Symplectic surgeries and normal surface singularities

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We show that every negative definite configuration of symplectic surfaces in a symplectic 4–manifold has a strongly symplectically convex neighborhood. We use this to show that if a negative definite configuration satisfies an additional negativity condition at each surface in the configuration and if the complex singularity with resolution diffeomorphic to a neighborhood of the configuration has a smoothing, then the configuration can be symplectically replaced by the smoothing of the singularity. This generalizes the symplectic rational blowdown procedure used in recent constructions of small exotic 4–manifolds.

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1 Introduction

Most of the recent examples in smooth 4–manifold topology have been constructed using the following "cut-and-paste" scheme: Suppose that the smooth closed 4–manifold X is decomposed along the embedded 3–manifold Y as

$$X = X_1 \cup_Y X_2$$

where X_1, X_2 are codimension–0 submanifolds of X with $\partial X_1 = -\partial X_2 = Y \neq \varnothing$. Suppose furthermore that Z_1 is a smooth 4–manifold with boundary ∂Z_1 diffeomorphic to $Y = \partial X_1$. Then a new 4–manifold

$$Z = Z_1 \cup_Y X_2$$

can be constructed by cutting $\mathrm{int}(X_1)$ out of X and gluing Z_1 back in. The topological type of Z might also depend on the gluing diffeomorphism $\varphi\colon \partial Z_1 \to Y$, but for simplicity we will suppress this dependence in the notation. For example, if X_1 is a tubular neighborhood of a torus of self-intersection 0 and $Z_1 = D^2 \times T^2$ then appropriate choices of φ give (generalized) logarithmic transformations and Luttinger surgeries.

The most important topological invariants of a closed smooth 4–manifold are the fundamental group π_1, the Euler characteristic χ and the signature σ. In fact, in

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the simply connected case χ and σ essentially determine the smooth 4–manifold up to homeomorphism by Freedman [7]. The change of χ and σ can be very easily determined in a cut-and-paste operation, since these quantities are additive, while the fundamental group can be computed using the Seifert–Van Kampen theorem. The determination of the smooth structure is, however, much more complicated. The most sensitive smooth invariant, the Seiberg–Witten function

$$\mathrm{SW}_Z\colon H^2(Z;\mathbb{Z}) \to \mathbb{Z},$$

is very hard to compute in general, and although a TQFT–type theory (the monopole Floer homology; see Kronheimer and Mrowka [10]) has been developed to compute the Seiberg–Witten invariants of the result of a cut-and-paste construction, such computations are extremely challenging in practice. Partial knowledge of SW_Z is provided by Taubes' famous theorem [19], stating that $\mathrm{SW}_Z(c_1(Z,\omega))$ is ± 1 provided $\omega \in \Omega^2(Z)$ is a symplectic form on Z (and $b_2^+(Z) > 1$). Therefore we are particularly interested in cut-and-paste constructions which can be performed within the symplectic category.

In this paper we will consider the following special case of the above cut-and-paste construction: Suppose that $C = C_1 \cup \cdots \cup C_n \subset (X,\omega)$ is a collection of closed symplectic 2–dimensional submanifolds of the closed symplectic 4–manifold (X,ω), intersecting each other ω–orthogonally according to the plumbing graph Γ. Recall that each vertex v of the plumbing graph Γ corresponds to a surface, hence is decorated by two integers, the genus g_v and the homological square (or self-intersection) s_v of the surface, and two vertices are connected by $m \geq 0$ edges if and only if the corresponding surfaces intersect each other transversely in m (positive) points. We will denote the number of edges emanating from a vertex v by d_v. Let X_1 be a tubular neighborhood νC of the configuration $C = C_1 \cup \cdots \cup C_n$. Assume that Γ is connected, negative definite (ie the corresponding intersection form is negative definite), and consider a normal surface singularity $(S_\Gamma, 0)$ with resolution graph Γ. (It is a result of algebraic geometry by Grauert [9] that such $(S_\Gamma, 0)$ exists for every connected, negative definite Γ, although the analytic structure on $(S_\Gamma, 0)$ might not be uniquely determined by Γ.) Suppose finally that Z_1 is the Milnor fiber of a smoothing of the singularity $(S_\Gamma, 0)$. Depending on $(S_\Gamma, 0)$, such smoothing may or may not exist. For example, if $(S_\Gamma, 0)$ is a hypersurface singularity (given by a single equation), or more generally it is a complete intersection (cf Section 2), then such smoothing always exists. The main result of this paper is:

Theorem 1.1 *Suppose that Γ is a connected, negative definite plumbing graph for which either*

(1) $-s_v \geq 2(d_v + g_v)$ *holds for every vertex v of Γ, or*

(2) Γ *is a tree and has $g_v = 0$ and $-s_v - d_v \geq 0$ for all vertices.*

Suppose furthermore that $C = C_1 \cup \cdots \cup C_n \subset (X, \omega)$ is a collection of closed symplectic 2–dimensional submanifolds of the closed symplectic 4–manifold (X, ω), intersecting each other ω–orthogonally according to Γ. Let $(S_\Gamma, 0)$ denote a singularity with resolution graph Γ and Z_1 the Milnor fiber of a smoothing of $(S_\Gamma, 0)$. If $X_1 \subset X$ is a closed tubular neighborhood of C in X, then the 4–manifold

$$Z = Z_1 \cup_Y (X - \mathrm{int}(X_1))$$

(with a suitable, naturally chosen gluing diffeomorphism φ specified later) admits a symplectic structure ω_Z, which can be assumed to agree with the given symplectic structure ω on $X - \mathrm{int}(X_1)$.

One way of interpreting this result is the following: Consider the singular 4–manifold X^{sing} we get by collapsing C to a point. If the singularity of X^{sing} is diffeomorphic to a holomorphic model admitting a smoothing, and Γ satisfies one of the additional hypotheses given in the theorem, then this smoothing can always be "globalized" in the symplectic category. Notice that we do not require the singular point to have a holomorphic model in X^{sing} as in McCarthy and Wolfson [14] (where the analytic structure near the singular point is also assumed to be modeled by the holomorphic situation) – we just require the existence of a diffeomorphism. For "globalizing" local deformations in the holomorphic category in a similar context, see Lee and Park [12].

According to Caubel, Némethi and Popescu-Pampu [2], the link $Y = \partial Z_1$ of the singularity $(S_\Gamma, 0)$ given by the (negative definite) plumbing graph Γ admits a unique (up to contactomorphism) Milnor fillable contact structure ξ_M, for which Z_1 (with its Stein structure originating from the deformation) provides a Stein filling. In fact, our proof will not use the fact that Z_1 is a smoothing of $(S_\Gamma, 0)$. Instead, we will rely on the fact that Z_1 admits a symplectic structure Ω such that (Z_1, Ω) is a strong symplectic filling of (Y, ξ_M). For this reason the chosen analytic structure on $(S_\Gamma, 0)$ is not relevant.

For the convenience of the reader, below we summarize the strategy we will use in the proof of Theorem 1.1. First we will show that the union $C \subset (X, \omega)$ of the symplectic surfaces (of arbitrary genera, intersecting each other ω–orthogonally and according to the connected, negative definite graph Γ) in the symplectic 4–manifold (X, ω) admits a compact ω–convex neighborhood U_C. This will be achieved by producing a model symplectic 4–manifold $(X_\Gamma, \omega_\Gamma)$ containing a configuration C_Γ of symplectic surfaces (intersecting each other ω_Γ–orthogonally and according to Γ), with the same areas and genera as the surfaces in C and with a neighborhood system of ω_Γ–convex neighborhoods of C_Γ, such that any neighborhood νC_Γ of C_Γ contains an element of this ω_Γ–convex neighborhood system. Then a Moser type argument shows that any small enough neighborhood $\nu C_\Gamma \subset X_\Gamma$ is symplectomorphic to a neighborhood νC of C in (X,ω), and hence νC contains an ω–convex neighborhood U_C. In the construction of $(X_\Gamma, \omega_\Gamma)$ we will use simple models for the surfaces which are symbolized by the vertices of the plumbing graph Γ (similarly to the approach we applied for the central vertex of a starshaped graph in our paper [8]) and will apply a toric construction for the edges of Γ (similarly to the construction along the legs in [8]). Since this construction might be of independent interest, we state it as:

Theorem 1.2 *If $C = C_1 \cup \cdots \cup C_n \subset (X,\omega)$ is a collection of symplectic surfaces in a symplectic 4–manifold (X,ω) intersecting each other ω–orthogonally according to the connected, negative definite plumbing graph Γ and $\nu C \subset X$ is an open set containing C, then C admits an ω–convex neighborhood $U_C \subset \nu C \subset (X,\omega)$. In particular, the complement $X - \operatorname{int} U_C$ is a strong concave filling of its contact boundary.*

Remark 1.3 Using Grauert's result [9] it is not hard to show that C admits a neighborhood which is a weak symplectic filling of an appropriate contact structure on its boundary. (A weak filling is one where the symplectic structure is positive on the contact planes on the boundary, as opposed to a strong filling, where the contact structure is induced by a Liouville vector field transverse to the boundary.) Therefore the complement of this neighborhood is a weak concave filling, and although in some cases weak convex fillings can be deformed to be strong (see Eliashberg [4]), no similar result for concave fillings is known. Weak fillings, however, are not suitable for the gluing constructions we will apply later, since in the weak case the contact structures do not determine the behavior of the symplectic forms near the boundaries. In the strong case, the Liouville vector fields allow us to glue symplectic forms when the contact forms agree. Hence we verify the existence of an ω–convex neighborhood, providing the desired strong concave filling of the boundary of the appropriate neighborhood. Notice also that in this first step the further assumptions on the plumbing graph Γ (listed in (1) and (2) of Theorem 1.1) are not necessary.

After finding the ω–convex neighborhood $U_C \subset (X,\omega)$ we would like to compare the induced contact structure ξ_C on ∂U_C to the Milnor fillable contact structure ξ_M on ∂Z_1 (given as the 2–plane field of complex tangencies on the link). To this end we describe an open book decomposition of ξ_C and (using a result of [2]) relate it to an open book decomposition of the Milnor fillable contact structure ξ_M. A natural open book decomposition compatible with ξ_C will be given only under the additional hypothesis that $-s_v - d_v \geq 0$ for each vertex v of Γ, and the relation to some open book decomposition compatible with ξ_M will be established in the two cases listed by Theorem 1.1. It is natural to conjecture, however, that these further technical assumptions are unnecessary, hence we state:

Conjecture 1.4 *The contact structures ξ_C and ξ_M are contactomorphic for any negative definite plumbing graph Γ, consequently the symplectic structure ω_Z on the 4–manifold Z of Theorem 1.1 exists for any connected, negative definite plumbing graph Γ.*

The paper is organized as follows: In Section 2 we recall some basics of normal surface singularities. Section 3 is devoted to the description of the ω–convex neighborhoods of the configuration $C \subset (X, \omega)$ and hence the proof of Theorem 1.2. In Section 4, under the additional assumption $-s_v - d_v \geq 0$ mentioned above, we describe an open book decomposition of (U_C, ξ_C) compatible with the contact structure induced on the boundary of the ω–convex neighborhood, while in Section 5 we prove Theorem 1.1.

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2 Generalities on normal surface singularities

For the sake of completeness, in this section we collect some of the basic results regarding normal surface singularities. For general reference see Laufer [11], Looijenga and Wahl [13], Némethi [15] and Wahl [20].

A *complex germ* $(V, 0)$ is an equivalence class of subsets of $\mathbb{C}^n$, where two subsets are equivalent if they agree on some open neighborhood of 0. A *germ* $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ of a holomorphic function is an equivalence class of holomorphic functions from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}, 0)$, where two functions are equivalent if they agree on some open neighborhood of $0 \in \mathbb{C}^n$. Note that the "inverse image of 0" under a germ of a holomorphic function is naturally a complex germ. Also note that all derivatives of a holomorphic germ are well defined at 0. The complex germ $(V, 0)$ is a *surface singularity* if there are germs of holomorphic functions $f_i\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ $(i = 1, \dots, m)$ such that

(2-1) $$(V, 0) = \{x \in \mathbb{C}^n \mid f_i(x) = 0 \;\; i = 1, \dots, m\},$$

and the rank $r(x)$ of the matrix

$$\left(\frac{\partial f_i}{\partial z_j}(x) \right)_{i=1,\dots,m;\, j=1,\dots,n}$$

is equal to $n - 2$ for generic points x of V. If $r(x) = n - 2$ for all $x \in V - 0$ and $r(0) < n - 2$ then the singularity is called *isolated*. $(V, 0)$ is *normal* if any bounded holomorphic function $f\colon V-\{0\}\to\mathbb{C}$ extends to a holomorphic function on V. A normal surface singularity is necessarily isolated. The singularity $(V,0)$ is a *complete intersection* if $m=n-2$ in (2-1), and it is a *hypersurface singularity* if $n=3$ and $m=1$.

The *link* L of the normal surface singularity $(V,0)$ is defined as the intersection of V and a sphere $S^{2n-1}_\epsilon=\{x\in\mathbb{C}^n \mid ||x||=\epsilon\}$. The 3–manifold L is independent of the embedding of V into $\mathbb{C}^n$, and (provided it is small enough) independent of ϵ.

A *resolution* of a singularity $(V,0)$ is a smooth complex surface $\widetilde{V}$ together with a proper holomorphic map $\pi\colon\widetilde{V}\to V$ such that π restricted to $\pi^{-1}(V-\{0\})$ is an isomorphism, that is, a diffeomorphism which is holomorphic in both directions. The resolution is *good* if $\pi^{-1}(0)$ is a normal crossing divisor, that is, in a decomposition of $\pi^{-1}(0)=E=E_1\cup\cdots\cup E_k$ into irreducible components all curves are smooth, intersect each other transversely and there is no triple intersection. Such a resolution always exists, but it is not unique. A resolution is called *minimal* if it does not contain any rational curve with self-intersection (-1). The minimal resolution is unique, but might not be good (in the above sense). The resolution can be assumed to be Kähler, in such a way that π is a biholomorphism away from $0\in V$. A good resolution can be described by its *dual graph*, where each irreducible component of E is symbolized by a vertex, each vertex is decorated by the genus and the self-intersection of the corresponding component, and two vertices are connected if the corresponding curves intersect each other. Notice that since the curves E_i are assumed to be smooth, the resulting graph contains no edge with coinciding endpoints. It is easy to see that the plumbing 3–manifold defined by the dual graph of a resolution is diffeomorphic to the link of the singularity at hand.

A resolution graph of a normal surface singularity is always negative definite, and according to a deep theorem of Grauert [9], any negative definite plumbing graph appears as the graph of a resolution of an appropriate (and not necessarily unique) normal surface singularity. Notice that the link L of the singularity $(V,0)$ admits a contact structure by considering the complex tangents along L. According to [2] this contact structure is unique up to contactomorphism. It is called the *Milnor fillable* contact structure on L. By a famous result of Bogomolov [1] the complex structure on a resolution $\widetilde{V}$ can be deformed to a (possible blow-up of a) Stein filling, hence Milnor fillable contact structures are necessarily Stein fillable.

A *smoothing* of $(V,0)$ consists of a germ of a complex 3–fold $(\mathfrak{V},0)$ together with a (germ of a) proper flat analytic map $f\colon(\mathfrak{V},0)\to(\Delta,0)$ (where $(\Delta,0)$ is the germ of an open disk in $\mathbb{C}$) and an isomorphism $i\colon(f^{-1}(0),0)\to(V,0)$ such that $\mathfrak{V}-\{0\}$ is nonsingular and $f|_{\mathfrak{V}-\{0\}}$ is a submersion. By the Ehresman fibration theorem it follows then that over $\Delta - \{0\}$ the map f is a fiber bundle whose fibers are smooth 2–dimensional Stein manifolds. The typical (nonsingular) fiber is called the *Milnor fiber* of the smoothing. Notice that its boundary is equal to the link of the singularity, and the contact structure induced on it by the complex tangencies is isotopic to the Milnor fillable contact structure of the link. Such smoothing does not necessarily exist for a given singularity; if it does, the Milnor fiber provides a further Stein filling of the Milnor fillable contact structure of the link of the singularity.

3 Construction of ω–convex neighborhoods

The aim of this section is to prove Theorem 1.2. We will always assume that the graph Γ does not admit an edge from a vertex back to itself; in other words, the symplectic surfaces $C_i \subset (X, \omega)$ are assumed to be embedded. The general case involving immersed surfaces can always be reduced to this situation by blow-ups.

By applying the following result (which is an application of Moser's method), the construction of the appropriate neighborhood relies on constructing model symplectic structures on the plumbing 4–manifold X_Γ determined by Γ. We start with recalling the Moser-type result.

Theorem 3.1 (Moser, cf also [8; 17]) *Suppose that ω_1 and ω_2 are symplectic forms on a 4–manifold M containing a configuration of smooth surfaces $C = C_1 \cup \cdots \cup C_n$ which are both ω_1– and ω_2–symplectic, with intersections which are both ω_1– and ω_2–orthogonal. Then C admits symplectomorphic neighborhoods (U_1, ω_1) and (U_2, ω_2) (via a symplectomorphism which is the identity on C) if and only if $\int_{C_i} \omega_1 = \int_{C_i} \omega_2$ for all $i = 1, \dots, n$.* □

The rest of the section is occupied by the construction of the model neighborhoods. Let Γ be a finite graph with vertex set $\{1, 2, \dots, n\}$, with each vertex v labelled with a self-intersection $s_v \in \mathbb{Z}$, an area $a_v \in \mathbb{R}^+$ and a genus $g_v \geq 0$. (As always, $\mathbb{R}^+$ denotes $(0, \infty)$.) Let $\mathbf{a} = (a_1, \dots, a_n)^T \in (\mathbb{R}^+)^n$. Assume that Γ has no edges from a vertex back to itself. Let Q be the associated $n \times n$ intersection matrix for Γ, so that $Q_{ii} = s_i$ and Q_{ij} is the number of edges from vertex i to vertex j. (Notice that the off-diagonals of Q are therefore all nonnegative.) The result we will prove will be slightly more general than needed because we will assume a condition more general than that Q is negative definite.

In [8] we defined a *neighborhood 5–tuple* as a 5–tuple (X, ω, C, f, V) such that (X, ω) is a symplectic 4–manifold, C is a collection of symplectic surfaces in X intersecting ω–orthogonally, $f\colon X \to [0, \infty)$ is a smooth function with no critical values in $(0, \infty)$ and with $f^{-1}(0) = C$, and V is a Liouville vector field on $X - C$ with $df(V) > 0$. From this it easily follows that, for small $t > 0$, $f^{-1}[0, t]$ is an ω–convex tubular neighborhood of C.

Proposition 3.2 *If there exists a vector $\mathbf{z} \in (\mathbb{R}^+)^n$ with $-Q\mathbf{z} = \frac{1}{2\pi}\mathbf{a}$ then there exists a neighborhood 5–tuple (X, ω, f, C, V) such that C is a configuration of symplectic surfaces $C_1 \cup \cdots \cup C_n$ intersecting ω–orthogonally according to the graph Γ, with $C_i \cdot C_i = s_i$, $\int_{C_i} \omega = a_i$ and* $\mathrm{genus}(C_i) = g_i$.

Before giving the proof we give a quick survey of the necessary facts about toric moment maps on symplectic 4–manifolds. These results are all standard except that here we suppress the importance of the torus action and focus instead on how the geometry of the moment map image determines the smooth and symplectic topology of the total space; from a 4–manifold topologist's point of view a useful exposition can be found in [18]. Suppose that $\mu\colon X \to \mathbb{R}^2$ is a toric moment map on a symplectic 4–manifold (X, ω) with connected fibers and with $\partial X = \varnothing$.

(1) Associated to μ we have coordinates (p_1, q_1, p_2, q_2) on X, with $p_i \in \mathbb{R}$ and $q_i \in \mathbb{R}/2\pi\mathbb{Z}$, such that $\mu(p_1, q_1, p_2, q_2) = (p_1, p_2)$ and $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$.

(2) The image $\mu(X) \subset \mathbb{R}^2$ has polygonal boundary with edges of rational slope. Where two edges with primitive integral tangent vectors $(a, b)^T$ and $(c, d)^T$ (oriented by $\partial\mu(X)$) meet at a vertex, we have the "Delzant condition":

$$\det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = 1.$$

(3) The fibers over interior points of $\mu(X)$ are tori (with coordinates (q_1, q_2)). The fiber above a point in the interior of an edge of $\partial\mu(X)$ with primitive integral tangent vector $(a, b)^T$ is a circle with coordinate $aq_1 + bq_2$, so that the $(-b, a)$–circles in a nearby (q_1, q_2)–torus bound disks. The fiber above a vertex of $\partial\mu(X)$ is a single point.

(4) Any other symplectic 4–manifold (X', ω') with toric moment map $\mu'\colon X' \to \mathbb{R}^2$ with connected fibers and with $\mu'(X') = \mu(X)$ is symplectomorphic to (X, ω) via a fiber-preserving symplectomorphism. Furthermore, the closure of any 2–dimensional submanifold B of $\mathbb{R}^2$ that has a rational slope polygonal boundary satisfying the Delzant conditions occurs as the image of a toric moment map on some symplectic 4–manifold (with connected fibers).

(5) Given any matrix $A \in GL(2, \mathbb{Z})$, there exists a toric moment map $\mu_A\colon (X, \omega) \to \mathbb{R}^2$ such that $\mu_A(X) = A\mu(X)$ and such that the coordinates (p_1', q_1', p_2', q_2') associated to μ_A are related to the coordinates (p_1, q_1, p_2, q_2) associated to μ via the

following transformation:

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = A \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} = A^{-T} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

(Here $A^{-T} = (A^{-1})^T$.)

(6) The vector field $x\partial_x + y\partial_y$ radiating out from the origin in $\mathbb{R}^2$ lifts to a Liouville vector field $V = p_1\partial_{p_1} + p_2\partial_{p_2}$ on $X - \mu^{-1}(\partial\mu(X))$. Given some $A \in GL(2,\mathbb{Z})$, the change of coordinates discussed in the preceding point transforms V to $V' = p_1'\partial_{p_1'} + p_2'\partial_{p_2'}$.

(7) Looking at a very specific case, if $R = (x_0, x_1) \times [y_0, y_1)$ is an open subset of $B = \mu(X)$ (hence $(x_0, x_1) \times \{y_0\} \subset \partial B$), then the set $\mu^{-1}(R)$ is diffeomorphic to $(x_0, x_1) \times S^1 \times D^2_\rho$, where D^2_ρ is an open disk in $\mathbb{R}^2$ of radius $\rho = \sqrt{2(y_1 - y_0)}$ centered at the origin. Furthermore, $\omega|_{\mu^{-1}(R)} = dt \wedge d\alpha + r dr \wedge d\theta$, where $t \in (x_0, x_1)$, $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ and (r, θ) are standard polar coordinates on D^2_ρ, and with these coordinates, $\mu(t, \alpha, r, \theta) = (t, \frac{1}{2}r^2 + y_0)$, ie $p_1 = t$, $q_1 = \alpha$, $p_2 = \frac{1}{2}r^2 + y_0$, $q_2 = \theta$. Then $\mu^{-1}(\partial R) = \mu^{-1}((x_0, x_1) \times \{y_0\})$ is a cylinder $(x_0, x_1) \times S^1 \times \{0\}$ with symplectic area $2\pi(x_1 - x_0)$. The Liouville vector field $p_1\partial_{p_1} + p_2\partial_{p_2}$ then becomes $V = t\partial_t + (\frac{1}{2}r + y_0/r)\partial_r$. (Note that V is clearly undefined at $r = 0$ except in the special case that $y_0 = 0$.)

Proof of Proposition 3.2 Fix a vector $\mathbf{z} = (z_1, \ldots, z_n)^T \in (\mathbb{R}^+)^n$ with $-Q\mathbf{z} = \frac{1}{2\pi}\mathbf{a}$. For each vertex v and for each edge e meeting v, choose an integer $s_{v,e}$ such that $\sum s_{v,e} = s_v$, where this sum and other similar sums below are taken over all edges meeting the given vertex v. Also, for each vertex v and each edge e meeting v, letting w be the vertex at the other end of e, let $x_{v,e} = -s_{v,e}z_v - z_w$. Note that, for each v we have $\sum x_{v,e} = (-Q\mathbf{z})_v = \frac{1}{2\pi}a_v > 0$. Choose a small positive constant ϵ, small enough so that for each v we have $\sum(x_{v,e} - \epsilon) > 0$. Also choose small positive constants δ and γ satisfying a constraint to be stated shortly.

Consider the first quadrant $P = [0,\infty)^2 \subset \mathbb{R}^2$ and let $g\colon P \to [0,\infty)$ be a smooth function satisfying the following properties (see Figure 1):

(1) 0 is the only critical value of g.

(2) $g^{-1}(0) = \partial P$.

(3) If $y - x \geq \gamma$ then $g(x, y) = x$.

(4) If $y - x \leq -\gamma$ then $g(x, y) = y$.

(5) For all x, y we have $g(x, y) = g(y, x)$.

(6) In the region $-\gamma \leq y-x \leq \gamma$, the level sets $g^{-1}(t)$, for $t>0$, are smooth curves symmetric about the line $y=x$, with slope changing monotonically as a function of $y-x$ from 0 to ∞.

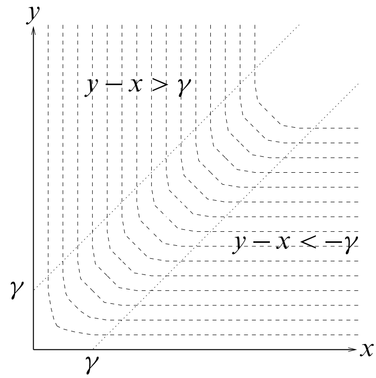

Figure 1: Contour plot of g

The constants δ and γ should satisfy the following constraint: For each vertex v and for each edge e incident to v, the line passing through $(0,\epsilon)$ with tangent vector $(1,-s_{v,e})$ should intersect $g^{-1}(\delta)$ in the region $y-x>\gamma$. By symmetry we will also have that the line passing through $(\epsilon,0)$ with tangent vector $(-s_{v,e},1)$ intersects $g^{-1}(\delta)$ in the region $y-x<-\gamma$. Note that if $s_{v,e}<0$, this constraint is simply the constraint that $\gamma<\epsilon$.

For each edge e we now construct a neighborhood 5–tuple $(X_e,\omega_e,f_e,C_e,V_e)$ as follows (see Figure 2): Consider the two vertices at the ends of e and arbitrarily label one v and the other v'. Let $g_e(x,y)=g(x-z_v,y-z_{v'})$, a function from $P+(z_v,z_{v'})$ to $[0,\infty)$. Let R_e be the open subset of $g_e^{-1}[0,\delta)$ between the line passing through $(z_v,z_{v'}+2\epsilon)$ with tangent vector $(1,-s_{v,e})$ and the line passing through $(z_v+2\epsilon,z_{v'})$ with tangent vector $(-s_{v',e},1)$. Let (X_e,ω_e) be the unique connected symplectic 4–manifold with toric moment map $\mu_e\colon X_e\to\mathbb{R}^2$ such that $\mu_e(X_e)=R_e$. Let $C_e=\mu_e^{-1}(\partial R_e)$, $f_e=g_e\circ\mu_e$ and let V_e be the Liouville vector field obtained by lifting the radial vector field emanating from the origin in $\mathbb{R}^2$, as in item (6) in the discussion of toric geometry above. Note that $df_e(V_e)>0$ because $dg_e(x\partial_x+y\partial_y)>0$, which is true because $z_v>0$ and $z_{v'}>0$. (Topologically, C_e is just a union of two disks meeting transversely at one point and X_e is a 4–ball neighborhood of C_e.)

Also let $R_{e,v}$ be the open subset of R_e between the parallel lines passing through $(z_v,z_{v'}+\epsilon)$ and $(z_v,z_{v'}+2\epsilon)$ with tangent vector $(1,-s_{v,e})$, and let $R_{e,v'}$ be the open subset of R_e between the parallel lines passing through $(z_v+\epsilon,z_{v'})$ and $(z_v+2\epsilon,z_{v'})$

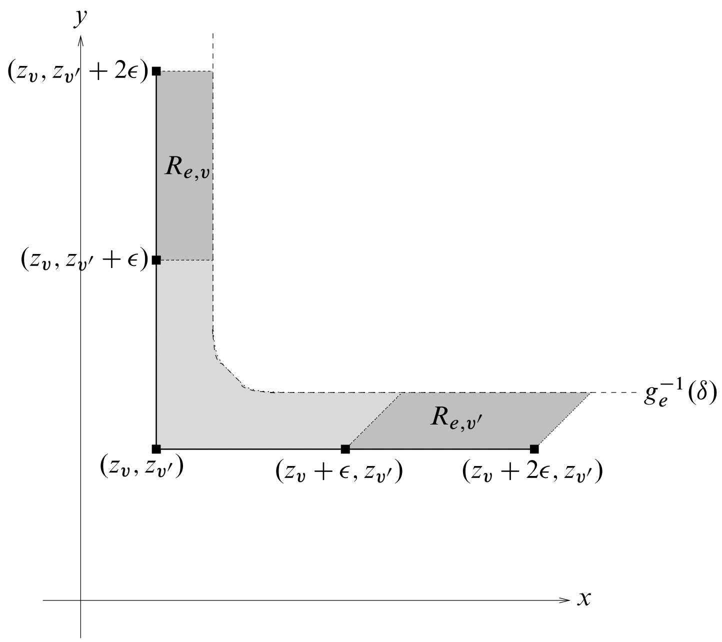

Figure 2: The moment map image R_e of (X_e, ω_e); in this example $s_{v,e} = 0$ and $s_{v',e} = -1$.

with tangent vector $(-s_{v',e}, 1)$. By the constraints on δ and γ, these are both parallelograms, open on three sides.

Now we introduce two reparametrizations of this neighborhood 5–tuple, one for each of the vertices v and v', using matrices $A_v, A_{v'} \in GL(2, \mathbb{Z})$ as in item (5) preceding this proof. These matrices are

$$A_v = \begin{pmatrix} -s_{v,e} & -1 \\ 1 & 0 \end{pmatrix}, \quad A_{v'} = \begin{pmatrix} -1 & -s_{v',e} \\ 0 & 1 \end{pmatrix}.$$

The reader should at this point verify that A_v transforms $R_{e,v}$ into the region

$$(x_{v,e} - 2\epsilon, x_{v,e} - \epsilon) \times [z_v, z_v + \delta)$$

and that $A_{v'}$ transforms $R_{e,v'}$ into the region

$$(x_{v',e} - 2\epsilon, x_{v',e} - \epsilon) \times [z_{v'}, z_{v'} + \delta).$$

Referring to item (7) in the toric discussion preceding this proof, we see that on $\mu_e^{-1}(R_{e,v})$ and on $\mu_e^{-1}(R_{e,v'})$ we can write everything down in particularly nice local coordinates as follows: On $\mu_e^{-1}(R_{e,v})$ we have:

- $\mu_e^{-1}(R_{e,v}) \cong (x_{v,e}-2\epsilon, x_{v,e}-\epsilon)\times S^1 \times D^2_{\sqrt{2\delta}}$ with corresponding coordinates (t,α,r,θ).
- In these coordinates, $\omega_e = dt\wedge d\alpha + r dr\wedge d\theta$.
- $C_e \cap \mu_e^{-1}(R_{e,v}) = (x_{v,e}-2\epsilon, x_{v,e}-\epsilon)\times S^1\times\{0\}$.
- $f_e = \frac{1}{2}r^2$.
- $V_e = t\partial_t + (\frac{1}{2}r + z_v/r)\partial_r$.

On $\mu_e^{-1}(R_{e,v'})$ we have exactly the same formulae but with each occurrence of v replaced with v'.

Now we will construct neighborhood 5–tuples associated to the vertices so that they can be glued to the neighborhoods constructed above using the explicit coordinates that we have just seen in the preceding paragraph. Lemma 2.4 from [8] tells us that for each vertex v we can find a compact surface Σ_v of genus g_v with a symplectic form β_v and Liouville vector field W_v (β_v and W_v both defined on all of Σ_v) such that Σ_v has one boundary component $\partial_{e,v}\Sigma_v$ for each edge e incident with v and such that there exists a collar neighborhood $N_{e,v}$ of each $\partial_{e,v}\Sigma_v$ parametrized as $(x_{v,e}-2\epsilon, x_{v,e}-\epsilon]\times S^1$ on which $\beta_v = dt\wedge d\alpha$ and $W_v = t\partial_t$. (Here we use the constraint we imposed on ϵ, namely that, for each vertex v we have $\sum(x_{v,e}-\epsilon)>0$.) Note that $\int_{\Sigma_v}\beta_v = 2\pi\sum(x_{v,e}-\epsilon)$. Then our neighborhood 5–tuple for the vertex v is

$$\begin{aligned}\Bigg(X_v &= (\Sigma_v - \partial\Sigma_v)\times D^2_{\sqrt{2\delta}},\\ \omega_v &= \beta_v + r dr\wedge d\theta,\\ C_v &= \Sigma_v - \partial\Sigma_v,\\ f_v &= \frac{1}{2}r^2,\\ V_v &= W_v + \left(\frac{1}{2}r + \frac{z_v}{r}\right)\partial_r\Bigg).\end{aligned}$$

These neighborhoods can then be glued to the neighborhoods for the edges as follows: For each edge e with incident vertices v and v', glue the end $(N_{e,v}-\partial_{e,v}\Sigma_v)\times D^2_{\sqrt{2\delta}}$ of X_v to the end $\mu_e^{-1}(R_{e,v})$ of X_e by identifying the (t,α,r,θ) coordinates, and similarly glue $(N_{e,v'}-\partial_{e,v'}\Sigma'_v)\times D^2_{\sqrt{2\delta}}$ to $\mu_e^{-1}(R_{e,v'})$. The result is the 5–tuple (X,ω,C,f,V).

We now verify that the areas and self-intersections of the surfaces in C are correct. For the areas, note that the closed surface $C_v\subset X$ is the union of $(\Sigma_v-\partial\Sigma_v)\times 0$ in X_v with the various disks $\mu_e^{-1}(\partial_v R_e)\subset X_e$, where $\partial_v R_e$ is one of the two edges making up ∂R_e. The area of $(\Sigma_v - \partial\Sigma_v)\times 0$ is $2\pi\sum(x_{v,e}-\epsilon)$, the area of each disk is $2\pi(2\epsilon)$ and the area of each overlapping cylinder is $2\pi\epsilon$, so the total area is $2\pi\sum x_{v,e} = a_v$. For the self-intersections, note that the boundary of a tubular neighborhood of C_v is a 3–manifold homeomorphic to $\Sigma_v \times S^1$ with the boundary components Dehn filled with solid tori. Looking at how the matrices A_v (or $A_{v'}$) transform the regions R_e, and following the argument at the end of the proof of [8, Proposition 2.3], we see that the $(1, s_{v,e})$ curves in each $\partial_{v,e}\Sigma_v \times S^1$ are filled in by disks. So this 3–manifold is the S^1–bundle over C_v of Euler class $\sum s_{v,e} = s_v$. □

In order to apply Proposition 3.2 in the proof of Theorem 3.1 we need to show that the symmetric matrix Q defined by the graph Γ of the symplectic surfaces $C_1 \cup \cdots \cup C_n \subset (X, \omega)$ satisfies the property that the equation

$$-Q\mathbf{z} = \frac{1}{2\pi}\mathbf{a}$$

admits a solution $\mathbf{z} = (z_1, \ldots, z_n) \in (\mathbb{R}^+)^n$ for any given $\mathbf{a} \in (\mathbb{R}^+)^n$. The basis of our argument is the following simple linear algebra observation:

Lemma 3.3 *Suppose that the bilinear form (x, y) is given by the negative definite symmetric matrix Q with only nonnegative off-diagonals in the basis $\{E_i\}$. If for a vector x the inequalities $(x, E_i) \leq 0$ $(i = 1, \ldots, n)$ are all satisfied, then all coordinates of x are nonnegative.*

Proof Let us expand x in the basis $\{E_i\}$ and denote the resulting n–tuple by x as well. Suppose that $x = x_1 - x_2$ where x_i has only nonnegative entries for $i = 1, 2$, and the supports of x_1 and x_2 are disjoint. Take E_i from the support of x_2. Then by the assumption

$$(x, E_i) = (x_1, E_i) - (x_2, E_i) \leq 0$$

implying that $(x_1, E_i) \leq (x_2, E_i)$. Summing for all basis vectors E_i in the support of x_2 and multiplying the inequalities with the positive coefficients they have in x_2 we get

$$(x_1, x_2) \leq (x_2, x_2).$$

Since the supports of x_1 and x_2 is disjoint (and the off-diagonals in Q are all non-negative, that is, $(E_i, E_j) \geq 0$ once $i \neq j$), we have that $(x_1, x_2) \geq 0$. On the other hand, Q is negative definite, so $(x_2, x_2) \leq 0$. This implies that $(x_2, x_2) = 0$, which by definiteness implies that $x_2 = 0$, hence $x = x_1$, verifying the lemma. □

Corollary 3.4 *For any $\mathbf{a} \in (\mathbb{R}^+)^n$ the vector $-Q^{-1}\mathbf{a}$ is in $(\mathbb{R}^+)^n$.*

Proof Suppose that $\mathbf{a}$ is in $(\mathbb{R}^+)^n$ and consider $\mathbf{b} = -Q^{-1}\mathbf{a}$. Then $-\mathbf{a} = Q\mathbf{b}$ is a vector with only nonpositive coordinates, that is, $(\mathbf{b}, E_i) \leq 0$ for all i. The application of Lemma 3.3 then finishes the proof. □

Proof of Theorem 1.2 By the above corollary and Proposition 3.2, there exists a neighborhood 5–tuple $(X_\Gamma, \omega_\Gamma, f_\Gamma, C_\Gamma, V_\Gamma)$ for the given plumbing graph Γ (decorated with $a_i = \int_{C_i} \omega$). By basic results in differential topology, there exists an open neighborhood U of C in X which is diffeomorphic to $f_\Gamma^{-1}(t)$ for some small $t > 0$, via a diffeomorphism sending C to C_Γ. By Theorem 3.1, we can make this diffeomorphism into a symplectomorphism, after possibly taking a smaller neighborhood of C and a smaller value for t. Since in the neighborhood 5–tuple every neighborhood of C_Γ contains an ω_Γ–convex neighborhood, its image under the symplectomorphism provides $U_C \subset (X, \omega)$. □

4 Open book decompositions on ∂U_C

Suppose that for every vertex v of the plumbing graph Γ with self-intersection (homological square) s_v and valency d_v the additional hypothesis

$$-s_v - d_v \geq 0$$

holds. In this section we describe an open book decomposition on ∂U_C compatible with the contact structure induced on it as an ω–convex neighborhood of C. We begin with a lemma about "open book decompositions" (OBDs) on 3–manifolds with boundary. By an OBD on a 3–manifold M with $\partial M \neq \varnothing$ we mean a pair (B, π), where $B \subset M - \partial M$ is a link and $\pi\colon M - B \to S^1$ is a fibration which behaves as open books usually behave near B and which restricts to ∂M to give an honest fibration of ∂M over S^1. When the pages are oriented, this induces an orientation on B as the boundary of a page.

Lemma 4.1 *Consider $M = [0,1] \times S^1 \times S^1$ with coordinates $t \in [0,1]$, $\alpha \in S^1$ and $\beta \in S^1$. Given a nonnegative integer m there exists an OBD (B, π) on M such that the following conditions hold:*

(1) $\pi|_{\{0\} \times S^1 \times S^1} = \beta$.

(2) $\pi|_{\{1\} \times S^1 \times S^1} = \beta + m\alpha$.

(3) *The pages $\pi^{-1}(\theta)$ are transverse to ∂_β.*

(4) *The binding B is tangent to ∂_β.*

(5) *B has m components $B_1, \dots, B_m$, which we can take to be $B_i = \{1/2\} \times \{(2\pi i)/m\} \times S^1$.*

(6) *When the pages are oriented so that ∂_β is positively transverse then $B_1, \dots, B_m$ are oriented in the positive ∂_β direction.*

Proof If $m = 0$ we use the map $\pi = \beta$ on all of M and have $B = \varnothing$. Otherwise the proof follows directly from the following observation which we leave to the reader to verify (with the aid of Figure 3): Consider $P = [0, 1] \times [0, 1] \times S^1$ with coordinates (x, y, θ). There is an OBD (B_P, π_P) on P with $B_P = \{1/2\} \times \{1/2\} \times S^1$, such that $f|_{\{0\}\times[0,1]\times S^1} = \theta$, $f|_{[0,1]\times\{0\}\times S^1} = \theta$, $f|_{[0,1]\times\{1\}\times S^1} = \theta$ and $f|_{\{1\}\times[0,1]\times S^1} = \theta + 2\pi y$. When the pages are oriented so that ∂_θ is positively transverse then B_P is oriented in the positive ∂_θ direction. Given this observation, the lemma can be proved

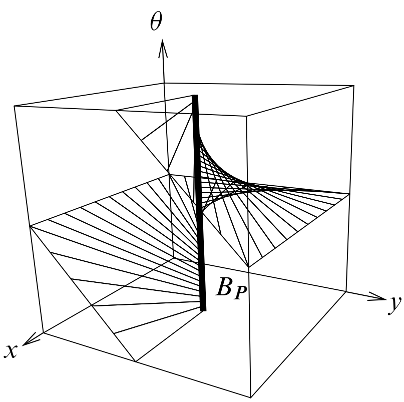

Figure 3: Building block for OBDs. The shaded surface indicates a page.

by stacking m of the above models side-by-side (in the y direction). Some trivial smoothing is required, of course. □

Recall that a plumbed 3–manifold $M = M_\Gamma$ constructed according to a plumbing graph Γ decomposes along a collection of tori $\{T_e\}$, indexed by the edges of Γ, into codimension-0 pieces $\{M_v\}$, indexed by the vertices of Γ. Each M_v fibers overs a compact surface Σ_v with each boundary component $\partial_{v,e} M_v$ of M_v fibering over a corresponding boundary component $\partial_{v,e}\Sigma_v$ of Σ_v. On each torus T_e there are thus two fibrations over S^1, coming from the vertices at the two ends of e. We say that an OBD on M is *horizontal* if the pages are transverse to the fibers on each M_v and transverse to both types of fibers on each T_e and if the binding components are disjoint from the T_e's and are fibers of the fibration of the corresponding M_v's. (Note that this definition depends on identifying M as a plumbed 3–manifold and specifying the fibrations on each M_v.) In addition, we can orient the binding components as boundary components of a page, with the page oriented so as to intersect fibers positively; we require this orientation to point in the positive fiber direction. (For more about horizontal OBDs, see Etgü and Ozbagci [5].)

Now we refer to the notation of Proposition 3.2 and its proof. For any small enough $t > 0$, $M = f^{-1}(t)$ is a plumbed 3–manifold. We may take the separating tori $\{T_e\}$ to be $T_e = \mu_e^{-1}(g_e^{-1}(t) \cap L)$, where L is the line $(y - z_{v'}) - (x - z_v) = 0$ in R_e. Let $\xi_C = \ker(\iota_V \omega|_M)$ be the contact structure induced on M by the Liouville vector field V and the symplectic structure ω.

Proposition 4.2 *Suppose that the plumbing graph Γ satisfies the additional hypothesis that $p_v = -s_v - d_v$ is nonnegative for every vertex v of Γ. Then there exists a horizontal OBD on M supporting ξ with p_v binding components in each fibered piece M_v. This OBD is independent of the areas $a_1, \ldots, a_n$ of the symplectic surfaces $C_1, \ldots, C_n$, and therefore the various contact structures induced by the different symplectic structures for different $\mathbf{a} \in (\mathbb{R}^+)^n$ are all isotopic (and will be denoted by ξ_C).*

Proof Referring to the proof of Proposition 3.2, we see that M is built by gluing the $f_v^{-1}(t)$'s to the $f_e^{-1}(t)$'s. Recall that $f_v^{-1}(t) = (\Sigma_v - \partial \Sigma_v) \times S^1_\rho$, where S^1_ρ is the circle of radius $\rho = \sqrt{2t}$. Each $f_e^{-1}(t)$ is a submanifold of X_e which has toric coordinates (p_1, q_1, p_2, q_2). The OBD we construct will be the S^1_ρ coordinate function θ on each $f_v^{-1}(t)$ and the function $q_1 + q_2$ on each $f_e^{-1}(t)$. We will put in binding components in the $(x_{v,e} - 2\epsilon, x_{v,e} - \epsilon) \times S^1 \times S^1_\rho$ overlaps where the gluing happens, in order to "interpolate" from θ to $q_1 + q_2$. In order to do this, we must transform the function $q_1 + q_2$ into the (t, α, θ) coordinates on each $(x_{v,e} - 2\epsilon, x_{v,e} - \epsilon) \times S^1 \times S^1_\rho$ and $(x_{v',e} - 2\epsilon, x_{v',e} - \epsilon) \times S^1 \times S^1_\rho$, using the transformations given by the matrices A_v and $A_{v'}$. We see that the change of coordinates associated with A_v at the end $R_{e,v}$, transforms $q_1 + q_2$ into the function $(-s_{v,e} - 1)\alpha + \theta$ and that the change associated with $A_{v'}$ transforms $q_1 + q_2$ into $(-s_{v',e} - 1)\alpha + \theta$. Thus using Lemma 4.1, we see that for each vertex v incident to an edge e, if we have nonnegative integers $p_{v,e}$ with $p_{v,e} = -s_{v,e} - 1$ we can interpolate from $q_1 + q_2$ to θ by introducing $p_{v,e}$ binding components. By suitably partitioning the p_v's into $p_{v,e}$'s we construct the desired OBD.

It remains to verify that this OBD is horizontal and supports ξ. The OBD is clearly horizontal on each $f_v^{-1}(t)$ and on the overlap regions where the binding components are put in. On each $f_e^{-1}(t)$, we need to see how the fiber directions ∂_θ coming from each vertex incident to e transform via the inverses of the transformations associated to

A_v and $A_{v'}$. This check is straightforward and we see that, at the v end, ∂_θ becomes ∂_{q_1} and at the v' end, ∂_θ becomes ∂_{q_2}. Both of these are transverse to the pages, ie the fibers of $q_1 + q_2$.

Lastly, we need to verify that the Reeb vector field for a contact form for ξ_C is transverse to the pages of this OBD and tangent to the bindings. However, this is clear because, on $f_v^{-1}(t)$ the Reeb vector field for the contact form induced by the Liouville vector field is a positive multiple of ∂_θ, and on $f_e^{-1}(t)$ the Reeb vector field for the contact form induced by the Liouville vector field is a positive multiple of $b_1\partial_{q_1} + b_2\partial_{q_2}$ where $dg_e = b_1 dx + b_2 dy$, and $b_1, b_2 > 0$ by construction of g_e. Notice that in this construction there was no dependence on the areas $\mathbf{a}$. □

5 The proof of Theorem 1.1

In order to apply the gluing scheme of symplectic 4–manifolds along hypersurfaces of contact type (as it is given in [6]) we have to verify that the contact structure ξ_C (given by the toric picture) and the Milnor fillable contact structure ξ_M (on the link of the singularity) are contactomorphic. Recall that in the previous section we saw that for a plumbing graph Γ with $-s_v - d_v \geq 0$ for every vertex v the toric approach produces isotopic contact structures for any input vector $\mathbf{a} \in (\mathbb{R}^+)^n$. In fact, in this case we have a detailed description of the binding of a compatible horizontal open book decomposition: it consists of $p_v = -s_v - d_v$ fibers in each fibered piece of the plumbed 3–manifold.

In the case of negative definite starshaped plumbing trees of spheres with three legs the above identification of contact structures relied on the classification of tight contact structures on the corresponding small Seifert fibered 3–manifolds [8]. Such classification is not available in general. Although we strongly believe that the two contact structures ξ_C and ξ_M are contactomorphic in general (which would lead to the verification of Conjecture 1.4), we could prove it only under strong restrictions on the plumbing graph Γ, giving the proof of Theorem 1.1.

Recall that each vertex v of the plumbing graph Γ is decorated by two integers: $g_v \geq 0$ denotes the genus of the surface Σ_v corresponding to the vertex v, while s_v is the Euler number of the normal disk bundle of Σ_v in the plumbing 4–manifold X_Γ (or alternatively the self-intersection of the homology class $[\Sigma_v]$). Since Γ is negative definite, we have that $s_v < 0$. As before, let d_v denote the valency of the vertex v, that is, the number of edges emanating from v. We will always assume from now on that $-s_v - d_v \geq 0$ holds for every vertex v.

The two contact structures ξ_C and ξ_M will be compared through open book decompositions compatible with them. The open book decompositions compatible with these contact structures, in turn, will be compared through their bindings, in the light of the following results.

Theorem 5.1 (Caubel–Némethi–Popescu-Pampu [2, Proposition 4.6]) *Suppose that Γ is a connected, negative definite plumbing graph, with associated plumbed manifold M_Γ. Suppose that two horizontal open book decompositions of M_Γ have isotopic bindings, such that each fibered component of M_Γ contains at least one binding component. Then the two horizontal open book decompositions are isotopic.* □

Theorem 5.2 (Stallings, cf Caubel–Popescu-Pampu [3, Proposition 2.2]) *Suppose that the 3–manifold M is a rational homology 3–sphere, that is, $b_1(M)=0$. Then two open book decompositions of M with isotopic bindings are isotopic.* □

With these results at hand, we turn to the identification of the contact structures ξ_C and ξ_M – at least under the additional hypotheses spelled out by Theorem 1.1.

Proposition 5.3 *Suppose that Γ is a connected, negative definite plumbing graph for which either*

(1) $-s_v \geq 2(d_v+g_v)$ *holds for every vertex v of Γ, or*

(2) Γ *is a tree and has $g_v=0$ and $-s_v-d_v\geq 0$ for all vertices.*

Then the contact structure ξ_C provided by Proposition 4.2 is contactomorphic to the Milnor fillable contact structure ξ_M.

Proof Suppose first that $-s_v\geq 2(d_v+g_v)$ holds for every vertex v. Since $g_v, d_v\geq 0$, this inequality implies, in particular, that $p_v=-s_v-d_v>0$ for all $v\in\Gamma$. Therefore by Proposition 4.2 the contact structure ξ_C is compatible with a horizontal open book decomposition having $p_v>0$ binding components in each fibered piece M_v of $M=M_\Gamma$.

Next we would like to find an open book decomposition on M which is compatible with ξ_M. In doing so, we would like to appeal to [2, Theorem 4.1]. Notice that by choosing $D=\sum_j E_j$ (ie, all $m_j=1$) the required inequality of the quoted theorem (in the notation of [2]) transforms to $(2E+K_\Sigma)\cdot E_i+2\leq 0$, which, in our convention translates to $s_v+2(d_v+g_v)\leq 0$. Since this is exactly the relation among s_v, d_v and g_v we hypothesized, [2, Theorem 4.1] can be applied. This argument therefore provides a function f, and through it (by [2, Theorem 3.9]) an open book decomposition of M compatible with the Milnor fillable contact structure ξ_M. According to [2, Example 4.5] this open book decomposition is horizontal and (again, in the notation of [2]) has $n_i = -D \cdot E_i$ binding components in each fibered piece of M. Since $D = \sum_j E_j$, we conclude that $D \cdot E_i = E_i^2 + \sum_{j \neq i} E_j \cdot E_i$, which quantity in our convention (with v being the i–th vertex of Γ) is equal to $s_v + d_v = -p_v$. In conclusion, we found a horizontal open book decomposition compatible with ξ_M which has the same binding as the horizontal open book decomposition (compatible with ξ_C) provided by Proposition 4.2. Since $p_v > 0$ for all $v \in \Gamma$, by Theorem 5.1 therefore the two open book decompositions are isotopic, hence the contact structures ξ_C and ξ_M are contactomorphic.

Assume now that Γ is a plumbing tree of spheres, satisfying $-s_v - d_v \geq 0$ for all vertices $v \in \Gamma$. In this case the function f of [2, Theorem 4.1] can be found directly: since Γ can be constructed by blowing up a point of $\mathbb{C}^2$ (ie, the graph Γ is *sandwich* in the sense of [16]), the pullback of the function vanishing in the blown-up point provides f corresponding to $D = \sum_j E_j$ as before. Notice that the corresponding open book decomposition has $p_v = -s_v - d_v = -D \cdot E_v$ binding components in each fibered piece, just as the open book decomposition found in Proposition 4.2 (and compatible with ξ_C does).

It follows from the assumption on Γ that $M = M_\Gamma$ is a rational homology 3–sphere, hence Theorem 5.2 implies that the open book decomposition corresponding to the above f and the one provided by Proposition 4.2 are isotopic. Therefore we get that the contact structures ξ_C and ξ_M are contactomorphic, concluding the proof of the proposition. (Notice that since we allowed $-s_v - d_v = 0$, we cannot appeal to [2, Theorem 4.6] anymore.) □

With the identification of the appropriate contact structures we are now ready to turn to the proof of Theorem 1.1.

Proof of Theorem 1.1 According to Proposition 5.3, under the assumptions of Theorem 1.1 the strong filling Z_1 of the Milnor fillable contact link (Y, ξ_M) and the strong concave filling $X - X_1$ of (Y, ξ_C) have contactomorphic contact structures on their boundaries, hence the gluing construction described in [6] applies (for a suitably chosen contactomorphism $\varphi\colon \partial(X - X_1) \to \partial(-Z_1)$), providing a symplectic structure on $Z = Z_1 \cup_Y (X - X_1)$. This concludes the proof of the main theorem. □

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