Computing knot Floer homology in cyclic branched covers

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We use grid diagrams to give a combinatorial algorithm for computing the knot Floer homology of the pullback of a knot $K \subset S^3$ in its m–fold cyclic branched cover $\Sigma_m(K)$, and we give computations when $m = 2$ for over fifty three-bridge knots with up to eleven crossings.

57R58; 57M12, 57M27

1 Introduction

Heegaard Floer knot homology, developed by Ozsváth and Szabó [15] and independently by Rasmussen [18], associates to a nulhomologous knot K in a three-manifold Y a group $\widehat{\mathrm{HFK}}(Y, K)$ that is an invariant of the knot type of K. If K is a knot in S^3, then the inverse image of K in $\Sigma_m(K)$, the m–fold cyclic branched cover of S^3 branched along K, is a nulhomologous knot $\widetilde{K}$ whose knot type depends only on the knot type of K, so the group $\widehat{\mathrm{HFK}}(\Sigma_m(K), \widetilde{K})$ is a knot invariant of K. In this paper, we describe an algorithm that can compute $\widehat{\mathrm{HFK}}(\Sigma_m(K), \widetilde{K})$ (with coefficients in $\mathbb{Z}/2$) for any knot $K \subset S^3$, and we give computations for a large collection of knots with up to eleven crossings.

Any knot $K \subset S^3$ can be represented by means of a *grid diagram*, consisting of an $n \times n$ grid in which the centers of certain squares are marked X or O, such that each row and each column contains exactly one X and one O. To recover a knot projection, draw an arc from the X to the O in each column and from the O to the X in each row, making the vertical strand pass over the horizontal strand at each crossing. We may view the diagram as lying on a standardly embedded torus $T^2 \subset S^3$ by making the standard edge identifications; the horizontal grid lines become α circles and the vertical ones β circles. Manolescu, Ozsváth, and Sarkar [12] showed that such diagrams can be used to compute $\widehat{\mathrm{HFK}}(S^3, K)$ combinatorially; we shall use them to give a combinatorial description of the chain complex for $\widehat{\mathrm{HFK}}(\Sigma_m(K), \widetilde{K})$ for any knot $K \subset S^3$.

Let $m \geq 2$ and let $\widetilde{T}$ be the surface obtained by gluing together m copies of T (denoted $T_0, \dots, T_{m-1}$) along branch cuts connecting the X and the O in each

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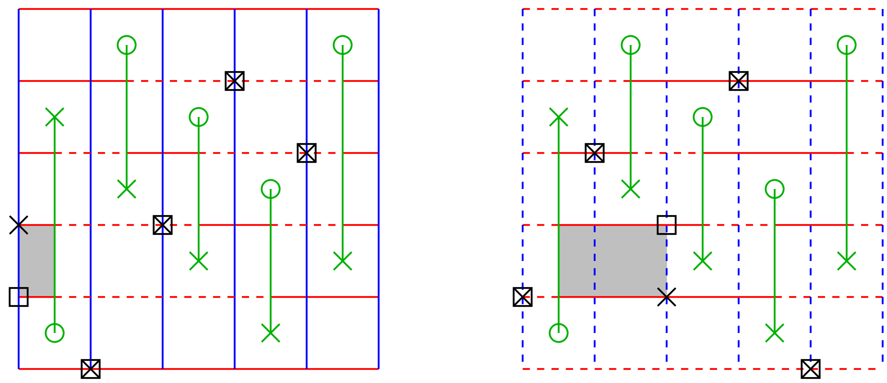

Figure 1: Heegaard diagram $\widetilde{D} = (\widetilde{T}, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\mathbf{w}}, \widetilde{\mathbf{z}})$ for $(\Sigma_2(K), \widetilde{K})$, where K is the right-handed trefoil. The solid and dashed lines represent different lifts of the α (horizontal/red) and β (vertical/blue) circles. The black squares and crosses represent two generators of $\widetilde{C} = \widehat{\mathrm{CFK}}(\widetilde{\mathcal{D}})$, and the shaded region is a disk that contributes to the differential.

column. Specifically, in each column, if the X is above the O, then glue the left side of the branch cut in T_k to the right side of the same cut in T_{k+1} (indices modulo m); if the O is above the X, then glue the left side of the branch cut in T_k to the right side of the same cut in T_{k-1}. The obvious projection $\pi\colon \widetilde{T} \to T$ is an m–fold cyclic branched cover, branched around the marked points. Each α and β circle in T intersects the branch cuts a total of zero times algebraically and therefore has m distinct lifts to T, and each lift of each α circle intersects exactly one lift of each β circle. (We will describe these intersections more explicitly in Section 4.)

Denote by $\mathcal{R}$ the set of embedded rectangles in T whose lower and upper edges are arcs of α circles, whose left and right edges are arcs of β circles, and which do not contain any marked points in their interior. Each rectangle in $\mathcal{R}$ has m distinct lifts to $\widetilde{T}$ (possibly passing through the branch cuts as in Figure 1); denote the set of such lifts by $\widetilde{\mathcal{R}}$.

Let $\mathcal{S}$ be the set of unordered mn–tuples $\mathbf{x}$ of intersection points between the lifts of α and β circles such that each such lift contains exactly one point of $\mathbf{x}$. (We will give a more explicit characterization of the elements of $\mathcal{S}$ later.) Let C be the $\mathbb{Z}/2$–vector space generated by $\mathcal{S}$. Define a differential ∂ on C by making the coefficient of $\mathbf{y}$ in $\partial\mathbf{x}$ nonzero if and only if the following conditions hold.

- All but two of the points in $\mathbf{x}$ are also in $\mathbf{y}$.

- There is a rectangle $R \in \widetilde{\mathcal{R}}$ whose lower-left and upper-right corners are in $\mathbf{x}$, whose upper-left and lower-right corners are in $\mathbf{y}$, and which does not contain any X, O, or point of $\mathbf{x}$ in its interior.

In Section 4, we shall define two gradings (Alexander and Maslov) on C, as well as a decomposition of C as a direct sum of complexes corresponding to spin^c structures on $\Sigma_m(K)$. We shall prove the following theorem.

Theorem 1.1 *The homology of the complex* (C, ∂) *is isomorphic as a bigraded group to* $\widehat{\mathrm{HFK}}(\Sigma_m(K), \widetilde{K}; \mathbb{Z}/2) \otimes V^{\otimes n-1}$, *where* $V \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *with generators in bigradings* $(0,0)$ *and* $(-1,-1)$.

In Section 2, we review the construction of knot Floer homology using multi-pointed Heegaard diagrams. In Section 3, we show how to obtain a Heegaard diagram for $(\Sigma_m(K), \widetilde{K})$ given one for (S^3, K), and we apply that discussion to grid diagrams in Section 4, proving Theorem 1.1. In Section 5, we give the values of $\widehat{\mathrm{HFK}}(\Sigma_m(K), \widetilde{K})$ for over fifty knots with up to eleven crossings. (Grigsby [6] has shown how to compute these groups for two-bridge knots, so our tables only include knots that are not two-bridge.) Finally, we make some observations and conjectures about these results in Section 6.

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2 Review of knot Floer homology

Let us briefly recall the basic construction of knot Floer homology using multiple basepoints (Ozsváth–Szabó [15], Manolescu–Ozsváth–Sarkar [12] and Sarkar–Wang [20]). For simplicity, we work with coefficients modulo 2. A *multi-pointed Heegaard diagram* $\mathcal{D} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ consists of an oriented surface Σ; two sets of closed, embedded, piecewise disjoint curves $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_{g+n-1}\}$ and $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_{g+n-1}\}$ (where $g = g(\Sigma)$ and $n \geq 1$), each of which spans a g–dimensional subspace of $H_1(\Sigma; \mathbb{Z})$; and two sets of basepoints, $\mathbf{w} = \{w_1, \dots, w_n\}$ and $\mathbf{z} = \{z_1, \dots, z_n\}$, such that each component of $\Sigma - \bigcup \alpha_i$ and each component of $\Sigma - \bigcup \beta_i$ contains exactly one point of $\mathbf{w}$ and one point of $\mathbf{z}$. We call the components of $\Sigma - \bigcup \alpha_i - \bigcup \beta_i$ *regions* and denote them $R_1, \dots, R_N$. The α and β curves specify a Heegaard decomposition

$H_\alpha \cup_\Sigma H_\beta$ for a 3–manifold Y, oriented so that Σ acquires its orientation as ∂H_α. We obtain a knot or link K by connecting the w (resp. z) basepoints to the z (resp. w) basepoints with arcs in the complement of the α (resp. β) curves and push those arcs into H_α (resp. H_β). The orientations are such that K intersects Σ positively at the z basepoints and negatively at the w basepoints. In terms of Morse theory, the Heegaard diagram corresponds to a self-indexing Morse function f on Y with n critical points of index 0, $g+n-1$ of index 1, $g+n-1$ of index 2, and n of index 3. Given a Riemannian metric g, the knot K is given as a union of gradient flowlines connecting the index 0 and 3 critical points through the w and z basepoints. We shall always assume that the knot K is nulhomologous.

Let $\widehat{\mathrm{CFK}}(\mathcal{D})$ be the $\mathbb{Z}/2$–vector space generated by the intersection points between the $(g+n-1)$–dimensional tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_{g+n-1}$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g+n-1}$ in the symmetric product $\mathrm{Sym}^{g+n-1}(\Sigma)$. The differential ∂ is defined by taking counts of holomorphic disks connecting intersection points:

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}) | \\ \mu(\phi)=1 \\ n_{\mathbf{w}}(\phi)=n_{\mathbf{z}}(\phi)=0}} \#(\widehat{\mathcal{M}}(\phi))\mathbf{y}.$$

Each homotopy class of Whitney disks $\phi \in \pi_2(\mathbf{x},\mathbf{y})$ has an associated *domain* in Σ: a 2–chain $D(\phi) = \sum a_i R_i$, such that ∂D is made of arcs of α curves that connect each point of $\mathbf{x}$ to a point of $\mathbf{y}$ and arcs of β curves that connect each point of $\mathbf{y}$ to a point of $\mathbf{x}$. Then $n_{\mathbf{w}}(\phi)$ (resp. $n_{\mathbf{z}}(\phi)$) is the sum of the multiplicities of the regions containing points of $\mathbf{w}$ (resp. $\mathbf{z}$). The *Maslov index* $\mu(\phi)$ can be given by formula due to Lipshitz [10]:

$$\mu(\phi) = e(D) + p_{\mathbf{x}}(\mathcal{D}) + p_{\mathbf{y}}(\mathcal{D}),$$

where $e(D)$ is the Euler measure of D and $p_{\mathbf{x}}(D)$ (resp. $p_{\mathbf{y}}(D)$) is the sum, taken over all points $x \in \mathbf{x}$ (resp. $y \in Y$), of the average of the multiplicities of the four domains that come together at x (resp. y). The coefficient of $\mathbf{y}$ represents the number of holomorphic representatives of ϕ and generally depends on the choice of almost complex structure on Σ.

Each generator $\mathbf{x}$ has an associated spin^c structure $\mathfrak{s}_{\mathbf{w}}(\mathbf{x}) \in \mathrm{Spin}^c(Y)$, obtained by considering the gradient of a compatible Morse function outside of regular neighborhoods of flowlines through the points of $\mathbf{x}$ and $\mathbf{w}$. Given two generators $\mathbf{x}$ and $\mathbf{y}$, let $\gamma_{\mathbf{x},\mathbf{y}}$ be any 1–cycle obtained by connecting $\mathbf{x}$ to $\mathbf{y}$ along the α circles and $\mathbf{y}$ to $\mathbf{x}$ along the β circles, and let $\epsilon(\mathbf{x},\mathbf{y})$ be its image in

$$H_1(Y) \cong H_1(\Sigma)/\,\mathrm{Span}([\alpha_i],[\beta_i] \mid i = 1,\ldots,g+n-1).$$

Then $\mathfrak{s}_w(\mathbf{x}) = \mathfrak{s}_w(\mathbf{y})$ if and only if $\epsilon(\mathbf{x},\mathbf{y}) = 0$. The complex $\widehat{\mathrm{CFK}}(\mathcal{D})$ splits as a direct sum over $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ of subcomplexes $\widehat{\mathrm{CFK}}(\mathcal{D},\mathfrak{s})$, each generated by those $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}_w(\mathbf{x}) = \mathfrak{s}$.

The restriction of $\mathfrak{s}_\mathbf{w}(\mathbf{x})$ to $Y - K$ extends uniquely to a spin^c structure $\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x})$ on the zero-surgery $Y_0(K)$. Given a Seifert surface F for K, we define the *Alexander grading* of $\mathbf{x}$ (relative to F) as $A(\mathbf{x}) = \frac{1}{2}\left\langle c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x})), [\widehat{F}]\right\rangle$, where $\widehat{F}$ is an extension of F to $Y_0(K)$. This quantity is in independent of the choice of F up to an additive constant, and it is completely well-defined if Y is a rational homology sphere. The relative Alexander grading between two generators $\mathbf{x}$ and $\mathbf{y}$, $A(\mathbf{x},\mathbf{y}) = A(\mathbf{x}) - A(\mathbf{y})$, can also be given as the linking number of $\gamma_{\mathbf{x},\mathbf{y}}$ and K (ie the intersection number of $\gamma_{\mathbf{x},\mathbf{y}}$ with F), or by the formula $A(\mathbf{x},\mathbf{y}) = n_\mathbf{z}(D) - n_\mathbf{w}(D)$ when $\mathbf{x}$ and $\mathbf{y}$ are in the same spin^c structure and $\mathcal{D}$ is any domain connecting $\mathbf{x}$ to $\mathbf{y}$. The latter formula shows that the complex $\widehat{\mathrm{CFK}}(\mathcal{D})$ splits according to Alexander gradings.

When Y is a rational homology sphere, the complex $\widehat{\mathrm{CFK}}(\mathcal{D})$ admits an absolute $\mathbb{Q}$–grading, the *Maslov grading*, which restricts to a relative $\mathbb{Z}$–grading on each $\widehat{\mathrm{CFK}}(\mathcal{D},\mathfrak{s})$.[1] The relative Maslov grading between two generators $\mathbf{x}$ and $\mathbf{y}$ with $\mathfrak{s}_\mathbf{w}(\mathbf{x}) = \mathfrak{s}_\mathbf{w}(\mathbf{y})$ is given by the integer $M(\mathbf{x},\mathbf{y}) = \mu(D) - 2n_\mathbf{w}(D)$, where D is any domain connecting $\mathbf{x}$ to $\mathbf{y}$. The differential lowers this grading by 1, so the grading descends to $\widehat{\mathrm{HFK}}(Y,K)$. The relative $\mathbb{Q}$–grading between generators in different spin^c structures can be computed using a formula of Lipshitz and Lee [9].

Theorem 2.1 ([15; 12; 20]) *For a suitable choice of complex structure, the homology of the complex* $(\widehat{\mathrm{CFK}}(\mathcal{D}),\partial)$ *is isomorphic to* $\widehat{\mathrm{HFK}}(Y,K) \otimes V^{\otimes n-1}$, *where* $V \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *with generators in bigradings* $(-1,-1)$ *and* $(0,0)$, *and* $\widehat{\mathrm{HFK}}(Y,K)$ *is an invariant of the knot type of* $K \subset Y$.

Call a diagram $\mathcal{D}$ *nice* if every elementary domain that does not contain a basepoint is either a bigon or a square. According to results of Manolescu–Ozsváth–Sarkar [12] and Sarkar–Wang [20], the holomorphic disks are easy to describe when $\mathcal{D}$ is nice.

Theorem 2.2 *Let* $\mathcal{D}$ *be a nice diagram, and let* $\phi \in \pi_2(\mathbf{x},\mathbf{y})$ *be a Whitney disk in* $\mathcal{D}$ *with* $\mu(\phi) = 1$. *Then* ϕ *admits a holomorphic representative if and only if* $D(\phi)$ *is either a bigon or a rectangle without any basepoint or point of* $\mathbf{x}$ *in its interior.*

It follows that when $\mathcal{D}$ is nice, the coefficients $\#\widehat{\mathcal{M}}(\phi)$ in the boundary map can be determined from the combinatorics of the diagram, without reference to the choice of complex structure on Σ, so $\widehat{\mathrm{HFK}}(Y,K)$ can be computed algorithmically.

[1] More generally, such a grading can be defined on $\widehat{\mathrm{CFK}}(\mathcal{D},\mathfrak{s})$ whenever $c_1(\mathfrak{s})$ is torsion.

If K is a knot in S^3, then a grid diagram for K, drawn on a torus as in Section 1, yields a Heegaard diagram $\mathcal{D} = (T^2, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ for the pair (S^3, K), where the α circles are the horizontal lines of the grid, the β circles are the vertical lines, and the w and z basepoints are the points marked O and X, respectively. Every region of this diagram is a rectangle, so $\widehat{\mathrm{HFK}}(S^3, K)$ can be computed combinatorially as above. Specifically, the generators correspond to permutations of the set $\{1, \dots, n\}$, and the Alexander and Maslov gradings of each generator can be given by simple formulae (discussed later). Using this diagram, Baldwin and Gillam [1] have computed $\widehat{\mathrm{HFK}}(S^3, K)$ for all knots with up to 12 crossings. Additionally, Manolescu, Ozsváth, Szabó, and Thurston [13] give a self-contained proof that this construction yields a knot invariant. (See also Sarkar and Wang [20], who show how to obtain good diagrams for knots in arbitrary 3–manifolds.)

3 Heegaard diagrams for cyclic branched covers of knots

Given a knot $K \subset S^3$ and an integer $m \geq 2$, the cover of $S^3 - K$ corresponding to the canonical homeomorphism $\pi_1(S^3 - K) \to \mathbb{Z}/m$ extends to an m–sheeted branched cover $\pi\colon \Sigma_m(K) \to S^3$, the *$m$–fold cyclic branched cover*, whose downstairs branch locus is K and whose upstairs branch locus is a knot $\widetilde{K} \subset \Sigma_m(K)$. The manifold $\Sigma_m(K)$ can be constructed explicitly from m copies of $S^3 - \operatorname{int} F$, where F is a Seifert surface for K, by connecting the negative side of a bicollar of F in the kth copy to the positive side in the $(k+1)$th (indices modulo m). The inverse image of K in $\Sigma_m(K)$ is a knot $\widetilde{K}$, which is nulhomologous because it bounds a Seifert surface (any of the lifts of the original Seifert surface F). This construction does not depend on the choice of Seifert surface. For details, see Rolfsen [19, chapters 6, 10].

The group of covering transformations of $\Sigma_m(K) \to S^3$ is cyclic of order m, generated by a map $\tau_m\colon \Sigma_m(K) \to \Sigma_m(K)$ that takes the kth copy of $S^3 - \operatorname{int} F$ to the $(k+1)$th (indices modulo m). If γ is a 1–cycle in S^3, then by using transfer homomorphisms, we see that for any lift $\widetilde{\gamma}$, the equation

$$\sum_{k=0}^{m-1} \tau_{m*}^k(\widetilde{\gamma}) = 0 \tag{1}$$

holds in $H_1(\Sigma_m(K); \mathbb{Z})$. In particular, when $m = 2$, we have $\tau_{2*}(\widetilde{\gamma}) = -\widetilde{\gamma}$.

When m is a power of a prime p, the group $H_1(\Sigma_m(K); \mathbb{Z})$ is then finite and contains no p^r–torsion for any r (Gordon [5, page 16]). The order of $H_1(\Sigma_m(K))$ is equal to $\prod_{j=0}^{m-1} \Delta_K(\omega^j)$, where Δ_K is the Alexander polynomial of K, and ω is a primitive mth root of unity (Fox [4, page 149]). In particular, note that the action of the deck transformation group on $H_1(\Sigma_m(K);\mathbb{Z})$ has no nonzero fixed points: if $\tau_{m*}(\alpha)=\alpha$, then

$$0=\alpha+\tau_{m*}(\alpha)+\cdots+\tau_{m*}^{m-1}(\alpha)=m\alpha,$$

by (1), so $\alpha=0$.

Let $\mathcal{D}=(S,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z})$ be a multi-pointed Heegaard diagram for $K\subset S^3$ with genus g and n basepoint pairs.[2] If $f\colon S^3\to\mathbb{R}$ is a self-indexing Morse function compatible with $\mathcal{D}$, then $\widetilde{f}=f\circ\pi\colon \Sigma_m(K)\to\mathbb{R}$ is a self-indexing Morse function for the pair $(\Sigma_m(K),\widetilde{K})$ whose critical points are simply the inverse images of the critical points of f. This function induces a Heegaard splitting $\Sigma_m(K)=\widetilde{H}_\alpha\cup_{\widetilde{S}}\widetilde{H}_\beta$ that projects onto the Heegaard splitting of S^3. A simple Euler characteristic argument shows that the genus of the new Heegaard surface $\widetilde{S}=\pi^{-1}(S)$ is $h=mg+(m-1)(n-1)$. Each α and β circle in S bounds a disk in S^3-K and hence has m distinct preimages in $\Sigma_m(K)$. Thus, we obtain a Heegaard diagram $\widetilde{\mathcal{D}}=(\widetilde{S},\widetilde{\boldsymbol{\alpha}},\widetilde{\boldsymbol{\beta}},\widetilde{\mathbf{w}},\widetilde{\mathbf{z}})$, where $\widetilde{S}$ is a surface of genus h and $\widetilde{\boldsymbol{\alpha}}$, $\widetilde{\boldsymbol{\beta}}$, $\widetilde{\mathbf{w}}$, and $\widetilde{\mathbf{z}}$ are the inverse images of the corresponding objects under the covering map.

The generators of the complex $\widehat{\mathrm{CFK}}(\widetilde{\mathcal{D}})$ may be described as follows.

Lemma 3.1 *Any generator $\mathbf{x}$ of $\widehat{\mathrm{CFK}}(\widetilde{\mathcal{D}})$ can be decomposed (non-uniquely) as $\mathbf{x}=\widetilde{\mathbf{x}}_1\cup\cdots\cup\widetilde{\mathbf{x}}_m$, where $\mathbf{x}_1,\dots,\mathbf{x}_m$ are generators of $\widehat{\mathrm{CFK}}(\mathcal{D})$, and $\widetilde{\mathbf{x}}_i$ is a lift of $\mathbf{x}_i$ to $\widetilde{\mathcal{D}}$.*

Proof Given a generator $\mathbf{x}$ of $\widehat{\mathrm{CFK}}(\widetilde{\mathcal{D}})$, let $G_{\mathbf{x}}$ be a graph with vertices denoted $\{a_1,\dots,a_{g+n-1},b_1,\dots,b_{g+n-1}\}$ and edges $\{e_x \mid x\in\mathbf{x}\}$, where e_x connects a_i to b_j if x is an intersection point between lifts of α_i and β_j. This is clearly a bipartite graph in which each vertex has incidence number m. By König's Theorem [3, Proposition 5.3.1], the edges of $G_{\mathbf{x}}$ can be partitioned (non-uniquely) into m perfect pairings, each of which corresponds to a lift of a generator of $\widehat{\mathrm{CFK}}(\mathcal{D})$. □

Example 3.2 As will be explained in Section 4, the diagram $\widetilde{D}$ in Figure 1 is the double branched cover of a grid diagram $\mathcal{D}$ for the right-handed trefoil in S^3. The generator $\mathbf{x}$ of $\widehat{\mathrm{CFK}}(\widetilde{D})$ indicated by the crosses can be decomposed either as lifts of the generators $\mathbf{x}_1=(20143)$ and $\mathbf{x}_2=(13240)$ or as lifts of $\mathbf{x}'_1=(23140)$ and $\mathbf{x}'_2=(10243)$ (where we identify generators of $\mathcal{D}$ with permutations of $\{0,1,2,3,4\}$ as described by Manolescu, Ozsváth, and Sarkar [12]). This provides an example of the non-uniqueness of decompositions beyond reordering of the $\mathbf{x}_i$.

[2] We denote the Heegaard surface by S rather than Σ to avoid confusion with the notation $\Sigma_m(K)$.

Given a generator $\mathbf{x}_0$ of $\widehat{\mathrm{CFK}}(\mathcal{D})$, let $L(\mathbf{x}_0)$ denote the generator of $\widehat{\mathrm{CFK}}(\widetilde{\mathcal{D}})$ consisting of all m lifts of each point of $\mathbf{x}_0$. Using the action of the deck transformation τ_m on $\mathcal{D}$, we may write $L(\mathbf{x}_0) = \widetilde{\mathbf{x}}_0 \cup \tau_m(\widetilde{\mathbf{x}}_0) \cup \cdots \cup \tau_m^{m-1}(\widetilde{\mathbf{x}}_0)$, where $\widetilde{\mathbf{x}}_0$ is any lift of $\mathbf{x}_0$ to $\widetilde{D}$.

Lemma 3.3 *All generators of $\widehat{\mathrm{CFK}}(\widetilde{\mathcal{D}})$ of the form $\mathbf{x} = L(\mathbf{x}_0)$ are in the same spin^c structure, denoted $\mathfrak{s}_0$ and called the* canonical spin^c structure *on $\Sigma_m(K)$.*

Proof (Adapted from Grigsby [7].) Let $\mathbf{x}_0$ and $\mathbf{y}_0$ be generators of $\widehat{\mathrm{CFK}}(\mathcal{D})$; we shall show that $L(\mathbf{x}_0)$ and $L(\mathbf{y}_0)$ are in the same spin^c structure. Let $\gamma_{\mathbf{x}_0,\mathbf{y}_0}$ be a 1–cycle joining $\mathbf{x}_0$ and $\mathbf{y}_0$ as in Section 2, and let $\widetilde{\gamma}_{\mathbf{x}_0,\mathbf{y}_0}$ be a lift of $\gamma_{\mathbf{x}_0,\mathbf{y}_0}$ to $\widetilde{S}$. Then the 1–cycle

$$\widetilde{\gamma}_{\mathbf{x}_0,\mathbf{y}_0} + \tau_{m*}(\widetilde{\gamma}_{\mathbf{x}_0,\mathbf{y}_0}) + \cdots + \tau_{m*}^{m-1}(\widetilde{\gamma}_{\mathbf{x}_0,\mathbf{y}_0})$$

connects $L(\mathbf{x}_0)$ and $L(\mathbf{y}_0)$. Then $\epsilon(L(\mathbf{x}_0), L(\mathbf{y}_0)) = 0$ by (1), so $L(\mathbf{x}_0)$ and $L(\mathbf{y}_0)$ are in the same spin^c structure. □

Remark 3.4 Note that the spin^c structure $\mathfrak{s}_0$ is fixed under the action of τ_m. To see this, if $f\colon S^3 \to \mathbb{R}$ is a self-indexing Morse function for (S^3, K), its pullback $\widetilde{f}\colon \Sigma_m(K) \to \mathbb{R}$ is τ_m–invariant. Using a Riemannian metric on $\Sigma_m(K)$ that is the pullback of a metric on S^3, the gradient $\vec{\nabla}\widetilde{f}$ is τ_m–invariant and projects onto $\vec{\nabla} f$, and the flowlines for $\widetilde{f}$ are precisely the lifts of flowlines for f. If N is the union of neighborhoods of flowlines through the points of $\mathbf{x}_0$ and $\mathbf{w}$, where $\mathbf{x}_0$ is a generator of $\widehat{\mathrm{CFK}}(\mathcal{D})$, then $\widetilde{N} = \pi^{-1}(N)$ is the union of neighborhoods of flowlines through the points of $L(\mathbf{x}_0)$ and $\widetilde{\mathbf{w}}$. By suitably modifying $\vec{\nabla}\widetilde{f}$ on $\widetilde{N}$, we may obtain a τ_m–invariant vector field that determines $\mathfrak{s}_{\widetilde{\mathbf{w}}}(L(\mathbf{x}_0)) = \mathfrak{s}_0$.[3] Now, if m is a prime power, then this property uniquely characterizes $\mathfrak{s}_0$, for if $\mathfrak{s}_0'$ is another spin^c structure fixed under the action of τ_m, then the difference between $\mathfrak{s}_0$ and $\mathfrak{s}_0'$ is a class in $H_1(\Sigma_m(K);\mathbb{Z})$ that is fixed by τ_m and hence equals zero. For more about the significance of $\mathfrak{s}_0$, see Grigsby, Ruberman, and Strle [8].

Proposition 3.5 *If $\mathbf{x} = \widetilde{\mathbf{x}}_1 \cup \cdots \cup \widetilde{\mathbf{x}}_m$ as in Lemma 3.1, then the Alexander grading of $\mathbf{x}$ (computed with respect to a lift of a Seifert surface for K) is equal to the average of the Alexander gradings of $\mathbf{x}_1, \dots, \mathbf{x}_m$.*[4] *In particular, for any generator $\mathbf{x}_0$ of $\widehat{\mathrm{CFK}}(\mathcal{D})$, we have $A(\mathbf{x}_0) = A(L(\mathbf{x}_0))$.*

[3] In general, spin^c structures can always be pulled back under a local diffeomorphism using the vector field interpretation. Specifically, if $F\colon M \to N$ is a local diffeomorphism and ξ is a nonvanishing vector field on N that determines a given spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(N)$, then $F^*(\mathfrak{s}) \in \mathrm{Spin}^c(M)$ is determined by the vector field $(F_*)^{-1}(\xi)$. The first Chern class is natural under this pullback.

[4] Note that we have specified a Seifert surface in order to define the Alexander grading. When m is a prime power, however, $\Sigma_m(K)$ is a rational homology sphere, so the Alexander grading does not depend at all on the choice of Seifert surface.

Proof We first consider the relative Alexander gradings. Let $F \subset S^3$ be a Seifert surface for K, and let $\widetilde{F}$ be a lift of F to $\Sigma_m(K)$. The translates $\widetilde{F}, \tau_m(\widetilde{F}), \dots, \tau_m^{m-1}(\widetilde{F})$ are all Seifert surfaces for $\widetilde{K}$. The relative Alexander grading between two generators does not depend on the choice of Seifert surface, so for generators $\mathbf{x}, \mathbf{y}$ of $\widehat{\mathrm{CFK}}(\widetilde{D})$, we have

$$mA(\mathbf{x}, \mathbf{y}) = \gamma_{\mathbf{x},\mathbf{y}} \cdot \widetilde{F} + \gamma_{\mathbf{x},\mathbf{y}} \cdot \tau_m(\widetilde{F}) + \cdots + \gamma_{\mathbf{x},\mathbf{y}} \cdot \tau_m^{m-1}(\widetilde{F}),$$

where $\gamma_{\mathbf{x},\mathbf{y}}$ is a 1–cycle connecting $\mathbf{x}$ and $\mathbf{y}$ as above. The projection $\pi_*(\gamma_{\mathbf{x},\mathbf{y}})$ is a 1–cycle in S that goes from points of $\pi(\mathbf{x})$ to points of $\pi(\mathbf{y})$ along α circles and from points of $\pi(\mathbf{y})$ to points of $\pi(\overline{\mathbf{x}})$ along β circles. Every intersection point of $\gamma_{\mathbf{x},\mathbf{y}}$ with one of the lifts of F corresponds to an intersection point of $\pi_*(\gamma_{\mathbf{x},\mathbf{y}})$ with F, so

$$\gamma_{\mathbf{x},\mathbf{y}} \cdot \widetilde{F} + \gamma_{\mathbf{x},\mathbf{y}} \cdot \tau_m(\widetilde{F}) + \cdots + \gamma_{\mathbf{x},\mathbf{y}} \cdot \tau_m^{m-1}(\widetilde{F}) = \pi_*(\gamma_{\mathbf{x},\mathbf{y}}) \cdot F.$$

The restriction of $\pi_*(\gamma_{\mathbf{x},\mathbf{y}})$ to any α or β circle consists of m (possibly constant or overlapping) arcs. By perhaps adding copies of the α or β circle, we can arrange that these arcs connect a point of $\mathbf{x}_1$ with a point of $\mathbf{y}_1$, a point of $\mathbf{x}_2$ with a point of $\mathbf{y}_2$, and so on. In other words,

$$\pi_*(\gamma_{\mathbf{x},\mathbf{y}}) \equiv \gamma_{\mathbf{x}_1,\mathbf{y}_1} + \cdots + \gamma_{\mathbf{x}_m,\mathbf{y}_m}$$

modulo the α and β circles in $\mathcal{D}$, whose intersection numbers with F are zero. We have:

$$\begin{aligned} A(\mathbf{x}, \mathbf{y}) &= \frac{1}{m}(\gamma_{\mathbf{x}_1,\mathbf{y}_1} + \dots, + \gamma_{\mathbf{x}_m,\mathbf{y}_m}) \cdot F \\ &= \frac{1}{m}(A(\mathbf{x}_1, \mathbf{y}_1) + \cdots + A(\mathbf{x}_m, \mathbf{y}_m)). \end{aligned}$$

Thus, the Alexander grading of a generator of $\widehat{\mathrm{CFK}}(\widetilde{\mathcal{D}})$ is given up to an additive constant by the average Alexander grading of its parts.

To pin down the additive constant, note that the branched covering map $\pi\colon \Sigma_m(K) \to S^3$ extends to an *unbranched* covering map from the zero-surgery on $\widetilde{K}$ to the zero-surgery on K, $\pi_0\colon Y_0(\widetilde{K}) \to S^3_0(K)$. Since this is a local diffeomorphism, it is possible to pull back spin^c structures. Let $\mathbf{x}_0$ be a generator of $\widehat{\mathrm{CFK}}(\mathcal{D})$ in Alexander grading 0, and let $\mathbf{x} = L(\mathbf{x}_0)$. (The symmetry $\widehat{\mathrm{HFK}}(S^3, K, i) \cong \widehat{\mathrm{HFK}}(S^3, K, -i)$ and the fact that rank $\widehat{\mathrm{HFK}}(S^3, K) \equiv \det(K) \equiv 1 \pmod 2$ [15] imply that such $\widehat{\mathrm{HFK}}(S^3, K, 0)$ has odd rank, so such a generator $\mathbf{x}_0$ always exists.) As in the discussion following Lemma 3.3, we may find a nonvanishing vector field that determines $\mathfrak{s}_{\widetilde{\mathbf{w}}}(\mathbf{x}) = \mathfrak{s}_0$ and is τ_m–equivariant. The unique extension (up to isotopy) of this vector field to $\Sigma_m(K)_0$ can also be made τ_m–invariant, so it is the pullback of an extension to S^3_0 of a vector field determining $\mathfrak{s}_{\mathbf{w}}(\mathbf{x}_0)$. It follows that $\underline{\mathfrak{s}}_{\widetilde{\mathbf{w}},\widetilde{\mathbf{z}}}(\mathbf{x}) = \pi_0^*(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x}_0))$. Now, if $\widehat{\widetilde{F}} \subset Y_0(\widetilde{K})$

is obtained by capping off $\widetilde{F}$ in the zero-surgery, then $\pi_{0*}[\widehat{\widetilde{F}}] = [\widehat{F}]$ in $H_2(S^0_3;\mathbb{Z})$. We therefore have:

$$\begin{aligned} A(\mathbf{x}) &= \frac{1}{2}\left\langle c_1(\underline{\mathfrak{s}}_{\widetilde{\mathbf{w}},\widetilde{\mathbf{z}}}(\mathbf{x})), [\widehat{\widetilde{F}}]\right\rangle \\ &= \frac{1}{2}\left\langle c_1(\pi_0^*(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x}_0))), [\widehat{\widetilde{F}}]\right\rangle \\ &= \frac{1}{2}\left\langle c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x}_0)), \pi_{0*}[\widehat{\widetilde{F}}]\right\rangle \\ &= \frac{1}{2}\left\langle c_1(\underline{\mathfrak{s}}_{\mathbf{w},\mathbf{z}}(\mathbf{x}_0)), [\widehat{F}]\right\rangle \\ &= 0 = A(\mathbf{x}_0). \end{aligned}$$

Thus, the additive constant C must equal 0. □

Remark 3.6 When K is a two-bridge knot and $m = 2$, Grigsby [7] shows that for a specific diagram $\mathcal{D}$, the map L is surjective and preserves the relative Maslov grading. Therefore, for any two-bridge knot K, $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K}, \mathfrak{s}_0) \cong \widehat{\mathrm{HFK}}(S^3, K)$, up to a possible shift in the absolute Maslov grading. It may be possible to extend this result to a wider class of knots, such as alternating knots. However, in general L is neither surjective nor Maslov-grading-preserving.

Finally, we consider the regions in $\widetilde{D}$. First, note that the preimage of any region R in $\mathcal{D}$ consists of either m distinct regions, each of which is projected diffeomorphically onto R, or a single region. (In the former case, we say that R is *evenly covered*.) In particular, when $\mathcal{D}$ is nice, each region of $\mathcal{D}$ that does not contain a basepoint is a simply-connected polygon that misses the branch set, so it is evenly covered. Thus, we obtain the following proposition.

Proposition 3.7 *Let $\mathcal{D}$ be a nice Heegaard diagram for (S^3, K), and let $\widetilde{D}$ be its m–fold cyclic branched cover. Then $\widetilde{\mathcal{D}}$ is nice.*

4 Grid diagrams and cyclic branched covers

Proof of Theorem 1.1 As described in Section 1, any oriented knot $K \subset S^3$ can be represented by means of a grid diagram. By drawing the grid diagram on a standardly embedded torus in S^3, we may think of the grid diagram as a genus 1, multi-pointed Heegaard diagram $\mathcal{D} = (T^2, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ for the pair (S^3, K), where the α circles are the horizontal lines of the grid, the β circles are the vertical lines, the w basepoints are in the regions marked O, and the z basepoints are in the regions marked X.

Note that the diagram $\mathcal{D}$ is nice, so the differential can be computed combinatorially as described in Section 2. Specifically, the coefficient of $\mathbf{y}$ in $\partial\mathbf{x}$ is 1 if all but two of the points of $\mathbf{x}$ and $\mathbf{y}$ agree and there is a rectangle embedded in the torus with points of $\mathbf{x}$ as its lower-left and upper-right corners, points of $\mathbf{y}$ as its lower-right and upper-left corners, and no basepoints or points of $\mathbf{x}$ in its interior, and 0 otherwise. Note that there cannot be two such rectangles, or else K would be a split link.

A Seifert surface for K may be seen as follows. We may isotope K to lie entirely within H_α by letting the arcs of $K \cap H_\beta$ fall onto the boundary torus. In fact, it lies within a ball contained in H_α since the knot projection in the grid diagram never passes through the left edge of the grid. Take a Seifert surface F contained in this ball, and then isotope F and K so that K returns to its original position. F then intersects the Heegaard surface T^2 in n arcs, one connecting the two basepoints in each column of the grid diagram, and it intersects H_β in strips that lie above these arcs. The orientations of K and S^3 imply that the positive side of a bicollar for F lies on the *right* of one of these strips when the X is above the O and on the *left* when the O is above the X.

If we construct $\Sigma_m(K)$ by gluing together m copies of $S^3 - \operatorname{int} F$ as in Section 3, the Heegaard surfaces in each copy are connected exactly to each other as described in Section 1 to form a surface $\widetilde{T}$. Hence, $\widetilde{\mathcal{D}} = (\widetilde{T}, \widetilde{\boldsymbol\alpha}, \widetilde{\boldsymbol\beta}, \widetilde{\mathbf{w}}, \widetilde{\mathbf{z}})$ is a Heegaard diagram for $(\Sigma_m(K), \widetilde{K})$ for which the results of Section 3 apply. In particular, it is a nice Heegaard diagram.

It remains to show that the domains that count for the differential in $\widehat{\mathrm{CFK}}(\widetilde{D})$ are precisely the lifts of those that count for the differential in $\widehat{\mathrm{CFK}}(\mathcal{D})$, as was asserted in Section 1. Since $\widetilde{D}$ is a nice diagram with no bigons, any domain that counts for the differential is an embedded rectangle R. The projection of R to $\mathcal{D}$, $\pi(R)$, is an immersed rectangle in $\widetilde{D}$ whose edges are contained in at most two α circles and two β circles. By lifting $\pi(R)$ to the universal cover of T^2, we see that $\pi(R)$ cannot intersect any α or β circle more than once, or else it would contain an entire column or row of the grid diagram and hence a basepoint. Therefore, $\pi(R)$ is an embedded rectangle that misses the basepoints, so it counts for the differential of $\widehat{\mathrm{CFK}}(\mathcal{D})$. □

We shall now give a more explicit description of the generators of $\widehat{\mathrm{CFK}}(\widetilde{D})$ and their gradings in order to facilitate computation.

In the grid diagram $\mathcal{D}$, we label the α circles $\alpha_0, \dots, \alpha_{n-1}$ from bottom to top and the β circles $\beta_0, \dots, \beta_{n-1}$ from left to right. Each α circle intersects each β circle exactly once: $\beta_i \cap \alpha_j = \{x_{ij}\}$. Generators of $\widehat{\mathrm{CFK}}(\mathcal{D})$ then correspond to permutations of the index set $\{0, \dots, n-1\}$ via the correspondence $\sigma \mapsto (x_{0,\sigma(0)}, \dots, x_{n-1,\sigma(n-1)})$.

For each grid point x, let $w(x)$ denote the winding number of the knot projection around x. Let $p_1, \dots, p_{8n}$ (repetitions allowed) denote the vertices of the $2n$ squares containing basepoints, and set

$$a = \frac{1-n}{2} + \frac{1}{8}\sum_{i=1}^{8n} w(p_i).$$

According to Manolescu, Ozsváth, and Sarkar [12], the Alexander grading of a generator $\mathbf{x}$ of $\widehat{\mathrm{CFK}}(\mathcal{D})$ is given by the formula

$$A(\mathbf{x}) = a - \sum_{x \in \mathbf{x}} w(x). \tag{2}$$

There is also a formula for the Maslov grading of a generator, but it is not relevant for our purposes.

The generators of $\widehat{\mathrm{CFK}}(\widetilde{\mathcal{D}})$ can be described easily as follows. For any $i = 0, \dots, n-1$ and $j = 0, \dots, n-1$, each lift of β_i meets exactly one lift of α_j. Specifically, let $\widetilde{\beta}_j^k$ denote the lift of β_j on the kth copy of $\mathcal{D}$ (for $k = 0, \dots, m-1$). Let $\widetilde{\alpha}_j^k$ denote the lift of α_j that intersects the leftmost edge of the kth grid diagram ($\widetilde{\beta}_0^k$). Let $\widetilde{x}_{i,j}^k$ denote the lift of $x_{i,j}$ on the kth diagram. Define a map $g\colon \mathbb{Z}/n \times \mathbb{Z}/n \times \mathbb{Z}/m \to \mathbb{Z}/m$ by $g(i,j,k) = k - w(x_{i,j}) \bmod m$. The lift of α_j that meets a particular $\widetilde{\beta}_i^k$ is given by the following lemma.

Lemma 4.1 *The point $\widetilde{x}_{i,j}^k$ is the intersection between $\widetilde{\beta}_i^k$ and $\widetilde{\alpha}_j^{g(i,j,k)}$.*

Proof We induct on i. For $i = 0$, we have $w(x_{0,j}) = 0$, and by construction $\widetilde{\alpha}_j^k$ meets $\widetilde{\beta}_0^k$. For the induction step, let S be the segment of α_j from $x_{i,j}$ to $x_{i+1,j}$. Note that $w(x_{i+1,j})$ is equal to $w(x_{i,j})+1$ if S passes below the X and above the O in its column, $w(x_{i,j})-1$ if it passes above X and below O, and $w(x_{i,j})$ otherwise. Similarly, if $\widetilde{x}_{i,j}^k$ lies on $\widetilde{\alpha}_j^l$, then by the previous discussion, $\widetilde{x}_{i+1,j}^k$ lies on $\widetilde{\alpha}_j^{l-1}$ in the first case, on $\widetilde{\alpha}_j^{l+1}$ in the second, and on $\widetilde{\alpha}_j^l$ in the third (upper indices modulo m). This proves the induction step. □

We may then identify the generators of $\widehat{\mathrm{CFK}}(\widetilde{\mathcal{D}})$ with the set of m–to-one maps

$$\phi\colon \{0, \dots, n-1\} \times \{0, \dots, m-1\} \to \{0, \dots, n-1\}$$

such that for each $j = 0, \dots, n-1$, the function $g(\cdot, j, \cdot)$ assumes all m possible values on $\phi^{-1}(j)$. In other words, if we shade the m lifts of each α circle with different colors as in Figure 1 and arrange the copies of $\mathcal{D}$ horizontally, a generator is a selection of mn grid points so each column contains one point and each row contains m points, one of each color. It is not difficult to enumerate such maps algorithmically.

To split up the generators of $\widehat{\mathrm{CFK}}(\widetilde{\mathcal{D}})$ according to spin^c structures, we simply need to express $\epsilon(\mathbf{x},\mathbf{y})$ in terms of a $\mathbb{Z}$–module presentation for $H_1(\Sigma_m(K);\mathbb{Z})$. We obtain such a presentation from the Heegaard decomposition of $\Sigma_m(K)$: the generators a_j^k $(0 \le j \le n-1,\ 0 \le k \le m-1)$ corresponding to the 1–handles dual to the α circles and relations corresponding to the 2–handles spanned by the β circles. By Lemma 4.1, the relations are

$$0 = [\widetilde{\beta}_i^k] = \sum_{j=1}^{n} a_j^{g(i,j,k)} \quad (0 \le i \le n-1, 0 \le k \le m-1).$$

To express $\epsilon(\mathbf{x},\mathbf{y})$ in terms of this basis, one simply counts the number of times that a representative $\gamma_{\mathbf{x},\mathbf{y}}$ crosses the α circles.

To compute the Alexander grading of a generator $\mathbf{x}$, we decompose it as $\mathbf{x} = \widetilde{\mathbf{x}}_1 \cup \cdots \cup \widetilde{\mathbf{x}}_m$ using Lemma 3.1 and then use Proposition 3.5 and (2) to write:

$$\begin{aligned} A(\mathbf{x}) &= \frac{1}{m}(A(\mathbf{x}_1) + \cdots + A(\mathbf{x}_m)) \\ &= \frac{1}{m}\sum_{k=1}^{m}\left(a - \sum_{x \in \mathbf{x}_k} w(x)\right) \\ &= a - \frac{1}{m}\sum_{k=1}^{m}\sum_{x \in \widetilde{\mathbf{x}}_k} w(\pi(x)) \\ &= a - \frac{1}{m}\sum_{x \in \mathbf{x}} w(\pi(x)). \end{aligned}$$

Computing the relative Maslov grading between two generators in the same spin^c structure requires finding a domain D connecting them, which is simply a matter of linear algebra, and then using the formula $M(\mathbf{x}) - M(\mathbf{y}) = \mu(D) - 2n_{\mathbf{w}}(D)$. The relative Maslov grading between generators in different spin^c structures can be computed similarly using the formula of Lee and Lipshitz [9]. Since all the basepoints in the Heegaard diagrams used in this paper are contained in $4m$–gonal regions, it is not possible to compute the absolute Maslov gradings or the spectral sequence from $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K})$ to $\widehat{\mathrm{HF}}(\Sigma_2(K))$ combinatorially. However, when $m = 2$, the groups $\widehat{\mathrm{HF}}(\Sigma_2(K))$, or at least the correction terms $d(\Sigma_2(K), \mathfrak{s})$, can in many instances be computed via other means (Ozsváth and Szabó [16]). In such cases, it is often possible to pin down the absolute Maslov gradings for $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K})$. Specifically, the relative Maslov $\mathbb{Q}$–grading and the action of $H_1(\Sigma_2(K))$ on $\mathrm{Spin}^c(\Sigma_2(K))$ usually provide enough information to match the groups $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K}, \mathfrak{s})$ up with the rational numbers $d(\Sigma_2(K), \mathfrak{s})$ that are computed via some other means. If there is a spin^c structure $\mathfrak{s}$ in which $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K}, \mathfrak{s})$ has rank 1, then the absolute Maslov grading of that group equals the corresponding d invariant, and the rest of the absolute gradings are completely determined.

5 Results

The following tables list the ranks for $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K}; \mathbb{Z}/2)$ by means of the Poincaré polynomials:

$$p_{\mathfrak{s}}(q,t) = \sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\mathrm{HFK}}_j(\Sigma_2(K), \widetilde{K}, \mathfrak{s}, i; \mathbb{Z}/2) t^i q^j .$$

The Maslov $\mathbb{Q}$–gradings are normalized so that in the canonical spin^c structure $\mathfrak{s}_0$, the nonzero elements in Alexander grading $g(K)$ have Maslov grading $g(K)$. For each knot, the first line gives $p_{\mathfrak{s}_0}(q,t)$, and each subsequent line gives $p_{\mathfrak{s}}(q,t)$ for a pair of conjugate spin^c structures. We identify spin^c structures with elements of $H_1(\Sigma_2(K);\mathbb{Z})$, which is either a cyclic group or the sum of two cyclic groups, taking $\mathfrak{s}_0$ to 0. (Of course, the choice of basis for $H_1(\Sigma_2(K);\mathbb{Z})$ is not canonical.) In each spin^c structure, most of the nonzero groups lie along a single diagonal; the terms corresponding to the groups not on that diagonal are underlined.

These results were computed using a program written in C++ and *Mathematica*, based on Baldwin and Gillam's program [1] for computing $\widehat{\mathrm{HFK}}(S^3, K)$. Most of the grid diagrams were obtained using Marc Culler's program *Gridlink* [2]. Using available computer resources, it was possible to compute $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K})$ for all the three-bridge knots with up to eleven crossings and arc index ≤ 9, and for many knots with arc index 10. (Grigsby [6] has a much more efficient algorithm for computing $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K})$ when K is two-bridge, so we do not list those knots here.)

K	$H_1(\Sigma_2(K);\mathbb{Z})$	$\mathfrak{s}$	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\mathrm{HFK}}_j(\Sigma_2(K), \widetilde{K}, \mathfrak{s}, i; \mathbb{Z}/2) t^i q^j$
8_5	$\mathbb{Z}/21$	0	$q^{-3}t^{-3} + 3q^{-2}t^{-2} + 4q^{-1}t^{-1} + 5 + 4qt + 3q^2t^2 + q^3t^3$
		±1	$q^{5/21}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
		±2	$q^{20/21}$
		±3	$q^{8/7}$
		±4	$q^{17/21}(q^{-1}t^{-1} + 1 + qt)$
		±5	$q^{20/21}$
		±6	$q^{4/7}$
		±7	$q^{2/3}(q^{-1}t^{-1} + 3 + qt)$
		±8	$q^{5/21}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
		±9	$q^{2/7}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2)$
		±10	$q^{17/21}(q^{-1}t^{-1} + 1 + qt)$

K	$H_1(\Sigma_2(K);\mathbb{Z})$	$\mathfrak{s}$	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\mathrm{HFK}}_j(\Sigma_2(K),\widetilde{K},\mathfrak{s},i;\mathbb{Z}/2)t^i q^j$
8_{10}	$\mathbb{Z}/27$	0	$q^{-3}t^{-3}+3q^{-2}t^{-2}+6q^{-1}t^{-1}+7+6qt+3q^2t^2+q^3t^3$
		± 1	$q^{7/27}(q^{-2}t^{-2}+3q^{-1}t^{-1}+5+3qt+q^2t^2)$
		± 2	$q^{1/27}$
		± 3	$q^{1/3}$
		± 4	$q^{4/27}(q^{-1}t^{-1}+1+qt)$
		± 5	$q^{13/27}$
		± 6	$q^{1/3}$
		± 7	$q^{-8/27}(q^{-1}t^{-1}+1+qt)$
		± 8	$q^{-11/27}(q^{-2}t^{-2}+q^{-1}t^{-1}+1+qt+q^2t^2)$
		± 9	$q^{-1}t^{-1}+1+qt$
		± 10	$q^{25/27}$
		± 11	$q^{10/27}(2q^{-1}t^{-1}+3+2qt)$
		± 12	$q^{1/3}(q^{-2}t^{-2}+2q^{-1}t^{-1}+3+2qt+q^2t^2)$
		± 13	$q^{22/27}(q^{-1}t^{-1}+1+qt)$
8_{15}	$\mathbb{Z}/33$	0	$3q^{-2}t^{-2}+8q^{-1}t^{-1}+11+8qt+3q^2t^2$
		± 1	$q^{13/33}(2q^{-1}t^{-1}+3+2qt)$
		± 2	$q^{-14/33}(q^{-2}t^{-2}+q^{-1}t^{-1}+1+qt+q^2t^2)$
		± 3	$q^{6/11}$
		± 4	$q^{10/33}$
		± 5	$q^{-5/33}(q^{-1}t^{-1}+1+qt)$
		± 6	$q^{2/11}$
		± 7	$q^{10/33}$
		± 8	$q^{7/33}(q^{-1}t^{-1}+1+qt)$
		± 9	$q^{10/11}$
		± 10	$q^{13/33}(2q^{-1}t^{-1}+3+2qt)$
		± 11	$q^{2/3}$
		± 12	$q^{-3/11}(q^{-1}t^{-1}+1+qt)$
		± 13	$q^{-14/33}(q^{-2}t^{-2}+q^{-1}t^{-1}+1+qt+q^2t^2)$
		± 14	$q^{7/33}(q^{-1}t^{-1}+1+qt)$
		± 15	$q^{-4/11}(q^{-2}t^{-2}+2q^{-1}t^{-1}+3+2qt+q^2t^2)$
		± 16	$q^{-5/33}(q^{-1}t^{-1}+1+qt)$
8_{16}	$\mathbb{Z}/35$	0	$q^{-3}t^{-3}+4q^{-2}t^{-2}+8q^{-1}t^{-1}+9+8qt+4q^2t^2+q^3t^3$
		± 1	$q^{16/35}(q^{-1}t^{-1}+1+qt)$
		± 2	$q^{29/35}$
		± 3	$q^{4/35}(q^{-1}t^{-1}+1+qt)$
		± 4	$q^{11/35}(q^{-2}t^{-2}+3q^{-1}t^{-1}+5+3qt+q^2t^2)$
		± 5	$q^{3/7}(q^{-1}t^{-1}+3+qt)$
		± 6	$q^{16/35}(q^{-1}t^{-1}+1+qt)$
		± 7	$q^{2/5}(2q^{-1}t^{-1}+3+2qt)$
		± 8	$q^{9/35}$
		± 9	$q^{1/35}$
		± 10	$q^{5/7}$
		± 11	$q^{11/35}(q^{-2}t^{-2}+3q^{-1}t^{-1}+5+3qt+q^2t^2)$
		± 12	$q^{29/35}$
		± 13	$q^{9/35}$
		± 14	$q^{3/5}$
		± 15	$q^{6/7}(q^{-1}t^{-1}+1+qt)$
		± 16	$q^{1/35}$
		± 17	$q^{4/35}(q^{-1}t^{-1}+1+qt)$

K	$H_1(\Sigma_2(K);\mathbb{Z})$	$\mathfrak{s}$	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\mathrm{HFK}}_j(\Sigma_2(K), \widetilde{K}, \mathfrak{s}, i; \mathbb{Z}/2) t^i q^j$
8_{17}	$\mathbb{Z}/37$	0	$q^{-3}t^{-3} + 4q^{-2}t^{-2} + 8q^{-1}t^{-1} + 11 + 8qt + 4q^2t^2 + q^3t^3$
		± 1	$q^{2/37}$
		± 2	$q^{8/37}(q^{-1}t^{-1} + 3 + qt)$
		± 3	$q^{18/37}$
		± 4	$q^{-5/37}(q^{-1}t^{-1} + 1 + qt)$
		± 5	$q^{13/37}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
		± 6	$q^{-2/37}$
		± 7	$q^{-13/37}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
		± 8	$q^{17/37}(q^{-1}t^{-1} + 1 + qt)$
		± 9	$q^{14/37}$
		± 10	$q^{-22/37}$
		± 11	$q^{-17/37}(q^{-1}t^{-1} + 1 + qt)$
		± 12	$q^{-8/37}(q^{-1}t^{-1} + 3 + qt)$
		± 13	$q^{5/37}(q^{-1}t^{-1} + 1 + qt)$
		± 14	$q^{22/37}$
		± 15	$q^{6/37}$
		± 16	$q^{-6/37}$
		± 17	$q^{-14/37}$
		± 18	$q^{-18/37}$
8_{18}	$\mathbb{Z}/3 \oplus \mathbb{Z}/15$	$(0,0)$	$q^{-3}t^{-3} + 5q^{-2}t^{-2} + 10q^{-1}t^{-1} + 13 + 10qt + 5q^2t^2 + q^3t^3$
		$\pm(0,1)$	$q^{7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(0,2)$	$q^{-2/15}$
		$\pm(0,3)$	$q^{1/5}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(0,4)$	$q^{7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(0,5)$	$q^{-2/3}$
		$\pm(0,6)$	$q^{-1/5}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(0,7)$	$q^{-2/15}$
		$\pm(1,0)$	$q^{-2/3}$
		$\pm(1,1)$	$q^{7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1,2)$	$q^{-7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1,3)$	$q^{-7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1,4)$	$q^{7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1,5)$	$q^{-2/3}$
		$\pm(1,6)$	$q^{2/15}$
		$\pm(1,7)$	$q^{-2/15}$
		$\pm(1,8)$	$q^{-7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1,9)$	$q^{2/15}$
		$\pm(1,10)$	$q^{2/3}$
		$\pm(1,11)$	$q^{2/15}$
		$\pm(1,12)$	$q^{-7/15}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1,13)$	$q^{-2/15}$
		$\pm(1,14)$	$q^{2/15}$
8_{19}	$\mathbb{Z}/3$	0	$q^{-3}t^{-3} + q^{-2}t^{-2} + q + q^2t^2 + q^3t^3$
		± 1	$q^{2/3}(q^{-1}t^{-1} + 1 + qt)$
8_{20}	$\mathbb{Z}/9$	0	$q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2$
		± 1	$q^{7/9}(q^{-1}t^{-1} + 1 + qt)$
		± 2	$q^{1/9}(q^{-1}t^{-1} + 1 + qt)$
		± 3	1
		± 4	$q^{4/9}$

K	$H_1(\Sigma_2(K);\mathbb{Z})$	$\mathfrak{s}$	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\mathrm{HFK}}_j(\Sigma_2(K), \widetilde{K}, \mathfrak{s}, i; \mathbb{Z}/2) t^i q^j$
8_{21}	$\mathbb{Z}/15$	1	$q^{-2}t^{-2}+4q^{-1}t^{-1}+5+4qt+q^2t^2$
		± 1	$q^{-2/15}(q^{-1}t^{-1}+1+qt)$
		± 2	$q^{7/15}$
		± 3	$q^{-1/5}(q^{-1}t^{-1}+3+qt)$
		± 4	$q^{-2/15}(q^{-1}t^{-1}+1+qt)$
		± 5	$q^{-1/3}$
		± 6	$q^{1/5}$
		± 7	$q^{7/15}$
9_{42}	$\mathbb{Z}/7$	0	$q^{-2}t^{-2}+2q^{-1}t^{-1}+2+\underline{q}+2qt+q^2t^2$
		± 1	$q^{3/7}$
		± 2	$q^{5/7}(q^{-1}t^{-1}+3+qt)$
		± 3	$q^{6/7}(q^{-1}t^{-1}+1+qt)$
9_{43}	$\mathbb{Z}/13$	0	$q^{-3}t^{-3}+3q^{-2}t^{-2}+2q^{-1}t^{-1}+1+2qt+3q^2t^2+q^3t^3$
		± 1	$q^{10/13}(q^{-1}t^{-1}+3+qt)$
		± 2	$q^{1/13}(q^{-1}t^{-1}+1+qt)$
		± 3	$q^{12/13}$
		± 4	$q^{4/13}(q^{-2}t^{-2}+q^{-1}t^{-1}+1+qt+q^2t^2)$
		± 5	$q^{16/13}$
		± 6	$q^{9/13}(q^{-1}t^{-1}+1+qt)$
9_{44}	$\mathbb{Z}/17$	0	$q^{-2}t^{-2}+4q^{-1}t^{-1}+7+4qt+q^2t^2$
		± 1	$q^{-8/17}$
		± 2	$q^{-15/17}(q^{-1}t^{-1}+1+qt)$
		± 3	$q^{-4/17}$
		± 4	$q^{8/17}$
		± 5	$q^{4/17}$
		± 6	$q^{-16/17}$
		± 7	$q^{-1/17}(q^{-1}t^{-1}+1+qt)$
		± 8	$q^{-2/17}(q^{-1}t^{-1}+3+qt)$
9_{45}	$\mathbb{Z}/23$	0	$q^{-2}t^{-2}+6q^{-1}t^{-1}+9+6qt+q^2t^2$
		± 1	$q^{-8/23}(2q^{-1}t^{-1}+3+2qt)$
		± 2	$q^{-9/23}$
		± 3	$q^{-3/23}(q^{-1}t^{-1}+3+qt)$
		± 4	$q^{-13/23}$
		± 5	$q^{7/23}$
		± 6	$q^{11/23}$
		± 7	$q^{-1/23}$
		± 8	$q^{-6/23}(q^{-1}t^{-1}+1+qt)$
		± 9	$q^{-4/23}(2q^{-1}t^{-1}+3+2qt)$
		± 10	$q^{-18/23}(q^{-1}t^{-1}+1+qt)$
		± 11	$q^{-2/23}(q^{-1}t^{-1}+1+qt)$
9_{46}	$\mathbb{Z}/3 \oplus \mathbb{Z}/3$	$(0,0)$	$2q^{-1}t^{-1}+5+2qt$
		$\pm(0,1)$	$q^{-2/3}(q^{-1}t^{-1}+3+qt)$
		$\pm(1,0)$	1
		$\pm(1,1)$	1
		$\pm(1,2)$	$q^{-4/3}$

K	$H_1(\Sigma_2(K);\mathbb{Z})$	$\mathfrak{s}$	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\mathrm{HFK}}_j(\Sigma_2(K), \widetilde{K}, \mathfrak{s}, i; \mathbb{Z}/2) t^i q^j$
9_{47}	$\mathbb{Z}/3 \oplus \mathbb{Z}/9$	$(0,0)$	$q^{-3}t^{-3} + 4q^{-2}t^{-2} + 6q^{-1}t^{-1} + 5 + 6qt + 4q^2t^2 + q^3t^3$
		$\pm(0,1)$	$q^{-1/9}(q^{-1}t^{-1} + 3 + qt)$
		$\pm(0,2)$	$q^{-4/9}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(0,3)$	$q^{-1}t^{-1} + 1 + qt$
		$\pm(0,4)$	$q^{-7/9}$
		$\pm(1,0)$	$q^{-1/3}$
		$\pm(1,1)$	$q^{-1/9}(q^{-1}t^{-1} + 3 + qt)$
		$\pm(1,2)$	$q^{-1/9}(q^{-1}t^{-1} + 3 + qt)$
		$\pm(1,3)$	$q^{-1/3}$
		$\pm(1,4)$	$q^{-7/9}$
		$\pm(1,5)$	$q^{-4/9}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1,6)$	$q^{-1/3}$
		$\pm(1,7)$	$q^{-4/9}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1,8)$	$q^{-7/9}$
9_{48}	$\mathbb{Z}/3 \oplus \mathbb{Z}/9$	$(0,0)$	$q^{-2}t^{-2} + 7q^{-1}t^{-1} + 11 + 7qt + q^2t^2$
		$\pm(0,1)$	$q^{-4/9}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(0,2)$	$q^{2/9}(2q^{-1}t^{-1} + 3 + 2qt)$
		$\pm(0,3)$	$q^{-1}t^{-1} + 1 + qt$
		$\pm(0,4)$	$q^{-1/9}$
		$\pm(1,0)$	$q^{1/3}$
		$\pm(1,1)$	$q^{2/9}(2q^{-1}t^{-1} + 3 + 2qt)$
		$\pm(1,2)$	$q^{2/9}(2q^{-1}t^{-1} + 3 + 2qt)$
		$\pm(1,3)$	$q^{1/3}$
		$\pm(1,4)$	$q^{-4/9}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1,5)$	$q^{-1/9}$
		$\pm(1,6)$	$q^{1/3}$
		$\pm(1,7)$	$q^{-1/9}$
		$\pm(1,8)$	$q^{-4/9}(q^{-1}t^{-1} + 1 + qt)$
9_{49}	$\mathbb{Z}/5 \oplus \mathbb{Z}/5$	$(0,0)$	$3q^{-2}t^{-2} + 6q^{-1}t^{-1} + 7 + 6qt + 3q^2t^2$
		$\pm(0,1)$	$q^{-2/5}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		$\pm(0,2)$	$q^{2/5}$
		$\pm(1,0)$	$q^{-2/5}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		$\pm(1,1)$	$q^{-1/5}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(1,2)$	$q^{1/5}(2q^{-1}t^{-1} + 3 + 2qt)$
		$\pm(1,3)$	$q^{1/5}(2q^{-1}t^{-1} + 3 + 2qt)$
		$\pm(1,4)$	$q^{-2/5}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		$\pm(2,0)$	$q^{2/5}$
		$\pm(2,1)$	$q^{1/5}(2q^{-1}t^{-1} + 3 + 2qt)$
		$\pm(2,2)$	$q^{-1/5}(q^{-1}t^{-1} + 1 + qt)$
		$\pm(2,3)$	$q^{2/5}$
		$\pm(2,4)$	$q^{-1/5}(q^{-1}t^{-1} + 1 + qt)$
10_{124}	$\{0\}$	0	$q^{-4}t^{-4} + q^{-3}t^{-3} + \underline{t^{-1} + q + q^2t} + q^3t^3 + q^4t^4$
10_{128}	$\mathbb{Z}/11$	0	$2q^{-3}t^{-3} + 3q^{-2}t^{-2} + q^{-1}t^{-1} + \underline{q} + qt + 3q^2t^2 + 2q^3t^3$
		± 1	$q^{8/11}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 2	$q^{10/11}(q^{-1}t^{-1} + 1 + qt)$
		± 3	$q^{6/11}(q^{-1}t^{-1} + 1 + qt)$
		± 4	$q^{-4/11}(q^{-2}t^{-2} + q^{-1}t^{-1} + \underline{q} + qt + q^2t^2)$
		± 5	$q^{2/11}(q^{-1}t^{-1} + 1 + qt)$

K	$H_1(\Sigma_2(K);\mathbb{Z})$	$\mathfrak{s}$	$\sum_{i,j}\dim_{\mathbb{Z}/2}\widehat{\mathrm{HFK}}_j(\Sigma_2(K),\widetilde{K},\mathfrak{s},i;\mathbb{Z}/2)t^iq^j$
10_{129}	$\mathbb{Z}/25$	0	$2q^{-2}t^{-2}+6q^{-1}t^{-1}+9+6qt+2q^2t^2$
		± 1	$q^{-8/25}(q^{-2}t^{-2}+2q^{-1}t^{-1}+3+2qt+q^2t^2)$
		± 2	$q^{-7/25}(q^{-1}t^{-1}+1+qt)$
		± 3	$q^{3/25}(2q^{-1}t^{-1}+3+2qt)$
		± 4	$q^{-3/25}(q^{-1}t^{-1}+1+qt)$
		± 5	1
		± 6	$q^{12/25}$
		± 7	$q^{8/25}$
		± 8	$q^{-12/25}$
		± 9	$q^{2/25}(q^{-1}t^{-1}+3+qt)$
		± 10	1
		± 11	$q^{7/25}(q^{-1}t^{-1}+1+qt)$
		± 12	$q^{23/25}(q^{-1}t^{-1}+1+qt)$
10_{130}	$\mathbb{Z}/17$	0	$2q^{-2}t^{-2}+4q^{-1}t^{-1}+5+4qt+2q^2t^2$
		± 1	$q^{4/17}(q^{-2}t^{-2}+2q^{-1}t^{-1}+3+2qt+q^2t^2)$
		± 2	$q^{16/17}$
		± 3	$q^{19/17}(q^{-1}t^{-1}+1+qt)$
		± 4	$q^{13/17}(2q^{-1}t^{-1}+3+2qt)$
		± 5	$q^{15/17}(q^{-1}t^{-1}+1+qt)$
		± 6	$q^{8/17}$
		± 7	$q^{9/17}(q^{-1}t^{-1}+1+qt)$
		± 8	$q^{1/17}(q^{-1}t^{-1}+1+qt)$
10_{131}	$\mathbb{Z}/31$	0	$2q^{-2}t^{-2}+8q^{-1}t^{-1}+11+8qt+2q^2t^2$
		± 1	$q^{-18/31}(q^{-1}t^{-1}+1+qt)$
		± 2	$q^{-10/31}(q^{-1}t^{-1}+1+qt)$
		± 3	$q^{-7/31}(q^{-1}t^{-1}+3+qt)$
		± 4	$q^{-9/31}$
		± 5	$q^{15/31}$
		± 6	$q^{3/31}$
		± 7	$q^{-14/31}(q^{-1}t^{-1}+1+qt)$
		± 8	$q^{-5/31}(2q^{-1}t^{-1}+5+2qt)$
		± 9	$q^{-1/31}$
		± 10	$q^{-2/31}(q^{-1}t^{-1}+1+qt)$
		± 11	$q^{-8/31}(q^{-2}t^{-2}+4q^{-1}t^{-1}+5+4qt+q^2t^2)$
		± 12	$q^{-19/31}(q^{-1}t^{-1}+3+qt)$
		± 13	$q^{-4/31}(2q^{-1}t^{-1}+3+2qt)$
		± 14	$q^{-25/31}$
		± 15	$q^{11/31}$
10_{132}	$\mathbb{Z}/5$	0	$q^{-2}t^{-2}+(2q^{-1}+\underline{1})t^{-1}+(2+\underline{q})+(2q+\underline{q^2})t+q^2t^2$
		± 1	$q^{2/5}$
		± 2	$q^{3/5}(q^{-1}t^{-1}+1+qt)$
10_{133}	$\mathbb{Z}/19$	0	$q^{-2}t^{-2}+5q^{-1}t^{-1}+7+5qt+q^2t^2$
		± 1	$q^{-3/19}$
		± 2	$q^{-12/19}(q^{-1}t^{-1}+1+qt)$
		± 3	$q^{-8/19}(q^{-1}t^{-1}+1+qt)$
		± 4	$q^{9/19}$
		± 5	$q^{1/19}$
		± 6	$q^{-13/19}(q^{-1}t^{-1}+3+qt)$
		± 7	$q^{5/19}$
		± 8	$q^{-2/19}(2q^{-1}t^{-1}+3+2qt)$
		± 9	$q^{-15/19}$

K	$H_1(\Sigma_2(K);\mathbb{Z})$	$\mathfrak{s}$	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\mathrm{HFK}}_j(\Sigma_2(K), \widetilde{K}, \mathfrak{s}, i; \mathbb{Z}/2) t^i q^j$
10_{134}	$\mathbb{Z}/23$	0	$2q^{-3}t^{-3} + 4q^{-2}t^{-2} + 4q^{-1}t^{-1} + 3 + 4qt + 4q^2t^2 + 2q^3t^3$
		± 1	$q^{8/23}(q^{-1}t^{-1} + 1 + qt)$
		± 2	$q^{9/23}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		± 3	$q^{3/23}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2)$
		± 4	$q^{-10/23}(q^{-3}t^{-3} + q^{-2}t^{-2} + \underline{q} + q^2t^2 + q^3t^3)$
		± 5	$q^{16/23}(q^{-1}t^{-1} + 1 + qt)$
		± 6	$q^{12/23}(q^{-1}t^{-1} + 1 + qt)$
		± 7	$q^{1/23}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		± 8	$q^{29/23}$
		± 9	$q^{4/23}(q^{-1}t^{-1} + 1 + qt)$
		± 10	$q^{18/23}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 11	$q^{25/23}$
10_{135}	$\mathbb{Z}/37$	0	$3q^{-2}t^{-2} + 9q^{-1}t^{-1} + 13 + 9qt + 3q^2t^2$
		± 1	$q^{14/37}$
		± 2	$q^{-18/37}$
		± 3	$q^{15/37}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 4	$q^{2/37}$
		± 5	$q^{17/37}(q^{-1}t^{-1} + 1 + qt)$
		± 6	$q^{-14/37}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2)$
		± 7	$q^{-17/37}(q^{-1}t^{-1} + 1 + qt)$
		± 8	$q^{8/37}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 5 + 3qt + q^2t^2)$
		± 9	$q^{-13/37}(q^{-1}t^{-1} + 1 + qt)$
		± 10	$q^{-6/37}$
		± 11	$q^{29/37}(q^{-1}t^{-1} + 1 + qt)$
		± 12	$q^{18/37}$
		± 13	$q^{-2/37}$
		± 14	$q^{6/37}(q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2)$
		± 15	$q^{5/37}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 16	$q^{-5/37}(q^{-1}t^{-1} + 1 + qt)$
		± 17	$q^{13/37}(q^{-1}t^{-1} + 1 + qt)$
		± 18	$q^{22/37}$
10_{136}	$\mathbb{Z}/15$	1	$q^{-2}t^{-2} + 4q^{-1}t^{-1} + 6 + \underline{q} + 4qt + q^2t^2$
		± 1	$q^{7/15}$
		± 2	$q^{13/15}(q^{-1}t^{-1} + 3 + qt)$
		± 3	$q^{1/5}$
		± 4	$q^{7/15}$
		± 5	$q^{2/3}(q^{-1}t^{-1} + 1 + qt)$
		± 6	$q^{4/5}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 7	$q^{13/15}(q^{-1}t^{-1} + 3 + qt)$
10_{139}	$\mathbb{Z}/3$	0	$q^{-4}t^{-4} + q^{-3}t^{-3} + \underline{2qt^{-1} + 3q^2 + 2q^3t} + q^3t^3 + q^4t^4$
		± 1	$q^{5/3}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
10_{140}	$\mathbb{Z}/9$	0	$q^{-2}t^{-2} + 2q^{-1}t^{-1} + 3 + 2qt + q^2t^2$
		± 1	$q^{11/9}(q^{-1}t^{-1} + 1 + qt)$
		± 2	$q^{8/9}$
		± 3	1
		± 4	$q^{5/9}(q^{-1}t^{-1} + 1 + qt)$
10_{142}	$\mathbb{Z}/15$	0	$q^{-3}t^{-3} + 2q^{-2}t^{-2} + 2q^{-1}t^{-1} + 1 + 2qt + 3q^2t^2 + 2q^3t^3$
		± 1	$q^{1/15}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		± 2	$q^{4/15}(q^{-1}t^{-1} + 1 + qt)$
		± 3	$q^{-2/5}(q^{-3}t^{-3} + q^{-2}t^{-2} + \underline{q} + q^2t^2 + q^3t^3)$
		± 1	$q^{1/15}(q^{-2}t^{-2} + q^{-1}t^{-1} + 1 + qt + q^2t^2)$
		± 6	$q^{2/3}(2q^{-1}t^{-1} + 3 + 2qt)$
		± 2	$q^{7/5}$
		± 2	$q^{4/15}(q^{-1}t^{-1} + 1 + qt)$

K	$H_1(\Sigma_2(K);\mathbb{Z})$	$\mathfrak{s}$	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\mathrm{HFK}}_j(\Sigma_2(K),\widetilde{K},\mathfrak{s},i;\mathbb{Z}/2)t^iq^j$
10_{145}	$\mathbb{Z}/3$	0	$q^{-2}t^{-2}+(q^{-1}+\underline{2q})t^{-1}+\underline{q+4q^2}+(q+\underline{2q^3})t+q^2t^2$
		± 1	$q^{4/3}(2q^{-1}t^{-1}+3+2qt)$
10_{147}	$\mathbb{Z}/27$	0	$2q^{-2}t^{-2}+7q^{-1}t^{-1}+9+7qt+2q^2t^2$
		± 1	$q^{7/27}(q^{-1}t^{-1}+3+qt)$
		± 2	$q^{1/27}$
		± 3	$q^{1/3}(2q^{-1}t^{-1}+5+2qt)$
		± 4	$q^{4/27}(q^{-2}t^{-2}+3q^{-1}t^{-1}+3+3qt+q^2t^2)$
		± 5	$q^{13/27}$
		± 6	$q^{1/3}$
		± 7	$q^{19/27}(q^{-1}t^{-1}+3+qt)$
		± 8	$q^{16/27}(q^{-1}t^{-1}+1+qt)$
		± 9	$q^{-1}t^{-1}+1+qt$
		± 10	$q^{25/27}$
		± 11	$q^{37/27}$
		± 12	$q^{1/3}$
		± 13	$q^{22/27}(2q^{-1}t^{-1}+3+2qt)$
10_{158}	$\mathbb{Z}/45$	0	$q^{-3}t^{-3}+4q^{-2}t^{-2}+10q^{-1}t^{-1}+15+10qt+4q^2t^2+q^3t^3$
		± 1	$q^{8/45}(q^{-1}t^{-1}+3+qt)$
		± 2	$q^{-13/45}(q^{-2}t^{-2}+3q^{-1}t^{-1}+3+3qt+q^2t^2)$
		± 3	$q^{-2/5}$
		± 4	$q^{38/45}$
		± 5	$q^{4/9}(q^{-1}t^{-1}+3+qt)$
		± 6	$q^{2/5}$
		± 7	$q^{-13/45}(q^{-2}t^{-2}+3q^{-1}t^{-1}+3+3qt+q^2t^2)$
		± 8	$q^{17/45}(q^{-2}t^{-2}+3q^{-1}t^{-1}+3+3qt+q^2t^2)$
		± 9	$q^{2/5}(2q^{-1}t^{-1}+5+2qt)$
		± 10	$q^{-2/9}(2q^{-1}t^{-1}+5+2qt)$
		± 11	$q^{-22/45}$
		± 12	$q^{-2/5}$
		± 13	$q^{2/45}$
		± 14	$q^{38/45}$
		± 15	$q^{-1}t^{-1}+3+qt$
		± 16	$q^{-22/45}$
		± 17	$q^{17/45}(q^{-2}t^{-2}+3q^{-1}t^{-1}+3+3qt+q^2t^2)$
		± 18	$q^{-2/5}$
		± 19	$q^{8/45}(q^{-1}t^{-1}+3+qt)$
		± 20	$q^{1/9}(q^{-1}t^{-1}+1+qt)$
		± 21	$q^{2/5}$
		± 22	$q^{2/45}$
10_{160}	$\mathbb{Z}/21$	0	$q^{-3}t^{-3}+4q^{-2}t^{-2}+4q^{-1}t^{-1}+3+4qt+4q^2t^2+q^3t^3$
		± 1	$q^{1/21}(q^{-1}t^{-1}+1+qt)$
		± 2	$q^{4/21}(q^{-2}t^{-2}+q^{-1}t^{-1}+1+qt+q^2t^2)$
		± 3	$q^{3/7}(q^{-1}t^{-1}+1+qt)$
		± 4	$q^{16/21}$
		± 5	$q^{4/21}(q^{-2}t^{-2}+q^{-1}t^{-1}+1+qt+q^2t^2)$
		± 6	$q^{5/7}(2q^{-1}t^{-1}+3+2qt)$
		± 7	$q^{4/3}$
		± 8	$q^{1/21}(q^{-1}t^{-1}+1+qt)$
		± 9	$q^{6/7}(q^{-1}t^{-1}+3+qt)$
		± 10	$q^{16/21}$
10_{161}	$\mathbb{Z}/5$	0	$q^{-3}t^{-3}+(q^{-2}+\underline{1})t^{-2}+\underline{2qt^{-1}+3q^2+2q^3t}+(q^2+\underline{q^4})t^2+q^3t^3$
		± 1	$q^{9/5}(2q^{-1}t^{-1}+3+2qt)$
		± 2	$q^{6/5}(q^{-2}t^{-2}+q^{-1}t^{-1}+1+qt+q^2t^2)$

K	$H_1(\Sigma_2(K);\mathbb{Z})$	$\mathfrak{s}$	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\mathrm{HFK}}_j(\Sigma_2(K), \widetilde{K}, \mathfrak{s}, i; \mathbb{Z}/2) t^i q^j$
10_{164}	$\mathbb{Z}/45$	0	$3q^{-2}t^{-2}+11q^{-1}t^{-1}+17+11qt+3q^2t^2$
		± 1	$q^{17/45}(q^{-1}t^{-1}+1+qt)$
		± 2	$q^{-22/45}$
		± 3	$q^{2/5}$
		± 4	$q^{2/45}$
		± 5	$q^{4/9}(q^{-1}t^{-1}+3+qt)$
		± 6	$q^{-2/5}$
		± 7	$q^{-2/45}$
		± 8	$q^{8/45}(q^{-2}t^{-2}+3q^{-1}t^{-1}+5+3qt+q^2t^2)$
		± 9	$q^{-2/5}(q^{-2}t^{-2}+2q^{-1}t^{-1}+3+2qt+q^2t^2)$
		± 10	$q^{-2/9}$
		± 11	$q^{-13/45}(q^{-1}t^{-1}+1+qt)$
		± 12	$q^{2/5}$
		± 13	$q^{38/45}$
		± 14	$q^{2/45}$
		± 15	$q^{-1}t^{-1}+3+qt$
		± 16	$q^{-13/45}(q^{-1}t^{-1}+1+qt)$
		± 17	$q^{8/45}(q^{-2}t^{-2}+3q^{-1}t^{-1}+5+3qt+q^2t^2)$
		± 18	$q^{2/5}$
		± 19	$q^{17/45}(q^{-1}t^{-1}+1+qt)$
		± 20	$q^{1/9}(2q^{-1}t^{-1}+3+2qt)$
		± 21	$q^{-2/5}$
		± 22	$q^{38/45}$
$11n_{12}$	$\mathbb{Z}/13$	0	$q^{-2}t^{-2}+(\underline{q^{-2}}+4q^{-1})t^{-1}+\underline{q^{-1}}+6+(\underline{1}+4q)t+q^2t^2$
		± 1	$q^{-2/13}$
		± 2	$q^{-8/13}(q^{-1}t^{-1}+3+qt)$
		± 3	$q^{-18/13}$
		± 4	$q^{-6/13}$
		± 5	$q^{-11/13}(2q^{-1}t^{-1}+3+2qt)$
		± 6	$q^{-7/13}(q^{-1}t^{-1}+1+qt)$
$11n_{19}$	$\mathbb{Z}/5$	0	$q^{-3}t^{-3}+2q^{-2}t^{-2}+(q^{-1}+\underline{1})t^{-1}+\underline{q}+(q+\underline{q^2})t+2q^2t^2+q^3t^3$
		± 1	$q^{4/5}(q^{-2}t^{-2}+q^{-1}t^{-1}+1+qt+q^2t^2)$
		± 2	$q^{6/5}(q^{-1}t^{-1}+3+qt)$
$11n_{20}$	$\mathbb{Z}/23$	0	$2q^{-2}t^{-2}+6q^{-1}t^{-1}+\underline{8+q}+6qt+2q^2t^2$
		± 1	$q^{17/23}(q^{-1}t^{-1}+3+qt)$
		± 2	$q^{-1/23}$
		± 3	$q^{-8/23}(q^{-2}t^{-2}+2q^{-1}t^{-1}+\underline{2+q}+2qt+q^2t^2)$
		± 4	$q^{19/23}(2q^{-1}t^{-1}+5+2qt)$
		± 5	$q^{11/23}$
		± 6	$q^{14/23}(q^{-1}t^{-1}+1+qt)$
		± 7	$q^{5/23}(q^{-1}t^{-1}+3+qt)$
		± 8	$q^{7/23}$
		± 9	$q^{20/23}(2q^{-1}t^{-1}+3+2qt)$
		± 10	$q^{21/23}(q^{-1}t^{-1}+3+qt)$
		± 11	$q^{10/23}(q^{-1}t^{-1}+1+qt)$
$11n_{38}$	$\mathbb{Z}/3$	0	$q^{-2}t^{-2}+(2q^{-1}+\underline{1})t^{-1}+\underline{2+3q}+(2q+\underline{q^2})t+q^2t^2$
		± 1	$q^{4/3}(q^{-1}t^{-1}+1+qt)$
$11n_{49}$	$\{0\}$	0	$q^{-2}t^{-2}+(\underline{4q^{-3}}+2q^{-1})t^{-1}+\underline{9q^{-2}}+2+(4q^{-1}+2q)t+q^2t^2$

K	$H_1(\Sigma_2(K);\mathbb{Z})$	$\mathfrak{s}$	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\mathrm{HFK}}_j(\Sigma_2(K), \widetilde{K}, \mathfrak{s}, i; \mathbb{Z}/2) t^i q^j$
$11n_{95}$	$\mathbb{Z}/33$	0	$q^{-3}t^{-3}+5q^{-2}t^{-2}+7q^{-1}t^{-1}+7+7qt+5q^2t^2+q^3t^3$
		± 1	$q^{-13/33}(q^{-1}t^{-1}+1+qt)$
		± 2	$q^{14/33}$
		± 3	$q^{5/11}(q^{-1}t^{-1}+1+qt)$
		± 4	$q^{-10/33}(q^{-2}t^{-2}+q^{-1}t^{-1}+1+qt+q^2t^2)$
		± 5	$q^{5/33}(2q^{-1}t^{-1}+3+2qt)$
		± 6	$q^{-2/11}(q^{-2}t^{-2}+q^{-1}t^{-1}+1+qt+q^2t^2)$
		± 7	$q^{-10/33}(q^{-2}t^{-2}+q^{-1}t^{-1}+1+qt+q^2t^2)$
		± 8	$q^{26/33}$
		± 9	$q^{1/11}(q^{-1}t^{-1}+1+qt)$
		± 10	$q^{-13/33}(q^{-1}t^{-1}+1+qt)$
		± 11	$q^{1/3}(q^{-1}t^{-1}+1+qt)$
		± 12	$q^{3/11}(2q^{-1}t^{-1}+3+2qt)$
		± 13	$q^{14/33}$
		± 14	$q^{26/33}$
		± 15	$q^{4/11}(q^{-1}t^{-1}+3+qt)$
		± 16	$q^{5/33}(2q^{-1}t^{-1}+3+2qt)$
$11n_{102}$	$\mathbb{Z}/3$	0	$q^{-2}t^{-2}+(5q^{-1}+\underline{2q})t^{-1}+7+\underline{4q^2}+(5q+\underline{2q^3})t+q^2t^2$
		± 1	$q^{1/3}(2q^{-1}t^{-1}+5+2qt)$
$11n_{116}$	$\{0\}$	0	$q^{-2}t^{-2}+(\underline{4q^{-3}}+2q^{-1})t^{-1}+\underline{9q^{-2}}+2+(4q^{-1}+2q)t+q^2t^2$
$11n_{117}$	$\mathbb{Z}/35$	0	$3q^{-2}t^{-2}+9q^{-1}t^{-1}+11+9qt+3q^2t^2$
		± 1	$q^{9/35}(2q^{-1}t^{-1}+5+2qt)$
		± 2	$q^{1/35}$
		± 3	$q^{11/35}(q^{-1}t^{-1}+3+qt)$
		± 4	$q^{4/35}(q^{-2}t^{-2}+3q^{-1}t^{-1}+3+3qt+q^2t^2)$
		± 5	$q^{3/7}(q^{-1}t^{-1}+3+qt)$
		± 6	$q^{9/35}(2q^{-1}t^{-1}+5+2qt)$
		± 7	$q^{-2/5}(q^{-2}t^{-2}+2q^{-1}t^{-1}+2+\underline{q}+2qt+q^2t^2)$
		± 8	$q^{16/35}(q^{-1}t^{-1}+1+qt)$
		± 9	$q^{29/35}$
		± 10	$q^{5/7}(2q^{-1}t^{-1}+5+2qt)$
		± 11	$q^{4/35}(q^{-2}t^{-2}+3q^{-1}t^{-1}+3+3qt+q^2t^2)$
		± 12	$q^{1/35}$
		± 13	$q^{16/35}(q^{-1}t^{-1}+1+qt)$
		± 14	$q^{7/5}$
		± 15	$q^{6/7}(2q^{-1}t^{-1}+3+2qt)$
		± 16	$q^{29/35}$
		± 17	$q^{11/35}(q^{-1}t^{-1}+3+qt)$
$11n_{118}$	$\mathbb{Z}/21$	0	$q^{-3}t^{-3}+4q^{-2}t^{-2}+4q^{-1}t^{-1}+3+4qt+4q^2t^2+q^3t^3$
		± 1	$q^{5/21}(q^{-1}t^{-1}+1+qt)$
		± 2	$q^{20/21}$
		± 3	$q^{1/7}(2q^{-1}t^{-1}+3+2qt)$
		± 4	$q^{-4/21}(q^{-2}t^{-2}+q^{-1}t^{-1}+1+qt+q^2t^2)$
		± 5	$q^{20/21}$
		± 6	$q^{4/7}$
		± 7	$q^{-1/3}(q^{-1}t^{-1}+1+qt)$
		± 8	$q^{5/21}(q^{-1}t^{-1}+1+qt)$
		± 9	$q^{2/7}(q^{-1}t^{-1}+3+qt)$
		± 10	$q^{-4/21}(q^{-2}t^{-2}+q^{-1}t^{-1}+1+qt+q^2t^2)$

K	$H_1(\Sigma_2(K);\mathbb{Z})$	$\mathfrak{s}$	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\mathrm{HFK}}_j(\Sigma_2(K), \widetilde{K}, \mathfrak{s}, i; \mathbb{Z}/2) t^i q^j$
$11n_{122}$	$\mathbb{Z}/27$	0	$2q^{-2}t^{-2}+7q^{-1}t^{-1}+9+2qt+2q^2t^2$
		±1	$q^{13/27}$
		±2	$q^{-2/27}(2q^{-1}t^{-1}+3+2qt)$
		±3	$q^{1/3}(2q^{-1}t^{-1}+5+2qt)$
		±4	$q^{-8/27}(q^{-1}t^{-1}+1+qt)$
		±5	$q^{1/27}$
		±6	$q^{1/3}$
		±7	$q^{-11/27}$
		±8	$q^{-5/27}(q^{-1}t^{-1}+3+qt)$
		±9	$q^{-1}t^{-1}+1+qt$
		±10	$q^{-23/27}$
		±11	$q^{-20/27}(q^{-2}t^{-2}+3q^{-1}t^{-1}+3+3qt+q^2t^2)$
		±12	$q^{1/3}$
		±13	$q^{-17/27}(q^{-1}t^{-1}+3+qt)$
$11n_{138}$	$\mathbb{Z}/15$	0	$2q^{-2}t^{-2}+4q^{-1}t^{-1}+(\underline{q^{-1}}+4)+4qt+2q^2t^2$
		±1	$q^{-7/15}$
		±2	$q^{-13/15}(q^{-1}t^{-1}+3+qt)$
		±3	$q^{-1/5}((q^{-2}+2q^{-1})t^{-1}+(q^{-1}+4)+(1+2q)t)$
		±4	$q^{-7/15}$
		±5	$q^{-2/3}(q^{-2}t^{-2}+3q^{-1}t^{-1}+3+3qt+q^2t^2)$
		±6	$q^{-9/5}$
		±7	$q^{-13/15}(q^{-1}t^{-1}+3+qt)$
$11n_{139}$	$\mathbb{Z}/9$	0	$2q^{-1}t^{-1}+5+2qt$
		±1	$q^{-4/9}$
		±2	$q^{-16/9}$
		±3	1
		±4	$q^{-10/9}(q^{-1}t^{-1}+3+qt)$
$11n_{141}$	$\mathbb{Z}/21$	0	$5q^{-1}t^{-1}+11+5qt$
		±1	$q^{-10/21}$
		±2	$q^{2/21}(2q^{-1}t^{-1}+5+2qt)$
		±3	$q^{-2/7}(2q^{-1}t^{-1}+5+2qt)$
		±4	$q^{8/21}(q^{-1}t^{-1}+3+qt)$
		±5	$q^{2/21}(2q^{-1}t^{-1}+5+2qt)$
		±6	$q^{6/7}$
		±7	$q^{2/3}(2q^{-1}t^{-1}+5+2qt)$
		±8	$q^{-10/21}$
		±9	$q^{10/7}$
		±10	$q^{8/21}(q^{-1}t^{-1}+3+qt)$
$11n_{142}$	$\mathbb{Z}/33$	0	$q^{-2}t^{-2}+8q^{-1}t^{-1}+15+8qt+q^2t^2$
		±1	$q^{2/33}(q^{-1}t^{-1}+3+qt)$
		±2	$q^{8/33}(2q^{-1}t^{-1}+5+2qt)$
		±3	$q^{6/11}(q^{-1}t^{-1}+3+qt)$
		±4	$q^{32/33}$
		±5	$q^{-16/33}$
		±6	$q^{2/11}(q^{-1}t^{-1}+3+qt)$
		±7	$q^{32/33}$
		±8	$q^{-4/33}$
		±9	$q^{10/11}(q^{-1}t^{-1}+3+qt)$
		±10	$q^{2/33}(q^{-1}t^{-1}+3+qt)$
		±11	$q^{4/3}$
		±12	$q^{8/11}(2q^{-1}t^{-1}+5+2qt)$
		±13	$q^{8/33}(2q^{-1}t^{-1}+5+2qt)$
		±14	$q^{-4/33}$
		±15	$q^{-4/11}(2q^{-1}t^{-1}+5+2qt)$
		±16	$q^{-16/33}$

K	$H_1(\Sigma_2(K);\mathbb{Z})$	$\mathfrak{s}$	$\sum_{i,j} \dim_{\mathbb{Z}/2} \widehat{\mathrm{HFK}}_j(\Sigma_2(K), \widetilde{K}, \mathfrak{s}, i; \mathbb{Z}/2) t^i q^j$
$11n_{143}$	$\mathbb{Z}/9$	0	$q^{-3}t^{-3} + (\underline{q^{-4}} + 3q^{-2})t^{-2} + (\underline{2q^{-3}} + 3q^{-1})t^{-1} + (\underline{2q^{-2}} + 3)$ $+(\underline{2q^{-1}} + 3q)t + (\underline{1} + 3q^2)t^2 + q^3t^3$
		± 1	$q^{-10/9}((q^{-1}+1)t^{-1} + (2+q) + (1+q^2)t)$
		± 2	$q^{-4/9}$
		± 3	1
		± 4	$q^{-7/9}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 3 + 3qt + q^2t^2)$
$11n_{145}$	$\mathbb{Z}/9$	0	$q^{-3}t^{-3} + (2q^{-2} + \underline{1})t^{-2} + (q^{-1} + \underline{4q})t^{-1} + \underline{7q^2} + (q + \underline{4q^3})t$ $+(2q^2 + \underline{t^4})t^2 + q^3t^3$
		± 1	$q^{10/9}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 5 + 3qt + q^2t^2)$
		± 2	$q^{22/9}$
		± 3	q^2
		± 4	$q^{16/9}(q^{-2}t^{-2} + 3q^{-1}t^{-1} + 5 + 3qt + q^2t^2)$

6 Observations

Grigsby [7] showed that when $K \subset S^3$ is a two-bridge knot, the Heegaard Floer knot homology of $\widetilde{K} \subset \Sigma_2(K)$ in the canonical spin^c structure is isomorphic as a bigraded $\mathbb{Z}/2$–vector space to that of $K \subset S^3$: ie, $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K}, \mathfrak{s}_0) \cong \widehat{\mathrm{HFK}}(S^3, K)$, up to an overall shift in the Maslov grading. Our results suggest that the same is true for a wider class of knots. Specifically, define the *δ–grading* on $\widehat{\mathrm{HFK}}(Y, K, \mathfrak{s})$ as the difference between the Alexander and Maslov gradings. We say that $\widehat{\mathrm{HFK}}(Y, K, \mathfrak{s})$ is *thin* if it is supported in a single δ–grading. We make the following conjecture.

Conjecture 6.1 *Let $K \subset S^3$ be a knot for which $\widehat{\mathrm{HFK}}(S^3, K)$ is thin. Then*

$$\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K}, \mathfrak{s}_0) \cong \widehat{\mathrm{HFK}}(S^3, K)$$

as bigraded groups, up to a possible shift in the absolute Maslov grading.

It is well-known (Ozsváth–Szabó [14] or Rasmussen [17]) that $\widehat{\mathrm{HFK}}(S^3, K)$ is thin whenever K is alternating (and hence for all two-bridge knots). More generally, let $\mathcal{Q}$ be the smallest set of link types such that:

- The unknot is in $\mathcal{Q}$.
- Suppose L admits a projection such that the two resolutions at some crossing, L_0 and L_1, are both in $\mathcal{Q}$ and satisfy $\det(L_0) + \det(L_1) = \det(L)$. Then L is in $\mathcal{Q}$.

The links in $\mathcal{Q}$ are called *quasi-alternating*; for instance, any alternating link is quasi-alternating. Manolescu and Ozsváth [11] that whenever L is quasi-alternating, both $\widehat{\mathrm{HFK}}(S^3, L)$ and the Khovanov homology of L are thin. Conjecture 6.1 would then imply that $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K}, \mathfrak{s}_0)$ is thin whenever K is quasi-alternating.

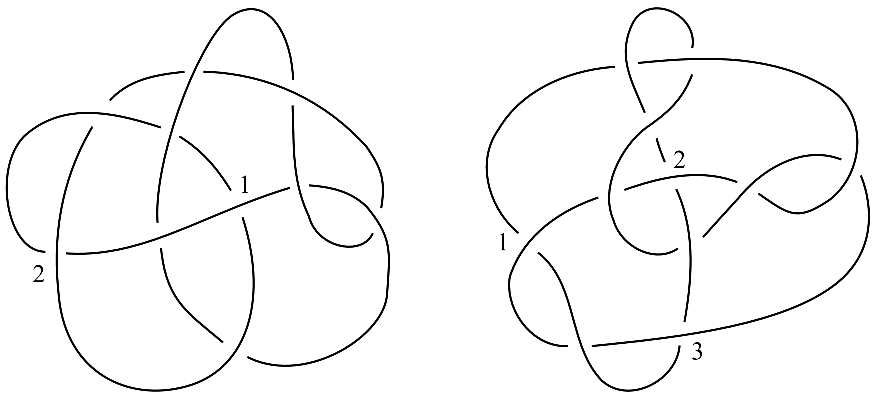

Figure 2: To see that the knots 10_{134} (left) and $11n_{117}$ (right) are quasi-alternating, resolve the marked crossings in the order indicated.

One may also ask under what conditions $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K}, \mathfrak{s})$ is thin for spin^c structures $\mathfrak{s} \neq \mathfrak{s}_0$. The knots 10_{134} and $11n_{117}$ both satisfy the hypothesis and conclusion of Conjecture 6.1. Indeed, they are both quasi-alternating, as illustrated in Figure 2. However, each one admits a spin^c structure $\mathfrak{s}$ on $\Sigma_2(K)$ in which $\widehat{\mathrm{HFK}}(S^2(K), \widetilde{K}, \mathfrak{s})$ is not thin. There are no known examples of alternating knots for which this phenomenon occurs, though.

On the other hand, when $\widehat{\mathrm{HFK}}(S^3, K)$ is not thin, the isomorphism between $\widehat{\mathrm{HFK}}(S^3, K)$ and $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K}, \mathfrak{s}_0)$ generally fails. A few patterns are worth mentioning. Note that for the knots considered here, in each Alexander grading i, the total rank of $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K}, \mathfrak{s}_0, i)$ is at least that of $\widehat{\mathrm{HFK}}(S^3, K, i)$, and the two ranks are congruent modulo 2. Some examples in which the ranks fail to be equal are $11n_{49}$, $11n_{102}$, and $11n_{116}$. Even when the total ranks of $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K}, \mathfrak{s}_0, i)$ and $\widehat{\mathrm{HFK}}(S^3, K, i)$ are the same for all i, the relative Maslov gradings can differ. A common pattern is that the Maslov gradings of all the groups in one δ–grading of $\widehat{\mathrm{HFK}}(S^3, K)$ are shifted by a constant amount in $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K}, \mathfrak{s}_0)$, such as with the knots 9_{42} and 10_{161}, where the groups are shifted by 2 and 3, respectively. However, there are also examples where the relative Maslov gradings in different Alexander gradings change in different ways. For example, for 10_{145}, the total ranks of $\widehat{\mathrm{HFK}}(S^3, K, i)$ and $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K}, \mathfrak{s}_0, i)$ are the same for each i, but $\widehat{\mathrm{HFK}}(S^3, K)$ is supported in two δ–gradings while $\widehat{\mathrm{HFK}}(\Sigma_2(K), K, \mathfrak{s}_0)$ is supported in three.

Finally, note that the pretzel knots $8_{20} = P(2, 3, -3)$ and $10_{140} = P(4, 3, -3)$ have identical knot Floer homology but can be distinguished by $\widehat{\mathrm{HFK}}(\Sigma_2(K), \widetilde{K})$. The relative Maslov gradings between spin^c structures are necessary in this case. For another such example, see Grigsby [7].

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