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FINITELY GENERATED SEMI-CONES IN PRODUCT RINGS

By

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Abstract. Semi-cones of rings determine the partial orders in the rings. We consider semi-cones in the direct product rings and the product extension rings, inducing finitely generated semi-cones. In particular, we give characterizations for semi-cones of the direct product rings and the basic product extension rings of the ring of integers.

1. Introduction

As a generalization of positive cones of integral domains, we introduced semi-cones of rings which determine partial orders in the rings ([4, 6]). In this paper, we consider semi-cones in the direct product rings and the product extension rings, inducing finitely generated semi-cones. In particular, we give characterizations for semi-cones of the direct product rings and the basic product extension rings of the ring of integers.

The symbol R means a non-zero commutative ring with the identity element denoted by 1.

The symbol $\mathbf{Z}$ means the ring of integers. Define $\mathbf{N} = \{1, 2, \dots\}$, and $\mathbf{Z}^* = \mathbf{N} \cup \{0\}$.

Let A, B be subsets of R. Define $-A = \{-x \mid x \in A\}$, $A + B = \{x + y \mid x \in A, y \in B\}$, $AB = \{xy \mid x \in A, y \in B\}$, $aB = \{a\}B$ for $a \in R$, and $A \backslash \{0\} = \{x \mid x \in A, x \neq 0\}$. Also, define the direct product set $A \times B = \{(x, y) \mid x \in A, y \in B\}$.

The single set $\{0\}$ (or $\{(0,0)\}$) is often denoted by 0.

As is well-known, for a partial order $\leq$ on R, $(R, \leq)$ is a *partially ordered ring* ([1]) if R satisfies the following conditions:

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(i) $a \le b$ implies $a + x \le b + x$ for all x, and

(ii) $a \le b$ and $0 \le x$ implies $ax \le bx$.

For a subset S of (a ring) R, let us call S a *semi-cone* (resp. *cone*) of R if it satisfies (i), (ii), and (iii) (resp. (i), (ii), (iii), and (iv)) below; see [4, 6].

(i) $S + S \subset S$, that is, S is additive.

(ii) $SS \subset S$, that is, S is multiplicative.

(iii) $S \cap (-S) = 0$.

(iv) $R = S \cup (-S)$.

A subset S of R satisfying (i), (ii) is called a *positive cone* ([8] (or [2])) if $R \backslash \{0\} = S \cup (-S)$.

We note that for a semi-cone S of R, we induce a partially ordered ring $(R, \le_S)$, defining $x \le_S y$ by $y - x \in S$. Conversely, for a partially ordered ring $(R, \le)$, putting $S = \{x \in R \mid 0 \le x\}$, S is a semi-cone of R, and $\le = \le_S$.

In view of the above, a ring R is a partially ordered ring; ordered ring; ordered integral domain iff it has a semi-cone; cone; positive cone, respectively.

For a ring R, let us recall the following product rings (I) and (II) on $R \times R$.

(I) The usual *direct product* $R \times R$ equipped with component-wise addition and multiplication (that is, for $(x, y), (x', y') \in R \times R$, $(x, y) + (x', y') = (x + x', y + y')$, and $(x, y) \cdot (x', y') = (xx', yy')$).

Let us call such a ring the *direct product ring* of R, and denote it the symbol $R \otimes R$ (as in [5, 6]).

(II) Let $(a, b) \in R \times R$. The ring $R \times R$ equipped with addition and multiplication by $(x, y) + (x', y') = (x + x', y + y')$ and $(x, y) * (x', y') = (xx' + ayy', xy' + yx' + byy')$.

Let us call such an extension ring of R the *product extension ring* of R, and denote it the symbol $(R \ltimes R; a, b)$ (as in [5, 6]). For example, an extension ring $(\mathbf{R} \ltimes \mathbf{R}; -1, 0)$ of the real number field $\mathbf{R}$ is isomorphic to the complex number field. (Algebraic structures of the rings $(R \ltimes R; a, b)$ and their ideals are observed in [5]). Especially, a basic ring $(R \ltimes R; 0, 0)$ is called the *trivial extension* of R by itself, denoted by $R \ltimes R$. As is well-known, this ring gives useful examples related to ring structures and order structures, or extensions ([9], for example).

2. Semi-cones

Lemma 2.1. *Let A, A' be subsets of R with $A, A' \ni 0$. If $A \times A'$ is multiplicative in $(R \ltimes R; a, b)$, then $AA \subset A$, $AA' \subset A'$, $aA'A' \subset A$, and $bA'A' \subset A'$. The converse holds if A and A' are additive.*

Proof. The first half holds, noting $(0,x')*(0,y') = (ax'y', bx'y')$ and $(x,x')*(y,0) = (xy, x'y)$ in $(R \ltimes R; a,b)$. The latter part is routine. □

Proposition 2.2. *Let A and A' be subsets of R. Then the following hold.*
(1) *$A \times A'$ is a semi-cone of $R \otimes R$ iff A and A' are semi-cones of R.*
(2) *$A \times A'$ is a semi-cone of $(R \ltimes R; a,b)$ iff A is a semi-cone of R, and A' is an additive set such that $A' \cap -A' = 0$, $AA' \subset A'$, $aA'A' \subset A$, and $bA'A' \subset A'$. In particular, $A \times A'$ is a semi-cone of $R \ltimes R$ iff A is a semi-cone of R, and A' is additive with $A' \cap -A' = 0$ and $AA' \subset A'$.*

Proof. (1) is routine. (2) is routinely shown, using Lemma 2.1. □

Remark 2.3. (1) In the only if part of Proposition 2.2(2), (i) for every $(R \ltimes R; a,b)$, A' need not be a semi-cone of R; while (ii) for $a \neq 0$, $A' \cap -A' = 0$ can be deleted under R being an integral domain. (Indeed, for (i), let $R = \mathbf{Z}$, and $A = \mathbf{Z}^*$ and $A' = -\mathbf{Z}^*$ in R. Then, for $a \geq 0$ and $b \leq 0$, $A \times A'$ is a semi-cone of $(R \ltimes R; a,b)$ by Proposition 2.2(2), but A' is not a semi-cone of R by $A'A' \not\subset A'$. For (ii), let $x \in A' \cap -A'$. Then $ax^2, -ax^2 \in A$ by $aA'A' \subset A$. Thus, $ax^2 = 0$ by $A \cap -A = 0$, thus $x = 0$. Hence $A' \cap -A' = 0$).

(2) For a semi-cone S of R with $SS \neq 0$, let $S_1 = S \times 0$ and $S_2 = 0 \times S$ be semi-cones of $R' = R \ltimes R$. Then $S_1 \times S_2$ is a semi-cone of $R' \ltimes R'$, but $S_2 \times S_1$ is not a semi-cone by Proposition 2.2(2), noting $S_1 * S_2 \subset S_2$, but $S_2 * S_1 \not\subset S_1$.

Let $p_1, p_2 : R \times R \to R$ be the projections defined by $p_1(x,y) = x$, and $p_2(x,y) = y$, unless otherwise stated.

For a semi-cone A of $R \otimes R$ or $R \ltimes R$, $p_i(A)$ need not be a semi-cone of R for each $i = 1,2$; see Remark 2.7 later. But, we have the following proposition which is routinely shown, here (2)(b) holds by Proposition 2.2(2).

Proposition 2.4. *The following hold.*
(1) *For a semi-cone A of $R \otimes R$,* (a) and (b) *below hold.*
  (a) *$p_i(A)$ is a semi-cone of R iff $p_i(A) \cap p_i(-A) = 0$ for each $i = 1,2$.*
  (b) *$p_1(A) \times p_2(A)$ is a semi-cone of $R \otimes R$ iff $p_i(A)$ are semi-cones of R (equivalently, $p_i(A) \cap p_i(-A) = 0$) for $i = 1,2$.*
(2) *For a semi-cone A of $R \ltimes R$,* (a) *and* (b) *below hold.*
  (a) *$p_1(A)$ is additive and multiplicative, and $p_2(A)$ is additive. In particular, $p_1(A)$ is a semi-cone of R iff $p_1(A) \cap p_1(-A) = 0$. While, $p_2(A)$ is a semi-cone of R iff $p_2(A) \cap p_2(-A) = 0$, and $p_2(A)p_2(A) \subset p_2(A)$.*

(b) $p_1(A) \times p_2(A)$ *is a semi-cone of* $R \ltimes R$ *iff* $p_i(A) \cap p_i(-A) = 0$ $(i = 1, 2)$, *and* $p_1(A)p_2(A) \subset p_2(A)$.

REMARK 2.5. In Proposition 2.4(2), if $p_2(A) \cap p_2(-A) = 0$ (resp. $p_2(A)p_2(A) \subset p_2(A)$) is deleted in (a), $p_2(A)$ need not be a semi-cone of R (by a cone $A = (\mathbf{N} \times \mathbf{Z}) \cup (0 \times \mathbf{Z}^*)$ (resp. a semi-cone $A = \mathbf{Z}^* \times -\mathbf{Z}^*$) of $\mathbf{Z} \ltimes \mathbf{Z}$. Also, if $p_2(A) \cap p_2(-A) = 0$ is deleted in (b), $p_1(A) \times p_2(A)$ need not be a semi-cone of $R \ltimes R$ (by the above cone A).

For a subset X of $R \otimes R$ (resp. $R \ltimes R$), the symbol $\operatorname{ann}(X)$ means the set $\{a \in R \mid (a, a)X = 0\}$ (resp. $\{a \in R \mid (a, 0) * X = 0\}$).

PROPOSITION 2.6. *Let* $A \subset R \times R$ *and* $A' = A \cap (0 \times R)$. *The following hold.*

(1) *If* A *is a semi-cone of* $R \otimes R$ *or* $R \ltimes R$, *and* $A' = 0$, *then* $p_1(A)$ *is a semi-cone of* R.

(2) *If* A *is a semi-cone of* $R \otimes R$ (*resp.* $R \ltimes R$), *and* $A' \neq 0$, *then* $p_2(A)$ (*resp.* $p_1(A)$) *is a semi-cone under* $\operatorname{ann}(A') = 0$.

PROOF. For (1), by Proposition 2.4, it suffices to show $p_1(A) \cap -p_1(A) = 0$, so let $x = p_1(x, y) = -p_1(x', y')$ with $(x, y), (x', y') \in A$. Then $(x, y) + (x', y') = (x + x', y + y') = (0, y + y') \in A \cap (0 \times R)$. Thus $y + y' = 0$ by $A' = 0$. Hence $(x, y) = (-x', -y') \in A \cap -A$, so $(x, y) = (0, 0)$ by $A \cap -A = 0$. Then $x = 0$.

For (2), we show $p_1(A)$ is a semi-cone in $R \ltimes R$ (similarly, $p_2(A)$ is a semi-cone in $R \otimes R$). Similarly as in (1), it suffices to show $p_1(A) \cap -p_1(A) = 0$, so let $x = p_1(x, y) = -p_1(x', y')$ with $(x, y), (x', y') \in A$. For any $(0, z) \in A'$, $(x, y) * (0, z) = (0, xz) \in A$, $(x', y') * (0, z) = (0, x'z) \in A$, and hence $(0, xz) = (0, -x'z) = -(0, x'z) \in A \cap -A$, which yields $(0, xz) = (0, 0)$. Then $(x, 0) * A' = (0, 0)$. Thus $x = 0$ by $\operatorname{ann}(A') = 0$ with $A' \neq 0$. □

REMARK 2.7. Related to Proposition 2.6, we have (1) and (2) below.

(1) For a semi-cone A of $R \otimes R$ and $R \ltimes R$ with $A' (= A \cap (0 \times R)) = 0$, $p_2(A)$ need not be a semi-cone of R (by a semi-cone $A = (\mathbf{N} \times \mathbf{Z}) \cup 0$ of $\mathbf{Z} \otimes \mathbf{Z}$ and $\mathbf{Z} \ltimes \mathbf{Z}$).

(2) For a semi-cone A of $R \otimes R$ (resp. $R \ltimes R$) with $A' \neq 0$, we have the following (a) (resp. (b)).

(a) (i) $p_1(A)$ need not be a semi-cone of R under $\operatorname{ann}(A') = 0$ (by a semi-cone $A = (\mathbf{Z} \times \mathbf{N}) \cup 0$ of $\mathbf{Z} \otimes \mathbf{Z}$). Also, (ii) $p_2(A)$ need not be a semi-cone of R (indeed, let $R = \mathbf{Z} \ltimes \mathbf{Z}$, and let $A_1 = 0 \times \mathbf{N}$, $A_2 = 0 \times \mathbf{Z}$. Then $A = (A_1 \times A_2) \cup$

$(0 \times (0 \times \mathbf{Z}^*))$ is a semi-cone of $R \otimes R$ with $\mathrm{ann}(A') \neq 0$, but $p_2(A)$ is not a semi-cone of R).

(b) (i) $p_1(A)$ need not be a semi-cone of R (indeed, let R, and A_1, A_2 be the same as (a)(ii). Then $A = (A_2 \times A_1) \cup 0$ is a semi-cone of $R \ltimes R$ with $\mathrm{ann}(A') \neq 0$, but $p_1(A)$ is not a semi-cone of R). Also, (ii) $p_2(A)$ need not be a semi-cone of R under $\mathrm{ann}(A') = 0$ (by the cone (or semi-cone) A of $\mathbf{Z} \ltimes \mathbf{Z}$ in Remark 2.5).

Proposition 2.8. *For a subset A of $\mathbf{Z}$, the following are equivalent.*

(1) *A is additive, and $A \cap -A = 0$.*

(2) *A is additive with $A \ni 0$, and $A \subset \mathbf{Z}^*$ or $A \subset -\mathbf{Z}^*$.*

(3) *$A = a_1\mathbf{Z}^* + \cdots + a_m\mathbf{Z}^*$ for some $a_1, \ldots, a_m$ with all $a_i \in \mathbf{Z}^*$ or all $a_i \in -\mathbf{Z}^*$.*

Proof. For (1) $\Rightarrow$ (2), suppose (2) doesn't hold. Then $m, -n \in A$ for some $m, n \in \mathbf{N}$, thus $A \cap -A \ni mn \neq 0$, a contradiction.

For (2) $\Rightarrow$ (3), for $A \subset \mathbf{Z}^*$, (3) holds in view of the proof of [3, Proposition 2.9], and thus (3) also holds for $A \subset -\mathbf{Z}^*$, putting $A' = -A$. (3) $\Rightarrow$ (1) is obvious. □

Corollary 2.9. *For a subset A of $\mathbf{Z}$, the following are equivalent* (*cf.* [3]).

(1) *A is a semi-cone of $\mathbf{Z}$.*

(2) *A is additive with $0 \in A \subset \mathbf{Z}^*$.*

(3) *$A = a_1\mathbf{Z}^* + \cdots + a_m\mathbf{Z}^*$ for some $a_1, \ldots, a_m \in \mathbf{Z}^*$.*

The following holds by Proposition 2.6 with Corollary 2.9.

Proposition 2.10. *Let R be an integral domain, and let A be a semi-cone of $R \otimes R$* (*resp. $R \ltimes R$*). *Then $p_1(A)$ or $p_2(A)$* (*resp. $p_1(A)$*) *is a semi-cone of R. In particular, for $R = \mathbf{Z}$, $p_1(A) \subset \mathbf{Z}^*$ or $p_2(A) \subset \mathbf{Z}^*$* (*resp. $p_1(A) \subset \mathbf{Z}^*$*).

The following holds by Propositions 2.2 and 2.8.

Proposition 2.11. *Let A and A' be subsets of $\mathbf{Z}$ with $A' \ni 0$. For $A' \neq 0$, $A \times A'$ is a semi-cone of $(\mathbf{Z} \ltimes \mathbf{Z}; a, b)$ iff A is a semi-cone of $\mathbf{Z}$, and A' is an additive set such that $aA'A' \subset A$, and $A' \subset \mathbf{Z}^*$ with $b \in \mathbf{Z}^*$ or $A' \subset -\mathbf{Z}^*$ with $b \in -\mathbf{Z}^*$.* (*For $A' = 0$, $A \times 0$ is a semi-cone of $(\mathbf{Z} \ltimes \mathbf{Z}; a, b)$ iff so is A of $\mathbf{Z}$*).

Corollary 2.12. *The following hold.*

(1) *For subsets* A *and* A' *of* $\mathbf{Z}$ *with* $A' \ni 0$, $A \times A'$ *is a semi-cone of* $\mathbf{Z} \ltimes \mathbf{Z}$ *iff* (i) A *is a semi-cone* (*equivalently*, A *is additive with* $0 \in A \subset \mathbf{Z}^*$), *and* (ii) A' *is additive with* $A' \subset \mathbf{Z}^*$ *or* $A' \subset -\mathbf{Z}^*$.

(2) *For a semi-cone* A *of* $\mathbf{Z} \ltimes \mathbf{Z}$, $p_1(A) \times p_2(A)$ *is a semi-cone of* $\mathbf{Z} \ltimes \mathbf{Z}$ *iff* $p_2(A) \subset \mathbf{Z}^*$ *or* $p_2(A) \subset -\mathbf{Z}^*$.

The following holds by Propositions 2.11, and 2.8 with Corollary 2.9.

Corollary 2.13. *For subsets* A *and* A' *of* $\mathbf{Z}$, $A \times A'$ *is a semi-cone of* $\mathbf{Z} \ltimes \mathbf{Z}$ *iff* $A = a_1\mathbf{Z}^* + \cdots + a_m\mathbf{Z}^*$ *for some* $a_1, \ldots, a_m \in \mathbf{Z}^*$, *and* $A' = b_1\mathbf{Z}^* + \cdots + b_n\mathbf{Z}^*$ *for some* $b_1, \ldots, b_n$ *with all* $b_i \in \mathbf{Z}^*$ *or all* $b_i \in -\mathbf{Z}^*$.

Corollary 2.14. *Corollaries* 2.12 *and* 2.13 *remain true in* $\mathbf{Z} \otimes \mathbf{Z}$, *but delete the part of* "$-\mathbf{Z}^*$" *in these corollaries.*

3. Finitely Generated Semi-cones

We shall introduce the concept of finitely generated semi-cones. We note that arbitrary intersections of semi-cones are semi-cones. Let X be a subset of R. When X is contained in some semi-cone, the intersection of all semi-cones which contain X is evidently the smallest semi-cone containing X. If there exists the smallest semi-cone containing X, then we shall call it the *semi-cone generated by* X, denoted by $\langle X \rangle$. Obviously, $\langle X \rangle = \langle X \cup 0 \rangle$.

For a finite subset $\{x_1, \ldots, x_n\}$ of R contained in some semi-cone, the symbol $\langle x_1, \ldots, x_n \rangle$ means $\langle \{x_1, \ldots, x_n\} \rangle$.

Let A be a semi-cone of R. We shall say that A is *finitely generated* if $A = \langle F \rangle$ for some finite subset F in A.

We note that every finitely generated semi-cone of R must be countable in view of Proposition 3.1 below. Also, note that every semi-cone A of R need not be finitely generated even if A is contained in a finitely generated semi-cone of R; see Proposition 3.8 (or Example 3.9) later.

The following basic proposition is routinely shown.

Proposition 3.1. *Let* $F = \{x_1, \ldots, x_n\}$ *be a finite subset of some semi-cone in* R *with all* $x_i \neq 0$, *and let* $x_i^0 = 1$. *Let* $\bar{F}$ *be the set of all finite sums of elements of the form* $cx_1^{v_1} \cdots x_n^{v_n}$, *where* $c, v_1, \ldots, v_n \in \mathbf{Z}^*$ *with some* $v_i > 0$. *Then* $\langle F \rangle = \bar{F}$.

For an element of the set $\bar{F}$ in the previous proposition, we will use a brief symbol

$$\sum c x_1^{v_1} \cdots x_n^{v_n}$$

under $c, v_1, \ldots, v_n \in \mathbf{Z}^*$ with some $v_i > 0$.

Proposition 3.2. *For a non-zero subset A of R, A is a semi-cone of R iff A has a cover $\mathscr{C}$ (i.e., $A = \bigcup\{X \mid X \in \mathscr{C}\}$) of semi-cones generated by any finitely many elements (or two elements) of A. In particular, for A being countable, we can take $\mathscr{C}$ to be an increasing countable cover of semi-cones generated by finitely many elements (or, a countable cover of semi-cones generated by any two elements) of A.*

Proof. For the only if part, let $\mathscr{F}$ be the collection of all finite sets (or two elements) in a semi-cone A. Let $\mathscr{C} = \{\langle F \rangle \mid F \in \mathscr{F}\}$. Then $\mathscr{C}$ is a cover of A, and each $\langle F \rangle$ is a finitely generated semi-cone. For the if part, note that any two elements x, y in A are contained in some semi-cone in A. Thus, A is a semi-cone. For A being countable, let $A = \{a_i \mid i \in \mathbf{N}\}$, and $F_n = \{a_1, a_2, \ldots, a_n\}$. Then $\mathscr{C} = \{\langle F_n \rangle \mid n \in \mathbf{N}\}$ is a desired increasing cover. □

Remark 3.3. Every union of two semi-cones generated by finitely many elements need not be a semi-cone (indeed, for finitely generated semi-cones $\mathbf{Z}^* \times 0$ and $0 \times \mathbf{Z}^*$ in $\mathbf{Z} \ltimes \mathbf{Z}$ (by Corollary 3.5 later), their union is not a semi-cone).

Theorem 3.4. *Let S, T be semi-cones of R. Then the following hold.*

(1) *$S \times T$ is a finitely generated semi-cone of $R \otimes R$ iff so are S and T of R.*

(2) *$S \times T$ is a finitely generated semi-cone of $R \ltimes R$ iff* (i) *S is finitely generated and* (ii) *$T = (Sy_1 + \cdots + Sy_n) + (\mathbf{Z}^* y_1 + \cdots + \mathbf{Z}^* y_n)$ for some $y_1, \ldots, y_n \in T$.*

Proof. For (1), it is routinely shown (as in the proof below).

For (2), note $S \times T$ is a semi-cone of $R \ltimes R$ iff $ST \subset T$ by Proposition 2.2(2). For the only if part of (2), let $S \times T = \langle (x_1, y_1), \ldots, (x_n, y_n) \rangle$ with all $(x_i, y_i) \neq 0$. Obviously, $S = \langle x_1, \ldots, x_n \rangle$, hence (i) holds. To see (ii), let $T' = Sy_1 + \cdots + Sy_n + \mathbf{Z}^* y_1 + \cdots + \mathbf{Z}^* y_n$. Let $y \in T$. Then $(0, y) \in S \times T$, so let $(0, y) = \sum c(x_1, y_1)^{v_1} * \cdots * (x_n, y_n)^{v_n} \in S \times T$ by Proposition 3.1. Note that for

$a \in \mathbf{Z}^*$ and $(s,t), (s',t') \in S \times T$, $a(s,t) = (as, at) \in S \times \mathbf{Z}^* t$, and $(s,t) * (s',t') = (ss', st' + s't) \in S \times (St + St')$. Then we show that $(0, y) \in S \times T'$, so $y \in T'$. Thus $T \subset T'$. While, $T' \subset T$ by $ST \subset T$. Hence $T' = T$. For the if part of (2), assume (i) and (ii) hold. Since $ST \subset T$, $S \times T$ is a semi-cone of $R \ltimes R$. Let $S = \langle x_1, \dots, x_m \rangle$. Let $F = \{(x_1, 0), \dots, (x_m, 0), (0, y_1), \dots, (0, y_n)\}$. To see $S \times T = \langle F \rangle$, let $(x, y) \in S \times T$. Let $x = \sum c x_1^{v_1} \cdots x_m^{v_m}$, and let $y = \sum_{j=1}^n s_j y_j + \sum_{j=1}^n c_j y_j$ $(s_j \in S,\ c_j \in \mathbf{Z}^*)$. Then,

$$\begin{aligned}
(x, 0) &= \left(\sum c x_1^{v_1} \cdots x_m^{v_m}, 0\right) = \sum c (x_1^{v_1} \cdots x_m^{v_m}, 0) \\
&= \sum c (x_1, 0)^{v_1} * \cdots * (x_m, 0)^{v_m} \in \langle F \rangle \\
(0, y) &= \left(0, \sum_{j=1}^n s_j y_j\right) + \left(0, \sum_{j=1}^n c_j y_j\right) \\
&= \sum_{j=1}^n (0, s_j y_j) + \sum_{j=1}^n c_j (0, y_j) \\
&= \sum_{j=1}^n (s_j, 0) * (0, y_j) + \sum_{j=1}^n c_j (0, y_j) \in \langle F \rangle,
\end{aligned}$$

noting each $(s_j, 0) \in \langle F \rangle$ by the above. Hence $(x, y) = (x, 0) + (0, y) \in \langle F \rangle$. Thus $S \times T = \langle F \rangle$. □

COROLLARY 3.5. *Let S be a semi-cone of R. Then the following hold.*
(1) *$S \times 0$ is a finitely generated semi-cone of $R \ltimes R$ iff so is S of R.*
(2) *$0 \times S$ is a finitely generated semi-cone of $R \ltimes R$ iff $S = \mathbf{Z}^* x_1 + \cdots + \mathbf{Z}^* x_n$ for some $x_1, \dots, x_n$ in S.*
(3) *$S \times S$ is a finitely generated semi-cone of $R \ltimes R$ iff so is S of R.*

PROOF. (1), (2), and (3) hold by Theorem 3.4. But, for (3), note $S = \sum_{i=1}^n S x_i + \sum_{i=1}^n \mathbf{Z}^* x_i$ if $S = \langle x_1, \dots, x_n \rangle$. □

COROLLARY 3.6. *Let S and T be semi-cones of R with $S \subset T$ (or, $1 \in T$). If $S \times T$ is a finitely generated semi-cone of $R \ltimes R$, then so are S and T of R.*

PROOF. Let $S \times T = \langle (x_1, y_1), \dots, (x_n, y_n) \rangle$ with all $(x_i, y_i) \in S \times T$. Let $F = \{x_i, y_j \mid i, j = 1, \dots, n\}$. Since $S \subset T$ (especially, $1 \in T$ by use of Lemma 2.1), $T = \langle F \rangle$ in view of the proof of Theorem 3.4(2). □

In the previous corollary, the converse need not hold (even if $S = 0$, $1 \in T$); see Example 3.9 later.

We will consider finitely generated semi-cones in $\mathbf{Z} \ltimes \mathbf{Z}$ (or $\mathbf{Z} \otimes \mathbf{Z}$).

Let us recall (lexicographic) sets $\boldsymbol{L} = (\mathbf{N} \times \mathbf{Z}) \cup (0 \times \mathbf{Z}^*)$ and $\boldsymbol{L}^* = (\mathbf{N} \times \mathbf{Z}) \cup (0 \times -\mathbf{Z}^*)$ in $\mathbf{Z} \times \mathbf{Z}$. We note that the cones of $\mathbf{Z} \ltimes \mathbf{Z}$ are precisely the sets $\boldsymbol{L}$ and $\boldsymbol{L}^*$ ([6]). While, $R \otimes R$ has no cones ([4]).

Proposition 3.7. *The following hold.*

(1) *All semi-cones of* $\mathbf{Z}$ *are finitely generated.*

(2) *The cones* $\boldsymbol{L}$ *and* $\boldsymbol{L}^*$ *of* $\mathbf{Z} \ltimes \mathbf{Z}$ *are finitely generated.*

Proof. (1) holds by Corollary 2.9. For (2), note $\boldsymbol{L} = \langle (1,-1), (0,1) \rangle$ and $\boldsymbol{L}^* = \langle (1,1), (0,-1) \rangle$ by the proof of [6, Proposition 2.12]. □

For a non-zero semi-cone S of R, let us define $S_0 = S \setminus \{0\}$, unless otherwise stated.

Proposition 3.8. *Let* S *and* T *be non-zero semi-cones of* $\mathbf{Z}$. *Then the following hold in* $\mathbf{Z} \otimes \mathbf{Z}$ *as well as* $\mathbf{Z} \ltimes \mathbf{Z}$.

(1) $S \times T$ *is a finitely generated semi-cone, while its subset*

(2) $A = (S_0 \times T_0) \cup 0$ *is a semi-cone, but* A *is not finitely generated.*

Proof. For (1), $S \times T$ is a semi-cone of $\mathbf{Z} \otimes \mathbf{Z}$ and $\mathbf{Z} \ltimes \mathbf{Z}$, and it is finitely generated by Theorem 3.4 with Corollary 2.9.

For (2), A is a semi-cone in $\mathbf{Z} \otimes \mathbf{Z}$ and $\mathbf{Z} \ltimes \mathbf{Z}$. To see the latter part in $\mathbf{Z} \otimes \mathbf{Z}$, suppose $A = \langle (a_1, b_1), \ldots, (a_n, b_n) \rangle$ with all $(a_i, b_i) \in S_0 \times T_0$. Let p be a prime number with $p > \max(a_1, \ldots, a_n)$. Take $a \in S_0$ and let $s = pa$. Then $s \in S_0$. Let $t = \min T_0 \in T_0$. Then $(s,t) \in A$, so let $(s,t) = \sum c(a_1, b_1)^{v_1} \cdots (a_n, b_n)^{v_n}$ by Proposition 3.1. Noting $x + y > t$ for all $x, y \in T_0$, $(s,t) = (a_1, b_1)^{v_1} \cdots (a_n, b_n)^{v_n}$ for some $v_1, \ldots, v_n$ with $c = 1$. Then $s = a_1^{v_1} \cdots a_n^{v_n}$, thus p is a divisor of some a_i, a contradiction. Hence A is not finitely generated. In $\mathbf{Z} \ltimes \mathbf{Z}$, A is also not finitely generated by the same way (or, taking $s \in S_0$ distinct from any a_i). □

As is seen above, every semi-cone which is a subset of a finitely generated semi-cone need not be finitely generated in $\mathbf{Z} \otimes \mathbf{Z}$ or $\mathbf{Z} \ltimes \mathbf{Z}$. Also, let us give a finitely generated semi-cone S of R such that (i) $S \times S$ is a finitely generated semi-cone of $R \ltimes R$, but (ii) a semi-cone $0 \times S$ of $R \ltimes R$ is not finitely generated. (In $R \otimes R$, such a semi-cone S of R does not exist in view of Theorem 3.4(1)).

EXAMPLE 3.9. For the cone $\boldsymbol{L}(=(\mathbf{N}\times\mathbf{Z})\cup(0\times\mathbf{Z}^*))$ of $R=\mathbf{Z}\ltimes\mathbf{Z}$, the above (i) and (ii) hold in $R\ltimes R$. Similarly, for the cone $\boldsymbol{L}^*$, (i) and (ii) also hold.

Indeed, (i) holds by Proposition 3.7(2) and Corollary 3.5(3). To see (ii), suppose that a semi-cone $0\times\boldsymbol{L}$ is finitely generated. Then, by Corollary 3.5(2), $\boldsymbol{L}=\mathbf{Z}^*(x_1,y_1)+\cdots+\mathbf{Z}^*(x_n,y_n)$ for some $(x_1,y_1),\dots,(x_n,y_n)$ in $\boldsymbol{L}$. Take $y_0\in\mathbf{Z}$ with $y_0<y_i$ for all y_i. Then $(1,y_0)\in\boldsymbol{L}$, but $(1,y_0)\notin\mathbf{Z}^*(x_1,y_1)+\cdots+\mathbf{Z}^*(x_n,y_n)$, a contradiction. To see this, assume $(1,y_0)=a_1(x_1,y_1)+\cdots+a_n(x_n,y_n)$ for some $a_1,\dots,a_n\in\mathbf{Z}^*$. Then $(1,y_0)=(a_1x_1+\cdots+a_nx_n,a_1y_1+\cdots+a_ny_n)$. Since all $x_i\geq 0$, all $a_ix_i\geq 0$. Thus $a_ix_i=1$ for some i. Hence, for $j\neq i$, $a_jx_j=0$, thus $a_jy_j=0$ for $a_j=0$, or $y_j\geq 0$ for $a_j\neq 0$ (by $(0,y_j)\in 0\times\mathbf{Z}^*$). Hence, $y_0=a_1y_1+\cdots+a_ny_n\geq y_i>y_0$. This is a contradiction. Then (ii) holds.

For a non-zero semi-cone S of R, recall the following subsets of $R\times R$ ([6]).

$D_0=\{(x,x)\mid x\in S\}$; $D_1=\{(x+y,x)\mid x,y\in S\}$; $D_2=\{(x,x+y)\mid x,y\in S\}$; and $L=(S_0\times R)\cup(0\times S)$; $L'=(R\times S_0)\cup(S\times 0)$.

In $R\otimes R$, D_0, D_1, D_2 (except L, L') are semi-cones. In $R\ltimes R$, D_2 is a semi-cone, and L (except D_0, D_1) is a semi-cone under R being an integral domain. But, L' is not a semi-cone in $R\ltimes R$. (For these, see [6]).

PROPOSITION 3.10. *Let S be a non-zero semi-cone of $\mathbf{Z}$. Then the following hold.*

(1) *D_0, D_1, D_2 are finitely generated semi-cones in $\mathbf{Z}\otimes\mathbf{Z}$, and so is D_2 in $\mathbf{Z}\ltimes\mathbf{Z}$.*

(2) *L is a finitely generated semi-cone of $\mathbf{Z}\ltimes\mathbf{Z}$ iff $1\in S$ (i.e., $L=\boldsymbol{L}$).*

PROOF. For (1), let $S=\sum_i\mathbf{Z}^*x_i$ with $x_1,\dots,x_n\in S$ by Corollary 2.9. We show D_2 is finitely generated in $\mathbf{Z}\otimes\mathbf{Z}$ (or $\mathbf{Z}\ltimes\mathbf{Z}$), for example. Let $x,y\in S$ with $x=\sum_i c_ix_i$, $y=\sum_i d_ix_i$ $(c_i,d_i\in\mathbf{Z}^*)$. Then $(x,x+y)=(x,x)+(0,y)=\sum_i c_i(x_i,x_i)+\sum_i d_i(0,x_i)$. Thus D_2 is generated by $\{(x_i,x_i),(0,x_i)\mid i=1,\dots,n\}$.

For (2), to see the if part, let $1\in S$. Then $S=\mathbf{Z}^*$, so $L=\boldsymbol{L}$. Hence, L is finitely generated by Proposition 3.7(2). For the only if part, let $L=\langle F\rangle$ for some finite set $F=\{(a_1,b_1),\dots,(a_k,b_k),(0,b_{k+1}),\dots,(0,b_n)\}$ with $a_1,\dots,a_k$, $b_{k+1},\dots,b_n\in S_0$. Suppose $1\notin S$. Let $a=\min(a_1,\dots,a_k)$. Then $1<a\leq a_i$ with $a\in S_0$. Take $b\in\mathbf{Z}$ with $b<b_i$ for all b_i. Then $(a,b)\in L$. But, $(a,b)\notin\langle F\rangle$ by Proposition 3.1, noting that $(a_i,b_i)*(a_j,b_j)=(a_ia_j,a_ib_j+b_ia_j)\neq(a,b)$ (by $a_ia_j>a$); and $(a_i,b_i)+(0,b_j)\neq(a,b)$, etc. This is a contradiction. Then $1\in S$. $\square$

4. Characterizations of Semi-cones in $\mathbf{Z} \otimes \mathbf{Z}$

The following lemma holds in view of the proof of Proposition 2.6, and Corollary 2.9.

LEMMA 4.1. *For a semi-cone A of $\mathbf{Z} \otimes \mathbf{Z}$, let $A' = A \cap (0 \times \mathbf{Z})$ and $A'' = A \cap (\mathbf{Z} \times 0)$. Then the following hold.*

(1) *If $A' = 0$, then $p_1(A)$ is a semi-cone of $\mathbf{Z}$ with $p_1(A) \subset \mathbf{Z}^*$. Also, if $A' \neq 0$, then $p_2(A)$ is a semi-cone of $\mathbf{Z}$ with $p_2(A) \subset \mathbf{Z}^*$.*

(2) *If $A'' = 0$, then $p_2(A)$ is a semi-cone of $\mathbf{Z}$ with $p_2(A) \subset \mathbf{Z}^*$. Also, if $A'' \neq 0$, then $p_1(A)$ is a semi-cone of $\mathbf{Z}$ with $p_1(A) \subset \mathbf{Z}^*$.*

Let $\mathscr{C}$ be the collection of all semi-cones of $\mathbf{Z} \otimes \mathbf{Z}$, and define the following subcollections $\mathscr{C}_i$ $(i = 1, 2, 3, 4)$ of $\mathscr{C}$ satisfying $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3 \cup \mathscr{C}_4$.

$\mathscr{C}_1 = \{A \in \mathscr{C} \mid A \cap (0 \times \mathbf{Z}) \neq 0,\ A \cap (\mathbf{Z} \times 0) \neq 0\}$,
$\mathscr{C}_2 = \{A \in \mathscr{C} \mid A \cap (0 \times \mathbf{Z}) \neq 0,\ A \cap (\mathbf{Z} \times 0) = 0\}$,
$\mathscr{C}_3 = \{A \in \mathscr{C} \mid A \cap (0 \times \mathbf{Z}) = 0,\ A \cap (\mathbf{Z} \times 0) \neq 0\}$,
$\mathscr{C}_4 = \{A \in \mathscr{C} \mid A \cap (0 \times \mathbf{Z}) = 0,\ A \cap (\mathbf{Z} \times 0) = 0\}$.

THEOREM 4.2. *Let A be an additive and multiplicative subset of $\mathbf{Z} \otimes \mathbf{Z}$ with $A \ni 0$. Then the following hold.*

(1) $A \in \mathscr{C}_1 \Leftrightarrow A \subset \mathbf{Z}^* \times \mathbf{Z}^*$, *but* $A \not\subset (\mathbf{Z} \times \mathbf{N}) \cup 0$ *and* $A \not\subset (\mathbf{N} \times \mathbf{Z}) \cup 0$.

(2) $A \in \mathscr{C}_2 \Leftrightarrow A \subset (\mathbf{Z} \times \mathbf{N}) \cup 0$, *but* $A \not\subset (\mathbf{N} \times \mathbf{N}) \cup 0$.

(3) $A \in \mathscr{C}_3 \Leftrightarrow A \subset (\mathbf{N} \times \mathbf{Z}) \cup 0$, *but* $A \not\subset (\mathbf{N} \times \mathbf{N}) \cup 0$.

(4) $A \in \mathscr{C}_4 \Leftrightarrow A \subset (\mathbf{N} \times \mathbf{N}) \cup 0$.

PROOF. In view of Lemma 4.1, $A \in \mathscr{C}_1$; $A \in \mathscr{C}_2$; $A \in \mathscr{C}_3$; $A \in \mathscr{C}_4$ implies $A \subset \mathbf{Z}^* \times \mathbf{Z}^*$; $A \subset (\mathbf{Z} \times \mathbf{N}) \cup 0$; $A \subset (\mathbf{N} \times \mathbf{Z}) \cup 0$; $A \subset (\mathbf{N} \times \mathbf{N}) \cup 0$, respectively. Also, any case implies A is a semi-cone by $A \cap -A = 0$.

For (1), to see $(\Rightarrow)$, by $A \cap (0 \times \mathbf{Z}) \neq 0$, there exists some $(x, y) \in A \cap (0 \times \mathbf{Z})$ with $(x, y) \neq (0, 0)$. Then $x = 0$ and $y > 0$. Thus $(0, y) \notin (\mathbf{N} \times \mathbf{Z}) \cup 0$, which yields $A \not\subset (\mathbf{N} \times \mathbf{Z}) \cup 0$. Similarly, $A \not\subset (\mathbf{Z} \times \mathbf{N}) \cup 0$ by $A \cap (\mathbf{Z} \times 0) \neq 0$. For $(\Leftarrow)$, by $A \not\subset (\mathbf{Z} \times \mathbf{N}) \cup 0$, there exists some $(x, y) \in A$ with $(x, y) \notin (\mathbf{Z} \times \mathbf{N}) \cup 0$. Then $y = 0$ and hence $x \neq 0$. Thus $(x, 0) \in A$ with $x \neq 0$. Hence $A \cap (\mathbf{Z} \times 0) \neq 0$. Similarly, $A \cap (0 \times \mathbf{Z}) \neq 0$ by $A \not\subset (\mathbf{N} \times \mathbf{Z}) \cup 0$. Hence $A \in \mathscr{C}_1$.

For (2), to see $(\Rightarrow)$, by $A \cap (0 \times \mathbf{Z}) \neq 0$, there exists some $(x, y) \in A \cap (0 \times \mathbf{Z})$ with $(x, y) \neq (0, 0)$. Then $(x, y) = (0, y) \notin (\mathbf{N} \times \mathbf{N}) \cup 0$. Thus $A \not\subset (\mathbf{N} \times \mathbf{N}) \cup 0$. For $(\Leftarrow)$, by $A \not\subset (\mathbf{N} \times \mathbf{N}) \cup 0$, there exists some $(x, y) \in A$ with

$(x, y) \notin \mathbf{N} \times \mathbf{N}$, $(x, y) \neq (0,0)$. Then $x \leq 0$ and $y > 0$. Since $(x, y)^2, -x(x, y) \in A$, $(x, y)^2 + (-x)(x, y) = (0, y^2 - xy) \in A$ with $y^2 - xy \neq 0$. Then $A \cap (0 \times \mathbf{Z}) \neq 0$. Hence $A \in \mathscr{C}_2$.

(3) is similarly shown as (2), and (4) is obvious. □

Corollary 4.3. *A subset A of $\mathbf{Z} \otimes \mathbf{Z}$ with $A \ni 0$ is a semi-cone iff it is additive and multiplicative, and* (i) $A \subset \mathbf{Z}^* \times \mathbf{Z}^*$, (ii) $A \subset (\mathbf{Z} \times \mathbf{N}) \cup 0$, *or* (iii) $A \subset (\mathbf{N} \times \mathbf{Z}) \cup 0$.

A semi-cone S of R is *maximal* if for any semi-cone T of R with $S \subset T$, $T = S$.

Corollary 4.4. *The maximal semi-cones of $\mathbf{Z} \otimes \mathbf{Z}$ are precisely the sets $\mathbf{Z}^* \times \mathbf{Z}^*$, $(\mathbf{Z} \times \mathbf{N}) \cup 0$, and $(\mathbf{N} \times \mathbf{Z}) \cup 0$.*

Theorem 4.5. *For a subset A of $\mathbf{Z} \otimes \mathbf{Z}$, the following are equivalent.*

(1) *A is a semi-cone of $\mathbf{Z} \otimes \mathbf{Z}$.*

(2) *A has an increasing cover $\{S_n \mid n \in \mathbf{N}\}$ of finitely generated semi-cones with types $\langle (a_1, b_1), \dots, (a_i, b_i) \rangle$, but all of these types satisfy one of the following*: (i) *all $a_j, b_j \in \mathbf{Z}^*$*, (ii) *all $a_j \in \mathbf{N}$, and* (iii) *all $b_j \in \mathbf{N}$.*

(3) *A has an increasing cover $\{S_n \mid n \in \mathbf{N}\}$ of finitely generated semi-cones.*

Proof. (1) $\Leftrightarrow$ (3) holds by Proposition 3.2. (2) $\Rightarrow$ (3) is clear, and (3) $\Rightarrow$ (2) holds by Corollary 4.3. □

5. Characterizations of Semi-cones in $\mathbf{Z} \ltimes \mathbf{Z}$

Lemma 5.1. *For an additive subset A of $\mathbf{Z} \ltimes \mathbf{Z}$ with $A \ni 0$, let $A' = A \cap (0 \times \mathbf{Z})$. If $A' \cap -A' = 0$, then $A' = \sum_{i=1}^{n} \mathbf{Z}^*(0, b_i)$ for some $b_1, \dots, b_n \in \mathbf{Z}$ with all $b_i \in \mathbf{Z}^*$ or all $b_i \in -\mathbf{Z}^*$.*

Proof. The lemma follows from Proposition 2.8, noting $\mathbf{Z} \cong 0 \times \mathbf{Z} (\subset \mathbf{Z} \ltimes \mathbf{Z})$ by $x \mapsto (0, x)$, as additive groups. □

Proposition 5.2. *The following hold.*

(1) *If A is a semi-cone of $\mathbf{Z} \ltimes \mathbf{Z}$, then $A' = A \cap (0 \times \mathbf{Z})$ and $A'' = (A \cap (\mathbf{N} \times \mathbf{Z})) \cup 0$ are semi-cones of $\mathbf{Z} \ltimes \mathbf{Z}$ with $A = A' \cup A'' = A' + A''$, and $A' = \sum_{i=1}^{n} \mathbf{Z}^*(0, b_i)$ for some $b_1, \dots, b_n \in \mathbf{Z}$ with all $b_i \in \mathbf{Z}^*$ or all $b_i \in -\mathbf{Z}^*$.*

(2) *If* A' *is an additive subset of* $0 \times \mathbf{Z}^*$ *or* $0 \times -\mathbf{Z}^*$ *with* $A' \ni 0$, *and* A'' *is an additive and multiplicative subset of* $(\mathbf{N} \times \mathbf{Z}) \cup 0$ *with* $A'' \ni 0$, *then* A', A'' *and* $A = A' + A''$ *are semi-cones of* $\mathbf{Z} \ltimes \mathbf{Z}$ *with* $A = A' \cup A''$.

Proof. For (1), A' and A'' are routinely semi-cones of $\mathbf{Z} \ltimes \mathbf{Z}$. Noting $A \subset \mathbf{Z}^* \times \mathbf{Z}$ by Proposition 2.10, $A = A' \cup A'' = A' + A''$. The later part holds by Lemma 5.1.

For (2), routinely, A', A'' are semi-cones, and A is additive. Obviously, A is multiplicative by $A' * A'' \subset A'$. Also, $A \cap -A = 0$ since A is a subset of the cone $\boldsymbol{L}$ or $\boldsymbol{L}^*$. Hence A is a semi-cone with $A = A' \cup A''$. □

Corollary 5.3. *The maximal semi-cones in* $\mathbf{Z} \ltimes \mathbf{Z}$ *are precisely the cones* $\boldsymbol{L}$ *and* $\boldsymbol{L}^*$.

Corollary 5.4. *A subset* A *of* $\mathbf{Z} \ltimes \mathbf{Z}$ *is a semi-cone of* $\mathbf{Z} \ltimes \mathbf{Z}$ *iff* $A = B + \sum_{i=1}^{n} \mathbf{Z}^*(0, b_i)$ *for some additive and multiplicative subset* B *of* $(\mathbf{N} \times \mathbf{Z}) \cup 0$ *with* $B \ni 0$ *and some* $b_1, \ldots, b_n$ *with all* $b_i \in \mathbf{Z}^*$ *or all* $b_i \in -\mathbf{Z}^*$.

For non-zero elements $(a_1, b_1), \ldots, (a_n, b_n)$ in $\mathbf{Z} \ltimes \mathbf{Z}$, let us define a condition (C): (i) $a_1, \ldots, a_k \in \mathbf{N}$, $a_{k+1} = \cdots = a_n = 0$ and (ii) $b_{k+1}, \ldots, b_n \in \mathbf{N}$ or $b_{k+1}, \ldots, b_n \in -\mathbf{N}$, where $0 \le k \le n$ (possibly, $k = 0$ or $k = n$).

Lemma 5.5. *Let* $F = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ *be a finite subset of* $\mathbf{Z} \ltimes \mathbf{Z}$. *Then the following hold.*

(1) *For* $a_1, \ldots, a_n \in \mathbf{N}$ *and* $b_1, \ldots, b_n \in \mathbf{Z}$, *the set* A *of all finite sums of elements of the form* $c(a_1, b_1)^{v_1} * \cdots * (a_n, b_n)^{v_n}$ $(c, v_1, \ldots, v_n \in \mathbf{Z}^*$ *with some* $v_i > 0)$ *is the semi-cone of* $\mathbf{Z} \ltimes \mathbf{Z}$ *generated by* F.
(2) *For* $a_1 = \cdots = a_n = 0$ *and* $b_1, \ldots, b_n \in \mathbf{Z}$ *with all* $b_i \in \mathbf{Z}^*$ *or all* $b_i \in -\mathbf{Z}^*$, $A = \sum_{i=1}^{n} \mathbf{Z}^*(0, b_i)$ *is the semi-cone of* $\mathbf{Z} \ltimes \mathbf{Z}$ *generated by* F.
(3) *For* $(a_1, b_1), \ldots, (a_n, b_n)$ *satisfying* (C), $A = \langle (a_1, b_1), \ldots, (a_k, b_k) \rangle + \sum_{j=k+1}^{n} \mathbf{Z}^*(0, b_j)$ *is the semi-cone of* $\mathbf{Z} \ltimes \mathbf{Z}$ *generated by* F.

Proof. For a case (1), (2), or (3), A is a semi-cone of $\mathbf{Z} \ltimes \mathbf{Z}$ by Proposition 5.2(2) (indeed, for (1), put $A'' = A$. For (2), put $A' = A$. For (3), let $A' = \sum_{j=k+1}^{n} \mathbf{Z}^*(0, b_j)$, and define A'' as A in (1), but $n = k$, then $A = A' + A''$). Thus, for each case, $A = \langle F \rangle$ by Proposition 3.1. □

Proposition 5.6. *For a finite subset* $F = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ *of* $\mathbf{Z} \ltimes \mathbf{Z}$, F *is contained in some semi-cone* A *of* $\mathbf{Z} \ltimes \mathbf{Z}$ *iff all* $a_i \in \mathbf{Z}^*$, *and* $\{b_i \mid a_i = 0\}$ *is a subset of* $\mathbf{Z}^*$ *or* $-\mathbf{Z}^*$.

PROOF. The if part holds by Lemma 5.5(3). The only if part holds, modifying the proof of Proposition 5.2(1). □

Let $\mathscr{F}$ be the collection of all finitely generated semi-cones of $\mathbf{Z} \ltimes \mathbf{Z}$. Let $\mathscr{F}_1 = \{A \in \mathscr{F} \mid A \cap (\mathbf{N} \times \mathbf{Z}) \neq \emptyset\}$; $\mathscr{F}_2 = \{A \in \mathscr{F} \mid A \cap (\mathbf{N} \times \mathbf{Z}) = \emptyset\}$; and, $\mathscr{F}_1^* = \{A \in \mathscr{F} \mid A \cap (0 \times \mathbf{Z}) \neq 0\}$; $\mathscr{F}_2^* = \{A \in \mathscr{F} \mid A \cap (0 \times \mathbf{Z}) = 0\}$.

Clearly, $\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2 = \mathscr{F}_1^* \cup \mathscr{F}_2^*$. Note that $\mathscr{F}_1 = \{A \in \mathscr{F} \mid p_1(A) \neq 0\}$; $\mathscr{F}_2 = \{A \in \mathscr{F} \mid p_1(A) = 0\}$ (by $p_1(A) \subset \mathbf{Z}^*$ in Proposition 2.10).

For $A \in \mathscr{F}$ with $A \neq 0$, let $A_0 = A \backslash \{0\}$. Then $\mathscr{F}_1^* = \{A \in \mathscr{F} \mid p_1(A_0) \ni 0\}$; $\mathscr{F}_2^* = \{A \in \mathscr{F} \mid p_1(A_0) \not\ni 0 \text{ or } A = 0\}$.

The following holds by Lemma 5.5 and Proposition 5.6.

PROPOSITION 5.7. *Let A be a subset of $\mathbf{Z} \ltimes \mathbf{Z}$. Then the following hold.*

(1) (a) *$A \in \mathscr{F}_1$ iff $A = \langle (a_1, b_1), \ldots, (a_k, b_k) \rangle + \sum_{j=k+1}^{n} \mathbf{Z}^*(0, b_j)$ for some $(a_1, b_1), \ldots, (a_n, b_n)$ in $\mathbf{Z} \ltimes \mathbf{Z}$ satisfying* (C), *but $1 \leq k \leq n$.*

(b) *$A \in \mathscr{F}_2$ iff $A = \sum_{i=1}^{n} \mathbf{Z}^*(0, b_i)$ for some $b_1, \ldots, b_n$ with all $b_i \in \mathbf{Z}^*$ or all $b_i \in -\mathbf{Z}^*$.*

(2) (a) *$A \in \mathscr{F}_1^*$ iff A is the same as in (1)(a), but $0 \leq k < n$.*

(b) *$A \in \mathscr{F}_2^*$ iff $A = \langle (a_1, b_1), \ldots, (a_n, b_n) \rangle$ for some $(a_1, b_1), \ldots, (a_n, b_n)$ in $\mathbf{N} \ltimes \mathbf{Z}$, or $A = 0$.*

For a semi-cone $A = \langle (a_1, b_1), \ldots, (a_k, b_k) \rangle + \sum_{j=k+1}^{n} \mathbf{Z}^*(0, b_j)$ of $\mathbf{Z} \ltimes \mathbf{Z}$ with $a_j \in \mathbf{N}$ and $0 \leq k < n$, let us say that A is *positive* (resp. *negative*) if all $b_{k+1}, \ldots, b_n$ are positive (resp. negative), for convenience.

THEOREM 5.8. *For a subset A of $\mathbf{Z} \ltimes \mathbf{Z}$, the following are equivalent.*

(1) *A is a semi-cone of $\mathbf{Z} \ltimes \mathbf{Z}$.*

(2) *$A = \bigcup_{n \in \mathbf{N}} A_n + \sum_{i=1}^{r} \mathbf{Z}^*(0, b_i)$ for some $A_n \in \mathscr{F}_2^*$ $(n \in \mathbf{N})$ with $A_n \subset A_{n+1}$, and some $b_1, \ldots, b_r$ with all $b_i \in \mathbf{Z}^*$ or all $b_i \in -\mathbf{Z}^*$.*

(3) *A has an increasing cover $\{S_n \mid n \in \mathbf{N}\}$ with all $S_n \in \mathscr{F}_1^*$ or all $S_n \in \mathscr{F}_2^*$, here for $S_n \in \mathscr{F}_1^*$, all S_n are positive or all are negative.*

(4) *A has an increasing cover $\{S_n \mid n \in \mathbf{N}\}$ of finitely generated semi-cones.*

PROOF. For (1) ⇒ (2), let A', A'' be same as in Proposition 5.2(1). Then A', A'' are semi-cones and $A = A' \cup A'' = A' + A''$. Also, there exists a finite subset $\{b_1, \ldots, b_r\}$ with all $b_i \in \mathbf{Z}^*$ or all $b_i \in -\mathbf{Z}^*$ such that $A' = \sum_{i=1}^{r} \mathbf{Z}^*(0, b_i)$. If $A'' = 0$, (2) holds, so assume $A'' \neq 0$. Let $A'' = \{z_1, z_2, \ldots\} \cup 0$ with all $z_i \in \mathbf{N} \times \mathbf{Z}$, and put $A_n = \langle z_1, \ldots, z_n \rangle$. Then $\{A_n \mid n \in \mathbf{N}\}$ is an increasing cover of A''

by finitely generated semi-cones in $\mathscr{F}_2^*$ (by Proposition 5.7(2)). This suggests (2) holds.

For (2) $\Rightarrow$ (3), put $S_n = A_n + \sum_{i=1}^{r} \mathbf{Z}^*(0, b_i)$ $(n \in \mathbf{N})$ in (2). Then $\{S_n \mid n \in \mathbf{N}\}$ is a desired cover of A in (3) in terms of Proposition 5.7(2).

(3) $\Rightarrow$ (4) is clear, and (4) $\Rightarrow$ (1) holds by Proposition 3.2. □

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