On deformations of hyperbolic 3–manifolds with geodesic boundary

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Let M be a complete finite-volume hyperbolic 3–manifold with compact non-empty geodesic boundary and k toric cusps, and let $\mathcal{T}$ be a geometric partially truncated triangulation of M. We show that the variety of solutions of consistency equations for $\mathcal{T}$ is a smooth manifold or real dimension $2k$ near the point representing the unique complete structure on M. As a consequence, the relation between deformations of triangulations and deformations of representations is completely understood, at least in a neighbourhood of the complete structure. This allows us to prove, for example, that small deformations of the complete triangulation affect the compact tetrahedra and the hyperbolic structure on the geodesic boundary only at the second order.

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The idea of constructing hyperbolic structures on manifolds by suitably gluing to each other geodesic polyhedra dates back to Thurston [15]. In the setting of cusped manifolds one employs ideal tetrahedra, which are parameterized by complex numbers, and tries to solve hyperbolicity equations. In [7] (written jointly with Petronio) we explained how this approach can be adapted to the case of non-empty geodesic boundary: in the bounded case one has to consider truncated tetrahedra, whose parameterization is more complicated, but basically the whole scheme extends.

The conditions under which a gluing of truncated tetrahedra defines a non-singular hyperbolic metric are encoded by *consistency equations*, while *completeness equations* translate the conditions ensuring that such a metric is complete. For our purposes it is crucial to control the number of consistency equations, and this is the reason why the equations described here are quite different from those introduced in [7]. The set of solutions of consistency equations naturally provides a deformation space for finite-volume hyperbolic structures with geodesic boundary on a fixed 3–manifold. Building on classical results in cohomology theory of representations, we prove that the complete structure is a smooth point of this deformation space and we explicitly describe local coordinates around it. This allows us to give a proof of Thurston's hyperbolic Dehn filling Theorem which applies to all the hyperbolic manifolds with geodesic boundary which admit a *good* geometric triangulation (see Definition 1.1). There is strong evidence that any complete finite-volume hyperbolic 3–manifold with

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geodesic boundary should admit a good triangulation (see the discussion preceding Conjecture 1.2); moreover, any such manifold admits a *partially flat* triangulation, and following Petronio and Porti [13] we could probably adapt our proof of Thurston's hyperbolic Dehn filling Theorem to deal also with this kind of triangulations. This would give a complete and self-contained proof of the filling Theorem via deformation theory of geometric triangulations.

In the last section we show that small deformations of the solution representing the complete structure affect the compact tetrahedra and the hyperbolic structure on the geodesic boundary only at the second order. These results are extensively used in [3] for studying small deformations in infinitely many concrete examples.

It is maybe worth mentioning that the deformation variety defined by the consistency equations for a cusped 3–manifold without boundary has already been studied by several authors (see eg, Neumann–Zagier [12], Petronio–Porti [13]). In particular, Choi has recently proved in [2] that in the cusped empty-boundary case the deformation variety is a smooth complex manifold at any point representing a non-degenerate (ie, neither partially flat nor partially negatively oriented) geodesic ideal triangulation.

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1 Triangulations and hyperbolicity equations

Let N be a complete finite-volume orientable hyperbolic 3–manifold with *compact* non-empty geodesic boundary (from now on we will usually summarize all this information saying just that N *is hyperbolic*). It is well-known that N consists of a compact portion containing ∂N together with several cusps of the form $T\times[0,\infty)$, where T is the torus, so N admits a natural compactification $\overline{N}$ obtained by adding some boundary tori. Since the components of ∂N are totally geodesic, they inherit a hyperbolic metric, and have therefore negative Euler characteristic.

1.1 Partially truncated tetrahedra

A *partially truncated tetrahedron* is a pair $(\Delta,\mathcal{I})$, where Δ is a tetrahedron and $\mathcal{I}$ is a set of vertices of Δ, that will be called *ideal vertices*. In the sequel we will always refer to Δ itself as a partially truncated tetrahedron, tacitly implying that $\mathcal{I}$ is also fixed. The *topological realization* Δ^* of Δ is obtained by removing from Δ the ideal vertices and small open stars of the non-ideal vertices. We call *lateral hexagon* and *truncation triangle* the intersection of Δ^* respectively with a face of Δ and with the link in Δ of a non-ideal vertex. The edges of the truncation triangles, which also belong to the lateral hexagons, are called *boundary edges*, and the other edges of Δ^* are called *internal edges*. If Δ has ideal vertices, a lateral hexagon of Δ^* may not be a hexagon, because some of its closed edges may be missing.

A *geometric realization* of Δ is an identification of Δ^* with a convex polyhedron in $\mathbb{H}^3$ such that the truncation triangles are geodesic triangles, the lateral hexagons are geodesic polygons with ideal vertices corresponding to missing edges, and truncation triangles and lateral hexagons lie at right angles to each other. An example of a geometric realization is shown in Figure 1, where truncation triangles are shadowed.

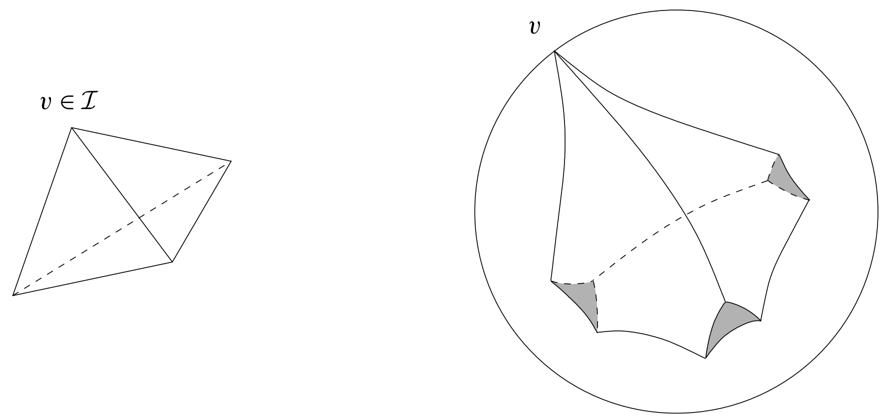

Figure 1: A geometric tetrahedron with one ideal vertex

1.2 Triangulations

Let $\bar{N}$ be a compact orientable manifold and let N be obtained from $\bar{N}$ by removing the toric components of $\partial\bar{N}$. We define a *partially truncated triangulation* of N to be a realization of N as a gluing of some Δ^*'s along a pairing of the lateral hexagons induced by a simplicial pairing of the faces of the Δ's. When N is endowed with a hyperbolic structure, a partially truncated triangulation of N is called *geometric* if, for each tetrahedron Δ of the triangulation, the pull-back to Δ^* of the Riemannian metric of N defines a geometric realization of Δ. Equivalently, the hyperbolic structure of N should be obtained by gluing geometric realizations of the Δ's along isometries of their lateral hexagons.

Definition 1.1 A partially truncated triangulation $\mathcal{T}$ of an orientable 3–manifold N is *good* if any tetrahedron in $\mathcal{T}$ has at most one ideal vertex.

Kojima proved in [10] that every hyperbolic N has a *canonical* decomposition into partially truncated *polyhedra*, rather than tetrahedra. Any polyhedron in a canonical decomposition has at most one ideal vertex, so triangulations arising as subdivisions of Kojima decompositions are examples of good triangulations. In the vast majority of cases the Kojima decomposition actually consists of tetrahedra, or at least can be subdivided into a geometric partially truncated triangulation. For instance, it is proved in Frigerio–Martelli–Petronio [6] that there exist exactly 5192 hyperbolic manifolds with non-empty geodesic boundary which can be (topologically) triangulated by at most four partially truncated tetrahedra: their Kojima decomposition can always be subdivided into a triangulation, and is itself a triangulation in 5108 cases. These facts strongly support the following:

Conjecture 1.2 *Any hyperbolic N with non-empty geodesic boundary admits a good geometric triangulation.*

1.3 Moduli for partially truncated tetrahedra

The following result implies that the dihedral angles can be used as moduli for geometric tetrahedra.

Theorem 1.3 *Let Δ be a partially truncated tetrahedron and let $\Delta^{(1)}$ be the set of edges of Δ. The geometric realizations of Δ are parameterized up to isometry by the dihedral angle assignments $\theta\colon \Delta^{(1)} \to (0,\pi)$ such that for each vertex v of Δ, if e_1, e_2, e_3 are the edges that emanate from v, then $\theta(e_1)+\theta(e_2)+\theta(e_3)$ is equal to π for ideal v and less than π for non-ideal v.*

Having introduced moduli for geometric tetrahedra, our next task is to determine, given a triangulated manifold, which values of moduli define a global hyperbolic structure on the manifold. The following well-known hyperbolic trigonometry formulae will prove useful later:

Lemma 1.4 *With notation as in Figure 2 we have*

$$\cosh a_1 = (\cos\alpha_2 \cdot \cos\alpha_3 + \cos\alpha_1)/(\sin\alpha_2 \cdot \sin\alpha_3),$$
$$\cosh b_1 = (\cosh c_2 \cdot \cosh c_3 + \cosh c_1)/(\sinh c_2 \cdot \sinh c_3).$$

Let now Δ be a partially truncated tetrahedron with edges $e_1, \ldots, e_6$ as in Figure 3. We fix a geometric realization θ of Δ determined by the dihedral angles $\theta_i = \theta(e_i)$ for $i = 1, \ldots, 6$ and we denote by L^θ the length with respect to this realization. The

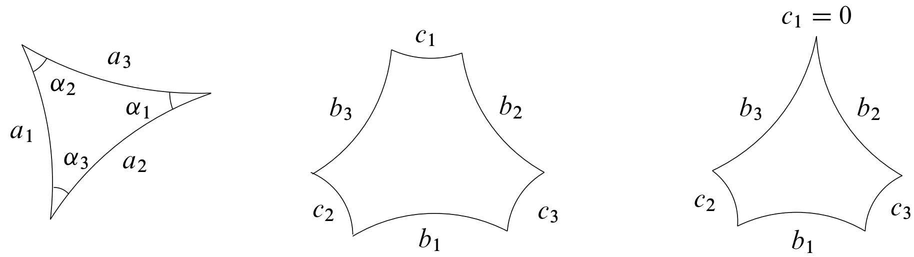

Figure 2: A triangle, a right-angled hexagon and a pentagon with four right angles and an ideal vertex

boundary edges of the lateral hexagons of Δ correspond to the pairs of distinct non-opposite edges $\{e_i, e_j\}$, and will be denoted by e_{ij}. Note that e_{ij} disappears towards infinity, so it has length 0, when the common vertex of e_i and e_j is ideal. Lemma 1.4 readily implies

$$\cosh L^{\theta}(e_{12}) = (\cos\theta_1 \cdot \cos\theta_2 + \cos\theta_3)/(\sin\theta_1 \cdot \sin\theta_2). \tag{1}$$

Note that this result is correct also when the common end of e_1 and e_2 is ideal.

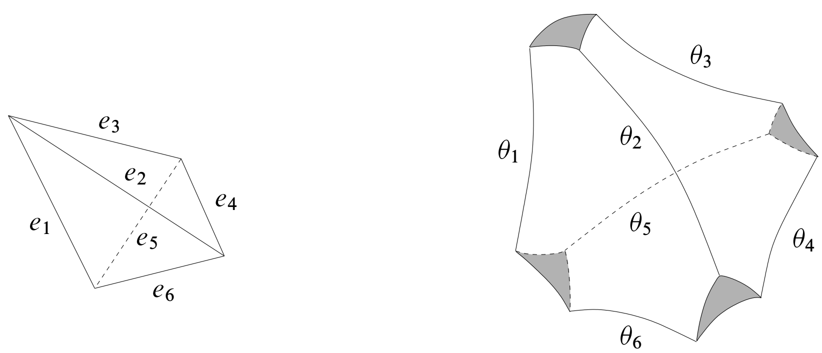

Figure 3: Notation for edges and dihedral angles of a truncated tetrahedron

Turning to the length of an internal edge, we note that the edge is an infinite half-line or an infinite line when one or both its ends are ideal, respectively. Otherwise the length is computed using Lemma 1.4. With notation as in Figure 3, and defining v_{ijk} as the vertex from which the edges e_i, e_j, e_k emanate, we set:

$$c^\theta(e_1) = \cos\theta_1 \cdot (\cos\theta_3 \cdot \cos\theta_6 + \cos\theta_2 \cdot \cos\theta_5) \\ + \cos\theta_2 \cdot \cos\theta_6 + \cos\theta_3 \cdot \cos\theta_5 + \cos\theta_4 \cdot \sin^2\theta_1;$$

$$d^\theta(v_{123}) = 2\cos\theta_1 \cdot \cos\theta_2 \cdot \cos\theta_3 + \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 - 1.$$

Proposition 1.5 *$d^\theta(v_{123}) = 0$ if and only if the vertex v_{123} is ideal. If v_{123} and v_{156} are both non-ideal then*

$$\cosh L^\theta(e_1) = c^\theta(e_1) \Big/ \sqrt{d^\theta(v_{123}) \cdot d^\theta(v_{156})}. \tag{2}$$

Remark 1.6 Let Δ be a partially truncated tetrahedron without ideal vertices. Then the geometric realizations of Δ are parameterized by the lengths of the internal edges. In fact, the map that associates to the dihedral angles of a geometric realization of Δ the lengths of its internal edges is a diffeomorphism between open subsets of $\mathbb{R}^6$.

1.4 Conditions for geometric gluing

Let N be obtained from an orientable compact $\overline{N}$ by removing all the tori in $\partial\overline{N}$ and let $\mathcal{T}$ be a partially truncated triangulation of N. Let also θ be a geometric realization of the tetrahedra in $\mathcal{T}$ and denote by L^θ the length with respect to this realization. We now describe the conditions under which the realization θ defines a hyperbolic structure on the whole of N. For our purposes it will be sufficient to deal only with *good* triangulations, so we assume from now on that $\mathcal{T}$ is good. The general case is treated in [7].

In order to define a global hyperbolic structure on N, the tetrahedra of $\mathcal{T}$ must satisfy two obvious necessary conditions, which in fact are also sufficient. Namely, we should be able to glue the lateral hexagons by isometries, and we should have a total dihedral angle of 2π around each edge of the manifold. The first condition ensures that the hyperbolic structure defined by θ on the complement of the 2–skeleton of $\mathcal{T}$ extends to the complement of the 1–skeleton. Since $\mathcal{T}$ is good, the second one ensures that the structure glues up without singularities also along the edges. The second condition is directly expressed in terms of moduli, and we will explain in a moment how to translate the first one into an equation on dihedral angles.

Remark 1.7 If $\mathcal{T}$ were not good, requiring the dihedral angles around each edge to sum up to 2π would not be sufficient to obtain a non-singular hyperbolic metric on the 1–skeleton of $\mathcal{T}$. The point is that when some geometric tetrahedra are arranged one after the other around an edge e with two ideal endpoints, the first face of the first

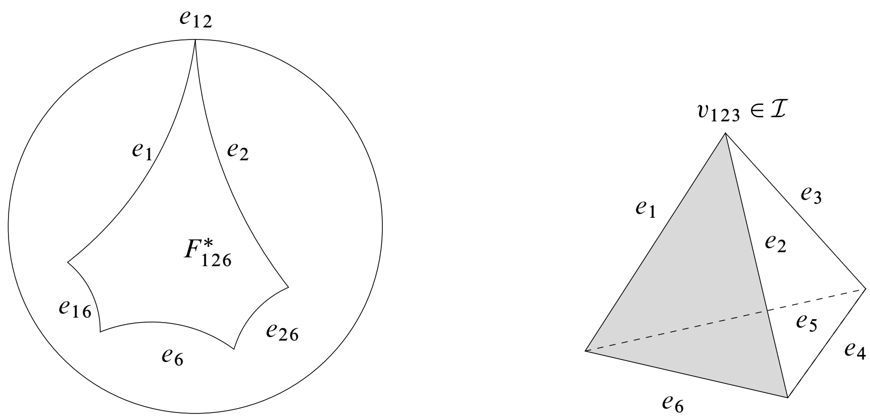

Figure 4: An exceptional hexagon

tetrahedron and the second face of the last tetrahedron may overlap without coinciding. Namely, the isometry which pairs these two faces may be a translation along e instead of being the identity. Of course the isometry has to be the identity if at least one endpoint of e is not ideal.

1.5 Exceptional hexagons

It is easily seen that gluings match ideal vertices to each other, because these notions are part of the initial topological information about a triangulation. When a pairing glues two *compact* lateral hexagons, to be sure that the gluing is an isometric one we may equivalently require the lengths of the internal edges or those of the boundary edges to match under the gluing. On the other hand, since $\mathcal{T}$ is a good triangulation, a non-compact lateral hexagon F is actually a pentagon with four right angles and an ideal vertex: we shall say in this case that F is an *exceptional* lateral hexagon. By Lemma 1.4, the isometry class of an exceptional lateral hexagon is determined by the lengths of its boundary edges. However, in order to end up with a non-redundant set of consistency equations, it is convenient to find an alternative approach to moduli for exceptional hexagons. To this aim we need now to be slightly more careful about orientation than we have been so far. Namely, we choose on the tetrahedra an orientation compatible with a global orientation of the manifold. As a result also the lateral hexagons have a fixed orientation, and the gluing maps reverse the orientation of the hexagons.

So, let us consider an exceptional hexagon F^*_{126} as in Figure 4, and recall that the hexagon is oriented and embedded in $\mathbb{H}^3$ by θ. We consider the horospheres O_1 and

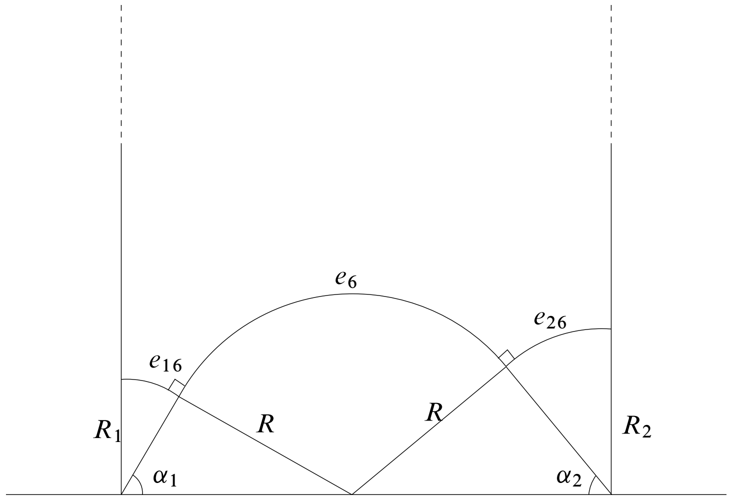

Figure 5: Notation for the proof of Proposition 1.8

O_2 centred at e_{12} and passing through the non-ideal ends of e_1 and e_2 respectively. We define $\sigma^\theta(F_{126})$ to be $\pm\mathrm{dist}(O_1, O_2)$, the sign being positive if e_2, e_{12}, e_1 are arranged positively on ∂F^*_{126} and O_1 is contained in the horoball bounded by O_2, or if e_2, e_{12}, e_1 are arranged negatively on ∂F^*_{126} and O_2 is contained in the horoball bounded by O_1, and negative otherwise. Together with equation (1), the following proposition allows to compute σ in terms of the dihedral angles.

Proposition 1.8 *We have*

$$\sigma^\theta(F_{126}) = \ln(\sinh L^\theta(e_{16})/\sinh L^\theta(e_{26})). \tag{3}$$

Proof Let α_1, α_2 and R, R_1, R_2 be the angles and lengths shown in Figure 5. An easy computation in the upper half-plane model of $\mathbb{H}^2$ shows that $L^\theta(e_{i6}) = \ln\cot(\alpha_i/2)$ for $i = 1, 2$. Moreover we have $R = R_1 \cdot \tan\alpha_1 = R_2 \cdot \tan\alpha_2$, so $\exp\sigma^\theta(F_{126}) = R_1/R_2 = \tan\alpha_2/\tan\alpha_1$. Now for $i = 1, 2$ we have $\tan\alpha_i = 2\cot(\alpha_i/2)/(\cot^2(\alpha_i/2) - 1) = 2\exp L^\theta(e_{i6})/(\exp(2L^\theta(e_{i6})) - 1)$. Combining these equations we finally get $\exp\sigma^\theta(F_{126}) = \sinh L^\theta(e_{16})/\sinh L^\theta(e_{26})$. $\square$

We now define $\ell^\theta(F_{126})$ to be the length of e_6. The next proposition shows that the functions σ and ℓ provide a parameterization of isometry classes of exceptional hexagons.

Proposition 1.9 *Let F and F' be paired exceptional lateral hexagons. Their pairing can be realized by an isometry if and only if $\sigma^\theta(F) + \sigma^\theta(F') = 0$ and $\ell^\theta(F) = \ell^\theta(F')$.*

Proof We concentrate on the "if" part of the statement, the "only if" part being obvious. Let f_1, f_2 and f_1', f_2' be the boundary edges of of F and F' respectively, and assume that the orientation-reversing pairing between F and F' glues f_1 to f_2' and f_1' to f_2. Since $\ell^\theta(F) = \ell^\theta(F')$, Lemma 1.4 easily implies that $L(f_1) \leqslant L(f_2')$ if and only if $L(f_2) \geqslant L(f_1')$. Moreover the assumption that $\sigma^\theta(F) = -\sigma^\theta(F')$ and Proposition 1.8 give $L(f_1) \leqslant L(f_2')$ if and only if $L(f_2) \leqslant L(f_1')$. This forces $L(f_1) = L(f_2')$ and $L(f_2) = L(f_1')$, whence the conclusion. □

Remark 1.10 Let Δ be a partially truncated tetrahedron and suppose that v is the unique ideal vertex of Δ. Let F_1, F_2, F_3 be the faces of Δ incident to v and for $i = 1, 2, 3$ let e_i be the edge of F_i not containing v. The isometry classes of the geometric realizations of Δ are parameterized by the lengths of the e_i's and the values taken by σ on the F_i's. More precisely, the map that associates to any geometric realization θ of Δ the point $(L^\theta(e_1), L^\theta(e_2), L^\theta(e_3), \sigma^\theta(F_1), \sigma^\theta(F_2), \sigma^\theta(F_3))$ defines a diffeomorphism between open subsets of two affine hyperplanes of $\mathbb{R}^6$.

1.6 Consistency equations

Recall now that we are considering a candidate hyperbolic 3–manifold N endowed with a good triangulation $\mathcal{T}$, and that we have fixed a geometric realization θ of the tetrahedra in $\mathcal{T}$. The above discussion implies the following:

Theorem 1.11 *The parameterization θ defines on N a hyperbolic structure with geodesic boundary if and only if the following conditions hold:*

(1) *the total dihedral angle along any edge of $\mathcal{T}$ in N is equal to 2π;*

(2) *$L^\theta(e) = L^\theta(e')$ for all pairs (e, e') of matching compact internal edges;*

(3) *$\sigma^\theta(F) + \sigma^\theta(F') = 0$ for all pairs (F, F') of matching exceptional hexagons.*

By Propositions 1.5, 1.8, conditions (1), (2), (3) of Theorem 1.11 translate into a set $\mathcal{C}(\mathcal{T})$ of smooth equations on θ, which are called *consistency equations*. Our next task is to compare the number of equations in $\mathcal{C}(\mathcal{T})$ with the dimension of the moduli space of geometric realizations of the tetrahedra of $\mathcal{T}$.

Let c (resp. p) be the number of compact (resp. non-compact) tetrahedra of $\mathcal{T}$. By Theorem 1.3, the number of parameters for the geometric realizations of $\mathcal{T}$ is equal to $6c + 5p = 6t - p$, where t is the total number of tetrahedra of $\mathcal{T}$. Let $e_1, \ldots, e_l$ be the edges of $\mathcal{T}$ without ideal endpoints, and for $i = 1, \ldots, l$ let x_i be the valence of e_i, ie, the number of tetrahedra of $\mathcal{T}$ incident to e_i, with multiplicity. Of course we have $\sum_{i=1}^{l} x_i = 6c + 3p$. If we denote by h the number of edges of $\mathcal{T}$ with exactly one ideal endpoint, then the union of the boundary tori of $\bar{N}$ admits a triangulation with h vertices and p triangles, so $h = p/2$. Let us now turn to the number of equations. Theorem 1.11 determines $l + h$ equations arising from conditions (1), $\sum_{i=1}^{l}(x_i - 1)$ equations arising from conditions (2) and $3p/2$ equations arising from conditions (3). Since

$$\begin{aligned} & l + h + \left(\sum_{i=1}^{l}(x_i - 1)\right) + 3p/2 && = h + \left(\sum_{i=1}^{l} x_i\right) + 3p/2 \\ = {} & p/2 + 6c + 3p + 3p/2 && = 6c + 5p, \end{aligned}$$

we can conclude that the number of equations in $\mathcal{C}(\mathcal{T})$ is equal to the dimension of the moduli space of geometric realizations of the tetrahedra of $\mathcal{T}$.

1.7 Reducing the number of equations

When N has cusps, some equations in $\mathcal{C}(\mathcal{T})$ turn out to be redundant. Let T be a fixed toric component of $\partial\bar{N}$ and let j be the number of tetrahedra asymptotic to T. Since any such tetrahedron contributes to the triangulation of T with a Euclidean triangle, the sum of all the dihedral angles along all the edges of $\mathcal{T}$ incident to T is equal to $j\pi$. So if we require condition (1) of Theorem 1.11 to hold for *all but one* edge incident to T, then the same condition is automatically satisfied also along the remaining edge. This allows us to discard from $\mathcal{C}(\mathcal{T})$ one equation for each cusp of N.

Moreover, let Δ be a partially truncated tetrahedron with an ideal vertex v incident to T, and let F_1, F_2, F_3 be the faces of Δ incident to v. By the very definition of σ it follows that $\sigma(F_1) + \sigma(F_2) + \sigma(F_3) = 0$. This implies that if $\mathcal{F}$ is the set of all the faces incident to T of tetrahedra of $\mathcal{T}$, then we have $\sum_{F\in\mathcal{F}} \sigma(F) = 0$. So if we require condition (3) of Theorem 1.11 to hold for *all but one* pair of matching exceptional hexagons incident to T, then the same condition also holds for the remaining pair. This means that another equation of $\mathcal{C}(\mathcal{T})$ for each cusp of N can be discarded. Suppose that N compactifies to an orientable $\bar{N}$ with k boundary tori. The above discussion is summarized by the following:

Proposition 1.12 *We can discard $2k$ equations from $\mathcal{C}(\mathcal{T})$ thus obtaining an equivalent set $\mathcal{C}^*(\mathcal{T})$ of $n - 2k$ equations, where n is the dimension of the moduli space of geometric realizations of the tetrahedra of $\mathcal{T}$.*

We have seen in the preceding subsection that the moduli space of geometric realizations of the tetrahedra of $\mathcal{T}$ is given by a subset $\mathcal{W}$ of $\mathbb{R}^{6t}$, where t is the number of tetrahedra of $\mathcal{T}$. More precisely, $\mathcal{W}$ is an open convex subset of an affine subspace of dimension $6t - p$, where p is the number of non-compact tetrahedra in $\mathcal{T}$. Recall that l is the number of compact edges (considered as subsets of N) of $\mathcal{T}$. The above computation implies that equations in $\mathcal{C}^*(\mathcal{T})$ corresponding to conditions (1) of Theorem 1.11 take the form $A(x) = 0$, where $A\colon \mathcal{W} \to \mathbb{R}^{l+p/2-k}$ is an affine map, while equations corresponding to conditions (2) and (3) take the form $F(x) = 0$, where $F\colon \mathcal{W} \to \mathbb{R}^{6t-l-(3/2)p-k}$ is constructed from formulae (2), (3), and is therefore smooth. From now on we denote by $\Omega(\mathcal{T}) = F^{-1}(0) \cap A^{-1}(0) \subset \mathcal{W} \subset \mathbb{R}^{6t}$ the set of solutions of consistency equations $\mathcal{C}^*(\mathcal{T})$.

1.8 Completeness

Let $T_1, \dots, T_k$ be the boundary tori of $\bar{N}$. From now on we denote by μ_i, λ_i a fixed basis of $H_1(T_i; \mathbb{Z}) \cong \pi_1(T_i)$, $i = 1, \dots, k$. Any point in $\Omega(\mathcal{T})$ naturally defines an Aff($\mathbb{C}$)–structure on T_i (see eg, [1; 5]). For $x \in \Omega(\mathcal{T})$, we denote by $a_i(x) \in \mathbb{C}$ (resp. by $b_i(x) \in \mathbb{C}$) the linear component of the holonomy of μ_i (resp. of λ_i) corresponding to the Aff($\mathbb{C}$)–structure defined by x on T_i. It is well-known that the hyperbolic structure defined by x on N induces a complete metric on the i^{th} cusp of N if and only if $a_i(x) = b_i(x) = 1$. Moreover, one can explicitly compute a_i and b_i in terms of the dihedral angles as follows.

Let Δ be a tetrahedron in $\mathcal{T}$, let v be an ideal vertex of Δ and $L_x(v)$ be the (similarity class of the) Euclidean triangle obtained by intersecting the geometric realization of Δ parameterized by x with a small horosphere centred at v. The tetrahedron being oriented, this triangle is also oriented, so, once a vertex p of $L_x(v)$ is fixed, we can associate to the similarity structure of $L_x(v)$ the unique complex number $z_x(L(v), p)$ such that $L_x(v)$ is carried to the Euclidean triangle with vertices $0, 1, z_x(L(v), p)$ by an orientation-preserving similarity sending p to 0. Suppose that e_1, e_2, e_3 are the internal edges emanating form v, and that they are positively arranged around v. If $p = L(v_{123}) \cap e_1$, then

$$z_x(L(v_{123}), p) = (\sin\theta_2 / \sin\theta_3) \cdot e^{i\theta_1}.$$

If γ is an oriented simplicial loop on T_i and q is a vertex of γ, then the set of all triangles touching γ in q and lying on the right of γ is well-defined and will be denoted by $R(\gamma, q)$. Moreover, we shall denote by $V(\gamma)$ the set of vertices of γ. Let $\widehat{\mu}_i, \widehat{\lambda}_i$ be simplicial loops on T_i representing μ_i, λ_i. The following result is proved in [15; 1].

Proposition 1.13 *We have*

$$\begin{aligned} a_i(x) &= (-1)^{\#V(\widehat{\mu}_i)} \cdot \prod_{q \in V(\widehat{\mu}_i)} \prod_{T \in R(\widehat{\mu}_i, q)} z_x(T, q), \\ b_i(x) &= (-1)^{\#V(\widehat{\lambda}_i)} \cdot \prod_{q \in V(\widehat{\lambda}_i)} \prod_{T \in R(\widehat{\lambda}_i, q)} z_x(T, q). \end{aligned}$$

As a consequence of Mostow–Prasad's rigidity Theorem for hyperbolic manifolds with geodesic boundary [7; 4] we get the following:

Theorem 1.14 *There exists at most one point in $\Omega(\mathcal{T})$ that defines on N a complete hyperbolic structure with geodesic boundary.*

Proof See [7]. $\square$

2 Smoothness at the complete structure

Suppose now that x_0 is the unique point in $\Omega(\mathcal{T})$ which defines on N a complete hyperbolic structure. For $i = 1, \dots, k$ and $x \in \Omega(\mathcal{T})$ let us define

$$u_i(x) = \ln a_i(x), \qquad v_i(x) = \ln b_i(x),$$

where ln is the branch of the complex logarithm defined on $\{z \in \mathbb{C} : \Re(z) > 0\}$ such that $\ln 1 = 0$. Since a non-trivial parabolic isometry does not commute with a non-trivial orientation-preserving non-parabolic isometry, we have the following:

Proposition 2.1 *In a neighbourhood of x_0 in $\Omega(\mathcal{T})$ we have $u_i(x) = 0 \Leftrightarrow v_i(x) = 0$ $\Leftrightarrow$ the hyperbolic structure defined by x on the i^{th} cusp of N is complete.*

Let $F\colon \mathcal{W} \to \mathbb{R}^{6t-l-(3/2)p-k}$, $A\colon \mathcal{W} \to \mathbb{R}^{l+p/2-k}$ be the smooth functions previously defined such that $\Omega(\mathcal{T}) = F^{-1}(0) \cap A^{-1}(0)$. We now set

$$G\colon \mathcal{W} \to \mathbb{R}^{6t-p-2k} \times \mathbb{C}^k, \quad G(x) = (F(x), A(x), u_1(x), \dots, u_k(x)).$$

By Theorem 1.14 and Proposition 2.1 we have $G^{-1}(0) = \{x_0\}$. This section is entirely devoted to the proof of our main result:

Theorem 2.2 *We have $\operatorname{Ker} dG_{x_0} = \{0\}$. Thus:*

(1) *G induces a diffeomorphism of an open neighbourhood of x_0 in $\mathcal{W}$ onto an open neighbourhood of 0 in $\mathbb{R}^{6t-p-2k} \times \mathbb{C}^k$;*

(2) *$\Omega(\mathcal{T})$ is a smooth manifold of real dimension $2k$ near x_0;*

(3) *the map*
$$u\colon \Omega(\mathcal{T}) \to \mathbb{C}^k, \quad u(x) = (u_1(x), \dots, u_k(x))$$
induces a diffeomorphism of an open neighbourhood of x_0 in $\Omega(\mathcal{T})$ onto an open neighbourhood of 0 in $\mathbb{C}^k$.

Any point in $\Omega_c(\mathcal{T}) := F^{-1}(0)$ defines a cone structure on N having cone singularities along the edges of $\mathcal{T}$. The following immediate consequence of Theorem 2.2 was also proved in [9; 11].

Corollary 2.3 *If N is compact then $\Omega_c(\mathcal{T})$ is parameterized in a neighbourhood of x_0 by the cone angles along the edges of $\mathcal{T}$.*

2.1 Deforming cone structures

We begin with the following:

Proposition 2.4 *The tangent map $dF_{x_0}\colon T_{x_0}(\mathcal{W}) \to \mathbb{R}^{6t-l-(3/2)p-k}$ has maximal rank, so $\Omega_c(\mathcal{T})$ is a manifold of real dimension $l + p/2 + k$ near x_0.*

Proof By Remarks 1.6, 1.10, the lengths of the compact internal edges and the values taken by σ on the exceptional lateral hexagons of the tetrahedra of $\mathcal{T}$ provide smooth coordinates on $\mathcal{W}$. It is easily seen that with respect to these coordinates the map F is affine and has maximal rank at x_0 (whence at any point of $\mathcal{W}$). $\square$

Let now $\alpha\colon (-\varepsilon, \varepsilon) \to \Omega_c(\mathcal{T})$ be a smooth arc with $\alpha(0) = x_0$. We will study the deformation associated to α using tools from the cohomology theory of representations: notation is as in Appendix A, where we give some basic definitions and results.

From now on we denote by N' the non-compact manifold obtained by drilling from N all the edges of $\mathcal{T}$. For any $t \in (-\varepsilon, \varepsilon)$ the point $\alpha(t)$ determines a smooth hyperbolic structure M_t on N', whose completion gives a hyperbolic cone structure on N. First of all we describe how to deduce the shape of the geometric tetrahedra corresponding to the point $\alpha(t)$ just from the geometric structure M_t on N'. To this end we fix for $t \in (-\varepsilon, \varepsilon)$ a developing map $D_t\colon \widetilde{N}' \to \mathbb{H}^3$ with associated holonomy representation $\rho_t\colon \pi_1(N') \to \mathrm{PSL}(2, \mathbb{C})$ (note that we can choose D_t and ρ_t to vary smoothly with t). Recall that N compactifies to a manifold $\bar{N}$ with k boundary tori, and denote by $\bar{N}'$ the non-compact manifold obtained by drilling from $\bar{N}$ the *closed* properly embedded arcs corresponding to the edges of $\mathcal{T}$. For $i = 1, \dots, k$ let T_i' be the punctured torus $T_i' = T_i \cap \bar{N}'$, and denote by $\mathcal{S}_{\mathrm{par}}$ the family of all the boundary components of the universal covering of $\bar{N}'$ projecting to some T_i'. We say that $P \subset \pi_1(N') \cong \pi_1(\bar{N}')$ is a *parabolic peripheral subgroup* of $\pi_1(N')$ if P is the stabilizer of some boundary component $S \in \mathcal{S}_{\mathrm{par}}$ of the universal covering of $\bar{N}'$ (so $\rho_0(P)$ is a $\mathbb{Z} + \mathbb{Z}$ parabolic subgroup of $\mathrm{PSL}(2, \mathbb{C})$). Let $\partial_{\mathrm{gd}} \bar{N}'$ be the portion of $\partial \bar{N}'$ corresponding to the geodesic boundary of N, ie, let $\partial_{\mathrm{gd}} \bar{N}' = \partial \bar{N}' \cap \partial N$, and denote by $\mathcal{S}_{\mathrm{gd}}$ the family of all the boundary components of the universal covering of

$\bar{N}'$ projecting to some component of $\partial_{\mathrm{gd}}\bar{N}'$. We say that $P \subset \pi_1(N')$ is a *Fuchsian peripheral subgroup* of $\pi_1(N')$ if P is the stabilizer of some component $S \in \mathcal{S}_{\mathrm{gd}}$ (so $\rho_0(P)$ is a Fuchsian subgroup of $\mathrm{PSL}(2,\mathbb{C})$).

2.2 Dual vectors to planes and horospheres

If $S \in \mathcal{S}_{\mathrm{gd}}$, by construction the image of S under the developing map D_t is contained in a totally geodesic immersed surface in $\mathbb{H}^3$. It is easily seen that such a surface must in turn be contained in a geodesic plane $\widehat{S}(t)$ of $\mathbb{H}^3$. We now need to associate to each $\widehat{S}(t)$ a suitable *ultra-ideal* point $b_S(t)$, which will be called the *dual* point of $\widehat{S}(t)$. Such point naturally lies in 4–dimensional Minkowsky space, so we fix some notation about this space.

We denote by $\mathbb{M}^{3+1}$ the space $\mathbb{R}^4$ with coordinates x_0, x_1, x_2, x_3 endowed with the Lorentzian inner product $\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$. We set

$$\begin{aligned}
\mathcal{H}^3_- &= \{x \in \mathbb{M}^{3+1} :\ \langle x, x\rangle = -1,\ x_0 > 0\},\\
\mathcal{H}^3_+ &= \{x \in \mathbb{M}^{3+1} :\ \langle x, x\rangle = 1\},\\
L^3_+ &= \{x \in \mathbb{M}^{3+1} :\ \langle x, x\rangle = 0,\ x_0 > 0\}.
\end{aligned}$$

We recall that $\mathcal{H}^3_-$ is the upper sheet of the two-sheeted hyperboloid, and that $\langle\cdot,\cdot\rangle$ restricts to a Riemannian metric on $\mathcal{H}^3_-$. With this metric, $\mathcal{H}^3_-$ is the so-called *hyperboloid model* $\mathbb{H}^3_{\mathrm{hyp}}$ of hyperbolic space. The one-sheeted hyperboloid $\mathcal{H}^3_+$ turns out to have a bijective correspondence with the set of hyperbolic half-spaces in $\mathbb{H}^3_{\mathrm{hyp}}$. Given $b \in \mathcal{H}^3_+$, the corresponding half-space, called the *dual* of b, is given by $\{v \in \mathcal{H}^3_- :\ \langle v, b\rangle \leqslant 0\}$. Similarly, the cone L^3_+ of future-oriented light-like vectors of $\mathbb{M}^{3+1}$ corresponds to the set of horospheres in $\mathbb{H}^3_{\mathrm{hyp}}$. The horosphere dual to $u \in L^3_+$ is given by $\{v \in \mathcal{H}^3_- :\ \langle v, u\rangle = -1\}$.

Note now that for any $S \in \mathcal{S}_{\mathrm{gd}}$ the set $D_t(\widetilde{N})$ locally lies on a definite side of $\widehat{S}(t)$, so we can define $b_S(t)$ to be the dual vector to the half-space that locally contains $D_t(\widetilde{N})$ and is bounded by $\widehat{S}(t)$.

When S belongs to $\mathcal{S}_{\mathrm{par}}$ a vector $b_S(t) \in L^3_+$ can also be defined as follows: take an oriented edge $\widetilde{f}$ of $\widetilde{\mathcal{T}}$ ending in S and set $b_S(t)$ to be the unique point in L^3_+ with $x_0(b_S(t)) = 1$ which projects to the endpoint of $D_t(\widetilde{f})$ in $\partial\mathbb{H}^3$. Such an endpoint exists because $D_t(\widetilde{f})$ is a geodesic, and is clearly independent of the choice of $\widetilde{f}$, so $b_S(t)$ is indeed well-defined. Since developing maps vary smoothly with respect to t we have the following:

Lemma 2.5 *For any $S \in \mathcal{S}_{\mathrm{gd}} \cup \mathcal{S}_{\mathrm{par}}$ the map $t \mapsto b_S(t)$ is smooth with respect to t.*

2.3 Lifting geometric tetrahedra

Let now Δ be a tetrahedron in $\mathcal{T}$, and denote by $\widetilde{\Delta}$ a lift of Δ to $\widetilde{N}'$. Let S_i be the boundary component of the universal covering of $\overline{N}'$ that corresponds to v_i, where $v_1, \dots, v_4$ are the vertices of $\widetilde{\Delta}$. If $\overline{\Delta}(t)$ is the convex hull of $b_{S_1}(t), \dots, b_{S_4}(t)$ in $\mathbb{M}^{3+1}$, then projecting $\overline{\Delta}(t)$ to $\mathbb{H}^3$ and truncating its infinite-volume ends with the corresponding $\widehat{S}_i(t)$'s gives back a truncated tetrahedron $\Delta^*(t)$ isometric to the geometric realization of Δ parameterized by $\alpha(t)$. It is easily seen that the dihedral angles of $\Delta^*(t)$ smoothly depend on $b_{S_1}(t), \dots, b_{S_4}(t)$, so Theorem 2.2 is now reduced to the following:

Proposition 2.6 *If $\dot{\alpha}(0) \in \operatorname{Ker} dG_{x_0}$, then we can choose the D_t's in such a way that $\dot{b}_S(0) = 0$ for all $S \in \mathcal{S}_{\mathrm{gd}} \cup \mathcal{S}_{\mathrm{par}}$.*

2.4 The tangent vector to ρ_t

Let us consider the double DN' of N' obtained by mirroring N' along its boundary (which is now given by some punctured surfaces of negative Euler characteristic). Since $\partial N'$ is totally geodesic with respect to the hyperbolic structure M_t, this structure can be doubled to a smooth hyperbolic metric DM_t on DN'.

Let $D\rho_t \colon \pi_1(DN') \to \mathrm{PSL}(2, \mathbb{C})$ be a holonomy representation corresponding to a developing map for DM_t. It is easily seen that we can assume $D\rho_t$ to vary smoothly with respect to t. We set

$$\dot{D\rho} \colon \pi_1(DN') \to \mathfrak{sl}(2, \mathbb{C}), \qquad \dot{D\rho}(\gamma) = \left.\frac{d}{dt}\right|_{t=0} \left(D\rho_t(\gamma) D\rho_0(\gamma)^{-1} \right).$$

As explained in Appendix A, we have $\dot{D\rho} \in Z^1(\pi_1(DN'); \mathfrak{sl}(2, \mathbb{C}); D\rho_0)$. This subsection is devoted to the proof of the following:

Proposition 2.7 *If $\dot{\alpha}(0) \in \operatorname{Ker} dG_{x_0}$, then $\dot{D\rho} \in B^1(\pi_1(DN'); \mathfrak{sl}(2, \mathbb{C}); D\rho_0)$.*

From now on we suppose $\dot{\alpha}(0) \in \operatorname{Ker} dG_{x_0}$. Let $f_1, \dots, f_m \in DN'$ be the doubles of the edges of $\mathcal{T}$ and for all $j = 1, \dots, m$ let ℓ_j be a small loop in DN' encircling f_j. We denote by γ_j an element in $\pi_1(DN')$ representing ℓ_j (such a γ_j is well-defined only up to conjugation). Now our hypothesis implies that if $\theta_j(t)$ is the cone angle of (the completion of) DM_t along f_j, then $\dot{\theta}_j(0) = 0$. Also observe that we have

$$D\rho_t(\gamma_j) = g_t \cdot \left[\begin{pmatrix} \exp(i\theta_j(t)/2) & 0 \\ 0 & \exp(-i\theta_j(t)/2) \end{pmatrix} \right] \cdot g_t^{-1}$$

for some smooth path $g\colon (-\varepsilon,\varepsilon)\to \mathrm{PSL}(2,\mathbb{C})$. Differentiating this relation we easily get $\dot{D\rho}(\gamma_j)=0$. Let K be the kernel of the map $\pi_1(DN')\to\pi_1(DN)$ induced by the inclusion, and observe that K is the smallest normal subgroup of $\pi_1(DN')$ generated by $\gamma_1,\dots,\gamma_m$. Let $D\overline{\rho}_0\colon \pi_1(DN)\to\mathrm{PSL}(2,\mathbb{C})$ be the natural representation associated to $D\rho_0$. Since $D\rho_0(\gamma)=1$ and $\dot{D\rho}(\gamma)=0$ for all $\gamma\in K$, Lemma A.1 implies the following:

Proposition 2.8 *A cocycle $z_\rho\in Z^1(\pi_1(DN);\mathfrak{sl}(2,\mathbb{C});D\overline{\rho}_0)$ is naturally induced by $\dot{D\rho}$. Moreover, $\dot{D\rho}$ belongs to $B^1(\pi_1(DN');\mathfrak{sl}(2,\mathbb{C});D\rho_0)$ if and only if z_ρ belongs to $B^1(\pi_1(DN);\mathfrak{sl}(2,\mathbb{C});D\overline{\rho}_0)$.*

Now if N is compact, ie, if no cusps are involved, Theorem A.2 directly applies concluding the proof of Proposition 2.7. When there are cusps, Proposition 2.7 can be deduced from Theorem A.3 and the following:

Proposition 2.9 *If N has cusps, then $[z_\rho]\in H^1_{\mathrm{par}}(\pi_1(DN);\mathfrak{sl}(2,\mathbb{C});D\overline{\rho}_0)$.*

Proof Let $\gamma\in\pi_1(DN)$ be such that $D\overline{\rho}_0(\gamma)$ is non-trivial parabolic, and let $\langle\gamma\rangle$ be the infinite cyclic group generated by γ. We have to check that z_ρ restricts to a coboundary in $B^1(\langle\gamma\rangle;\mathfrak{sl}(2,\mathbb{C});D\overline{\rho}_0\circ i)$, where $i\colon\langle\overline{\gamma}\rangle\to\pi_1(DN)$ is the natural inclusion.

Without loss of generality we can suppose $\gamma\in\pi_1(T_i)\subset\pi_1(N)\subset\pi_1(DN)$ for some $i=1,\dots,k$. Recall that a preferred element $\mu_i\in\pi_1(T_i)$ was previously fixed, set $u_i(t)=u_i(\alpha(t))$ and observe that since $\dot\alpha(0)\in\operatorname{Ker} dG_{x_0}$ we have $\dot u_i(0)=0$. Let μ_i',γ' be elements in $\pi_1(T_i')$ projecting respectively to μ_i,γ. By Lemma 2.5 a smooth path $g\colon(-\varepsilon,\varepsilon)\to\mathrm{PSL}(2,\mathbb{C})$ exists such that both $g_t^{-1}\cdot\rho_t(\mu_i')\cdot g_t$ and $g_t^{-1}\cdot\rho_t(\gamma')\cdot g_t$ fixes $\infty\in\partial\mathbb{H}^3$ for $t\in(-\varepsilon,\varepsilon)$, so that

$$\begin{aligned} D\rho_t(\mu_i') &= g_t\cdot\left[\begin{pmatrix}\exp(u_i(t)/2) & \tau_i(t)\\ 0 & \exp(-u_i(t)/2)\end{pmatrix}\right]\cdot g_t^{-1}\\ D\rho_t(\gamma') &= g_t\cdot\left[\begin{pmatrix}a(t) & b(t)\\ 0 & a(t)^{-1}\end{pmatrix}\right]\cdot g_t^{-1}\end{aligned}$$

where $\tau_i,a,b\colon(-\varepsilon,\varepsilon)\to\mathbb{C}$ are smooth arcs with $a(0)=1, b(0)\neq 0, \tau_i(0)\neq 0$.

Since $\rho_0(\pi_1(T_i'))\cong\mathbb{Z}+\mathbb{Z}$ is Abelian, an element $k\in K$ exists such that $\mu_i'\cdot\gamma'=k\cdot\gamma'\cdot\mu_i'$ in $\pi_1(DN')$. By Proposition 2.8, this readily implies

$$\frac{d}{dt}\Big|_{t=0}\big(D\rho_t(\mu_i')D\rho_t(\gamma')\big)=\frac{d}{dt}\Big|_{t=0}\big(D\rho_t(\gamma')D\rho_t(\mu_i')\big)$$

which after some computations gives $\dot{a}(0)=0$ (here we use $\dot{u}_i(0)=0$). Let us consider the deformation $\varphi_t\colon \langle\gamma'\rangle \to \mathrm{PSL}(2,\mathbb{C})$ defined by $\varphi_t((\gamma')^n) = D\rho_t((\gamma')^n)$ for any $n\in\mathbb{Z}$.

We claim that $\dot{\varphi}=0$ in $H^1(\langle\gamma'\rangle;\mathfrak{sl}(2,\mathbb{C});\varphi_0)$. This will easily give that z_ρ restricts to a coboundary in $B^1(\langle\gamma\rangle;\mathfrak{sl}(2,\mathbb{C});D\overline{\rho}_0\circ i)$, whence the conclusion. Since derivatives of conjugated deformations differ by a coboundary, we can suppose

$$\varphi_t(\gamma') = \left[\begin{pmatrix} a(t) & b(t) \\ 0 & a(t)^{-1}\end{pmatrix}\right].$$

Setting

$$v = \begin{pmatrix} \frac{\dot{b}(0)}{2b(0)} & 0 \\ 0 & -\frac{\dot{b}(0)}{2b(0)}\end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})$$

an easy computation shows that $\dot{\varphi}(\gamma') = v - \mathrm{Ad}(\varphi_0(\gamma'))(v)$. This readily implies that $\dot{\varphi}$ is a coboundary. □

2.5 The final step

For $\gamma\in\pi_1(DN')$ let $\mathrm{tr}_\gamma\colon(-\varepsilon,\varepsilon)\to\mathbb{C}$ be the map defined as follows: $\mathrm{tr}_\gamma(t) = \mathrm{trace}(D\widehat{\rho}_t(\gamma))$, where $t\mapsto D\widehat{\rho}_t(\gamma)\in\mathrm{SL}(2,\mathbb{C})$ is a smooth lift of $t\mapsto D\rho_t(\gamma)\in\mathrm{PSL}(2,\mathbb{C})$ (so tr_γ is well-defined only up to the sign). Also recall that D_t and ρ_t are respectively a developing map and a holonomy representation for the hyperbolic structure M_t on N'. The following result can be easily deduced from the proof of the previous proposition.

Lemma 2.10 *Let $\gamma\in\pi_1(DN')$ be such that $D\rho_0(\gamma)$ is non-trivial parabolic. Then $\ddot{\mathrm{tr}}_\gamma(0)=0$.*

As a consequence of Proposition 2.7 and Lemma 2.10 we obtain the following:

Proposition 2.11 *D_t and ρ_t can be chosen in such a way that*

$$\frac{d}{dt}\Big|_{t=0}\rho_t(\gamma)=0$$

for all $\gamma\in\pi_1(N')$. Moreover, if γ is a non-trivial element of a peripheral parabolic subgroup of $\pi_1(N')$, then $\ddot{\mathrm{tr}}_\gamma(0)=0$.

Let $\gamma\in\pi_1(N')$ be such that $\rho_0(\gamma)\neq 1$ and suppose $r_\gamma\colon(-\varepsilon,\varepsilon)\to\partial\mathbb{H}^3$ is a smooth path such that $r_\gamma(t)$ is a fixed point for $\rho_t(\gamma)$ for any $t\in(-\varepsilon,\varepsilon)$. We want to study how the derivative of $r(t)$ is related to the derivative of $\rho_t(\gamma)$. We identify $\partial\mathbb{H}^3$ with $\mathbb{C}\cup\{\infty\}$ and we set

$$\rho_t(\gamma) = \left[\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}\right].$$

Without loss of generality we can suppose that $\rho_0(\gamma)$ does not fix ∞ and that $r_\gamma(0) = 0 \in \mathbb{C} \subset \partial\mathbb{H}^3$, so $b(0) = 0$, $c(0) \neq 0$. An easy computation now shows that

$$r_\gamma(t) = \left(a(t) - d(t) \pm \sqrt{(\mathrm{tr}_\gamma(t)+2)(\mathrm{tr}_\gamma(t)-2)}\right) / (2c(t)).$$

From this formula we can readily deduce the following lemmas.

Lemma 2.12 *If $\rho_0(\gamma)$ is non-trivial loxodromic and $\dot\rho(\gamma) = 0$, then $\dot r_\gamma(0) = 0$. On the other hand, let $\rho_0(\gamma)$ be non-trivial parabolic. Also assume that $\dot\rho(\gamma) = 0$ and $\ddot{\mathrm{tr}}_\gamma(0) = 0$. Then $\dot r_\gamma(0) = 0$.*

We can now conclude the proof of Proposition 2.6. Choose D_t and ρ_t as in the statement of Proposition 2.11. Let S be a boundary component of the universal covering of $\overline{N}'$ which belongs to $\mathcal{S}_{\mathrm{par}}$, denote by P_S the stabilizer of S in $\pi_1(N')$ and choose an element $\gamma \in P_S$ with $\rho_0(\gamma) \neq 1$. By construction the projection of $b_S(t)$ to $\partial\mathbb{H}^3$ is fixed by $\rho_t(\gamma)$, so Lemma 2.12 applies ensuring $\dot b_S(0) = 0$.

Suppose now that S belongs to $\mathcal{S}_{\mathrm{gd}}$, and let P_S be the stabilizer of S in $\pi_1(N')$. For $\gamma \in P_S$ with $\rho_0(\gamma) \neq 1$ let $p_\gamma(t), q_\gamma(t)$ be the fixed points of $\rho_t(\gamma)$ on $\partial\mathbb{H}^3$. Note that since $\rho_0(\gamma)$ is loxodromic we can choose p_γ, q_γ to vary smoothly with respect to t, at least in a small neighbourhood of 0. This gives $\dot p_\gamma = \dot q_\gamma = 0$ by Lemma 2.12. Now a standard result in Kleinian group theory ensures that the set $\{p_\gamma(0), q_\gamma(0) : \gamma \in P_S,\ \rho_0(\gamma) \neq 1\}$ is dense in the closure at infinity of $D_0(S) \subset \mathbb{H}^3$, and this easily implies $\dot b_S(0) = 0$.

3 Dehn filling

Once the smoothness of $\Omega(\mathcal{T})$ at x_0 is established, one can prove Thurston's hyperbolic Dehn filling Theorem just by following the strategy described in [12]. At this stage, this argument applies only to those hyperbolic manifolds with geodesic boundary which admit a good geodesic triangulation (but see Conjecture 1.2). The following result is taken from [12].

Lemma 3.1 *Let $j \in \{1, \dots, k\}$. Then there exists a complex number τ_j with non-zero imaginary part such that if $\{y_n\}_{n\in\mathbb{N}} \subset \Omega(\mathcal{T})$ is a sequence with $\lim_{n\to\infty} y_n = x_0$ and $u_j(y_n) \neq 0$ for every $n \in \mathbb{N}$, then $\lim_{n\to\infty} v_j(y_n)/u_j(y_n) = \tau_j$.*

3.1 Thurston's hyperbolic Dehn filling Theorem

Let U be a sufficiently small neighbourhood of x_0 in $\Omega(\mathcal{T})$ and let $x \in U$. For $j = 1, \dots, k$, we define the *j–Dehn filling coefficient* $(p_j(x), q_j(x)) \in \mathbb{R}^2 \cup \{\infty\}$ as follows: if $u_j(x) = 0$, then $(p_j(x), q_j(x)) = \infty$; otherwise, $p_j(x), q_j(x)$ are the unique real solutions of the equation

$$p_j(x)u_j(x) + q_j(x)v_j(x) = 2\pi i.$$

(Existence and uniqueness of such solutions near x_0 can be easily deduced from Theorem 2.2 and Lemma 3.1.)

Let us set

$$d = (d_1, \dots, d_k)\colon U \to \prod_{i=1}^{k} S^2, \quad d_j(x) = (p_j(x), q_j(x)) \in S^2 = \mathbb{R}^2 \cup \{\infty\}.$$

As a consequence of Theorem 2.2 and Lemma 3.1 we have the following:

Theorem 3.2 *If U is small enough, the map d defines a diffeomorphism onto an open neighbourhood of $(\infty, \dots, \infty)$ in $S^2 \times \cdots \times S^2$.*

For $x \in \Omega(\mathcal{T})$ we denote by $N(x)$ the hyperbolic structure induced on N by x, and by $\widehat{N}(x)$ the metric completion of $N(x)$. Recall that a preferred basis μ_i, λ_i of $H_1(T_i; \mathbb{Z})$ is fixed for every $i = 1, \dots, k$. We also set

$$I\Omega(\mathcal{T}) = \big\{x \in U \subset \Omega(\mathcal{T}) : \text{each Dehn filling coefficient corresponding to } x \\ \text{is equal either to } \infty \text{ or to a pair of coprime integers}\big\}.$$

Theorem 3.3 *If U is sufficiently small and x belongs to $I\Omega(\mathcal{T}) \cap U$, then $\widehat{N}(x)$ admits a complete finite-volume smooth hyperbolic structure which is obtained by adding to $N(x)$ a closed geodesic at any cusp with non-infinite Dehn filling coefficient. From a topological point of view, $\widehat{N}(x)$ is obtained by Dehn filling the i^{th} cusp of N along the slope $p_i(x)\mu_i + q_i(x)\lambda_i$ if $(p_i(x), q_i(x)) \neq \infty$, and by leaving the i^{th} cusp of N unfilled if $(p_i(x), q_i(x)) = \infty$, $i = 1, \dots, k$.*

Proof See eg, Thurston [15], Neumann–Zagier [12], Benedetti–Petronio [1], Frigerio [5]. □

The following proposition will prove useful in the last subsection.

Proposition 3.4 *Let X be any smooth manifold and let $g\colon \Omega(\mathcal{T}) \to X$ be a smooth map. Suppose that there exists a small neighbourhood U of x_0 in $\Omega(\mathcal{T})$ such that for all $x, x' \in U \cap I\Omega(\mathcal{T})$ with $d(x) = -d(x')$ we have $g(x) = g(x')$. Then $dg_{x_0} = 0$.*

Proof Since $d\colon U \to \prod_{i=1}^{k} S^2$ defines a chart around x_0, it is sufficient to observe that $U \cap I\Omega(\mathcal{T})$ accumulates to x_0 along any direction in $T_{x_0}\Omega(\mathcal{T})$. □

3.2 Infinitesimal deformations of compact tetrahedra

We know from Mostow's rigidity Theorem that compact hyperbolic 3–manifolds do not admit deformations. The following results seem to suggest that in the non-compact case deformations take place mostly near the cusps: even if it has to be affected by any non-trivial deformation, the compact core offers resistance to changing its shape. More precisely, we now show that deformations of $\mathcal{T}$ near the complete structure affect compact tetrahedra only at the second order.

Let f be any *compact* internal edge of $\mathcal{T}$. For $x \in \Omega(\mathcal{T})$ we denote by $\ell^f(x)$ the length of f with respect to the metric structure defined by x. By Lemma 1.4 the map $\ell^f\colon \Omega(\mathcal{T}) \to \mathbb{R}$ is smooth.

Proposition 3.5 *We have $d\ell^f_{x_0} = 0$.*

Proof Let $x, x' \in U \cap I\Omega(\mathcal{T})$ be such that $d(x) = -d(x')$. Then the identity of N extends to a homeomorphism between $\widehat{N}(x)$ and $\widehat{N}(x')$. By Mostow–Prasad's rigidity Theorem, such a homeomorphism is homotopic to an isometry $\psi\colon \widehat{N}(x) \to \widehat{N}(x')$ via a homotopy which preserves the geodesic boundary (see eg, [4]). For $y \in U \cap I\Omega(\mathcal{T})$ let $f(y) \subset N(y) \subset \widehat{N}(y)$ be the geodesic segment corresponding to f. From the above discussion it follows that $\psi(f(x))$ is homotopic to $f(x')$ relatively to $\partial\widehat{N}(x')$. Since both $\psi(f(x))$ and $f(x')$ intersect $\partial\widehat{N}(x')$ perpendicularly, this easily implies that $\psi(f(x)) = f(x')$, whence $\ell^f(x) = \ell^f(x')$. Now the conclusion follows from Proposition 3.4. □

The lengths of the boundary edges of a compact lateral hexagon smoothly depend on the lengths of its internal edges, and the dihedral angles of a compact truncated tetrahedron smoothly depend on the lengths of its internal edges. Thus Proposition 3.5 implies the following results.

Corollary 3.6 *Fix a boundary edge f lying on a* compact *lateral hexagon of some tetrahedron of $\mathcal{T}$, and let $\ell^f\colon \Omega(\mathcal{T}) \to \mathbb{R}$ be the function which associates to $x \in \Omega(\mathcal{T})$ the length of f in the geometric realization parameterized by x. Then $d\ell^f_{x_0} = 0$.*

Corollary 3.7 *Fix an internal edge f in a* compact *tetrahedron of $\mathcal{T}$ and let $a^f\colon \Omega(\mathcal{T}) \to \mathbb{R}$ be the function that associates to $x \in \Omega(\mathcal{T})$ the dihedral angle assigned to f by x. Then $da^f_{x_0} = 0$.*

3.3 Infinitesimal deformations of the geodesic boundary

Let $\mathrm{Teich}(\partial N)$ be the Teichmüller space of hyperbolic structures on ∂N, ie, the space of equivalence classes of hyperbolic metrics on ∂N, where two such metrics are considered equivalent if they are isometric through a diffeomorphism homotopic to the identity of ∂N. For $x \in \Omega(\mathcal{T})$ we denote by $B(x) \in \mathrm{Teich}(\partial N)$ the equivalence class of the hyperbolic structure induced by $N(x)$ on ∂N. It is well-known that $\mathrm{Teich}(\partial N)$ admits a structure of differentiable manifold such that $B\colon \Omega(\mathcal{T}) \to \mathrm{Teich}(\partial N)$ is smooth. As a consequence of Mostow–Prasad's rigidity Theorem and of Proposition 3.4 we get the following:

Proposition 3.8 *We have $dB_{x_0} = 0$.*

Appendix A Cohomology theory of representations

A.1 The tangent space to a representation

Let G be a Lie group with associated Lie algebra $\mathfrak{g}$ and Γ be any group, and denote by $\mathcal{R}(\Gamma, G)$ the set of representations of Γ in G. We say that a path $\{\rho_t \in \mathcal{R}(\Gamma, G) : t \in (-\varepsilon, \varepsilon)\}$ is *smooth* if $\rho_t(\gamma)$ is a smooth function of t for any $\gamma \in \Gamma$. If $\{\rho_t\}$ is a smooth path of representations, the tangent vector to the map $t \mapsto \rho_t(\gamma)$ at 0 gives an element in $T_{\rho_0(\gamma)}G$. Identifying this tangent space with $\mathfrak{g} = T_1 G$ by *right* translation we get an element $\dot\rho(\gamma)$ of the Lie algebra $\mathfrak{g}$:

$$\dot\rho(\gamma) = \frac{d}{dt}\Big|_{t=0} \left(\rho_t(\gamma)\rho_0(\gamma)^{-1}\right).$$

Differentiating the homomorphism relation $\rho_t(\gamma_1\gamma_2) = \rho_t(\gamma_1)\rho_t(\gamma_2)$ we see that $\dot\rho\colon \Gamma \to \mathfrak{g}$ satisfies the so-called *cocycle* relation

$$\dot\rho(\gamma_1\gamma_2) = \dot\rho(\gamma_1) + \mathrm{Ad}(\rho_0(\gamma_1))(\dot\rho(\gamma_2)),$$

where $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is the usual adjoint representation.

Consider now a *trivial* deformation of ρ_0, ie, let $t \mapsto g_t$ be a smooth path in G starting at the identity and set $\rho_t(\gamma) = g_t\rho_0(\gamma)g_t^{-1}$ for all $\gamma \in \Gamma$. Then differentiation shows that

$$\dot\rho(\gamma) = \dot g - \mathrm{Ad}(\rho_0(\gamma))(\dot g)$$

for any $\gamma \in \Gamma$, where $\dot{g} \in \mathfrak{g}$ is the tangent vector to $t \mapsto g_t$ at $t = 0$. We now set:

$$\begin{aligned} Z^1(\Gamma; \mathfrak{g}; \rho_0) &= \{c\colon \Gamma \to \mathfrak{g} :\ c(\gamma_1\gamma_2) = c(\gamma_1) + \mathrm{Ad}(\rho_0(\gamma_1))(c(\gamma_2))\} \\ B^1(\Gamma; \mathfrak{g}; \rho_0) &= \{b\colon \Gamma \to \mathfrak{g} :\ b(\gamma) = m - \mathrm{Ad}(\rho_0(\gamma))(m) \text{ for some } m \in \mathfrak{g}\} \\ H^1(\Gamma; \mathfrak{g}; \rho_0) &= Z^1(\Gamma; \mathfrak{g}; \rho_0)/B^1(\Gamma; \mathfrak{g}; \rho_0) \end{aligned}$$

The above discussion shows that $Z^1(\Gamma; \mathfrak{g}; \rho_0)$ corresponds in some sense to the tangent space of $\mathcal{R}(\Gamma, G)$ at ρ_0. Under this identification the module $B^1(\Gamma; \mathfrak{g}; \rho_0)$ should represent the tangent space to trivial deformations of ρ_0, so $H^1(\Gamma; \mathfrak{g}; \rho_0)$ should give the tangent space of $\mathcal{R}(\Gamma, G)/G$ at $[\rho_0]$ (however, this holds true only in the setting of algebraic schemes).

Let $c \in Z^1(\Gamma; \mathfrak{g}; \rho)$, and suppose that Γ_0 is a normal subgroup of Γ such that $\rho(\gamma) = 1_G$, $c(\gamma) = 0$ for all $\gamma \in \Gamma_0$. Let also $\overline{\rho}\colon \Gamma/\Gamma_0 \to G$ be the representation induced by ρ.

Lemma A.1 *The map $\overline{c}\colon \Gamma/\Gamma_0 \to M$ defined by $\overline{c}([\gamma]) = c(\gamma)$ is well-defined and gives a cocycle $\overline{c} \in Z^1(\Gamma/\Gamma_0; \mathfrak{g}; \overline{\rho})$. Moreover, we have $\overline{c} \in B^1(\Gamma/\Gamma_0; \mathfrak{g}; \overline{\rho})$ if and only if $c \in B^1(\Gamma; \mathfrak{g}; \rho)$.*

A.2 Classical rigidity results

Let N be a smooth 3–manifold without boundary and suppose $\rho_0\colon \pi_1(N) \to \mathrm{PSL}(2, \mathbb{C})$ is the holonomy representation for a complete finite-volume hyperbolic structure on N. The following result is due to Weil [16], and can be considered as a local version of Mostow's rigidity Theorem for compact hyperbolic 3–manifolds.

Theorem A.2 *Suppose N is compact. Then $H^1(\pi_1(N); \mathfrak{sl}(2, \mathbb{C}); \rho_0) = 0$.*

Suppose now that N compactifies to a manifold $\overline{N}$ with non-empty boundary $\partial\overline{N} = T_1 \sqcup \ldots \sqcup T_k$. In this case ρ_0 admits non-trivial deformations, so we cannot expect $H^1(\pi_1(N); \mathfrak{sl}(2, \mathbb{C}); \rho_0)$ to be trivial. If K is a subgroup of $\pi_1(N)$, the natural injection $i_K\colon K \to \pi_1(N)$ induces a map on cohomology

$$i_K^*\colon H^1(\pi_1(N); \mathfrak{sl}(2, \mathbb{C}); \rho_0) \to H^1(K; \mathfrak{sl}(2, \mathbb{C}); \rho_0 \circ i_K).$$

For $\gamma \in \pi_1(B)$ we denote by $\langle\gamma\rangle \subset \pi_1(N)$ the cyclic subgroup generated by γ. If $P = \{\gamma \in \pi_1(N) :\ \rho_0(\gamma) \text{ is parabolic}\}$, we set

$$H^1_{\mathrm{par}}(\pi_1(N); \mathfrak{sl}(2, \mathbb{C}); \rho_0) = \bigcap_{\gamma \in P} \mathrm{Ker}\, i^*_{\langle\gamma\rangle} \subset H^1(\pi_1(N); \mathfrak{sl}(2, \mathbb{C}); \rho_0).$$

The naïve correspondence between $H^1(\pi_1(N);\mathfrak{sl}(2,\mathbb{C});\rho_0)$ and the set of conjugacy classes of infinitesimal deformations of ρ_0 restricts to an identification between $H^1_{\mathrm{par}}(\pi_1(N);\mathfrak{sl}(2,\mathbb{C});\rho_0)$ and the set of classes of infinitesimal deformations through holonomies for *complete* structures on N. Thus the following result [8; 14] can be considered an infinitesimal version of Mostow–Prasad's rigidity Theorem for complete finite-volume hyperbolic 3–manifolds:

Theorem A.3 *We have* $H^1_{\mathrm{par}}(\pi_1(N);\mathfrak{sl}(2,\mathbb{C});\rho_0)=0$.

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