-\mathcal{V}$” in the sense that F and G together give a map $X\times P\times Q\to N\times N$ which is transverse to the diagonal. The preimage of the diagonal in $X\times P\times Q$ is a submanifold, and it is embedded in X by the projection. Call its image Y. The normal bundle of Y in X is the pullback of $TN/(TP\oplus TQ)$ by ϕ.

Now inverting the equivalence

$$\mathrm{hocolim}(X - Y \leftarrow P(Y, X) \to Y) \to X$$

and composing with the obvious map

$$\mathrm{hocolim}(X - Y \leftarrow P(Y, X) \to Y) \to \mathrm{hocolim}(\mathcal{M} - \mathcal{B} \leftarrow P(\mathcal{B} - \mathcal{V}, \mathcal{M} - \mathcal{V}) \to \mathcal{M} - \mathcal{V})$$

we get

$$X \to \mathrm{hocolim}(\mathcal{M} - \mathcal{B} \leftarrow P(\mathcal{B} - \mathcal{V}, \mathcal{M} - \mathcal{V}) \to \mathcal{M} - \mathcal{V}),$$

a lifting (up to homotopy) of ϕ. Essentially the same argument serves to lift a homotopy and prove the injectivity.

Taking homotopy fibers over $\mathcal{M} - \mathcal{V}$ all around, we obtain the needed equivalence F.

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