

Chevalley p -local finite groups

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We describe the spaces of homotopy fixed points of unstable Adams operations acting on p -compact groups and also of unstable Adams operations twisted with a finite order automorphism of the p -compact group. We obtain new exotic p -local finite groups.

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1 Introduction

The main purpose of this paper is the description of the structure of the spaces of homotopy fixed points of unstable Adams operations ψ^q acting on p -compact groups and also of unstable Adams operations twisted by automorphisms of p -compact groups $\tau\psi^q$.

In the classical case, for a prime number p , a prime power q , prime to p , a compact connected Lie group G , and a finite order automorphism τ of G , Friedlander showed that there is a homotopy pullback diagram

$$\begin{array}{ccc} B {}^{\tau}G(q)_p^{\wedge} & \xrightarrow{f} & BG_p^{\wedge} \\ f \downarrow & & \downarrow \Delta \\ BG_p^{\wedge} & \xrightarrow{(1, \tau\psi^q)} & BG_p^{\wedge} \times BG_p^{\wedge} \end{array}$$

where ${}^{\tau}G(q)$ is the twisted Chevalley group over \mathbb{F}_q of type G , Δ is the diagonal map, and ψ^q an unstable Adams operation of exponent q [33; 34]. Here and throughout, p -completion is understood in the sense of Bousfield and Kan [11].

The concept of a p -compact group was introduced by Dwyer and Wilkerson in [25] as a p -local homotopy theoretic analogue of a compact Lie group. A p -compact group is a triple (X, BX, e) , where $H^*(X; \mathbb{F}_p)$ is finite, BX is a pointed p -complete space, and $e : X \rightarrow \Omega BX$ is a homotopy equivalence. We will usually refer to a p -compact group simply as X . BX is then understood as its classifying space, a concrete loop

space structure imposed in the underlying space X . If G is a compact connected Lie group, then the p -completion of its classifying space BG_p^\wedge is a p -compact group. A p -compact group that cannot be obtained in this way is called *exotic*. We postpone till [Section 2](#) a more detailed description of the theory of p -compact groups.

Unstable Adams operations ψ^q , for any p -adic unit q , can be defined for any connected p -compact group X (see [Section 2](#)). Following the above pattern, if $\tau\psi^q$ a twisted Adams operation, then the space $B^\tau X(q)$ is defined by the homotopy pullback square

$$(1) \quad \begin{array}{ccc} B^\tau X(q) & \xrightarrow{f} & BX \\ f \downarrow & & \downarrow \Delta \\ BX & \xrightarrow{(1, \tau\psi^q)} & BX \times BX. \end{array}$$

Thus if X is obtained as the p -completion of a compact Lie group G , and τ is a finite order automorphism of G , $B^\tau X(q)$ is homotopy equivalent to the p -completed classifying space of the twisted Chevalley group ${}^\tau G(q)$.

The concept of p -local finite group has been recently introduced by Broto, Levi and Oliver [\[14\]](#) as algebraic objects that are modeled on the p -local structure of finite groups and as such they have classifying spaces which are p -complete spaces. In turn, the classifying space of a p -local finite group determines its algebraic structure. Every finite group G determines a p -local finite group at a prime p with classifying space BG_p^\wedge . Like in the case of p -compact groups, p -local finite groups that do not arise in this way for any finite group G are called *exotic*. We refer to [Section 3](#) for the precise definition and main properties of p -local finite groups. Our main result shows that $B^\tau X(q)$ is the classifying space of a p -local finite group. We will also determine the cases in which they are exotic p -local finite groups.

Theorem A *Let p be an odd prime. If X is a 1-connected p -compact group, q is a prime power, prime to p , and τ is an automorphism of X of finite order prime to p , then the space of homotopy fixed points of BX by the action of $\tau\psi^q$, denoted $B^\tau X(q)$, is the classifying space of a p -local finite group.*

By analogy with the classical case, we will call the p -local finite group $X(q)$ (${}^\tau X(q)$) with classifying space $BX(q)$ ($B^\tau X(q)$) obtained in [Theorem A](#) a (twisted) *Chevalley p -local finite group of type X* .

Our arguments concentrate on the exotic p -compact groups at odd primes and break into two separate steps. One deals with actions of finite groups of order not divisible by p on p -compact groups and the results obtained have an independent interest. The

other step deals with the action of unstable Adams operations ψ^q where $q \equiv 1 \pmod p$ and it is the one leading to the new exotic examples of p -local finite groups.

Group actions will be understood in the weak sense of *proxy* actions; that is, we will say that an action of a group G on a space M is a fibration

$$M \xrightarrow{i} M_{hG} \xrightarrow{\text{pr}} BG$$

(see Dwyer [25]). The total space M_{hG} is referred to as the homotopy quotient space. The space of homotopy fixed points is the space

$$M^{hG} = \{BG \xrightarrow{s} M_{hG} \mid \text{pr} \circ s = \text{id}_{BG}\}$$

of sections. Two actions will be considered equivalent if they are defined by fiber homotopy equivalent fibrations. If M is a G -space in the usual sense then M_{hG} is the Borel construction and M^{hG} is homeomorphic to the space $\text{Map}_G(EG, M)$ of equivariant maps where EG is a contractible free G -space. When we specialize to p -compact groups X , an outer action of G is a homomorphism $\rho: G \rightarrow \text{Out}(X)$, where $\text{Out}(X)$ is the group of outer automorphisms of the p -compact group X , in other words, unpointed homotopy classes of self-equivalences of BX . By obstruction theory, it turns out that if G has finite order prime to p , then an outer action on a connected p -compact group X determines a unique action, up to equivalence, and the space of homotopy fixed points is again a connected p -compact group.

The space $B^\tau X(q)$ defined by pullback square (1) can also be viewed as a homotopy fixed point space $BX^{h\langle \tau\psi^q \rangle}$ for the action of the infinite cyclic group generated by $\tau\psi^q \in \text{Out}(X)$. More details are given in Section 6.

Theorem B *Let X be a connected p -compact group. If G is a finite group of order prime to p and $\rho: G \rightarrow \text{Out}(X)$ an outer action, then the following holds:*

- (1) ρ lifts to a unique action of G on X , up to equivalence.
- (2) X^{hG} is a connected p -compact group with

$$H^*(BX^{hG}; \mathbb{Q}_p) \cong S[QH^*(BX; \mathbb{Q}_p)_G],$$

the symmetric algebra generated on the coinvariants $QH^*(BX; \mathbb{Q}_p)_G$.

- (3) (Harper splitting) $X^{hG} \rightarrow X$ is a p -compact group monomorphism, there is a homotopy equivalence

$$X \simeq X^{hG} \times X/X^{hG}$$

and X/X^{hG} is an H -space.

- (4) Assume that p is odd. If $H^*(BX; \mathbb{F}_p)$ is a polynomial ring, then $H^*(BX^{hG}; \mathbb{F}_p)$ is also a polynomial ring.

Here and throughout, $H^*(-; \mathbb{Q}_p)$ stands for $H^*(-; \mathbb{Z}_p) \otimes \mathbb{Q}$, and $QH^*(BX; \mathbb{Q}_p)$ denotes the module of the indecomposables in $H^*(BX; \mathbb{Q}_p)$.

Some interesting cases to which [Theorem B](#) applies are F_4 at the prime 3 and E_8 at the prime 5, where the p -compact groups \mathbf{X}_{12} , respectively, \mathbf{X}_{31} split off (see [Section 2](#) for notation). In the first case, Friedlander's exceptional isogeny φ of F_4 at the prime 3 gives rise to an automorphism of order 2 and the homotopy fixed point p -compact group $F_4^{hC_2}$ is the p -compact group $\mathbf{X}_{12} = DI_2$ whose cohomology realizes the rank 2 Dickson algebra $H^*(B\mathbf{X}_{12}; \mathbb{F}_3) \cong \mathbb{F}_3[x_{12}, x_{16}]$ (subscripts of cohomology classes indicate degrees) over \mathbb{F}_3 . This case was already considered in our previous work [\[16\]](#). In the second case, a cyclic group of order 4 generated by the unstable Adams operation ψ^i , $i = \sqrt{-1}$, acts on E_8 . The homotopy fixed point p -compact group $E_8^{hC_4}$ is the p -compact group \mathbf{X}_{31} corresponding to the reflection group number 31 on the Clark–Ewing list, and its mod 5 cohomology ring is $H^*(B\mathbf{X}_{31}; \mathbb{F}_5) = \mathbb{F}_5[x_{16}, x_{24}, x_{40}, x_{48}]$ (see [Section A.12](#)).

It turns out that \mathbf{X}_{12} and \mathbf{X}_{31} are the two exotic p -compact groups originally constructed by Zabrodsky [\[71\]](#), and later included in the Aguadé family [\[2\]](#). Zabrodsky used the actions of these same automorphisms, φ and ψ^i , on the homotopy groups of BF_4 and BE_8 , respectively, and realized the invariant subgroups as homotopy groups of new spaces, $B\mathbf{X}_{12}$ and $B\mathbf{X}_{31}$.

The corresponding splittings are $F_4 \simeq DI_2 \times F_4/DI_2$ at the prime 3, first discovered by Harper [\[36\]](#), and $E_8 \simeq X_{31} \times E_8/X_{31}$ at the prime 5, which was obtained by Wilkerson [\[68\]](#). Other examples appear in [Example 5.4](#).

Our second step deals with the action of unstable Adams operations ψ^q of exponent $q \equiv 1 \pmod{p}$, $q \neq 1$, on connected p -compact groups X . These automorphisms have infinite order and the effect now is opposite in some sense to the case of finite groups of order prime to p . The spaces of homotopy fixed points $BX(q)$ have the same p -rank as the original p -compact groups X , but the maximal tori $T^n \simeq ((S^1)^n)^\wedge_p$ are cut down to finite maximal tori $T_\ell^n \cong (\mathbb{Z}/p^\ell)^n$, $\ell = v_p(1 - q)$, keeping the same Weyl group (see [Proposition 7.5](#) and [Proposition 7.6](#)).

We restrict our calculations in this part to p -compact groups for which the mod p cohomology ring $H^*(BX; \mathbb{F}_p)$ is a polynomial ring. For simplicity, we will refer to them as polynomial p -compact groups. At odd primes, these include all irreducible exotic examples and will therefore suffice for our purposes.

Theorem C *Let q be a p -adic unit such that $q \equiv 1 \pmod{p}$, $q \neq 1$. If X is an irreducible 1-connected polynomial p -compact group, then $BX(q)$ is the classifying space of a p -local finite group.*

The proof is based on the classification theorem for p -compact groups at odd primes [7]; see Section 2. The irreducible polynomial p -compact groups are

- (1) $BSU(n)_p^\wedge$ (family 1 in the Clark–Ewing list),
- (2) the generalized Grassmannians (family 2a in the Clark–Ewing list),
- (3) the Clark–Ewing p -compact groups (p -compact groups with Weyl group of order prime to p), and
- (4) the Aguadé family \mathbf{X}_{12} , \mathbf{X}_{29} , \mathbf{X}_{31} , \mathbf{X}_{34} at primes $p = 3, 5, 5,$ and 7 , respectively, and of rank $p - 1$. (The subscripts indicate the number of the Weyl group in the Clark–Ewing list.)

Theorem C is proved by considering separately these four cases in Remark 11.1, Theorem 11.4, Theorem 9.8, and Theorem 10.3, respectively.

In cases (1) and (3) we always obtain that $BX(q)$ is the p -completed classifying space of a finite group. The other two families contain the new exotic examples of p -local finite groups.

A complete description of the structure of the p -local finite groups $\mathbf{X}_i(q)$, $i = 12, 29, 31, 34$, is obtained in Section 10. Fix $q \equiv 1 \pmod{p}$ and let $v_3(1 + 2^{2n+1}) = v_3(1 - q)$. For $\mathbf{X}_{12}(q)$, $p = 3$, we obtain that $B\mathbf{X}_{12}(q) \simeq B(^2F_4(2^{2n+1}))_3^\wedge$ (see Example 10.7). For $\mathbf{X}_{31}(q)$, $p = 5$, it turns out that if $v_5(1 + 2^{4m+2}) = v_5(1 - q)$, then $B\mathbf{X}_{31}(q) \simeq BE_8(2^{2m+1})_5^\wedge$ (see Example 10.8). In particular, we can obtain the p -compact groups \mathbf{X}_{12} and \mathbf{X}_{31} as telescopes of a sequence of p -completed classifying spaces of finite groups (see Remark 10.9):

$$B\mathbf{X}_{12} \simeq \operatorname{hocolim}_m B(^2F_4(2^{3^m}))_3^\wedge,$$

$$B\mathbf{X}_{31} \simeq \operatorname{hocolim}_m BE_8(2^{5^m})_5^\wedge.$$

The cases $B\mathbf{X}_{29}(q)$ and $B\mathbf{X}_{34}(q)$ at primes 5 and 7, respectively, are classifying spaces of exotic p -local finite groups (Example 10.6).

Family 2a in the Clark–Ewing list consists of the reflection groups $G(m, r, n)$ with $r|m|(p - 1)$ generated in $GL(n, \mathbb{Z}_p)$ by the permutation matrices together with the diagonal matrices $\operatorname{diag}(a_1, a_2, \dots, a_n)$ with $a_i^m = 1$ and $(a_1 a_2 \dots a_n)^{m/r} = 1$. We denote $\mathbf{X}(m, r, n)$ the p -compact group of rank n with Weyl group $G(m, r, n)$. We also prove that $B\mathbf{X}(m, r, n)(q)$ is the classifying space of an exotic p -local finite group provided $n \geq p$ and $r > 2$ (Proposition 11.5).

Theorem D For $q \equiv 1 \pmod{p}$, $q \neq 1$, the following are classifying spaces of exotic p -local finite groups:

- $B\mathbf{X}_{29}(q)$ and $B\mathbf{X}_{34}(q)$ at primes $p = 5$ and $p = 7$, respectively, and
- $B\mathbf{X}(m, r, n)(q)$ for $n \geq p$ and $r > 2$.

Our next theorem provides the necessary arguments in order to deduce the general case of [Theorem A](#) from the two steps.

Theorem E Let p be an odd prime and X a 1-connected p -compact group, τ an automorphism of X of order prime to p , and ψ^q an unstable Adams operation of exponent a p -adic unit.

- (1) If $q \equiv 1 \pmod{p}$, $q \neq 1$, then $B^\tau X(q) \simeq BX^{h(\tau)}(q)$.
- (2) If q' is another p -adic unit such that q and q' have the same multiplicative order mod p and such that $v_p(1 - q^r) = v_p(1 - q'^r)$, where r is the order of q and q' mod p , then $BX(q) \simeq BX(q')$.

Since we can decompose a p -adic unit q as $q = \zeta q_0$ where ζ is a $(p - 1)$ -st root of unity and $q_0 \equiv 1 \pmod{p}$, part (1) of [Theorem E](#) will reduce the question of computing $BX(q)$ to the case where $q \equiv 1 \pmod{p}$ which turns out to be easier to handle in abstract calculations and concrete examples. The second part of the theorem tells us that $BX(q)$ does only depend on the order r of q mod p and the p -adic valuation $v_p(1 - q^r)$, so we can change the exact value of q at our convenience if we keep those parameters fixed.

Part (2) of [Theorem E](#) also explains the often observed fact that finite Chevalley groups $G(q)$ and $G(q')$ have same cohomology ring or identical p -local structure when q and q' are prime powers, with $q^r \equiv q'^r \equiv 1 \pmod{p}$ and $v_p(1 - q^r) = v_p(1 - q'^r)$, for some r , $1 \leq r \leq p - 1$. We plan to investigate this phenomenon closer in a future paper.

Proof of [Theorem A](#) Consider $B^\tau X(q)$ as the homotopy fixed point space $BX^{h(\tau\psi^q)}$ for the action on BX of the group generated by $\tau\psi^q$.

If we write $q = \zeta q_0$, where ζ is a $(p - 1)$ -th root of unity and $q_0 \equiv 1 \pmod{p}$, $q_0 \neq 1$, so that $\tau\psi^q = \tau\psi^\zeta\psi^{q_0}$, then we have

$$B^\tau X(q) = BX^{h(\tau\psi^q)} \simeq BX^{h(\tau\psi^\zeta)}(q_0),$$

according to [Theorem E](#).

$X^{h\langle\tau\psi^\xi\rangle}$ is a 1-connected p -compact group by [Theorem B](#), hence it splits as a product of irreducible 1-connected p -compact groups [\[27; 59\]](#)

$$BX^{h\langle\tau\psi^\xi\rangle} \simeq BX_1 \times \cdots \times BX_s,$$

and then also, $BX^{h\langle\tau\psi^\xi\rangle}(q_0) \simeq BX_1(q_0) \times \cdots \times BX_s(q_0)$. It remains to show that each $BX_i(q_0)$ is the classifying space of a p -local finite group.

If X_i is polynomial, [Theorem C](#) applies and $BX_i(q_0)$ is the classifying space of a p -local finite group.

If X_i is the p -completion of a compact Lie group G , then we can find a prime number q'_0 with $q'_0 \equiv q_0 \equiv 1 \pmod p$ and $v_p(1-q_0) = v_p(1-q'_0)$, and then $BX_i(q_0) \simeq BX_i(q'_0)$ by [Theorem E](#) (cf [Remark 6.6](#)), and this last is the p -completed classifying space of a finite Chevalley group of type G , by the classical result of Friedlander [\[34\]](#).

By the classification theorem of p -compact groups at odd primes [\[7\]](#) (see [Section 2](#)), every irreducible, simply connected p -compact group is either polynomial or the p -completion of a compact Lie group, hence the proof is complete. \square

Many authors have been interested in the cohomology rings of finite Chevalley groups at primes different from the defining characteristic. Quillen [\[61, Theorem 4\]](#) shows that for an odd prime p and a prime power q prime to p , if m is the order of $q \pmod p$ and $\ell = v_p(1-q^m)$, then

$$H^*(BGL(n, q); \mathbb{F}_p) \cong P[x_1, \dots, x_{\lfloor \frac{n}{m} \rfloor}] \otimes E[y_1, \dots, y_{\lfloor \frac{n}{m} \rfloor}]$$

where $\deg(x_i) = 2mi$ and $\deg(y_i) = 2mi - 1$.

Fiedorowicz and Priddy [\[31; 30\]](#) computed the cohomology rings of Chevalley groups of classical type. Kleinerman [\[39\]](#) has computed the cohomology of Chevalley groups of exceptional Lie type at large primes. M Mimura, M Tezuka and S Tsukuda [\[45\]](#) have recently approached the cohomology rings of finite Chevalley groups at torsion primes, by newly constructing a spectral sequence of Eilenberg–Moore type.

The result that we include here is essentially due to L Smith; at least part (1) already appears in [\[64\]](#). We include it here for the convenience of the reader, as it is an important step in our arguments.

Theorem F *Let X be a polynomial p -compact group with*

$$H^*(BX; \mathbb{F}_p) \cong P[x_1, \dots, x_n]$$

and q a p -adic unit with $q \equiv 1 \pmod p$, $q \neq 1$. Then:

- (1) $H^*(BX(q); \mathbb{F}_p) \cong P[x_1, \dots, x_n] \otimes E[y_1, \dots, y_n]$ with higher Bockstein relations $\beta_{(\ell_i)}(y_i) = x_i$, $\ell_i = v_p(1 - q^{d_i})$, $2d_i = \deg x_i$, $2d_i - 1 = \deg y_i$, and
- (2) the inclusion of the maximal finite torus $i: BT_\ell^n \rightarrow BX(q)$, $\ell = v_p(1 - q)$, induces a monomorphism $i^*: H^*(BX(q); \mathbb{F}_p) \rightarrow H^*(BT_\ell^n; \mathbb{F}_p)^{W_X}$.

The inclusion $i^*: H^*(BX(q); \mathbb{F}_p) \rightarrow H^*(BT_\ell^n; \mathbb{F}_p)^{W_X}$ is an isomorphism in many cases. This is checked by direct calculation of the relevant invariant rings. In cases in which X is a Clark–Ewing p -compact group or a generalized Grassmannian, i^* is an isomorphism (see [Section 9](#)). It is also an isomorphism in the case of the Aguadé p -compact groups $\mathbf{X}_i(q)$, $i = 29, 31, 34$. However, i^* is not an epimorphism in case of $\mathbf{X}_{12}(q)$, for which we obtain $H^*(\mathbf{B}\mathbf{X}_{12}(q); \mathbb{F}_3) \cong P[x_{12}, x_{16}] \otimes E[y_{11}, y_{15}]$, while

$$H^*(BT_\ell^2; \mathbb{F}_3)^{W_{\mathbf{X}_{12}}} \cong P[x_{12}, x_{16}] \otimes E[y_{10}, y_{11}, y_{15}] / (y_{11}y_{15} - x_{16}y_{10}, y_{10}y_{11}, y_{10}y_{15})$$

(see [Example 9.7](#)).

We have restricted our calculations at odd primes, although some of the results are also valid at the prime two. The classification of 2-compact groups [\[54; 6\]](#) implies that the Dwyer–Wilkerson 2-compact group DI_4 is the only irreducible exotic 2-compact group. The Chevalley 2-local finite groups of type DI_4 , named $BSol(q)$, for odd prime powers q , have been first considered by Benson [\[9\]](#) and then by Levi and Oliver [\[40\]](#) who proved that they are classifying spaces of 2-local finite groups and their 2-local structure is in fact a system of fusion relations studied by Solomon [\[65\]](#) and defined over the Sylow 2-subgroup of $\text{Spin}(7, q)$.

The paper is organized as follows. In [Section 2](#) and [Section 3](#) we review the definitions and main results from the theory of p -compact groups and p -local finite groups. In [Section 4](#) we further develop some aspects of the theory of p -local finite groups concerning the homotopy characterization of classifying spaces of p -local finite groups. The main results in [Sections 10](#) and [11](#) state that $BX(q)$ is the classifying space of a p -local finite group if X is a p -compact group in the Aguadé family or a generalized Grassmannian are based in this homotopy characterization of classifying spaces.

[Section 5](#) deals with what we have called *first step*. We discuss different ways in which we can understand an action of a group on a p -compact group and prove [Theorem B](#). This theorem states that a homotopy fixed point space X^{hG} is again a p -compact group if X was a connected p -compact group and G is a finite group of order prime to p . Identifying X^{hG} with a p -compact group in the classification list requires a close look at the restriction of the action to the maximal torus normalizer. This will be considered in [Appendix A](#). In particular, [Corollary A.6](#) contains a criterion for the

recognition of the homotopy fixed point p -compact group by action of unstable Adams operations of finite order. This is applied to many examples through the Clark–Ewing list at the end of this appendix in [Section A.7](#) through [Section A.12](#).

[Section 6](#) is devoted to the proof of [Theorem E](#). It reduces the analysis of the structure of a general homotopy fixed point space $B^{\tau}X(q)$ to first analyzing a homotopy fixed point p -compact group and then a homotopy fixed point space by the action of an unstable Adams operation $\psi^{q'}$ of exponent $q' \equiv 1 \pmod{p}$. This allows us to complete the argument for the proof of [Theorem A](#) from steps one and two.

The *second step* starts in [Sections 7, 8, and 9](#), where we analyze the general subgroup structure of spaces $BX(q)$, where $q \equiv 1 \pmod{p}$, $q \neq 1$, and their cohomological properties. [Theorem F](#) is proved in [Section 8](#). Some technical results concerning the Bousfield–Kan spectral sequence for the cohomology of a homotopy colimit are postponed to [Appendix B](#).

Finally, [Section 10](#) and [Section 11](#) are devoted to the more specific properties of the p -compact groups in the Aguadé family and the generalized Grassmannians, respectively. With them, we complete the proof of [Theorem C](#) and [Theorem D](#).

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2 p -compact groups

A p -compact group is a triple (X, BX, e) where X is a space, BX is a p -complete connected pointed space, $H^*(X; \mathbb{F}_p)$ is finite, and $e: X \rightarrow \Omega BX$ is a homotopy equivalence from X to the space ΩBX of based loops in BX .

Throughout the paper, and when no confusion is possible, we will simply denote a p -compact group (X, BX, e) as X . We shall say that X is *connected* if $\pi_0(X)$ is a point and *simply connected* if also $\pi_1(X)$ is trivial. These spaces were introduced by Dwyer and Wilkerson in 1994 as p -local homotopy theoretic versions of compact Lie groups [25]. We present here a short summary of the theory of p -compact groups and refer to the surveys of the second author [48], Notbohm [57] and Dwyer [21] for more information. Examples of p -compact groups include all simply connected p -complete spaces with polynomial \mathbb{F}_p -cohomology, and the p -completed classifying spaces of all compact Lie groups G such that $\pi_0(G)$ is a finite p -group. The p -compact group obtained in this way from a torus is called a p -compact torus. Thus a p -compact torus BT of rank n is simply a $K(\mathbb{Z}_p, 2)^n$ and we have that $H_2(BT; \mathbb{Z}_p) = \mathbb{Z}_p^n$ is a finitely generated free \mathbb{Z}_p -module. A *maximal torus* of a p -compact group BX is a pointed map $BT \rightarrow BX$, satisfying an injectivity and a maximality condition, of a p -compact torus into BX . The *Weyl group* W of the maximal torus $BT \rightarrow BX$, which we may assume is a fibration, is the monoid of fiber homotopy classes $BT \rightarrow BT$ over BX . It turns out that all elements of W are invertible so that W is actually a group. Equivalently, the Weyl group is the group of components of the *Weyl space* which is the associative topological monoid of self-maps of BT over BX . The Borel construction, BN , for the action of the Weyl space on BT is called the *normalizer of the maximal torus*. The monomorphism $BT \rightarrow BX$ extends to a monomorphism $BN \rightarrow BX$ [25, 9.2, 9.8].

Theorem 2.1 (Existence of maximal tori [25, 9.7]) *Any p -compact group X admits a maximal torus $BT_X \rightarrow BX$ and a Weyl group W_X . When X is connected, the Weyl group W_X acts faithfully on the finitely generated free \mathbb{Z}_p -module $L_X = H_2(BT_X; \mathbb{Z}_p)$, the pair (W_X, L_X) is a \mathbb{Z}_p -reflection group, and*

$$H^*(BX; \mathbb{Z}_p) \otimes \mathbb{Q} \rightarrow (H^*(BT_X; \mathbb{Z}_p) \otimes \mathbb{Q})^{W_X}$$

is an isomorphism.

This theorem introduces a relationship between p -compact groups and \mathbb{Z}_p -reflection groups as defined below.

An automorphism of a finitely generated free \mathbb{Z}_p -module is a *reflection* if it acts as the identity on a hyperplane. A \mathbb{Z}_p -*reflection group* is a pair (W, L) where L is a

finitely generated free \mathbb{Z}_p -module and W a subgroup of $\text{Aut}_{\mathbb{Z}_p}(L) = GL(L)$ that is generated by the reflections that it contains. A *morphism* between two \mathbb{Z}_p -reflection groups, (W_1, L_1) and (W_2, L_2) , is a pair (α, θ) consisting of a group homomorphism $\alpha: W_1 \rightarrow W_2$ and an α -linear \mathbb{Z}_p -module homomorphism $\theta: L_1 \rightarrow L_2$ [53, 4.1]. The \mathbb{Z}_p -reflection group (W, L) is *irreducible* if $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an irreducible $\mathbb{Q}_p W$ -module. Using the Shephard–Todd classification of irreducible complex reflection groups [63], Clark and Ewing [19] produced the list of all finite irreducible \mathbb{Z}_p -reflection groups [53, 11.18]. At odd primes the list is as follows:

- Family 1: $(\Sigma_{n+1}, S(\mathbb{Z}_p^{n+1}))$ where the symmetric group Σ_{n+1} permutes the $n+1$ factors of \mathbb{Z}_p^{n+1} and $S(\mathbb{Z}_p^{n+1}) = \{(x_1, \dots, x_{n+1}) \in \mathbb{Z}_p^{n+1} \mid \sum x_i = 0\}$.
- Family 2a: Let $r \geq 1$ and $m \geq 2$ natural numbers such that $r \mid m \mid p-1$. The cyclic group C_m of order m is contained in the group of units \mathbb{Z}_p^\times for \mathbb{Z}_p . The \mathbb{Z}_p -reflection group $(G(m, r, n), \mathbb{Z}_p^n)$, $n \geq 2$, is the group generated by the subgroup Σ_n of all permutations of the n coordinates and the subgroup

$$A(m, r, n) = \{\text{diag}(\theta_1, \dots, \theta_n) \in C_m^n \mid (\theta_1 \cdots \theta_n)^{m/r} = 1\}$$

consisting of diagonal matrices.

- Family 2b: (D_{2m}, \mathbb{Z}_p^2) , $m > 2$, when $m \equiv \pm 1 \pmod{p}$ or $m = 3, 6$ if $p = 3$ is the dihedral group of order $2m$, generated by matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & \theta + \theta^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where θ is a primitive m -th root of unity. It is also usual to call them $G(m, m, 2)$, following the notation of Shephard and Todd [63].

- Family 3: (C_m, \mathbb{Z}_p) when $m \mid p-1$ and C_m is the order m cyclic subgroup of \mathbb{Z}_p^\times .
- Sporadic groups: 34 sporadic \mathbb{Z}_p -reflection groups G_i , $4 \leq i \leq 37$.

See Andersen [5] for a more detailed description of this list of all irreducible \mathbb{Z}_p -reflection groups.

The automorphism group of the \mathbb{Z}_p -reflection group (W, L) is isomorphic to the group $N_{GL(L)}(W)$. There is an obvious homomorphism from this group to the group of trace preserving automorphisms of W . The kernel is the group $\text{Aut}_{\mathbb{Z}_p W}(L)$ of automorphisms of the $\mathbb{Z}_p W$ -module L . Using this we get an exact sequence of groups [53, 3.14–16]

$$(2) \quad 1 \rightarrow \text{Aut}_{\mathbb{Z}_p W}(L)/Z(W) \rightarrow N_{GL(L)}(W)/W \rightarrow \text{Out}_{\text{tr}}(W)$$

where the group to the right is the group

$$\text{Out}_{\text{tr}}(W) = \{\alpha \in \text{Out}(W) \mid \forall w \in W: \text{tr}(\alpha(w)) = \text{tr}(w)\}$$

of trace preserving outer automorphisms of $W < GL(L)$. Observe that there is a group homomorphism

$$\psi: \mathbb{Z}_p^\times \rightarrow N_{GL(L)}(W)/W$$

that takes the p -adic unit $u \in \mathbb{Z}_p^\times$ to scalar multiplication, $\psi^u: L \rightarrow L$, by u on L . The kernel of ψ is the finite subgroup $\mathbb{Z}_p^\times \cap Z(W)$ of $W < GL(L)$.

If (W, L) is irreducible, $\text{Aut}_{\mathbb{Z}_p} W(L) = \mathbb{Z}_p^\times$ consists only of the scalar matrices ψ^u according to Schur's lemma so that (2) takes the form

$$(3) \quad 1 \rightarrow \mathbb{Z}_p^\times / Z(W) \xrightarrow{\psi} N_{GL(L)}(W)/W \rightarrow \text{Out}_{\text{tr}}(W).$$

Moreover, an explicit case-by-case computation shows that the group $\text{Out}_{\text{tr}}(W)$ is trivial for all irreducible \mathbb{Z}_p -reflection groups except for a few of the dihedral groups $G(m, m, 2)$ and for the sporadic \mathbb{Z}_p -reflection groups G_5 , G_7 , and $G_{28} = W(F_4)$, and in these cases it consists of elements that lift to finite order elements τ in $N_{GL(L)}(W)/W$. We conclude that if (W, L) is irreducible then $N_{GL(L)}(W)/W$ consists only of elements of the form $\tau\psi^u$ where τ has finite order.

Theorem 2.2 (Classification of p -compact groups at odd primes [53; 7]) *Let p be an odd prime. The assignment $X \rightsquigarrow (W_X, L_X)$ gives a bijective correspondence between isomorphism classes of connected p -compact groups X and isomorphism classes of \mathbb{Z}_p -reflection groups (W, L) . We have*

$$\text{Out}(X) \cong N_{GL(L)}(W)/W$$

where (W, L) is the \mathbb{Z}_p -reflection group assigned to the connected p -compact group X .

The *irreducible* p -compact groups, which are the p -compact groups corresponding to the irreducible \mathbb{Z}_p -reflection groups of the Clark–Ewing classification table [19] (see also Dwyer, Miller and Wilkerson [23, 1.5]) are as follows:

- Family 1: $BSU(n+1)_p^\wedge$ (the special unitary groups)
- Family 2a: $B\mathbf{X}(m, r, n)$, $(m, r, n) \neq (m, m, 2)$, (the generalized Grassmannians)
- Family 2b: $B\mathbf{X}(m, m, 2)$, $m \geq 3$
- Family 3: $B\widehat{S}_p^{2m-1}$ (the Sullivan spheres)
- Sporadic groups: 34 sporadic p -compact groups $B\mathbf{X}_i$, $4 \leq i \leq 37$.

Among the generalized Grassmannians we find

$$BX(2, 1, n) = BSO(2n + 1)_p^\wedge, \quad BX(2, 2, n) = BSO(2n)_p^\wedge,$$

in family 2b

$$BX(3, 3, 2) = BPU(3)_p^\wedge, \quad BX(6, 6, 2) = (BG_2)_p^\wedge,$$

and among the sporadic cases we find

$$BX_{28} = (BF_4)_p^\wedge, \quad BX_{35} = (BE_6)_p^\wedge, \quad BX_{36} = (BE_7)_p^\wedge, \quad BX_{37} = (BE_8)_p^\wedge.$$

Any simply connected p -compact group splits as a product of irreducible p -compact groups [27; 59], and, in general, any connected p -compact group is locally isomorphic to the product of finitely many irreducible simply connected p -compact groups and a p -compact torus [50, 2.8].

If $H^*(BX; \mathbb{F}_p)$ is a polynomial \mathbb{F}_p -algebra, we say that BX is a *polynomial* p -compact group. Observe that all the irreducible p -compact groups are either polynomial or of the form BG_p^\wedge where G is an irreducible compact connected Lie group [53, 7.4].

The polynomial irreducible p -compact groups, which include all irreducible p -compact groups that are exotic, can be constructed as homotopy colimits of diagrams whose nodes are the p -subgroups of the Weyl group [53, 7.8]. We mention these special cases for later reference:

- Clark–Ewing p -compact groups: The p -compact groups corresponding to the \mathbb{Z}_p -reflection groups (W, L) where the order of W is prime to p [19]. They have the form

$$BX = (B(\check{T} \rtimes W))_p^\wedge$$

where $\check{T} \rtimes W$ is the semidirect product for the action of the Weyl group on the discrete maximal torus $\check{T} = (L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) / L \cong (\mathbb{Z} / p^\infty)^r$ where r is the rank. The Sullivan spheres (family 3)

$$B\hat{S}_p^{2m-1} \equiv B(\mathbb{Z} / p^\infty \rtimes C_m)_p^\wedge,$$

where $m|(p-1)$, are special cases of this construction. Also family 2b for $p > 3$ is included here.

- Aguadé p -compact groups: The four p -compact groups, \mathbf{X}_{12} at $p = 3$, \mathbf{X}_{29} at $p = 5$, \mathbf{X}_{31} at $p = 5$, and \mathbf{X}_{34} at $p = 7$ constructed by Aguadé [2] in a uniform way as homotopy colimits of diagrams

$$Z(W)^{\text{op}} \hookrightarrow BSU(r + 1) \xleftarrow{\Sigma_{r+1}^{\text{op}} \setminus W^{\text{op}}} BT \hookrightarrow W^{\text{op}}$$

with two nodes where $r = 2, 4, 4, 6$, respectively, is the rank and $Z(W)$, cyclic of order $2, 4, 4, 6$, respectively, is the center of the Weyl group W . In all four cases p divides the order of the Weyl group exactly once. The two cases \mathbf{X}_{12} , \mathbf{X}_{31} had been constructed by Zabrodsky using different methods [71].

- Generalized Grassmannians: The p -compact groups $\mathbf{X}(m, r, n)$ corresponding to the \mathbb{Z}_p -reflection groups $G(m, r, n)$ where $r|m|(p-1)$. The cases $r = 1$ were constructed by Quillen as p -completed classifying spaces of general linear groups over suitable infinite fields for characteristic prime to p . The cases with $r > 1$ were later obtained by Oliver; see Notbohm [58]. See also Møller [53, 7.10].

Theorem 2.2 describes $\text{Out}(X)$, the group of invertible elements of the monoid $[BX, BX]$ of unpointed homotopy classes of self-maps of BX , in purely algebraic terms as the “Weyl group of the Weyl group”, $N_{GL(L)}(W)/W$. In particular, we may regard the automorphism ψ^u of (W, L) as the homotopy class of a self-homotopy equivalence of BX . The map $\psi^u: BX \rightarrow BX$ is called an *unstable Adams operation* of exponent $u \in \mathbb{Z}_p^\times$.

Classically, unstable Adams operations were first defined by Sullivan [67] on $BU(n)$, for $q \in \mathbb{Z}$, $(p, q) = 1$, $q > n$, as restrictions of Adams operations defined on BU . Then extended by Wilkerson to all compact Lie groups [68]. In [38] the second author showed that p -completed classifying spaces of compact connected Lie groups admit unstable Adams operations ψ^q of exponent a p -adic unit $q \in \mathbb{Z}_p^*$. This is extended to p -compact groups for odd primes p in [53].

3 p -local finite groups

The concept of *p -local finite group* has been introduced by Broto, Levi and Oliver [14] (see also their survey [15]). A p -local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a finite p -group, \mathcal{F} a saturated fusion system over S , and \mathcal{L} a centric linking system associated to \mathcal{F} . We will state here again all necessary definitions for the convenience of the reader.

A *fusion system* over a finite group S consists of a set $\text{Hom}_{\mathcal{F}}(P, Q)$ of monomorphisms for every pair of subgroups P, Q of S , such that it contains at least those monomorphisms induced by conjugation by elements of S and all together form a category where every morphism factors as an isomorphism followed by an inclusion. A fusion system is *saturated* if it satisfies certain additional axioms formulated by L Puig [60] (see also Broto, Levi and Oliver [14, Section 1]). Two subgroups P, P' of S are called *\mathcal{F} -conjugate* if there is an isomorphism between them in \mathcal{F} .

Definition 3.1 Let \mathcal{F} be a fusion system over a p -group S .

- (1) A subgroup $P \leq S$ is *fully centralized in \mathcal{F}* if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which is \mathcal{F} -conjugate to P .
- (2) A subgroup $P \leq S$ is *fully normalized in \mathcal{F}* if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is \mathcal{F} -conjugate to P .
- (3) \mathcal{F} is a *saturated fusion system* if the following two conditions hold:
 - (i) For each $P \leq S$ which is fully normalized in \mathcal{F} , P is fully centralized in \mathcal{F} and $\text{Aut}_S(P)$ is a Sylow p -subgroup of $\text{Aut}_{\mathcal{F}}(P)$.
 - (ii) If $P \leq S$ and $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$ are such that φP is fully centralized, and if we set

$$N_{\varphi} = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \text{Aut}_S(\varphi P)\},$$

then there is $\bar{\varphi} \in \text{Hom}_{\mathcal{F}}(N_{\varphi}, S)$ such that $\bar{\varphi}|_P = \varphi$.

A subgroup P of S is *\mathcal{F} -centric* if $C_S(P') \leq P'$ for every subgroup $P' \leq S$ which is \mathcal{F} -conjugate to P . \mathcal{F}^c denotes the full subcategory whose objects are the *\mathcal{F} -centric* subgroups of S .

A subgroup $P \leq S$ is *\mathcal{F} -radical* if $\text{Out}_{\mathcal{F}}(P) = \text{Aut}_{\mathcal{F}}(P)/\text{Inn}(P)$ is p -reduced, namely, it does not contain nontrivial normal p -subgroups.

Definition 3.2 Let \mathcal{F} be a fusion system over the p -group S . A *centric linking system associated to \mathcal{F}* is a category \mathcal{L} whose objects are the \mathcal{F} -centric subgroups of S , together with a functor

$$\pi: \mathcal{L} \longrightarrow \mathcal{F}^c,$$

and *distinguished* monomorphisms $\delta_P: P \longrightarrow \text{Aut}_{\mathcal{L}}(P)$ for each \mathcal{F} -centric subgroup $P \leq S$, which satisfy the following conditions.

- (A) π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\text{Mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \text{Aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\text{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \text{Hom}_{\mathcal{F}}(P, Q).$$

- (B) For each \mathcal{F} -centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in \text{Aut}_{\mathcal{L}}(P)$ to $c_g \in \text{Aut}_{\mathcal{F}}(P)$.

- (C) For each $f \in \text{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, the following square commutes in \mathcal{L} :

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \delta_P(g) \downarrow & & \downarrow \delta_Q(\pi(f)(g)) \\ P & \xrightarrow{f} & Q. \end{array}$$

The *classifying space* of the p -local finite group $(S, \mathcal{F}, \mathcal{L})$ is defined as the p -completion $|\mathcal{L}|_p^\wedge$ of the nerve of the category \mathcal{L} . The classifying space determines the p -local finite group in the sense that two p -local finite groups are isomorphic if and only if they have homotopy equivalent classifying spaces. Actually, the complete structure of a p -local finite group can be recovered from its classifying space by homotopy theoretic methods.

Finite groups are the main source of examples and motivation for p -local finite group theory.

Example 3.3 (The p -local finite group $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ of a finite group G) If G is a finite group and S a Sylow p -subgroup, the monomorphisms from $P \leq S$ to $Q \leq S$ induced by conjugation in G , $\text{Hom}_G(P, Q) \cong N_G(P, Q)/C_G(P)$, where $N_G(P, Q) = \{x \in G \mid xPx^{-1} \leq Q\}$, form a saturated fusion system over S , $\mathcal{F}_S(G)$. The $\mathcal{F}_S(G)$ -centric subgroups of S are the subgroups $P \leq S$ which are p -centric in G . A p -subgroup $P \leq G$ is p -centric if its center, $Z(P)$, is the Sylow p -subgroup of $C_G(P)$, or, equivalently, if the centralizer splits as the product of the center of P and a group $C'_G(P)$ of order prime to p , $C_G(P) = Z(P) \times C'_G(P)$.

Now, we define $\mathcal{L}_S^c(G)$ as the category with objects all subgroups of S which are p -centric in G , and morphisms $\text{Mor}_{\mathcal{L}}(P, Q) \cong N_G(P, Q)/C'_G(P)$, where $C'_G(P)$ is the p' -complement in $C_G(P)$ of the center of P , which is well defined because P is p -centric. $\mathcal{L}_S^c(G)$ is a centric linking system associated to $\mathcal{F}_S(G)$, and $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ is a p -local finite group with classifying space $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$ [13; 14].

A p -subgroup P of G is called p -radical if it is the maximal normal p -subgroup of $N_G(P)$, $P = O_p(N_G(P))$, or, equivalently, if $N_G(P)/P$ is p -reduced [35], whereas being $\mathcal{F}_S(G)$ -radical means that $\text{Out}_{\mathcal{F}_S(G)}(P) \cong N_G(P)/PC_G(P) = \text{Out}_G(P)$ is p -reduced. However, if $P \leq S$ is $\mathcal{F}_S(G)$ -centric and $\mathcal{F}_S(G)$ -radical, then it is p -centric and p -radical in G : Assume that P is not p -radical in G , then there is another p -subgroup Q with $P \triangleleft Q \triangleleft N_G(P)$ and $Q \neq P$. Since P is p -centric, $C_G(P) = Z(P) \times C'_G(P)$, where $C'_G(P)$ is a p' -group, hence $C'_G(P) \cap Q = 1$.

Therefore $P \triangleleft Q \triangleleft N_G(P)/C'_G(P)$ and $Q/P \triangleleft N_G(P)/PC'_G(P) = N_G(P)/PC_G(P)$, hence $\text{Out}_G(P)$ is not p -reduced. The converse it is not always true.

Alperin's fusion theorem for saturated fusion systems [14, A.10] establishes that morphisms in a saturated fusion system \mathcal{F} are composites of automorphisms of fully normalized, \mathcal{F} -centric, and \mathcal{F} -radical subgroups of the system, or restrictions of those. Hence in order to describe a saturated fusion system \mathcal{F} over a finite p -group S it is enough to describe $\text{Aut}_{\mathcal{F}}(Q_i)$ for a set Q_1, \dots, Q_r of fully normalized representatives of \mathcal{F} -conjugacy classes of \mathcal{F} -centric, \mathcal{F} -radical subgroups of S in \mathcal{F} . This motivates the next construction.

If \mathcal{F}_0 is a fusion system over S , and Q_1, \dots, Q_r are subgroups of S , and Δ_i a group of automorphisms such that $\text{Inn}(Q_i) \leq \Delta_i \leq \text{Aut}(Q_i)$, for each i , then we denote by $\mathcal{F}_{Q_i}(\Delta_i)$ the fusion system over Q_i whose morphisms are restrictions of elements of Δ_i , and define

$$\mathcal{F} = \langle \mathcal{F}_0; \mathcal{F}_{Q_1}(\Delta_1), \dots, \mathcal{F}_{Q_r}(\Delta_r) \rangle$$

the fusion system over S whose morphisms are composites of morphisms belonging to any of the generating fusion systems (cf [14, Section 9]).

Thus, in particular, if \mathcal{F} is a saturated fusion system over a finite p -group S and Q_1, \dots, Q_r is a set of fully normalized representatives of \mathcal{F} -conjugacy classes of \mathcal{F} -centric, \mathcal{F} -radical subgroups of S in \mathcal{F} , then

$$\mathcal{F} = \langle \mathcal{F}_S(\text{Aut}_{\mathcal{F}}(S)); \mathcal{F}_{Q_1}(\text{Aut}_{\mathcal{F}}(Q_1)), \dots, \mathcal{F}_{Q_r}(\text{Aut}_{\mathcal{F}}(Q_r)) \rangle.$$

We now describe the fusion systems of $GL_p(q)$ and $SL_p(q)$ over the respective Sylow p -subgroups, where p is a prime number and q is a prime power $q \equiv 1 \pmod p$. This will be useful in later sections.

Example 3.4 (The fusion system of $GL_p(q)$, p odd) We will describe the fusion system of $GL_p(q)$ over a Sylow p -subgroup, for p a prime and q a prime power such that $q \equiv 1 \pmod p$. We can use the Alperin-Fong description of p -radical subgroups of general linear groups [4]. The elements

$$B = \text{diag}(1, \zeta, \zeta^2, \dots, \zeta^{p-1}), \quad C = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

where ζ be a primitive p -th root of unity in \mathbb{F}_q^* , generate an extraspecial subgroup $\Gamma_1 = \langle B, C \rangle \leq GL_p(q)$ of order p^3 and exponent p .

R	$N_{GL_p(q)}(R)$	$\text{Out}_{GL_p(q)}(R)$
Z_ℓ	$GL_p(q)$	1
T_ℓ^p	$(\mathbb{F}_q^*)^p \rtimes \Sigma_p$	Σ_p
\bar{S}	$(\mathbb{F}_q^*)^p \rtimes (\mathbb{Z}/p \rtimes \mathbb{Z}/p-1)$	$\mathbb{Z}/p-1$
Γ_ℓ	$(\mathbb{F}_q^*) \cdot \Gamma_\ell \cdot SL_2(p)$	$SL_2(p)$
$U_{\ell+1}$	$\mathbb{F}_{q^p}^* \rtimes \mathbb{Z}/p$	\mathbb{Z}/p

Table 1: p -radical subgroups of $GL_p(q)$ for $q \equiv 1 \pmod p$

R	$\text{Out}_{GL_p(q)}(R)$
T_ℓ^p	Σ_p
\bar{S}	$\mathbb{Z}/p-1$
Γ_ℓ	$SL_2(p)$

Table 2: \mathcal{F} -centric, \mathcal{F} -radical subgroups in the fusion system of $GL_p(q)$

The p -primary part of the multiplicative group of units \mathbb{F}_q^* is isomorphic to \mathbb{Z}/p^ℓ where $\ell = v_p(1 - q)$. Let $T_\ell^p \cong (\mathbb{Z}/p^\ell)^p$, the maximal finite torus, be the group of diagonal matrices of p -power order. Then $\bar{S} = T_\ell^p \rtimes \langle C \rangle \cong \mathbb{Z}/p^\ell \wr \mathbb{Z}/p$ is a Sylow p -subgroup of $GL_p(q)$.

Define the subgroup $\Gamma_\ell = Z_\ell \circ \Gamma_1 \leq GL_p(q)$ to be the central product over the center of Γ_1 of the center $Z_\ell \cong \mathbb{Z}/p^\ell$ of $GL_p(q)$ and Γ_1 .

There is an standard inclusion $\mathbb{F}_{q^p}^* \subseteq GL_p(q)$, obtained by letting $\mathbb{F}_{q^p}^*$ act on \mathbb{F}_{q^p} by multiplication and considering \mathbb{F}_{q^p} as \mathbb{F}_q -vector space. We define $U_{\ell+1}$ as the image in $GL_p(q)$ of the cyclic group $\mathbb{Z}/p^{\ell+1} \leq \mathbb{F}_{q^p}^*$ of all roots of unity of p -power order in \mathbb{F}_{q^p} .

With this notation and according to Alperin and Fong [4], if R is a p -radical subgroup of $GL_p(q)$ then R is conjugate to one of the subgroups displayed in Table 1.

It is now easy to extract from Table 1 the \mathcal{F} -centric, \mathcal{F} -radical subgroups of \bar{S} in the fusion system $\mathcal{F} = \mathcal{F}_{\bar{S}}(GL_p(q))$ of $GL_p(q)$ over \bar{S} . Notice that Z_ℓ is clearly not \mathcal{F} -centric and $U_{\ell+1}$ clearly not \mathcal{F} -radical. This leads to Table 2.

Example 3.5 (The fusion system of $SL_p(q)$, p odd) We proceed now by describing the fusion system of $SL_p(q)$ over a Sylow p -subgroup, for p a prime and q a prime power such that $q \equiv 1 \pmod p$. Let $\ell = v_p(1 - q)$ as in the previous example.

P	$\text{Out}_{SL_p(q)}(P)$	Conditions
$T_\ell^{(p-1)}$	Σ_p	$p > 3$
S	$\mathbb{Z}/p-1$	
$\Gamma_1(\xi^r)$	$SL_2(p)$	$r = 0$ if $\ell = 1, p = 3$; $r = 0, 1, \dots, p-1$ if $\ell > 1$ or $p > 3$,

Table 3: \mathcal{F} -centric \mathcal{F} -radical subgroups in the fusion system of $SL_p(q)$

We first show that every p -radical subgroup of $SL_p(q)$ is the intersection $Q \cap SL_p(q)$ of a p -radical subgroup Q of $GL_p(q)$ with $SL_p(q)$. For a given p -radical p -subgroup P of $SL_p(q)$ define $Q = O_p(N_{GL_p(q)}(P))$. $Q \cap SL_p(q)$ is a normal subgroup of $N_{SL_p(q)}(P)$ and since P is the maximal normal p -subgroup of $N_{SL_p(q)}(P)$, we have $Q \cap SL_p(q) \leq P$. Same argument with $N_{GL_p(q)}(P)$ shows that $P \leq Q$ and therefore $Q \cap SL_p(q) \leq P$.

Every element $g \in GL_p(q)$ normalizes $SL_p(q)$, so if g normalizes Q it also normalizes $Q \cap SL_p(q) \leq P$, so $N_{GL_p(q)}(Q) \leq N_{GL_p(q)}(P)$. But, by definition of Q , this is normal in $N_{GL_p(q)}(P)$, hence we actually have $N_{GL_p(q)}(Q) = N_{GL_p(q)}(P)$. So, therefore, $Q = O_p(N_{GL_p(q)}(Q))$ is p -radical.

Fix the Sylow p -subgroup $S = \bar{S} \cap SL_p(q)$ of $SL_p(q)$, and let $\mathcal{F} = \mathcal{F}_S(SL_p(q))$ be the fusion system of $SL_p(q)$ over S . Assume that $P \leq S$ is \mathcal{F} -centric and \mathcal{F} -radical. Then P is p -centric and p -radical in $SL_p(q)$. In particular $P = Q \cap SL_p(q)$ where Q is p -radical in $GL_p(q)$, hence conjugate by an element $g \in GL_p(q)$ to a p -subgroup in the Table 1. Among those intersections, only $S = \bar{S} \cap SL_p(q)$, $T_\ell^{(p-1)} = S \cap T_\ell^p$, and $\Gamma_1 = S \cap \Gamma_\ell$ are also p -centric. Hence, the complete list of conjugacy classes of p -centric and p -radical subgroups of $SL_p(q)$, is obtained by conjugating these three subgroups by elements $g \in GL_p(q)$: where $\Gamma_1(\xi^r)$, $r = 0, 1, \dots, (p-1)$ are subgroups of $SL_p(q)$, defined as the conjugates of Γ_1 in $GL_p(q)$, $\Gamma_1(\xi^r) = x_r \Gamma_1 x_r^{-1}$, where $x_r = \text{diag}(\xi^r, 1, \dots, 1) \in GL_p(q)$, ξ a $(q-1)$ -st root of unity. Notice that for $g \in GL_p(q)$, gSg^{-1} lies in S if and only if it is exactly S and the same happens with $T_\ell^{(p-1)}$. In the case of Γ_1 we just need to check which of the subgroups $\Gamma_1(\xi^r)$ are conjugate in $SL_p(q)$. In fact, Alperin's fusion theorem [14, A.10], together with the list of p -radical p -centric subgroups that we have obtained so far, tells us that if two subgroups $\Gamma_1(\xi^r)$ and $\Gamma_1(\xi^s)$ are conjugate in $SL_p(q)$ they are already conjugate in $N_{SL_p(q)}(S)$, hence we obtain the Table 3 by direct calculation as a list of p -centric and p -radical subgroups but, by inspection, this coincides with the list of \mathcal{F} -centric \mathcal{F} -radical subgroups.

An p -local finite group that is not of the form $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ for any finite group G is called *exotic*. Examples of exotic p -local finite groups are already shown in [14]. Recently, Levi and Oliver have obtained a family of exotic 2-local finite groups, $B \text{Sol}(q)$ [40], based on fusion systems originally described by Solomon [65].

Definition 3.6 (a) For any saturated fusion system \mathcal{F} over a p -group S , and any $P \leq S$, fully centralized in \mathcal{F} , the *centralizer* fusion system $C_{\mathcal{F}}(P)$ over $C_S(P)$ is defined by setting

$$\text{Hom}_{C_{\mathcal{F}}(P)}(Q, Q') = \{(\varphi|_Q) \mid \varphi \in \text{Hom}_{\mathcal{F}}(PQ, PQ'), \varphi(Q) \leq Q', \varphi|_P = \text{Id}_P\}$$

for all $Q, Q' \leq C_S(P)$.

(b) For a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ and $P \leq S$ fully centralized in \mathcal{F} , we define the category $C_{\mathcal{L}}(P)$ whose objects are $C_{\mathcal{F}}(P)$ -centric subgroups $Q \leq C_S(P)$ and where

$$\text{Mor}_{C_{\mathcal{L}}(P)}(Q, Q') = \{\varphi \in \text{Hom}_{\mathcal{L}}(PQ, PQ') \mid \pi(\varphi)|_P = \text{Id}_P, \pi(\varphi)(Q) \leq Q'\}.$$

It is proved in [14, Section 2] that if $(S, \mathcal{F}, \mathcal{L})$ is a p -local finite group and $P \leq S$ is fully centralized in \mathcal{F} , then $(C_S(P), C_{\mathcal{F}}(P), C_{\mathcal{L}}(P))$ is a p -local finite group.

In [40] Levi and Oliver have obtained necessary and sufficient conditions for a fusion system to be saturated. We reproduce here their result for the convenience of the reader. We will write $C_{\mathcal{F}}(x) = C_{\mathcal{F}}(\langle x \rangle)$ for $x \in S$.

Proposition 3.7 [40] *Let \mathcal{F} be any fusion system over a p -group S . Then \mathcal{F} is saturated if and only if there is a set \mathfrak{X} of elements of order p in S such that the following conditions hold:*

- (a) *Each $x \in S$ of order p is \mathcal{F} -conjugate to some element of \mathfrak{X} .*
- (b) *If x and y are \mathcal{F} -conjugate and $y \in \mathfrak{X}$, then there is some homomorphism $\psi \in \text{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$ such that $\psi(x) = y$.*
- (c) *For each $x \in \mathfrak{X}$, $C_{\mathcal{F}}(x)$ is a saturated fusion system over $C_S(x)$.*

4 Recognition of classifying spaces of p -local finite groups

In [14] it is shown that a p -local finite group can be completely recovered from its classifying space by homotopy theoretic methods. Also, a recognition principle for classifying spaces of p -local finite groups is provided in [14, Theorem 7.5]. We will

briefly describe these methods and derive an inductive method that will be useful in our situation.

We will first recall how a fusion system $\mathcal{F}_{(S,f)}(X)$ and a linking system $\mathcal{L}_{(S,f)}(X)$ are attached to a space X equipped with a map $f: BS \rightarrow X$, where S is a finite p -group.

If (S, f) is a p -subgroup of a space X we can define a fusion system over S , $\mathcal{F}_{(S,f)}(X)$, by declaring

$$\mathrm{Hom}_{\mathcal{F}_{(S,f)}(X)}(P, Q) = \{ \varphi \in \mathrm{Hom}(P, Q) \mid f|_{BP} \simeq f|_{BQ} \circ B\varphi \}$$

for all $P, Q \leq S$, where $f|_{BP}$ denotes the composition $BP \xrightarrow{Bi_P} BS \xrightarrow{f} X$. Next, we define the category $\mathcal{L}_{(S,f)}(X)$ that has objects the subgroups of S and

$$\mathrm{Mor}_{\mathcal{L}_{(S,f)}(X)}(P, Q) = \{ (\varphi, [H]) \mid \begin{array}{l} \varphi \in \mathrm{Hom}(P, Q) \text{ and} \\ [H] \text{ is the homotopy class of a homotopy} \\ \text{from } f|_{BP} \text{ to } f|_{BQ} \circ B\varphi \}, \end{array}$$

and the full subcategory $\mathcal{L}_{(S,f)}^c(X)$ whose objects are $\mathcal{F}_{(S,f)}(X)$ -centric subgroups $P \leq S$.

The important question and the aim of the rest of this section is to find sufficient conditions on a space X and a p -subgroup (S, f) under which

$$(S, \mathcal{F}_{(S,f)}(X), \mathcal{L}_{(S,f)}^c(X))$$

is a p -local finite group and X is its classifying space $|\mathcal{L}_{(S,f)}^c(X)|_p^\wedge \simeq X$.

One first important case is that of $X = |\mathcal{L}|_p^\wedge$, the classifying space itself of a given p -local finite group $(S, \mathcal{F}, \mathcal{L})$. The distinguished homomorphism $\delta_S: S \rightarrow \mathrm{Aut}_{\mathcal{L}}(S)$ provides a functor $BS \rightarrow \mathcal{L}$, where BS denotes the category that has one object and its group of automorphisms is S . In turn, this functor induces a map between the respective nerves $|BS| \rightarrow |\mathcal{L}|$. Finally, composing with the p -completion of $|\mathcal{L}|$ we obtain a canonical map for $(S, \mathcal{F}, \mathcal{L})$:

$$\theta_S: |BS| \rightarrow |\mathcal{L}|_p^\wedge,$$

where we can identify $|BS| \simeq BS$. It turns out that $(S, \mathcal{F}_{(S,\theta_S)}(|\mathcal{L}|_p^\wedge), \mathcal{L}_{(S,\theta_S)}^c(|\mathcal{L}|_p^\wedge))$ is isomorphic to the original $(S, \mathcal{F}, \mathcal{L})$ [14, 7.3]. This is how a p -local finite group is completely recovered from its classifying space.

The basic tool in order to show that these systems define a p -local finite group with classifying space X is [14, Theorem 7.5]. In order to apply this theorem in our situation we face two main difficulties, namely, to show that the p -completed nerve of

$\mathcal{L}_{(S,f)}(X)$ is homotopy equivalent to X and to show that $\mathcal{F}_{(S,f)}(X)$ is a saturated fusion system. In order to overcome these difficulties, we develop in this section an inductive method mainly based on the centralizer decomposition of p -local finite groups.

Definition 4.1 Given spaces X and Y , we say that a map $\alpha: X \rightarrow Y$ is a homotopy monomorphism at p if the homotopy fiber of α , F , over any connected component of Y , is p -quasifinite; that is, the inclusion $F \rightarrow \text{Map}(B\mathbb{Z}/p, F)$ as constant maps is a weak homotopy equivalence.

Given two maps $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, where g is a homotopy monomorphism at p , it is not hard to prove f is also a homotopy monomorphism at p if and only if the composition $g \circ f$ is so.

Definition 4.2 Let X be a space. A finite p -subgroup of X is a pair (P, f) , where P is a finite p -group and $f: BP \rightarrow X$ is a homotopy monomorphism at p . A p -subgroup (S, f) of X is called a Sylow p -subgroup of X if for any other p -subgroup (Q, g) of X , $g: BQ \rightarrow X$ factors through $f: BS \rightarrow X$, up to homotopy. If (P, f) is a p -subgroup of X , then we denote $BC_X(P, f) = \text{Map}(BP, X)_f$.

Our basic example comes from p -local finite groups. If $(S, \mathcal{F}, \mathcal{L})$ is a p -local finite group, then (S, θ_S) is a Sylow p -subgroup of $|\mathcal{L}|_p^\wedge$. The map $\theta_S: |\mathcal{B}S| \rightarrow |\mathcal{L}|_p^\wedge$ satisfies the required conditions by [14, Theorem 4.4].

We will need later the next technical lemma.

Lemma 4.3 Assume that X and Y are spaces for which $\text{Map}(B\mathbb{Z}/p, X)_{ct} \simeq X$ and $\text{Map}(B\mathbb{Z}/p, Y)_{ct} \simeq Y$. Let $f: X \rightarrow Y$ be a homotopy monomorphism at p and $\mu: BP \rightarrow X$ a finite p -subgroup of X , then each map in the diagram

$$(4) \quad \begin{array}{ccc} BC_X(P, \mu) & \xrightarrow{\text{ev}} & X \\ f_\# \downarrow & & \downarrow f \\ BC_Y(P, f \circ \mu) & \xrightarrow{\text{ev}} & Y \end{array}$$

is a homotopy monomorphism at p .

Proof Let F be the homotopy fiber of the evaluation map

$$BC_X(P, \mu) = \text{Map}(BP, X)_\mu \xrightarrow{\text{ev}} X.$$

There is an induced fibration

$$\mathrm{Map}(B\mathbb{Z}/p, F) \rightarrow \mathrm{Map}(B\mathbb{Z}/p, \mathrm{Map}(BP, X)_\mu)_{\tilde{ct}} \rightarrow \mathrm{Map}(B\mathbb{Z}/p, X)_{ct}$$

where \tilde{ct} stands for all components mapping down to the component of the constant map in $\mathrm{Map}(B\mathbb{Z}/p, X)$. Since $\mathrm{Map}(B\mathbb{Z}/p, X)_{ct} \simeq X$, also

$$\mathrm{Map}(B\mathbb{Z}/p, \mathrm{Map}(BP, X)_\mu)_{\tilde{ct}} \simeq \mathrm{Map}(BP, \mathrm{Map}(B\mathbb{Z}/p, X)_{ct})_{\tilde{\mu}} \simeq \mathrm{Map}(BP, X)_\mu,$$

and therefore $\mathrm{Map}(B\mathbb{Z}/p, F) \simeq F$; that is, F is p -quasifinite and $\mathrm{ev}: C_X(P, \mu) \rightarrow X$ is a homotopy monomorphism at p . Similarly, $\mathrm{ev}: BC_Y(P, f \circ \mu) \rightarrow Y$ is a homotopy monomorphism at p . Finally, since all other maps in diagram (4) are homotopy monomorphisms at p , then also $f_{\#}$ is a homotopy monomorphism at p . \square

The next is a useful result that provides conditions on the space X and a Sylow p -subgroup (S, f) under which the fusion system $\mathcal{F}_{(S, f)}(X)$ is saturated. An element $x \in S$ of order p determines a homomorphism $i_x: \mathbb{Z}/p \rightarrow S$ and then a map $f \circ Bi_x: B\mathbb{Z}/p \rightarrow X$. We write $BC_X(x) = \mathrm{Map}(B\mathbb{Z}/p, X)_x$, the connected component that contains the map $f \circ Bi_x$, and $f_x: BC_S(x) \rightarrow BC_X(x)$ the map induced by f .

Proposition 4.4 *Let X be a space, (S, f) a Sylow p -subgroup of X , and \mathfrak{X} a set of elements of order p in S . Assume that:*

- (1) $\mathrm{Map}(B\mathbb{Z}/p, X)_{ct} \simeq X$.
- (2) For all $x \in \mathfrak{X}$, the natural map $f_x: BC_S(x) \rightarrow BC_X(x)$ is a Sylow p -subgroup for $BC_X(x)$.
- (3) For all $x \in \mathfrak{X}$, $\mathcal{F}_{(C_S(x), f_x)}(BC_X(x))$ is a saturated fusion system over $C_S(x)$.
- (4) For all $x \in S$ of order p , there is $\varphi \in \mathrm{Hom}_{\mathcal{F}_{(S, f)}(X)}(\langle x \rangle, S)$ such that $\varphi(x) \in \mathfrak{X}$.

Then $\mathcal{F}_{(S, f)}(X)$ is a saturated fusion system over S and $C_{\mathcal{F}_{(S, f)}(X)}(x)$ coincides with $\mathcal{F}_{(C_S(x), f_x)}(BC_X(x))$ as fusion systems over $C_S(x)$, for all $x \in \mathfrak{X}$.

Proof Write $\mathcal{F} = \mathcal{F}_{(S, f)}(X)$ for short. Clearly, \mathcal{F} is a fusion system over S . Condition (a) of Proposition 3.7 holds by (4); and it remains to show that conditions (b) and (c) of Proposition 3.7 hold.

Condition (b) of Proposition 3.7 Fix $x, y \in S$ of order p such that $y \in \mathfrak{X}$, and such that there is $\psi_0 \in \mathrm{Hom}_{\mathcal{F}}(\langle x \rangle, \langle y \rangle)$ with $\psi_0(x) = y$. We must show that ψ_0 extends to some $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$.

Since x and y are \mathcal{F} -conjugate,

$$[f \circ Bi_x] = [f \circ Bi_y] \in [B\mathbb{Z}/p, X],$$

so $\text{Map}(B\mathbb{Z}/p, X)_x = \text{Map}(B\mathbb{Z}/p, X)_y$. Since $C_S(y)$ is a Sylow p -subgroup of $\text{Map}(B\mathbb{Z}/p, X)_y$ by (2), the natural map $BC_S(x) \rightarrow \text{Map}(B\mathbb{Z}/p, X)_x$ factors through $BC_S(y)$. In other words, there is some $\psi \in \text{Hom}(C_S(x), C_S(y))$ such that the following square commutes up to homotopy

$$(5) \quad \begin{array}{ccc} BC_S(x) \times B\mathbb{Z}/p & \xrightarrow{f \circ B(\text{incl} \times i_x)} & X \\ B\psi \times \text{Id} \downarrow & & \parallel \\ BC_S(y) \times B\mathbb{Z}/p & \xrightarrow{f \circ B(\text{incl} \times i_y)} & X. \end{array}$$

Thus $\psi \in \text{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$. If $\rho, \rho' \in \text{Hom}(C_S(x) \times \mathbb{Z}/p, S)$ denote the homomorphisms $\rho(g, t) = gx^t$ and $\rho'(g, t) = \psi(g)y^t$, then $f \circ B\rho \simeq f \circ B\rho'$ by (5), and hence $\text{Ker}(\rho) = \text{Ker}(\rho')$ by [14, Proposition 5.4(d)] (and point (1)). And this implies that $\psi(x) = y$.

Condition (c) of Proposition 3.7 Fix some $x \in \mathfrak{X}$; we must show that $C_{\mathcal{F}}(x)$ is a saturated fusion system. By (3), the fusion system $\mathcal{F}' \stackrel{\text{def}}{=} \mathcal{F}_{(C_S(x), f_x)}(BC_X(x))$ is saturated, so it suffices to show that these two fusion systems over $C_S(x)$ are equal.

To see this, fix $P, Q \leq C_S(x)$, and let $\varphi \in \text{Hom}(P, Q)$ be any monomorphism. Set $\bar{P} = P \cdot \langle x \rangle$ and $\bar{Q} = Q \cdot \langle x \rangle$. Let $\rho \in \text{Hom}(P \times \mathbb{Z}/p, S)$ and $\rho' \in \text{Hom}(Q \times \mathbb{Z}/p, S)$ be defined by $\rho(g, t) = gx^t$ and $\rho'(g, t) = gx^t$. Then $\varphi \in \text{Hom}_{\mathcal{F}'}(P, Q)$ if and only if the following square commutes up to homotopy

$$(6) \quad \begin{array}{ccc} BP \times B\mathbb{Z}/p & \xrightarrow{f \circ B\rho} & X \\ B\varphi \times \text{Id} \downarrow & & \parallel \\ BQ \times B\mathbb{Z}/p & \xrightarrow{f \circ B\rho'} & X. \end{array}$$

This holds if and only if $K \stackrel{\text{def}}{=} \text{Ker}(\rho) = \text{Ker}(\rho' \circ (\varphi \times \text{Id}))$ by (1) and [14, Proposition 5.4(d)], and the induced maps from $B((P \times \mathbb{Z}/p)/K)$ to X are homotopic. The kernels are equal if and only if φ extends to a monomorphism $\bar{\varphi}$ from \bar{P} to \bar{Q} which sends x to itself. And in this case, the induced maps on $B((P \times \mathbb{Z}/p)/K)$ are homotopic if and only if $f|_{B\bar{P}} \simeq f|_{B\bar{Q}} \circ B\bar{\varphi}$, if and only if $\varphi \in \text{Hom}_{C_{\mathcal{F}}(x)}(P, Q)$.

Now, Proposition 3.7 implies that $\mathcal{F}_{(S, f)}(X)$ is a saturated fusion system over S and the argument for condition (c) already contains the proof that $C_{\mathcal{F}}(x)$ coincides with $\mathcal{F}' = \mathcal{F}_{(C_S(x), f_x)}(BC_X(x))$ as fusion systems over $C_S(x)$. □

We derive now another characterization that will be useful in the specific cases in which we are interested or more generally in cases in which there is a good knowledge of elementary abelian p -subgroups of X and of their centralizers.

Theorem 4.5 Let X be a p -complete space and (S, f) a p -subgroup of X . Assume the following conditions hold:

- (1) $\text{Map}(B\mathbb{Z}/p, X)_{ct} \simeq X$.
- (2) For each nontrivial element $x \in S$ of order p
 - (a) $BC_X(x)$ is the classifying space of a p -local finite group, and
 - (b) if (H, g) is a Sylow p -subgroup for $BC_X(x)$, there is a group homomorphism $\rho: H \rightarrow S$ that makes the diagram

$$\begin{array}{ccc}
 BH & \xrightarrow{B\rho} & BS \\
 g \downarrow & & \downarrow f \\
 BC_X(x) & \xrightarrow{\text{ev}} & X
 \end{array}$$

commutative up to homotopy,

Then (S, f) is a Sylow p -subgroup for X and

$$(S, \mathcal{F}_{(S,f)}(X), \mathcal{L}_{(S,f)}^c(X))$$

is a p -local finite group.

Furthermore, $X \simeq |\mathcal{L}_{(S,f)}(X)|_p^\wedge$ if and only if the natural map induced by evaluation

$$\text{hocolim}_{\mathcal{F}_{(S,f)}^e(X)^{\text{op}}} \text{Map}(BE, X)_{f|_{BE}} \longrightarrow X$$

is a mod p homology equivalence. Here $\mathcal{F}_{(S,f)}^e(X)$ denotes the full subcategory of $\mathcal{F}_{(S,f)}(X)$ consisting of nontrivial fully centralized (Definition 3.1) elementary abelian p -subgroups of S .

Proof The proof is divided in five steps. First, we prove that (S, f) is a Sylow p -subgroup of X . Next, that the fusion system of X over (S, f) , $\mathcal{F}_{(S,f)}(X)$ is saturated. In the third step we show that for each $\mathcal{F}_{(S,f)}(X)$ -centric subgroup $P \leq S$ the map $f|_{BP}$ is centric. A map $g: BP \rightarrow X$ is called centric if the induced map $f_\# : \text{Map}(BP, BP)_{\text{Id}} \rightarrow \text{Map}(BP, X)_g$ is a weak homotopy equivalence.

These two last steps are the hypothesis (a) and (c) of [14, Theorem 7.5]. According to the remarks after the proof of this theorem in [14], this suffices in order to conclude that $(S, \mathcal{F}_{(S,f)}(X), \mathcal{L}_{(S,f)}^c(X))$ is a p -local finite group. This is the first part of the theorem.

The second part states that $X \simeq |\mathcal{L}_{(S,f)}(X)|_p^\wedge$ if and only if the natural map induced by evaluation

$$\text{hocolim}_{\mathcal{F}_{(S,f)}^e(X)^{\text{op}}} \text{Map}(BE, X)_{f|_{BE}} \longrightarrow X$$

is a mod p homology equivalence. This is proved in steps 4 and 5. Notice that $X \simeq |\mathcal{L}_{(S,f)}(X)|_p^\wedge$ is condition (b) in [14, Theorem 7.5].

Step 1 ((S, f) is a Sylow p -subgroup for X .) Let (P, μ) be a finite p -subgroup of X . Choose a central element x of order p in P . It determines a homomorphism $i_x: \mathbb{Z}/p \rightarrow P$ for which $C_P(\mathbb{Z}/p) = P$, and a map $\mu \circ Bi_x: B\mathbb{Z}/p \rightarrow X$. According to our hypothesis, $BC_X(x)$ is the classifying space of a p -local finite group, and if (H, g) is its Sylow p -subgroup, there are homomorphisms $\rho: H \rightarrow S$ and $\varphi: C_P(\mathbb{Z}/p) \rightarrow H$ that make the diagram

$$\begin{array}{ccccc}
 & & B\varphi & & \\
 & \swarrow & & \searrow & \\
 BC_P(\mathbb{Z}/p) & \xrightarrow{\mu_\#} & BC_X(\mathbb{Z}/p, \mu \circ Bi_x) & \xleftarrow{g} & BH \\
 \downarrow \simeq \text{ev} & & \downarrow \text{ev} & & \downarrow B\rho \\
 BP & \xrightarrow{\mu} & X & \xleftarrow{f} & BS
 \end{array}$$

commutative up to homotopy. Hence, $\rho \circ \varphi: P = C_P(\mathbb{Z}/p) \rightarrow S$ provides the factorization of (P, μ) through (S, f) .

Step 2 (The fusion system of X over (S, f) , $\mathcal{F}_{(S,f)}(X)$ is saturated.) This part of the proof will be based on Proposition 4.4. Define

$$\mathfrak{X} = \left\{ x \in S \mid x \text{ of order } p \text{ and } f_x: BC_S(x) \rightarrow BC_X(x) \right. \\
 \left. \text{is a Sylow } p\text{-subgroup for } BC_X(x) \right\}.$$

Notice now that conditions (1) and (2) of Proposition 4.4 are satisfied by our hypothesis and by definition of the class \mathfrak{X} . Condition (3) is easily verified, too. In fact, by hypothesis, for each $x \in \mathfrak{X}$, $BC_X(x)$ is the classifying space of a p -local finite group and since $f_x: BC_S(x) \rightarrow BC_X(x)$ is a Sylow p -subgroup for $BC_X(x)$, the fusion system $\mathcal{F}_{(C_S(x), f_x)}(BC_X(x))$ is saturated.

It remains to verify condition (4); that is, that every element $x \in S$ of order p is $\mathcal{F}_{(S,f)}(X)$ -conjugate to an element of the class \mathfrak{X} .

Assume that $x \in S$ has order p . It gives a homomorphism $i_x: \mathbb{Z}/p \rightarrow S$ and a map $f \circ Bi_x: B\mathbb{Z}/p \rightarrow X$. There is an evaluation map $\text{ev}: B\mathbb{Z}/p \times BC_X(x) \rightarrow X$. Let

(H, g) be a Sylow p -subgroup of $BC_X(x)$. Since (S, f) is a Sylow p -subgroup of X , there is a homomorphism $\rho: \mathbb{Z}/p \times H \rightarrow S$ making the diagram

$$\begin{array}{ccc} B\mathbb{Z}/p \times BH & \xrightarrow{B\rho} & BS \\ \downarrow 1 \times g & & \downarrow f \\ B\mathbb{Z}/p \times BC_X(x) & \xrightarrow{\text{ev}} & X \end{array}$$

commutative up to homotopy.

Let $\varphi = \rho|_{\mathbb{Z}/p}$ the restriction of ρ to the first component \mathbb{Z}/p . From the above diagram we deduce that $\varphi \in \text{Hom}_{\mathcal{F}_{(S,f)}(X)}(\mathbb{Z}/p, S)$. Let $y = \varphi(x)$.

Then, ρ induces

$$BH \xrightarrow{B\tilde{\rho}} BC_S(y) \xrightarrow{f_y} BC_X(y) \xrightarrow{\text{ev}} X$$

where all maps are homotopy monomorphisms at p . The first one because $\tilde{\rho}$ is a monomorphism, the others by [Lemma 4.3](#).

Now, φ induces a homotopy equivalence $BC_X(y) \simeq BC_X(x)$, hence also an isomorphism between the respective Sylow p -subgroups. Since (H, g) is a Sylow p -subgroup for $C_X(x)$, it follows from the above sequence of maps that $(C_S(y), f_y)$ is a Sylow p -subgroup for $C_X(y)$. Hence $y = \varphi(x) \in \mathfrak{X}$.

Step 3 ($f|_{BP}$ is a p -centric map for each $\mathcal{F}_{(S,f)}(X)$ -centric subgroup $P \leq S$.) Suppose that $P \leq S$ is $\mathcal{F}_{(S,f)}(X)$ -centric. Choose a central element $x \in S$ of order p . Since P is $\mathcal{F}_{(S,f)}(X)$ -centric, $x \in P$ and we have a sequence of homotopy monomorphisms at p

$$BP \xrightarrow{B \text{incl}} BS \xrightarrow{f_x} BC_X(x) \xrightarrow{\text{ev}} X.$$

By hypothesis, $BC_X(x)$ is the classifying space of a p -local finite group, and from the above sequence of maps we easily obtain that (S, f_x) is a Sylow p -subgroup for $BC_X(x)$. Furthermore, P is also $\mathcal{F}_{(S,f_x)}(BC_X(x))$ -centric, and then $f_x|_{BP}$ is a p -centric map. There is a sequence of equivalences

$$\begin{aligned} (7) \quad \text{Map}(BP, BP)_{\text{Id}} &\simeq \text{Map}(BP, BC_X(x))_{f_x|_{BP}} \\ &\simeq \text{Map}(BP \times B\mathbb{Z}/p, X)_{f|_{BP \circ Bm}} \simeq \text{Map}(BP, X)_{f|_{BP}} \end{aligned}$$

where $m: P \times \mathbb{Z}/p \rightarrow P$ denotes multiplication by x , the generator of $\mathbb{Z}/p = \langle x \rangle$. The last equivalence is implied by the Zabrodsky's lemma (cf [\[20, Proposition 3.5\]](#))

applied to the fibration $B\mathbb{Z}/p \rightarrow BP \times B\mathbb{Z}/p \xrightarrow{Bm} BP$. The homotopy equivalence (7) shows that $f|_{BP}$ is a p -centric map.

Step 4 (There is a map $\kappa: |\mathcal{L}_{(S,f)}(X)|_p^\wedge \rightarrow X$ that induces homotopy equivalences

$$\kappa_P: \text{Map}(BP, |\mathcal{L}_{(S,f)}(X)|_p^\wedge)_{|\delta_S|_{BP}} \xrightarrow{\simeq} \text{Map}(BP, X)_{f|_{BP}},$$

for each nontrivial subgroup $P \leq S$.) The construction of the map $\kappa: |\mathcal{L}_{(S,f)}^c(X)| \rightarrow X$ requires some technical constructions and will be explained in Proposition 4.6. Indeed, it will be shown that there is a homotopy commutative diagram

(8)

$$\begin{array}{ccc} & BS & \\ \theta_S \swarrow & & \searrow f \\ |\mathcal{L}_{(S,f)}^c(X)| & \xrightarrow{\kappa} & X \end{array}$$

where we have identified $BS \simeq |\mathcal{B}S|$.

We will show that the induced map

(9)

$$\kappa_P: \text{Map}(BP, |\mathcal{L}_{(S,f)}^c(X)|_p^\wedge)_{\theta_S|_{BP}} \longrightarrow \text{Map}(BP, X)_{f|_{BP}}.$$

is a homotopy equivalence by induction on the order of the group P .

If $P = \langle x \rangle$, for some $x \in S$ of order p , then $BC_X(x) = \text{Map}(BP, X)_{f|_{BP}}$ is the classifying space of a finite p -local group, by hypothesis. According to Step 2 above, we can assume without loss of generality that $x \in \mathfrak{X}$, and so, the induced map $f_x: BC_S(x) \rightarrow BC_X(x)$ is the inclusion of a Sylow p -subgroup, and the fusion system $\mathcal{F}_{(C_S(x), f_x)}(BC_X(x))$ coincides with $C_{\mathcal{F}_{(S,f)}(X)}(x)$ by Proposition 4.4.

Now, diagram (8) induces the new homotopy commutative diagram

(10)

$$\begin{array}{ccc} & BC_S(x) & \\ \theta_\# \swarrow & & \searrow f_x \\ \text{Map}(BP, |\mathcal{L}_{(S,f)}^c(X)|_p^\wedge)_{\theta|_{BP}} & \xrightarrow{\kappa_P} & BC_X(x) \end{array}$$

where, according to [14, 6.3], the map $\theta_\#$ is the inclusion of a Sylow p -subgroup of the mapping space $\text{Map}(BP, |\mathcal{L}_{(S,f)}^c(X)|_p^\wedge)_{\theta|_{BP}}$ which is the classifying space of a centralizer p -local finite group with fusion system $C_{\mathcal{F}_{(S,f)}(X)}(x)$. Furthermore, κ_P induces an equivalence of fusion systems, and therefore a homotopy equivalence.

For an arbitrary nontrivial subgroup $P \leq S$, we fix an element x of order p in the center of P . Again, we can assume that x belongs to \mathfrak{X} . There is a diagram

$$\begin{array}{ccc}
 \text{Map}(BP, |\mathcal{L}_{(S,f)}^c(X)|)_{\theta|_{BP}} & \longrightarrow & \text{Map}(BP \times B\langle x \rangle, |\mathcal{L}_{(S,f)}^c(X)|)_{\theta|_{BP \circ Bm}} \longrightarrow \\
 \downarrow \kappa_P & & \downarrow \kappa_{P \times \langle x \rangle} \\
 \text{Map}(BP, X)_{f|_{BP}} & \longrightarrow & \text{Map}(BP \times B\langle x \rangle, X)_{f|_{BP \circ Bm}} \longrightarrow \\
 & & \longrightarrow \text{Map}(BP, \text{Map}(B\langle x \rangle, |\mathcal{L}_{(S,f)}^c(X)|)_{B \text{ incl}})_{\theta|_{BP}} \\
 & & \downarrow \text{Map}(1, \kappa_{\langle x \rangle}) \\
 & & \longrightarrow \text{Map}(BP, \text{Map}(B\langle x \rangle, X)_{f|_{BP}})
 \end{array}$$

where horizontal arrows are homotopy equivalences, by adjunction and by Zabrodsky's lemma (cf [20, Proposition 3.5]) applied to the fibration

$$B\mathbb{Z}/p \rightarrow BP \times B\langle x \rangle \xrightarrow{Bm} BP,$$

where we identify \mathbb{Z}/p with the kernel of multiplication $m: P \times \langle x \rangle \rightarrow P$. Also, $\text{Map}(1, \kappa_{\langle x \rangle})$ is a homotopy equivalence. That concludes the proof that κ_P in Equation (9) is a natural mod p homology equivalence for subgroups $P \leq S$.

Step 5 ($X \simeq |\mathcal{L}_{(S,f)}(X)|_p^\wedge$ if and only if the natural map

$$\text{hocolim}_{\mathcal{F}_{(S,f)}^e(X)^{\text{op}}} \text{Map}(BE, X)_{f|_{BE}} \longrightarrow X$$

induced by evaluation is a mod p homology equivalence.) Diagram (8) induces an isomorphism of fusion systems over S : $\mathcal{F}_{(S,\theta)}(|\mathcal{L}_{(S,f)}^c(X)|) = \mathcal{F}_{(S,f)}(X)$. We will consider the full subcategories of nontrivial fully centralized elementary abelian p -subgroups $E \leq S$. In order to simplify the notation, we will write \mathcal{F}^e in place of $\mathcal{F}_{(S,\theta)}^e(|\mathcal{L}_{(S,f)}^c(X)|) = \mathcal{F}_{(S,f)}^e(X)$.

For every elementary abelian subgroup $E \leq S$, the map κ_E , as defined in step 4, fits in a commutative diagram

$$\begin{array}{ccc}
 \text{Map}(BE, |\mathcal{L}_{(S,f)}^c(X)|)_{\theta|_{BE}} & \xrightarrow{\kappa_E} & \text{Map}(BE, X)_{f|_{BE}} \\
 \text{ev} \downarrow & & \downarrow \text{ev} \\
 |\mathcal{L}_{(S,f)}^c(X)| & \xrightarrow{\kappa} & X
 \end{array}$$

where vertical maps are induced by evaluation at the base point. As a consequence, we obtain a map between the corresponding homotopy colimits together with compatible

maps induced by evaluation:

$$(11) \quad \begin{array}{ccc} \text{hocolim}_{(\mathcal{F}^e)^{\text{op}}} \text{Map}(BE, |\mathcal{L}_{(S,f)}^c(X)|_p^\wedge)_{\theta|_{BE}} & \xrightarrow{\widehat{\kappa}} & \text{hocolim}_{(\mathcal{F}^e)^{\text{op}}} \text{Map}(BE, X)_{f|_{BE}} \\ \text{ev} \downarrow & & \downarrow \text{ev} \\ |\mathcal{L}_{(S,f)}^c(X)|_p^\wedge & \xrightarrow{\kappa} & X \end{array}$$

where $\widehat{\kappa} = \text{hocolim}_{(\mathcal{F}^e)^{\text{op}}} \kappa_E$ is the induced map between the respective homotopy colimits. It turns out that $\widehat{\kappa}$ is a homotopy equivalence because all κ_E are homotopy equivalences according to step 4. Also, the left vertical map of is a homotopy equivalence by [14, 2.6 and 6.3].

Hence, the right vertical map ev in (11) is a homotopy equivalence if and only if κ is a homotopy equivalence. This proves step 5. \square

Notice also, that, reciprocally, if X is the classifying space of a p -local finite group with Sylow p -subgroup (S, f) , then all conditions of Theorem 4.5 are satisfied according to [14, Section 7].

There seems to be no natural way to construct a map between X and $|\mathcal{L}_{(S,f)}^c(X)|$ in either direction. This problem was solved by Broto, Levi and Oliver [13] by means of some auxiliary constructions. For the convenience of the reader we shall reproduce the argument here. For this aim we will introduce a variation of the categories $\mathcal{F}_{(S,f)}(X)$ and $\mathcal{L}_{(S,f)}(X)$, independent of the choice of a Sylow p -subgroup.

For a space X , we denote $\mathcal{F}_p(X)$ the category in which the objects are finite p -subgroups (P, f) of X , and the morphisms are defined

$$\text{Mor}_{\mathcal{F}_p(X)}((P, f), (Q, g)) = \{ \varphi \in \text{Hom}(P, Q) \mid f \simeq g \circ B\varphi \}.$$

Similarly, $\mathcal{L}_p(X)$ is the category in which the objects are the p -subgroups (P, f) of X and morphisms are defined as

$$\text{Mor}_{\mathcal{L}_p(X)}((P, f), (Q, g)) = \{ (\varphi, [H]) \mid \varphi \in \text{Hom}(P, Q) \text{ and } [H] \text{ is the homotopy class of a homotopy from } f \text{ to } g \circ B\varphi \}.$$

Note that if (S, f) is a p -subgroup of X , then there are obvious functors $\mathcal{F}_{(S,f)}(X) \rightarrow \mathcal{F}_p(X)$ and $\mathcal{L}_{(S,f)}(X) \rightarrow \mathcal{L}_p(X)$, sending and sending an object P of $\mathcal{F}_{(S,f)}(X)$ (resp. $\mathcal{L}_{(S,f)}(X)$) to the map $f|_{BP}: BP \rightarrow X$ considered as an object of $\mathcal{F}_p(X)$ (resp. $\mathcal{L}_p(X)$). Furthermore, if (S, f) is a Sylow p -subgroup, then these are equivalences of categories.

Proposition 4.6 *Let X be a space, S a finite p -group, and $f: BS \rightarrow X$ a map. Assume that (S, f) is a Sylow p -subgroup of X and that for each $\mathcal{F}_{(S,f)}(X)$ -centric subgroup $P \leq S$, $f|_{BP}$ is a centric map, then there is a homotopy equivalence $\kappa_S: |\mathcal{B}S| \rightarrow BS$ and a map $\kappa_X: |\mathcal{L}_{(S,f)}^c(X)| \rightarrow X$ such that the diagram*

$$\begin{array}{ccc} |\mathcal{B}S| & \xrightarrow[\simeq]{\kappa_S} & BS \\ \theta_S \downarrow & & \downarrow f \\ |\mathcal{L}_{(S,f)}^c(X)| & \xrightarrow{\kappa_X} & X. \end{array}$$

is homotopy commutative.

Proof We will sketch here the necessary constructions in order to obtain the map $\kappa_X: |\mathcal{L}_{(S,f)}^c(X)| \rightarrow X$. We refer to Broto, Levi and Oliver [13, Section 4] for full details.

We denote $\mathcal{L}_p^c(X)$ the full subcategory of $\mathcal{L}_p(X)$ whose objects are the p -subgroups (P, f) of X where f is a centric map. The hypothesis on (S, f) and on $\mathcal{F}_{(S,f)}(X)$ -centric subgroups imply that the functor $\mathcal{L}_{(S,f)}(X) \rightarrow \mathcal{L}_p(X)$ defined above restricts to an equivalence of categories

$$\mathcal{L}_{(S,f)}^c(X) \rightarrow \mathcal{L}_p^c(X).$$

In order to connect the nerve of $\mathcal{L}_p^c(X)$ and X , in [13], it is defined the simplicial space $M_\bullet^c(X)$ where n -simplices are maps $\eta: \Delta(\mathbf{P}) \rightarrow X$, where

$$\mathbf{P} = (P_0 \xrightarrow{\varphi_1} P_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_n} P_n)$$

is a sequence of p -subgroups of S and monomorphisms, and $\Delta(\mathbf{P})$ can be regarded as the homotopy colimit of the sequence

$$BP_0 \xrightarrow{B\varphi_1} BP_1 \xrightarrow{B\varphi_2} \dots \xrightarrow{B\varphi_n} BP_n,$$

with the condition that the restriction of η to any BP_i is a centric map.

The inclusion of base points in BP_i provides a map $\iota_{\mathbf{P}}: \Delta^n \rightarrow \Delta(\mathbf{P})$, and then an evaluation map

$$ev_X: |M_\bullet^c(X)| \rightarrow X,$$

where $ev_X(t, \eta) = \eta(\iota_{\mathbf{P}}(t))$.

For each i , the mapping cylinder of $BP_{i-1} \xrightarrow{\varphi_i} BP_i$ embeds naturally in $\Delta(\mathbf{P})$ and the restriction of η to this mapping cylinder can be interpreted as a homotopy between $\eta|_{BP_{i-1}}$ and $\eta|_{BP_i} \circ B\varphi_i$, thus, a morphism of $\mathcal{L}_p^c(X)$ from $\eta|_{BP_{i-1}} \rightarrow X$

to $\eta|_{BP_i} \rightarrow X$. In this way, the n -simplex $\eta: \Delta(\mathbf{P}) \rightarrow X$ determines an n -simplex in $N_\bullet(\mathcal{L}_p^c(X))$ and gives rise to a simplicial map from $M_\bullet^c(X)$ to the nerve of $\mathcal{L}_p^c(X)$, and therefore a map between the respective geometric realizations:

$$\tau_X: |M_\bullet^c(X)| \longrightarrow |\mathcal{L}_p^c(X)|.$$

Each object $\alpha: BP \rightarrow X$ of $\mathcal{L}_p^c(X)$ is a centric map. In particular, $\text{Map}(BP, X)_\alpha \simeq \text{Map}(BP, BP)_{\text{Id}} \simeq BZ(P)$ is aspherical, and so, according to [13, Lemma 4.2] (see its proof), $\tau_X: |M_\bullet^c(X)| \rightarrow |\mathcal{L}_p^c(X)|$ is a homotopy equivalence. Then, choosing a homotopy inverse of τ_X we can define $\kappa_X: |\mathcal{L}_{(S,f)}^c(X)| \rightarrow X$ as the composition

$$(12) \quad |\mathcal{L}_{(S,f)}^c(X)| \xrightarrow{\simeq} |\mathcal{L}_p^c(X)| \xleftarrow[\simeq]{\tau_X} |M_\bullet^c(X)| \xrightarrow{ev_X} X.$$

In case $X = BS$, Proposition 2.7, Lemma 4.2 and Lemma 4.3 of [13] provide homotopy equivalences

$$(13) \quad |\mathcal{L}_p^c(S)| \xrightarrow{\simeq} |\mathcal{L}_p^c(BS)| \xleftarrow{\simeq} |M_\bullet^c(BS)| \xrightarrow{\simeq} BS$$

hence, the key to finish the proof of the proposition lies in the naturality properties of this construction with respect to $f: BS \rightarrow X$. However, in general, a subgroup $P \leq S$ which is centric in S , need not be centric when regarded as a p -subgroup of X by considering the restriction $f|_{BP}: BP \rightarrow X$ of f to BP . For this reason, we will have to restrict $M_\bullet^c(BS)$ to the subspace $M_\bullet^S(BS)$ of simplices $\eta: \Delta(\mathbf{P}) \rightarrow BS$ of $M_\bullet^c(BS)$ where every group in the sequence \mathbf{P} is S itself. Accordingly, we call $\mathcal{L}_p^S(BS)$ the full subcategory of $\mathcal{L}_p^c(BS)$ with objects the homotopy equivalences $g: BS \rightarrow BS$. With this notation we have a diagram of homotopy equivalences

$$(14) \quad \begin{array}{ccccccc} |BS| & \xrightarrow{\simeq} & |\mathcal{L}_p^S(BS)| & \xleftarrow{\simeq} & |M_\bullet^S(BS)| & \xrightarrow[\simeq]{ev_{BS}} & BS \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \parallel \\ |\mathcal{L}_p^c(S)| & \xrightarrow{\simeq} & |\mathcal{L}_p^c(BS)| & \xleftarrow{\simeq} & |M_\bullet^c(BS)| & \xrightarrow[\simeq]{ev_{BS}} & BS \end{array}$$

where same arguments as in [13] for the sequence (13) are used.

Now, for every equivalence $g: BS \rightarrow BS$, the composition $BS \xrightarrow{g} BS \xrightarrow{f} X$ defines a centric p -subgroup of X , and then f induces a well defined map of simplicial spaces

$M_\bullet^S(BS) \rightarrow M_\bullet^c(X)$, that makes commutative the diagram

$$(15) \quad \begin{array}{ccccccc} |\mathcal{B}S| & \xrightarrow{\simeq} & |\mathcal{L}_p^S(BS)| & \xleftarrow{\simeq} & |M_\bullet^S(BS)| & \xrightarrow[\simeq]{ev_{BS}} & BS \\ \theta_S \downarrow & & \downarrow f_\# & & \downarrow f_\# & & \downarrow f \\ |\mathcal{L}_{(S,f)}^c(X)| & \xrightarrow{\simeq} & |\mathcal{L}_p^c(X)| & \xleftarrow[\simeq]{\tau_X} & |M_\bullet^c(X)| & \xrightarrow{ev_X} & X. \end{array}$$

Then $\kappa_S: |\mathcal{B}S| \xrightarrow{\simeq} BS$ is the composite homotopy equivalence in the top row of the above diagram, and this finishes proof. \square

5 Homotopy fixed point p -compact groups

Let M be a space and G a discrete group. An action of the group G on the space M is group homomorphism

$$\rho: G \rightarrow \text{aut}(M)$$

where $\text{aut}(M)$ is the topological monoid of self-homotopy equivalences of M .

Dwyer and Wilkerson introduced [25, Section 10] the homotopy theoretic notion of proxy actions. A proxy action of G on M is defined as a fibration

$$(16) \quad M \longrightarrow M_{hG} \xrightarrow{p} BG.$$

Now, this is classified up to fiber homotopy equivalence by a map

$$BG \rightarrow B \text{aut}(M).$$

Any action $\rho: G \rightarrow \text{aut}(M)$ of G on M determines a proxy action by taking $M_{hG} = M \times_G EG$ to be the Borel construction and the classifying map is $B\rho: BG \rightarrow B \text{aut}(M)$. Conversely, the proxy action (16) produces a rigid action of G on a space homotopy equivalent to M by turning $M \rightarrow M_{hG}$ into a covering space.

We will adopt the more flexible notion of proxy actions throughout this paper and by abuse of language will call just an action to a proxy action. In this setting, the total space M_{hG} of (16) is called the *homotopy quotient space* and the *homotopy fixed point space* is defined as the space M^{hG} of sections of fibration (16). In this section we use obstruction theory to develop some basic structure results for M^{hG} , and we apply them in the case where $M = BX$ is the classifying space of a p -compact group and to the proof of [Theorem B](#).

We will show conditions under which M^{hG} is nonempty, and if this is the case, a way to describe the set of path-components. Fibration (16) induces an action of G on the

set of path-components of M and $\pi_0(M)^G$ denotes the set of path-components of M that remain fixed under this action. Then, evaluation of a section at the base point $b \in BG$ induces a map

$$(17) \quad \pi_0(M^{hG}) \xrightarrow{\pi_0(\text{ev})} \pi_0(M)^G.$$

Thus, a necessary condition for M^{hG} being nonempty is that $\pi_0(M)^G$ is nonempty.

Fix now a point $m \in M$ which represents a G -invariant path-component of M , then there is a short exact sequence

$$(18) \quad 1 \rightarrow \pi_1(M, m) \rightarrow \pi_1(M_{hG}, m) \rightarrow \pi_1(BG, b) \rightarrow 1$$

of fundamental groups, where $b = p(m)$. If $m \in \pi_0(M)^G$ happens to be in the image of the evaluation map (17), then $s(b) = m$ for some homotopy fixed point $s \in M^{hG}$ and then the exact sequence (18) does have a section, namely $\pi_1(s)$.

Define $H^1(G; \pi_1(M, m))$ [62] to be the set (possibly empty) of $\pi_1(M, m)$ -conjugacy classes of sections $\pi_1(BG, b) \rightarrow \pi_1(M_{hG}, m)$ of the exact sequence (18). Then, the argument in the previous paragraph produces a well defined map $\pi_0(\text{ev})^{-1}([m]) \rightarrow H^1(G; \pi_1(M, m))$. In next Lemma it will be shown that, under certain conditions, this is a bijection for every $[m] \in \pi_0(M)^G$.

Since $\pi_1(M_{hG}, m)$ acts on the homotopy groups $\pi_i(M, m)$ of the fiber, also $G = \pi_1(BG, b)$ acts on $\pi_i(M, m)$ through $\pi_1(s)$, for a given element $s \in M^{hG}$. We let $\pi_i(M, m)^{s*G}$, $i \geq 1$, denote the fixed point group for this action.

Lemma 5.1 *Suppose that G is a finite group of order prime to p and that $\pi_i(M, m)$ is a module over the ring $\mathbb{Z}_{(p)}$ of p -local integers for all $i \geq 2$ and all base points $m \in \pi_0(M)^G$. Then the following hold:*

- (1) *A class $[m] \in \pi_0(M)^G$ is in the image of the evaluation map (17) if and only if the exact sequence (18) splits.*
- (2) *If $[m] \in \pi_0(M)^G$ is in the image of the evaluation map (17), then there is an exact sequence of pointed sets*

$$* \rightarrow H^1(G; \pi_1(M, m)) \rightarrow \pi_0(M^{hG}) \xrightarrow{\pi_0(\text{ev})} \pi_0(M)^G$$

where $[m]$ is the base point of $\pi_0(M)^G$.

- (3) *If $s \in M^{hG}$ is a homotopy fixed point with $s(b) = m$ then*

$$\pi_i(M^{hG}, s) \cong \pi_i(M, m)^{s*G}$$

for all $i \geq 1$.

Proof The Postnikov functors P_r , defined as nullification with respect to S^{r-1} (see Farjoun [29]), determine a tower of fibrations

$$M_{hG} \rightarrow \cdots \rightarrow P_r M_{hG} \rightarrow P_{r-1} M_{hG} \rightarrow \cdots \rightarrow P_1 M_{hG} \rightarrow BG$$

so that M^{hG} is the homotopy inverse limit of a sequence

$$\cdots \rightarrow (P_r M)^{hG} \rightarrow (P_{r-1} M)^{hG} \rightarrow \cdots \rightarrow (P_1 M)^{hG}$$

of Postnikov homotopy fixed point spaces.

Note that $\pi_0(P_1 M_{hG}) = \pi_0(M_{hG})$ and that each path-component of $P_1 M_{hG}$ is aspherical with fundamental group $\pi_1(P_1 M_{hG}, m) = \pi_1(M_{hG}, m)$ for all $m \in P_1 M$. It is now easy to see that $H^1(G; \pi_1(M, m))$ is the fiber over $[m] \in \pi_0(P_1 M)^G = \pi_0(M)^G$ of the evaluation map $\pi_0(P_1 M^{hG}) \rightarrow \pi_0(M)^G$ and also that $\pi_1(P_1 M^{hG}, s) = \pi_1(M, m)^{s*G}$ for any $s \in P_1 M^{hG}$ with $s(b) = m$; cf [47, Section 6]. Obstruction theory implies that $\pi_0(M^{hG}) = \pi_0(P_1 M^{hG})$. This proves the first two items.

For the third item, suppose that the homotopy fixed point space is nonempty and let $s \in M^{hG}$ be a homotopy fixed point. Then the component (M^{hG}, s) containing s is the homotopy inverse limits of the corresponding components

$$\cdots \rightarrow (P_r M^{hG}, s_r) \rightarrow (P_{r-1} M^{hG}, s_{r-1}) \rightarrow \cdots \rightarrow (P_1 M^{hG}, s_1)$$

of the Postnikov homotopy fixed point spaces. To finish the proof, observe [47, 3.1] that the fiber of $(P_r M^{hG}, s_r) \rightarrow (P_{r-1} M^{hG}, s_{r-1})$ is the Eilenberg–Mac Lane space $K(\pi_r(M, m)^{s*G}, r)$. \square

Theorem 5.2 *Let M be any simply connected p -complete space, G a finite group of order prime to p , and*

$$M \rightarrow M_{hG} \rightarrow BG$$

an action of G on M . Then the homotopy fixed point space M^{hG} is nonempty, $\pi_i(M^{hG}) = \pi_i(M)^G$ for all $i \geq 0$, and there is a homotopy equivalence

$$\Omega M \xrightarrow{\cong} \Omega(M^{hG}) \times \text{Fib}(M^{hG} \rightarrow M).$$

In particular, the fiber $\text{Fib}(M^{hG} \rightarrow M)$ of the evaluation map $M^{hG} \rightarrow M$ is an H -space.

Proof The space of sections M^{hG} is nonempty, connected, and $\pi_*(M^{hG}) = \pi_*(M)^G$ according to Lemma 5.1 since M is simply connected and p -complete. We will show

first how to turn this action with a homotopy fixed point into an honest action of G on a space homotopy equivalent to M and with a fixed point. The pullback diagram

$$\begin{array}{ccc} M & \xrightleftharpoons{\quad} & EG \\ \downarrow & & \downarrow \\ M_{hG} & \xrightleftharpoons{\quad} & BG \end{array}$$

realizes $M \rightarrow M_{hG}$ as a regular covering space with G acting on M . Liftings of sections $BG \rightarrow M_{hG}$ provide G -equivariant maps $EG \rightarrow M$. Let $M/EG = M \cup C(EG)$ be the homotopy cofibre of any such G -map. Then $M \rightarrow M/EG$ is a G -equivariant homotopy equivalence and the G -action on M/EG has a fixed point.

Now, we can assume that there is an honest G -action on M with a fixed point. Let ΩM denote the loop space based at any G -fixed point. There is a fibration sequence

$$\dots \rightarrow \Omega M^{hG} \rightarrow \Omega M \rightarrow \text{Fib}(M^{hG} \rightarrow M) \rightarrow M^{hG} \rightarrow M$$

and it suffices to construct a homotopy left inverse for $\Omega M^{hG} \rightarrow \Omega M$.

Define $\text{tr}: \Omega M \rightarrow \Omega M$ to be the map that takes any loop ω to the product $\prod g\omega$ of the loops $g\omega$ where g runs through the elements of G in some fixed order. The image of the induced map $\text{tr}_*: \pi_*(\Omega M) \rightarrow \pi_*(\Omega M)$, which takes a homotopy class α to $\sum_{g \in G} g_*\alpha$, is contained in the fixed group $\pi_*(\Omega M)^G$ and the composition

$$\pi_*(\Omega M)^G \rightarrow \pi_*(\Omega M) \xrightarrow{\text{tr}_*} \pi_*(\Omega M)^G$$

is an isomorphism. This implies that the composition $\Omega M^{hG} \rightarrow \Omega M \rightarrow T$, where T is the mapping telescope of

$$\Omega M \xrightarrow{\text{tr}} \Omega M \xrightarrow{\text{tr}} \dots,$$

is a (weak) homotopy equivalence and we have the left inverse we were looking for. \square

Let (X, BX, e) be a p -compact group or, more generally, a loop space. The above arguments suggest the following definition of a (proxy) action of a discrete group G on X .

Definition 5.3 Let (X, BX, e) be a loop space and G a group. A *proxy action* of G on (X, BX, e) is a fibration

$$(19) \quad BX \xrightarrow{i} BX_{hG} \xrightleftharpoons[s]{p} BG$$

with a section, fixed up to vertical homotopy.

When it is clear from the context that we refer to an action in the sense of this definition, we will simply say that G acts on the loop space X . The section in (19) guarantees an induced action of G on the space X , compatible with the loop structure. In fact, the homotopy quotient for this action on X is defined as the pullback space in the diagram

$$(20) \quad \begin{array}{ccc} X_{hG} & \xrightarrow{\bar{p}} & BG \\ \bar{p} \downarrow & & \downarrow s \\ BG & \xrightarrow{s} & BX_{hG}. \end{array}$$

This diagram turns out to be a diagram of spaces over BG . The homotopy fiber of \bar{p} is X , and it has a canonical section \bar{s} defined by the pullback diagram (20) that we can interpret as the *homotopy constant loop*

$$X \xrightarrow{\bar{i}} X_{hG} \xleftarrow[\bar{s}]{\bar{p}} BG.$$

The action of G on X depends on the section $s: BG \rightarrow BX_{hG}$, and for this action we obtain that the homotopy fixed point space X^{hG} is a loop space with classifying space $B(X^{hG}) \simeq (BX)_s^{hG}$, the connected component of $(BX)^{hG}$ with base point the section s . Furthermore, the evaluation map $X^{hG} \rightarrow X$ is seen to be the loop map of the evaluation map $(BX)_s^{hG} \rightarrow BX$, thus we have a sequence of fibrations

$$X^{hG} \xrightarrow{\text{ev}} X \longrightarrow X/X^{hG} \longrightarrow (BX)_s^{hG} \xrightarrow{\text{ev}} BX$$

where we write X/X^{hG} for the homotopy fiber of the evaluation map $(BX)_s^{hG} \rightarrow BX$.

In Section 2 we have introduced $\text{Out}(X)$ as the group of invertible elements of the monoid $[BX, BX]$ of unbased homotopy classes of unbased self-maps of BX . By analogy with discrete group theory, we call *outer* action of G on X a homomorphism of groups $\rho: G \rightarrow \text{Out}(X)$. Since $\text{Out}(X)$ is well understood (see Theorem 2.2), outer actions will be a source for group actions on p -compact groups provided we can lift outer actions to actions in the sense of Definition 5.3. Theorem B solves the problem in case of finite groups of order prime to p .

Proof of Theorem B Fix a finite group G of order prime to p and $\rho: G \rightarrow \text{Out}(X)$ an outer action of G on a connected p -compact group X . Recall that we have a fibration sequence

$$B^2Z(X) \rightarrow \text{Baut}(BX) \rightarrow \text{BOut}(X)$$

and that the center of X , $Z(X)$, is p -local. By obstruction theory we obtain a unique lifting of ρ to a map $\varphi: BG \rightarrow \text{Baut}(BX)$, that determines an action $BX \rightarrow BX_{hG} \rightarrow$

BG . Furthermore, since $\pi_1(BX) = 1$, Lemma 5.1.(2) implies that $\pi_0(BX^{hG}) = *$; that is, there is a unique section

$$(21) \quad BX \longrightarrow BX_{hG} \longleftarrow BG$$

up to fiber homotopy equivalence; in other words, ρ lifts to a unique action of G on X .

This is part (1) of the theorem. Theorem 5.2 provides the splitting $X \simeq X^{hG} \times X/X^{hG}$. It follows that X/X^{hG} is an \mathbb{F}_p -finite H -space, X^{hG} is a loop space with classifying space BX^{hG} and it is also \mathbb{F}_p -finite. Furthermore, BX^{hG} is p -complete because BX is p -complete [25, 11.13], hence X^{hG} is a connected p -compact group.

The rational cohomology algebra $H^*(BY; \mathbb{Q}_p)$ is polynomial for any connected p -compact group Y and it follows that the Hurewicz homomorphism induces an isomorphism

$$QH^*(BY; \mathbb{Q}_p) \rightarrow \pi_*(BY)^\vee \otimes \mathbb{Q}$$

between the indecomposables and the rationalized dual ($\pi^\vee = \text{Hom}_{\mathbb{Z}_p}(\pi, \mathbb{Z}_p)$) of the homotopy groups of the simply connected space BY [8, Theorem 3.2.3]. For the connected fixed point p -compact group BX^{hG} , in particular, we have

$$QH^*(BX^{hG}; \mathbb{Q}_p) \cong \pi_*(BX^{hG})^\vee \otimes \mathbb{Q} \cong (\pi_*(BX)^\vee \otimes \mathbb{Q})_G \cong (QH^*(BX; \mathbb{Q}_p))_G$$

for $\pi_*(BX^{hG}) = \pi_*(BX)^G$ as the order of G is prime to p . This proves points (2) and (3).

We finish by proving point (4). Assume p is odd. If X is a polynomial p -compact group

$$H^*(X; \mathbb{F}_p) \cong H^*(X^{hG}; \mathbb{F}_p) \otimes H^*(X/X^{hG}; \mathbb{F}_p)$$

is an exterior algebra, hence $H^*(X^{hG}; \mathbb{F}_p)$ is an exterior algebra, too. Therefore, $H^*(BX^{hG}; \mathbb{F}_p)$ is a polynomial algebra. \square

Example 5.4 At any odd prime, let C_2 act on E_6 through the unstable Adams operation ψ^{-1} . Since the fixed point p -compact group $BE_6^{hC_2}$ is the p -compact group BF_4 (Section A.12), there is a splitting

$$E_6 \simeq F_4 \times E_6/F_4$$

of homogeneous spaces. This splitting is due to Harris [37]. Also, $BPE_6^{hC_2} \simeq BF_4$, where PE_6 is the adjoint form of E_6 (Section A.12) thus there is also a splitting $PE_6 \simeq F_4 \times PE_6/F_4$.

Let p be an odd prime and m a divisor of $p-1$ so that the cyclic group C_m of order m acts on $BSU(mn+s)$, $0 \leq s < m$, through unstable Adams operations. Since the

fixed point p -compact group $BSU(mn+s)^{hC_m}$ is (Proposition A.9) the generalized Grassmannian $BX(m, 1, n)$ with polynomial cohomology $H^*(BX(m, 1, n); \mathbb{F}_p) = \mathbb{F}_p[x_m, \dots, x_{nm}]$, $|x_{im}| = 2im$, there is a splitting

$$SU(mn+s) \simeq X(m, 1, n) \times SU(mn+s)/X(m, 1, n)$$

of homogeneous spaces. This splitting is originally due to Mimura, Nishida and Toda [44]), although the recognition of $X(m, 1, n)$ as a loop space is due to Quillen [61] (see also Stasheff [66], Zabrodsky [71] and Castellana [18]). The case $m = 2$ is the classical splitting $SU(2n) \simeq Sp(n) \times SU(2n)/Sp(n)$. Similar splittings for central quotients of $SU(n)$ can be worked out.

Similarly, at $p = 5$, let C_4 act on BE_8 through unstable Adams operations. The fixed point p -compact group $BE_8^{hC_4}$ is the p -compact group $B\mathbf{X}_{31}$ corresponding to \mathbb{Z}_p -reflection group number 31 on the Clark–Ewing list (see Section A.12), $H^*(B\mathbf{X}_{31}; \mathbb{F}_p) = \mathbb{F}_p[x_{16}, x_{24}, x_{40}, x_{48}]$ where subscripts indicate degrees, there is a splitting

$$E_8 \simeq \mathbf{X}_{31} \times E_8/\mathbf{X}_{31}$$

of homogeneous spaces, that was obtained in [68].

At $p = 3$, BF_4 admits an exceptional isogeny of order 2 and (see [16]) the fixed point group $BF_4^{hC_2}$ is the p -compact group $B DI_2$ whose cohomology realizes the Dickson algebra $\mathbb{F}_3[x_{12}, x_{16}]$. The corresponding splitting

$$F_4 \simeq DI_2 \times F_4/DI_2$$

was first obtained by Harper [36]. Later proofs of this splitting were obtained independently by Wilkerson and by Kono, using Friedlander’s exceptional isogeny of F_4 localized away from two.

In these last two cases, it was Zabrodsky [71, 4.3], who first recognized the factors $\mathbf{X}_{12} = DI_2$ and \mathbf{X}_{31} as loop spaces. Later, Aguadé gave a nice uniform construction of a family of modular p -compact groups including these cases [2].

6 Homotopy fixed point spaces of twisted unstable Adams operations

In this section we prove Theorem E. Part (1) of follows from Proposition 6.2 and Remark 6.3, while Part (2) is Proposition 6.5.

Let X be a connected p -compact group and set $\alpha: X \rightarrow X$ a p -compact group automorphism. The homotopy pullback diagram

$$(22) \quad \begin{array}{ccc} BX^{h\alpha} & \xrightarrow{\iota} & BX \\ \downarrow \iota & & \downarrow \Delta \\ BX & \xrightarrow{(1,\alpha)} & BX \times BX \end{array}$$

serves as the definition of the space $BX^{h\alpha}$. If α is homotopic to α' , then $BX^{h\alpha} \simeq BX^{h\alpha'}$.

In the special case where $\alpha = \tau\psi^q$ is a twisted unstable Adams operation with $q \in \mathbb{Z}_p$, $q \neq 1$, and $q \not\equiv 0 \pmod p$, we also write $B^r X(q) = BX^{h\tau\psi^q}$, or just $BX(q)$, if $\tau = 1$. For $q = 1$ we trivially obtain $BX(1) \simeq \Lambda(BX)$, the free loop space.

Assume that α represents an element of finite order r in $\text{Out}(X)$, with r prime to p , and X is a connected p -compact group. According to [Theorem B](#), it defines an action of the cyclic group C_r on X . The next proposition shows that the natural map $\Lambda(BX^{hC_r}) \rightarrow BX^{h\alpha}$ is a homotopy equivalence.

Proposition 6.1 *Assume that X is a connected p -compact group. If $\beta: BX \rightarrow BX$ represents an element of $\text{Out}(X)$ of finite order r , prime to p , then $BX^{h\beta}$ is homotopy equivalent to the space of free loops on BX^{hC_r} , where the action of the cyclic group C_r on BX is given by β .*

Proof According to [Theorem B](#), β defines an action of C_r on X ,

$$BX \xrightarrow{i} BX_{hC_r} \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} BC_r.$$

Evaluation at the base point of BC_r induces a map $\text{ev}: BX^{hC_r} \rightarrow BX_{hC_r}$ that makes the triangle

$$\begin{array}{ccc} & & BX \\ & \nearrow \text{ev} & \downarrow \beta \\ BX^{hC_r} & & \\ & \searrow \text{ev} & \downarrow \\ & & BX \end{array}$$

commutative up to homotopy. Therefore, we can form a homotopy commutative diagram

$$(23) \quad \begin{array}{ccccc} \Lambda(BX^{hC_r}) & \xrightarrow{\quad} & BX^{hC_r} & \xrightarrow{\Delta} & BX^{hC_r} \times BX^{hC_r} \\ \downarrow & \searrow & \downarrow & \searrow & \downarrow \\ & BX^{hC_r} & \xrightarrow{\Delta} & & BX^{hC_r} \times BX^{hC_r} \\ \downarrow & \downarrow & \downarrow & \searrow & \downarrow \\ BX^{h\beta} & \xrightarrow{\quad} & BX & \xrightarrow{\Delta} & BX \times BX \\ & \searrow & \downarrow & \searrow & \downarrow \\ & BX & \xrightarrow{(1,\beta)} & & BX \times BX \end{array}$$

We will show that $\Lambda(BX^{hC_r}) \rightarrow BX^{h\beta}$ is a homotopy equivalence. According to [Theorem 5.2](#), BX^{hC_r} is the classifying space of a connected p -compact group and by [Lemma 5.1](#) the map $\text{ev}: BX^{hC_r} \rightarrow BX$ induces an identification of the homotopy groups of BX^{hC_r} with the invariant elements in the homotopy groups of BX by the action of C_r : $\pi_i(BX^{hC_r}) \cong \pi_i(BX)^{C_r} \hookrightarrow \pi_i(BX)$. There is a Mayer-Vietoris long exact sequence for the homotopy groups of $BX^{h\beta}$

$$\dots \rightarrow \pi_i(BX^{h\beta}) \rightarrow \pi_i(BX) \xrightarrow{1-\beta_*} \pi_i(BX) \rightarrow \pi_{i-1}(BX^{h\beta}) \rightarrow \dots$$

and for the homotopy groups of the free loop space:

$$\dots \rightarrow \pi_i(\Lambda(BX^{hC_r})) \rightarrow \pi_i(BX^{hC_r}) \xrightarrow{0} \pi_i(BX^{hC_r}) \rightarrow \pi_{i-1}(\Lambda(BX^{hC_r})) \rightarrow \dots$$

Both long exact sequences together give

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_{i+1}(BX)^{C_r} & \longrightarrow & \pi_i(\Lambda(BX^{hC_r})) & \longrightarrow & \pi_i(BX)^{C_r} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Coker}\{1 - \beta_*\} & \longrightarrow & \pi_i(BX^{h\beta}) & \longrightarrow & \text{Ker}\{1 - \beta_*\} \longrightarrow 0. \end{array}$$

Now, $\text{Ker}(1 - \beta_*) = \pi_i(BX)^{C_r}$ and $\text{Coker}(1 - \beta_*) = \pi_{i+1}(BX)_{C_r}$. Since r is prime to p , and the homotopy groups $\pi_i(BX)$ are $\mathbb{Z}_{(p)}$ -modules for every $i \geq 2$, the composition $\pi_{i+1}(BX)^{C_r} \rightarrow \pi_{i+1}(BX) \rightarrow \pi_{i+1}(BX)_{C_r}$ is an isomorphism. Hence also the middle vertical map $\pi_i(\Lambda(BX^{hC_r})) \rightarrow \pi_i(BX^{h\beta})$ is an isomorphism. \square

Our next result contains [Proposition 6.1](#) as a special case and it will reduce, in many cases, the question of describing $BX^{h\alpha}$ to two separate steps. The computation of the homotopy fixed point space BX^{hC_r} , for elements α of order r prime to p , and the case in which $\alpha = \psi^q$ is an unstable Adams operation of exponent $q \equiv 1 \pmod p$ (see [Theorem 2.2](#) and formula (3) in [Section 2](#)). It is one of the two claims of [Theorem E](#).

Proposition 6.2 *Let X be a connected p -compact group. Suppose α is an automorphism of X that factors $\alpha = \psi^q \beta$ satisfying*

- (1) $q \equiv 1 \pmod p$, and $(\psi^q)^*: H^*(X; \mathbb{F}_p) \rightarrow H^*(X; \mathbb{F}_p)$ is the identity, and
- (2) β is an automorphism of X that represents an element of finite order r , prime to p , in $\text{Out}(X)$.

Then $BX^{h\alpha} \simeq BX^{hC_r}(q)$ where $C_r = \langle \beta \rangle \subset \text{Out}(X)$ is the cyclic group of order r generated by the homotopy class of β .

Proof Let $BY = BX^{hC_r}$ denote the homotopy fixed point p -compact group for the action of the cyclic group $C_r = \langle \beta \rangle \subset \text{Out}(X)$ and $i: BY \rightarrow BX$ the evaluation map. Now, β restricts trivially to BY , as in the proof of Proposition 6.1, and then since ψ^q commutes with β , up to homotopy, we have a homotopy commutative diagram

$$\begin{array}{ccc}
 BY & \xrightarrow{\psi^q} & BY \\
 i \downarrow & & \downarrow i \\
 BX & \xrightarrow{\alpha = \psi^q \beta} & BX
 \end{array}$$

that extends to

$$\begin{array}{ccccc}
 BY(q) & \longrightarrow & BY & & \\
 \downarrow & \searrow & \downarrow & \Delta & \\
 & & BY & \xrightarrow{(1, \psi^q)} & BY \times BY \\
 & & \downarrow & & \downarrow \\
 BX^{h\alpha} & \longrightarrow & BX & \xrightarrow{\Delta} & BX \times BX \\
 & \searrow & \downarrow & & \downarrow \\
 & & BX & \xrightarrow{(1, \alpha)} & BX \times BX
 \end{array}$$

where the top and bottom faces are homotopy pullback diagrams, and the front face commutes up to homotopy. Consequently, the homotopy fibres of the vertical maps form another homotopy pullback diagram:

$$\begin{array}{ccc}
 (X/Y)^{h\alpha} & \longrightarrow & X/Y \\
 \downarrow & & \downarrow \Delta \\
 X/Y & \xrightarrow{(1, \alpha)} & X/Y \times X/Y
 \end{array}$$

with $(X/Y)^{h\alpha} \simeq \text{hofib}(BY(q) \rightarrow BX^{h\alpha})$, and where we still denote by α the self-equivalence of X/Y induced by $\alpha: BX \rightarrow BX$. Theorem B says that X/Y is a

connected H -space and then we can also describe $(X/Y)^{h\alpha}$ as the homotopy fiber of $1 - \alpha: X/Y \rightarrow X/Y$. It also implies that the map $(\psi^q)^*: H^*(X/Y; \mathbb{F}_p) \rightarrow H^*(X/Y; \mathbb{F}_p)$ can be read off the map $(\psi^q)^*$ defined on $H^*(X; \mathbb{F}_p)$, which by hypothesis is the identity. This fact easily implies that $(1 - \alpha)^* = (1 - \beta)^*$ on $H^*(X/Y; \mathbb{F}_p)$.

According to [Proposition 6.1](#), the homotopy fiber, $(X/Y)^{h\beta}$, of $1 - \beta$ is contractible, hence $(1 - \beta)^*$ is an automorphism of $H^*(X/Y; \mathbb{F}_p)$. Thus, a spectral sequence argument shows that $(X/Y)^{h\alpha}$ is mod p acyclic. Finally, it is easy to see that $(X/Y)^{h\alpha}$ is p -complete, hence contractible, and therefore $BY(q) \simeq BX^{h\alpha}$. \square

Remark 6.3 If X polynomial, the effect of $\Omega\psi^q$, $q \equiv 1 \pmod{p}$, on mod p cohomology of X is determined by the effect of ψ^q on $H^*(BX, \mathbb{F}_p)$ and this is in turn determined by the effect of ψ^q on $H^*(BT_X; \mathbb{F}_p)$ which is multiplication by q , hence the identity. For $X = F_4, E_6, E_7, E_8$ at the prime 3 or $X = E_8$ at the prime 5, we also obtain that $\Omega\psi^q$, $q \equiv 1 \pmod{p}$, acts trivially on $H^*(X; \mathbb{F}_p)$. In order to check this, we can look at the Serre spectral sequence for the path-loop fibration $X \rightarrow PBX \rightarrow BX$. It turns out that the generators for $H^*(X; \mathbb{F}_p)$ either transgress to elements detected in the maximal torus of BX , or are linked to such elements by Steenrod operations (cf [\[46, Chapter 7\]](#)). In particular, [Proposition 6.2](#) applies to all 1-connected p -compact groups, p odd, according to the classification theorem [\[7\]](#).

In particular, $BX(\zeta q) = BX^{h(\zeta)}(q)$ when ζ is a $(p-1)$ -th root of unity and $q \equiv 1 \pmod{p}$ satisfies the conditions of [Proposition 6.2](#). If $q = 1$ we obtain [Proposition 6.1](#), $BX(\zeta) = BX^{h(\zeta)}(1) = \Lambda(BX^{h(\zeta)})$, as a special case.

For the next result, we need to interpret $BX^{h\alpha}$ as homotopy fixed point set by the action of \mathbb{Z} generated by $\alpha \in \text{Out}(X)$. In fact, given $\alpha \in \text{Out}(X)$, we denote again by α a representative homotopy equivalence $\alpha: BX \rightarrow BX$. The mapping torus is defined $BX_{h\alpha} = BX \times I / \sim$, where $I = [0, 1]$ is the unit interval and $(x, 0) \sim (\alpha(x), 1)$. There is a fibration, up to homotopy,

$$BX \rightarrow BX_{h\alpha} \rightarrow S^1$$

given by projection onto the second component. This fibration is classified by a loop $\omega_\alpha: S^1 \rightarrow B \text{aut}(BX)$ that represents $\alpha \in \pi_1(B \text{aut}(BX)) = \text{Out}(X)$.

The space of sections for this fibration clearly coincides with $BX^{h\alpha}$ as defined in diagram [\(22\)](#), so we can interpret $BX_{h\alpha}$ and $BX^{h\alpha}$ as the homotopy quotient space $BX_{h\mathbb{Z}}$ and the homotopy fixed point space $BX^{h\mathbb{Z}}$, respectively, for the action of \mathbb{Z} on BX determined by $\alpha \in \text{Out}(X)$. Notice that since X is connected, so is $BX^{h\alpha} \simeq BX^{h\mathbb{Z}}$

and therefore, there is a unique lifting, up to equivalence, of the action of \mathbb{Z} on BX to an action of \mathbb{Z} on X , in the sense of [Definition 5.3](#).

We will use this point of view in order to proof the second claim of [Theorem E](#). We will see that an action of \mathbb{Z} on BX generated by an unstable Adams operation ψ^q of exponent $q \equiv 1 \pmod p$, extends to an action of \mathbb{Z}_p on BX , and that this implies that the homotopy type of the homotopy fixed point space $BX(q) = BX^{h\mathbb{Z}}$ depends only on the p -adic valuation $v_p(1 - q)$.

Lemma 6.4 *Suppose that $BX_{h\mathbb{Z}_p} \rightarrow B\mathbb{Z}_p$ is a fibration over $B\mathbb{Z}_p$ with fiber BX . The p -completion map $\ell: B\mathbb{Z} \rightarrow B\mathbb{Z}_p$ induces a homotopy equivalence $BX^{h\mathbb{Z}_p} \rightarrow BX^{h\mathbb{Z}}$ of spaces of sections.*

Proof The maps $B\mathbb{Z} \longrightarrow B\mathbb{Z}_p \longleftarrow BX_{h\mathbb{Z}_p}$ determine a commutative diagram

$$\begin{array}{ccc} \text{Map}(B\mathbb{Z}_p, BX_{h\mathbb{Z}_p}) & \longrightarrow & \text{Map}(B\mathbb{Z}, BX_{h\mathbb{Z}_p}) \\ \downarrow & & \downarrow \\ \text{Map}(B\mathbb{Z}_p, B\mathbb{Z}_p) & \xrightarrow{\simeq} & \text{Map}(B\mathbb{Z}, B\mathbb{Z}_p) \end{array}$$

which is a pullback diagram since $\ell: B\mathbb{Z} \rightarrow B\mathbb{Z}_p$ is an \mathbb{F}_p -equivalence [[10](#), [12.2](#)]. (To see that $BX_{h\mathbb{Z}_p} \rightarrow B\mathbb{Z}_p$ is an $H_*\mathbb{F}_p$ -fibration observe that the action of \mathbb{Z}_p on $H_i(BX; \mathbb{F}_p)$, $i \geq 0$, is nilpotent because it factors through a finite quotient of \mathbb{Z}_p .) Thus the fiber of the left fibration over the identity map of $B\mathbb{Z}_p$, $BX_{h\mathbb{Z}_p}$, is homotopy equivalent to the fiber of the right fibration over the p -completion map $\ell: B\mathbb{Z} \rightarrow B\mathbb{Z}_p$, $BX^{h\mathbb{Z}}$. □

Using the description of $\text{Out}(X)$ in [Section 2](#) we will see that actions of \mathbb{Z} on connected p -compact groups given by Adams operations ψ^q extend to the p -adics precisely when $q \equiv 1 \pmod p$. The inclusion of Adams operations in $\text{Out}(X)$, described as $q \in \mathbb{Z}_p^\times \mapsto \psi^q \in \text{Out}(X)$ induces a diagram of group homomorphisms

$$\begin{array}{ccc} \text{Hom}(\mathbb{Z}_p, \mathbb{Z}_p^\times) & \xrightarrow{\text{res}} & \text{Hom}(\mathbb{Z}, \mathbb{Z}_p^\times) \\ \downarrow & & \downarrow \\ \text{Hom}(\mathbb{Z}_p, \text{Out}(X)) & \xrightarrow{\text{res}} & \text{Hom}(\mathbb{Z}, \text{Out}(X)) \end{array}$$

where the horizontal homomorphisms are given by restriction.

Recall that, for an odd prime p , $\mathbb{Z}_p^\times \cong \mathbb{Z}/p - 1 \times \mathbb{Z}_p$, where $\mathbb{Z}/p - 1$ corresponds to the subgroup of \mathbb{Z}_p^\times of roots of unity and \mathbb{Z}_p is identified with the subgroup of

elements $q \in \mathbb{Z}_p^\times$, with $q \equiv 1 \pmod p$, via the exponential map:

$$a \in \mathbb{Z}_p \mapsto \exp(pa) \in \mathbb{Z}_p^\times$$

(\exp defined by the usual expansion $\exp(pa) = 1 + pa + \dots$). Since there are no nontrivial homomorphisms $\mathbb{Z}_p \rightarrow \mathbb{Z}/p - 1$, an action of \mathbb{Z} on BX determined by an Adams operation ψ^q can only be the restriction of an action of \mathbb{Z}_p if $q \equiv 1 \pmod p$. On the other hand, if $q \equiv 1 \pmod p$, then we can write $q = 1 + pm_q$ ($m_q = \frac{1}{p} \log(q)$), and the homomorphism $\omega_q: \mathbb{Z} \rightarrow \mathbb{Z}_p^\times$ that maps 1 to q is clearly the restriction to \mathbb{Z} of the homomorphism $\bar{\omega}_q: \mathbb{Z}_p \rightarrow \mathbb{Z}_p^\times$ defined $\omega_q(x) = \exp(xpm_q)$.

Now, we can prove the second claim of [Theorem E](#).

Proposition 6.5 *If $q, q' \in \mathbb{Z}_p^\times$, both are of multiplicative order $r \pmod p$, and $v_p(1 - q^r) = v_p(1 - q'^r)$, then $BX(q) \simeq BX(q')$, for any 1-connected p -compact group X .*

Proof The proof is divided in two steps. First, we will consider the case $q \equiv q' \equiv 1 \pmod p$ ($r = 1$). In these cases, the actions of \mathbb{Z} given by ψ^q and $\psi^{q'}$, respectively, extend to actions of the p -adics described by $m_q = (1/p) \log(q)$ and $m_{q'} = (1/p) \log(q')$, respectively. The homotopy fixed point space $BX^{h\mathbb{Z}_p}$ depends only of the image of the action $\mathbb{Z}_p \rightarrow \text{Out}(X)$. The image of the two actions are clearly the same if and only if m_q and $m_{q'}$ differ by a p -adic unit; that is, if and only if $v_p(m_q) = v_p(m_{q'})$, if and only if $v_p(1 - q) = v_p(1 - q')$, in which case, we have

$$BX(q) \simeq BX^{h\psi^q} \simeq BX^{h\mathbb{Z}_p} \simeq BX^{h\psi^{q'}} \simeq BX(q').$$

In the general case, we can decompose $q = \zeta \cdot q_0$ and $q' = \zeta' \cdot q'_0$, where ζ and ζ' are primitive r -th roots of unity and $q_0 \equiv q'_0 \equiv 1 \pmod p$. Since ζ and ζ' generate the same subgroup of \mathbb{Z}_p^\times we have that

$$BX(q) \simeq BX^{h\langle \zeta \rangle}(q_0) \simeq BX^{h\langle \zeta' \rangle}(q_0) \simeq BX^{h\langle \zeta' \rangle}(q'_0) \simeq BX(q'). \quad \square$$

Remark 6.6 If q is a p -adic unit, we can find a prime number q_0 such that $q \equiv q_0 \pmod p$ and $v_p(1 - q^r) = v_p(1 - q_0^r)$, where r is the order of $q \pmod p$, and then

$$BX(q) \simeq BX(q_0)$$

by [Proposition 6.5](#).

In fact, we can assume that q is an integer, otherwise change it by the sum of enough first terms in its p -adic expansion. Then, by Dirichlet's theorem there is a prime number q_0 of the form $q_0 = p^N c + q$, with $N > v_p(1 - q^r)$, satisfying the above conditions.

7 General structure of Chevalley p -local finite groups

In this section we will study some general properties of the spaces $BX(q)$, obtained as homotopy fixed point spaces for the action of unstable Adams operations on classifying spaces of connected p -compact groups. The main results being the identification of the maximal finite torus, the Weyl group, and the fusion category of elementary abelian p -subgroups of $BX(q)$.

Proposition 7.1 *Let X be a connected p -compact group and α a self homotopy equivalence of X . Then we have the following:*

- (1) $BX^{h\alpha}$ is connected and p -complete.
- (2) $\iota: BX^{h\alpha} \rightarrow BX$ is a homotopy monomorphism at p .
- (3) For any finite p -group P , $\text{Map}(BP, BX^{h\alpha})_c \simeq BX^{h\alpha}$.

Proof From the definition we obtain a fibration $X \rightarrow BX^{h\alpha} \xrightarrow{\iota} BX$ where X and BX are p -complete, X is connected and BX is simply connected. It follows that $BX^{h\alpha}$ is connected and p -complete.

For any finite p -group P , $\text{Map}(BP, BX)_c \simeq BX$, and $\text{Map}_*(BP, X) \simeq X$ for any choice of base point. It then follows that $\iota: BX^{h\alpha} \rightarrow BX$ is a homotopy monomorphism at p , and from the induced fibration

$$\text{Map}(BP, X) \rightarrow \text{Map}(BP, BX^{h\alpha})_c \rightarrow \text{Map}(BP, BX)_c$$

it follows that $\text{Map}(BP, BX^{h\alpha})_c \simeq BX^{h\alpha}$. □

Lemma 7.2 *Let X be a p -compact group, α a self homotopy equivalence of BX , and (P, ν) an object of $\mathcal{F}_p(BX)$ fixed by α up to homotopy; that is, $\nu \simeq \alpha \circ \nu$. If $C_X(P, \nu)$ is connected, then there is a unique lifting of $\nu: BP \rightarrow BX$ to a homotopy monomorphism $g: BP \rightarrow BX^{h\alpha}$, and*

$$\begin{array}{ccc} \text{Map}(BP, BX^{h\alpha})_g & \longrightarrow & \text{Map}(BP, BX)_\nu \\ \downarrow & & \downarrow \Delta \\ \text{Map}(BP, BX)_\nu & \xrightarrow{1 \times \alpha_\sharp} & \text{Map}(BP, BX)_\nu \times \text{Map}(BP, BX)_\nu \end{array}$$

is a homotopy pullback diagram.

Proof Since (22) is a homotopy pullback diagram, there is at least a lifting of ν , $g: BP \rightarrow BX^{h\alpha}$.

The homotopy fiber of

$$\text{Map}(BP, BX)_\nu \xrightarrow{\Delta} \text{Map}(BP, BX)_\nu \times \text{Map}(BP, BX)_\nu$$

is $C_X(P, \nu) = \Omega \text{Map}(BP, BX)_\nu$, hence pulling back along $1 \times \alpha_\#$ we obtain a fibration, up to homotopy,

$$C_X(P, \nu) \rightarrow \text{Map}(BP, BX^{h\alpha})_{\hat{\nu}} \xrightarrow{l_\#} \text{Map}(BP, BX)_\nu$$

where $\text{Map}(BP, BX^{h\alpha})_{\hat{\nu}}$ consists of all possible liftings of ν up to homotopy. The base space consists of just one connected component, hence if we assume that the fiber $C_X(P, \nu)$ is also connected, then the total space must be connected, and therefore any other lifting of ν is homotopic to g . \square

The following lemma will help us determine the restriction of α to the centralizers.

Lemma 7.3 *Let X be a connected p -compact group and α a self-equivalence of BX . Let $T(\alpha)$ be a given restriction of α to the maximal torus $T = T_X$, and (P, ν) an object of $\mathcal{F}_p(BX)$.*

Suppose that $\nu: BP \rightarrow BX$ admits a factorization $\mu: BP \rightarrow BT$ through the maximal torus $j: BT \rightarrow BX$. Then, the object (P, ν) is fixed by α if and only if $T(\alpha)\mu = w\mu$ for an element w of the Weyl group. If this is the case, the restriction to the maximal torus of the induced self homotopy equivalence $\alpha|_{C_X(P, \nu)}$ of the centralizer $C_X(P, \nu)$ is $T(\alpha|_{C_X(P, \nu)}) = w^{-1} \circ T(\alpha)$.

Proof (P, ν) is fixed by α means that $\nu \simeq \alpha \circ B\nu$, and if ν factors as $j \circ \mu$, that is to say, $j \circ B\mu \simeq \alpha \circ j \circ \mu \simeq j \circ T(\alpha) \circ \mu$, and according to [55, 4.1; 50, 3.4] this is equivalent to the existence of w , in the Weyl group of X , such that $w \circ \mu \simeq BT(\alpha) \circ \mu$.

Now assuming the existence of such element w , we read from the commutative diagram

$$\begin{array}{ccccc}
 BT & \xrightarrow{T(\alpha)} & BT & \xleftarrow{w} & BT \\
 \simeq \uparrow \text{ev} & & \simeq \uparrow \text{ev} & & \simeq \uparrow \text{ev} \\
 \text{Map}(BP, BT)_\mu & \xrightarrow{T(\alpha)_\#} & \text{Map}(BP, BT)_{w\mu} & \xleftarrow{w_\#} & \text{Map}(BP, BT)_\mu \\
 \simeq \downarrow j_\# & & \simeq \downarrow j_\# & \swarrow \simeq j_\# & \\
 \text{Map}(BP, BX)_\nu & \xrightarrow{\alpha_\#} & \text{Map}(BP, BX)_\nu & &
 \end{array}$$

that the restriction of $\alpha|_{C_X(P,v)} = \alpha_{\sharp}$ to the maximal torus of the centralizer $C_X(V, v)$ is $w^{-1} \circ T(\alpha)$. □

If the centralizer $C_X(V, v)$ is connected, this determines the restriction $\alpha|_{C_X(V,v)}$ (see Section 2).

Corollary 7.4 *Let X be a p -compact group and $v: BV \rightarrow BX$ a toral elementary abelian p -subgroup such that its centralizer $C_X(V, v)$ is connected. If ψ^q is an unstable Adams operation of exponent $q \equiv 1 \pmod p$, $q \neq 1$, then*

- (a) *there is a unique lift of v to $g: BV \rightarrow BX(q)$,*
- (b) *$\psi^q|_{C_X(V,v)}$ is an unstable Adams operation of exponent q , and*
- (c) *the centralizer of (V, g) in $X(q)$ is $C_{X(q)}(V, g) \cong C_X(V, v)(q)$.*

Proof In particular, when $v: BV \rightarrow BX$ is a toral elementary abelian p -group in X and $\alpha = \psi^q$ is an Adams operation of exponent $q \equiv 1 \pmod p$, then we can write $T(\psi^q) = \psi^q$, the q -th power map in the maximal torus $T = T_X$ and $\psi^q \circ \mu \simeq \mu$, where $\mu: BV \rightarrow BT$ is a lift to BT of $v: BV \rightarrow BX$, so, by Lemma 7.3, there is a commutative diagram

$$\begin{array}{ccc}
 BT & \xrightarrow{T(\psi^q)=\psi^q} & BT \\
 \downarrow & & \downarrow \\
 BC_X(V, v) & \xrightarrow{\psi^q|_{C_X(V,v)}} & BC_X(V, v) \\
 \downarrow & & \downarrow \\
 BX & \xrightarrow{\psi^q} & BX .
 \end{array}$$

This proves (b), namely, $\psi^q|_{BC_X(V,v)}$ is, as well, an unstable Adams map ψ^q .

Now, (a) and (c) follow from Lemma 7.2. □

We will now restrict our attention to cases with $q \equiv 1 \pmod p$, $q \neq 1$. According to Proposition 6.2, the general case can be reduced to this one, in the cases that are of interest to us (see Remark 6.3). Hence, essentially, there will be no loss of generality in our assumption.

Proposition 7.5 *Let X be a connected p -compact group, p an odd prime, and ψ^q an unstable Adams operation of exponent $q \in \mathbb{Z}_p^*$, with $q \equiv 1 \pmod p$, $q \neq 1$. Then the*

inclusion $\nu: Bt_X \rightarrow BX$ of the subgroup of elements of order p in the maximal torus T_X has a unique lift to $g: Bt_X \rightarrow BX(q)$ and its centralizer is

$$C_{X(q)}(t_X, g) = T_X(q).$$

Proof Since $C_X(t_X, \nu) = T_X$ [50, 3.2] and $\psi^q|_{T_X} = T(\psi^q) = \psi^q$ this follows from Lemma 7.3 (see Corollary 7.4). \square

The group $T_X(q) \cong T_\ell^n \cong (\mathbb{Z}/p^\ell)^n$, where n is the rank of X and $\ell = \nu_p(q - 1)$, established in Proposition 7.5, embeds in $BX(q)$

$$i: BT_\ell^n \rightarrow BX(q)$$

as a subgroup (T_ℓ^n, i) that will be referred to as the *maximal finite torus* of $X(q)$. When no confusion is possible we will simply write T_ℓ^n for the maximal finite torus of $BX(q)$. Notice that T_ℓ^n is self-centralizing in $BX(q)$. Then, we define the Weyl group of $BX(q)$ as the automorphism group

$$W_{X(q)}(T_\ell^n, i) = \text{Aut}_{\mathcal{F}_p(BX(q))}(T_\ell^n) = \{ \varphi \in \text{aut}(T_\ell^n) \mid i \simeq i \circ B\varphi \}$$

of (T_ℓ^n, i) in the category $\mathcal{F}_p(BX(q))$. The group T_ℓ^n affords a faithful representation $W_{X(q)}(T_\ell^n, i) \rightarrow GL_n(\mathbb{Z}/p^\ell)$.

The Weyl group of $BX(q)$ can also be interpreted as the set of connected components of $\text{Map}_*(BT_\ell^n, BT_\ell^n)$ that lie over the connected component of $i: BT_\ell^n \rightarrow BX(q)$ through the map $\text{Map}_*(BT_\ell^n, BT_\ell^n) \rightarrow \text{Map}(BT_\ell^n, BX(q))$. The *normalizer* of the maximal finite torus of $BX(q)$, $BN_{X(q)}(T_\ell^n)$, is defined by its classifying space, the Borel construction for the action of $W_{X(q)}(T_\ell^n)$ on $\text{Map}(BT_\ell^n, BX(q))_i$, together with the inclusion

$$\bar{i}: BN_{X(q)}(T_\ell^n) = (\text{Map}(BT_\ell^n, BX(q))_i)_{hW_{X(q)}(T_\ell^n)} \longrightarrow BX(q)$$

induced by evaluation at the base point of BT_ℓ^n .

Proposition 7.6 *Let X be a connected p -compact group, p an odd prime, and ψ^q an unstable Adams operation of exponent $q \in \mathbb{Z}_p^*$, with $q \equiv 1 \pmod p$, $q \neq 1$. If (T_ℓ^n, i) is the maximal finite torus of $BX(q)$, then its Weyl group is*

$$W_{X(q)}(T_\ell^n) \cong W_X,$$

the Weyl group of X , with action on T_ℓ^n given by the mod p^ℓ reduction of the p -adic representation of W_X . The normalizer of the maximal finite torus is the split

extension $N_{X(q)}(T_\ell^n) = T_\ell^n \rtimes W_{X(q)}(T_\ell^n)$, and its classifying space fits in a homotopy commutative diagram

$$(24) \quad \begin{array}{ccc} BN_{X(q)}(T_\ell^n) & \xrightarrow{\tilde{\tau}} & BN \\ \bar{i} \downarrow & & \downarrow j \\ BX(q) & \xrightarrow{\iota} & BX \end{array}$$

where $j: BN \rightarrow BX$ is the inclusion of the maximal torus normalizer of X .

Proof We will first see that the automorphism in $\mathcal{F}_p(BX)$ of T_ℓ^n as a subgroup of BX via the composition $\iota \circ i: BT_\ell^n \rightarrow BX$, $W_X(T_\ell^n)$, coincides with the Weyl group W_X . In fact, an element $w \in W_X$ is a homotopy equivalence of BT over BX . Its restriction to BT_ℓ^n , factors again to give a homotopy equivalence \bar{w} of BT_ℓ^n and a homotopy commutative diagram

$$(25) \quad \begin{array}{ccc} BT_\ell^n & \longrightarrow & BT \\ \bar{w} \downarrow & & \downarrow w \quad \searrow j \\ BT_\ell^n & \longrightarrow & BT \xrightarrow{j} BX \end{array}$$

where $BT \simeq K(\mathbb{Z}_p^n, 2)$ and the map $BT_\ell^n \rightarrow BT$ classifies the extension class of the exact sequence $(\mathbb{Z}_p)^n \xrightarrow{p^\ell} (\mathbb{Z}_p)^n \rightarrow (\mathbb{Z}/p^\ell)^n$. Hence, if w is represented by a certain matrix in $GL_n(\mathbb{Z}_p)$, then \bar{w} is represented by its mod p^ℓ reduction in $GL_n(\mathbb{Z}/p^\ell)$. We have produced a homomorphism $W_X(T_\ell^n) \rightarrow W_X$ which is injective because W_X is finite and mod p^ℓ reduction has torsion free kernel in $GL_n(\mathbb{Z}_p)$. Furthermore, since $\text{Map}(BT_\ell^n, BX)_{\iota \circ i} \cong BT$, it turns out that every homotopy equivalence of BT_ℓ^n over BX can be extended to a diagram like (25) and therefore we actually have an isomorphism $W_X(T_\ell^n) \cong W_X$.

Next, we compare $W_{X(q)}(T_\ell^n)$ and $W_X(T_\ell^n)$. By composition with $\iota: BX(q) \rightarrow BX$, every homotopy equivalence \bar{w} of BT_ℓ^n over $BX(q)$ can also be considered over BX

$$\begin{array}{ccc} BT_\ell^n & & \\ \bar{w} \downarrow & \searrow i & \\ BT_\ell^n & \xrightarrow{i} & BX(q) \xrightarrow{\iota} BX \end{array}$$

which gives an inclusion $W_{X(q)}(T_\ell^n) \hookrightarrow W_X(T_\ell^n)$. Now, Lemma 7.2 implies that this is actually an isomorphism.

Finally, the natural maps

$$\text{Map}(BT_\ell^n, BX(q))_i \xrightarrow{\iota_\#} \text{Map}(BT_\ell^n, BX(q))_{\iota \circ i} \xleftarrow{\simeq} \text{Map}(BT_\ell^n, BX(q))_j$$

induced by composition with $\iota: BX(q) \rightarrow BX$ and with the inclusion $BT_\ell^n \rightarrow BT$, respectively, are equivariant for the respective actions of $W_{X(q)}(T_\ell^n)$, $W_X(T_\ell^n)$ and W_X , respectively, induced by the natural actions on the first component. Applying the Borel construction, we obtain a map

$$BN_{X(q)}(T_\ell^n) \xrightarrow{\tilde{\tau}} (\text{Map}(BT_\ell^n, BX(q))_{\iota \circ i})_{hW_X(T_\ell^n)} \simeq BN$$

and diagram (24) is induced by evaluation at base points. Moreover this maps extends the map between classifying spaces of tori to give a diagram of fibrations

$$\begin{array}{ccccc} T & \xrightarrow{\simeq} & T & & \\ \downarrow & & \downarrow & & \\ BT_\ell^n & \longrightarrow & BN_{X(q)}(T_\ell^n) & \longrightarrow & BW_{X(q)}(T_\ell^n) \\ \downarrow & & \downarrow \tilde{\tau} & & \downarrow \simeq \\ BT & \longrightarrow & BN & \longrightarrow & BW_X \end{array}$$

where $T \simeq \Omega BT \simeq K((\mathbb{Z}_p)^n, 1)$. By [5, 1.2] the bottom row fibration has a section and by [5, 3.3] this section lifts to a section of the fibration in the middle row. It follows that $N_{X(q)}(T_\ell^n)$ is a split extension. \square

For X a p -compact group and α a self equivalence, the inclusion $\iota: BX^{h\alpha} \rightarrow BX$ induces a functor between the respective fusion categories

$$\iota_\#: \mathcal{F}_p(BX^{h\alpha}) \longrightarrow \mathcal{F}_p(BX)$$

and Lemma 7.2 above gives some useful information in order to compare the morphism sets. Thus, for instance,

$$(26) \quad \text{Mor}_{\mathcal{F}_p(BX^{h\alpha})}((P, g), (Q, h)) \longrightarrow \text{Mor}_{\mathcal{F}_p(BX)}((P, \iota \circ g), (Q, \iota \circ h))$$

is a bijection provided $C_X(P, \iota \circ g)$ is connected. It rarely happens that those centralizers are connected for a general p -group P , but it is not so unusual if we restrict to some particular classes of small groups. For a space Y , we denote $\mathcal{F}_p^e(Y)$ the full subcategory of $\mathcal{F}_p(Y)$ whose objects are the elementary abelian subgroups of Y .

Corollary 7.7 *Let p be an odd prime. If X is a connected polynomial p -compact group and α a self homotopy equivalence, then the functor*

$$\iota_{\#}: \mathcal{F}_p^e(BX^{h\alpha}) \longrightarrow \mathcal{F}_p^e(BX)$$

is both full and faithful.

Proof If X is a connected polynomial p -compact group, then centralizers of elementary abelian p -subgroups are connected and [Lemma 7.2](#) applies. In fact, if (E, ν) is an elementary abelian p -subgroup of X , then the centralizer $C_X(E, \nu)$ is also a polynomial p -compact group, hence $H^1(BC_X(E, \nu); \mathbb{F}_p) = 0$ and therefore $C_X(E, \nu)$ is connected (see Dwyer and Wilkerson [\[28, 1.3\]](#)) and the map [\(26\)](#) is a bijection for every elementary abelian p -subgroups (P, g) and (Q, h) of $BX^{h\alpha}$. \square

Corollary 7.8 *Let p be an odd prime. If X is a connected polynomial p -compact group and ψ^q an unstable Adams operation of exponent $q \in \mathbb{Z}_p^*$, with $q \equiv 1 \pmod{p}$, then*

$$\iota_{\#}: \mathcal{F}_p^e(BX(q)) \longrightarrow \mathcal{F}_p^e(BX)$$

is an equivalence of categories.

Proof By [Corollary 7.7](#) we only have to check that $\iota_{\#}$ induces in this case a bijection between isomorphism classes of objects, and this follows from [Proposition 7.5](#), because in a polynomial p -compact group every elementary abelian subgroup is toral. \square

Let X be a polynomial p -compact group with trivial center and $q \in \mathbb{Z}_p^*$ a p -adic unit with $q \equiv 1 \pmod{p}$, $q \neq 1$. Putting $BC_{X(q)}(V, g) = \text{Map}(BV, BX(q))_g$ for any object (V, g) of $\mathcal{F}_p^e(BX(q))$ we get a functor from $\mathcal{F}_p^e(BX(q))^{\text{op}}$ to topological spaces. There is natural map

$$\text{hocolim}_{\mathcal{F}_p^e(BX(q))^{\text{op}}} BC_{X(q)} \rightarrow BX(q)$$

from the homotopy colimit of this functor. When $C_X(V, g)$ is connected, we have

$$BC_{X(q)}(V, g) \simeq BC_X(V, \iota \circ g)(q)$$

according to [Lemma 7.3](#) and [Corollary 7.4](#).

Let T_X be the maximal torus and W_X the Weyl group of a p -compact group X , p odd. As usually, we denote by t_X the group of all elements of order p in T_X , and $g: Bt_X \rightarrow X(q)$ the inclusion. For any nontrivial elementary abelian p -subgroup $E \leq T$, write $W_X(E)$ for the pointwise stabilizer subgroup of E .

Proposition 7.9 *Let X be a polynomial p -compact group with trivial center, p odd, and $q \in \mathbb{Z}_p^*$ a p -adic unit with $q \equiv 1 \pmod{p}$, $q \neq 1$. Assume that*

$$H^*(BX(q); \mathbb{F}_p) \cong H^*(BT_X(q); \mathbb{F}_p)^{W_X}$$

and that $H^*(BC_{X(q)}(E, g|_{BE}); \mathbb{F}_p) \cong H^*(BT_X(q); \mathbb{F}_p)^{W_X(E)}$

for any nontrivial, subgroup E of t_X . Then, the natural map

$$(27) \quad \operatorname{hocolim}_{\mathcal{F}_p^e(BX(q))^{\text{op}}} BC_{X(q)} \rightarrow BX(q)$$

is an \mathbb{F}_p -equivalence.

A similar statement holds with $\mathcal{F}_p^e(BX(q))$ replaced by the full subcategory generated by all objects of the form $(t_X)^P$ where P runs through the subgroups of a Sylow p -subgroup of W_X .

Proof The functor from $\mathcal{F}_p^e(BX(q)) = \mathcal{F}_p^e(BX)$ to the category of \mathbb{F}_p -vector spaces that takes an object E to $H^*(BC_{X(q)}(E); \mathbb{F}_p) = H^*(BT_X(q); \mathbb{F}_p)^{W_X(E)}$ is acyclic in the sense that $\lim^0 = H^*(BT_X(q); \mathbb{F}_p)^{W_X}$ and the higher limits vanish [24, 8.1]. Therefore, the Bousfield–Kan spectral sequence for the cohomology of the homotopy colimit

$$\operatorname{hocolim}_{\mathcal{F}_p^e(BX(q))^{\text{op}}} BC_{X(q)}$$

[11, XII.4.5] collapses at E_2 -term, and then it shows that (27) is an \mathbb{F}_p -equivalence. The same conclusion holds if we replace the category $\mathcal{F}_p^e(BX(q))$ by its full subcategory generated by all objects of the form $(t_X)^P$ where P runs through the subgroups of a Sylow p -subgroup of W_X [53, 2.16]. \square

This result motivates the research on the cohomology rings $H^*(BX(q); \mathbb{F}_p)$ and on the invariant rings $H^*(BT_X(q); \mathbb{F}_p)^{W_X}$, in the next two sections.

8 Cohomology rings

This section is devoted to the proof of [Theorem F](#). The Eilenberg–Moore spectral sequence is used in order to get a hold of the cohomology rings of the spaces $BX(q)$ of fixed points of unstable Adams operations acting on polynomial p -compact groups BX . We follow the arguments of Smith [64] that already contain the first part of the theorem.

Proof of Theorem F Part (1) is due to L Smith [64]. We will sketch his arguments here and then will continue with the proof of the second part of the theorem.

There is an Eilenberg–Moore spectral sequence associated to the pullback diagram

$$(28) \quad \begin{array}{ccc} BX(q) & \xrightarrow{\iota} & BX \\ \downarrow \iota & & \downarrow \Delta \\ BX & \xrightarrow{1 \times \psi^q} & BX \times BX . \end{array}$$

This is a second quadrant spectral sequence with

$$E_2^{s,t} \cong \text{Tor}_{H^*(BX; \mathbb{F}_p)}^{s,t} (H^*(BX; \mathbb{F}_p), H^*(BX; \mathbb{F}_p)) \implies H^{s+t}(BX(q); \mathbb{F}_p)$$

converging to a graded ring associated of $H^*(BX(q); \mathbb{F}_p)$.

For simplicity, we will write $P[x_i] = P[x_1, \dots, x_n] \cong H^*(BX; \mathbb{F}_p)$. The Koszul complex

$$\mathcal{E}(x_i) = P[x_i] \otimes P[x_i] \otimes E[sx_1, \dots, sx_n]$$

with $\text{bideg}(sx_i) = (-1, 2d_i)$ and $d(sx_i) = x_i \otimes 1 - 1 \otimes x_i$, is a free resolution of $P[x_i]$ as $(P[x_i] \otimes P[x_i])$ -module, with module structure given by the multiplication $m = \Delta^*$. Then, $\text{Tor}_{P[x_i] \otimes P[x_i]}^{**}(P[x_i], P[x_i])$ is the homology of the complex

$$P[x_i] \otimes_{P[x_i] \otimes P[x_i]} \mathcal{E}(x_i) \cong P[x_i] \otimes E[sx_1, \dots, sx_n]$$

where now the action of $P[x_i] \otimes P[x_i]$ on the left hand side term $P[x_i]$ is given by the algebra map $(1 \times \psi^q)^*$, hence one obtains the expression $d(sx_i) = x_i - q^{d_i} x_i$ for the differential, but since $q \equiv 1 \pmod p$, we actually have $d(sx_i) = 0$ for all $i = 1, \dots, n$. This yields

$$E_2^{**} \cong \text{Tor}_{P[x_i] \otimes P[x_i]}^{**}(P[x_i], P[x_i]) \cong P[x_1, \dots, x_n] \otimes E[sx_1, \dots, sx_n]$$

and, since the algebra generators appear in filtration degrees 0 and -1 , the spectral sequence collapses at the E_2 -page and then we can find elements y_i in $H^*(BX(q); \mathbb{F}_p)$ representing sx_i in the graded associated ring, with

$$H^*(BX(q); \mathbb{F}_p) \cong P[x_1, \dots, x_n] \otimes E[y_1, \dots, y_n].$$

Let T_X be the maximal torus of X and W_X the Weyl group. Since X is polynomial, the mod p cohomology ring of BX coincides with the invariants by the action of the Weyl group on the mod p cohomology of BT_X , $H^*(BT_X; \mathbb{F}_p)^{W_X} \cong H^*(BX; \mathbb{F}_p) \cong P[x_1, \dots, x_n]$.

According to [Corollary 7.4](#) and [Proposition 7.5](#), the classifying space of maximal finite torus of $X(q)$ is $BT(q) \cong BT_\ell^n$ and it is obtained from a pullback diagram

$$(29) \quad \begin{array}{ccc} BT_\ell^n & \xrightarrow{\iota} & BT \\ \downarrow \iota & & \downarrow \Delta \\ BT & \xrightarrow{1 \times \psi^q} & BT \times BT. \end{array}$$

Furthermore, the Weyl group is W_X by [Proposition 7.6](#). Hence, the restriction map

$$i^*: H^*(BX(q); \mathbb{F}_p) \rightarrow H^*(BT_\ell^n; \mathbb{F}_p)$$

has image in the invariant subring by the action of the Weyl group, W_X . It remains to show that this restriction map is injective.

The pullback diagram (29) yields another Eilenberg–Moore spectral sequence:

$$\bar{E}_2^{s,t} \cong \text{Tor}_{H^*(BT; \mathbb{F}_p)}^{s,t}(H^*(BT; \mathbb{F}_p), H^*(BT; \mathbb{F}_p)) \implies H^{s+t}(BT_\ell^n; \mathbb{F}_p).$$

We will pay special attention to the map between the two spectral sequences $i^*: E_r^{**} \rightarrow \bar{E}_r^{**}$ induced by the natural map from diagram (29) to diagram (28) given by inclusion of the maximal torus. In order to describe the induced map at the level of E_2 -pages, we need some elementary algebraic considerations.

Again for simplicity, we will write $P[t_i] = P[t_1, \dots, t_n] \cong H^*(BT_X; \mathbb{F}_p)$. The kernel of the multiplication $m: P[t_i] \otimes P[t_i] \rightarrow P[t_i]$ is a Borel ideal

$$\text{Ker } m = (t_1 \otimes 1 - 1 \otimes t_1, \dots, t_n \otimes 1 - 1 \otimes t_n)$$

and then we can define derivations

$$\partial_i: P[t_i] \rightarrow P[t_i]$$

for $i = 1, \dots, n$, in the following way. For any homogeneous polynomial $f \in P[t_i]$, $f \otimes 1 - 1 \otimes f \in \text{Ker } m$, hence we can find an expression $f \otimes 1 - 1 \otimes f = \sum_i c_i(f)(t_i \otimes 1 - 1 \otimes t_i)$, with coefficients $c_i(f) \in P[t_i] \otimes P[t_i]$, and then define $\partial_i(f) = m(c_i(f)) \in P[t_i]$. A routine calculation shows:

- (1) ∂_i is well defined and does not depend on the choice of coefficients $c_1(f), \dots, c_n(f)$,
- (2) ∂_i is a derivation of $P[t_i]$, and
- (3) $\partial_i(t_i) = 1$ and $\partial_i(t_j) = 0$ if $j \neq i$.

These properties show that these are the partial derivatives:

$$\partial_i(f) = \frac{\partial f}{\partial t_i} .$$

After these considerations we can easily describe the map between the respective E_2 -pages and show that it is injective. In order to compute the \bar{E}_2^{**} , we define now the Koszul complex

$$\mathcal{E}(t_i) = P[t_i] \otimes P[t_i] \otimes E[st_1, \dots, st_n]$$

with $\text{bideg}(st_i) = (-1, 2)$ and $d(st_i) = t_i \otimes 1 - 1 \otimes t_i$. As before, we obtain that

$$(30) \quad \begin{aligned} \bar{E}_2^{**} &\cong \text{Tor}_{P[t_i] \otimes P[t_i]}^{**}(P[t_i], P[t_i]) \cong P[t_i] \otimes_{P[t_i] \otimes P[t_i]} \mathcal{E}(t_i) \\ &\cong P[t_i] \otimes E[st_1, \dots, st_n] \end{aligned}$$

since the differential in this complex turns out to be trivial, again, because $q \equiv 1 \pmod p$. Also as before, the algebra generators of \bar{E}_2^{**} appear in filtration degree 0 and -1 and therefore the spectral sequence \bar{E}_r^{**} collapses at the E_2 -page.

Now, the inclusion $i^*: P[x_i] \rightarrow P[t_i]$ extends to a map of Koszul complexes

$$i^*: \mathcal{E}(x_i) \rightarrow \mathcal{E}(t_i)$$

which is a $P[x_i] \otimes P[x_i]$ -module map defined by

$$i^*(sx_i) = \sum_j c_j(x_i) \otimes st_j$$

on generators. Then the induced map

$$\begin{aligned} i^*: \text{Tor}_{P[x_i] \otimes P[x_i]}^{**}(P[x_i], P[x_i]) &\cong P[x_i] \otimes E[sx_1, \dots, sx_n] \\ &\longrightarrow \text{Tor}_{P[t_i] \otimes P[t_i]}^{**}(P[t_i], P[t_i]) \cong P[t_i] \otimes E[st_1, \dots, st_n] \end{aligned}$$

is determined by $i^*(sx_i) = \sum_j \partial_j(x_i) \otimes st_j = \sum_j \frac{\partial x_i}{\partial t_j} \otimes st_j$.

Now, i^* is injective because the Jacobian determinant is nontrivial,

$$J = \det \left(\frac{\partial x_i}{\partial t_j} \right) \neq 0,$$

by [69]. Since both spectral sequences collapse at the E_2 -page, it follows that the induced homomorphism $i^*: H^*(BX(q); \mathbb{F}_p) \rightarrow H^*(BT_\ell^n; \mathbb{F}_p)$ is also injective. \square

Remark 8.1 The argument with the Eilenberg–Moore spectral sequence used in the proof of part (1) of [Theorem F](#) applies more generally to the case of any unstable Adams operation ψ^q of arbitrary exponent $q \in \mathbb{Z}_p^*$ acting on a polynomial p -compact group [\[64\]](#). Under these more general hypothesis we obtain that if $H^*(BX) \cong P[x_1, \dots, x_n]$ then the cohomology of $BX(q)$ is

$$H^*(BX(q); \mathbb{F}_p) \cong P[x_{i_1}, \dots, x_{i_k}] \otimes E[y_{i_1}, \dots, y_{i_k}]$$

where the polynomial generators x_{i_j} correspond to those x_i with degree $2d_i = \deg x_i$ where $m|d_i$, if m is the order of $q \bmod p$, and $2d_i - 1 = \deg y_i$.

Notice that we can write $q = \zeta q'$ where ζ is an m -th root of one in \mathbb{Z}_p and $q' \equiv 1 \pmod p$. Hence $\psi^q = \psi^{q'} \circ \psi^\zeta$, and ψ^ζ has finite order m as automorphism of the p -compact group X . It follows from [Proposition 6.2](#) and [Remark 6.3](#), that $BX(q') \simeq BX(q)$ if $BX = BX^{h\psi^\zeta}$. Moreover, by [Theorem B](#), $Y = X^{h\psi^\zeta}$ is again a polynomial p -compact group. According to [Theorem F](#) the cohomology of BX must be

$$H^*(BY; \mathbb{F}_p) \cong P[x_{i_1}, \dots, x_{i_k}].$$

9 Invariant theory

Let X be a polynomial p -compact group of rank n and let q be a p -adic unit, $q \equiv 1 \pmod p$, $q \neq 1$, and $\ell = v_p(1 - q)$. In the second part of [Theorem F](#) we obtained a monomorphism $i^*: H^*(BX(q); \mathbb{F}_p) \hookrightarrow H^*(BT_\ell^n; \mathbb{F}_p)^{W_X}$, where T_ℓ^n is the maximal finite torus of $BX(q)$ and W_X the Weyl group (see [Proposition 7.5](#) and [Proposition 7.6](#)). Whether or not i^* is an isomorphism, $H^*(BX(q); \mathbb{F}_p) \cong H^*(BT_\ell^n; \mathbb{F}_p)^{W_X}$, is now a question of invariant theory and this is the subject of this section.

We recollect the necessary results from invariant theory and apply that in a case by case discussion, based on the Clark–Ewing list, and restricted to our cases of interest, namely:

- (1) Nonmodular groups: This consists of groups represented in a characteristic p that does not divide the order of the group. ([Example 9.2](#).)
- (2) Family 1 in the Clark–Ewing list: These are the symmetric groups Σ_{n+1} represented as Weyl groups of $SU(n+1)$. ([Examples 9.3](#) and [9.4](#).)
- (3) Family 2a en the Clark–Ewing list: The groups $G(m, r, n)$, $r|m|(p-1)$, $m > 1$, $(m, r, n) \neq (m, m, 2)$. ([Example 9.5](#).)
- (4) Family 2b en the Clark–Ewing list: Dihedral groups $D_{2m} = G(m, m, 2)$, $m \geq 3$. ([Example 9.6](#).)

- (5) The Aguadé family: These are the groups G_{12} , G_{29} , G_{31} , and G_{34} in the Clark–Ewing list (see Aguadé [2]). (Example 9.7.)

We obtain that $i^*: H^*(BX(q); \mathbb{F}_p) \hookrightarrow H^*(BT_\ell^n; \mathbb{F}_p)^{W_X}$ is an isomorphism in all cases except for Σ_3 at the prime 3 (included in class (2) above) and W_{G_2} , the Weyl group of G_2 and G_{12} , at the prime 3. It is also excluded the case 2b with $m = 3$ and $p = 3$, that corresponds to $PU(3)$ at prime 3.

From here one easily derives the structure of $BX(q)$ for Clark–Ewing p -compact groups and this is done in Theorem 9.8. The Aguadé family and 2a family are our cases of main interest and the discussion is postponed to Section 10 and Section 11, respectively. All of the other cases correspond to compact Lie groups.

At the end of the section we illustrate this methods with some examples going from Example 9.9 to Example 9.13.

Continuing with the notation of the preceding section we write $V = t_X$ for the elements of order p in the maximal finite torus and identify the dual vector space with the two dimensional primitive elements in the cohomology of BT_ℓ^n , $V^* \cong PH^2(BT_\ell^n; \mathbb{F}_p)$. The Bockstein operations provide a vector space isomorphism $PH^2(BT_\ell^n; \mathbb{F}_p) \cong H^1(BT_\ell^n; \mathbb{F}_p)$, that we will denote as $d: V^* \rightarrow dV^*$, of degree (-1) . If $P(V^*)$ is the symmetric algebra on V^* and $E(dV^*)$ the exterior algebra on dV^* , we can describe the algebra structure of $H^*(BT_\ell^n; \mathbb{F}_p)$ as

$$K(V^*) = P(V^*) \otimes E(dV^*) = P[x_1, \dots, x_n] \otimes E[dx_1, \dots, dx_n],$$

and d extends to an algebra derivation on $K(V^*)$. Moreover, any subgroup $G \leq GL(V)$ of linear substitutions acts on $K(V^*)$ in a natural way that commutes with the derivation d , hence $K(V^*)^G$ is still a differential algebra.

Assume that the ring of invariants, $P(V^*)^G = P[\rho_1, \dots, \rho_n]$ is a polynomial algebra; in particular, G is a reflection group. Then $d\rho_1, \dots, d\rho_n$ are also invariant under the action of G . The purpose of the next theorem is to establish the cases in which $\{\rho_1, \dots, \rho_n, d\rho_1, \dots, d\rho_n\}$ is a free system of generators for $K(V^*)^G$.

An element f of $P(V^*)$ is *invariant relative to* \det^{-1} if $g \cdot f = \det^{-1}(g)f$ for all $g \in G \leq GL(V)$. The subspace of relative invariant elements, $P(V^*)_{\det^{-1}}^G$, is a module over the invariant ring $P(V^*)^G$. In fact, $P(V^*)_{\det^{-1}}^G = f_{\det^{-1}} \cdot P(V^*)^G$ is a free module on one generator $f_{\det^{-1}} \in P(V^*)$, unique up to an invertible of \mathbb{F}_p [17]. For instance, if we write $d\rho_i = \sum_{j=1}^n a_{ij} dx_j$, then the Jacobian $J = \det(a_{ij}) \in P(V^*)$, of degree $\deg J = \sum_{i=1}^n (\deg \rho_i - 2)$, is invariant relative to \det^{-1} . In particular, $f_{\det^{-1}}$ divides J in $P(V^*)$ and $\deg f_{\det^{-1}} \leq \sum_{i=1}^n (\deg \rho_i - 2)$.

Theorem 9.1 [12] *Let V be a vector space of dimension n over a field of characteristic $p \neq 2$. Assume that $G \leq GL(V)$ is a group of linear substitutions such that $P(V^*)^G = P[\rho_1, \dots, \rho_n]$ is a polynomial algebra, then*

$$K(V^*)^G = P[\rho_1, \dots, \rho_n] \otimes E[d\rho_1, \dots, d\rho_n]$$

if and only if $f_{\det^{-1}}$ has degree $\deg f_{\det^{-1}} = \sum_{i=1}^n (\deg \rho_i - 2)$.

Proof Since $P(V^*)^G = P[\rho_1, \dots, \rho_n]$ is a polynomial ring of invariants, the Jacobian is nonzero, $J \neq 0$ [69], and this implies that the homomorphism $P[\rho_1, \dots, \rho_n] \otimes E[d\rho_1, \dots, d\rho_n] \rightarrow K(V^*)$ defined from the free anticommutative algebra to the subalgebra of $K(V^*)^G$ by mapping the variable ρ_i to the polynomial ρ_i of $P(V^*)^G$ and $d\rho_i$ to the differential of ρ_i in $K(V^*)$ is injective.

If $I = (i_1, \dots, i_k)$ is an ordered sequence of integers $1 \leq i_1 < \dots < i_k \leq n$, we write $d\rho_I = d\rho_{i_1} d\rho_{i_2} \dots d\rho_{i_k}$ and also $dx_I = dx_{i_1} dx_{i_2} \dots dx_{i_k}$. Let $FP(V^*)$ be the graded field of fractions of $P(V^*)$. Then, $FK(V^*) = FP(V^*) \otimes_{P(V^*)} K(V^*)$ is a vector space over $FP(V^*)$ spanned by $\{dx_I\}_I$. And then $\{d\rho_I\}_I$ is also a base of $FK(V^*)$.

Assume that $\deg f_{\det^{-1}} = \sum_{i=1}^n (\deg \rho_i - 2)$. This is the degree of the Jacobian J , hence $J = f_{\det^{-1}}$, up to an invertible of \mathbb{F}_p . Let $w \in K(V^*)^G$ be an arbitrary element. We can write $w = \sum_I w_I d\rho_I$, with $w_I \in FP(V^*)$ and then we will show that actually, for each index I , $w_I \in P(V^*)$. We choose I_0 of minimal length such that $w_{I_0} \neq 0$. Let I'_0 be the complementary sequence, then

$$w d\rho_{I'_0} = w_{I_0} d\rho_{I_0} d\rho_{I'_0} = \pm w_{I_0} d\rho_1 \dots d\rho_n = \pm w_{I_0} J dx_1 \dots dx_n$$

is an element of $K(V^*)^G$, and, since $dx_1 \dots dx_n$ is invariant relative to \det , $w_{I_0} J \in P(V^*)^G_{\det^{-1}} = f_{\det^{-1}} P(V^*)^G$. So, our assumption implies that $w_{I_0} \in P(V^*)^G$. Now we can repeat the argument with $w - w_{I_0} d\rho_{I_0} \in K(V^*)^G$. It follows that each w_I belongs to $P(V^*)^G$ and then $w \in P[\rho_1, \dots, \rho_n] \otimes E[d\rho_1, \dots, d\rho_n]$.

Assume otherwise that $\deg f_{\det^{-1}} \neq \sum_{i=1}^n (\deg \rho_i - 2)$; that is, $J = \iota f_{\det^{-1}}$ for some element $\iota \in P(V^*)^G$ of positive degree, then

$$w = \frac{d\rho_1 \dots d\rho_n}{\iota} = f_{\det^{-1}} dx_1 \dots dx_n$$

is an element of the algebra $K(V^*)^G$ which does not belong to $P[\rho_1, \dots, \rho_n] \otimes E[d\rho_1, \dots, d\rho_n]$. \square

In the examples below, we explore the invariants $K(V^*)^G$ for all groups G in the Clark–Ewing list that have polynomial invariants. We proceed by looking at the different

families, as they are listed at the beginning of this section, and we isolate the cases G where $K(V^*)^G$ is not a free graded anticommutative algebra; namely, Σ_3 (from Family 1 in the Clark–Ewing list), D_{12} (from Family 2b in the Clark–Ewing list), and G_{12} , all of them at $p = 3$.

Example 9.2 (G a nonmodular group [3]) If $G \leq GL(V)$ is a reflection group of order not divisible by p , then it is known that $P(V^*)^G = P[\rho_1, \dots, \rho_n]$ is a polynomial algebra and also that the degree of $f_{\det^{-1}}$ is twice the number of reflections in G . On the other hand, the number of reflections in G is $\sum_{i=1}^n (\deg \rho_i / 2 - 1)$. Hence $\deg f_{\det^{-1}} = \sum_{i=1}^n (\deg \rho_i - 2)$ and then Theorem 9.1 implies

$$K(V^*)^G = P[\rho_1, \dots, \rho_n] \otimes E[d\rho_1, \dots, d\rho_n].$$

For a group $G \leq GL(V)$ we denote $[x] = \{gx \mid g \in G\}$ the orbit of an element $x \in V^*$. The coefficients c_i of the polynomial $\prod_{y \in [x]} (X - y) = X^m + c_1 X^{m-1} + \dots + c_{m-1} X + c_m$ are the Chern classes of the orbit $[x]$ and belong to $P(V^*)^G$. The element $c_m = \prod_{y \in [x]} y$ is also called the Euler element of $[x]$. If we choose just one element $z_L \in L \cap [x]$ for each 1–dimensional vector subspace L of V^* that intersects the orbit $[x]$ nontrivially, $E[x] = \prod z_L$ is the pre-Euler element of the orbit $[x]$, defined up to a nonzero scalar. This is a relative invariant respect a linear character χ of G that we can associate to the orbit $[x]$ by the equation $g(E[x]) = \chi(g) \cdot E[x]$, for all $g \in G$. (See the articles by the first author [12] and the first author, Smith and Stong [17].)

Example 9.3 (Family 1 in the Clark–Ewing list: Σ_{n+1} , except Σ_3 at $p = 3$) The symmetric group Σ_{n+1} acts on the integral lattice of $SU(n + 1)$ that we can describe as $V = \mathbb{Z}\{(\hat{t}_1 - \hat{t}_{n+1}), (\hat{t}_2 - \hat{t}_{n+1}), \dots, (\hat{t}_n - \hat{t}_{n+1})\}$ where Σ_{n+1} permutes the letters $\hat{t}_1, \dots, \hat{t}_{n+1}$. Dually, V^* is generated by classes t_1, t_2, \dots, t_n , and Σ_{n+1} permutes $t_1, t_2, \dots, t_n, t_{n+1}$ with the relation $t_1 + t_2 + \dots + t_n + t_{n+1} = 0$.

The orbit of t_1 is $[t_1] = \{t_1, t_2, \dots, t_n, t_{n+1}\}$, and the Chern classes of this orbit, obtained as the coefficients of the polynomial $\prod_{i=1}^{n+1} (X - t_i)$, are, up to a sign, the generators c_i of the invariant ring $P(V^*)^{\Sigma_{n+1}} = P[c_2, \dots, c_{n+1}]$.

The orbit of $t_1 - t_2$ is

$$\begin{aligned} [t_1 - t_2] &= \{(t_i - t_j) \mid 1 \leq i, j \leq n + 1, i \neq j\} \\ &= \{\pm(t_i - t_j) \mid 1 \leq i \neq j \leq n + 1\} \\ &= \{\pm(t_i - t_j) \mid 1 \leq i \neq j \leq n\} \cup \{\pm(t_1 + \dots + 2t_i + \dots + t_n) \mid 1 \leq i \leq n\}, \end{aligned}$$

thus the pre-Euler element associated to this orbit is

$$E = E[t_1 - t_2] = \prod_{1 \leq i \neq j \leq n} (t_i - t_j) \prod_{1 \leq i \leq n} (t_1 + \dots + 2t_i + \dots + t_n).$$

Notice here the exception $n = 2$ at $p = 3$, in which case $E[t_1 - t_2] = (t_1 - t_2)$. With this exception, we can check that the linear character associated to the pre-Euler element is precisely the determinant ($\det = \det^{-1}$ in this case) and also that the degree of E , $n^2 + n$, coincides with the degree of the Jacobian J . Thus for $(n, p) \neq (2, 3)$, we have

$$K(V^*)^{\Sigma_{n+1}} = P[c_2, \dots, c_{n+1}] \otimes E[dc_2, \dots, dc_{n+1}].$$

Example 9.4 (Σ_3 at the prime 3) The integral lattice of $SU(3)$ is $\pi_2(T_{SU(3)}) = \mathbb{Z}\{(\hat{t}_1 - \hat{t}_3), (\hat{t}_2 - \hat{t}_3)\}$ with the action of Σ_3 that permutes \hat{t}_1 , \hat{t}_2 , and \hat{t}_3 . If Σ_3 is generated by the 3-cycle σ and the transposition τ , the representation afforded by $\pi_2(BT_{SU(3)})$ is determined by

$$\sigma \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The dual action in mod 3 cohomology $V^* = H^2(BT_{SU(3)}; \mathbb{F}_3) = \mathbb{F}_3\{t_1, t_2\}$ gives $P(V^*)^{\Sigma_3} \cong P[x_4, x_6]$, where $x_4 = t_1^2 + t_1 t_2 + t_2^2$ and $x_6 = t_1 t_2 (t_1 + t_2)$. This is the particular case of [Example 9.3](#) with $n = 2$ at the prime 3.

The action extends to $K(V^*) = P[t_1, t_2] \otimes E[dt_1, dt_2]$ where we obtain invariant elements

$$\begin{aligned} y_3 &= dx_4 = (t_2 - t_1)dt_1 + (t_1 - t_2)dt_2 \\ y_5 &= dx_6 = (t_2^2 - t_1 t_2)dt_1 + (t_1^2 - t_1 t_2)dt_2 \\ y_4 &= (t_2 - t_1)dt_1 dt_2 \end{aligned}$$

so that $y_3 y_5 = (t_1^2 + t_1 t_2 + t_2^2)(t_2 - t_1)dt_1 dt_2 = x_4 y_4$.

These elements together with the polynomial invariants generate the invariant ring $K(V^*)^{\Sigma_3}$:

$$(31) \quad K(V^*)^{\Sigma_3} \cong \frac{P[x_4, x_6] \otimes E[y_3, y_4, y_5]}{(y_3 y_5 - x_4 y_4, y_3 y_4, y_4 y_5)}.$$

The proof follows the method of [Theorem 9.1](#). In this particular case $1, dt_1, dt_2, dt_1 dt_2$ is a basis of $K(V^*)$ as a free $P(V^*)$ -module, while $1, y_3, y_5, y_3 y_5$ or $1, y_3, y_4, y_5$ are basis of $FK(V^*)$ as graded $FP(V^*)$ vector spaces.

Assume that w is an element of $K(V^*)^{\Sigma_3}$ of even degree. We can write $w = w_0 + w_1 y_4$, with $w_0, w_1 \in FP(V^*)$. First, multiply the equality by y_4 : $w y_4 \in K(V^*)^{\Sigma_3}$ and $w y_4 = w_0 y_4 = w_0 (t_2 - t_1) dt_1 dt_2$. Then

$$w_0 (t_2 - t_1) \in P(V^*)_{\det^{-1}}^{\Sigma_3} = (t_2 - t_1) P(V^*)^{\Sigma_3},$$

hence $w_0 \in P(V^*)^{\Sigma_3}$. Now, we also have $w_1 y_4 \in K(V^*)^{\Sigma_3}$, hence the same argument implies that $w_1 \in P(V^*)^{\Sigma_3}$.

Next, assume that w is an element of $K(V^*)^{\Sigma_3}$ of odd degree. In this case, $w = w_2 y_3 + w_3 y_5$ with $w_2, w_3 \in FP(V^*)$. If we multiply this equality by $y_5 \in K(V^*)^{\Sigma_3}$ we get $w y_5 \in K(V^*)^{\Sigma_3}$ and $w y_5 = w_2 y_3 y_5 = w_2 x_4 y_4$, and then again the equality $w_2 x_4 y_4 = w_2 x_4 (t_2 - t_1) dt_1 dt_2 \in K(V^*)^{\Sigma_3}$ implies that $w_2 x_4 \in P(V^*)^{\Sigma_3}$. Since $P(V^*)^{\Sigma_3} = P[x_4, x_6]$, we can write $w_2 = q_2 + \lambda(x_6^r/x_4)$, $q_2 \in P(V^*)^{\Sigma_3}$ and $\lambda \in \mathbb{F}_3$, $r \geq 0$. A similar argument, in which we multiply w by y_3 , implies that $w_3 = q_3 + \mu(x_6^s/x_4)$, $q_3 \in P(V^*)^{\Sigma_3}$ and $\mu \in \mathbb{F}_3$, $s \geq 0$. If we substitute these expressions in $w = w_2 y_3 + w_3 y_5$ we can easily check that this element can only belong to $K(V^*)$ provided $\lambda = \mu = 0$. It follows that $w_2 = q_2 \in P(V^*)^{\Sigma_3}$ and $w_3 = q_3 \in P(V^*)^{\Sigma_3}$. This proves the isomorphism (31).

Example 9.5 (Family 2a in the Clark–Ewing list: $G = G(m, r, n)$, $r | m | p - 1$ [12])

$G(m, r, n)$ is the subgroup of $GL_n(\mathbb{Z}_p)$ generated by the permutation matrices and the diagonal matrices $\text{diag}(\theta_1, \dots, \theta_n)$, where $\theta_i^m = 1$ and $(\theta_1 \dots \theta_n)^{m/r} = 1$. In particular, $G(m, 1, n)$ is isomorphic to the semidirect product $(\mathbb{Z}/m)^n \rtimes \Sigma_n$. In this case we clearly have $P(V^*)^{G(m,1,n)} = P[\rho_1, \dots, \rho_n]$, where $1 + \rho_1 + \dots + \rho_n = \prod_{i=1}^n (1 + x_i^m)$, if we write $P(V^*) = P[x_1, \dots, x_n]$. Now, $\rho_n = (x_1 \dots x_n)^m$ is the Euler element associated to the orbit of $x_1, [x_1]$. The pre-Euler element is $E_1 = E[x_1] = x_1 \dots x_n$. It carries an associated linear character χ_1 , defined by $\chi_1(\text{diag}(\theta_1, \dots, \theta_n)) = \theta_1 \dots \theta_n$ and $\chi_1(\sigma) = 1$ if $\sigma \in \Sigma_n$ is a permutation matrix. Notice that $G(m, r, n) = \text{Ker } \chi_1^{m/r}$ and

$$P(V^*)^{G(m,r,n)} = P[\rho_1, \dots, \rho_{n-1}, E_1^{m/r}].$$

The orbit of $(x_1 - x_2)$ is $[x_1 - x_2] = \{ \theta_1 x_i - \theta_2 x_j \mid \theta_1^m = \theta_2^m = 1, i < j \}$ and its pre-Euler element is $E_2 = \prod_{i < j} (x_i^m - x_j^m)$. In this case the associated character is χ_2 defined by $\chi_2(\text{diag}(\theta_1, \dots, \theta_n)) = 1$ and $\chi_2(\sigma) = \text{sg}(\sigma)$ is the sign of the permutation. We clearly have $\det = \chi_1 \chi_2$ and then $\det^{-1} = \chi_1^{m/r-1} \chi_2$. It follows that $f_{\det^{-1}} = E_1^{m/r-1} E_2$. Counting degrees, we obtain $\sum_{i=1}^{n-1} (\deg \rho_i - 2) + \deg(E_1^{m/r}) - 2 = \sum_{i=1}^{n-1} (2im - 2) + 2n(m/r) - 2 = n(n-1)m + 2n(m/r - 1) = \deg f_{\det^{-1}}$. Hence, Theorem 9.1 implies

$$K(V^*)^{G(m,r,n)} = P[\rho_1, \dots, \rho_{n-1}, E_1^{m/r}] \otimes E[d\rho_1, \dots, d\rho_{n-1}, d(E_1^{m/r})].$$

Example 9.6 (D_{12} at the prime 3) In family 2b there are two modular cases at odd primes, namely, D_6 and D_{12} at $p = 3$. The first one is the Weyl group of $PU(3)$ which is not polynomial at $p = 3$, the second case corresponds to the Weyl group of

the exceptional Lie group G_2 . The action of D_{12} on $\pi_2(BT_{G_2})$ gives a representation

$$\omega \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The dual action in mod 3 cohomology $V^* = H^2(BT_{G_2}; \mathbb{F}_3) = \mathbb{F}_3\{t_1, t_2\}$ gives $P(V^*)^{D_{12}} \cong P[x_4, x_{12}]$, where $x_4 = t_1^2 + t_1t_2 + t_2^2$ and $x_{12} = (t_1t_2(t_1 + t_2))^2$.

The extension of this action to $K(V^*)$ gives now

$$(32) \quad K(V^*)^{D_{12}} \cong \frac{P[x_4, x_{12}] \otimes E[y_3, y_{10}, y_{11}]}{(y_3y_{11} - x_4y_{10}, y_3y_{10}, y_{10}y_{11})}$$

with elements $y_3 = dx_4$ and $y_{11} = dx_{12}$, so that $y_3y_{11} = (t_1^2 + t_1t_2 + t_2^2)t_1t_2(t_1^2 - t_2^2)dt_1dt_2 = x_4y_{10}$, which serves as definition for y_{10} . The isomorphism (32) is proved with same arguments of [Example 9.4](#).

Actually, the inclusion of $SU(3)$ as maximal subgroup of G_2 , induces an inclusion $\Sigma_3 \hookrightarrow D_{12}$, identifying the generator τ and σ with ω^2 . The induced inclusion $K(V^*)^{D_{12}} \hookrightarrow K(V^*)^{\Sigma_3}$ identifies the generators x_4 and y_3 and maps x_{12} to x_6^2 , y_{10} to $-x_6y_4$ and y_{11} to $-x_6y_5$.

Example 9.7 (G_{12} , G_{29} , G_{31} , and G_{34} in the Clark–Ewing list at modular primes)

The groups G_{12} (rank 2, $p=3$), G_{29} (rank 4, $p=5$), G_{31} (rank 4, $p=5$), and G_{34} (rank 6, $p=7$), of the Clark–Ewing list have polynomial invariants [[2](#); [1](#); [70](#)].

We obtain by direct calculation that the generator of the \det^{-1} -relative invariants $f_{\det^{-1}}$ has the same degree as the corresponding Jacobian in cases G_{29} , G_{31} , and G_{34} , and then [Theorem 9.1](#) applies.

The case $G_{12} = GL(2, 3)$ is special. Notice that all those groups contain a copy of the symmetric group of the same rank affording the representation of [Example 9.3](#). G_{12} contains Σ_3 as described in [Example 9.4](#). The invariant ring $K(V^*)^{GL(2,3)}$ was computed by Mui [[56](#)] (alternatively, use the arguments in [Example 9.4](#)):

$$K(V^*)^{GL(2,3)} \cong \frac{P[x_{12}, x_{16}] \otimes E[y_{10}, y_{11}, y_{15}]}{(y_{11}y_{15} - x_{16}y_{10}, y_{10}y_{11}, y_{10}y_{15})}$$

where $x_{12} = \frac{(t_1t_2^9 - t_2t_1^9)}{(t_1t_2^3 - t_2t_1^3)}$, $x_{16} = \mathcal{P}^1(x_{12}) = \frac{(t_1^3t_2^9 - t_2^3t_1^9)}{(t_1t_2^3 - t_2t_1^3)}$,

$y_{11} = dx_{12}$, $y_{15} = dx_{16}$, and y_{10} is defined by the relation $y_{11}y_{15} = x_{16}y_{10}$.

We can easily obtain the description of the inclusion

$$K(V^*)^{GL(2,3)} \hookrightarrow K(V^*)^{\Sigma_3}$$

as $R: \frac{P[x_{12}, x_{16}] \otimes E[y_{10}, y_{11}, y_{15}]}{(y_{11}y_{15} - x_{16}y_{10}, y_{10}y_{11}, y_{10}y_{15})} \longrightarrow \frac{P[x_4, x_6] \otimes E[y_3, y_4, y_5]}{(y_3y_5 - x_4y_4, y_3y_4, y_4y_5)}$

mapping

$$\begin{aligned} x_{12} &\mapsto x_4^3 + x_6^2, \\ x_{16} &\mapsto x_6^2x_4, \\ y_{15} &\mapsto x_6^2y_3 - x_4x_6y_5, \\ y_{11} &\mapsto -x_6y_5, \\ y_{10} &\mapsto x_6y_4. \end{aligned}$$

Let X be a Clark–Ewing p –compact group; that is, a connected p –compact group for which p does not divide the order of the Weyl group. Models for these p –compact groups were constructed by Clark and Ewing [19]. If W_X is the Weyl group of X , the action of W_X on the maximal torus T_X is determined by the induced representation $\rho: W_X \rightarrow GL_n(\mathbb{Z}_p)$, where n is the rank of X . This representation gives W_X the structure of a \mathbb{Z}_p –reflection group, thus product of irreducibles listed in [19]. It turns out that $BX \simeq (BT_{hW_X})_p^\wedge$, where the action of W_X on BT is given by ρ [23]. Our next result is a similar description of $X(q)$, for $q \equiv 1 \pmod p$.

Theorem 9.8 *Let X be a Clark–Ewing p –compact group and $q \equiv 1 \pmod p$, $q \neq 1$, then*

$$BX(q) \simeq ((BT_X(q))_{hW_X})_p^\wedge \simeq B(T_\ell^n \rtimes W_X)_p^\wedge$$

with $T_\ell^n \cong (\mathbb{Z}/p^\ell)^n$, where n is the rank of X and $\ell = v_p(q - 1)$.

Proof In Proposition 7.5 we have obtained a map $BT_\ell^n \xrightarrow{\simeq} \text{Map}(BV, BX(q))_{Bi} \rightarrow BX(q)$ and according to Proposition 7.6 we have a factorization

$$BT_\ell^n \simeq \text{Map}(BV, BX(q))_{Bi} \longrightarrow (\text{Map}(BV, BX(q))_{Bi})_{hW_X} \longrightarrow BX(q).$$

The induced maps in cohomology are

$$H^*(BX(q); \mathbb{F}_p) \longrightarrow H^*((BT_\ell^n)_{hW_X}) \xrightarrow{\cong} H^*(BT_\ell^n)^{W_X}$$

where the second arrow is an isomorphism because the order of W_X is prime to p and the composition is a monomorphism by Theorem F.

According to Theorems F and 9.1, $H^*(BX(q); \mathbb{F}_p)$ and $H^*(BT_\ell^n)^{W_X}$ has the same Poincaré series, hence $H^*(BX(q); \mathbb{F}_p) \cong H^*((BT_\ell^n)_{hW_X})$ and the result follows. \square

Example 9.9 ($SU(2)$ at odd primes) The Weyl group of $SU(2)$ is $\mathbb{Z}/2$ acting on the maximal torus $S^1 \subset \mathbb{C}$ by sign change, that is, as ψ^{-1} . Then, [Theorem 9.8](#) applies. All spaces will be considered completed at $p > 2$.

Let ψ^q be an Adams map of exponent $q \in \mathbb{Z}_p^*$, $q \neq 1$. We have that $BSU(2)(q) = BSU(2)(-q)$ as $\psi^{-q} = \psi^{-1} \circ \psi^q = \psi^q$ because ψ^{-1} is the identity. For $q \equiv 1 \pmod p$, define $\ell = v_p(1 - q)$, and then $BSU(2)(q)$ has maximal finite torus \mathbb{Z}/p^ℓ and Weyl group $\mathbb{Z}/2$, acting by sign change, so

$$BSU(2)(q) \simeq (B\mathbb{Z}/p^\ell)_{h\mathbb{Z}/2}$$

is an equivalence at the prime p .

Notice that if $q \not\equiv \pm 1 \pmod p$, then we can write $\psi^q = \psi^\zeta \circ \psi^{q'}$, where ζ is a $(p-1)$ -th root of 1, different than ± 1 , and $q' \equiv 1 \pmod p$. Then by [Proposition 6.2](#), $BSU(2)(q) \simeq BSU(2)^{h\psi^\zeta}(q')$, and according to [Proposition A.5](#) (see [Proposition A.8](#)), $BSU(2)^{h\psi^\zeta}$ is trivial, hence $BSU(2)(q)_p^\wedge$ is also trivial.

For q a prime power, prime to p , $SU(2)(q)$ is equivalent at p to the finite Chevalley group $SL_2(q)$ and $SU(2)(-q)$ to $SU_2(q)$. This agrees with the above calculations, for in any case $\ell = v_p(1 - q^2)$.

Example 9.10 (Sullivan spheres S^{2m-1} , $m \mid p-1$) This generalizes the previous example. When $m \geq 2$ divides $p-1$, the cyclic group C_m of order m acts on \mathbb{Z}/p^∞ . The Sullivan sphere BS^{2m-1} is the p -completion of the classifying space of the semidirect product $\mathbb{Z}/p^\infty \rtimes C_m$ for this action and $H^*(BS^{2m-1}; \mathbb{F}_p) = P[x_{2m}]$. If u is any p -adic unit then

$$BS^{2m-1}(u) = \begin{cases} \Lambda(BS^{2m-1}) & u^m = 1 \\ B(\mathbb{Z}/p^\ell \rtimes C_m) & u^m \neq 1, u^m \equiv 1 \pmod p, \ell = v_p(u^m - 1) \\ * & u^m \not\equiv 1 \pmod p. \end{cases}$$

All spaces are understood to be completed at p . To see this, note that $BS^{2m-1}(u)$ is contractible if $u^m \not\equiv 1 \pmod p$ by [Theorem B](#). Otherwise, if $u^m \equiv 1 \pmod p$, then $u = \zeta q$ with $\zeta \in C_m \subset C_{p-1}$, $q \equiv 1 \pmod p$ and $BS^{2m-1}(u) = (BS^{2m-1})^{h(\zeta)}(q) = BS^{2m-1}(q) = B(\mathbb{Z}/p^\ell \rtimes C_m)$ by [Proposition 6.2](#), [Proposition A.8](#), and [Theorem 9.8](#), because $v_p(q-1) = v_p(q^m-1) = v_p(u^m-1)$.

Example 9.11 ($SU(3)(q)$ at the prime 3) Fix q a 3-adic integer with $0 < \ell = v_3(1 - q) < \infty$. According to [Theorem F](#)

$$H^*(SU(3)(q); \mathbb{F}_3) \cong P[x_4, x_6] \otimes E[y_3, y_5],$$

with $\beta_{(\ell)}(y_3) = x_4$ and $\beta_{(\ell)}(y_5) = x_6$.

According to Propositions 7.5 and 7.6, $T_\ell^2 \cong (\mathbb{Z}/3^\ell)^2$ is the maximal finite torus of $SU(3)(q)$ with Weyl group Σ_3 . Now, the invariant ring

$$H^*(T_\ell^2; \mathbb{F}_3)^{\Sigma_3} \cong \frac{P[x_4, x_6] \otimes E[y_3, y_4, y_5]}{(y_3 y_5 - x_4 y_4, y_3 y_4, y_4 y_5)}$$

computed in Example 9.4 turns out to differ from $H^*(SU(3)(q); \mathbb{F}_3)$. The natural map $H^*(SU(3)(q); \mathbb{F}_3) \hookrightarrow H^*(T_\ell^2; \mathbb{F}_3)^{\Sigma_3}$ (see Theorem F) has cokernel isomorphic to $P[x_6]y_4$.

Example 9.12 (G_2 at the prime 3) The exceptional Lie group G_2 has rank two and the Weyl group is dihedral $D_{12} \cong \Sigma_3 \times C_2$, listed in family 2b for $m = 6$ in the Clark–Ewing list. The category $\mathcal{F}_3^e(G_2)$ of nontrivial elementary abelian 3–subgroups of G_2 has an isomorphism class of rank two elementary abelian 3–subgroups with automorphism group D_{12} , the Weyl group of G_2 , and two classes of elementary abelian 3–subgroups of rank one, with automorphism group of order two. It is equivalent to the category $\mathbb{I}(2)$ of Appendix B, with $G = D_{12}$, $H_1 = \Sigma_3$, and $H_2 = \Sigma_2$. The centralizer diagram for elementary abelian 3–subgroups is

$$C_2 \circlearrowleft BSU(3) \xleftarrow{(\Sigma_3)^{\text{op}} \setminus (D_{12})^{\text{op}}} BT^2 \xrightarrow{(\Sigma_2)^{\text{op}} \setminus (D_{12})^{\text{op}}} BU(2) \circlearrowright C_2 .$$

$\begin{array}{c} \uparrow \\ (D_{12})^{\text{op}} \end{array}$

By Corollary 7.8 the categories of nontrivial elementary abelian 3–subgroups of G_2 and $G_2(q)$ coincide: $\mathcal{F}_3^e(G_2(q)) \cong \mathcal{F}_3^e(G_2)$, and furthermore, for every object (E, ν) of $\mathcal{F}_p^e(G_2)$, $BC_{G_2(q)}(E, \nu) \simeq BC_{G_2}(E, \nu)(q)$, thus the centralizer diagram of elementary abelian subgroups of $G_2(q)$ is

(33)

$$C_2 \circlearrowleft BSU(3)(q) \xleftarrow{(\Sigma_3)^{\text{op}} \setminus (D_{12})^{\text{op}}} BT_\ell^2 \xrightarrow{(\Sigma_2)^{\text{op}} \setminus (D_{12})^{\text{op}}} BU(2)(q) \circlearrowright C_2$$

$\begin{array}{c} \uparrow \\ (D_{12})^{\text{op}} \end{array}$

and there is a natural map $\text{hocolim}_{\mathcal{F}_p^e(G_2(q))^{\text{op}}} BC_{G_2(q)} \rightarrow BG_2(q)$ which is a sharp homology decomposition [22]; that is, the Bousfield–Kan spectral sequence for the homotopy colimit collapses at the E_2 –term and gives

$$H^*(G_2(q); \mathbb{F}_3) \cong \varprojlim^0_{\mathcal{F}_p^e(G_2(q))} H^*(BC_{G_2(q)}; \mathbb{F}_3) \cong P[x_4, x_{12}] \otimes E[y_3, y_{11}].$$

This result can also be obtained by direct calculation from [Proposition B.1](#) using the invariant theory calculations in [Examples 9.4](#) and [9.6](#). Notice that [Proposition 7.9](#) does not apply to $BG_2(q)$ at the prime 3 (see [Example 9.11](#)).

Example 9.13 (G_2 at primes $p > 3$) Let p be a prime > 3 . The compact Lie group G_2 , at the prime p , is a Clark–Ewing p -compact group and $H^*(BG_2; \mathbb{F}_p) = P[x_4, x_{12}]$. The Weyl group has order $|W(G_2)| = 12$ and the center is cyclic of order two. Let $u \neq \pm 1$ be a p -adic unit, and let r denote the order of $u \bmod p$. Then

$$BG_2(u) = \begin{cases} BG_2(u^2) = B(T_\ell^2 \rtimes W(G_2)) & r \in \{1, 2\}, \ell = v_p(u^2 - 1) \\ BS^{11}(u^6) = B(\mathbb{Z}/p^\ell \rtimes C_6) & r \in \{3, 6\}, \ell = v_p(u^6 - 1) \\ * & \text{otherwise,} \end{cases}$$

where it is understood that all spaces are completed at p . To see this, write $u = \zeta q$ where ζ is a $(p - 1)$ -th root of unity and $q \equiv 1 \pmod p$. Note first that $BG_2(u) = BG_2(\pm u)$ as the Weyl group of G_2 contains -1 . In case $u^2 \equiv 1 \pmod p$ ($u^2 \neq 1$), we have that $u = \pm q$ so that $BG_2(u) = BG_2(\pm u) = BG_2(u^2) = B(T_\ell^2 \rtimes W(G_2))$ by [Theorem 9.8](#) and [Proposition 6.5](#). If $u^2 \not\equiv 1 \pmod p$, $u^6 \equiv 1 \pmod p$, then $u = \pm \sigma q$ where $\sigma^3 = 1$ and $BG_2(u) = BG_2(\pm \sigma q) = BG_2(\sigma q) = BG_2^{h(\sigma)}(q) = BS^{11}(q) = BS^{11}(u^6)$ by [Proposition 6.2](#) and [Proposition A.10](#); the last equality follows from [Proposition 6.5](#) since $v_p(u^6 - 1) = v_p(q^6 - 1) = v_p(q - 1)$. If $u^6 \not\equiv 1 \pmod p$ then $BG_2(u)$ is contractible by [Theorem B](#). It follows that

$$H^*(BG_2(u); \mathbb{F}_p) = \begin{cases} P[x_4, x_{12}] \otimes E(y_3, y_{11}) & r \in \{1, 2\} \\ P[x_{12}] \otimes E(y_{11}) & r \in \{3, 6\} \\ \mathbb{F}_p & \text{otherwise} \end{cases}$$

with higher order Bocksteins as explained in [Theorem F](#). This provides the geometric explanation of Kleiner’s computation [[39](#), Theorem 1-1] of cohomology rings of finite Chevalley groups of type G_2 .

10 Chevalley p -local finite groups from Aguadé p -compact groups

In [[2](#)], Aguadé constructed the exotic p -compact groups \mathbf{X}_i , $i = 12, 29, 31, 34$, with Weyl groups the groups G_{12} (rank 2, $p = 3$), G_{29} (rank 4, $p = 5$), G_{31} (rank 4, $p = 5$), and G_{34} (rank 6, $p = 7$), on the Sheppard–Todd and Clark–Ewing lists, respectively. All four of them are obtained as the homotopy colimit of a diagram that we proceed by describing.

Write G_i to denote one of the groups G_{12} , G_{29} , G_{31} , or G_{34} , and Z its center, namely, $Z \cong \mathbb{Z}/2$ for G_{12} , $Z \cong \mathbb{Z}/4$ for G_{29} , $Z \cong \mathbb{Z}/4$ for G_{31} , $Z \cong \mathbb{Z}/6$ for G_{34} , in all cases represented by diagonal matrices with entries $p - 1$ roots of unity. In all four cases we also fix a subgroup isomorphic to Σ_p . Then, the index category is the opposite category of $\mathbb{I}(1)$, with two objects 0 and 1, and

$$\begin{aligned} \text{Aut}_{\mathbb{I}(1)}(0) &= G_i, \\ \text{Aut}_{\mathbb{I}(1)}(1) &= N_{G_i}(\Sigma_p) / \Sigma_p \cong Z, \\ \text{Mor}_{\mathbb{I}(1)}(1, 0) &= \Sigma_p \setminus G_i, \\ \text{Mor}_{\mathbb{I}(1)}(0, 1) &= \emptyset. \end{aligned}$$

The functor assigns BT^{p-1} to 0 and $BSU(p)$ to 1, up to homotopy, where the center of G_i , Z , acts on $BSU(p)$ via unstable Adams operations. The diagram is described in the following picture:

$$Z \left(\begin{array}{c} \text{BSU}(p) \xleftarrow{(\Sigma_p)^{\text{op}} \setminus (G_i)^{\text{op}}} BT^{p-1} \end{array} \right) (G_i)^{\text{op}}.$$

Each \mathbf{X}_i is a p -compact group with maximal torus $T_{\mathbf{X}_i} = T^{p-1}$ and Weyl group $W_{\mathbf{X}_i} = G_i$. The respective cohomology rings coincide with the isomorphic invariant rings $H^*(B\mathbf{X}_i; \mathbb{F}_p) \cong H^*(BT_{\mathbf{X}_i}; \mathbb{F}_p)^{G_i}$, and these are the polynomial rings [2; 1; 70]:

$$\begin{aligned} H^*(B\mathbf{X}_{12}; \mathbb{F}_3) &\cong P[x_{12}, x_{16}], \\ H^*(B\mathbf{X}_{29}; \mathbb{F}_5) &\cong P[x_8, x_{16}, x_{24}, x_{40}], \\ H^*(B\mathbf{X}_{31}; \mathbb{F}_5) &\cong P[x_{16}, x_{24}, x_{40}, x_{48}], \\ H^*(B\mathbf{X}_{34}; \mathbb{F}_7) &\cong P[x_{12}, x_{24}, x_{36}, x_{48}, x_{60}, x_{84}]. \end{aligned}$$

Throughout this section we fix an unstable Adams operation ψ^q of exponent $q \in \mathbb{Z}_p^*$ with $q \equiv 1 \pmod p$, $q \neq 1$. We will describe the p -local structure of the spaces $B\mathbf{X}_i(q)$ and will show that they are classifying spaces of p -local finite groups. In particular, cases $i = 29, 34$ provide new exotic examples of p -local finite groups.

The first results on the p -local structure of $B\mathbf{X}_i(q)$ are given by Propositions 7.5 and 7.6. Set $\ell = v_p(1 - q)$. The maximal elementary abelian p -subgroup of \mathbf{X}_i , $(t_{\mathbf{X}_i}, \nu)$, factors as a p -subgroup $(t_{\mathbf{X}_i}, g)$ of $\mathbf{X}_i(q)$, and the centralizer of this group

$$C_{\mathbf{X}_i(q)}(t_{\mathbf{X}_i}, g) \simeq T_\ell^{p-1} \cong (\mathbb{Z}/p^\ell)^{p-1}$$

is the maximal finite torus of $\mathbf{X}_i(q)$. All elementary abelian p -subgroups of $\mathbf{X}_i(q)$ factor through this one. Moreover, the Weyl group is $W_{\mathbf{X}_i(q)}(T_\ell^{p-1}) = G_i$, and the normalizer $N_{\mathbf{X}_i(q)}(T_\ell^{p-1}) = T_\ell^{p-1} \rtimes G_i$ sits in the maximal torus normalizer of $\mathbf{X}_i(q)$,

making homotopy commutative the diagram

$$\begin{array}{ccc} BN_{\mathbf{X}_i(q)}(T_\ell^{p-1}) & \longrightarrow & BN_{\mathbf{X}_i}(T^{p-1}) \\ \downarrow & & \downarrow \\ B\mathbf{X}_i(q) & \longrightarrow & B\mathbf{X}_i. \end{array}$$

Now, we fix the Sylow p -subgroup $S = (\mathbb{Z}/p^\ell)^{(p-1)} \rtimes \mathbb{Z}/p$ of $N_{\mathbf{X}_i(q)}(T_\ell^{p-1})$, generated by T_ℓ^{p-1} and a p -cycle of $\Sigma_p \leq G_i$. We will denote by $f: BS \rightarrow B\mathbf{X}_i(q)$ the homotopy monomorphism obtained as the composition $BS \rightarrow BN_{\mathbf{X}_i(q)}(T_\ell^{p-1}) \rightarrow B\mathbf{X}_i(q)$. Then (S, f) is a p -subgroup of $B\mathbf{X}_i(q)$, and it will play the role of a Sylow p -subgroup.

Since $\mathbf{X}_i, i = 12, 29, 31, 34$, are polynomial p -compact groups, according to Corollary 7.8, $\iota: B\mathbf{X}_i(q) \rightarrow B\mathbf{X}_i$ induces an equivalence of categories

$$\iota_\# : \mathcal{F}_p^e(B\mathbf{X}_i(q)) \longrightarrow \mathcal{F}_p^e(B\mathbf{X}_i).$$

Thus, we obtain that every elementary abelian p -subgroup (E, μ) of $B\mathbf{X}_i(q)$ factors as a subgroup of $t_{\mathbf{X}_i}: E \leq t_{\mathbf{X}_i}$ and $\mu \simeq v|_{BE}$. There is a distinguished subgroup $\mathbb{Z}/p \cong Z \leq t_{\mathbf{X}_i}$ such that, $Z \leq t_{\mathbf{X}_i} \leq T_{\mathbf{X}_i} \leq SU(p) \cong C_{\mathbf{X}_i}(Z, v|_{BZ})$. If $E \leq t_{\mathbf{X}_i}$ is not conjugate to Z in \mathbf{X}_i , then the centralizer $C_{\mathbf{X}_i}(E, v|_{BE})$ is a p -compact group whose Weyl group, the pointwise stabilizer of $E \leq T_{\mathbf{X}_i}$, $W_{\mathbf{X}_i}(E)$, has order not divisible by p . In $\mathbf{X}_i(q)$, we obtain:

Proposition 10.1 *There is one conjugacy class of elements of order p in $\mathbf{X}_i(q)$, $(Z, g|_{BZ})$, such that the centralizer is*

$$C_{\mathbf{X}_i(q)}(Z, g|_{BZ}) \simeq SU(p)(q)$$

and contains (S, f) :

$$\begin{array}{ccc} BS & & \\ \mathbf{B} \text{ incl} \downarrow & \searrow f & \\ BSU(p)(q) & \longrightarrow & B\mathbf{X}_i(q) \end{array}$$

as Sylow p -subgroup of $SU(p)(q)$.

If $E \leq t_{\mathbf{X}_i}$ represents another conjugacy class of elementary abelian p -subgroups, then

$$C_{\mathbf{X}_i(q)}(E, g|_{BE}) \simeq T_\ell^{p-1} \rtimes W_{\mathbf{X}_i}(E)$$

where the order of $W_{X_i}(E)$ is not divisible by p . Furthermore, the diagram

$$\begin{array}{ccc}
 BT_\ell^{(p-1)} & \xrightarrow{B \text{ incl}} & BS \\
 B \text{ incl} \downarrow & & \downarrow f \\
 BC_{X_i(q)}(E, g|_{BE}) & \xrightarrow{j} & BX_i(q)
 \end{array}$$

is commutative up to homotopy, where $j: BC_{X_i(q)}(E, g|_{BE}) \rightarrow BX_i(q)$ is the natural map induced by evaluation.

Proof For $Z \leq t_{X_i}$, we have $C_{X_i(q)}(Z, g|_{BZ}) \cong SU(p)(q)$ by Corollary 7.4.

If $E \leq t_{X_i}$ be another subgroup, not conjugate to Z , then the centralizer in X_i is the Clark–Ewing p -compact group $BC_{X_i}(E, \nu|_{BE}) \simeq B(T_{X_i} \rtimes W_{X_i}(E))_p^\wedge$, and then first, Corollary 7.4 implies that $BC_{X_i(q)}(E, g|_{BE}) \simeq BC_{X_i}(E, \nu|_{BE})(q)$, and secondly, Theorem 9.8 gives $BC_{X_i}(E, \nu|_{BE})(q) \simeq B(T_\ell^{p-1} \rtimes W_{X_i}(E))_p^\wedge$.

Finally, we use the inclusions $BE \rightarrow Bt_{X_i} \rightarrow BS \xrightarrow{f} BX_i(q)$ in order to compare the centralizers of E and t_{X_i} in S and $X_i(q)$:

$$\begin{array}{ccccc}
 BT_\ell^{p-1} \simeq BC_S(t_{X_i}) & \xrightarrow{\simeq} & BC_S(E) & \longrightarrow & BS \\
 \simeq \downarrow f_\# & & \downarrow f_\# & & \downarrow f \\
 BC_{X_i(q)}(t_{X_i}, g) & \longrightarrow & BC_{X_i(q)}(E, g|_{BE}) & \longrightarrow & BX_i(q).
 \end{array}$$

The proof follows. □

Proposition 10.2 For $i = 12, 29, 31, 34$, the natural map

$$(34) \quad \text{hocolim}_{\mathcal{F}_p^c(BX_i(q))^{\text{op}}} BC_{X_i(q)} \rightarrow BX_i(q)$$

is a mod p homology equivalence.

Proof According to Theorem F the cohomology rings of $BX_i(q)$ are:

$$\begin{aligned}
 H^*(BX_{12}(q); \mathbb{F}_3) &\cong P[x_{12}, x_{16}] \otimes E[y_{11}, y_{15}], \\
 H^*(BX_{29}(q); \mathbb{F}_5) &\cong P[x_8, x_{16}, x_{24}, x_{40}] \otimes E[y_7, y_{15}, y_{23}, y_{39}], \\
 H^*(BX_{31}(q); \mathbb{F}_5) &\cong P[x_{16}, x_{24}, x_{40}, x_{48}] \otimes E[y_{15}, y_{23}, y_{39}, y_{47}], \\
 H^*(BX_{34}(q); \mathbb{F}_7) &\cong P[x_{12}, x_{24}, x_{36}, x_{48}, x_{60}, x_{84}] \\
 &\quad \otimes E[y_{11}, y_{23}, y_{35}, y_{47}, y_{59}, y_{83}],
 \end{aligned}$$

and they embed in the invariant rings $H^*(B\mathbf{X}_i(q); \mathbb{F}_p) \subseteq H^*(BT_\ell^{p-1}; \mathbb{F}_p)^{G_i}$. These invariant rings are described in the [Example 9.7](#). It turns out that the above inclusion is an isomorphism if $i = 29, 31, 34$, but it is not surjective when $i = 12$.

The centralizers of elementary abelian p -subgroups of $B\mathbf{X}_i(q)$ are described in [Proposition 10.1](#). The centralizer, $C_{\mathbf{X}_i(q)}(E, g|_{BE})$, of an elementary abelian p -subgroup $E \leq t_{\mathbf{X}_i}$ in $\mathbf{X}_i(q)$ is either $SU(p)(q)$ or $C(q)$ where C is a Clark–Ewing p -compact group.

In cases $i = 29, 31, 34$, $H^*(C_{\mathbf{X}_i(q)}(E, g|_{BE}); \mathbb{F}_p) \cong H^*(BT_{\mathbf{X}_i}; \mathbb{F}_p)^{W(E)}$ is satisfied by [Theorem F](#) and [Examples 9.2](#) and [9.3](#), hence we meet the conditions of [Proposition 7.9](#) and the map (34) is a mod p homology equivalence.

In the case $i = 12$, [Proposition 7.9](#) does not apply, so we will need a separate analysis. The p -compact group \mathbf{X}_{12} , $p = 3$, is also denoted DI_2 , because $G_{12} \cong GL(2, 3)$ and $H^*(BDI_2; \mathbb{F}_3) \cong H^*(BT^2; \mathbb{F}_3)^{GL(2,3)} \cong \mathbb{F}_3[x_{12}, x_{16}]$ is the rank two Dickson algebra at $p = 3$. It admits two conjugacy classes of elementary abelian p -subgroups, one of rank one and another of rank two, hence so does $BDI_2(q)$, as well. We have equivalences of categories

$$\mathcal{F}_p^e(BDI_2) \cong \mathcal{F}_p^e(BDI_2(q)) \cong \mathbb{1}(1)$$

with $\text{Aut}_{\mathbb{1}(1)}(0) = GL(2, 3)$, $\text{Aut}_{\mathbb{1}(1)}(1) = N_{GL(2,3)}(\Sigma_3)/\Sigma_3 \cong \mathbb{Z}/2$, where the group $N_{GL(2,3)}(\Sigma_3) = \Sigma_3 \times \mathbb{Z}/2$, and $\text{Mor}_{\mathbb{1}(1)}(1, 0) = \Sigma_3 \setminus GL(2, 3)$ and $\text{Mor}_{\mathbb{1}(1)}(0, 1) = \emptyset$. The centralizers diagram $BC_{DI_2(q)}$ is described in the picture

$$(35) \quad \mathbb{Z}/2 \curvearrowright BSU(3)(q) \xleftarrow{\Sigma_3^{\text{op}} \setminus GL(2,3)^{\text{op}}} BT_\ell^2 \curvearrowright GL(2,3)^{\text{op}} .$$

The Bousfield–Kan spectral sequence

$$E_2^{i,j} \cong \varprojlim_{\mathbb{1}(1)}^i H^j(BC_{DI_2(q)}; \mathbb{F}_3) \implies H^{i+j}(\text{hocolim}_{\mathbb{1}(1)^{\text{op}}} BC_{DI_2(q)}; \mathbb{F}_3)$$

computes the cohomology of the homotopy colimit $\text{hocolim}_{\mathbb{1}(1)^{\text{op}}} BC_{DI_2(q)}$.

The computation of the E_2 -term follows from [Proposition B.1](#). Since $N_{GL(2,3)}(\Sigma_3) \cong \Sigma_3 \times \mathbb{Z}/2$ and $H^*(GL(2, 3); A) \cong H^*(N_{GL(2,3)}(\Sigma_3); A) \cong H^*(\Sigma_3; A)$, for any $GL(2, 3)$ -module A , there is an exact sequence

$$(36) \quad 0 \rightarrow \varprojlim_{\mathbb{1}(1)}^0 H^*(BC_{DI_2(q)}; \mathbb{F}_3) \rightarrow H^*(BSU(3)(q); \mathbb{F}_3)^{\mathbb{Z}/2} \oplus H^*(BT_\ell^2; \mathbb{F}_3)^{GL(2,3)} \\ \rightarrow H^*(BT_\ell^2; \mathbb{F}_3)^{\Sigma_3 \times \mathbb{Z}/2} \rightarrow \varprojlim_{\mathbb{1}(1)}^1 H^*(BC_{DI_2(q)}; \mathbb{F}_3) \rightarrow 0,$$

and $\varprojlim_{\mathbb{I}(1)}^i BC_{DI_2}(q) = 0$ if $i \geq 2$.

The invariant rings $H^*(BT_\ell^2; \mathbb{F}_3)^{GL(2,3)}$ and $H^*(BT_\ell^2; \mathbb{F}_3)^{\Sigma_3}$ as well as the restriction

$$R: H^*(BT_\ell^2; \mathbb{F}_3)^{GL(2,3)} \hookrightarrow H^*(BT_\ell^2; \mathbb{F}_3)^{\Sigma_3}$$

have been described in Examples 9.4 and 9.7. The cohomology of $BSU(3)(q)$ is identified with the subalgebra $P[x_4, x_6] \otimes E[y_3, y_5]$ of $H^*(BT_\ell^2; \mathbb{F}_3)^{\Sigma_3}$. The cokernel of the inclusion is isomorphic to $P[x_6]y_4$, and then the exact sequence (36) is simplified to

$$0 \rightarrow \varprojlim_{\mathbb{I}(1)}^0 H^*(BC_{DI_2}(q); \mathbb{F}_3) \rightarrow \frac{P[x_{12}, x_{16}] \otimes E[y_{10}, y_{11}, y_{15}]}{(y_{11}y_{15} - x_{16}y_{10}, y_{10}y_{11}, y_{10}y_{15})} \xrightarrow{\bar{R}} (P[x_6]y_4)^{\mathbb{Z}/2} \rightarrow \varprojlim_{\mathbb{I}(1)}^1 H^*(BC_{DI_2}(q); \mathbb{F}_3) \rightarrow 0,$$

and $(P[x_6]y_4)^{\mathbb{Z}/2} = P[x_6^2](x_6y_4)$ which is in the image of \bar{R} . It follows that

$$\varprojlim_{\mathbb{I}(1)}^0 H^*(BC_{DI_2}(q); \mathbb{F}_3) \cong P[x_{12}, x_{16}] \otimes E[y_{11}, y_{15}]$$

and $\varprojlim_{\mathbb{I}(1)}^i BC_{DI_2}(q) = 0$ if $i \geq 1$, so, therefore the Bousfield–Kan spectral sequence collapses to an isomorphism

$$H^*(\text{hocolim}_{\mathbb{I}(1)^{\text{op}}} BC_{DI_2}(q); \mathbb{F}_3) \cong \varprojlim_{\mathbb{I}(1)}^0 H^*(BC_{DI_2}(q); \mathbb{F}_3) \cong P[x_{12}, x_{16}] \otimes E[y_{11}, y_{15}];$$

that is, $\text{hocolim}_{\mathbb{I}(1)^{\text{op}}} BC_{DI_2}(q) \rightarrow BG_2(q)$ is a sharp homology decomposition at the prime 3 and

$$H^*(DI_2(q); \mathbb{F}_3) \cong \varprojlim_{\mathbb{I}(1)}^0 H^*(BC_{DI_2}(q); \mathbb{F}_3) \cong P[x_{12}, x_{16}] \otimes E[y_{11}, y_{15}]. \quad \square$$

Theorem 10.3 Fix $q \in \mathbb{Z}_p$, $1 \neq q \equiv 1 \pmod p$. Then, (S, f) is a Sylow p -subgroup for $B\mathbf{X}_i(q)$, the fusion system $\mathcal{F}_{(S, f)}(B\mathbf{X}_i(q))$ of the space $B\mathbf{X}_i(q)$ over the p -subgroup (S, f) is saturated, and

$$(S, \mathcal{F}_{(S, f)}(B\mathbf{X}_i(q)), \mathcal{L}_{(S, f)}(B\mathbf{X}_i(q)))$$

is a p -local finite group with classifying space

$$|\mathcal{L}_{(S, f)}(B\mathbf{X}_i(q))|_p^\wedge \simeq B\mathbf{X}_i(q).$$

Proof It is a consequence of Theorem 4.5, using the Propositions 10.1 and 10.2. \square

Now, we will go deeper into the structure of the fusion system $\mathcal{F} = \mathcal{F}_{(S, \mathcal{F})}(B\mathbf{X}_i(q))$. According to Remark 6.6 we may assume that q is an integer and a prime power. We have seen that the fusion category of elementary abelian p -subgroups is equivalent to that of the p -compact group \mathbf{X}_i ; in particular, every elementary abelian p -subgroup is toral; that is, \mathcal{F} -conjugate to a subgroup of $T_\ell^{(p-1)}$. If we denote $Z = Z(S)$ the center of S , then 10.1 $BC_{\mathbf{X}_i(q)}(Z) = BSU(p)(q)_p^\wedge \simeq BSL_p(q)_p^\wedge$, so, the centralizer fusion system $C_{\mathcal{F}}(Z)$ over $C_S(Z) = S$ coincides with the fusion system of $SL_p(q)$ over S . Hence, we can identify S with the Sylow p -subgroup of $SL_p(q)$ and then use the notation of Example 3.5. Recall from Example 3.5 that any centric radical subgroup of S in $C_{\mathcal{F}}(Z)$ is conjugate to either S , $T_\ell^{(p-1)}$, or an extraspecial group $\Gamma_1(\xi^r)$, $r = 0, \dots, p - 1$.

Proposition 10.4 Any centric radical subgroup of S in $\mathcal{F} = \mathcal{F}_{(S, \mathcal{F})}(B\mathbf{X}_i(q))$ is conjugate to one of the groups in the table:

Q	$\text{Out}_{\mathcal{F}}(Q)$	Conditions
$T_\ell^{(p-1)}$	G_i	
S	$\mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1)$	
Γ_1	$GL_2(p)$	
$\Gamma_1(\xi)$	$SL_2(p)$	if $\ell > 1$ or $p > 3$.

Proof The proof is divided in four steps, where we first determine a set of representatives for centric radical subgroups of S in \mathcal{F} , and then refine it to a minimal set of representatives and compute their automorphisms groups in \mathcal{F} .

Step 1 (Toral and nontoral centric radical subgroups) T_ℓ^{p-1} is centric in \mathcal{F} and $\text{Out}_{\mathcal{F}}(T_\ell^{p-1}) \cong G_i$ is p -reduced, hence T_ℓ^{p-1} is also radical in \mathcal{F} . No other subgroup of T_ℓ^{p-1} is centric, so for any other centric and radical subgroup $Q \leq S$ in \mathcal{F} , there is a morphism of extensions

$$(38) \quad \begin{array}{ccccc} Q_0 & \longrightarrow & Q & \longrightarrow & \mathbb{Z}/p \\ \downarrow & & \downarrow & & \parallel \\ T_\ell^{p-1} & \longrightarrow & S & \longrightarrow & \mathbb{Z}/p \end{array}$$

where $Q_0 = T_\ell^{p-1} \cap Q$.

We are assuming that Q is centric, hence the center $Z \cong \mathbb{Z}/p$ of S should be contained in Q_0 . But if $Q_0 = Z$, then $Q \cong \mathbb{Z}/p \times \mathbb{Z}/p$ is elementary abelian and then toral in \mathcal{F} , hence it would not be centric. Thus $Z \neq Q_0$ and the center of Q is $Z(Q) = Q_0^{\mathbb{Z}/p} = Z$.

In particular, every automorphism of Q restricts to an automorphism of Z , so we obtain a homomorphism $\text{Aut}_{\mathcal{F}}(Q) \rightarrow \text{Aut}_{\mathcal{F}}(Z)$. The kernel is composed of automorphisms of Q that restrict to the identity in Z ; that is, automorphisms of Q in the centralizer fusion system $C_{\mathcal{F}}(Z)$, hence we have an exact sequence

$$(39) \quad 1 \rightarrow \text{Aut}_{C_{\mathcal{F}}(Z)}(Q) \rightarrow \text{Aut}_{\mathcal{F}}(Q) \rightarrow \text{Aut}_{\mathcal{F}}(Z)$$

where $\text{Aut}_{\mathcal{F}}(Z) \leq \mathbb{Z}/p-1$ lifts to $\text{Aut}_{\mathcal{F}}(T_{\ell}^{(p-1)})$ and $\text{Aut}_{\mathcal{F}}(S)$ as unstable Adams operations (the center of G_i). Thus, if Q is radical in $C_{\mathcal{F}}(Z)$, then it is radical in \mathcal{F} .

Step 2 (Nonabelian centric radical subgroups, all of which abelian characteristic subgroups are cyclic) Assume that all abelian characteristic subgroups of Q are cyclic, then a theorem of Hall implies that Q is the central product of an extraspecial group Γ of exponent p and a cyclic group C , where the elements of order p in C , $\Omega_1(C)$, coincide with the center $Z(\Gamma)$ of Γ (cf Gorenstein [35, Chapter 5, 4.9, 5.3]).

The faithful irreducible representations of the central product of an extraspecial group Γ of order p^{1+2r} and a cyclic group of order p^{ℓ} over the algebraic closure of a field of q elements, $(q, p) = 1$, have degree p^r , and there are exactly $p^{\ell-1}(p-1)$ inequivalent representations in this degree.

Hence, only the case $r = 1$ can appear in $GL_p(q)$. We denote Γ_1 the extraspecial group of order p^3 and exponent p , and Γ_k the central product $\mathbb{Z}/p^k \circ \Gamma_1$. The different irreducible faithful representations of Γ_k in $GL_p(q)$ are obtained by composing with the extension to Γ_k of the automorphisms of \mathbb{Z}/p^k , $(\mathbb{Z}/p^k)^*$. Thus, there is at most one subgroup isomorphic to Γ_k in $GL_p(q)$, up to conjugation. A subgroup of $GL_p(q)$ isomorphic to Γ_1 is described in Example 3.4. Since $C_{GL_p(q)}(\Gamma_1) = Z(GL_p(q)) \cong GL_1(q)$, Γ_k is a subgroup of $GL_p(q)$ if and only if $\mathbb{Z}/p^k < GL_1(q)$. Hence Γ_{ℓ} , $\ell = v_p(1-q)$, is the biggest one that can occur in $GL_p(q)$ (see Example 3.4).

Finally, the intersection of Γ_{ℓ} with $SL_p(q)$, and hence, of any conjugate of Γ_{ℓ} , is isomorphic to Γ_1 , and there are exactly p conjugacy classes of such subgroups $\Gamma_1(\xi^r)$ (see Example 3.5). These are radical in $C_{\mathcal{F}}(Z)$, and so, therefore, they are also radical in \mathcal{F} .

Step 3 (Nonabelian centric radical subgroups having noncyclic abelian characteristic subgroups) Assume now that Q contains a noncyclic abelian characteristic group. If Q is radical in $C_{\mathcal{F}}(Z)$, then it is radical in \mathcal{F} . Now, we assume also that Q is not radical in $C_{\mathcal{F}}(Z)$.

We can view $Q \leq S$ as subgroups of $SL_p(q)$ and $GL_p(q)$, for an appropriate prime power q such that S is the Sylow p -subgroup of $SL_p(q)$: $\ell = v_p(1-q)$. Write

$N = N_{GL_p(q)}(Q)$. The arguments of [4, (4A)] show that (up to conjugacy in $GL_p(q)$)

$$Q \leq N \cap (\mathbb{Z}/p^k \wr \mathbb{Z}/p) \triangleleft N$$

for some $k \leq \ell$, or, taking the intersection with $SL_p(q)$

$$Q \leq \bar{N} \cap S_k \triangleleft \bar{N}$$

where $S_k = (\mathbb{Z}/p^k \wr \mathbb{Z}/p) \cap SL_p(q) \leq S$ and $\bar{N} = N \cap SL_p(q) = N_{SL_p(q)}(Q)$. Then

$$\text{Inn } Q \leq (\bar{N} \cap S_k)/Z(Q) \triangleleft \text{Aut}_{C_{\mathcal{F}}(Z)}(Q)$$

where $\bar{N}/C_{SL_p(q)}(Q) = \text{Aut}_{C_{\mathcal{F}}(Z)}(Q)$. We will see that $(\bar{N} \cap S_k)/Z(Q)$ is still normal in $\text{Aut}_{\mathcal{F}}(Q)$.

Assume that $\varphi \in \text{Aut}_{\mathcal{F}}(Q)$ restricts to Z as the unstable Adams operation ψ^ξ , ξ a $(p-1)$ -st root of unity. If $\psi^{1/\xi}(Q) = Q' \leq S$, then $\psi^{1/\xi} \circ \varphi: Q \rightarrow Q'$ is a morphism of \mathcal{F} , that restricted to Z is trivial, hence a morphism of $C_{\mathcal{F}}(Z)$. Since, we have assumed that Q is not radical in $C_{\mathcal{F}}(Z)$, $\psi^{1/\xi} \circ \varphi$ should be obtained as composition of restrictions of automorphisms of centric radical subgroups of $C_{\mathcal{F}}(Z)$, by Alperin fusion theorem [14, A.10]. This is the fusion system of $SL_p(q)$, and the Sylow p -subgroup S itself is the only centric radical that contains Q , hence, there is $\chi \in \text{Aut}_{C_{\mathcal{F}}(Z)}(S)$ with $\chi|_Q = \psi^{1/\xi} \circ \varphi$, hence $\varphi = \psi^\xi \circ \chi|_Q$ extends to an automorphism $\psi^\xi \circ \chi$ of $\text{Aut}_{\mathcal{F}}(S)$. Notice that $\psi^\xi(S_k) = S_k$ and also $\chi(S_k) = S_k$, hence, if $g \in S_k$ normalizes Q , we have $\varphi \circ c_g \circ \varphi^{-1} = c_{\varphi(g)}$, with $\varphi(g) \in \bar{N} \cap S_k$. This proves that we have

$$\text{Inn } Q \leq (\bar{N} \cap S_k)/Z(Q) \triangleleft \text{Aut}_{\mathcal{F}}(Q)$$

and since Q is radical in \mathcal{F} , $Q = S_k$.

We claim that only the case $S_k = S$ is radical. First we compute the normalizer of $\mathbb{Z}/p^k \wr \mathbb{Z}/p$ in $GL_p(q)$. The subgroup $(\mathbb{Z}/p^k)^p$ is a characteristic subgroup of $\mathbb{Z}/p^k \wr \mathbb{Z}/p$, for it is the only abelian subgroup of index p , hence we have $N_{GL_p(q)}(\mathbb{Z}/p^k \wr \mathbb{Z}/p) \leq N_{GL_p(q)}((\mathbb{Z}/p^k)^p)$. It is not difficult to compute that $N_{GL_p(q)}((\mathbb{Z}/p^k)^p) = GL_1(q) \wr \Sigma_p$, the group of invertible matrices with only one nontrivial entry in each line and column. By direct computation one can obtain that $N_{GL_p(q)}(\mathbb{Z}/p^k \wr \mathbb{Z}/p) = GL_1(q) \cdot (\mathbb{Z}/p^k \wr N_{\Sigma_p}(\mathbb{Z}/p))$, where $GL_1(q)$ is identified with the subgroup of all diagonal matrices of $GL_p(q)$; that is, the center of $GL_p(q)$.

Call $N_k = N_{GL_p(q)}(\mathbb{Z}/p^k \wr \mathbb{Z}/p) \cap SL_p(q)$. We have $N_k \cong B_k \rtimes N_{\Sigma_p}(\mathbb{Z}/p)$, with

$$B_k = \{ (z \cdot x_1, \dots, z \cdot x_p) \in GL_1(q)^p \mid x_i \in \mathbb{Z}/p^k, z^p x_1 \dots x_p = 1 \}$$

and $N_{SL_p(q)}(S_k) = N_k$. Notice that, when $k < \ell$, S_k has index p in the Sylow p -subgroup $B_k \rtimes \mathbb{Z}/p$, and this is normal in N_k , hence only $S = S_\ell$ is radical in $SL_p(q)$.

The centralizer of S_k in $SL_p(q)$ is $C_{SL_p(q)}(S_k) = Z \cong \mathbb{Z}/p$. So $\text{Aut}_{C_{\mathcal{F}}(Z)}(S_k) \cong \text{Aut}_{SL_p(q)}(S_k) \cong N_k/Z$. $(B_k/Z) \rtimes \mathbb{Z}/p$ is normal in N_k/Z , and, since the Adams operations ψ^ζ , ζ a $(p-1)$ -st root of unity, act internally in B_k , $(B_k/Z) \rtimes \mathbb{Z}/p$ is also normal in $\text{Aut}_{\mathcal{F}}(S_k)$

$$\text{Inn } S_k = S_k/Z/p \triangleleft (B_k/Z/p) \rtimes \mathbb{Z}/p \triangleleft \text{Aut}_{\mathcal{F}}(S_k)$$

thus, S_k is radical in the fusion system \mathcal{F} if and only if $k = \ell$; that is, only the case $S_k = S$ is radical. In this case we have obtained $\text{Aut}_{\mathcal{F}}(S) \cong N_\ell/Z \rtimes \mathbb{Z}/(p-1)$, where $\mathbb{Z}/(p-1)$ on the right is generated by the Adams operations of exponent a primitive $(p-1)$ -st root of unity, and $\text{Out}_{\mathcal{F}}(S) \cong \mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1)$, given by the Adams operations and $N_{\Sigma_p}(\mathbb{Z}/p)/\mathbb{Z}/p$.

Step 4 (Minimal set of representatives and automorphism groups) It remains to check which of those are \mathcal{F} -conjugate to one of the others in the list and also to compute their \mathcal{F} -automorphisms.

For $Q = S$ the restriction $\text{Aut}_{\mathcal{F}}(Q) \rightarrow \text{Aut}_{\mathcal{F}}(Z)$ is split because unstable Adams operations extend to S . Moreover, since they are realized by the center of G_i , the \mathcal{F} -automorphisms of S are given by conjugation in the normalizer $N_{\ell,i}$ of the maximal finite torus $T_\ell^{(p-1)}$. We have seen already that the same is true for $Q = T_\ell^{(p-1)}$.

Finally, we analyze the case $Q = \Gamma_1(\xi^r)$, $r = 0, \dots, p-1$. Assume that $\varphi \in \text{Aut}_{\mathcal{F}}(Q)$ and that the restriction to the center Z is the unstable Adams operation ψ^z . This extends to an \mathcal{F} -automorphism of S . Write $Q' = \psi^z(Q)$. Then $\chi = \psi^z \circ \varphi^{(-1)}: Q \rightarrow Q'$ is a homomorphism of \mathcal{F} that restricts to the identity in Z , hence it belongs to the centralizer fusion system $C_{\mathcal{F}}(Z)$. In other words, every automorphism $\varphi \in \text{Aut}_{\mathcal{F}}(Q)$ is the composite of an isomorphism $\chi: Q \rightarrow Q'$ of $C_{\mathcal{F}}(Z)$ and a unstable Adams operation ψ^z .

It is then enough to compute the effect of unstable Adams operations on the family of subgroups $\Gamma_1(\xi^r)$. It turns out that unstable Adams operations restrict to automorphisms of $\Gamma_1 = \Gamma_1(\xi^0)$ so that $\text{Out}_{\mathcal{F}}(\Gamma_1) = GL_p(q)$, while, for $p > 3$ or $\ell > 1$, they conjugate $\Gamma_1(\xi^r)$ for $r = 1, \dots, p-1$ to each other and $\text{Out}_{\mathcal{F}}(\Gamma_1(\xi)) = SL_p(q)$. □

Corollary 10.5 *The fusion system of $BX_i(q)$ is*

$$\mathcal{F}_{(S,f)}(BX_i(q)) = \langle \mathcal{F}_{N_{\ell,i}}(S); \mathcal{F}_{\Gamma_1}(GL_2(p)), \mathcal{F}_{\Gamma_1(\xi)}(SL_2(p)) \rangle,$$

for $p > 3$ or $\ell > 1$, and $\mathcal{F}_{(S,f)}(B\mathbf{X}_{12}(q)) = \langle \mathcal{F}_{N_{1,i}}(S); \mathcal{F}_{\Gamma_1}(GL_2(p)) \rangle$, for $p = 3$ and $\ell = 1$, where $N_{\ell,i} = N_{\mathbf{X}_i(q)}(T_\ell^{(p-1)}) \cong T_\ell^{(p-1)} \rtimes G_i$.

Proof It is a consequence of Proposition 10.4 and Alperin’s fusion theorem for saturated fusion systems (see Section 3). \square

We end this section with a case by case study in order to determine which spaces $B\mathbf{X}_i(q)$ are p -completed classifying spaces of finite groups and which cases correspond to exotic examples of p -local finite groups.

As we shall see, S contains no proper strongly closed subgroups in $\mathcal{F} = \mathcal{F}_{(S,f)}(B\mathbf{X}_i(q))$ and so, according to [14, 9.2], if \mathcal{F} is the p -completed classifying space of a finite group, this group is almost simple.

In fact, a strongly closed subgroup of S in \mathcal{F} is a normal subgroup P of S such that no element of P is \mathcal{F} -conjugate to any element in $S \setminus P$. Now, if P is nontrivial it contains at least an element of order p , and this is \mathcal{F} -conjugate to an element of order p in $T_\ell^{(p-1)}$. Now, the maximal elementary abelian p -subgroup t of $T_\ell^{(p-1)}$ turns out to be an irreducible G_i -module, hence $t \subset P$ and since the cycle of order p generating $S/T_\ell^{(p-1)}$ is conjugate to an element of t , the extension of t by this cycle is in P . Thus we have a diagram of extensions

$$\begin{array}{ccccc} P_T & \longrightarrow & P & \longrightarrow & \mathbb{Z}/p \\ \downarrow & & \downarrow & & \parallel \\ T_\ell^{(p-1)} & \longrightarrow & S & \longrightarrow & \mathbb{Z}/p \end{array}$$

where $t \leq P_T = P \cap T$. Now $S/P \cong T_\ell^{(p-1)}/P_T$ is abelian. The abelianization of S is seen to be $\mathbb{Z}/p \times \mathbb{Z}/p$, and then we obtain that $T_\ell^{(p-1)}/P_T$ is either trivial or has order p . It follows that all elements of order up to $p^{\ell-1}$ of $T_\ell^{(p-1)}$ belong to P_T . Taking the quotient by this subgroup we obtain an inclusion of G_i -modules

$$\overline{P_T} \leq \overline{T_\ell^{(p-1)}},$$

but again, this last is an irreducible G_i -module, hence

$$\overline{P_T} = \overline{T_\ell^{(p-1)}},$$

and then $P = S$.

Example 10.6 $B\mathbf{X}_{29}(q)$ at $p = 5$ and $B\mathbf{X}_{34}(q)$ at $p = 7$ are classifying spaces of exotic p -local finite groups. We have seen that the Sylow subgroup does not

contain any proper strongly closed subgroup in $\mathcal{F}_{(S,f)}(B\mathbf{X}_i(q))$, hence if this is the p -completed classifying space of a finite group G , then G is almost simple [14, Lemma 9.2]. A complete list of almost simple groups with a Sylow subgroup isomorphic to S is provided by [14, Proposition 9.5]. No group in the list contains G_{29} or G_{34} as automorphisms of $T_\ell^{(p-1)}$ induced by conjugation in the group. Hence $\mathbf{X}_{29}(q)$ at $p = 5$ and $\mathbf{X}_{34}(q)$ at $p = 7$ are exotic.

Example 10.7 $BX_{12}(q)$ at $p = 3$ is the 3-completed classifying space of a twisted Chevalley group of type F_4 . More precisely, $BX_{12}(q) = B(^2F_4(2^{3^{\ell-1}}))_3^\wedge$ where $\ell = v_3(q^2 - 1)$.

The 3-completed classifying space of the twisted Chevalley group ${}^2F_4(2^{2n+1})$ can be described at $p = 3$ as $B(^2F_4(2^{2n+1})) \simeq BF_4^\alpha$, for $\alpha = \varphi \circ \psi^{2^n}$, where φ is the Friedlander's exceptional isogeny of F_4 [32]. φ has the effect of reflecting the Dynkin diagram of F_4 by sending the short roots to the long roots and the long roots to 2 times short roots. Furthermore, $\varphi^2 \simeq \psi^2$, and then we can choose ζ a square root of -2 in \mathbb{Z}_3 such that $\beta = \varphi \circ \psi^{1/\zeta}$ is a self equivalence of BF_4 at $p = 3$ of order two and $2^n \zeta \equiv 1 \pmod{3}$. We can write $\alpha = \beta \circ \psi^{2^n \zeta}$, and then by Proposition 6.2, $BF_4^\alpha \simeq (BF_4)^{h\beta}(2^n \zeta)$. In [16] it is shown that $(BF_4)^{h\beta} \simeq BX_{12}$, hence $BX_{12}(2^n \zeta) \simeq B(^2F_4(2^{2n+1}))_3^\wedge$. Since ψ^{-1} belongs to the Weyl group of X_{12} , $BX_{12}(q) \simeq BX_{12}(-q)$, and then according to Theorem E, the homotopy type of $BX_{12}(\pm q)$ does only depend on $\ell = v_3(q^2 - 1)$, thus, if we choose n with $\ell = v_3(q^2 - 1) = v_3(1 - 2^n \zeta) = v_3(1 + 2^{2n+1})$, then we have

$$BX_{12}(q) \simeq BX_{12}(2^n \zeta) \simeq B(^2F_4(2^{2n+1}))_3^\wedge.$$

In particular, $BX_{12}(q) \simeq B(^2F_4(2^{3^{\ell-1}}))_3^\wedge$. The local structure of ${}^2F_4(2^{2n+1})$, also called Ree groups of characteristic two, was studied by Malle [43].

Example 10.8 For any 5-adic unit, $q \in \mathbb{Z}_5^*$, $BX_{31}(q)$ at $p = 5$ is the 5-completed classifying space of a Chevalley group of type E_8 , namely, $BX_{31}(q) \simeq BE_8(2^{2m+1})_5^\wedge$ if $v_5(q^4 - 1) = v_5(1 + 2^{4m+2})$.

Let $i = \sqrt{-1}$ be a primitive 4th root of unity. Since ψ^i belongs to the Weyl group of X_{31} , we can assume that $q \equiv 1 \pmod{5}$ for otherwise we can multiply q by an appropriate power of i and still have $BX_{31}(q) \simeq BX_{31}(i^r q)$. Moreover, according to Theorem E, the homotopy type of $BX_{31}(q)$ will only depend on $\ell = v_5(q^4 - 1)$.

We fix a prime power q_0 with $q_0 \equiv \pm 2 \pmod{5}$ and $\ell = v_5(\pm i q_0 - 1) = v_5(q_0^4 - 1) = v_5(q_0^2 + 1)$, where we choose $+i$ or $-i$ in order that the equality makes sense.

We can write $q_0 = i \cdot (-i \cdot q_0)$, where now $-i \cdot q_0 \equiv \pm 1 \pmod{p}$. Since ψ^{-1} belongs to the Weyl group of E_8 , we can apply [Proposition 6.2](#) and get $BE_8(q_0) \simeq (BE_8)^{h\psi^i}(-iq_0)$. Now we have seen in [Section A.12\(2\)](#), that $(BE_8)^{h\psi^i} \simeq BX_{31}$, so, therefore

$$BE_8(q_0) \simeq BX_{31}(-iq_0) \simeq BX_{31}(q_0),$$

and this last is homotopy equivalent to $BX_{31}(q)$ by our choice of q_0 with $v_5(q_0^4 - 1) = v_5(q^4 - 1)$.

Similar considerations can be made, more generally, at any prime p such that $p \equiv 1 \pmod{4}$; that is, any prime at which X_{31} can be defined, and then obtain that $BE_8(q_0) \simeq BX_{31}(q_0)$ for a prime power q_0 with $q_0^2 + 1 \equiv 0 \pmod{p}$.

The local structure of $E_8(q)$ was described in [\[41\]](#).

Remark 10.9 One can easily obtain natural maps $B\mathbf{X}_i(q^{p^n}) \rightarrow B\mathbf{X}_i(q^{p^{n+1}})$, that at the level of maximal finite tori induce inclusions $T_{\ell+n}^{(p-1)} \leq T_{\ell+n+1}^{(p-1)}$, and then obtain that the p -compact group \mathbf{X}_i can be reconstructed by means of a telescope construction

$$B\mathbf{X}_i \simeq \operatorname{hocolim}_n B\mathbf{X}_i(q^{p^n}).$$

In particular, we may obtain the p -compact groups $B\mathbf{X}_{12}$ (at $p = 3$) and $B\mathbf{X}_{31}$ (at $p = 5$) as telescopes

$$B\mathbf{X}_{12} = \operatorname{hocolim} B\mathbf{X}_{12}(4^{3^n}) = \operatorname{hocolim} B(^2F_4(2^{3^n}))$$

$$B\mathbf{X}_{31} = \operatorname{hocolim} B\mathbf{X}_{31}(16^{5^n}) = \operatorname{hocolim} BE_8(2^{5^n})$$

of p -completed classifying spaces of finite Chevalley groups.

11 Chevalley p -local finite groups from generalized Grassmannians

We discuss here the Chevalley p -local finite groups of type $\mathbf{X}(m, r, n)$. Let p be an odd prime, $m \geq 1$, $r \geq 1$, and $n \geq 1$ with $r|m|(p-1)$. The simply connected polynomial irreducible p -compact group $\mathbf{X}(m, r, n)$ has Weyl group $G(m, r, n)$ (see [Section 2](#)) and its cohomology is the invariant ring

$$H^*(B\mathbf{X}(m, r, n); \mathbb{F}_p) = H^*(BT(\mathbf{X}(m, r, n)); \mathbb{F}_p)^{G(m, r, n)} \cong P[x_1, \dots, x_{n-1}, e]$$

with $\deg(x_i) = 2mi$ and $\deg(e) = \frac{2mn}{r}$. See Quillen [\[61\]](#), Notbohm [\[58\]](#) and Møller [\[53\]](#) for the construction of these spaces. We are here interested in the associated spaces $B\mathbf{X}(m, r, n)(q)$ defined by the pullback diagram [\(22\)](#) with $\alpha = \psi^q$ where q is a p -adic unit.

Remark 11.1 Many cases already appear in the literature; see Fiedorowicz and Priddy [31], Friedlander [34] and Quillen [61]. We can extract the following equivalences, up to p -completion, for a prime power q , prime to p :

- (1) $BSU(n + 1)(q) \simeq BSL_{n+1}(q)$.
- (2) $BU(n)(q) \simeq BX(1, 1, n)(q) \simeq BGL_n(q)$.
- (3) $BX(m, 1, n)(q) \simeq BGL_{mn}(q)$.
- (4) $BX(2, 2, n)(q) \simeq BSO(2n)(q) \simeq BSO_{2n}^+(q)$.

By Remark 6.6, we have that, also for any p -adic unit q , the spaces $BSU(n + 1)(q)$, $BX(m, 1, n)(q)$ and $BX(2, 2, n)(q)$ are homotopy equivalent to classifying spaces of finite groups, up to p -completion.

These also include the cases $BX(m, 2, n)(q)$, that can be reduced to $BX(2, 2, n)(q')$ using Propositions A.10 and 6.2, so they are also equivalent, up to p -completion, to classifying spaces of orthogonal groups over finite fields.

The above observations will be used as the starting point of the induction arguments that we will develop in the rest of this section in order to study the structure of $BX(m, r, n)(q)$, for $q \equiv 1 \pmod p$, $q \neq 1$, and general values of m , r , and n .

Fix $q \equiv 1 \pmod p$, $q \neq 1$. The p -compact groups $X(m, r, n)$ are polynomial, hence Proposition 7.5 and Proposition 7.6 apply. The maximal elementary abelian p -subgroup of $X(m, r, n)$, (t_X, v) , factors as a p -subgroup, (t_X, g) , of $X(m, r, n)(q)$, and the maximal finite torus of $X(m, r, n)(q)$ is

$$BT_\ell^n \simeq BC_{X(m,r,n)(q)}(t_X, g)$$

where $\ell = v_p(q - 1)$. The Weyl group is $W_{X(m,r,n)(q)}(T_\ell^n) \cong G(m, r, n)$, and the extension $N_{X(m,r,n)(q)}(T_\ell^n) \cong T_\ell^n \rtimes G(m, r, n)$ sits in the maximal torus normalizer of $X(m, r, n)$, making the following diagram homotopy commutative:

$$\begin{CD} BN_{X(m,r,n)(q)}(T_\ell^n) @>>> BN_{X(m,r,n)}(T^n) \\ @VVV @VVV \\ BX(m, r, n)(q) @>\iota>> BX(m, r, n). \end{CD}$$

Corollary 7.7 implies that the functor

$$(40) \quad \iota_\# : \mathcal{F}_p^e(X(m, r, n)(q)) \rightarrow \mathcal{F}_p^e(X(m, r, n))$$

is an equivalence of categories. The next result is a description of the centralizers of elementary abelian p -subgroups.

Proposition 11.2 [53, 7.11] *Let p be an odd prime, $m \geq 1$, $r \geq 1$, $n \geq 1$ with $r|m|(p-1)$, and $q \equiv 1 \pmod{p}$, $q \neq 1$. Then*

- (1) *any elementary abelian p -subgroup $h: BE \rightarrow \mathbf{BX}(m, r, n)(q)$, factors through the maximal finite torus, and*
- (2) *for any subgroup $E \leq t_x \leq T_\ell^n$, the centralizer of $(E, g|_{BE})$ in $\mathbf{X}(m, r, n)(q)$,*

$$BC_{\mathbf{X}(m, r, n)(q)}(E, g|_{BE}) \simeq \mathbf{BX}(m, r, n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q),$$

$$n = n_0 + n_1 + \cdots + n_s, \text{ is determined by the pointwise stabilizer of } E \leq T_\ell^n \text{ in the Weyl group } G(m, r, n), G(m, r, n)(E) \cong G(m, r, n_0) \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}.$$

Proof All elementary abelian p -subgroups of $\mathbf{X}(m, r, n)$ are toral, hence the same is true for $\mathbf{X}(m, r, n)(q)$ by the equivalence (40). If $E \leq t_X$, by Corollary 7.4, the restriction of ψ^q to the centralizer of $(E, g|_{BE})$, is ψ^q again, $\psi^q|_{C_{\mathbf{X}(m, r, n)(q)}(E, g|_{BE})} = \psi^q$, and

$$BC_{\mathbf{X}(m, r, n)(q)}(E, g|_{BE}) \simeq BC_{\mathbf{X}(m, r, n)}(E, v|_{BE})(q).$$

The centralizers $C_{\mathbf{X}(m, r, n)}(E, v|_{BE})$ are known to be connected p -compact groups of maximal rank, with Weyl group $G(m, r, n_0) \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$, the pointwise stabilizer of E in T^n by the action of the Weyl group $G(m, r, n)$:

$$BC_{\mathbf{X}(m, r, n)}(E, v|_{BE}) \simeq \mathbf{BX}(m, r, n_0) \times BU(n_1) \times \cdots \times BU(n_s),$$

thus, $BC_{\mathbf{X}(m, r, n)(q)}(E, v|_{BE})(q) \simeq \mathbf{BX}(m, r, n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)$

contains the same maximal finite torus T_ℓ^n as $\mathbf{X}(m, r, n)(q)$, $\ell = v_p(q-1)$, $n = n_0 + n_1 + \cdots + n_s$ and the Weyl group is $G(m, r, n_0) \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$ (see Proposition 7.5 and Proposition 7.6). \square

Proposition 11.3 *Let p be an odd prime, $m \geq 1$, $r \geq 1$, $n \geq 1$ with $r|m|(p-1)$, and $q \equiv 1 \pmod{p}$, $q \neq 1$. The natural map*

$$\text{hocolim}_{\mathcal{F}_p^e(\mathbf{X}(m, r, n)(q))^{\text{op}}} BC_{\mathbf{X}(m, r, n)(q)} \longrightarrow \mathbf{BX}(m, r, n)(q)$$

is a mod p homology equivalence.

Proof According to Theorem F and Example 9.5,

$$\begin{aligned} H^*(\mathbf{BX}(m, r, n)(q); \mathbb{F}_p) &\cong H^*(BT_\ell^n; \mathbb{F}_p)^{G(m, r, n)} \\ &\cong P[x_1, \dots, x_{n-1}, e] \otimes E[y_1, \dots, y_{n-1}, u] \end{aligned}$$

with $\deg(x_i) = 2mi$, $\deg(e) = \frac{2mn}{r}$, $\deg(y_i) = 2mi - 1$, and $\deg(u) = \frac{2mn}{r} - 1$.

Since this is true for all values of m, r, n , we obtain from [Proposition 11.2](#) that also, for every elementary abelian p -subgroup $E \leq t_X$,

$$H^*(BC_{\mathbf{X}(m,r,n)(q)}(E, g|_{BE}); \mathbb{F}_p) \cong H^*(BT_\ell^n; \mathbb{F}_p)^{G(m,r,n)(E)}$$

where $G(m, r, n)(E)$ is the pointwise stabilizer of E in T_ℓ^n , by the action of the Weyl group $G(m, r, n)$. So, then the result follows from [Proposition 7.9](#). \square

Fix a Sylow p -subgroup of $N_{\mathbf{X}(m,r,n)(q)}(T_\ell^n)$, $S_{n,\ell} \cong \mathbb{Z}/p^\ell \wr S_n$, where S_n is the Sylow p -subgroup of the symmetric group Σ_n . Call f the composition $BS_{n,\ell} \rightarrow BN_{\mathbf{X}(m,r,n)(q)}(T_\ell^n) \rightarrow B\mathbf{X}(m, r, n)(q)$. Thus the group $(S_{n,\ell}, f)$ is a p -subgroup of $B\mathbf{X}(m, r, n)(q)$.

We will denote by

$$\mathcal{F}(m, r, n, q) = \mathcal{F}_{(S_{n,\ell}, f)}(B\mathbf{X}(m, r, n)(q))$$

the fusion system of $B\mathbf{X}(m, r, n)(q)$ over $(S_{n,\ell}, f)$ and by

$$\mathcal{L}(m, r, n, q) = \mathcal{L}_{(S_{n,\ell}, f)}(B\mathbf{X}(m, r, n)(q)),$$

the associated centric linking system. Recall that the underlying category of the system $\mathcal{F}(m, r, n, q)$ is equivalent to $\mathcal{F}_p(B\mathbf{X}(m, n, r)(q))$.

Theorem 11.4 *If q is a p -adic unit such that $q \equiv 1 \pmod p$, $q \neq 1$, and $\ell = v_p(1 - q)$, then $(S_{n,\ell}, f)$ is a Sylow p -subgroup for $B\mathbf{X}(m, r, n)(q)$ and*

$$(S_{n,\ell}, \mathcal{F}(m, r, n, q), \mathcal{L}(m, r, n, q))$$

is a p -local finite group with classifying space

$$|\mathcal{L}(m, r, n, q)|_p^\wedge \simeq B\mathbf{X}(m, r, n)(q).$$

Proof We proceed by induction on n , the p -rank of $\mathbf{X}(m, r, n)(q)$. For $n < p$, $\mathbf{X}(m, r, n)$ is a Clark–Ewing p -compact group, and then $\mathbf{X}(m, r, n)(q)$ is the p -completed classifying space of a finite group (see [Theorem 9.8](#)). Also, for $BX(1, 1, n) \simeq BU(n)_p^\wedge$, [Remark 11.1](#) characterizes $BX(1, 1, n)(q)$ as p -completed classifying spaces of finite groups. In all that cases, the conclusion of the theorem is clearly satisfied (see [Section 3](#)).

Assume that n is large and that the theorem holds for every $n_0 < n$. That is, for every $n_0 < n$, the space $BX(m, r, n_0)(q)$ is the classifying space of the p -local finite group $(S_{n_0,\ell}, \mathcal{F}(m, r, n_0, q), \mathcal{L}(m, r, n_0, q))$. The result about $B\mathbf{X}(m, r, n)(q)$ will follow from [Theorem 4.5](#). We will show that the space $B\mathbf{X}(m, r, n)(q)$ and its p -subgroup

$(S_{n,\ell}, f)$ meet the conditions of [Theorem 4.5](#). Condition (1) of [Theorem 4.5](#) is satisfied by [Proposition 7.1](#).

Condition (2a) of [Theorem 4.5](#) amounts to show that if $E \leq t_X$, then the centralizer $BC_{\mathbf{X}(m,r,n)(q)}(E, g|_{BE})$ is the classifying space of a p -local finite group. This follows by the induction hypothesis. In fact, by [Proposition 11.2](#), there is a homotopy equivalence $BC_{\mathbf{X}(m,r,n)(q)}(E, g|_{BE}) \simeq B\mathbf{X}(m, r, n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)$, for $n = n_0 + n_1 + \cdots + n_s$, a nontrivial decomposition of n into positive summands, and by the induction hypothesis and [\[14, 1.4\]](#) this is the classifying space of the p -local finite group defined as the product

$$(S_{n_0,\ell}, \mathcal{F}(m, r, n_0, q), \mathcal{L}(m, r, n_0, q)) \times (S_{n_1,\ell}, \mathcal{F}(1, 1, n_1, q), \mathcal{L}(1, 1, n_1, q)) \times \cdots \times (S_{n_s,\ell}, \mathcal{F}(1, 1, n_s, q), \mathcal{L}(1, 1, n_s, q)).$$

Condition (2b) of [Theorem 4.5](#) establishes that Sylow p -subgroups of centralizers of elementary abelian subgroups of $B\mathbf{X}(m, r, n)(q)$ factor through $(S_{n,\ell}, f)$. This is proved by reducing the question to unitary groups, obtained as centralizers of the center of $S_{n,\ell}$.

Let $Z \cong \mathbb{Z}/p$ denote the diagonal elements of order p in $T_\ell^n \cong (\mathbb{Z}/p^\ell)^n \leq S_{n,\ell}$. Then, the pointwise stabilizer of Z in T_ℓ^n by the action of $G(m, r, n)$ is Σ_n and therefore, according to [Proposition 11.2](#), $BC_{B\mathbf{X}(m,r,n)(q)}(Z, g|_{BZ}) \simeq BU(n)(q)$.

By naturality of the construction of the normalizer of the maximal finite torus, we obtain a diagram

$$\begin{array}{ccc} BN_{U(n)(q)}(T_\ell^n) & \longrightarrow & BN_{\mathbf{X}(m,r,n)(q)}(T_\ell^n) \\ \downarrow & & \downarrow \\ BU(n)(q) & \xrightarrow{Bj_n} & B\mathbf{X}(m, r, n)(q) \end{array}$$

hence a factorization of $(S_{n,\ell}, f)$:

$$(41) \quad \begin{array}{ccc} & BS_{n,\ell} & \\ f' \swarrow & & \searrow f \\ BU(n)(q) & \xrightarrow{Bj_n} & B\mathbf{X}(m, r, n)(q) \end{array}$$

Choose any other subgroup $E \leq t_X \leq S_{n,\ell}$. Assume that the pointwise stabilizer of E in T_ℓ^n by the action of $G(m, r, n)$ is $G(m, r, n)(E) \cong G(m, r, n_0) \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$. Define $E' = Z \cdot E \leq t_X$, then the pointwise stabilizer of E' will be $G(m, r, n)(E') \cong$

$\Sigma_{n_0} \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$. The inclusions $E \leq E' \geq Z$ induce a commutative diagram of centralizers

$$(42) \quad \begin{array}{ccc} BC_{\mathbf{X}(m,r,n)(q)}(E', g|_{BE'}) & \xrightarrow{Bj_n^\#} & BC_{\mathbf{X}(m,r,n)(q)}(E, g|_{BE}) \\ \downarrow & & \downarrow \\ BC_{\mathbf{X}(m,r,n)(q)}(Z, g|_{BZ}) & \xrightarrow{Bj_n} & B\mathbf{X}(m, r, n)(q). \end{array}$$

Now,

$$BC_{\mathbf{X}(m,r,n)(q)}(E, g|_{BE}) \simeq BX(m, r, n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)$$

with Sylow p -subgroup $S_{n_0,\ell} \times \cdots \times S_{n_s,\ell}$ while

$$BC_{\mathbf{X}(m,r,n)(q)}(E', g|_{BE'}) \simeq BU(n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q),$$

and from the above discussion we have a factorization

$$(43) \quad \begin{array}{ccc} B(S_{n_0,\ell} \times \cdots \times S_{n_s,\ell}) & \longrightarrow & BU(n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q) \\ & \searrow & \downarrow B j_n^\# \simeq B j_{n_0} \times 1 \times \cdots \times 1 \\ & & BX(m, r, n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q). \end{array}$$

Diagrams (41), (42), and (43) provide a homotopy commutative diagram

$$\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ B(S_{n_0,\ell} \times \cdots \times S_{n_s,\ell}) & \longrightarrow & BC_{\mathbf{X}(m,r,n)(q)}(E', g|_{BE'}) & \xrightarrow{Bj_n^\#} & BC_{\mathbf{X}(m,r,n)(q)}(E, g|_{BE}) \\ \downarrow B\rho & & \downarrow & & \downarrow \\ BS_{n,\ell} & \longrightarrow & BC_{\mathbf{X}(m,r,n)(q)}(Z, g|_{BZ}) & \xrightarrow{Bj_n} & B\mathbf{X}(m, r, n)(q) \\ & & & & \end{array}$$

where the existence of the homomorphism $\rho: S_{n_0,\ell} \times \cdots \times S_{n_s,\ell} \rightarrow S_{n,\ell}$ making homotopy commutative the left square is obtained because $S_{n,\ell}$ is a Sylow p -subgroup of $U(n)(q)$.

We have proved that $B\mathbf{X}(m, r, n)(q)$ and $(S_{n,\ell}, f)$ satisfy the conditions (1) and (2) of Theorem 4.5, and therefore, that $(S_{n,\ell}, f)$ is a Sylow p -subgroup of $B\mathbf{X}(m, r, n)(q)$ and $(S_{n,\ell}, \mathcal{F}(m, r, n, q), \mathcal{L}(m, r, n, q))$ is a p -local finite group.

Finally, $B\mathbf{X}(m, r, n)(q)$ is the classifying space $|\mathcal{L}(m, r, n, q)|_p^\wedge$ according to Proposition 11.3 and Theorem 4.5. □

Proposition 11.5 For $q \equiv 1 \pmod{p}$, $q \neq 1$, $\mathbf{X}(m, r, n)(q)$ is a exotic p -local finite group if $r > 2$, $n \geq p$.

Notice that in the above hypothesis $r|(p-1)$, thus $r > 2$ can only occur with $p \geq 5$, so that we are implicitly assuming also that $p \geq 5$.

Proof We will first reduce the question to the rank p -case. Then we classify the centric radical subgroups in the fusion system of $B\mathbf{X}(m, r, p)(q)$ and show that they coincide with the p -local finite groups of [14, Example 9.4].

There is an elementary abelian p -subgroup $E \leq t_X$, in $\mathbf{X}(m, r, n)(q)$, of rank $n - p$ such that

$$C_{\mathbf{X}(m, r, n)(q)}(E, g|_{BE}) \cong \mathbf{X}(m, r, p)(q) \times U(1)_p^\wedge(q)^{n-p}$$

(see Proposition 11.2). If we assume that there is a finite group G such that $BG_p^\wedge \simeq B\mathbf{X}(m, r, n)(q)$, then the map $Bg|_{BE}: BE \rightarrow B\mathbf{X}(m, r, n)(q) \simeq BG_p^\wedge$ is induced by a homomorphism $\varphi: E \rightarrow G$, and

$$BC_G(\varphi(E))_p^\wedge \simeq B\mathbf{X}(m, r, p)(q) \times BU(1)_p^\wedge(q)^{n-p}.$$

Since $BU(1)_p^\wedge(q) \simeq B\mathbb{Z}/p^\ell$, the projection $BC_G(\varphi(E))_p^\wedge \rightarrow BU(1)_p^\wedge(q)^{n-p}$ is the p -completion of the map induced by a homomorphism $\rho: C_G(\varphi(E)) \rightarrow (\mathbb{Z}/p^\ell)^{n-p}$. It has a section, also induced by a homomorphism $\sigma: (\mathbb{Z}/p^\ell)^{n-p} \rightarrow C_G(\varphi(E))$, hence ρ is an epimorphism. Therefore, we have a short exact sequence $\text{Ker } \rho \rightarrow C_G(\varphi(E)) \rightarrow (\mathbb{Z}/p^\ell)^{n-p}$ and an induced fibration $B(\text{Ker } \rho)_p^\wedge \rightarrow BC_G(\varphi(E))_p^\wedge \rightarrow B(\mathbb{Z}/p^\ell)^{n-p}$, from which we obtain an equivalence $B(\text{Ker } \rho)_p^\wedge \simeq B\mathbf{X}(m, r, p)(q)$. This reduces the question to showing that $\mathbf{X}(m, r, p)(q)$ is an exotic p -local finite group.

We will show now that $\mathbf{X}(m, r, p)(q)$ coincide with the p -local finite groups constructed in [14, Example 9.4] in purely algebraic terms. For this aim we will need to describe the centric and radical p -subgroups of $\mathbf{X}(m, r, p)(q)$.

Recall that $T_\ell^p \cong (\mathbb{Z}/p^\ell)^p$ is the maximal finite torus of $\mathbf{X}(m, r, p)(q)$ with Weyl group $G(m, r, p)$ and they form a split extension

$$T_\ell^p \rightarrow N_{\mathbf{X}(m, r, p)(q)}(T_\ell^p) \rightarrow G(m, r, p)$$

that contains $S_{p, \ell} = T_\ell^p \rtimes \mathbb{Z}/p \leq N_{\mathbf{X}(m, r, p)(q)}(T_\ell^p)$, which is a Sylow p -subgroup of $\mathbf{X}(m, r, p)(q)$. For simplicity we will denote $\mathcal{F} = \mathcal{F}(m, r, p, q)$, the fusion system of $B\mathbf{X}(m, r, p)(q)$ over $(S_{p, \ell}, f)$.

The center of the Sylow p -subgroup is $Z(S_{p, \ell}) \cong \mathbb{Z}/p^\ell$ embedded diagonally in T_ℓ^p , and, if we write $Z(t_X)$ for the elements of order p in $Z(S_{p, \ell})$, then we obtain $BC_{\mathbf{X}(m, r, p)(q)}(Z(S_{p, \ell})) \simeq BC_{\mathbf{X}(m, r, p)(q)}(Z(t_X)) \simeq BU(p)_p^\wedge(q)$ (see Proposition 11.2).

We also know (see Remark 11.1) that $BU(p)\hat{^p}_p(q) \simeq BGL_p(q_0)\hat{^p}_p$ for a prime power q_0 with $\ell = v_p(1 - q) = v_p(1 - q_0)$, hence we conclude that the centralizer fusion system $C_{\mathcal{F}}(Z(S_{p,\ell}))$ coincides with the fusion system of $GL_p(q_0)$, that has been described in Example 3.4.

The Sylow p -subgroup $S_{p,\ell}$ is clearly centric and radical. T_ℓ^P is centric and we have $\text{Out}_{\mathcal{F}}(T_\ell^P) = G(m, r, p)$ hence it is also radical ($p \geq 5$). Proper subgroups of T_ℓ^P are not centric, so we will look at subgroups $Q \leq S_{p,\ell}$ not contained in T_ℓ^P . such a subgroup fits in an extension

$$\begin{array}{ccccc} Q_0 & \longrightarrow & Q & \longrightarrow & \mathbb{Z}/p \\ \downarrow & & \downarrow & & \parallel \\ T_\ell^P & \longrightarrow & S_{p,\ell} & \longrightarrow & \mathbb{Z}/p \end{array}$$

where $Q_0 = Q \cap T_\ell^n$, and since Q is centric, $Z(S_{p,\ell}) \leq Q_0$. It turns out that this is actually a characteristic subgroup of Q , Hence there is an exact sequence of groups:

$$1 \rightarrow \text{Aut}_{C_{\mathcal{F}}(Z(S_{p,\ell}))}(Q) \rightarrow \text{Aut}_{\mathcal{F}}(Q) \rightarrow \text{Aut}_{\mathcal{F}}(Z(S_{p,\ell}))$$

where $\text{Aut}_{\mathcal{F}}(Z(S_{p,\ell})) \cong \mathbb{Z}/(m/r)$ is given by the action of the Adams operations of exponents a (m/r) -th root of unity.

Assume that Q is abelian. Then $Q_0 = Z(S_{p,\ell})$ and Q is either $\mathbb{Z}/p \times Z(S_{p,\ell})$ or cyclic $\mathbb{Z}/p^{\ell+1}$. In the first case, Q is \mathcal{F} -conjugate to a subgroup of T_ℓ^n , hence it is not centric while in the second case, it is conjugate to the group $U_{\ell+1}$ described in Example 3.4. Adams operations do not act internally in $U_{\ell+1}$, hence $\text{Out}_{\mathcal{F}}(U_{\ell+1}) \cong \text{Aut}_{C_{\mathcal{F}}(Z(S_{p,\ell}))}(U_{\ell+1}) \cong \mathbb{Z}/p$ and then $U_{\ell+1}$ is not radical in \mathcal{F} .

Assume that Q is nonabelian. The same arguments as in Corollary 10.5 show that Q is either $S_{p,\ell}$ or Γ_ℓ , and both are radical in $C_{\mathcal{F}}(Z(S_{p,\ell}))$. Thus we obtain that they complete the list of conjugacy classes of centric radical subgroups of $S_{p,\ell}$ in \mathcal{F} .

In order to complete the picture it remains to compute the \mathcal{F} -automorphisms of Γ_ℓ . We have $\text{Aut}_{C_{\mathcal{F}}(Z(S_{p,\ell}))}(\Gamma_\ell) \cong SL_2(p)$. Now, the Adams operations act internally in Γ_ℓ and we get $\text{Out}_{\mathcal{F}}(\Gamma_\ell) \cong SL_2(p).(m/r)$.

By Alperin’s fusion theorem, a fusion system over S is generated by the automorphisms of its fully normalized centric radical subgroups in S . Since in our case all the automorphisms of T_ℓ^P are induced by conjugation in $N_{\mathbf{X}(m,r,p)(q)}(T_\ell^P)$, we can write

$$\mathcal{F}(m, r, p, q) = \langle \mathcal{F}_{N_{\mathbf{X}(m,r,p)(q)}(T_\ell^P)}(S_{p,\ell}); \mathcal{F}_{\Gamma_\ell}(SL_2(p).(m/r)) \rangle$$

(see Section 3) but this is precisely the definition of the fusion systems in [14, Example 9.4]. □

The cases $B\mathbf{X}(m, r, n)(q)$ with $r = 1, 2$ or $n < p$, are homotopy equivalent to p -completed classifying spaces of finite groups according to [Theorem 9.8](#) and [Remark 11.1](#).

Appendix A Recognition of homotopy fixed point p -compact groups

The objective of this appendix is to obtain a recognition principle for the homotopy fixed point p -compact group BX^{hG} where p is an odd prime, X a connected p -compact group, G a finite group of order prime to p , and $\rho: G \rightarrow \text{Out}(X)$ and outer action.

Let $N \rightarrow X$ be the maximal torus normalizer for the p -compact group X . The short exact sequence of topological monoids

$$BZ(N) = \text{aut}(BN)_1 \rightarrow \text{aut}(BN) \rightarrow \text{Out}(N)$$

induces a fibration sequence

$$B^2Z(N) \rightarrow \text{Baut}(BN) \rightarrow \text{BOut}(N)$$

which shows that equivalence classes of fibrations over BG with fiber BN is in one-to-one correspondence with

$$[BG, \text{BOut}(N)] = \text{Hom}(G, \text{Out}(N)).$$

Also, we know from [Theorem B](#) that equivalence classes of fibrations over BG with fiber BX is in one-to-one correspondence with

$$[BG, \text{BOut}(X)] = \text{Hom}(G, \text{Out}(X)).$$

However, $\text{Out}(X) \cong \text{Out}(N)$ [[53](#); [7](#)] and therefore there is a bijective correspondence between fibrations with fiber BX over BG and fibrations with fiber BN over BG . We shall now make this correspondence more explicit.

Define the group-like topological monoid $\text{aut}(Bj)$ to be the submonoid of $\text{aut}(BN) \times \text{aut}(BX)$ consisting of all pairs $(a, b) \in \text{aut}(BN) \times \text{aut}(BX)$ such that the following diagram commutes.

$$\begin{array}{ccc} BN & \xrightarrow{a} & BN \\ Bj \downarrow & & \downarrow Bj \\ BX & \xrightarrow{b} & BX \end{array}$$

Lemma A.1 Assume that p is odd. The forgetful homomorphisms

$$\text{aut}(BN) \longleftarrow \text{aut}(Bj) \longrightarrow \text{aut}(BX)$$

are homotopy equivalences.

Proof The group homomorphisms $\pi_0 \text{aut}(BN) \leftarrow \pi_0 \text{aut}(Bj) \rightarrow \pi_0 \text{aut}(BX)$ are injective because X has N -determined automorphisms [53; 7]. The group homomorphism to the left is surjective because X is N -determined and the one to the right is surjective because any self-homotopy equivalence of BX lifts to a self-homotopy equivalence of BN [51, Section 3]. The identity components fit into a map of fibrations [26, 11.10]

$$\begin{array}{ccccc} \text{aut}_{BX}(Bj)_1 & \longrightarrow & \text{aut}(Bj)_1 & \longrightarrow & \text{aut}(BX)_1 \\ \parallel & & \downarrow & & \downarrow \simeq \\ \text{aut}_{BX}(Bj)_1 & \longrightarrow & \text{aut}(BN)_1 & \longrightarrow & \text{Map}(BN, BX)_{Bj} \end{array}$$

where the right vertical map, defined by composition with Bj , is a homotopy equivalence [26, 7.5, 1.3; 25, 9.1; 51, 3.4]. The fiber, consisting of the space of maps $BN \rightarrow BN$ over BX and vertically homotopic to the identity map of BN , is (one component) of the space $(X/N)^{hN}$ which is contractible [50, 5.1]. \square

Thus we have bijections

$$[B, \text{Baut}(BN)] = [B, \text{Baut}(Bj)] = [B, \text{Baut}(BX)]$$

for any space B and this means BN -fibrations and BX -fibrations over B are in bijective correspondence.

Proposition A.2 Let X be a connected p -compact group with maximal torus normalizer $N \rightarrow X$. If G is a finite group of order prime to p , then any outer action $\rho: G \rightarrow \text{Out}(X)$, lifts to a unique G -action on BX and unique G -action on BN . Moreover, these actions make the map $BN \rightarrow BX$ G -equivariant; that is, the diagram

$$\begin{array}{ccccc} BN & \longrightarrow & BN_{hG} & \longrightarrow & BG \\ \downarrow & & \downarrow & & \parallel \\ BX & \longrightarrow & BX_{hG} & \longrightarrow & BG \end{array}$$

is homotopy commutative.

Proof Let us say that our input is an outer action

$$(44) \quad \rho: G \rightarrow \text{Out}(X) = W \backslash N_{GL(L)}(W) = \text{Out}(N)$$

of the finite group G on X and N . By [Theorem B](#), ρ lifts to a unique action of G on BX , and by [Lemma A.1](#) the same is true for BN . In particular, ρ determines a unique map, up to homotopy,

$$\tilde{B\rho}: BG \longrightarrow \text{Baut}(Bj)$$

inducing ρ on fundamental groups.

Over $\text{Baut}(Bj)$ there are two related fibrations

$$\begin{array}{ccc} BN_{h\text{aut}(Bj)} & \xrightarrow{\quad} & BX_{h\text{aut}(Bj)} \\ & \searrow & \swarrow \\ & \text{Baut}(Bj) & \end{array}$$

with fiber BN and BX , respectively. Pull back these two related fibrations along the map $\tilde{B\rho}$ to obtain the commutative diagram of the proposition. \square

Next, we need to lift the action of G on BN and BX to an action on the loop spaces N and X (see [Definition 5.3](#)), such that the inclusion $N \rightarrow X$ is still equivariant.

[Lemma 5.1](#) applies to show that the fibration $BX \rightarrow BX_{hG} \rightarrow BG$ admits a section, unique up to vertical homotopy, when X is connected; that is, there is a unique lifting of the action on BX to an action on X . However, BN is not simply connected as $\pi_1(BN) \cong W$ and then [Lemma 5.1](#) ensures neither the existence nor the uniqueness of a lifting of the action of G on BN to an action of G on N . Instead, it leads to the next description of the possible actions.

Proposition A.3 *If a finite group G of order prime to p acts on BN with outer action $\rho: G \rightarrow W \backslash N_{GL(L)}(W) \cong \text{Out}(N)$, then there are natural one-to-one correspondences between the sets:*

- (1) $\pi_0(BN^{hG})$,
- (2) W -conjugacy classes of lifts in the diagram

$$\begin{array}{ccc} & N_{GL(L)}(W) & \\ & \nearrow & \downarrow \\ G & \xrightarrow{\rho} & W \backslash N_{GL(L)}(W). \end{array}$$

If these sets are nonempty, then they are also in one-to-one correspondence with $H^1(G; W)$.

Proof An action of G on BN is by definition a fibration

$$(45) \quad BN \rightarrow BN_{hG} \rightarrow BG,$$

and according to Proposition A.2 this action of G on BN is uniquely determined by ρ .

Next, we map $\pi_0(BN^{hG})$ directly to the set (2). Let $\varphi: BG \rightarrow B \operatorname{aut}(BN)$ be a classifying map for the fibration (45). Thus, φ extends to a map of fibrations

$$\begin{array}{ccccc} BN & \longrightarrow & BN_{hG} & \longrightarrow & BG \\ \parallel & & \downarrow & & \downarrow B\rho \\ BN & \longrightarrow & B \operatorname{aut}_*(BN) & \longrightarrow & B \operatorname{aut}(BN) \end{array}$$

into the universal BN -fibration. Here, $\operatorname{aut}_*(BN)$ is the topological monoid of based self-homotopy equivalences of BN . On the level of fundamental groups we get an induced morphism

$$(46) \quad \begin{array}{ccccc} W & \longrightarrow & \pi_0(N_{hG}) & \longrightarrow & G \\ \parallel & & \downarrow & & \downarrow \rho \\ W & \longrightarrow & N_{GL(L)}(W) & \longrightarrow & W \backslash N_{GL(L)}(W) \end{array}$$

of group extensions. Here we use the short exact sequence from [49, 5.2] in combination with the vanishing results from [5, 3.3].

We have seen (Lemma 5.1) that the existence of an action of G on N lifting the action on BN is equivalent to the existence of a section of the exact sequence on the top row of (46), and the diagram shows that this is equivalent to the existence of a lifting of ρ to a homomorphism $\sigma: G \rightarrow N_{GL(L)}(W)$. This gives the bijection between $\pi_0(BN^{hG})$ and the set (2).

Finally, if these sets are nonempty, then obstruction theory as in Lemma 5.1 shows that they are in one-to-one correspondence with the set $H^1(G; W) = H^1(G; \pi_1(BN))$. \square

Proposition A.4 *Let X be a connected p -compact group with Weyl group W and maximal torus normalizer $N \rightarrow X$. If G is a finite group of order prime to p and*

$$\rho: G \rightarrow \operatorname{Out}(X) \cong W \backslash N_{GL(L)}(W)$$

is an outer action, then ρ lifts to a unique action of G on X , and each lift

$$\sigma: G \rightarrow N_{GL(L)}(W)$$

determines a unique action of G on N such that the inclusion $N \rightarrow X$ is G -equivariant.

Proof The first part was proved in Proposition A.2. According to Proposition A.3, the actions of G on N that lift the given outer action are in one-to-one correspondence with lifts of ρ to $N_{GL(L)}(W)$. If we view one of these actions as a sectioned fibration

$$BN \longrightarrow BN_{hG} \overset{\longleftarrow}{\underset{\longrightarrow}{\rightleftarrows}} BG$$

it clearly induces an action on X that makes $N \rightarrow X$ equivariant:

$$\begin{array}{ccc} BN_{hG} & \xrightarrow{\quad} & BX_{hG} \\ & \swarrow \text{dotted} & \searrow \text{dotted} \\ & BG & \end{array}$$

The proposition follows because there is only one action of G on X inducing ρ . \square

Proposition A.5 Let p be an odd prime and G a finite group of order prime to p . Assume that G acts on the connected p -compact group X and that

$$\bar{\rho}: G \rightarrow N_{GL(L)}(W)$$

is a lift of the given outer action. Suppose Y is a connected p -compact group that satisfies the following conditions:

- (1) $W^{\bar{\rho}G}$ contains a subgroup \bar{W} , complementary to the kernel of the map $W^{\bar{\rho}G} \rightarrow GL(L^{\bar{\rho}G})$, such that $(\bar{W}, L(X)^{\bar{\rho}G})$ is a \mathbb{Z}_p -reflection group isomorphic to $(W(Y), L(Y))$.
- (2) $QH^*(BY; \mathbb{Q}_p) \cong QH^*(BX; \mathbb{Q}_p)_G$.

Then $BY = BX^{hG}$.

Proof By the classification theorem for p -compact groups at odd primes [53; 7], it suffices [52, 1.2] to find a map $BN(Y) \rightarrow BX^{hG}$ that induces an isomorphism on $H^*(-; \mathbb{Q}_p)$ and restricts to monomorphism on the p -normalizer $N_p(Y)$, is a p -monomorphism. The homomorphism $\bar{\rho}$ corresponds (Proposition A.4) to compatible

G -actions $BG \rightarrow BN(X)_{hG} \rightarrow BX_{hG}$ on $N(X)$ and X . Taking homotopy fixed points we obtain a commutative diagram of loop space morphisms

$$\begin{array}{ccc} N(X)^{hG} & \longrightarrow & X^{hG} \\ \downarrow & & \downarrow \\ N(X) & \longrightarrow & X \end{array}$$

which shows that $N(X)^{hG} \rightarrow X^{hG}$ is a p -monomorphism. Since the discrete approximation to $N(X)$, $N(X)^{hG}$, and $N(Y)$ are semidirect products [5], there is a p -monomorphism $N(Y) \rightarrow N(X)^{hG}$ for $W(Y)$ is a subgroup $W^{\bar{\rho}G} = \pi_0 N(X)^{hG}$ by the first condition. By the second condition, $H^*(BY; \mathbb{Q}_p) = H^*(BN(Y); \mathbb{Q}_p)$ and $H^*(BX^{hG}; \mathbb{Q}_p)$ are abstractly isomorphic graded vector spaces. Therefore, Y and X^{hG} have the same rank [25, 5.9] so that $T(Y) \rightarrow N(X)^{hG} \rightarrow X^{hG}$ is a maximal torus and $H^*(BX^{hG}; \mathbb{Q}_p) \rightarrow H^*(BN(Y); \mathbb{Q}_p)$ is injective [25, 9.7], hence bijective. \square

A special case arises when G acts through unstable Adams operations so that the action $\pi_0\rho: G \rightarrow \text{Out}(N) \rightarrow \text{Out}(W)$ is trivial. Then the image of G in $\text{Out}(N) = W \backslash N_{GL(L)}(W)$ is contained in the subgroup $Z(W) \backslash C_{GL(L)}(W)$ [53, 3.16] and we have a morphism

$$\begin{array}{ccccc} W & \longrightarrow & \pi_0(N_{hG}) & \longrightarrow & G \\ \parallel & & \downarrow & & \downarrow B\rho \\ W & \longrightarrow & W.C_{GL(L)}(W) & \longrightarrow & Z(W) \backslash C_{GL(L)}(W) \end{array}$$

of group extensions. The possible extensions occurring in the upper line, realizing the trivial action $G \rightarrow \text{Out}(W)$, are classified by $H^2(G; Z(W))$; they are all isomorphic to

$$W \rightarrow Z(W) \backslash (D \times W) \rightarrow G$$

for some central extension $Z(W) \rightarrow D \rightarrow G$ [42, IV.Section 8]. If $Z(W)$ is trivial, $\pi_0(N_{hG}) = G \times W$ and $H^1(G; W) = \text{Rep}(G, W)$.

Assume that $G = C_r$ is a cyclic group of order r , and the outer action of G on X , $\rho: C_r \rightarrow \text{Out}(X)$, is given by an Adams operation $\rho(\lambda) = \psi^\lambda$, where $\lambda \in \mathbb{Z}_p^\times$ is a p -adic unit of order $r|(p-1)$. We can lift $\psi^\lambda \in Z(W) \backslash C_{GL(L)}(W)$ to an element $\zeta \in C_{GL(L)}(W)$, such that $\zeta^r \in Z(W)$. If there is a choice of ζ with $\zeta^r = 1$, then $\bar{\rho}\lambda = \zeta$ provides a lifting of ρ .

Assume, otherwise, that ζ^r has order s in $Z(W)$. Since p is odd, $Z(W)$ has order prime to p , hence s is prime to p . Now, even if there is no lift of the action of C_r

on X to an action on N , we can reduce the problem by extending the action of C_r to an action of C_{sr} on X determined by $\rho'(\lambda) = \psi^\lambda \in Z(W) \setminus C_{GL(L)}(W) \subset \text{Out}(X)$, that now admits the lift $\bar{\rho}'(\lambda) = \zeta$. Notice that $C_s = \langle \lambda^r \rangle$ acts trivially on X , so that $BX^{hC_s} \simeq BX$, and then $BX^{hC_{sr}} \cong BX^{hC_r}$, so we can still determine BX^{hC_r} by analyzing the equivariant action of C_{sr} on N and X .

Notice also, that if W is irreducible, then $C_{GL(L)}(W)$ consists of diagonal matrices and therefore ζ is an Adams operation.

Corollary A.6 *Let $\lambda \in \mathbb{Z}_p^\times$ be a p -adic unit of order $r|(p-1)$. Consider the outer action $\rho: C_r = \langle \lambda \rangle \rightarrow W \setminus N_{GL(L)}(W)$ through unstable Adams operations given by $\rho(\lambda) = \psi^\lambda$. Then, if ρ admits a lift $\bar{\rho}: C_r \rightarrow N_{GL(L)}(W)$, then all possible lifts are parameterized by $H^1(C_r; W) = \text{Rep}(C_r, W)$, the set of conjugacy classes of order r elements w of W , and*

$$(W^{\bar{\rho}C_r}, L^{\bar{\rho}C_r}) = (C_W(w), L^{\langle \lambda w \rangle})$$

for the lift $\bar{\rho}(\lambda) = \lambda w$ corresponding to w .

Proof The lifts

$$\begin{array}{ccc} & & W.C_{GL(L)}(W) \\ & \nearrow \bar{\rho} & \downarrow \\ C_r = \langle \lambda \rangle & \xrightarrow{\rho} & Z(W) \setminus C_{GL(L)}(W) \end{array}$$

are given by $\bar{\rho}(\lambda) = w\psi^\lambda$ where $w \in W$ is any element of order r . □

We next apply the recognition principle [Corollary A.6](#) in some concrete cases.

A.7 The three infinite families

We identify the fixed point p -compact groups for the actions of finite cyclic groups of order prime to p through unstable Adams operations on the p -compact groups of the three infinite families of irreducible p -compact groups, namely the projective or special unitary groups, the generalized Grassmannians, and the Sullivan spheres (as defined in [Section 2](#)).

Proposition A.8 (Sullivan spheres) *Let p be an odd prime. Suppose that m and $r > 1$ divide $p-1$. Consider the outer action through unstable Adams operations*

$\psi: C_r \rightarrow \text{Out}(S^{2m-1}) \cong \mathbb{Z}_p^\times / C_m$ of the cyclic group $C_r \leq \mathbb{Z}_p^\times$ on the Sullivan sphere S^{2m-1} . Then the homotopy fixed point group is

$$(S^{2m-1})^{hC_r} = \begin{cases} S^{2m-1} & r \mid m \\ * & \text{otherwise,} \end{cases}$$

Proof Let λ be a primitive r -th root of unity, so that $C_r = \langle \lambda \rangle \leq \mathbb{Z}_p^*$. According to [Theorem B](#), $(S^{2m-1})^{h\langle \lambda \rangle}$ is a connected polynomial p -compact group. If r does not divide m , $H^{2m}(\psi^\lambda) = \lambda^m$ is nontrivial, so that the vector space of covariants $QH^*(BS^{2m-1}; \mathbb{Q}_p)_{\langle \lambda \rangle}$ vanishes in positive degrees, and the fixed point p -compact group is trivial. If r does divide m , ψ^λ acts trivially on S^{2m-1} , because the kernel of ψ is C_m which contains C_r , and the fixed point p -compact group is again S^{2m-1} . \square

Proposition A.9 (Special unitary groups) *Let p be an odd prime. Suppose that $m > 1$ divides $p - 1$, and let $C_m = \langle \lambda \rangle \subset \mathbb{Z}_p^\times$ be the cyclic group generated by a primitive m -th root of unity acting through unstable Adams operations. Then*

$$X(mn + s)^{hC_m} = U(mn + s)^{hC_m} = \begin{cases} X(m, 1, n) & n > 0 \\ * & n = 0 \end{cases}$$

for any p -compact group $X(mn + s)$ locally isomorphic to $SU(mn + s)$, $0 \leq s < m$.

Proof In the rational cohomology algebras

$$H^*(BU(mn + s); \mathbb{Q}_p) = \mathbb{Q}_p[c_1, \dots, c_{mn+s}]$$

and $H^*(BX(mn + s); \mathbb{Q}_p) = \mathbb{Q}_p[c_2, \dots, c_{mn+s}]$

we have c_i is preserved by $H^{2i}(\psi^\lambda) \iff m \mid i$

and therefore

$$\begin{aligned} QH^*(BU(mn + s); \mathbb{Q}_p)_{C_m} &= \mathbb{Q}_p\{c_m, \dots, c_{mn}\} = QH^*(BX(m, 1, n); \mathbb{Q}_p) \\ &= QH^*(BX(mn + s); \mathbb{Q}_p)_{C_m}. \end{aligned}$$

The Weyl group $W = \Sigma_{mn+s}$ is the symmetric group in its natural representation on $L = \mathbb{Z}_p^{mn+s}$. Let e_1, \dots, e_{mn+s} be the canonical basis vectors of L . The permutation

$$w = (1 \cdots m)(m + 1 \cdots 2m) \cdots (m(n - 1) + 1 \cdots mn) \in \Sigma_{mn+s}$$

has order m and

$$(C_{\Sigma_{mn+s}}(w), L^{\langle \lambda w \rangle}) \\ = (C_m \wr \Sigma_n \times \Sigma_s, \mathbb{Z}_p \{ \lambda e_1 + \lambda^2 e_2 + \cdots + \lambda^m e_m, \dots, \lambda e_{m(n-1)+1} + \cdots + \lambda^m e_{mn} \})$$

contains the \mathbb{Z}_p -reflection group $G(m, 1, n) = C_m \wr \Sigma_n$ as a subgroup complementary to the kernel, Σ_s , of the action of $(C_{\Sigma_{mn+s}}(w)$ on $L^{\langle \lambda w \rangle}$. This means by [Corollary A.6](#) that the fixed point p -compact group $U(mn+s)^{hC_m} = X(m, 1, n)$.

From the two short exact sequences of $\mathbb{Z}_p \Sigma_{mn+s}$ -modules [[53](#), Section 10]

$$0 \rightarrow \mathbb{Z}_p \xrightarrow{\Delta} L \rightarrow LPU(mn+s) \rightarrow 0, \quad 0 \rightarrow LX(mn+s) \rightarrow LPU(mn+s) \rightarrow \check{\pi} \rightarrow 0$$

where Δ is the diagonal and $\check{\pi}$ a subgroup of $\pi_1(PU(mn+s)) = \mathbb{Z}_p/\mathbb{Z}_p(mn+s)$ (with trivial Σ_{mn+s} -action), we get that

$$L^{\langle \lambda w \rangle} = LPU(mn+s)^{\langle \lambda w \rangle} = LX(mn+s)^{\langle \lambda w \rangle}$$

as $\mathbb{Z}_p C_{\Sigma_{mn+s}}(w)$ -modules. □

The proof of [Proposition A.10](#) will make use of the following elementary facts:

- For arbitrary natural numbers m and n we write m_n for $m/\gcd(m, n)$. Then $m_n n = \text{lcm}(m, n)$ and $m_n n_m = \text{lcm}(m, n)/\gcd(m, n)$.
- $C_{\text{lcm}(q, m)} = \langle \lambda, \mu \mid \lambda^q = 1, \mu^m = 1, \lambda\mu = \mu\lambda, \lambda^{qm} = \mu^{mq} \rangle$.
- Let $A(a, t) \in GL(\mathbb{Z}_p, t)$ denote the linear automorphism

$$A(a, t)(x_1, \dots, x_t) = (ax_t, x_1, \dots, x_{t-1})$$

where $a \in \mathbb{Z}_p^\times$ is a unit. The i -th power $A(a, t)^i$ has characteristic polynomial $(x^{ti} - a^{it})^{t/t_i}$ and $A(a, t)^t = aE$.

- If $\lambda \in \mathbb{Z}_p^\times$ has order q , then $A(\lambda^{-qm}, q_m)$ also has order q for $A(\lambda^{-qm}, q_m)^{qm} = \lambda^{-qm} E$ has order $\gcd(q, m)$. The λ^{-1} eigenspace of $A(\lambda^{-qm}, q_m)$ has rank one and $A(\lambda^{-qm}, q_m)^{-1}$ acts on it as multiplication by λ .
- In the exact sequence $1 \rightarrow A^{\langle g \rangle} \rightarrow C_{A \rtimes G}(a, g) \rightarrow C_G(g)$ the image in $C_G(g)$ consists of those $h \in C_G(g)$ that fix $a \in A/(1-g)A$.

Proposition A.10 (Generalized Grassmannians) *Let $\mathbf{X}(m, r, n)$, $m \geq 2$, $r \geq 1$, $n \geq 2$, $r \mid m \mid p-1$, be the irreducible polynomial p -compact group corresponding to the imprimitive \mathbb{Z}_p -reflection group $G(m, r, n)$. Suppose that the natural number ℓ divides*

$p - 1$ and let the cyclic group $C_\ell \subset C_{\text{lcm}(\ell, m)} \subset \mathbb{Z}_p^\times$ act on $\mathbf{X}(m, r, n)$ through unstable Adams operations. The homotopy fixed point group for this action is

$$\mathbf{X}(m, r, n)^{hC_\ell} = \begin{cases} \mathbf{X}(\text{lcm}(\ell, m), r, n/\ell_m) & r\ell \mid mn \\ \mathbf{X}(\text{lcm}(\ell, m), 1, n/\ell_m - 1) & r\ell \nmid mn, \ell \mid mn, \\ \mathbf{X}(\text{lcm}(\ell, m), 1, \lceil n/\ell_m \rceil) & \ell \nmid mn \end{cases}$$

where $\ell_m = \ell / \text{gcd}(\ell, m)$ and $\lceil n/\ell_m \rceil$ is the biggest integer $\leq n/\ell_m$. (By convention, $G(m, r, 1)$ is cyclic of order m/r and $G(m, r, 0)$ is the trivial group.)

Proof Let $\lambda \in \mathbb{Z}_p^\times$ be a primitive ℓ -th root of unity. In the rational cohomology algebra $H^*(B\mathbf{X}(m, r, n); \mathbb{Q}_p) \cong \mathbb{Q}_p[x_1, \dots, x_{n-1}, e]$ the degrees $|x_i| = 2im$ and $|e| = 2\frac{m}{r}n$ so that

$$\begin{aligned} x_i \text{ is preserved by } H^{2im}(\psi^\lambda) &= \lambda^{im} \Leftrightarrow \ell \mid im \Leftrightarrow \ell_m \mid i \\ e \text{ is preserved by } H^{2\frac{m}{r}n}(\psi^\lambda) &= \lambda^{\frac{m}{r}n} \Leftrightarrow \ell \mid nm/r \Leftrightarrow \ell_{m/r} \mid n \end{aligned}$$

and thus $QH^*(B\mathbf{X}(m, r, n); \mathbb{Q}_p)_{C_\ell}$ is isomorphic to the indecomposables of the rational cohomology algebra of the p -compact group on the right hand side of the equation.

We have $r\ell \mid mn \Leftrightarrow \ell_{m/r} \mid n$, $\ell \mid mn \Leftrightarrow \ell_m \mid n$, and $\ell_m \mid \ell_{m/r} \mid \ell \mid p - 1$.
 $\ell_{m/r} \mid n$: The element

$$w = \text{diag} \left(\underbrace{A(\lambda^{-\ell_m}, \ell_m), \dots, A(\lambda^{-\ell_m}, \ell_m)}_{n/\ell_m} \right) \in G(m, r, n)$$

has order ℓ . Since $((\lambda^{-\ell_m})^{n/\ell_m})^{m/r} = \lambda^{-mn/r} = 1$ because $\ell \mid (mn/r)$ by assumption, w does indeed belong to the index r subgroup $G(m, r, n)$ of $G(m, 1, n) = C_m \wr \Sigma_n$. Let $\{e_1, \dots, e_n\}$ be the canonical basis for the free \mathbb{Z}_p -module $L = \mathbb{Z}_p^n$ on which $G(m, r, n)$ acts. The free \mathbb{Z}_p -module

$$\begin{aligned} L^{\langle \lambda w \rangle} &= \langle e_1 + \lambda e_2 + \dots + \lambda^{\ell_m - 1} e_{\ell_m}, \dots, e_{(n - \ell_m) + 1} \\ &\quad + \lambda e_{(n - \ell_m) + 2} + \dots + \lambda^{\ell_m - 1} e_n \rangle \end{aligned}$$

has rank n/ℓ_m . We shall now compute the centralizer of w . Let ζ be a generator of the cyclic group $C_{\text{lcm}(\ell, m)} \subset \mathbb{Z}_p^\times$ so that $C_m = \langle \mu \rangle$ and $C_\ell = \langle \lambda \rangle$ with $\mu = \zeta^{\ell_m}$ and $\lambda = \zeta^{m\ell}$. The homomorphisms $A(\ell, 1, n/\ell_m) \rightarrow CG(m, 1, n)(w) \leftarrow A(m, 1, n/\ell_m)$

defined by

$$\lambda_i \longrightarrow \text{diag} \left(\underbrace{E, \dots, E}_{i-1}, A(\lambda^{-\ell_m}, \ell_m)^{-1}, E, \dots, E \right),$$

$$\text{diag} \left(\underbrace{E, \dots, E}_{i-1}, \mu E, E, \dots, E \right) \longleftarrow \mu_i$$

combine to a homomorphism defined on $A(\text{lcm}(\ell, m), 1, n/\ell_m)$ since they agree on their common domain $A(\text{gcd}(\ell, m), 1, n/\ell_m) = \langle \mu^{m\ell} \rangle^{n/\ell_m} = \langle \lambda^{\ell m} \rangle^{n/\ell_m}$. Observe that $(\lambda^{a_1}, \dots, \lambda^{a_{n/\ell_m}}) \in A(\ell, 1, n/\ell_m)$ lies in the subgroup $A(\text{lcm}(\ell, m), r, n/\ell_m)$ if and only if its image lies in $G(m, r, n)$ and that $(\mu^{a_1}, \dots, \mu^{a_{n/\ell_m}}) \in A(m, 1, n/\ell_m)$ lies in the subgroup $A(m, r, n/\ell_m)$ if and only if its image lies in $G(m, r, n)$. Together with the diagonal $\Delta: \Sigma_{n/\ell_m} \rightarrow \Sigma_n$ given by $\Delta(\sigma)((i-1)\ell_m + j) = (\sigma(i)-1)\ell_m + j$, $1 \leq \ell i \leq n/\ell_m$, $1 \leq j \leq \ell_m$, we obtain a group isomorphism

$$G(\text{lcm}(\ell, m), 1, n/\ell_m) \cong C_{G(m, 1, n)}(w)$$

that restricts to a group isomorphism $G(\text{lcm}(\ell, m), r, n/\ell_m) \cong C_{G(m, r, n)}(w)$ between index r subgroups. This isomorphism identifies the pair $(C_{G(m, r, n)}(w), L^{\langle \lambda w \rangle})$ and the imprimitive \mathbb{Z}_p -reflection group $(G(\text{lcm}(\ell, m), r, n/\ell_m), \mathbb{Z}_p^{n/\ell_m})$.

Case $\ell_m/r \nmid n$, $\ell_m \mid n$ It will suffice to consider the case of $G(m, m, n)$ where $\ell \nmid n$ and $\ell_m \mid n$. The element

$$w = \text{diag} \left(\underbrace{A(\lambda^{-\ell_m}, \ell_m), \dots, A(\lambda^{-\ell_m}, \ell_m)}_{n/\ell_m - 1}, A(\lambda^{-\ell_m}, \ell_m)^{1-n/\ell_m} \right) \in G(m, m, n)$$

has order ℓ . Note that λ^{-1} is not an eigenvalue for $A(\lambda^{-\ell_m}, \ell_m)^{1-n/\ell_m}$ because

$$\begin{aligned} A(\lambda^{-\ell_m}, \ell_m)^{1-n/\ell_m} \text{ has eigenvalue } \lambda^{-1} &\Leftrightarrow A(\lambda^{-\ell_m}, \ell_m)^{n/\ell_m - 1} \text{ has eigenvalue } \lambda \\ &\Leftrightarrow \lambda^{(\ell_m)n/\ell_m - 1} = \lambda^{-\ell_m(n/\ell_m - 1)\ell_m} \\ &\Leftrightarrow \ell \mid (\ell_m)n/\ell_m - 1 + \ell_m(n/\ell_m - 1)\ell_m \\ &\Leftrightarrow \ell \mid n/\text{gcd}(\ell_m, n/\ell_m - 1) \\ &\Rightarrow \ell \mid n \Rightarrow \ell_m \mid n \end{aligned}$$

which is not the case. Therefore the λ^{-1} -eigenspace

$$L^{\langle \lambda w \rangle} = \left\langle e_1 + \lambda e_2 + \dots + \lambda^{\ell_m - 1} e_{\ell_m}, \dots, e_{(n-2\ell_m)+1} + \lambda e_{(n-2\ell_m)+2} + \dots + \lambda^{\ell_m - 1} e_{n-\ell_m} \right\rangle$$

has rank $n/\ell_m - 1$. The two monomorphisms $A(\ell, 1, n/\ell_m - 1) \rightarrow C_{G(m,m,n)}(w)$ and $C_{G(m,m,n)}(w) \leftarrow A(m, 1, n/\ell_m - 1)$ given by

$$\lambda_i \rightarrow \text{diag} \left(\underbrace{E, \dots, E}_{i-1}, A(\lambda^{-\ell_m}, \ell_m)^{-1}, E, \dots, E, A(\lambda^{-\ell_m}, \ell_m) \right)$$

$$\text{diag} \left(\underbrace{E, \dots, E}_{i-1}, \mu E, E, \dots, E, \mu^{-1} E \right) \leftarrow \mu_i$$

agree on their common domain $A(\text{gcd}(\ell, m), 1, n/\ell_m - 1)$ and together with the monomorphism

$$\Sigma_{n/\ell_m-1} \xrightarrow{\Delta} \Sigma_{n-\ell_m} \hookrightarrow \Sigma_m$$

they define a homomorphism on the group $A(\text{lcm}(\ell, m), 1, n/\ell_m - 1) \rtimes \Sigma_{n/\ell_m-1}$ such that the composition

$$A(\text{lcm}(\ell, m), 1, n/\ell_m - 1) \rtimes \Sigma_{n/\ell_m-1} \hookrightarrow C_{G(m,m,n)}(w)$$

$$\twoheadrightarrow \text{Im} \left(C_{G(m,m,n)}(w) \rightarrow GL(L^{(\lambda w)}) \right)$$

is an isomorphism whose image is isomorphic to $G(\text{lcm}(\ell, m), 1, n/\ell_m - 1)$ as a \mathbb{Z}_p -reflection group.

Case $\ell_m \nmid n$ It will suffice to consider the case of $G(m, m, n)$. The element

$$w = \text{diag} \left(\underbrace{A(\lambda^{-\ell_m}, \ell_m), \dots, A(\lambda^{-\ell_m}, \ell_m)}_{[n/\ell_m]}, \underbrace{\lambda^{\ell_m [n/\ell_m]}, 1, \dots, 1}_{n-\ell_m [n/\ell_m]} \right) \in G(m, m, n)$$

has order ℓ . Note that λ^{-1} is not an eigenvalue for $\lambda^{\ell_m [n/\ell_m]}$ because

$$\lambda^{\ell_m [n/\ell_m]} = \lambda^{-1} \Leftrightarrow \ell \mid \ell_m [n/\ell_m] + 1 \Leftrightarrow \ell_m \text{gcd}(\ell, m) \mid \ell_m [n/\ell_m] + 1 \Rightarrow \ell_m \mid 1$$

which is not the case as $\ell_m > 1$. Therefore the λ^{-1} eigenspace $L^{(\lambda w)}$ has rank $[n/\ell_m]$. The two monomorphisms $A(\ell, 1, [n/\ell_m]) \rightarrow C_{G(m,m,n)}(w) \leftarrow A(m, 1, [n/\ell_m])$ given by

$$\lambda_i \rightarrow \text{diag} \left(\underbrace{E, \dots, E}_{(i-1)\ell_m}, A(\lambda^{-\ell_m}, \ell_m)^{-1}, E, \dots, E, \underbrace{\lambda^{-\ell_m}, 1, \dots, 1}_{n-\ell_m [n/\ell_m]} \right)$$

$$\text{diag} \left(\underbrace{E, \dots, E}_{(i-1)\ell_m}, \mu E, E, \dots, E, \underbrace{\mu^{-\ell_m}, 1, \dots, 1}_{n-\ell_m [n/\ell_m]} \right) \leftarrow \mu_i$$

agree on their common domain $A(\text{gcd}(\ell, m), 1, [n/\ell_m])$ and together with the inclusion of permutation groups

$$\Sigma_{[n/\ell_m]} \xrightarrow{\Delta} \Sigma_{\ell_m [n/\ell_m]} \hookrightarrow \Sigma_m,$$

they define a homomorphism on the group $A(\text{lcm}(\ell, m), 1, [n/\ell_m]) \rtimes \Sigma_{[n/\ell_m]}$ such that the composition

$$A(\text{lcm}(\ell, m), 1, [n/\ell_m]) \rtimes \Sigma_{[n/\ell_m]} \hookrightarrow C_{G(m,m,n)}(w) \twoheadrightarrow \text{Im} (C_{G(m,m,n)}(w) \rightarrow GL(L^{(\lambda w)}))$$

is an isomorphism with image isomorphic to $G(\text{lcm}(\ell, m), 1, [n/\ell_m])$ as \mathbb{Z}_p -reflection group. \square

The outer automorphism group of $\mathbf{X}(m, r, n)$ is isomorphic to $A(m, r, n) \backslash \mathbb{Z}_p^\times A(m, 1, n)$ except in the cases $(m, r, n) \in \{(2, 1, 2), (4, 2, 2), (3, 3, 3), (2, 2, 4)\}$ [58, Section 6; 53, 7.14]. The (exotic) homotopy action

$$\rho: C_m = \langle \mu \rangle \rightarrow \text{Out}(\mathbf{X}(m, r, n)) \cong A(m, r, n) \backslash \mathbb{Z}_p^\times A(m, 1, n)$$

that takes the generator μ of C_m to $A(m, r, n)(\mu, 1, \dots, 1)$ is distinct from the actions through unstable Adams operations of Proposition A.10 when $\text{gcd}(r, n) > 1$ [53, 7.14].

Proposition A.11 (Exotic actions on generalized Grassmannians) *Assume that $m \geq 2$, $r \geq 1$, $n \geq 2$, and $(m, r, n) \notin \{(2, 1, 2), (4, 2, 2), (3, 3, 3), (2, 2, 4)\}$. Then the homotopy fixed point p -compact group*

$$\mathbf{X}(m, r, n)^{hC_m} = \mathbf{X}(m, 1, n - 1)$$

for the above exotic homotopy action on $\mathbf{X}(m, r, n)$.

Proof The second assumption of Proposition A.5 is clearly satisfied as the action preserves the generators x_1, \dots, x_{n-1} but does not preserve the generator e . To verify the first assumption, take $\bar{\rho}: C_m \rightarrow N_{GL(L)}(G(m, r, n)) = \mathbb{Z}_p^\times G(m, 1, n)$ to be the obvious choice $\bar{\rho}(\mu) = (\mu, 1, \dots, 1)$. Then

$$G(m, r, n)^{\bar{\rho}C_m} = A(m, r, n) \rtimes \Sigma_{n-1}, \quad L^{\bar{\rho}C_m} = \mathbb{Z}_p^{n-1}$$

and the composition

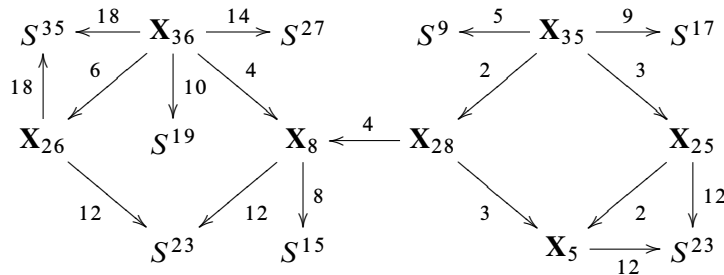
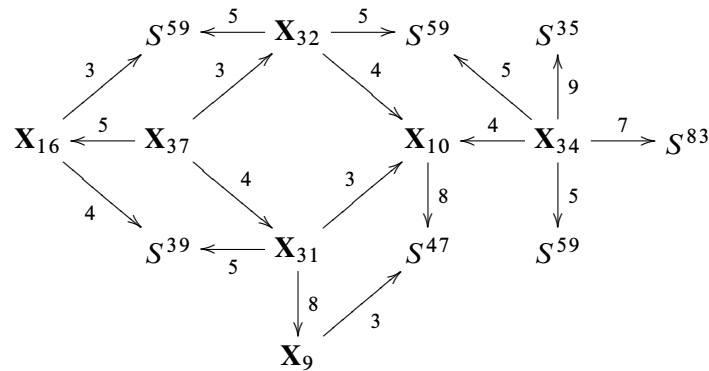
$$A(m, 1, n - 1) \rtimes \Sigma_{n-1} \hookrightarrow G(m, r, n)^{\bar{\rho}C_m} \twoheadrightarrow \text{Im} (G(m, r, n)^{\bar{\rho}C_m} \rightarrow GL(L^{\bar{\rho}C_m}))$$

where the first morphism is $(\mu_2, \dots, \mu_m) \rightarrow ((\mu_2 \cdots \mu_n)^{-1}, \mu_2, \dots, \mu_n)$, $\Sigma_{n-1} \hookrightarrow \Sigma_n$, identifies the group to the right as the \mathbb{Z}_p -reflection group $G(m, 1, n - 1)$. \square

The results of Proposition A.9 and Proposition A.11 were obtained by Castellana [18] using different methods.

A.12 The sporadic p -compact groups

As in Section 2 we write G_i , $4 \leq i \leq 37$, for the sporadic irreducible and simply connected \mathbb{Z}_p -reflection group with number i in the Clark–Ewing classification table, and \mathbf{X}_i for the corresponding simply connected p -compact group. When \mathbf{X}_i is defined at the odd prime p and r divides $p - 1$, $\mathbf{X}_i^{hC_r}$ denotes the fixed point p -compact group for the homotopy action $\psi: C_r \rightarrow \text{Out}(\mathbf{X}_i)$ through unstable Adams operations on the p -compact group \mathbf{X}_i . We identify the fixed point p -compact groups for actions through unstable Adams operations on the 34 sporadic irreducible p -compact groups. We may summarize our results in the following diagrams:



Here, for instance, $\mathbf{X}_{32} \xrightarrow{4} \mathbf{X}_{10}$ means that $\mathbf{X}_{32}^{hC_4} = \mathbf{X}_{10}$ (when $p \equiv 1 \pmod{12}$) and $\mathbf{X}_{32} \xrightarrow{5} S^{59}$ means that $\mathbf{X}_{32}^{hC_5} = S^{59}$ (when $p \equiv 1 \pmod{30}$). The relevant primes are mentioned in the more detailed explanations below but not displayed in the above diagrams. We use Corollary A.6 to identify the homotopy fixed point groups. With a computer algebra program it is quite easy to find eigenspaces for the elements of these \mathbb{Z}_p -reflection groups. We used the program MAGMA.

- (1) ($G_{37} = W(E_8), C_3, G_{32}, p \equiv 1 \pmod{3}$) There is an element $w \in G_{37}$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{37}}(w), L_{37}^{\langle \lambda w \rangle}) = (G_{32}, L_{32})$$

meaning that $E_8^{hC_3} = \mathbf{X}_{37}^{hC_3} = \mathbf{X}_{32}$.

- (2) ($G_{37} = W(E_8), C_4, G_{31}, p \equiv 1 \pmod{4}$) There is an element $w \in G_{37}$ of order 4 such that

$$(C_{G_{37}}(w), L_{37}^{\langle iw \rangle}) = (G_{31}, L_{31})$$

meaning that $E_8^{hC_4} = \mathbf{X}_{37}^{hC_4} = \mathbf{X}_{31}$.

- (3) ($G_{37} = W(E_8), C_5, G_{16}, p \equiv 1 \pmod{5}$) There is an element $w \in G_{37}$ of order 5 and a primitive 5-th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{37}}(w), L_{37}^{\langle \lambda w \rangle}) = (G_{16}, L_{16})$$

meaning that $E_8^{hC_5} = \mathbf{X}_{37}^{hC_5} = \mathbf{X}_{16}$.

- (4) ($G_{34}, C_4, G_{10}, p \equiv 1 \pmod{12}$). There exists an element $w \in G_{34}$ of order 4, a (index 4) subgroup G of $C_{G_{34}}(w)$, and a primitive 4-th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(G, L_{34}^{\langle \lambda w \rangle}) = (G_{10}, L_{10})$$

meaning that $\mathbf{X}_{34}^{hC_4} = \mathbf{X}_{10}$.

- (5) ($G_{32}, C_4, G_{10}, p \equiv 1 \pmod{12}$) There is an element $w \in G_{32}$ of order 4 and a primitive 4-th root of unity $i \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{32}}(w), L_{32}^{\langle iw \rangle}) = (G_{10}, L_{10})$$

which means that $\mathbf{X}_{32}^{hC_4} = \mathbf{X}_{10}$.

- (6) ($G_{32}, C_{30}, C_5, p \equiv 1 \pmod{30}$) There is an element $w \in G_{32}$ of order 5 and a primitive 5-th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{32}}(w), L_{32}^{\langle \lambda w \rangle}) = (C_{30}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_{32}^{hC_5} = S^{59}$.

- (7) ($G_{31}, C_3, G_{10}, p \equiv 1 \pmod{12}$). There exists an element $w \in G_{31}$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{31}}(w), L_{31}^{\langle \lambda w \rangle}) = (G_{10}, L_{10}).$$

This means that $\mathbf{X}_{31}^{hC_3} = \mathbf{X}_{10}$. (The group that the computer finds is G_{10} and not G_{15} (of the same rank and the same degrees) because the elements of order 8 square to central elements [63, p 281].)

- (8) $(G_{31}, C_8, G_9, p \equiv 1 \pmod{24})$. There exists an element $w \in G_{31}$ of order 8 and a primitive 8–th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that the \mathbb{Z}_p –reflection group

$$(C_{G_{31}}(w), L_{31}^{(\lambda w)}) = (G_9, L_9)$$

which means that $\mathbf{X}_{31}^{hC_8} = \mathbf{X}_9$.

- (9) $(G_{10}, C_8, C_{24}, p \equiv 1 \pmod{24})$ There is an element $w \in G_{10}$ of order 8 and a primitive 8–th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{10}}(w), L_{10}^{(\lambda w)}) = (C_{24}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_{10}^{hC_8} = S^{47}$.

- (10) $(G_9, C_3, C_{24}, p \equiv 1 \pmod{24})$ There is an element $w \in G_9$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_9}(w), L_9^{(\lambda w)}) = (C_{24}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_9^{hC_3} = S^{47}$.

- (11) $(G_{34}, C_9, C_{18}, p \equiv 1 \pmod{18})$ There is an element $w \in G_{34}$ of order 9 and a primitive 9–th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{34}}(w), L_{34}^{(\lambda w)}) = (C_{18}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_{34}^{hC_9} = S^{37}$.

- (12) $(G_{36} = W(E_7), C_6, G_{26}, p \equiv 1 \pmod{6})$ There is an element $w \in G_{36}$ of order 6 and a primitive 6–th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{36}}(w), L_{36}^{(\lambda w)}) = (G_{26}, L_{26})$$

which means that $E_7^{hC_6} = \mathbf{X}_{36}^{hC_6} = \mathbf{X}_{26}$.

- (13) $(G_{36} = W(E_7), C_4, G_8, p \equiv 1 \pmod{8})$. There is an element $w \in G_{36}$ of order 4, a subgroup $\bar{W} < C_{G_{36}}(w)$ of index 8, faithfully represented in $L_{36}^{(iw)}$, and a primitive 4–th root of unity $i \in \mathbb{Z}_p^\times$ such that

$$(\bar{W}, L_{36}^{(iw)}) = (G_8, L_8)$$

which means that $E_7^{hC_4} = \mathbf{X}_{36}^{hC_4} = G_8$. (The \mathbb{Z}_p -reflection group \bar{W} contains elements of order 8 with central square so it is not isomorphic to G_{13} [63, p. 281].)

- (14) ($G_{36} = W(E_7), C_{14}, C_{14}, p \equiv 1 \pmod{14}$) There is an element $w \in G_{36}$ of order 14 and a primitive 14-th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{36}}(w), L_{36}^{(\lambda w)}) = (C_{14}, \mathbb{Z}_p)$$

which means that $E_7^{hC_{14}} = \mathbf{X}_{36}^{hC_{14}} = S^{27}$.

- (15) ($G_{36} = W(E_7), C_{18}, C_{18}, p \equiv 1 \pmod{18}$) There is an element $w \in G_{36}$ of order 18 and a primitive 18-th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{36}}(w), L_{36}^{(\lambda w)}) = (C_{18}, \mathbb{Z}_p)$$

which means that $E_7^{hC_{18}} = \mathbf{X}_{36}^{hC_{18}} = S^{35}$.

- (16) ($G_{26}, C_{18}, C_{18}, p \equiv 1 \pmod{18}$) There is an element $w \in G_{26}$ of order 18 and a primitive 18-th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{26}}(w), L_{26}^{(\lambda w)}) = (C_{18}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_{26}^{hC_{18}} = S^{35}$.

- (17) ($G_8, C_{12}, C_{12}, p \equiv 1 \pmod{12}$) There is an element $w \in G_8$ of order 12 and a primitive 12-th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_8}(w), L_8^{(\lambda w)}) = (C_{12}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_8^{hC_{12}} = S^{23}$.

- (18) ($G_8, C_8, C_8, p \equiv 1 \pmod{8}$) There is an element $w \in G_8$ of order 8 and a primitive 8-th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_8}(w), L_8^{(\lambda w)}) = (C_8, \mathbb{Z}_p)$$

which means that $\mathbf{X}_8^{hC_8} = S^{15}$.

- (19) ($G_{35} = W(E_6), C_2, G_{28} = W(F_4), p \equiv 1 \pmod{2}$) There is an element $w \in G_{35}$ of order 2 such that

$$(C_{G_{35}}(w), L_{35}^{(-w)}) = (G_{28}, L_{28})$$

which means that $E_6^{hC_2} = \mathbf{X}_{35}^{hC_2} = F_4$.

- (20) ($G_{35} = W(E_6), C_3, G_{25}, p \equiv 1 \pmod{3}$) There is an element $w \in G_{35}$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{35}}(w), L_{35}^{\langle \lambda w \rangle}) = (G_{25}, L_{25})$$

which means that $E_6^{hC_3} = \mathbf{X}_{35}^{hC_3} = G_{25}$.

- (21) ($G_{35} = W(E_6), C_5, G_{25}, p \equiv 1 \pmod{5}$) There is an element $w \in G_{35}$ of order 5 and a primitive 5–th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{35}}(w), L_{35}^{\langle \lambda w \rangle}) = (C_5, \mathbb{Z}_p)$$

which means that $E_6^{hC_5} = \mathbf{X}_{35}^{hC_5} = S^9$.

- (22) ($G_{35} = W(E_6), C_4, G_8, p \equiv 1 \pmod{4}$) There is an element $w \in G_{35}$ of order 4 and a primitive 4–th root of unity $i \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{35}}(w), L_{35}^{\langle iw \rangle}) = (G_8, L_8)$$

which means that $E_6^{hC_4} = \mathbf{X}_{35}^{hC_4} = G_8$.

- (23) ($G_{25}, C_2, G_5, p \equiv 1 \pmod{6}$) There is an element $w \in G_{25}$ of order 2 such that

$$(C_{G_{25}}(w), L_{25}^{\langle -w \rangle}) = (G_5, L_5)$$

which means that $\mathbf{X}_{25}^{hC_2} = \mathbf{X}_5$.

- (24) ($G_{28} = W(F_4), C_3, G_5, p \equiv 1 \pmod{6}$) There is an element $w \in G_{28}$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{28}}(w), L_{28}^{\langle \lambda w \rangle}) = (G_5, L_5)$$

which means that $F_4^{hC_3} = \mathbf{X}_{28}^{hC_3} = \mathbf{X}_5$.

- (25) ($G_{28} = W(F_4), C_4, G_4, p \equiv 1 \pmod{4}$) There is an element $w \in G_{28}$ of order 4 and a primitive 4–th root of unity $i \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{28}}(w), L_{28}^{\langle iw \rangle}) = (G_8, L_8)$$

which means that $F_4^{hC_4} = \mathbf{X}_{28}^{hC_4} = \mathbf{X}_8$.

- (26) ($G_{25}, C_{12}, C_{12}, p \equiv 1 \pmod{12}$) There is an element $w \in G_{25}$ of order 12 and a primitive 12–th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{25}}(w), L_{25}^{\langle \lambda w \rangle}) = (C_{12}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_{25}^{hC_{12}} = S^{23}$.

- (27) $(G_{25}, C_{12}, C_{12}, p \equiv 1 \pmod{12})$ There is an element $w \in G_{25}$ of order 12 and a primitive 12-th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_5}(w), L_5^{\langle \lambda w \rangle}) = (C_{12}, \mathbb{Z}_p)$$

which means that $\mathbf{X}_{25}^{hC_{12}} = S^{23}$.

Appendix B Derived functors of inverse limit functor

In this appendix we discuss higher limits over some finite categories of a special type.

Given a finite group G and subgroups $H_1, H_2, \dots, H_k \leq G$, we define a finite category $\mathbb{I}(k)$ with objects $\{0, 1, 2, \dots, k\}$, where G is the group of automorphisms of 0 and for each $i > 0$, $H_i \setminus G = \text{Hom}_{\mathbb{I}(k)}(i, 0)$ as G -sets and $\text{Aut}_{\mathbb{I}(k)}(i) = N_G(H_i)/H_i$, and other morphism sets are empty. Those categories appear in the context of the Aguadé p -compact groups and other compact Lie groups, as categories of elementary abelian subgroups. The next result is essentially contained in [2; 53].

Proposition B.1 *Let \mathbf{M} be a given diagram of \mathbb{Z}_p -modules index by the category $\mathbb{I}(k)$. Assume that*

- (a) *restriction gives an isomorphism $H^j(G; A) \cong H^j(H_1; A)$, for any \mathbb{Z}_p - G -module A and $j \geq 1$, and*
- (b) *$p \nmid |N_G(H_i)|$ and $M_i = M_0^{H_i}$, for every $i \geq 2$.*

Then there is an exact sequence

$$0 \rightarrow \varprojlim_{\mathbb{I}(k)}^0 \mathbf{M} \rightarrow M_1^{N_G(H_1)/H_1} \oplus M_0^G \rightarrow M_0^{N_G(H_1)} \rightarrow \varprojlim_{\mathbb{I}(k)}^1 \mathbf{M} \rightarrow 0,$$

and $\varprojlim_{\mathbb{I}(k)}^j \mathbf{M} = 0$ if $j \geq 2$.

Proof We consider a star-shaped category $\mathbb{I}(k)$ with $k + 1$ objects $\{0, 1, 2, \dots, k\}$. There is an exact sequence of the following form [53]:

$$\begin{aligned} 0 \rightarrow \varprojlim^0 \mathbf{M} &\rightarrow M_0^G \times \prod_{i>0} M_i^{N_G(H_i)/H_i} \rightarrow \prod_{i>0} M_0^{N_G(H_i)} \\ &\rightarrow \varprojlim^1 \mathbf{M} \rightarrow H^1(G; M_0) \times \prod_{i>0} H^1(N_G(H_i)/H_i; M_i) \rightarrow \prod_{i>0} H^1(N_G(H_i); M_0) \\ &\rightarrow \varprojlim^2 \mathbf{M} \rightarrow H^2(G; M_0) \times \prod_{i>0} H^2(N_G(H_i)/H_i; M_i) \rightarrow \prod_{i>0} H^2(N_G(H_i); M_0) \\ &\rightarrow \varprojlim^3 \mathbf{M} \rightarrow \dots \end{aligned}$$

Under condition (b) this exact sequence reduces to the exact sequence:

$$\begin{aligned} 0 \rightarrow \lim^0 \mathbf{M} &\rightarrow M_0^G \times M_1^{N_G(H_1)/H_1} \rightarrow M_0^{N_G(H_1)} \\ &\rightarrow \lim^1 \mathbf{M} \rightarrow H^1(G; M_0) \times H^1(N_G(H_1)/H_1; M_1) \rightarrow H^1(N_G(H_1); M_0) \\ &\rightarrow \lim^2 \mathbf{M} \rightarrow H^2(G; M_0) \times H^2(N_G(H_1)/H_1; M_1) \rightarrow H^2(N_G(H_1); M_0) \\ &\rightarrow \lim^3 \mathbf{M} \rightarrow \dots \end{aligned}$$

Condition (a) implies that H_1 and G have the same Sylow p -subgroup. Hence p does not divide $|N_G(H_1)/H_1|$ and so therefore $H^*(N_G(H_1); A) \cong H^*(H_1; A)^{N_G(H_1)/H_1}$. Now, in the diagram given by restrictions

$$H^j(G; A) \rightarrow H^j(N_G(H_1); A) \rightarrow H^j(H_1; A), \quad j \geq 1,$$

the composition is an isomorphism and the second arrow is a monomorphism, hence both arrows are isomorphisms

$$H^j(G; A) \cong H^j(N_G(H_1); A) \cong H^j(H_1; A), \quad j \geq 1,$$

and the proposition follows. \square

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