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A function on the homology of 3–manifolds

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In analogy with the Thurston norm, we define for an orientable 3–manifold M a numerical function on $H_2(M;\mathbb{Q}/\mathbb{Z})$. This function measures the minimal complexity of folded surfaces representing a given homology class. A similar function is defined on the torsion subgroup of $H_1(M;\mathbb{Z})$. These functions are estimated from below in terms of abelian torsions of M.

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1 Introduction

One of the most beautiful invariants of a 3–dimensional manifold M is the Thurston semi-norm on $H_2(M;\mathbb{Q})$ [9]. The geometric idea leading to this semi-norm is to consider the minimal genus of a surface in M realizing any given 2–homology class of M. Thurston's definition of the semi-norm uses a suitably normalized Euler characteristic of the surface rather than the genus. The Thurston semi-norm is uninteresting for a rational homology sphere M, since then $H_2(M;\mathbb{Q})=0$. However, a rational homology sphere may have nontrivial 2–homology with coefficients in $\mathbb{Q}/\mathbb{Z}$. Homology classes in $H_2(M;\mathbb{Q}/\mathbb{Z})$ can be realized by folded surfaces, locally looking like unions of several half-planes in $\mathbb{R}^3$ with common boundary line. It is natural to consider "smallest" folded surfaces in a given homology class.

We use this train of ideas to define for an arbitrary orientable 3–manifold M (not necessarily a rational homology sphere) a function

$$\theta=\theta_M\colon H_2(M;\mathbb{Q}/\mathbb{Z})\to\mathbb{R}_+=\{r\in\mathbb{R}\mid r\geq 0\}.$$

This function measures the "minimal" normalized Euler characteristic of a folded surface representing a given class in $H_2(M;\mathbb{Q}/\mathbb{Z})$.

Using the boundary homomorphism

$$d\colon H_2(M;\mathbb{Q}/\mathbb{Z})\to H_1(M)=H_1(M;\mathbb{Z}),$$

whose image is equal to $\operatorname{Tors} H_1(M)$, we derive from θ a function

$$\Theta=\Theta_M\colon \operatorname{Tors} H_1(M)\to\mathbb{R}_+$$

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by $\Theta(u) = \inf_{x \in d^{-1}(u)} \theta(x)$ for any $u \in \operatorname{Tors} H_1(M)$. One can view $\Theta(u)$ as a "normalized minimal genus" of oriented knots in M representing u. If M is a rational homology sphere, then d is an isomorphism and $\Theta = \theta \circ d^{-1}$.

We give an estimate of the function θ from above in terms of the Thurston semi-norm on knot complements in M. This estimate implies that θ is bounded from above and is upper semi-continuous with respect to a natural topology on $H_2(M;\mathbb{Q}/\mathbb{Z})$. (I do not know whether θ is continuous.) The functions θ and Θ are also estimated from below using abelian torsions of M. These estimates are parallel to the McMullen estimate [6] of the Thurston semi-norm in terms of the Alexander polynomial.

A simple example of nonzero functions θ and Θ is provided by the lens space $M = L(5,1)$. We identify $H_1(M) = \mathbb{Z}/5\mathbb{Z}$ so that the core circles of the two solid tori forming M represents $\pm 1 \in \mathbb{Z}/5\mathbb{Z}$. It is shown in Section 2.3, Section 2.4, and Section 6.1 that $\Theta_M(\pm 1) = \Theta_M(0) = 0$ and $\Theta_M(2) = \Theta_M(-2) \geq 1/5$. In this example, the function Θ takes nonzero values only on $\pm 2 \in \mathbb{Z}/5\mathbb{Z}$. This shows that, in contrast to the Thurston semi-norm, the function Θ may not satisfy the triangle inequality and may be nonhomogeneous, that is in general $\Theta(kx) \neq k\,\Theta(x)$ for $k \in \mathbb{Z}$ and $x \in H_1(M)$. The same remarks apply to θ since in this example $H_2(M;\mathbb{Q}/\mathbb{Z}) = H_1(M)$ and $\theta = \Theta \circ d$.

The Thurston semi-norm of a 3–manifold M is fully determined by the Heegaard-Floer homology of M (see Ozsváth and Szabó [8]), and by the Seiberg–Witten monopole homology of M (see Kronheimer and Mrowka [4]). It would be interesting to obtain similar computations of the functions θ and Θ.

The organization of the paper is as follows. We introduce the functions θ and Θ in Section 2 and estimate them from above in Section 3. In Section 4 these functions are estimated from below in the case where the first Betti number of the 3–manifold is nonzero. A similar estimate for rational homology spheres is given in Section 5. In Section 6 we describe a few examples. In Section 7 we make several miscellaneous remarks.

Throughout the paper, the unspecified group of coefficients in homology is $\mathbb{Z}$.

2 Folded surfaces and the functions θ, Θ

2.1 Folded surfaces

By a *folded surface* (without boundary), we mean a compact 2–dimensional polyhedron such that each point has a neighborhood homeomorphic to a union of several half-planes in $\mathbb{R}^3$ with common boundary line. Such a neighborhood is homeomorphic to $\mathbb{R} \times \Gamma_n$ where n is a positive integer and Γ_n is a union of n closed intervals with one common endpoint and no other common points.

The *interior* $\mathrm{Int}(X)$ of a folded surface X consists of the points of X which have neighborhoods homeomorphic to $\mathbb{R}^2$. Clearly, $\mathrm{Int}(X)$ is a 2–dimensional manifold. The *singular set* $\mathrm{sing}(X) = X - \mathrm{Int}(X)$ of X consists of a finite number of disjoint circles. A neighborhood of a component of $\mathrm{sing}(X)$ in X fibers over this component with fiber Γ_n for some $n \neq 2$.

Cutting out X along $\mathrm{sing}(X)$ we obtain a compact 2–manifold (with boundary) X_{cut}. Each component of $\mathrm{Int}(X)$ is the interior of a component of X_{cut}. Set

$$\chi_-(X) = \sum_Y \chi_-(Y),$$

where Y runs over all components of X_{cut} and

$$\chi_-(Y) = \max(-\chi(Y), 0).$$

The number $\chi_-(X) \geq 0$ measures the complexity of X. It is equal to zero if and only if all components of X_{cut} belong to the following list: spheres, tori, projective planes, annuli, Möbius bands, disks.

By an *orientation* of a folded surface X, we mean an orientation of the 2–manifold $\mathrm{Int}(X)$. An orientation of X allows us to view X as a singular 2–chain with integer coefficients. This 2–chain is denoted by the same letter X. Its boundary expands as $\sum_K i(K)\,\langle K\rangle$ where K runs over connected components of $\mathrm{sing}(X)$, the symbol $\langle K\rangle$ denotes a 1–cycle on K representing a generator of $H_1(K) \cong \mathbb{Z}$ and $i(K) \in \mathbb{Z}$. Multiplying, if necessary, both $\langle K\rangle$ and $i(K)$ by -1, we can assume that $i(K) \geq 0$. In this way the integer $i(K)$ is uniquely determined by K. It is called the *index* of K in X. For K with $i(K) \neq 0$, the 1–cycle $\langle K\rangle$ determines an orientation of K. We say that this orientation is *induced* by the one on X.

We call a folded surface X *simple* if it is oriented, the set $\mathrm{sing}(X)$ is homeomorphic to a circle, and its index in X is nonzero. This index is denoted i_X. Note that X is not required to be connected; however, all components of X but one are closed oriented 2–manifolds.

2.2 Representation of 2–homology by folded surfaces

Let M be an orientable 3–manifold. By a folded surface in M, we mean a folded surface *embedded* in M. Given a simple folded surface X in M, the 2–chain $(i_X)^{-1} X$

with rational coefficients is a 2–cycle modulo $\mathbb{Z}$. This cycle represents a homology class in $H_2(M;\mathbb{Q}/\mathbb{Z})$ denoted $[X]$.

The short exact sequence of groups of coefficients $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ induces an exact homology sequence:

$$\cdots \to H_2(M;\mathbb{Q}) \to H_2(M;\mathbb{Q}/\mathbb{Z}) \to H_1(M) \to H_1(M;\mathbb{Q}) \to \cdots \tag{1}$$

The homomorphism $H_2(M;\mathbb{Q}/\mathbb{Z}) \to H_1(M)$ in this sequence will be denoted d_M and called the *boundary homomorphism*. The exactness of (1) implies that the image of d_M is equal to the group $\operatorname{Tors} H_1(M)$ consisting of all elements of $H_1(M)$ of finite order.

For a simple folded surface X in M, the homomorphism d_M sends $[X]$ into the homology class in $H_1(M)$ represented by the circle $\operatorname{sing}(X)$ with orientation induced by the one on X.

For example, if $X \subset M$ is a compact oriented 2–manifold with connected nonvoid boundary, then X is a simple folded surface with $\operatorname{sing}(X) = \partial X$, $i_X = 1$, and $[X] = 0$. Another example: consider an unknotted circle K lying in a 3–ball in M and pick $n \neq 2$ closed 2–disks bounded by K in this ball and having no other common points. We orient these disks so that the induced orientations on K are the same. The union of these disks, $X = X(n)$, is a simple folded surface with $\operatorname{sing}(X) = K$, $i_X = n$, and $[X] = 0$.

Lemma 2.1 *Any homology class $x \in H_2(M;\mathbb{Q}/\mathbb{Z})$ can be represented by a simple folded surface.*

Proof Set $d = d_M \colon H_2(M;\mathbb{Q}/\mathbb{Z}) \to H_1(M)$. We can represent $d(x) \in \operatorname{Tors} H_1(M)$ by an oriented embedded circle $K \subset \operatorname{Int}(M) = M - \partial M$. Pick an integer $n \geq 1$ such that $n\,d(x) = 0$. The standard arguments, using the Poincaré duality and transversality, show that there is a simple folded surface X in M such that $\operatorname{sing}(X) = K$ and $i_X = n$.

Since both X and M are orientable, the 1–dimensional normal bundle of $\operatorname{Int}(X)$ in M is trivial. Keeping $\operatorname{sing}(X)$ and pushing $X - \operatorname{sing}(X)$ in a normal direction, we obtain a "parallel" copy X_1 of X such that $X \cap X_1 = \operatorname{sing}(X_1) = \operatorname{sing}(X) = K$. The orientation of X induces an orientation of X_1 in the obvious way. Repeating this process $k \geq 1$ times, we can obtain k parallel copies $X_1, X_2, \ldots, X_k$ of X meeting each other exactly at K. Then $X^{(k)} = X_1 \cup X_2 \cup \ldots \cup X_k$ is a simple folded surface such that $\operatorname{sing}(X^{(k)}) = K$ and $i_{X^{(k)}} = nk$. It follows from the construction that $[X^{(k)}] = [X] \in H_2(M;\mathbb{Q}/\mathbb{Z})$ for all $k \geq 1$.

The equalities $d(x)=[K]=d([X])$ imply that $x-[X]\in \operatorname{Ker} d=\operatorname{Im} j$, where j is the homomorphism $H_2(M;\mathbb{Q})\to H_2(M;\mathbb{Q}/\mathbb{Z})$ induced by the projection $\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}$. Pick $y\in j^{-1}(x-[X])\subset H_2(M;\mathbb{Q})$. There is an integer $k\geq 1$ such that ky lies in the image of the coefficient homomorphism $H_2(M;\mathbb{Z})\to H_2(M;\mathbb{Q})$. *A fortiori*, the homology class nky lies in this image. We represent nky by a closed oriented (possibly nonconnected) surface $\Sigma\subset M$. Since $d(x)\in \operatorname{Tors} H_1(M)$, the intersection number $\Sigma\cdot K=\Sigma\cdot d(x)$ is 0. Applying if necessary surgeries of index 1 to Σ, we can assume that $\Sigma\cap K=\varnothing$. Then y is represented by the 2–cycle $(nk)^{-1}\Sigma$ in $M-K$ and $x=[X]+j(y)=[X^{(k)}]+j(y)$ is represented by the 2–cycle $(nk)^{-1}(X^{(k)}+\Sigma)$ mod $\mathbb{Z}$. Applying to $X^{(k)}$ and Σ the usual cut and paste technique, we can transform their union into a simple folded surface Z such that $\operatorname{sing}(Z)=\operatorname{sing}(X^{(k)})=K$ and $i_Z=nk$. Clearly, $[Z]=x$. □

2.3 Functions θ and Θ

For an orientable 3–dimensional manifold M, we define a function $\theta=\theta_M$ from $H_2(M;\mathbb{Q}/\mathbb{Z})$ to $\mathbb{R}_+$ by

$$\theta(x)=\inf_X \frac{\chi_-(X)}{i_X}, \tag{2}$$

where $x\in H_2(M;\mathbb{Q}/\mathbb{Z})$ and X runs over all simple folded surfaces in M representing x. In particular, the class $x=0$ can be represented by the simple folded surface $X=X(n)\subset M$ with $n\neq 2$, constructed before Lemma 2.1. The equality $\chi_-(X)=0$ implies that $\theta(0)=0$.

For a simple folded surface X, denote by $-X$ the same simple folded surface with opposite orientation in its interior. The obvious equalities

$$[-X]=-[X], \quad \chi_-(-X)=\chi_-(X), \quad i_{-X}=i_X$$

imply that $\theta(-x)=\theta(x)$ for all $x\in H_2(M;\mathbb{Q}/\mathbb{Z})$.

We define a function $\Theta=\Theta_M\colon \operatorname{Tors} H_1(M)\to\mathbb{R}_+$ by

$$\Theta(u)=\inf_{x\in d^{-1}(u)}\theta(x)=\inf_X\frac{\chi_-(X)}{i_X}, \tag{3}$$

where $u\in \operatorname{Tors} H_1(M)$, X runs over all simple folded surfaces in M such that the circle $\operatorname{sing}(X)$ represents u, and $d\colon H_2(M;\mathbb{Q}/\mathbb{Z})\to H_1(M)$ is the boundary homomorphism. In (3), we can restrict ourselves to connected X. Indeed, all components of X disjoint from $\operatorname{sing}(X)$ are closed oriented surfaces. They may be removed from X without increasing $\chi_-(X)$.

The properties of θ imply that $\Theta(0) = 0$ and $\Theta(-u) = \theta(u)$ for all $u \in \operatorname{Tors} H_1(M)$. By the very definition of Θ, for all $x \in H_2(M;\mathbb{Q}/\mathbb{Z})$,

$$\theta(x) \geq \Theta(d(x)).$$

Using folded surfaces with boundary, we can similarly define relative versions

$$H_2(M,\partial M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{R}_+ \quad \text{and} \quad \operatorname{Tors} H_1(M,\partial M;\mathbb{Z}) \to \mathbb{R}_+$$

of the functions θ and Θ. We will not study them in this paper.

2.4 Constructions and examples

2.4.1 Let Σ be a closed connected 2–manifold embedded in an oriented 3–manifold M. Let $K \subset \Sigma$ be a simple closed curve such that $\Sigma - K$ has an orientation which switches to the opposite when one crosses K in Σ. (Such an orientation exists when Σ is orientable and K splits Σ into two surfaces or when Σ is nonorientable and K represents the Stiefel–Whitney class $w^1(\Sigma) \in H^1(\Sigma;\mathbb{Z}/2\mathbb{Z}) = H_1(\Sigma;\mathbb{Z}/2\mathbb{Z})$.) The orientations of M and $\Sigma - K$ induce an orientation of the normal bundle of $\Sigma - K$ in M. Keeping K and pushing $\Sigma - K$ in the corresponding normal direction, we obtain a copy Σ' of Σ such that Σ' transversely meets Σ along K. The union $X = \Sigma \cup \Sigma'$ is a simple folded surface such that $\operatorname{sing}(X) = K$ and $i_X = 4$. Then $\theta([X]) \leq (1/4)\,\chi_-(X) = (1/2)\,\chi_-(\Sigma - K)$.

For example, we can apply this construction to the projective plane $\Sigma = \mathbb{R}P^2$ in $\mathbb{R}P^3$ taking as K a projective circle on $\mathbb{R}P^2$. The resulting simple folded surface X represents the only nonzero element x of $H_2(\mathbb{R}P^3;\mathbb{Q}/\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ because $\operatorname{sing}(X)$ represents the nonzero element of $H_1(\mathbb{R}P^3) = \mathbb{Z}/2\mathbb{Z}$. The equality $\chi_-(\Sigma - K) = 0$ implies that $\theta_{\mathbb{R}P^3} = 0$ and $\Theta_{\mathbb{R}P^3} = 0$.

2.4.2 Consider the 3–dimensional lens space $M = L(p,q)$, where p,q are coprime integers with $p \geq 2$. The manifold M splits as a union of two solid tori with common boundary. It is easy to exhibit a folded surface $X \subset M$ such that $\operatorname{sing}(X)$ is the core circle of one of the solid tori and $X - \operatorname{sing}(X)$ is a disjoint union of p open 2–disks. This implies that the function Θ_M annihilates the elements of $H_1(M)$ represented by the core circles of the solid tori. Under an appropriate isomorphism $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$, these elements correspond to $1 \pmod p$ and $q \pmod p$. This implies that $\Theta_M = 0$ if $p = 2$ or $p = 3$ or $p = 5, q = 2$. For $p = 2$, we recover the previous example, since $L(2,1) = \mathbb{R}P^3$.

2.4.3 Let K be an oriented homologically trivial knot in an oriented 3–manifold N. Let M be obtained by a (p,q)–surgery on K where p,q are coprime integers with $p \geq 2$. Thus, M is obtained by cutting out a tubular neighborhood $U \subset N$ of K and gluing it back along a homeomorphism $\partial U \to \partial U$ mapping the meridian $\mu \subset \partial U$ of K onto a curve on ∂U homological to $p\mu + q\lambda$, where $\lambda \subset \partial U$ is the longitude of K homologically trivial in $N - K$. The element $u \in H_1(M)$ represented by the (oriented) core circle of the solid torus $U \subset M$ has finite order. This follows from the fact that the p–th power of the core circle is homotopic in $U \subset M$ to $\lambda \subset \partial U$. We claim that $\Theta(u) = 0$ if K is a trivial knot in N and $\Theta(u) \leq p^{-1}(2g-1)$ if K is a nontrivial knot of genus $g \geq 1$. Indeed, the longitude λ bounds in $N - \mathrm{Int}(U)$ an embedded compact connected oriented surface of genus g. This surface extends in the obvious way to a simple folded surface X in M such that $\mathrm{sing}(X)$ is the core circle of $U \subset M$ and $i_X = p$. Clearly, $\chi_-(X) = \max(2g-1, 0)$. This implies our claim. (For $p = 2$, one should "double" X along $\mathrm{sing}(X)$ as in Section 2.4.1.) As we shall see below, if K is a nontrivial fibred knot and $p \geq 4g-2$, then $\Theta(u) = p^{-1}(2g-1)$.

3 Estimates from above and semi-continuity

In this section we estimate the function $\theta = \theta_M$ from above using the Thurston norm. Throughout this section, M is a connected orientable 3–manifold (possibly, noncompact).

3.1 Comparison with the Thurston norm

Recall first the definition of the Thurston semi-norm $\|\cdot\|_M$ on $H_2(M;\mathbb{Q})$. The Poincaré duality (applied to compact submanifolds of M) implies that the abelian group $H_2(M) = H_2(M;\mathbb{Z})$ has no torsion. We shall view $H_2(M)$ as a lattice in the $\mathbb{Q}$–vector space $H_2(M;\mathbb{Q}) = \mathbb{Q} \otimes_{\mathbb{Z}} H_2(M)$. For any $x \in H_2(M;\mathbb{Q})$, there is an integer $n \geq 1$ such that $nx \in H_2(M)$. Then $\|x\|_M = n^{-1} \min_\Sigma \chi_-(\Sigma) \in \mathbb{Q}$, where Σ runs over all closed oriented embedded surfaces in M representing nx. The number $\|x\|_M$ does not depend on the choice of n and is always realized by a certain Σ. Using surfaces in M with boundary on ∂M, one similarly defines the Thurston semi-norm on $H_2(M, \partial M;\mathbb{Q})$.

Lemma 3.1 *Let j be the coefficient homomorphism $H_2(M;\mathbb{Q}) \to H_2(M;\mathbb{Q}/\mathbb{Z})$. Then $\theta(j(x)) \leq \|x\|_M$ for any $x \in H_2(M;\mathbb{Q})$.*

Proof Let Σ be a closed oriented embedded surface in M representing $nx \in H_2(M)$ with $n \geq 3$. The surface Σ is an oriented folded surface with empty singular set.

Consider a folded surface $X = X(n)$ inside a 3–ball in $M - \Sigma$, as constructed before Lemma 2.1. The union $Z = X \cup \Sigma$ is a simple folded surface representing x and $i_Z = i_X = n$. By the definition of θ,

$$\theta(j(x)) \leq n^{-1}\, \chi_-(Z) = n^{-1}\, \chi_-(\Sigma).$$

Therefore $\theta(j(x)) \leq \|x\|_M$. □

Lemma 3.2 *Let K be an oriented knot in M. Set $N = M - K$ and let ι be the inclusion homomorphism $H_2(N;\mathbb{Q}) \to H_2(M;\mathbb{Q})$. Let j be the coefficient homomorphism $H_2(M;\mathbb{Q}) \to H_2(M;\mathbb{Q}/\mathbb{Z})$. Then for any simple folded surface X in M with sing$(X) = K$ and any $y \in H_2(N;\mathbb{Q})$,*

$$\theta\big([X] + j\iota(y)\big) \leq (i_X)^{-1}\chi_-(X) + \|y\|_N. \tag{4}$$

Proof Set $n = i_X$ and let k be a positive integer such that $ky \in H_2(N) \subset H_2(N;\mathbb{Q})$. It is enough to prove that for any closed oriented surface $\Sigma \subset N$ representing nky,

$$\theta\big([X] + j\iota(y)\big) \leq n^{-1}\chi_-(X) + (nk)^{-1}\chi_-(\Sigma). \tag{5}$$

This can be reformulated in terms of the simple folded surface $X^{(k)}$ as

$$\theta\big([X^{(k)}] + j\iota(y)\big) \leq (nk)^{-1}\big(\chi_-(X^{(k)}) + \chi_-(\Sigma)\big).$$

Therefore it is enough to prove that for any simple folded surface T in M with sing$(T) = K$ and $i_T = nk$,

$$\theta([T] + j\iota(y)) \leq (nk)^{-1}\big(\chi_-(T) + \chi_-(\Sigma)\big). \tag{6}$$

Suppose first that T is *compressible* in $N = M - K$ in the sense that there is an embedded closed 2–disk $D \subset N$ such that $T \cap D = \partial D \subset T - K$ and the circle ∂D does not bound a 2–disk in $T - K$. The surgery on T along D yields a simple folded surface T_D with $[T_D] = [T]$ and $\chi_-(T_D) < \chi_-(T)$. Applying this procedure several times, we can reduce (6) to the case where T is incompressible, ie T admits no disks D as above. By the same reasoning, we can assume that Σ is incompressible in N (it may be compressible in M). The homology class $[T] + j\iota(y) \in H_2(M;\mathbb{Q}/\mathbb{Z})$ is represented by the 2–cycle $(nk)^{-1}\, T \cup \Sigma$ (mod $\mathbb{Z}$). Deforming Σ in N so that it meets T transversely and applying to $T \cup \Sigma$ the usual cut and paste technique, we can transform $T \cup \Sigma$ into a simple folded surface Z with sing$(Z) =$ sing$(T) = K$ and $i_Z = nk$. Clearly, $[Z] = [T] + j\iota(y)$. The folded surface Z may have spherical components (that is components homeomorphic to S^2) created from pieces of $T - K$ and Σ by cutting and pasting. One of these pieces will necessarily be a 2–disk D such that either $D \subset T - K$ and $D \cap \Sigma = \partial D$ or $D \subset \Sigma$ and $D \cap (T - K) = \partial D$.

In the first case the incompressibility of Σ implies that the circle ∂D bounds a disk on Σ. The surgery on Σ along D yields a surface $\Sigma_+ \approx \Sigma \amalg S^2$ homological to Σ in N. Then $\chi_-(\Sigma_+) = \chi_-(\Sigma)$ and the 1–manifold $T \cap \Sigma_+$ has one component less than $T \cap \Sigma$. Similarly, if $D \subset \Sigma$, then the incompressibility of $T - K$ implies that ∂D bounds a disk on $T - K$. The surgery on T along D yields a simple folded surface $T_+ \approx T \amalg S^2$ such that $[T_+] = [T]$, $\chi_-(T_+) = \chi_-(T)$, and the 1–manifold $T_+ \cap \Sigma$ has one component less than $T \cap \Sigma$. Continuing in this way, we can reduce ourselves to the case where Z does not have spherical components except the spherical components of T disjoint from Σ and the spherical components of Σ disjoint from T. A similar argument allows us to assume that the components of $Z - K$ are not disks except the disk components of $T - K$ disjoint from Σ. Then the additivity of the Euler characteristic under cutting and pasting implies that $\chi_-(Z) = \chi_-(T) + \chi_-(\Sigma)$. Therefore

$$\theta\big([T] + j\iota(y)\big) \leq (nk)^{-1}\chi_-(Z) = (nk)^{-1}\big(\chi_-(T) + \chi_-(\Sigma)\big).$$

This proves (6), (5), and (4). □

Theorem 3.3 *If M is compact, then there is a number $C > 0$ (depending on M) such that $\theta(x) \leq C$ for all $x \in H_2(M;\mathbb{Q}/\mathbb{Z})$.*

Proof Set $d = d_M\colon H_2(M;\mathbb{Q}/\mathbb{Z}) \to H_1(M)$. Since the group $\operatorname{Im} d = \operatorname{Tors} H_1(M)$ is finite, it is enough to prove that for every $u \in \operatorname{Tors} H_1(M)$, the values of θ on the elements of the set $d^{-1}(u)$ are bounded from above.

Consider first the case $u = 0$. Then $d^{-1}(u) = \operatorname{Im} j$ where j is the coefficient homomorphism $H_2(M;\mathbb{Q}) \to H_2(M;\mathbb{Q}/\mathbb{Z})$. We need to prove that the values of $\theta \circ j$ are bounded from above. Since M is compact, the group $H_2(M)$ is finitely generated. Pick a basis $a_1, \ldots, a_n$ in $H_2(M)$ and let $Q \subset H_2(M;\mathbb{Q})$ be the cube consisting of the vectors $r_1 a_1 + \ldots + r_n a_n$ with rational nonnegative $r_1, \ldots, r_n \leq 1$. The supremum $s = \sup_{x \in Q} \|x\|_M$ is a finite number, because the Thurston semi-norm extends to a continuous semi-norm on $H_2(M;\mathbb{R})$ and the closure of Q in $H_2(M;\mathbb{R})$ is compact. We claim that $\theta(j(x)) \leq s$ for any $x \in H_2(M;\mathbb{Q})$. Indeed, there is $a \in H_2(M)$ such that $x + a \in Q$. Then $j(x) = j(x+a)$ and $\theta(j(x)) = \theta(j(x+a)) \leq s$.

Consider now the case $u \neq 0$. Pick an oriented knot $K \subset M$ representing u and a simple folded surface X in M with $\operatorname{sing}(X) = K$. Then $d^{-1}(u) = \{[X] + j\iota(y)\}_y$ where ι is the inclusion homomorphism $H_2(M - K;\mathbb{Q}) \to H_2(M;\mathbb{Q})$ and y runs over $H_2(M - K;\mathbb{Q})$. The rest of the argument goes as in the case $u = 0$ using Lemma 3.2. □

3.2 Semi-continuity

For compact M, the group $H_2(M;\mathbb{Q}/\mathbb{Z})$ has a natural topology as follows. The image of the coefficient homomorphism $j\colon H_2(M;\mathbb{Q}) \to H_2(M;\mathbb{Q}/\mathbb{Z})$ can be identified with the quotient $H_2(M;\mathbb{Q})/H_2(M)$. Provide $\mathrm{Im}\,(j)$ with the quotient topology induced by the standard topology in the finite dimensional $\mathbb{Q}$–vector space $H_2(M;\mathbb{Q})$. This extends to a topology in $H_2(M;\mathbb{Q}/\mathbb{Z})$ by declaring a set $U \subset H_2(M;\mathbb{Q}/\mathbb{Z})$ open if $(a+U)\cap \mathrm{Im}\,(j)$ is open in $\mathrm{Im}\,(j)$ for all $a \in H_2(M;\mathbb{Q}/\mathbb{Z})$. Recall that an $\mathbb{R}$–valued function f on a topological space A is *upper semi-continuous* if for any point $a \in A$ and any real $\varepsilon > 0$, there is a neighborhood $U \subset A$ of a such that $f(U) \subset (-\infty, f(a)+\varepsilon)$.

Lemma 3.4 *For compact M, the function $\theta = \theta_M$ is upper semi-continuous.*

Proof Let $a \in H_2(M;\mathbb{Q}/\mathbb{Z})$ and $\varepsilon > 0$. Let X be a simple folded surface in M representing a and such that $(i_X)^{-1}\chi_-(X) \le \theta(a) + \varepsilon/2$. Set $K = \mathrm{sing}(X)$ and $N = M - K$. Let $\iota\colon H_2(N;\mathbb{Q}) \to H_2(M;\mathbb{Q})$ be the inclusion homomorphism. Put

$$V = \{y \in H_2(N;\mathbb{Q}) \mid \|y\|_N < \varepsilon/2\}.$$

The set V is open in $H_2(N;\mathbb{Q})$ since the Thurston norm is continuous. The set $\iota(V)$ is open in $H_2(M;\mathbb{Q})$ since ι is an epimorphism. The set $j\iota(V)$ is open in $\mathrm{Im}\,(j)$ by definition of the topology in $\mathrm{Im}\,(j)$. Finally, the set $U = a + j\iota(V)$ is an open neighborhood of a in $H_2(M;\mathbb{Q}/\mathbb{Z})$ by definition of the topology in $H_2(M;\mathbb{Q}/\mathbb{Z})$. By (4), we have $\theta(U) \subset (-\infty, \theta(a)+\varepsilon)$. Hence θ is upper semi-continuous. □

4 Estimates from below: the case $b_1 \ge 1$

In this section we give an estimate from below for the functions $\theta = \theta_M$ and $\Theta = \Theta_M$ of a 3–manifold M with nonzero first Betti number $b_1(M)$. We begin with preliminaries on group rings and abelian torsions of 3–manifolds.

4.1 Preliminaries

Let H be a finitely generated abelian group written in multiplicative notation. Any element a of the group ring $\mathbb{Q}[H]$ expands uniquely in the form $a = \sum_{h\in H} a_h h$, where $a_h \in \mathbb{Q}$ and $a_h = 0$ for all but finitely many h. We say that an element $h \in H$ is *a–basic* if $a_h \neq 0$. The (finite) set of a–basic elements of H is denoted B_a. The element $\sum_{h\in \mathrm{Tors}\, H} h$ of $\mathbb{Q}[H]$ will be denoted Σ_H. Clearly, $B_{\Sigma_H} = \mathrm{Tors}\, H$.

The classical ring of quotients of $\mathbb{Q}[H]$ that is, the (commutative) ring obtained by inverting all nonzero-divisors of $\mathbb{Q}[H]$ is denoted $Q(H)$. It is known that $\mathbb{Q}[H]$ splits as a direct sum of domains. Therefore $Q(H)$ splits as a direct sum of fields and the natural ring homomorphism $\mathbb{Q}[H] \to Q(H)$ is an embedding. We identify $\mathbb{Q}[H]$ with its image under this embedding. Note that if H is a finite abelian group, then $Q(H) = \mathbb{Q}[H]$.

Let M be a compact connected 3–manifold. From now on, we use multiplicative notation for the group operation in $H = H_1(M)$. In particular, the neutral element of H is denoted 1. The manifold M gives rise to a *maximal abelian torsion* $\tau(M)$ which is an element of $Q(H)$ defined up to multiplication by -1 and elements of H (see Turaev [11] and Nicolaescu [7]). If $b_1(M) \geq 2$, then all representatives of $\tau(M)$ belong to $\mathbb{Z}[H] \subset \mathbb{Q}[H] \subset Q(H)$. We express this by writing $\tau(M) \in \mathbb{Z}[H]$. If $b_1(M) = 1$ and $\partial M \neq \varnothing$, then $\tau(M) \in \mathbb{Z}[H] + \Sigma_H \cdot Q(H)$. This implies that $(h-1)\,\tau(M) \in \mathbb{Z}[H]$ for all $h \in \operatorname{Tors} H$ (indeed $(h-1)\Sigma_H = 0$).

If M is oriented and $b_1(M) \geq 2$, then the Thurston semi-norm $\|\cdot\|_M$ on $H_2(M, \partial M; \mathbb{Q})$ can be estimated in terms of $\tau(M)$ as follows (see [11]): for any $s \in H_2(M, \partial M; \mathbb{Q})$ and any representative $a \in \mathbb{Z}[H]$ of $\tau(M)$,

$$\|s\|_M \geq \max_{h,h' \in B_a} |h \cdot s - h' \cdot s|, \tag{7}$$

where $h \cdot s \in \mathbb{Z}$ is the intersection index of h and s. Note that the right hand side of (7) does not depend on the choice of a in $\tau(M)$.

4.2 An estimate for θ_M

The function θ will be estimated in terms of spans of subsets of $\mathbb{Q}/\mathbb{Z}$. The *span* $\operatorname{spn}(A)$ of a finite set $A \subset \mathbb{Q}/\mathbb{Z}$ is a rational number defined as the minimal length of an interval in $\mathbb{Q}/\mathbb{Z}$ containing A, that is the minimal rational number $t \geq 0$ such that for some $r \in \mathbb{Q}$, the projection of the set $[r, r+t] \cap \mathbb{Q}$ into $\mathbb{Q}/\mathbb{Z}$ contains A. Clearly, $1 > \operatorname{spn}(A) \geq 0$ and $\operatorname{spn}(A) = 0$ if and only if A is empty or has only one element.

Given an oriented 3–manifold M and a homology class $x \in H_2(M; \mathbb{Q}/\mathbb{Z})$, we set for any $a \in \mathbb{Q}[H_1(M)]$,

$$\operatorname{spn}_x(a) = \operatorname{spn}(\{h \cdot x\}_{h \in B_a}),$$

where $h \cdot x \in \mathbb{Q}/\mathbb{Z}$ is the intersection index of h and x. Clearly, $1 > \operatorname{spn}_x(a) \geq 0$.

Theorem 4.1 *Let M be a compact connected oriented 3–manifold with $b_1(M) \geq 1$. Set $H = H_1(M)$ and let $\tau \in Q(H)$ be a representative of the torsion $\tau(M)$. Let $x \in H_2(M; \mathbb{Q}/\mathbb{Z})$ and $u = d_M(x) \in H$. Then $(u-1)\,\tau \in \mathbb{Z}[H]$ and*

$$\theta_M(x) \geq \operatorname{spn}_x((u-1)\,\tau). \tag{8}$$

Proof If $b_1(M) \geq 2$, then $\tau \in \mathbb{Z}[H]$ and $(u-1)\,\tau \in \mathbb{Z}[H]$. The inclusion $u \in \operatorname{Tors} H$ and the remarks in Section 4.1 imply that $(u-1)\,\tau \in \mathbb{Z}[H]$ for $b_1(M) = 1$ as well.

We prove (8). Let X be a simple folded surface in M representing x. The knot $\operatorname{sing}(X) \subset M$ endowed with orientation induced from the one on X represents the class $u \in \operatorname{Tors} H$. Let E be the exterior of this knot in M. The homological sequence of the pair (M, E) and the inclusion $u \in \operatorname{Tors} H$ imply that $b_1(E) \geq b_1(M) + 1 \geq 2$. Therefore $\tau(E) \in \mathbb{Z}[H_1(E)]$. Pick a representative $a \in \mathbb{Z}[H_1(E)]$ of $\tau(E)$. Denote by ι the inclusion homomorphism $H_1(E) \to H_1(M) = H$ and denote by ι_* the induced ring homomorphism $\mathbb{Z}[H_1(E)] \to \mathbb{Z}[H]$. By [11, Theorem VII.1.4], we have $\iota_*(a) = (u-1)\,b$ where b is a representative of $\tau(M)$. Note that the right hand side of (8) does not depend on the choice of τ in $\tau(M)$. Therefore without loss of generality we can assume that $\tau = b$.

Deforming, if necessary, X in M, we can assume that $S = X \cap E$ is the complement in X of a regular neighborhood of $\operatorname{sing}(X)$. Then S is a proper surface in E and $\chi_-(X) = \chi_-(S)$. The orientation of $\operatorname{Int}(X)$ induces an orientation of S. The oriented surface S represents a relative homology class $s \in H_2(E, \partial E)$. By (7),

$$\chi_-(X) = \chi_-(S) \geq \max_{h,h' \in B_a} |h \cdot s - h' \cdot s|,$$

where $B_a \subset H_1(E)$ is the set of a–basic elements. Let $r \in \mathbb{Q}$ be the minimal element of the set $\{h \cdot s\}_{h \in B_a}$. Then

$$\{h \cdot s\}_{h \in B_a} \subset [r, r + \chi_-(X)].$$

Denote the projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ by π. Observe that for any $h \in H_1(E)$,

$$\iota(h) \cdot x = \pi\left(\frac{h \cdot s}{i_X}\right).$$

Therefore $$\{\iota(h) \cdot x\}_{h \in B_a} \subset \pi\left(\left[\frac{r}{i_X}, \frac{r + \chi_-(X)}{i_X}\right]\right).$$

The equality $\iota_*(a) = (u-1)\,\tau$ implies that $B_{(u-1)\,\tau} \subset \iota(B_a)$. Hence

$$\{g \cdot x\}_{g \in B_{(u-1)\,\tau}} \subset \{\iota(h) \cdot x\}_{h \in B_a} \subset \pi\left(\left[\frac{r}{i_X}, \frac{r + \chi_-(X)}{i_X}\right]\right).$$

Therefore $$\operatorname{spn}_x((u-1)\,\tau) \leq (i_X)^{-1}\,\chi_-(X).$$

Since this holds for all simple folded surfaces X representing x, we have (8). □

4.3 An estimate for Θ_M

Let M and H be as in Theorem 4.1. To estimate the function $\Theta_M\colon \operatorname{Tors} H \to \mathbb{Q}/\mathbb{Z}$, we need the linking form $L_M\colon \operatorname{Tors} H \times \operatorname{Tors} H \to \mathbb{Q}/\mathbb{Z}$ of M. It is defined by $L_M(h,g) = h \cdot x \in \mathbb{Q}/\mathbb{Z}$ where x is an arbitrary element of $H_2(M;\mathbb{Q}/\mathbb{Z})$ mapped to g by the boundary homomorphism $d\colon H_2(M;\mathbb{Q}/\mathbb{Z}) \to H$. The pairing L_M is well defined, bilinear, and symmetric.

Given $u \in \operatorname{Tors} H$ and $a \in \mathbb{Q}[H]$, set

$$\operatorname{spn}_u(a) = \operatorname{spn}(\{L_M(h,u)\}_{h \in B_a \cap \operatorname{Tors} H}).$$

Clearly, $\operatorname{spn}_x(a) \geq \operatorname{spn}_{d(x)}(a)$ for any $x \in H_2(M;\mathbb{Q}/\mathbb{Z})$ and any $a \in \mathbb{Q}[H]$. This and Theorem 4.1 imply that, under the conditions of this theorem,

$$\Theta_M(u) \geq \operatorname{spn}_u((u-1)\,\tau), \tag{9}$$

for any $u \in \operatorname{Tors} H$ and any representative τ of $\tau(M)$. Generally speaking, the right-hand side of (9) depends on the choice of τ.

Remark Estimate (7) strengthens the McMullen estimate [6] of the Thurston norm via the Alexander polynomial. For recent more general estimates of this type, see Friedl [3].

5 Estimates from below: the case of $\mathbb{Q}$–homology spheres

For $\mathbb{Q}$–homology spheres, the functions θ and Θ contain the same information and it is enough to give an estimate for Θ. We begin with preliminaries on refined torsions and $\mathbb{Q}$–homology spheres, referring for details to [11, Chapters I and X].

5.1 Refined torsions

The maximal abelian torsion $\tau(M)$ of a compact connected 3–manifold M admits a refinement $\tau(M,e,\omega) \in Q(H_1(M))$ depending on an orientation ω in the vector space $H_*(M;\mathbb{Q}) = \oplus_{i\geq 0} H_i(M;\mathbb{Q})$ and an Euler structure e on M. An Euler structure on M is determined by a nonsingular vector field on M directed outside on ∂M. Two such vector fields determine the same Euler structure if for a point $x \in \operatorname{Int}(M)$, the restrictions of these fields to $M - \{x\}$ are homotopic in the class of nonsingular vector field on $M - \{x\}$ directed outside on ∂M. The set of Euler structures on M is denoted $\operatorname{Eul}(M)$. This set admits a canonical free transitive action of the group $H_1(M)$. The torsion $\tau(M,e,\omega)$ satisfies $\tau(M,he,\pm\omega) = \pm h\,\tau(M,e,\omega)$ for any $e \in \operatorname{Eul}(M), h \in H_1(M)$. The unrefined torsion $\tau(M)$ is just the set $\{\pm\tau(M,e,\omega)\}_{e \in \operatorname{Eul}(M)}$. If $\partial M = \varnothing$, then the set $\operatorname{Eul}(M)$ can be identified with the set of Spin^c–structures on M.

5.2 Homology spheres

Let M be an oriented 3–dimensional $\mathbb{Q}$–homology sphere. Denote ω_M the orientation in $H_*(M;\mathbb{Q}) = H_0(M;\mathbb{Q}) \oplus H_3(M;\mathbb{Q})$ determined by the following basis: (the homology class of a point, the fundamental class of M).

The group $H = H_1(M)$ is finite and the linking form $L_M\colon H \times H \to \mathbb{Q}/\mathbb{Z}$ is nondegenerate in the sense that the adjoint homomorphism $H \to \mathrm{Hom}(H, \mathbb{Q}/\mathbb{Z})$ is an isomorphism. Recall that we use multiplicative notation for the group operation in H. Every Euler structure $e \in \mathrm{Eul}(M)$ determines a torsion $\tau(M, e, \omega_M) \in Q(H) = \mathbb{Q}[H]$. The linking form L_M can be computed from this torsion by

(10) $$L_M(h,g) = -\pi\Big(\big((1-h)(1-g)\,\tau(M,e,\omega_M)\big)_1\Big) \in \mathbb{Q}/\mathbb{Z}$$

for all $h, g \in H$, where π is the projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ and for any $a \in \mathbb{Q}[H]$, the symbol $a_1 \in \mathbb{Q}$ denotes the coefficient of the neutral element $1 \in H$ in the expansion of a as a formal linear combination of elements of H with rational coefficients. The Euler structure e determines a function $q_e\colon H \to \mathbb{Q}/\mathbb{Z}$ by

(11) $$q_e(u) = \pi\Big(\big((1-u)\,\tau(M,e,\omega_M)\big)_1\Big),$$

for any $u \in H$. It follows from (10) and (11) that q_e is quadratic in the sense that $q_e(hg) = q_e(h) + q_e(g) + L_M(h,g)$ for all $h, g \in H$. Formula (11) also implies that

(12) $$q_{he}(u) = q_e(u) + L_M(h,u),$$

for any $h \in H$.

If $u \in H$ has order n (ie n is the minimal positive integer such that $u^n = 1$), then by [11, Section X.4.3] there is a unique residue $K(e,u) \in \mathbb{Z}/2n\mathbb{Z}$ such that

(13) $$q_e(u) = \frac{K(e,u)}{2n} + \frac{1}{2} \pmod{\mathbb{Z}}.$$

Formula (12) implies that the residue $K(e,u) \pmod 2$ does not depend on e. We say that u is *even* if this residue is 0 and *odd* if it is 1.

Every homology class $u \in H$ gives rise to a group

$$G = G_u = \{g \in H \mid L_M(u,g) = 0\} \subset H.$$

The nondegeneracy of L_M implies that the quotient H/G is a finite cyclic group whose order is equal to the order, n, of u in H. Moreover, there is an element $v = v_u \in H$ such that $L_M(u,v) = n^{-1} \pmod{\mathbb{Z}}$. Such v is determined by u uniquely up to multiplication by elements of G. The inclusion $v^n \in G$ implies that the order, p, of v is divisible by n (in particular, $p \geq n$). Set

$$\alpha_v = \frac{1+2v+3v^2+\ldots+pv^{p-1}}{p} - \frac{p+1}{2}\cdot\frac{1+v+\ldots+v^{p-1}}{p}. \tag{14}$$

This element of $\mathbb{Q}[H]$ can be uniquely characterized by the following property: if φ is any ring homomorphism from $\mathbb{Q}[H]$ to a field, then $\varphi(v)=1 \Rightarrow \varphi(\alpha_v)=0$ and $\varphi(v)\neq 1 \Rightarrow \varphi(\alpha_v)=(\varphi(v)-1)^{-1}$. In [11] we used the notation $(v-1)^{-1}_{\mathrm{par}}$ for α_v.

Theorem 5.1 *Let M be an oriented 3–dimensional $\mathbb{Q}$–homology sphere. Let u be an element of $H=H_1(M)$ of order $n\geq 1$. Set $G=\{g\in H \mid L_M(u,g)=0\}$ and $\Sigma_G=\sum_{g\in G} g \in \mathbb{Z}[G]\subset\mathbb{Z}[H]$. Pick any $v\in H$ such that $L_M(u,v)=n^{-1} \pmod 1$. For $e\in \mathit{Eul}(M)$, set*

$$a_e(u)=(u-1)\,\tau(M,e,\omega_M)-\frac{v^{K(e,u)/2}(v+1)}{2}\alpha_v\,\Sigma_G\in\mathbb{Q}[H],$$

if u is even and

$$a_e(u)=(u-1)\,\tau(M,e,\omega_M)-v^{(K(e,u)+1)/2}\alpha_v\,\Sigma_G\in\mathbb{Q}[H],$$

if u is odd. Then for any $e\in \mathit{Eul}(M)$,

$$\Theta_M(u)\geq \mathit{spn}_u(a_e(u))=\mathit{spn}(\{L_M(h,u)\}_{h\in B_{a_e(u)}}).$$

Proof If u is even (resp. odd), then $K(e,u)\in\mathbb{Z}_{2n}$ is even (resp. odd). Therefore the power of v in the definition of $a_e(u)$ is well defined up to multiplication by v^n. However, $v^n\in G$ and $v^n\Sigma_G=\Sigma_G$. Therefore the right hand sides of the formulas for $a_e(u)$ are well defined. If v' is another element of H such that $L_M(u,v')=n^{-1} \pmod 1$, then $v'\in vG$ and $v^k\Sigma_G=(v')^k\Sigma_G$ for all $k\in\mathbb{Z}$. Therefore $a_e(u)$ does not depend on the choice of v. It is easy to see that $a_{he}(u)=h\,a_e(u)$ for all $h\in H$. Therefore the number $\mathrm{spn}_u(a_e(u))$ does not depend on e.

Consider a simple folded surface $X\subset M$ which represents the 2–homology class $x=d_M^{-1}(u)\in H_2(M;\mathbb{Q}/\mathbb{Z})$. The knot $K=\mathrm{sing}(X)$ with orientation induced from the one on X represents $u\in H_1(M)$. Let E be the exterior of K in M. Clearly $b_1(E)=1$. Fix an orientation ω in $H_*(E;\mathbb{Q})$ and an Euler structure e_K on E. The torsion $\tau(E,e_K,\omega)\in Q(H_1(E))$ can be canonically expanded as a sum of a certain $[\tau]=[\tau](E,e_K,\omega)\in\mathbb{Q}[H_1(E)]$ with an element of $Q(H_1(E))$ given by an explicit formula using solely ω and the Chern class of e_K [11, Section II.4.5]. The inclusion homomorphism $\mathbb{Q}[H_1(E)]\to\mathbb{Q}[H_1(M)]$ sends $[\tau]$ to $\pm a_e(u)$ for some $e\in\mathrm{Eul}(M)$ [11, Formula X.4.d]. The inequality (7) holds for any $s\in H_2(E,\partial E;\mathbb{Q})$ and $a=[\tau]$

[11, Chapter IV]. The rest of the argument goes as the proof of Theorem 4.1 with τ replaced by $[\tau]$. This gives $(i_X)^{-1}\chi_-(X) \geq \mathrm{spn}_x(a_e(u)) = \mathrm{spn}_u(a_e(u))$. Since this holds for all X representing x, we have $\Theta_M(u) = \theta_M(x) \geq \mathrm{spn}_u(a_e(u))$. □

Remarks 1. Let $\frac{1}{2}\mathbb{Z}$ be the additive group of integers and half-integers. Then in Theorem 5.1, $a_e(u) \in \mathbb{Z}[H]$ if u is even and $a_e(u) \in \frac{1}{2}\mathbb{Z}[H]$ if u is odd. This follows from the proof of this theorem and the inclusion $[\tau] \in \mathbb{Z}[H_1(E)]$ if u is even and $[\tau] \in \frac{1}{2}\mathbb{Z}[H_1(E)]$ if u is odd.

2. It is proven by Deloup and Massuyeau [2] that the function $q_e\colon H \to \mathbb{Q}/\mathbb{Z}$ derived from the torsion coincides with the quadratic function defined geometrically by Looijenga and Wahl [5] and Deloup [1].

6 Examples

6.1 Lens spaces

The computation of the abelian torsions for the lens space $M = L(p,q)$ goes back to K Reidemeister. See, for instance, [10] for an introduction to the theory of torsions. Let t, t^q be the generators of $H = H_1(M)$ represented by the core circles of the two solid tori forming M. For an appropriate choice of an orientation on M and an Euler structure e on M, we have $\tau(M,e,\omega_M) = \alpha_t\,\alpha_{t^q}$, where $\alpha_v \in \mathbb{Q}[H]$ is defined by (14) for any $v \in H$. This allows us to compute $a_e(u)$ for any $u \in H$ and to apply Theorem 5.1. We give here a few examples.

Consider the lens space $M = L(5,1)$. By Section 2.3 and Section 2.4.2, Θ satisfies $\Theta(t^4) = \Theta(t) = \Theta(1) = 0$ and $\Theta(t^2) = \Theta(t^3)$. We show that $\Theta(t^2) \geq 1/5$. We have

$$\alpha_t = \frac{-2 - t + t^3 + 2t^4}{5}.$$

Then
$$\tau = \tau(M,e,\omega_M) = \alpha_t^2 = \frac{t + t^2 - 2t^4}{5}.$$

A direct computation shows that

$$L_M(t,t) = (-(1-t)^2\tau)_1 = 1/5, \quad q_e(t^2) = ((1-t)\,\tau)_1 = 0.$$

Note that $u = t^2$ has order 5 in H. From (13), we obtain that $K(e,u) = 5 \pmod{10}$. Therefore u is odd. The associated group G_u is trivial, $v = v_u = t^3$, and

$$a_e(u) = (u-1)\,\tau - v^3\alpha_v = t^4 - t.$$

Since $L_M(t^4, u) = 3/5 \pmod 1$ and $L_M(t, u) = 2/5 \pmod 1$, the span of the set $\{L_M(h, u)\}_{h \in B_{a_e(u)}}$ is equal to $1/5$. By Theorem 5.1, $\Theta(t^2) \geq 1/5$.

Consider the lens space $M = L(6, 1)$. Then

$$\alpha_t = \frac{-5 - 3t - t^2 + t^3 + 3t^4 + 5t^5}{12},$$
$$\tau = \alpha_t^2 = \frac{-5 + 13t + 19t^2 + 13t^3 - 5t^4 - 35t^5}{72},$$

and $L_M(t, t) = 1/6$. For $u = t^2$, the computations similar to the ones above give $q_e(u) = 0 \pmod 1$, $K(e, u) = 3 \pmod 6$, $G_u = \{1, t^3\}$, $v_u = t$, and $a_e(u) = t^5 - t$. Theorem 5.1 yields $\Theta(t^2) \geq 1/3$. For $u = t^3$, we similarly obtain $q_e(u) = 3/4 \pmod 1$, $K(e, u) = 1 \pmod 4$, $G_u = \{1, t^2, t^4\}$, $v_u = t$, and $a_e(u) = t^5 - t^2$. Theorem 5.1 yields $\Theta(t^3) \geq 1/2$.

6.2 Surgeries on knots

Let L be an oriented knot in an oriented 3–dimensional $\mathbb{Z}$–homology sphere N. Let M be the closed oriented 3–manifold obtained by surgery on N along L with framing $p \geq 2$. Let $u \in H = H_1(M)$ be the homology class of the meridian of L whose linking number with L is $+1$. Clearly, H is a cyclic group of order p with generator u and $L_M(u, u) = p^{-1} \pmod 1$. We explain now how to estimate $\Theta(u)$ in terms of the Alexander polynomial of L. We will see that in some cases this estimate is exact.

Recall that the *span* $\operatorname{spn}(\Delta)$ of a nonzero Laurent polynomial $\Delta = \sum_i a_i t^i \in \mathbb{Z}[t^{\pm 1}]$ is the number $\max\{i \mid a_i \neq 0\} - \min(\{i \mid a_i \neq 0\})$. Let $\Delta = \Delta_L(t)$ be the Alexander polynomial of L normalized so that $\Delta(t^{-1}) = \Delta(t)$ and $\Delta(1) = 1$. Expand $\Delta(t) = 1 + (t-1)\,\beta(t)$ where $\beta(t) \in \mathbb{Z}[t^{\pm 1}]$. We claim the expression $a_e(u) \in \mathbb{Q}[H]$ defined in Theorem 5.1 is equal to $\beta(u)$ for an appropriate Euler structure e on M. By Theorem 5.1, this will imply that $\Theta(u) \geq \operatorname{spn}_u(\beta(u))$. For example, if $p \geq 2\operatorname{spn}(\beta)$, then $\operatorname{spn}_u(\beta(u)) = p^{-1}\operatorname{spn}(\beta) = p^{-1}(\operatorname{spn}(\Delta) - 1)$. Therefore $\Theta(u) \geq p^{-1}(\operatorname{spn}(\Delta) - 1)$. On the other hand, by Section 2.4.3, $\Theta(u) \leq p^{-1}(2g - 1)$, where g is the genus of K. In particular, if $\operatorname{spn}(\Delta) = 2g > 0$ (for instance, if K is a nontrivial fibred knot) and $p \geq 4g - 2$, then $\Theta(u) = p^{-1}(2g - 1)$.

We now verify the claim above. Set $\tau = \alpha_u^2\, \Delta(u) \in \mathbb{Q}[H]$. It is easy to deduce from the multiplicativity of the torsions that $\tau(M, e, \omega_M) = \tau$ for a certain orientation on M and a certain Euler structure e on M (for details, see [11, Formula X.5.e]). Set $\sigma = 1 + u + u^2 + \cdots + u^{p-1} \in \mathbb{Z}[H]$. Clearly, $\sigma u^k = \sigma$ for any integer k. Therefore for any integer 1–variable polynomial f, the product $\sigma f(u)$ is equal to $\operatorname{aug}(f)\,\sigma$

where $\operatorname{aug}(f) = f(1)$ is the sum of coefficients of f. Since $\operatorname{aug}(\alpha_u) = 0$, we have $\sigma\alpha_u = 0$. A direct computation shows that $(1-u)\alpha_u = \sigma/p - 1$. Hence

$$\begin{aligned}(1-u)\,\tau &= (1-u)\,\alpha_u^2\,\Delta(u) = (\sigma/p-1)\,\alpha_u\,\Delta(u) = -\alpha_u\,\Delta(u)\\ &= -\alpha_u + \alpha_u\,(1-u)\,\beta(u) = -\alpha_u + (\sigma/p-1)\,\beta(u) = -\alpha_u - \beta(u),\end{aligned}$$

where we use the equality $\operatorname{aug}(\beta) = 0$ which follows from the symmetry of Δ. Thus,

$$q_e(u) = ((1-u)\tau)_1 = -(\alpha_u)_1 = (p-1)/2p \pmod 1.$$

Formula (13) implies that $K(e,u) = -1 \pmod{2p}$. In particular, u is odd.

We also have

$$(1-u)^2\tau = (1-u)(-\alpha_u - \beta(u)) = 1 - \sigma/p - (1-u)\beta(u).$$

Hence
$$L_M(u,u) = -((1-u)^2\tau)_1 = p^{-1} \pmod 1.$$

This shows that the orientation of M chosen so that $\tau(M,e,\omega_M) = \tau$ is actually the one induced from the orientation on N. The equality $L_M(u,u) = p^{-1} \pmod 1$ implies that $v_u = u$ and $G_u = 1$. We conclude that

$$a_e(u) = (u-1)\,\tau - \alpha_u = \alpha_u + \beta(u) - \alpha_u = \beta(u).$$

6.3 Surgeries on 2–component links

Let M be a closed oriented 3–manifold obtained by surgery on a 2–component oriented link $L = L_1 \cup L_2$ in an oriented 3–dimensional $\mathbb{Z}$–homology sphere N. Suppose that the linking number of L_1, L_2 in N is 0, the framing of L_1 is $p \neq 0$, and the framing of L_2 is 0. Then $H = H_1(M) = (\mathbb{Z}/p\mathbb{Z})u_1 \oplus \mathbb{Z}u_2$, where $u_i \in H$ is the homology class of the meridian of L_i whose linking number with L_i is $+1$, for $i = 1, 2$. The Alexander polynomial of L has the form

$$\Delta_L(t_1,t_2) = f(t_1,t_2)(t_1-1)(t_2-1)$$

for some Laurent polynomial $f(t_1,t_2) \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]$. Both Δ_L and f are defined only up to multiplication by -1 and monomials on t_1, t_2. By [11, Formula VIII.4.e], the torsion $\tau(M)$ is represented by

$$\tau = f(u_1,u_2) \pm \Delta_{L_2}(u_2)\,u_2^n\,(u_2-1)^{-2}\,\Sigma_H \in Q(H)$$

for an appropriate sign $\pm$ and an integer n, both depending on the choice of f. Here Δ_{L_2} is the Alexander polynomial of L_2 normalized as in Section 6.2. Pick

$x \in H_2(M;\mathbb{Q}/\mathbb{Z})$ and set $u = d(x) \in \operatorname{Tors} H$. Since $(u-1)\Sigma_H = 0$, Theorem 4.1 implies that

$$\theta(x) \geq \operatorname{spn}_x((u-1)\, f(u_1, u_2)). \tag{15}$$

For sufficiently big p, the span on the right hand side does not depend on p.

Note another curious phenomenon. Suppose for simplicity that $f(t_1,t_2) = 1$ (a constant polynomial). Then $\theta(x) \geq \operatorname{spn}_x(u-1)$. If $u = d(x) \neq 1$, then the set $B_{u-1} \subset H$ consists of two elements $u, 1$ and

$$\operatorname{spn}_x(u-1) = \operatorname{spn}(\{u \cdot x, 0\}) = \operatorname{spn}(\{L_M(u,u), 0\}).$$

For $u = u_1^k$ with $k \in \{0, 1, ..., n-1\}$, we have $L_M(u,u) = k^2/n \pmod 1$. For $k < \sqrt{n/2}$, we obtain $\operatorname{spn}(\{L_M(u,u), 0\}) = k^2/n$. Thus $\Theta(u_1^k) \geq k^2/n$. This suggests that the number $\Theta(u_1^k)$, considered as a function of k, may behave like a quadratic function for small values of k.

7 Miscellaneous

7.1 Quasi-simple folded surfaces

One can use a larger class of folded surfaces to represent 2–homology classes. Let us call a folded surface X *quasi-simple* if it is oriented, $\operatorname{sing}(X) \neq \varnothing$, and the indices of all components of $\operatorname{sing}(X)$ in X are equal to each other and nonzero. Denote the common value of these indices i_X. In particular, simple folded surfaces are quasi-simple.

For a quasi-simple folded surface X in a 3–manifold M, the 2–chain $(i_X)^{-1} X$ is a 2–cycle mod $\mathbb{Z}$ representing a homology class $[X] \in H_2(M;\mathbb{Q}/\mathbb{Z})$. We claim that

$$\theta([X]) \leq i_X^{-1}\, \chi_-(X) + b_0(\operatorname{sing}(X)) - 1, \tag{16}$$

where $b_0(\operatorname{sing}(X))$ is the number of components of $\operatorname{sing}(X)$. Indeed, X can be modified in a neighborhood of $\operatorname{sing}(X)$ so that each point of $\operatorname{sing}(X)$ is adjacent to exactly i_X local branches of $\operatorname{Int}(X)$ (which then induce the same orientation on $\operatorname{sing}(X)$). Let Γ be a graph with two vertices and i_X edges connecting these vertices. Given an embedded arc in M with endpoints on different components of $\operatorname{sing}(X)$ and with interior in $M - X$, we can modify X by cutting it out along $\operatorname{sing}(X)$ near the endpoints and gluing in $\Gamma \times [0,1]$ along the arc. This gives a quasi-simple folded surface, Z, such that

$$b_0(\operatorname{sing}(Z)) = b_0(\operatorname{sing}(X)) - 1, \quad i_Z = i_X, \quad [Z] = [X], \quad \text{and} \quad \chi_-(Z) \leq \chi_-(X) + i_X.$$

Modifying X in this way, we can reduce ourselves to the case where $\mathrm{sing}(X)$ is connected. In this case (16) follows from the definition of θ. It may happen that there are no distinct components of $\mathrm{sing}(X)$ connected by an arc with interior in $M-X$. This occurs if each arc joining distinct components of $\mathrm{sing}(X)$ has to cross the closed 2–manifold X_0 formed by the components of X disjoint from $\mathrm{sing}(X)$. To circumvent this obstruction, we first modify X_0 so that $X-X_0$ is contained in a connected component of $M-X_0$; cf [11, p 60].

Formula (16) implies that for any $x \in H_2(M;\mathbb{Q}/\mathbb{Z})$,

$$\theta(x) = \inf_X \left(\frac{\chi_-(X)}{i_X} + b_0(\mathrm{sing}(X)) \right) - 1, \tag{17}$$

where X runs over all quasi-simple folded surfaces in M representing x.

7.2 Coverings

Let M be a compact oriented 3–manifold and $p\colon \widetilde{M} \to M$ be an n–fold (unramified) covering. Let $p^*\colon H_2(M;\mathbb{Q}/\mathbb{Z}) \to H_2(\widetilde{M};\mathbb{Q}/\mathbb{Z})$ be the following composition of the duality isomorphisms and the pull back

$$H_2(M;\mathbb{Q}/\mathbb{Z}) \cong H^1(M,\partial M;\mathbb{Q}/\mathbb{Z}) \to H^1(\widetilde{M},\partial\widetilde{M};\mathbb{Q}/\mathbb{Z}) \cong H_2(\widetilde{M};\mathbb{Q}/\mathbb{Z}).$$

Then for any $x \in H_2(M;\mathbb{Q}/\mathbb{Z})$,

$$\theta_{\widetilde{M}}(p^*(x)) + 1 \leq n(\theta_M(x)+1).$$

This follows from (17) and the fact that if a simple folded surface X in M represents x, then $p^{-1}(X) \subset \widetilde{M}$ is a quasi-simple folded surface representing $p^*(x)$.

7.3 Norms associated with links

A link L in an oriented 3–manifold M determines a semi-norm $\|\cdot\|_{M,L}$ on $H_2(M;\mathbb{Q})$ as follows. Let $U \subset M$ be a regular neighborhood of L and $E = \overline{M-U}$ the exterior of L. We can embed $H_2(M;\mathbb{Q})$ into $H_2(E,\partial E;\mathbb{Q})$ via the inclusion homomorphism

$$H_2(M;\mathbb{Q}) \hookrightarrow H_2(M,L;\mathbb{Q}) \cong H_2(M,U;\mathbb{Q}) \cong H_2(E,\partial E;\mathbb{Q}).$$

Restricting the Thurston semi-norm on $H_2(E,\partial E;\mathbb{Q})$ to $H_2(M;\mathbb{Q})$, we obtain the semi-norm $\|\cdot\|_{M,L}$. The arguments as above allow us to estimate the latter semi-norm from below for compact M. Namely, if L has $m \geq 1$ components and $h_1,\dots,h_m$ are their homology classes in $H = H_1(M)$, then

$$\|x\|_{M,L} \geq \mathrm{spn}_x\Big(\prod_{i=1}^m (h_i - 1)\,\tau\Big)$$

for any $x \in H_2(M;\mathbb{Q})$ and any $\tau \in Q(H)$ representing $\tau(M)$ in the case $b_1(M) \geq 2$ and representing $[\tau](M)$ in the case $b_1(M) = 1$. A similar construction can be used to derive a function on $H_2(M;\mathbb{Q}/\mathbb{Z})$ from the function θ on $H_2(E,\partial E;\mathbb{Q}/\mathbb{Z})$. It would be interesting to see whether these semi-norms and functions may be used to distinguish nonisotopic links.

7.4 Open questions

Is the infimum in (2) realizable by a simple folded surface? Does θ take only rational values? A positive answer to the first question certainly implies a positive answer to the second one. Similar questions can be asked for Θ.

It would be interesting to compute the function Θ for the lens spaces. Is it true that for the lens spaces, the inequality in Theorem 5.1 is an equality?

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