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REALIZATIONS OF INNER AUTOMORPHISMS OF ORDER 4 AND FIXED POINTS SUBGROUPS BY THEM ON THE CONNECTED COMPACT EXCEPTIONAL LIE GROUP E_8, PART I

By

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Abstract. The compact simply connected Riemannian 4-symmetric spaces were classified by J. A. Jiménez as the type of Lie algebra. Needless to say, these spaces as homogeneous manifolds are of the form G/H, where G is a connected compact simple Lie group with an automorphism $\tilde{\gamma}$ of oder 4 on G and H is a fixed points subgroup G^{γ} of G. In the present article, as Part I, for the connected compact exceptional Lie group E_8, we give the explicit form of automorphism $\tilde{\sigma}'_4$ of order 4 on E_8 induced by the C-linear transformation σ'_4 of 248-dimensional vector space $\mathfrak{e}_8^C$ and determine the structure of the group $(E_8)^{\sigma'_4}$. This amounts to the global realization of one of seven cases with an automorphism of order 4 corresponding to the Lie algebra $\mathfrak{h} = \mathfrak{so}(6) \oplus \mathfrak{so}(10)$.

1. Introduction

Let G be a Lie group and H a compact subgroup of G. A homogeneous space G/H with G-invariant Riemannian metric g is called a Riemannian 4-symmetric space if there exists an automorphism $\tilde{\gamma}$ of order 4 on G such that $(G^{\gamma})_0 \subset H \subset G^{\gamma}$, where G^{γ} and $(G^{\gamma})_0$ is the fixed points subgroup of G by $\tilde{\gamma}$ and its identity component, respectively.

Now, for the exceptional compact Lie group E_8, as in Table 1 below, there exist seven cases of the compact simply connected Riemannian 4-symmetric spaces ([2]). The compact simply connected Riemannian 4-symmetric spaces were

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classified by J. A. Jiménez as mentioned in abstract. Accordingly, our interest is to realize the groupfication for the classification as Lie algebra. In a sense, at the stage when its groupfication is completed, the author would like to say the final completion of the classification.

Our results of groupfication corresponding to the Lie algebra $\mathfrak{h}$ in Table 1 are as follows.

Table 1

Case	$\mathfrak{h}$	$\tilde{\gamma}$	$H = G^{\gamma}$
1	$\mathfrak{so}(6) \oplus \mathfrak{so}(10)$	$\tilde{\sigma}'_4$	$(Spin(6) \times Spin(10))/\boldsymbol{Z}_4$
2	$i\boldsymbol{R} \oplus \mathfrak{su}(8)$	$\tilde{w}_4$	$(U(1) \times SU(8))/\boldsymbol{Z}_{24}$
3	$i\boldsymbol{R} \oplus \mathfrak{so}(14)$	$\tilde{\kappa}_4$	$(U(1) \times Spin(14))/\boldsymbol{Z}_4$
4	$\mathfrak{su}(2) \oplus i\boldsymbol{R} \oplus \mathfrak{so}(12)$	$\tilde{\varepsilon}_4$	$(SU(2) \times U(1) \times Spin(12))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$
5	$i\boldsymbol{R} \oplus \mathfrak{e}_7$	$\tilde{\upsilon}_4$	$(U(1) \times E_7)/\boldsymbol{Z}_2$
6	$\mathfrak{su}(2) \oplus \mathfrak{su}(8)$	$\tilde{\mu}_4$	$(SU(2) \times SU(8))/\boldsymbol{Z}_2$
7	$\mathfrak{su}(2) \oplus i\boldsymbol{R} \oplus \mathfrak{e}_6$	$\tilde{\omega}_4$	$(SU(2) \times U(1) \times E_6)/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_3)$

The realizations of groupfication in Table 1 have already been completed as original results by the author. In the present article, we state about the realization of the group H of Case 1 beginning from Section 3, and hereafter as for Cases 2–7 we will announce as an article in order.

We use the same notations as in [6], [7], [8] or [9]. Finally, the author would like to say that the features of this article are to give elementary proofs of the isomorphism of groups using the homomorphism theorem except several proofs and of the connectedness of groups as topological spaces.

2. Preliminaries

Let $\mathfrak{J}(3, \mathfrak{C}^C)$ and $\mathfrak{J}(3, \mathfrak{C})$ be the exceptional C- and $\boldsymbol{R}$-Jordan algebras, respectively. In $\mathfrak{J}(3, \mathfrak{C}^C)$, the Jordan multiplication $X \circ Y$, the inner product (X, Y) and a cross multiplication $X \times Y$, called the Freudenthal multiplication, are defined by

$$X \circ Y = \frac{1}{2}(XY + YX), \quad (X, Y) = \mathrm{tr}(X \circ Y),$$

$$X \times Y = \frac{1}{2}(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\,\mathrm{tr}(Y) - (X, Y))E),$$

respectively, where E is the 3×3 unit matrix. Moreover, we define the trilinear form (X, Y, Z), the determinant $\det X$ by

$$(X, Y, Z) = (X, Y \times Z), \quad \det X = \frac{1}{3}(X, X, X),$$

respectively, and briefly denote $\mathfrak{J}(3, \mathfrak{C}^C)$ and $\mathfrak{J}(3, \mathfrak{C})$ by $\mathfrak{J}^C$ and $\mathfrak{J}$, respectively. In $\mathfrak{J}$, we can also define the relational formulas above.

The connected complex Lie group F_4^C and the connected compact Lie group F_4 are defined by

$$\begin{aligned} F_4^C &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, \alpha(X \circ Y) = \alpha X \circ \alpha Y\} \\ &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, \det \alpha X = \det X, (\alpha X, \alpha Y) = (X, Y)\} \\ &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, \det \alpha X = \det X, \alpha E = E\}, \\ F_4 &= \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}) \,|\, \alpha(X \circ Y) = \alpha X \circ \alpha Y\}, \end{aligned}$$

respectively. Let τ be the complex conjugation in $\mathfrak{J}^C$. Then we have $F_4 = (F_4^C)^\tau$ (see [6, Section 2.4] in detail). Moreover, the Lie algebra $\mathfrak{f}_4^C$ of the group F_4^C is given by

$$\mathfrak{f}_4^C = \{\delta \in \mathrm{Hom}_C(\mathfrak{J}^C) \,|\, \delta(X \circ Y) = \delta X \circ Y + X \circ \delta Y\}.$$

We note the following as for the subalgebra $\mathfrak{d}_4^C$ of the Lie algebra $\mathfrak{f}_4^C$. The subalgebra $\mathfrak{d}_4^C$ is defined by

$$\mathfrak{d}_4^C = \{\delta \in \mathfrak{f}_4^C \,|\, \delta E_k = 0, k = 1, 2, 3\}.$$

Then the subalgebra $\mathfrak{d}_4^C$ is isomorphic to the Lie algebra $\mathfrak{so}(8, C) = \mathfrak{so}(\mathfrak{C}^C)$ by the correspondence

$$g : \mathfrak{so}(8, C) \to \mathfrak{d}_4^C, \quad g(D_1) = \delta, \quad \delta \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & D_3 x_3 & \overline{D_2 x_2} \\ \overline{D_3 x_3} & 0 & D_1 x_1 \\ D_2 x_2 & \overline{D_1 x_1} & 0 \end{pmatrix},$$

where D_2, D_3 are elements of $\mathfrak{so}(8, C)$ uniquely determined by the Principle of triality $(D_1 x)y + x(D_2 y) = \overline{D_3(\overline{xy})}$, $x, y \in \mathfrak{C}$ for $D_1 \in \mathfrak{so}(8, C)$. From now on, we identify $D_1 \in \mathfrak{so}(8, C)$ with $\delta = (D_1, D_2, D_3) \in \mathfrak{d}_4^C \subset \mathfrak{f}_4^C$.

Any element δ of the Lie algebra $\mathfrak{f}_4^C$ can be uniquely expressed by

$$\mathfrak{f}_4^C = \{D + \tilde{A}_1(a_1) + \tilde{A}_2(a_2) + \tilde{A}_3(a_3) \,|\, D \in \mathfrak{so}(8, C), a_k \in \mathfrak{C}^C, k = 1, 2, 3\},$$

where $\tilde{A}_k(a_k)$ is the C-linear mapping of $\mathfrak{J}^C$ (see [9, Subsection 2.4] in detail).

We define an $\boldsymbol{R}$-linear transformation σ_4' of $\mathfrak{J}$ by

$$\sigma_4' X = \sigma_4' \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 e_1 & \overline{e_1 x_2} \\ -\overline{x_3 e_1} & \xi_2 & -e_1 x_1 e_1 \\ e_1 x_2 & -\overline{e_1 x_1 e_1} & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J},$$

where the element e_1 is one of the basis of $\mathfrak{C} = \{e_0 = 1, e_1, e_2, \dots, e_7\}_{\boldsymbol{R}}$. Hereafter, a symbol e_k means one of the basis of $\mathfrak{C}$ or $\mathfrak{C}^C$. Then we have that $\sigma_4' \in Spin(8) \subset F_4 \subset F_4^C$, $(\sigma_4')^4 = 1$, $(\sigma_4')^2 = \sigma$, where an $\boldsymbol{R}$-linear transformation $\sigma : \mathfrak{J} \to \mathfrak{J}$ is defined by

$$\sigma X = \sigma \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}.$$

Note that the $\boldsymbol{R}$-linear transformation σ of $\mathfrak{J}$ is naturally extended to the C-linear transformation of $\mathfrak{J}^C$. Hence σ_4' induces the automorphisms $\tilde{\sigma}_4'$ of order 4 on $F_4 : \tilde{\sigma}_4'(\alpha) = \sigma_4'^{-1} \alpha \sigma_4'$, $\alpha \in F_4$, and using inclusion $F_4 \subset F_4^C$, the $\boldsymbol{R}$-linear transformation σ_4' of $\mathfrak{J}$ is naturally extended to the C-linear transformation of $\mathfrak{J}^C$. Hence σ_4' induces the automorphisms $\tilde{\sigma}_4'$ of order 4 on $F_4^C : \tilde{\sigma}_4'(\alpha) = \sigma_4'^{-1} \alpha \sigma_4'$, $\alpha \in F_4^C$.

The simply connected complex Lie group E_6^C is defined by

$$\begin{aligned} E_6^C &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, \det \alpha X = \det X\} \\ &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, (\alpha X, \alpha Y, \alpha Z) = (X, Y, Z)\}. \end{aligned}$$

Then we have naturally the inclusion $F_4^C \subset E_6^C$, and it is well known that $(E_6^C)_E = F_4^C$ (see [9, Definitions in Subsections 2.13, 3.1] in detail). Moreover, the Lie algebra $\mathfrak{e}_6^C$ of the group E_6^C is given by

$$\mathfrak{e}_6^C = \{\phi = \delta + \tilde{T} \,|\, \delta \in \mathfrak{f}_4^C, T \in (\mathfrak{J}^C)_0\},$$

where $(\mathfrak{J}^C)_0 = \{X \in \mathfrak{J}^C \,|\, \mathrm{tr}(X) = 0\}$ and the C-linear mapping $\tilde{T}$ of $\mathfrak{J}^C$ is defined by $\tilde{T}X = T \circ X$, $X \in \mathfrak{J}^C$ (see [9, Proposition 2.4.1, Theorem 3.2.1] in detail).

Let $\mathfrak{P}^C$ be the 56-dimensional Freudenthal C-vector space

$$\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C,$$

in which the Freudenthal cross operation $P \times Q$, $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$, is defined as follows:

$$P \times Q := \Phi(\phi, A, B, \nu), \quad \begin{cases} \phi = -\dfrac{1}{2}(X \vee W + Z \vee Y) \\ A = -\dfrac{1}{4}(2Y \times W - \xi Z - \zeta X) \\ B = \dfrac{1}{4}(2X \times Z - \eta W - \omega Y) \\ \nu = \dfrac{1}{8}((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta)), \end{cases}$$

where $X \vee W \in \mathfrak{e}_6^C$ is defined by

$$X \vee W = [\tilde{X}, \tilde{W}] + \left(X \circ W - \frac{1}{3}(X, W)E\right)^{\sim},$$

here the C-linear mappings $\tilde{X}$, $\tilde{W}$ of $\mathfrak{J}^C$ are same ones as in E_6^C.

The simply connected complex Lie group E_7^C is defined by

$$E_7^C = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \,|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\}.$$

Moreover, the Lie algebra $\mathfrak{e}_7^C$ of the group E_7^C is given by

$$\mathfrak{e}_7^C = \{\Phi(\phi, A, B, \nu) \,|\, \phi \in \mathfrak{e}_6^C, A, B \in \mathfrak{J}^C, \nu \in C\}.$$

For $\alpha \in E_6^C$, the mapping $\tilde{\alpha} : \mathfrak{P}^C \to \mathfrak{P}^C$ is defined by

$$\tilde{\alpha}(X, Y, \xi, \eta) = (\alpha X, {}^t\alpha^{-1} Y, \xi, \eta),$$

then we have $\tilde{\alpha} \in E_7^C$, and so α and $\tilde{\alpha}$ will be identified. The group E_7^C contains E_6^C as a subgroup by

$$E_6^C = (E_7^C)_{\dot{1}, \underset{.}{1}} (= \{\alpha \in (E_7^C \,|\, \alpha\dot{1} = \dot{1}, \alpha\underset{.}{1} = \underset{.}{1}\}),$$

where the symbols $\dot{1}$, $\underset{.}{1}$ are defined in Subsection 3.2. Hence we have the inclusion $F_4^C \subset E_6^C \subset E_7^C$. Using these inclusions, the C-linear transformation σ_4' of $\mathfrak{J}^C$ is naturally extended to the C-linear transformation of $\mathfrak{P}^C$:

$$\sigma_4'(X, Y, \xi, \eta) = (\sigma_4' X, \sigma_4' Y, \xi, \eta), \quad (X, Y, \xi, \eta) \in \mathfrak{P}^C.$$

Hence we see $\sigma_4' \in E_7^C$, and so σ_4' induces the automorphisms $\tilde{\sigma}_4'$ of order 4 on $E_7^C : \tilde{\sigma}_4'(\alpha) = \sigma_4'^{-1}\alpha\sigma_4'$, $\alpha \in E_7^C$.

Let $\mathfrak{e}_8^C$ be the 248-dimensional C-vector space

$$\mathfrak{e}_8^C = \mathfrak{e}_7^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C.$$

We define a Lie bracket $[R_1, R_2]$, $R_1 = (\Phi_1, P_1, Q_1, r_1, s_1, t_1)$, $R_2 = (\Phi_2, P_2, Q_2, r_2, s_2, t_2)$, by

$$[R_1, R_2] := (\Phi, P, Q, r, s, t),$$

$$\begin{cases} \Phi = [\Phi_1, \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1 \\ P = \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + s_1 Q_2 - s_2 Q_1 \\ Q = \Phi_1 Q_2 - \Phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + t_1 P_2 - t_2 P_1 \\ r = -\dfrac{1}{8}\{P_1, Q_2\} + \dfrac{1}{8}\{P_2, Q_1\} + s_1 t_2 - s_2 t_1 \\ s = \dfrac{1}{4}\{P_1, P_2\} + 2r_1 s_2 - 2r_2 s_1 \\ t = -\dfrac{1}{4}\{Q_1, Q_2\} - 2r_1 t_2 + 2r_2 t_1. \end{cases}$$

Then the C-vector space $\mathfrak{e}_8^C$ becomes a complex simple Lie algebra of type E_8.

We define a C-linear transformation λ_ω of $\mathfrak{e}_8^C$ by

$$\lambda_\omega(\Phi, P, Q, r, s, t) = (\lambda\Phi\lambda^{-1}, \lambda Q, -\lambda P, -r, -t, -s),$$

where a C-linear transformation λ of $\mathfrak{P}^C$ on the right-hand side is defined by $\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi)$. As in $\mathfrak{J}^C$, the complex conjugation in $\mathfrak{e}_8^C$ is denoted by τ:

$$\tau(\Phi, P, Q, r, s, t) = (\tau\Phi\tau, \tau P, \tau Q, \tau r, \tau s, \tau t).$$

The connected complex Lie group E_8^C and the connected compact Lie group E_8 are defined by

$$E_8^C = \{\alpha \in \mathrm{Iso}_C(\mathfrak{e}_8^C) \mid \alpha[R, R'] = [\alpha R, \alpha R']\},$$

$$E_8 = \{\alpha \in E_8^C \mid \tau\lambda_\omega\alpha\lambda_\omega\tau = \alpha\} = (E_8^C)^{\tau\lambda_\omega},$$

respectively. Moreover, the Lie algebra $\mathfrak{e}_8$ of the group E_8 is given by

$$\mathfrak{e}_8 = \{(\Phi, P, -\tau\lambda P, r, s, -\tau s) \mid \Phi \in \mathfrak{e}_7, P \in \mathfrak{J}^C, r \in i\boldsymbol{R}, s \in C\}.$$

For $\alpha \in E_7^C$, the mapping $\tilde{\alpha} : \mathfrak{e}_8^C \to \mathfrak{e}_8^C$ is defined by

$$\tilde{\alpha}(\Phi, P, Q, r, s, t) = (\alpha\Phi\alpha^{-1}, \alpha P, \alpha Q, r, s, t),$$

then we have $\tilde{\alpha} \in E_8^C$, and so α and $\tilde{\alpha}$ will be identified. The group E_8^C contains E_7^C as a subgroup by

$$E_7^C = \{\tilde{\alpha} \in E_8^C \mid \alpha \in E_7^C\}$$
$$= (E_8^C)_{\tilde{1},1^-,1_-} (= \{\alpha \in (E_8^C) \mid \alpha\tilde{1} = \tilde{1}, \alpha 1^- = 1^-, \alpha 1_- = 1_-\}),$$

where the symbols $\tilde{1}$, 1^-, 1_- are defined in subsection 3.3. Hence we have the inclusion $E_7^C \subset E_8^C$. Using this inclusion, since the C-linear transformation σ_4' of $\mathfrak{P}^C$ is naturally extended to the C-linear transformation of $\mathfrak{e}_8^C$:

$$\sigma_4'(\Phi, P, Q, r, s, t) = (\sigma_4'^{-1}\Phi\sigma_4', \sigma_4' P, \sigma_4' Q, r, s, t), \quad (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C,$$

we have $\sigma_4' \in E_8^C$, and so σ_4' induces the automorphisms $\tilde{\sigma}_4'$ of order 4 on $E_8^C : \tilde{\sigma}_4'(\alpha) = \sigma_4'^{-1}\alpha\sigma_4'$, $\alpha \in E_8^C$, and so is E_8.

In the last of this section, we state two useful lemmas, the Principle of triality on $SO(8, C)$ as theorem and proposition related to its theorem.

Lemma 2.1. *For Lie groups G, G', let a mapping $\varphi : G \to G'$ be a homomorphism of Lie groups. When G' is connected, if* Ker φ *is discrete and* $\dim(\mathfrak{g}) = \dim(\mathfrak{g}')$, *$\varphi$ is surjection.*

Proof. The proof is omitted (cf. [6, Lemma 0.6 (2)]). □

Lemma 2.2 (E. Cartan-Raševskii). *Let G be a simply connected Lie group with a finite order automorphism σ of G. Then G^σ is connected.*

Proof. The proof is omitted (cf. [6, Lemma 0.7]). □

After this, using these lemmas without permission each times, we often prove lemma, proposition or theorem.

Theorem 2.3 (Principle of triality on $SO(8, C)$). *For any $\alpha_3 \in SO(8, C)$, there exist $\alpha_1, \alpha_2 \in SO(8, C)$ such that*

$$(\alpha_1 x)(\alpha_2 y) = \alpha_3(xy), \quad x, y \in \mathfrak{C}^C.$$

Moreover, α_1, α_2 are determined uniquely up to the sign for α_3, that is, for α_3, these α_1, α_2 have to be α_1, α_2 or $-\alpha_1$, $-\alpha_2$.

Proof. The proof is omitted (cf. [9, Theorem 1.14.2]).

Proposition 2.4. *If* $\alpha_1, \alpha_2, \alpha_3 \in O(8, C)$ *satisfy the relational formula*

$$(\alpha_1 x)(\alpha_2 y) = \alpha_3(xy), \quad x, y \in \mathfrak{C}^C,$$

then $\alpha_1, \alpha_2, \alpha_3 \in SO(8, C)$.

Proof. The proof is omitted (cf. [9, Proposition 1.14.4]).

3. Case 1. The Automorphism σ'_4 of Order 4 and the Group $(E_8)^{\sigma'_4}$

The main purpose of this section is to give the automorphism $\tilde{\sigma}'_4$ of order 4 on E_8 explicitly and to determine the structure of the fixed points subgroup $(E_8)^{\sigma'_4}$ by $\tilde{\sigma}'_4$, where the structure of the group $(E_8)^{\sigma'_4}$ is as follows:

$$(E_8)^{\sigma'_4} \cong (Spin(6) \times Spin(10))/\boldsymbol{Z}_4.$$

Here, the spinor groups $Spin(6)$ and $Spin(10)$ above are respectively realized as the subgroup $(F_4)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$ and the subgroup $(E_8)^{\sigma'_4, \mathfrak{so}(6)}$ of $(E_8)^{\sigma'_4}$, where the definitions or the details of $(F_4)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$ and $(E_8)^{\sigma'_4, \mathfrak{so}(6)}$ are shown later.

Moreover, we would like to state about the group $(E_8)^{\sigma'_4, \mathfrak{so}(6)}$. The essential part to prove the isomorphism as a group is to show the connectedness of the group $(E_8)^{\sigma'_4, \mathfrak{so}(6)}$. In order to obtain this end, we need to treat the complex case as follows:

$$(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1} \cong Spin(6, C),$$

$$(E_7^C)^{\sigma'_4} \cong (SL(2, C) \times Spin(6, C) \times Spin(6, C))/\boldsymbol{Z}_4,$$

$$(E_7^C)^{\sigma'_4, \mathfrak{so}(6, C)} \cong SL(2, C) \times Spin(6, C),$$

$$(E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)} \cong Spin(10, C),$$

$$(E_8^C)^{\sigma'_4} \cong (Spin(6, C) \times Spin(10, C))/\boldsymbol{Z}_4,$$

and the connectedness of the group $(E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)}$, the definitions or the details of the group on the left-hand side in each row above are also shown later.

In this section, in order to study the subgroups of E_8 as mentioned above, since we need to have some knowledge of the their complexification G^C of $G = F_4, E_6, E_7$ or E_8, we state these as detailed as possible, however as for insufficient parts, refer to [1], [5], [6], [7], [8] or [9].

3.1. The Group $(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$

The aim of this section is to determine the structure of the group $(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$:

$$(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1} = \{\alpha \in F_4^C \mid \alpha E_i = E_i, i = 1, 2, 3, \alpha F_1(e_k) = F_1(e_k), k = 0, 1\}.$$

Now, we start to make some preparations.

We define groups $(F_4^C)_{E_1, E_2, E_3}$ and $Spin(8, C)$ by

$$(F_4^C)_{E_1, E_2, E_3} = \{\alpha \in F_4^C \mid \alpha E_i = E_i, i = 1, 2, 3\},$$

$$Spin(8, C) = \{(\alpha_1, \alpha_2, \alpha_3) \in SO(8, C)^{\times 3} \mid (\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})}, x, y \in \mathfrak{C}^C\},$$

respectively. Then we have the following theorem.

THEOREM 3.1. *The group $(F_4^C)_{E_1, E_2, E_3}$ is isomorphic to $Spin(8, C)$: $(F_4^C)_{E_1, E_2, E_3} \cong Spin(8, C)$.*

PROOF. We define a mapping $\varphi : Spin(8, C) \to (F_4^C)_{E_1, E_2, E_3}$ by

$$\varphi((\alpha_1, \alpha_2, \alpha_3))X = \begin{pmatrix} \xi_1 & \alpha_3 x_3 & \overline{\alpha_2 x_2} \\ \overline{\alpha_3 x_3} & \xi_2 & \alpha_1 x_1 \\ \alpha_2 x_2 & \overline{\alpha_1 x_1} & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}^C.$$

This homomorphism φ induces the isomorphism between $(F_4^C)_{E_1, E_2, E_3}$ and $Spin(8, C)$ (cf. [9, Theorem 2.7.1]). □

As necessary, we denote any element $\alpha \in (F_4^C)_{E_1, E_2, E_3}$ by $(\alpha_1, \alpha_2, \alpha_3) \in Spin(8, C)$, that is, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

We define an $\boldsymbol{R}$-linear transformation δ_1 of $\mathfrak{C}$ by

$$\delta_1 : e_0 \to e_6,\ e_1 \to e_7,\ e_i \to e_i,\ i = 2, 3, 4, 5,\ e_6 \to e_0,\ e_7 \to e_1,$$

basiswisely. Using matrix representation, the explicit form of δ_1 is as follows:

$$\delta_1 = \begin{pmatrix} 0 & & & & & & 1 & 0 \\ & 0 & & & & & 0 & 1 \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ 1 & 0 & & & & & 0 & \\ 0 & 1 & & & & & & 0 \end{pmatrix} \in M(8, \boldsymbol{R}),$$

where the blanks are 0. Then we easily see that $\delta_1 \in SO(8)$. The $\boldsymbol{R}$-linear transformation δ_1 is naturally extended to the C-linear transformation of $\mathfrak{C}^C$.

We consider groups $(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1,2,3,4,5}$, $(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=2,3,4,5,6,7}$:

$$(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1,2,3,4,5} = \left\{ \alpha \in F_4^C \,\middle|\, \begin{array}{l} \alpha E_i = E_i,\ i = 1,2,3, \\ \alpha F_1(e_k) = F_1(e_k),\ k = 0,1,2,3,4,5 \end{array} \right\},$$

$$(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=2,3,4,5,6,7} = \left\{ \alpha \in F_4^C \,\middle|\, \begin{array}{l} \alpha E_i = E_i,\ i = 1,2,3, \\ \alpha F_1(e_k) = F_1(e_k),\ k = 2,3,4,5,6,7 \end{array} \right\}.$$

Hereafter, we often denote $k = 0,1,2,3,4,5$ by abbreviated form $k = 0,\dots,5$, and also often denote these groups above by abbreviated forms $(F_4^C)_{E_{1,2,3},F_1(0,\dots,5)}$, $(F_4^C)_{E_{1,2,3},F_1(2,\dots,7)}$ as example. The other cases are similar to these.

Proposition 3.2. *The group* $(F_4^C)_{E_{1,2,3},F_1(0,\dots,5)}$ *is isomorphic to the group* $(F_4^C)_{E_{1,2,3},F_1(2,\dots,7)}$: $(F_4^C)_{E_{1,2,3},F_1(0,\dots,5)} \cong (F_4^C)_{E_{1,2,3},F_1(2,\dots,7)}$.

Proof. We define a mapping $\varphi : (F_4^C)_{E_{1,2,3},F_1(0,\dots,5)} \to (F_4^C)_{E_{1,2,3},F_1(2,\dots,7)}$ by

$$\varphi(\alpha) = \delta^{-1}\alpha\delta,$$

where $\delta = (\delta_1,\delta_2,\delta_3) \in Spin(8) \subset Spin(8,C) \cong (F_4^C)_{E_1,E_2,E_3}$ (Theorem 3.1), here δ_1 is defined above, and note that for this δ_1 there exist $\delta_2,\delta_3 \in SO(8)$ by the Principle of triality on $SO(8)$ (Theorem 2.3). From $\alpha,\delta \in (F_4^C)_{E_1,E_2,E_3}$, it is easy to verify that $\varphi(\alpha) \in (F_4^C)_{E_1,E_2,E_3}$.

Moreover, we have that

$$\begin{aligned} \varphi(\alpha)F_1(e_6) &= (\delta^{-1}\alpha\delta)F_1(e_6) = (\delta^{-1}\alpha)F_1(\delta_1 e_6) \\ &= (\delta^{-1}\alpha)F_1(e_0) = \delta^{-1}F_1(e_0) = F_1(\delta_1^{-1}e_0) = F_1(e_6). \end{aligned}$$

Similarly, we have $\varphi(\alpha)F_1(e_7) = F_1(e_7)$, and it is clear that $\varphi(\alpha)F_1(e_k) = F_1(e_k)$, $k = 2,3,4,5$. Hence we have $\varphi(\alpha) \in (F_4^C)_{E_{1,2,3},F_1(2,\dots,7)}$, that is, φ is well-defined. From the definition of the mapping φ, it is clear that φ is bijection.

Therefore we have the required isomorphism

$$(F_4^C)_{E_{1,2,3},F_1(0,\dots,5)} \cong (F_4^C)_{E_{1,2,3},F_1(2,\dots,7)}. \qquad \square$$

We define a mapping $\kappa : SO(8,C) \to SO(8,C)$ by

$$\kappa(\alpha)x = \overline{\alpha\overline{x}}, \quad x \in \mathfrak{C}^C.$$

It is easily to verify that κ is well-defined and a homomorphism, moreover we see $\kappa^2 = 1$ (cf. [9, Theorem 1.16.4]).

Let the complex unitary group $U(1, \boldsymbol{C}^C) = \{\theta \in \boldsymbol{C}^C \mid \theta\bar{\theta} = 1\}$. Then we have the following theorem.

Theorem 3.3. *The group* $(F_4^C)_{E_{1,2,3}, F_1(2,\dots,7)}$ *is isomorphic to* $U(1, \boldsymbol{C}^C)$: $(F_4^C)_{E_{1,2,3}, F_1(2,\dots,7)} \cong U(1, \boldsymbol{C}^C)$.

Proof. We define a mapping $\phi : U(1, \boldsymbol{C}^C) \to (F_4^C)_{E_{1,2,3}, F_1(2,\dots,7)}$ by

$$\phi(\theta)X = \begin{pmatrix} \xi_1 & x_3\theta & \overline{\theta x_2} \\ \overline{x_3\theta} & \xi_2 & \bar{\theta}x_1\bar{\theta} \\ \theta x_2 & \theta\bar{x}_1\theta & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}^C.$$

Then ϕ is well-defined. Indeed, by using the relational formula $\mathrm{Re}(x(yz)) = \mathrm{Re}(y(zx)) = \mathrm{Re}(z(xy))$, $x, y, z \in \mathfrak{C}^C$, we have that

$$\begin{aligned}
\det(\phi(\theta)X) &= \xi_1\xi_2\xi_3 + 2\,\mathrm{Re}((\bar{\theta}x_1\bar{\theta})(\theta x_2)(x_3)\theta)) - \xi_1(\bar{\theta}x_1\bar{\theta})(\theta\bar{x}_1\theta) \\
&\quad - \xi_2(\theta x_2)(\overline{\theta_2 x_2}) - \xi_3(x_3\theta)(\overline{x_3\theta}) \\
&= \xi_1\xi_2\xi_3 + 2\,\mathrm{Re}((\bar{\theta}x_1\bar{\theta})(\theta(x_2x_3)\theta)) - \xi_1|\bar{\theta}x_1\bar{\theta}|^2 - \xi_2|\theta x_2|^2 - \xi_3|x_3\theta|^2 \\
&= \xi_1\xi_2\xi_3 + 2\,\mathrm{Re}(\theta(\bar{\theta}x_1\bar{\theta}))(\theta(x_2x_3)) - \xi_1|x_1|^2 - \xi_2|x_2|^2 - \xi_3|x_3|^2 \\
&= \xi_1\xi_2\xi_3 + 2\,\mathrm{Re}(x_1\bar{\theta})(\theta(x_2x_3)) - \xi_1|x_1|^2 - \xi_2|x_2|^2 - \xi_3|x_3|^2 \\
&= \xi_1\xi_2\xi_3 + 2\,\mathrm{Re}((x_2x_3)((x_1\bar{\theta})\theta) - \xi_1|x_1|^2 - \xi_2|x_2|^2 - \xi_3|x_3|^2 \\
&= \xi_1\xi_2\xi_3 + 2\,\mathrm{Re}((x_2x_3)(x_1) - \xi_1|x_1|^2 - \xi_2|x_2|^2 - \xi_3|x_3|^2 \\
&= \xi_1\xi_2\xi_3 + 2\,\mathrm{Re}(x_1x_2x_3) - \xi_1x_1\bar{x}_1 - \xi_2x_2\bar{x}_2 - \xi_3x_3\bar{x}_3 \\
&= \det X,
\end{aligned}$$

and it is clear that $(\phi(\theta)X, \phi(\theta)Y) = (X, Y)$, $X, Y \in \mathfrak{J}^C$ and $\phi(\theta)E_i = E_i$, $i = 1, 2, 3$. Hence we see that $\phi(\theta) \in (F_4^C)_{E_{1,2,3}}$. Moreover, from $e_i a = \bar{a}e_i$, $i = 2, \dots, 7$, $a \in U(1, \boldsymbol{C}^C)$, we have that $\phi(\theta)F_1(e_i) = F_1(e_i)$, $i = 2, \dots, 7$, that is, $\phi(\theta) \in (F_4^C)_{E_{1,2,3}, F_1(2,\dots,7)}$. Needless to say, ϕ is a homomorphism. We shall show that ϕ is surjection. Let $\alpha \in (F_4^C)_{E_{1,2,3}, F_1(2,\dots,7)}$. Here, set

$$(\mathfrak{J}^C)_k = \{F_k(x) \mid x \in \mathfrak{C}^C\} = \{X \in \mathfrak{J}^C \mid 2E_{k+1} \circ X = 2E_{k+2} \circ X = X\}, \quad k = 1, 2, 3,$$

where the indices are considered as mod 3. Then from $\alpha E_k = E_k$, $k = 1, 2, 3$, we have $\alpha X \in (\mathfrak{J}^C)_k$ for $X \in (\mathfrak{J}^C)_k$, and so α induces C-isomorphisms

$$\alpha : (\mathfrak{J}^C)_k \to (\mathfrak{J}^C)_k, \quad \alpha_k : \mathfrak{C}^C \to \mathfrak{C}^C$$

satisfying the conditions $\alpha F_k(x) = F_k(\alpha_k x)$, $x \in \mathfrak{C}^C$, $k = 1, 2, 3$.

Applying α on $F_k(x) \circ F_k(y) = (x, y)(E_{k+1} + E_{k+2})$, that is, $\alpha F_k(x) \circ \alpha F_k(y) = (x, y)(E_{k+1} + E_{k+2})$, on the other hand we have

$$\alpha F_k(x) \circ \alpha F_k(y) = F_k(\alpha_k x) \circ F_k(\alpha_k y) = (\alpha_k x, \alpha_k y)(E_{k+1} + E_{k+2}).$$

Hence we have $(\alpha_k x, \alpha_k y) = (x, y)$, $x, y \in \mathfrak{C}^C$, that is, $\alpha_k \in O(8, C)$, $k = 1, 2, 3$. Moreover, applying α on $F_1(x) \circ F_2(y) = (1/2)F_3(\overline{xy})$, we have $(\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})}$. Indeed, apply α on the left-hand side:

$$\alpha(F_1(x) \circ F_2(y)) = \alpha F_1(x) \circ \alpha F_2(y) = F_1(\alpha_1 x) \circ F_2(\alpha_2 y) = \frac{1}{2} F_3(\overline{(\alpha_1 x)(\alpha_2 y)}),$$

on the other hand, apply α on the right-hand side: $\alpha((1/2)F_3(\overline{xy})) = (1/2)F_3(\alpha_3(\overline{xy}))$. Hence we have $F_3(\overline{(\alpha_1 x)(\alpha_2 y)}) = F_3(\alpha_3(\overline{xy}))$, that is, $(\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})} (= (\kappa\alpha_3)(xy))$.

Since $\alpha_1, \alpha_2, \alpha_3 \in O(8, C)$ satisfy the relational formula $(\alpha_1 x)(\alpha_2 y) = (\kappa\alpha_3)(xy)$ above, we see that $\alpha_1, \alpha_2, \kappa\alpha_3 \in SO(8, C)$ (Proposition 2.4). Hence we have $\alpha_1, \alpha_2, \alpha_3 \in SO(8, C)$. Indeed, in general if $\alpha \in SO(8, C)$, so is $\kappa\alpha \in SO(8, C)$. Hence, now since α_3 holds the condition $\kappa\alpha_3 \in SO(8, C)$, from $\kappa^2 = 1$ we have $\alpha_3 = \kappa(\kappa\alpha_3) \in SO(8, C)$. Moreover from $\alpha F_1(e_i) = F_1(e_i)$ we have $\alpha_1 e_i = e_i$, $i = 2, \ldots, 7$. Hence since we can confirm that α_1 induces C-isomorphism of $\boldsymbol{C}^C \subset \mathfrak{C}^C$, there exists $\theta \in U(1, \boldsymbol{C}^C)$ such that $\alpha_1 x = \bar{\theta} x \bar{\theta}$, $x \in \mathfrak{C}^C$. For this θ, by the Principle of triality on $SO(8, C)$ we can set $\alpha_2 x = \theta x$, $\alpha_3 x = x\theta$, $x \in \mathfrak{C}^C$ (Theorem 2.3). The proof of surjection is completed. Finally, it is easy to obtain that $\operatorname{Ker} \phi = \{1\}$.

Therefore we have the required isomorphism

$$(F_4^C)_{E_{1,2,3}, F_1(2,\ldots,7)} \cong U(1, \boldsymbol{C}^C). \qquad \square$$

From Proposition 3.2 and Theorem 3.3, we have the following proposition.

PROPOSITION 3.4. *The group* $(F_4^C)_{E_{1,2,3}, F_1(0,\ldots,5)}$ *is isomorphic* $U(1, \boldsymbol{C}^C)$: $(F_4^C)_{E_{1,2,3}, F_1(0,\ldots,5)} \cong U(1, \boldsymbol{C}^C)$.

In particular, the group $(F_4^C)_{E_{1,2,3}, F_1(0,\ldots,5)}$ *is connected.*

PROOF. Since the group $U(1, \boldsymbol{C}^C)$ is isomorphic to the general linear group $GL(1, C)$ as a Lie group, the group $U(1, \boldsymbol{C}^C)$ is connected. Hence the group $(F_4^C)_{E_{1,2,3}, F_1(0,\dots,5)}$ is connected. □

We shall construct $Spin(3, C)$ in F_4^C.

Now, we consider a group $(F_4^C)_{E_{1,2,3}, F_1(0,\dots,4)}$:

$$(F_4^C)_{E_{1,2,3}, F_1(0,\dots,4)} = \left\{ \alpha \in F_4^C \,\middle|\, \begin{array}{l} \alpha E_i = E_i,\ i = 1,2,3, \\ \alpha F_1(e_k) = F_1(e_k),\ k = 0,1,2,3,4 \end{array} \right\}.$$

LEMMA 3.5. *The Lie algebra* $(\mathfrak{f}_4^C)_{E_{1,2,3}, F_1(0,\dots,4)}$ *of the group* $(F_4^C)_{E_{1,2,3}, F_1(0,\dots,4)}$ *is given by*

$$\begin{aligned}(\mathfrak{f}_4^C)_{E_{1,2,3}, F_1(0,\dots,4)} &= \left\{ \delta \in \mathfrak{f}_4^C \,\middle|\, \begin{array}{l} \delta E_i = 0,\ i = 1,2,3, \\ \delta F_1(e_k) = 0,\ k = 0,1,2,3,4 \end{array} \right\} \\ &= \{\delta = d_{56}G_{56} + d_{57}G_{57} + d_{67}G_{67} \,|\, d_{kl} \in C\}.\end{aligned}$$

In particular, $\dim_C((\mathfrak{f}_4^C)_{E_{1,2,3}, F_1(0,\dots,4)}) = 3$.

PROOF. By doing simple computation, this lemma is proved easily (As for G_{ij}, $i, j = 5, 6, 7$, see [9, Subsection 1.3]). □

We define a 3-dimensional C-vector subspace $(V^C)^3$ of $\mathfrak{J}^C$ by

$$\begin{aligned}(V^C)^3 &= \left\{ X \in \mathfrak{J}^C \,\middle|\, \begin{array}{l} E_1 \circ X = 0,\ (E_2, X) = (E_3, X) = 0, \\ (F_1(e_k), X) = 0,\ k = 0,1,2,3,4 \end{array} \right\} \\ &= \{X = F_1(t) \,|\, t = t_5e_5 + t_6e_6 + t_7e_7,\ t_k \in C\}\end{aligned}$$

with the norm $(X, X) = 2(t_5^2 + t_6^2 + t_7^2)$. Obviously, the group $(F_4^C)_{E_{1,2,3}, F_1(0,\dots,4)}$ acts on $(V^C)^3$.

PROPOSITION 3.6. *The homogeneous space* $(F_4^C)_{E_{1,2,3}, F_1(0,\dots,4)}/U(1, \boldsymbol{C}^C)$ *is homeomorphic to the complex sphere* $(S^C)^2$: $(F_4^C)_{E_{1,2,3}, F_1(0,\dots,4)}/U(1, \boldsymbol{C}^C) \simeq (S^C)^2$.

In particular, the group $(F_4^C)_{E_{1,2,3}, F_1(0,\dots,4)}$ *is connected.*

PROOF. We define a 2-dimensional complex sphere $(S^C)^2$ by

$$\begin{aligned}(S^C)^2 &= \{X \in (V^C)^3 \,|\, (X, X) = 2\} \\ &= \{X = F_1(t) \,|\, t = t_5e_5 + t_6e_6 + t_7e_7, t_5^2 + t_6^2 + t_7^2 = 1, t_k \in C\}.\end{aligned}$$

Then the group $(F_4^C)_{E_{1,2,3},F_1(0,\dots,4)}$ acts on $(S^C)^2$, obviously. We shall show that this action is transitive. In order to prove this, it is sufficient to show that any element $F_1(t) \in (S^C)^2$ can be transformed to $F_1(e_5) \in (S^C)^2$.

Now, for a given $X = F_1(t) \in (S^C)^2$, we choose $s_0 \in \boldsymbol{R}$, $0 \le s_0 \le \pi$ such that $\tan s_0 = \mathrm{Re}(t_6)/\mathrm{Re}(t_5)$ (if $\mathrm{Re}(t_5) = 0$, let $s_0 = \pi/2$).

Operate $g_{56}(s_0) := \exp(s_0 G_{56}) \in ((F_4^C)_{E_{1,2,3},F_1(0,\dots,4)})_0$ on $X = F_1(t)$ (Lemma 3.5), then we have that

$$\begin{aligned} g_{56}(s_0)X &= g_{56}(s_0)F_1(t) \\ &= F_1(((\cos s_0)t_5 + (\sin s_0)t_6)e_5 + ((\cos s_0)t_6 - (\sin s_0)t_5)e_6 + t_7e_7) \\ &= F_1(((\cos s_0)t_5 + (\sin s_0)t_6)e_5 \\ &\qquad + i((\cos s_0)\,\mathrm{Im}(t_6) - (\sin s_0)\,\mathrm{Im}(t_5))e_6 + t_7e_7) \\ &= F_1(t_5^{(1)}e_5 + ir_6e_6 + t_7e_7) =: X^{(1)}, \end{aligned}$$

where $t_5^{(1)} := (\cos s_0)t_5 + (\sin s_0)t_6 \in \boldsymbol{C}$, $r_6 := (\cos s_0)\,\mathrm{Im}(t_6) - (\sin s_0)\,\mathrm{Im}(t_5) \in \boldsymbol{R}$.

Moreover, we choose $s_1 \in \boldsymbol{R}$, $0 \le s_1 \le \pi$ such that $\tan s_1 = \mathrm{Re}(t_7)/\mathrm{Re}(t_5^{(1)})$ (if $\mathrm{Re}(t_5^{(1)}) = 0$, let $s_1 = \pi/2$). Operate $g_{57}(s_1) := \exp(s_1 G_{57}) \in ((F_4^C)_{E_{1,2,3},F_1(0,\dots,4)})_0$ on X' (Lemma 3.5), then we have that

$$\begin{aligned} g_{57}(s_1)X^{(1)} &= g_{57}(s_1)F_1((t_5^{(1)}e_5 + ir_6e_6 + t_7e_7) \\ &= F_1(((\cos s_1)t_5^{(1)} + (\sin s_1)t_7)e_5 + ir_6e_6 + ((\cos s_1)t_7 - (\sin s_1)t_5^{(1)})e_7) \\ &= F_1(((\cos s_1)t_5^{(1)} + (\sin s_1)t_7)e_5 + ir_6e_6 \\ &\qquad + i((\cos s_1)\,\mathrm{Im}(t_7) - (\sin s_1)\,\mathrm{Im}(t_5^{(1)}))e_7) \\ &= F_1(t_5^{(2)}e_5 + ir_6e_6 + ir_7e_7) =: X^{(2)}, \end{aligned}$$

where $t_5^{(2)} := (\cos s_1)t_5^{(1)} + (\sin s_1)t_7 \in \boldsymbol{C}$, $r_7 := (\cos s_1)\,\mathrm{Im}(t_7) - (\sin s_1)\,\mathrm{Im}(t_5^{(1)}) \in \boldsymbol{R}$.

Additionally, we choose $s_2 \in \boldsymbol{R}$, $0 \le s_2 \le \pi$ such that $\tan s_2 = r_7/r_6$ (if $r_6 = 0$, let $s_2 = \pi/2$). Operate $g_{67}(s_2) := \exp(s_2 G_{67}) \in ((F_4^C)_{E_{1,2,3},F_1(0,\dots,4)})_0$ on $X^{(2)}$ (Lemma 3.5), then we have that

$$\begin{aligned} g_{67}(s_2)X^{(2)} &= g_{67}(s_2)F_1(t_5^{(2)}e_5 + ir_6e_6 + ir_7e_7) \\ &= F_1(t_5^{(2)}e_5 + i((\cos s_2)r_6 + (\sin s_2)r_7)e_6 + i((\cos s_2)r_7 - (\sin s_2)r_6)e_7) \\ &= F_1(t_5^{(2)}e_5 + ir_6^{(1)}e_6) =: X^{(3)}, \end{aligned}$$

where $r_6^{(1)} := (\cos s_2)r_6 + (\sin s_2)r_7 \in \boldsymbol{R}$.

Here, note that it follows from $X^{(3)} = F_1(t_5^{(2)}e_5 + ir_6^{(1)}e_6) \in (S^C)^2((t_5^{(2)})^2 + (ir_6^{(1)})^2 = 1)$, that is, $(t_5^{(2)})^2 = 1 + (r_6^{(1)})^2$, $t_5^{(2)} \in C$, $r_6^{(1)} \in \boldsymbol{R}$ that we have $t_5^{(2)} \in \boldsymbol{R} - \{0\}$ and $-1 < r_6^{(1)}/t_5^{(2)} < 1$.

So, we can choose $s_3 \in \boldsymbol{R}$ such that $\tan(is_3) = ir_6^{(1)}/t_5^{(2)} (\in \boldsymbol{iR})$. Indeed, because of $\tan(is_3) = -i(1 - 2/(e^{-2s_3} + 1))(-1 < 1 - 2/(e^{-2s_3} + 1) < 1)$, together with $-1 < r_6^{(1)}/t_5^{(2)} < 1$, we can choose $s_3 \in \boldsymbol{R}$. As in the first case above, operate $g_{56}(is_3)$ on $X^{(3)}$, then we have that

$$\begin{aligned} g_{56}(is_3)X^{(3)} &= g_{56}(is_3)F_1(t_5^{(2)}e_5 + ir_6^{(1)}e_6) \\ &= F_1((\cos(is_3)t_5^{(2)} + \sin(is_3)(ir_6^{(1)}))e_5 + (\cos(is_3)(ir_6^{(1)}) - \sin(is_3)t_5^{(2)})e_6) \\ &= F_1((\cos(is_3)t_5^{(2)} + \sin(is_3)(ir_6^{(1)}))e_5) \\ &= F_1(t_5^{(3)}e_5), \end{aligned}$$

where $t_5^{(3)} := \cos(is_3)t_5^{(2)} + \sin(is_3)(ir_6^{(1)}) \in C$. Hence, from $F_1(t_5^{(3)}e_5) \in (S^C)^2$ we have that $t_5^{(3)} = 1$ or $t_5^{(3)} = -1$. In the latter case, again operate $g_{57}(\pi)$ on $F_1(-e_5)$, then we have that $g_{57}(\pi)F_1(-e_5) = F_1(e_5)$. This shows the transitivity of action to $(S^C)^2$ by the group $(F_4^C)_{E_{1,2,3},F_1(0,\dots,4)}$. The isotropy subgroup of the group $(F_4^C)_{E_{1,2,3},F_1(0,\dots,4)}$ at $F_1(e_5)$ is the group $(F_4^C)_{E_{1,2,3},F_1(0,\dots,5)} \cong U(1, \boldsymbol{C}^C)$ (Proposition 3.4). Thus we have the required homeomorphism

$$(F_4^C)_{E_{1,2,3},F_1(0,\dots,4)}/U(1, \boldsymbol{C}^C) \simeq (S^C)^2.$$

Therefore we see that the group $(F_4^C)_{E_{1,2,3},F_1(0,\dots,4)}$ is connected. □

THEOREM 3.7. *The group* $(F_4^C)_{E_{1,2,3},F_1(0,\dots,4)}$ *is isomorphic to* $Spin(3, C)$: $(F_4^C)_{E_{1,2,3},F_1(0,\dots,4)} \cong Spin(3, C)$.

PROOF. Let $O(3, C) = O((V^C)^3) = \{\beta \in \mathrm{Iso}_C((V^C)^3) \mid (\beta X, \beta Y) = (X, Y)\}$. We consider the restriction $\beta = \alpha|_{(V^C)^3}$ of $\alpha \in (F_4^C)_{E_{1,2,3},F_1(0,\dots,4)}$ to $(V^C)^3$, then we have $\beta \in O(3, C)$. Hence we define a homomorphism $p : (F_4^C)_{E_{1,2,3},F_1(0,\dots,4)} \to O(3, C) = O((V^C)^3)$ by

$$p(\alpha) = \alpha|_{(V^C)^3}.$$

Moreover since the mapping p is continuous and the group $(F_4^C)_{E_{1,2,3},F_1(0,\dots,4)}$ is connected (Proposition 3.6), the mapping p induces a homomorphism

$$p : (F_4^C)_{E_{1,2,3},F_1(0,\dots,4)} \to SO(3, C) = SO((V^C)^3).$$

It is not difficult to obtain that Ker $p = \{1, \sigma\} \cong \boldsymbol{Z}_2$. Indeed, Let $\alpha \in \operatorname{Ker} p$. Then, since $\alpha \in (F_4^C)_{E_{1,2,3}, F_1(0,\dots,4)} \subset (F_4^C)_{E_{1,2,3}} \cong Spin(8, C)$, we can set $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ (Theorem 3.1). Moreover, from $\alpha F_1(e_i) = F_1(e_i)$, $i = 0, \dots, 4$ and $\alpha|_{(V^C)^3} = 1$, we have $\alpha_1 x = x$ for all $x \in \mathfrak{C}^C$, that is, $\alpha_1 = 1$. Hence, from the principle triality on $SO(8)$ (Theorem 2.3) we have that

$$\alpha = (1, 1, 1) \quad \text{or} \quad \alpha = (1, -1, -1) = \sigma,$$

that is, $\operatorname{Ker} p \subset \{1, \sigma\}$ and vice versa. Thus we obtain $\operatorname{Ker} p = \{1, \sigma\} \cong \boldsymbol{Z}_2$. We shall show that p is surjection. From Lemma 3.5, we have that $\dim_C((\mathfrak{f}_4^C)_{E_{1,2,3}, F_1(0,\dots,4)}) = 3 = \dim_C(\mathfrak{so}(3, C))$, and in addition to this, $SO(3, C)$ is connected and $\operatorname{Ker} p$ is discrete. Hence p is surjection. Thus we have the isomorphism

$$(F_4^C)_{E_{1,2,3}, F_1(0,\dots,4)}/\boldsymbol{Z}_2 \cong SO(3, C).$$

Therefore the group $(F_4^C)_{E_{1,2,3}, F_1(0,\dots,4)}$ is isomorphic to $Spin(3, C)$ as the universal covering group of $SO(3, C)$, that is, $(F_4^C)_{E_{1,2,3}, F_1(0,\dots,4)} \cong Spin(3, C)$. □

Continuously, we shall construct $Spin(4, C)$ in F_4^C.

Now, we consider a group $(F_4^C)_{E_{1,2,3}, F_1(0,\dots,3)}$:

$$(F_4^C)_{E_{1,2,3}, F_1(0,\dots,3)} = \left\{ \alpha \in F_4^C \;\middle|\; \begin{array}{l} \alpha E_i = E_i,\ i = 1, 2, 3, \\ \alpha F_1(e_k) = F_1(e_k),\ k = 0, 1, 2, 3 \end{array} \right\}.$$

Lemma 3.8. *The Lie algebra* $(\mathfrak{f}_4^C)_{E_{1,2,3}, F_1(0,\dots,3)}$ *of the group* $(F_4^C)_{E_{1,2,3}, F_1(0,\dots,3)}$ *is given by*

$$\begin{aligned} (\mathfrak{f}_4^C)_{E_{1,2,3}, F_1(0,\dots,3)} &= \{\delta \in \mathfrak{f}_4^C \mid \delta E_i = 0, i = 1, 2, 3, \delta F_1(e_k) = 0, k = 0, 1, 2, 3\} \\ &= \left\{ \begin{array}{l} \delta = d_{45} G_{45} + d_{46} G_{46} + d_{47} G_{47} \\ \qquad + d_{56} G_{56} + d_{57} G_{57} + d_{67} G_{67} \end{array} \;\middle|\; d_{kl} \in C \right\}. \end{aligned}$$

In particular, $\dim_C((\mathfrak{f}_4^C)_{E_{1,2,3}, F_1(0,\dots,3)}) = 6$.

Proof. By doing simple computation, this lemma is proved easily (Again, as for G_{ij}, $i, j = 4, 5, 6, 7$, see [9, Subsection 1.3]). □

We define a 4-dimensional C-vector subspace $(V^C)^4$ of $\mathfrak{J}^C$ by

$$(V^C)^4 = \left\{ X \in \mathfrak{J}^C \,\middle|\, \begin{array}{l} E_1 \circ X = 0,\ (E_2, X) = (E_3, X) = 0, \\ (F_1(e_k), X) = 0,\ k = 0, 1, 2, 3 \end{array} \right\}$$

$$= \{X = F_1(t) \mid t = t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7, t_k \in C\}$$

with the norm $(X, X) = 2(t_4^2 + t_5^2 + t_6^2 + t_7^2)$. Obviously, the group $(F_4^C)_{E_{1,2,3}, F_1(0,\ldots,3)}$ acts on $(V^C)^4$.

Proposition 3.9. *The homogeneous space $(F_4^C)_{E_{1,2,3}, F_1(0,\ldots,3)}/Spin(3, C)$ is homeomorphic to the complex sphere $(S^C)^3$: $(F_4^C)_{E_{1,2,3}, F_1(0,\ldots,3)}/Spin(3, C) \simeq (S^C)^3$.*

In particular, the group $(F_4^C)_{E_{1,2,3}, F_1(0,\ldots,3)}$ is connected.

Proof. We define a 3-dimensional complex sphere $(S^C)^3$ by

$$(S^C)^3 = \{X \in (V^C)^4 \mid (X, X) = 2\}$$

$$= \{X = F_1(t) \mid t = t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7, t_4^2 + t_5^2 + t_6^2 + t_7^2 = 1, t_k \in C\}.$$

Then the group $(F_4^C)_{E_{1,2,3}, F_1(0,\ldots,3)}$ acts on $(S^C)^3$, obviously. We shall show that this action is transitive. In order to prove this, it is sufficient to show that any element $F_1(t) \in (S^C)^5$ can be transformed to $F_1(e_4) \in (S^C)^3$.

Now, for a given $X = F_1(t) \in (S^C)^3$, we choose $s_0 \in \boldsymbol{R}$, $0 \le s_0 \le \pi$ such that $\tan s_0 = -\mathrm{Re}(t_4)/\mathrm{Re}(t_5)$ (if $\mathrm{Re}(t_5) = 0$, let $s_0 = \pi/2$). Operate $g_{45}(s_0) := \exp(s_0 G_{45}) \in ((F_4^C)_{E_{1,2,3}, F_1(0,\ldots,3)})_0$ on $X = F_1(t)$ (Lemma 3.8), then we have that

$$\begin{aligned} g_{45}(s_0)X &= g_{45}(s_0)F_1(t) \\ &= F_1(((\cos s_0)t_4 + (\sin s_0)t_5)e_4 + ((\cos s_0)t_5 - (\sin s_0)t_4)e_5 + t_6e_6 + t_7e_7) \\ &= F_1(i((\cos s_0)\,\mathrm{Im}(t_4) + (\sin s_0)\,\mathrm{Im}(t_5))e_4 \\ &\qquad + ((\cos s_0)t_5 - (\sin s_0)t_4)e_5 + t_6e_6 + t_7e_7) \\ &= F_1(ir_4^{(1)}e_4 + t_5^{(1)}e_5 + t_6e_6 + t_7e_7) =: X^{(1)}, \end{aligned}$$

where $r_4^{(1)} := (\cos s_0)\,\mathrm{Im}(t_4) + (\sin s_0)\,\mathrm{Im}(t_5) \in \boldsymbol{R}$, $t_5^{(1)} := (\cos s_0)t_5 - (\sin s_0)t_4 \in C$.

Moreover, we choose $s_1 \in \boldsymbol{R}$, $0 \le s_1 \le \pi$ such that $\tan s_1 = -\mathrm{Re}(t_5)/\mathrm{Re}(t_6)$ (if $\mathrm{Re}(t_6) = 0$, let $s_1 = \pi/2$). Operate $g_{56}(s_1) := \exp(s_1 G_{56}) \in ((F_4^C)_{E_{1,2,3}, F_1(0,\ldots,3)})_0$ on $X^{(1)}$ (Lemma 3.8), then we have that

$$
\begin{aligned}
g_{56}(s_1)X^{(1)} &= g_{34}(s_1)F_1(ir_4^{(1)}e_4 + t_5^{(1)}e_5 + t_6e_6 + t_7e_7) \\
&= F_1(ir_4^{(1)}e_4 + ((\cos s_1)t_5 + (\sin s_1)t_6)e_5 \\
&\qquad + ((\cos s_1)t_6 - (\sin s_1)t_5)e_6 + t_7e_7) \\
&= F_1(ir_4^{(1)}e_4 + i((\cos s_1)\,\mathrm{Im}(t_5) + (\sin s_1)\,\mathrm{Im}(t_6))e_5 \\
&\qquad + ((\cos s_1)t_6 - (\sin s_1)t_5)e_6 + t_7e_7) \\
&= F_1(ir_4^{(1)}e_4 + ir_5^{(1)}e_5 + t_6^{(1)}e_6 + t_7e_7) =: X^{(2)},
\end{aligned}
$$

where $r_5^{(1)} := (\cos s_1)\,\mathrm{Im}(t_5) + (\sin s_1)\,\mathrm{Im}(t_6) \in \boldsymbol{R}$, $t_6^{(1)} := (\cos s_1)t_6 - (\sin s_1)t_5 \in C$.

Additionally, we choose $s_2 \in \boldsymbol{R}$, $0 \le s_2 \le \pi$ such that $\tan s_2 = -(r_4^{(1)})/(r_5^{(1)})$ (if $r_5^{(1)} = 0$, let $s_2 = \pi/2$). Again, operate $g_{45}(s_2) = \exp(s_2G_{45}) \in ((F_4^C)_{E_{1,2,3},F_1(0,\dots,3)})_0$ on $X^{(2)}$, then we have that

$$
\begin{aligned}
g_{45}(s_2)X^{(2)} &= g_{45}(s_2)F_1(ir_4^{(1)}e_4 + ir_5^{(1)}e_5 + t_6^{(1)}e_6 + t_7e_7) \\
&= F_1(i((\cos s_2)r_4^{(1)} + (\sin s_2)r_5^{(1)})e_4 \\
&\qquad + i((\cos s_2)r_5^{(1)} - (\sin s_2)r_4^{(1)})e_5 + t_6^{(1)}e_6 + t_7e_7) \\
&= F_1(i((\cos s_2)r_5^{(1)} - (\sin s_2)r_4^{(1)})e_5 + t_6^{(1)}e_6 + t_7e_7) \\
&= F_1(t_5^{(2)}e_5 + t_6^{(1)}e_6 + t_7e_7) =: X' \in (S^C)^2,
\end{aligned}
$$

where $t_5^{(2)} := i((\cos s_2)r_5^{(1)} - (\sin s_2)r_4^{(1)}) \in$.

Since $Spin(3,C) \cong (F_4^C)_{E_{1,2,3},F_1(0,\dots,4)} (\subset (F_4^C)_{E_{1,2,3},F_1(0,\dots,3)})$ acts transitively on $(S^C)^2$ (Proposition 3.6), there exists $\alpha \in Spin(3,C)$ such that

$$\alpha X' = F_1(e_5), \quad X' \in (S^C)^2.$$

Again, operate $g_{45}(\pi/2) \in (F_4^C)_{E_{1,2,3},F_1(0,\dots,3)}$ on $F_1(e_5)$, then we have that

$$g_{45}\left(\frac{\pi}{2}\right)F_1(e_5) = F_1(e_4).$$

This shows the transitivity of action to $(S^C)^3$ by the group $(F_4^C)_{E_{1,2,3},F_1(0,\dots,3)}$. The isotropy subgroup of the group $(F_4^C)_{E_{1,2,3},F_1(0,\dots,3)}$ at $F_1(e_4)$ is $(F_4^C)_{E_{1,2,3},F_1(0,\dots,4)} \cong Spin(3,C)$ (Theorem 3.7). Thus we have the required homeomorphism

$$(F_4^C)_{E_{1,2,3},F_1(0,\dots,3)}/Spin(3,C) \simeq (S^C)^3.$$

Therefore we see that the group $(F_4^C)_{E_{1,2,3},F_1(0,\dots,3)}$ is connected. □

THEOREM 3.10. *The group* $(F_4^C)_{E_{1,2,3},F_1(0,\dots,3)}$ *is isomorphic to* $Spin(4,C)$: $(F_4^C)_{E_{1,2,3},F_1(0,\dots,3)} \cong Spin(4,C)$.

PROOF. Since we can prove this theorem as in Theorem 3.7, this proof is omitted. □

Continuously, we shall construct $Spin(5,C)$ in F_4^C.

Now, we consider a group $(F_4^C)_{E_{1,2,3},F_1(0,1,2)}$:

$$(F_4^C)_{E_{1,2,3},F_1(0,1,2)} = \left\{ \alpha \in F_4^C \,\middle|\, \begin{array}{l} \alpha E_i = E_i, i = 1,2,3, \\ \alpha F_1(e_k) = F_1(e_k), k = 0,1,2 \end{array} \right\}.$$

LEMMA 3.11. *The Lie algebra* $(\mathfrak{f}_4^C)_{E_{1,2,3},F_1(0,1,2)}$ *of the group* $(F_4^C)_{E_{1,2,3},F_1(0,1,2)}$ *is given by*

$$\begin{aligned}(\mathfrak{f}_4^C)_{E_{1,2,3},F_1(0,1,2)} &= \{\delta \in \mathfrak{f}_4^C \,|\, \delta E_i = 0, i = 1,2,3, \delta F_1(e_k) = 0, k = 0,1,2\} \\ &= \left\{ \begin{array}{l} \delta = d_{34}G_{34} + d_{35}G_{35} + d_{36}G_{36} + d_{37}G_{37} + d_{45}G_{45} \\ \qquad + d_{46}G_{46} + d_{47}G_{47} + d_{56}G_{56} + d_{57}G_{57} + d_{67}G_{67} \end{array} \,\middle|\, d_{kl} \in C \right\}.\end{aligned}$$

In particular, $\dim_C((\mathfrak{f}_4^C)_{E_{1,2,3},F_1(0,1,2)}) = 10$.

PROOF. By doing simple computation, this lemma is proved easily (As for G_{ij}, $i,j = 3,4,5,6,7$, see [9, Subsection 1.3]). □

We define a 5-dimensional C-vector subspace $(V^C)^5$ of $\mathfrak{J}^C$ by

$$\begin{aligned}(V^C)^5 &= \left\{ X \in \mathfrak{J}^C \,\middle|\, \begin{array}{l} E_1 \circ X = 0, \ (E_2, X) = (E_3, X) = 0, \\ (F_1(e_k), X) = 0, \ k = 0,1,2 \end{array} \right\} \\ &= \{X = F_1(t) \,|\, t = t_3e_3 + t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7, t_k \in C\}\end{aligned}$$

with the norm $(X,X) = 2(t_3^2 + t_4^2 + t_5^2 + t_6^2 + t_7^2)$. Obviously, the group $(F_4^C)_{E_{1,2,3},F_1(0,1,2)}$ acts on $(V^C)^5$.

PROPOSITION 3.12. *The homogeneous space* $(F_4^C)_{E_{1,2,3},F_1(0,1,2)}/Spin(4,C)$ *is homeomorphic to the complex sphere* $(S^C)^4$: $(F_4^C)_{E_{1,2,3},F_1(0,1,2)}/Spin(4,C) \simeq (S^C)^4$.

In particular, the group $(F_4^C)_{E_{1,2,3},F_1(0,1,2)}$ *is connected.*

Proof. We define a 4-dimensional complex sphere $(S^C)^4$ by

$$(S^C)^4 = \{X \in (V^C)^5 \mid (X, X) = 2\}$$
$$= \left\{ X = F_1(t) \,\middle|\, \begin{array}{l} t = t_3 e_3 + t_4 e_4 + t_5 e_5 + t_6 e_6 + t_7 e_7, \\ t_3^2 + t_4^2 + t_5^2 + t_6^2 + t_7^2 = 1,\ t_k \in C \end{array} \right\}.$$

Then the group $(F_4^C)_{E_{1,2,3}, F_1(0,1,2)}$ acts on $(S^C)^4$, obviously. We shall show that this action is transitive. In order to prove this, it is sufficient to show that any element $F_1(t) \in (S^C)^4$ can be transformed to $F_1(e_3) \in (S^C)^4$.

Now, for a given $X = F_1(t) \in (S^C)^4$, we choose $s_0 \in \boldsymbol{R}$, $0 \le s_0 \le \pi$ such that $\tan s_0 = -\mathrm{Re}(t_3)/\mathrm{Re}(t_4)$ (if $\mathrm{Re}(t_4) = 0$, let $s_0 = \pi/2$). Operate $g_{34}(s_0) := \exp(s_0 G_{34}) \in ((F_4^C)_{E_{1,2,3}, F_1(0,1,2)})_0$ on $X = F_1(t)$ (Lemma 3.11), then we have that

$$\begin{aligned} g_{34}(s_0)X &= g_{34}(s_0)F_1(t) \\ &= F_1(((\cos s_0)t_3 + (\sin s_0)t_4)e_3 \\ &\qquad + ((\cos s_0)t_4 - (\sin s_0)t_3)e_4 + t_5 e_5 + t_6 e_6 + t_7 e_7) \\ &= F_1(i((\cos s_0)\ \mathrm{Im}(t_3) + (\sin s_0)\ \mathrm{Im}(t_4))e_3 \\ &\qquad + ((\cos s_0)t_4 - (\sin s_0)t_3)e_4 + t_5 e_5 + t_6 e_6 + t_7 e_7) \\ &= F_1(ir_3^{(1)}e_3 + t_4^{(1)}e_4 + t_5 e_5 + t_6 e_6 + t_7 e_7) =: X^{(1)}, \end{aligned}$$

where $r_3^{(1)} := (\cos s_0)\ \mathrm{Im}(t_3) + (\sin s_0)\ \mathrm{Im}(t_4) \in \boldsymbol{R}$, $t_4^{(1)} := (\cos s_0)t_4 - (\sin s_0)t_3 \in C$.

Moreover, we choose $s_1 \in \boldsymbol{R}$, $0 \le s_1 \le \pi$ such that $\tan s_1 = -\mathrm{Re}(t_4)/\mathrm{Re}(t_5)$ (if $\mathrm{Re}(t_5) = 0$, let $s_1 = \pi/2$). Operate $g_{45}(s_1) := \exp(s_1 G_{45}) \in ((F_4^C)_{E_{1,2,3}, F_1(0,1,2)})_0$ on $X^{(1)}$ (Lemma 3.11), then we have that

$$\begin{aligned} g_{45}(s_1)X^{(1)} &= g_{45}(s_1)F_1(ir_3^{(1)}e_3 + t_4^{(1)}e_4 + t_5 e_5 + t_6 e_6 + t_7 e_7) \\ &= F_1(ir_3^{(1)}e_3 + ((\cos s_1)t_4 + (\sin s_1)t_5)e_4 \\ &\qquad + ((\cos s_1)t_5 - (\sin s_1)t_4)e_5 + t_6 e_6 + t_7 e_7) \\ &= F_1(ir_3^{(1)}e_3 + i((\cos s_1)\ \mathrm{Im}(t_4) + (\sin s_1)\ \mathrm{Im}(t_5))e_4 \\ &\qquad + ((\cos s_1)t_5 - (\sin s_1)t_4)e_5 + t_6 e_6 + t_7 e_7) \\ &= F_1(ir_3^{(1)}e_3 + ir_4^{(1)}e_4 + t_5^{(1)}e_5 + t_6 e_6 + t_7 e_7) := X^{(2)}, \end{aligned}$$

where $r_4^{(1)} := (\cos s_1)\ \mathrm{Im}(t_4) + (\sin s_1)\ \mathrm{Im}(t_5) \in \boldsymbol{R}$, $t_5^{(1)} := (\cos s_1)t_5 - (\sin s_1)t_4 \in C$.

Additionally, we choose $s_2 \in \boldsymbol{R}$, $0 \le s_2 \le \pi$ such that $\tan s_2 = -(r_3^{(1)})/(r_4^{(1)})$ (if $r_4^{(1)} = 0$, let $s_2 = \pi/2$). Again, operate $g_{34}(s_2) = \exp(s_2 G_{34}) \in ((F_4^C)_{E_{1,2,3}, F_1(0,1,2)})_0$ on $X^{(2)}$ (Lemma 3.11), then we have that

$$\begin{aligned} g_{34}(s_2)X^{(2)} &= g_{34}(s_2)F_1(ir_3^{(1)}e_3 + ir_4^{(1)}e_4 + t_5^{(1)}e_5 + t_6e_6 + t_7e_7) \\ &= F_1(i((\cos s_2)r_3^{(1)} + (\sin s_2)r_4^{(1)})e_3 + i((\cos s_2)r_4^{(1)} - (\sin s_2)r_3^{(1)})e_4 \\ &\qquad + t_5e_5 + t_5e_5 + t_6e_6 + t_7e_7) \\ &= F_1(i((\cos s_2)r_4^{(1)} - (\sin s_2)r_3^{(1)})e_4 + t_5e_5 + t_6e_6 + t_7e_7) \\ &= F_1(t_4^{(2)}e_4 + t_5e_5 + t_6e_6 + t_7e_7) =: X' \in (S^C)^3, \end{aligned}$$

where $t_4^{(2)} := i((\cos s_2)r_4^{(1)} - (\sin s_2)r_3^{(1)}) \in C$.

Since $Spin(4, C) \cong (F_4^C)_{E_{1,2,3}, F_1(0,\dots,3)} (\subset (F_4^C)_{E_{1,2,3}, F_1(0,1,2)})$ acts transitively on $(S^C)^3$ (Proposition 3.9), there exists $\alpha \in Spin(4, C)$ such that

$$\alpha X' = F_1(e_4), \quad X' \in (S^C)^3.$$

Again, operate $g_{34}(\pi/2) \in ((F_4^C)_{E_{1,2,3}, F_1(0,1,2)})_0$ on $F_1(e_4)$, then we have that

$$g_{34}\left(\frac{\pi}{2}\right)(F_1(e_4)) = F_1(e_3).$$

This shows the transitivity of action to $(S^C)^4$ by the group $(F_4^C)_{E_{1,2,3}, F_1(0,1,2)}$. The isotropy subgroup of the group $(F_4^C)_{E_{1,2,3}, F_1(0,1,2)}$ at $F_1(e_3)$ is $(F_4^C)_{E_{1,2,3}, F_1(0,\dots,3)} \cong Spin(4, C)$ (Theorem 3.10). Thus we have the required homeomorphism

$$(F_4^C)_{E_{1,2,3}, F_1(0,1,2)}/Spin(4, C) \simeq (S^C)^4.$$

Therefore we see that the group $(F_4^C)_{E_{1,2,3}, F_1(0,1,2)}$ is connected. □

THEOREM 3.13. *The group* $(F_4^C)_{E_{1,2,3}, F_1(0,1,2)}$ *is isomorphic to* $Spin(5, C)$: $(F_4^C)_{E_{1,2,3}, F_1(0,1,2)} \cong Spin(5, C)$.

PROOF. Since we can also prove this theorem as in Theorem 3.7, this proof is omitted. □

Now, we determine the structure of the group $(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$ as the aim of this subsection.

LEMMA 3.14. *The Lie algebra* $(\mathfrak{f}_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$ *of the group* $(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$ *is given by*

$$(\mathfrak{f}_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1} = \{\delta \in \mathfrak{f}_4^C \,|\, \delta E_i = 0, i = 1,2,3, \delta F_1(e_k) = 0, k = 0,1\}$$

$$= \left\{ \begin{array}{l} \delta = d_{23}G_{23} + d_{24}G_{24} + d_{25}G_{25} + d_{26}G_{26} \\ \quad + d_{27}G_{27} + d_{34}G_{34} + d_{35}G_{35} + d_{36}G_{36} \\ \quad + d_{37}G_{37} + d_{45}G_{45} + d_{46}G_{46} + d_{47}G_{47} \\ \quad + d_{56}G_{56} + d_{57}G_{57} + d_{67}G_{67} \end{array} \middle| \, d_{kl} \in C \right\}.$$

In particular, $\dim_C((\mathfrak{f}_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1}) = 15$.

Proof. By doing simple computation, this lemma is proved easily (As for G_{ij}, $i, j = 2,3,4,5,6,7$, see [9, Section 1.3]). □

We define a 6-dimensional C-vector subspace $(V^C)^6$ of $\mathfrak{J}^C$ by

$$(V^C)^6 = \left\{ X \in \mathfrak{J}^C \,\middle|\, \begin{array}{l} E_1 \circ X = 0,\ (E_2, X) = (E_3, X) = 0, \\ (F_1(e_k), X) = 0,\ k = 0,1 \end{array} \right\}$$

$$= \{X = F_1(t) \,|\, t = t_2e_2 + t_3e_3 + t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7, t_k \in C\}$$

with the norm $(X, X) = 2(t_2^2 + t_3^2 + t_4^2 + t_5^2 + t_6^2 + t_7^2)$. The group $(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1}$ acts on $(V^C)^6$, obviously.

Proposition 3.15. *The homogeneous space* $(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1}/Spin(5,C)$ *is homeomorphic to the complex sphere* $(S^C)^5$: $(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1}/Spin(5,C) \simeq (S^C)^5$.

In particular, the group $(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1}$ *is connected.*

Proof. We define a 5-dimensional complex sphere $(S^C)^5$ by

$$(S^C)^5 = \{X \in (V^C)^6 \,|\, (X, X) = 2\}$$

$$= \left\{ X = F_1(t) \,\middle|\, \begin{array}{l} t = t_2e_2 + t_3e_3 + t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7, \\ t_2^2 + t_3^2 + t_4^2 + t_5^2 + t_6^2 + t_7^2 = 1,\ t_k \in C \end{array} \right\}.$$

Then the group $(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1}$ acts on $(S^C)^5$, obviously. We shall show that this action is transitive. In order to prove this, it is sufficient to show that any element $F_1(t) \in (S^C)^5$ can be transformed to $F_1(e_2) \in (S^C)^5$.

Now, for a given $X = F_1(t) \in (S^C)^5$, we choose $s_0 \in \boldsymbol{R}$, $0 \le s_0 \le \pi$ such that $\tan s_0 = -\mathrm{Re}(t_2)/\mathrm{Re}(t_3)$ (if $\mathrm{Re}(t_3) = 0$, let $s_0 = \pi/2$). Operate $g_{23}(s_0) := \exp(s_0 G_{23}) \in ((F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1})_0$ on $X = F_1(t)$ (Lemma 3.14), then we have that

$$\begin{aligned} g_{23}(s_0)X &= g_{23}(s_0)F_1(t) \\ &= F_1(((\cos s_0)t_2 + (\sin s_0)t_3)e_2 + ((\cos s_0)t_3 - (\sin s_0)t_2)e_3 \\ &\qquad + t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7) \\ &= F_1(i((\cos s_0)\operatorname{Im}(t_2) + (\sin s_0)\operatorname{Im}(t_3))e_2 \\ &\qquad + ((\cos s_0)t_3 - (\sin s_0)t_2)e_3 + t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7) \\ &= F_1(ir_2^{(1)}e_2 + t_3^{(1)}e_3 + t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7) =: X^{(1)}, \end{aligned}$$

where $r_2^{(1)} := (\cos s_0)\operatorname{Im}(t_2) + (\sin s_0)\operatorname{Im}(t_3) \in \boldsymbol{R}$, $t_3^{(1)} := (\cos s_0)t_3 - (\sin s_0)t_2 \in C$.

Moreover, we choose $s_1 \in \boldsymbol{R}$, $0 \le s_1 \le \pi$ such that $\tan s_1 = -\operatorname{Re}(t_3)/\operatorname{Re}(t_4)$ (if $\operatorname{Re}(t_4) = 0$, let $s_1 = \pi/2$). Operate $g_{34}(s_1) := \exp(s_1 G_{34}) \in ((F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1})_0$ on $X^{(1)}$ (Lemma 3.14), then we have that

$$\begin{aligned} g_{34}(s_1)X^{(1)} &= g_{34}(s_1)F_1(ir_2^{(1)}e_2 + t_3^{(1)}e_3 + t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7) \\ &= F_1(ir_2^{(1)}e_2 + ((\cos s_1)t_3 + (\sin s_1)t_4)e_3 \\ &\qquad + ((\cos s_1)t_4 - (\sin s_1)t_3)e_4 + t_5e_5 + t_6e_6 + t_7e_7) \\ &= F_1(ir_2^{(1)}e_2 + i((\cos s_1)\operatorname{Im}(t_3) + (\sin s_1)\operatorname{Im}(t_4))e_3 \\ &\qquad + ((\cos s_1)t_4 - (\sin s_1)t_3)e_4 + t_5e_5 + t_6e_6 + t_7e_7) \\ &= F_1(ir_2^{(1)}e_2 + ir_3^{(1)}e_3 + t_4^{(1)}e_4 + t_5e_5 + t_6e_6 + t_7e_7) =: X^{(2)}, \end{aligned}$$

where $r_3^{(1)} := (\cos s_1)\operatorname{Im}(t_3) + (\sin s_1)\operatorname{Im}(t_4) \in \boldsymbol{R}$, $t_4^{(1)} := (\cos s_1)t_4 - (\sin s_1)t_3 \in C$.

Additionally, we choose $s_2 \in \boldsymbol{R}$, $0 \le s_2 \le \pi$ such that $\tan s_2 = -(r_2^{(1)})/(r_3^{(1)})$ (if $r_3^{(1)} = 0$, let $s_2 = \pi/2$). Again, operate $g_{23}(s_2) = \exp(s_2 G_{23}) \in ((F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1})_0$ on $X^{(2)}$, then we have that

$$\begin{aligned} g_{23}(s_2)X^{(2)} &= g_{23}(s_2)F_1(ir_2^{(1)}e_2 + ir_3^{(1)}e_3 + t_4^{(1)}e_4 + t_5e_5 + t_6e_6 + t_7e_7) \\ &= F_1(i((\cos s_2)r_2^{(1)} + (\sin s_2)r_3^{(1)})e_2 + i((\cos s_2)r_3^{(1)} - (\sin s_2)r_2^{(1)})e_3 \\ &\qquad + t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7) \\ &= F_1(i((\cos s_2)r_3^{(1)} - (\sin s_2)r_2^{(1)})e_3 + t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7) \\ &= F_1(ir_3^{(2)}e_3 + t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7) =: X' \in (S^C)^4, \end{aligned}$$

where $r_3^{(2)} := (\cos s_2)r_3^{(1)} - (\sin s_2)r_2^{(1)} \in \boldsymbol{R}$.

Since $Spin(5, C) \cong (F_4^C)_{E_{1,2,3}, F_1(0,\dots,2)} (\subset (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1})$ acts transitively on $(S^C)^4$ (Proposition 3.12), there exists $\alpha \in Spin(5, C)$ such that

$$\alpha X' = F_1(e_3), \quad X' \in (S^C)^4.$$

Again, operate $g_{23}(\pi/2) \in ((F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1})_0$ on $F_1(e_3)$, then we have that

$$g_{23}\left(\frac{\pi}{2}\right) F_1(e_3) = F_1(e_2).$$

This shows the transitivity of action to $(S^C)^5$ by the group $(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$. The isotropy subgroup of the group $(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$ at $F_1(e_2)$ is $Spin(5, C)$ (Theorem 3.13). Thus we have the required homeomorphism

$$(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1} / Spin(5, C) \simeq (S^C)^5.$$

Therefore we see that the group $(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$ is connected. □

Theorem 3.16. *The group* $(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$ *is isomorphic to* $Spin(6, C)$: $(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1} \cong Spin(6, C)$.

Proof. This proof is proved by an argument similar to the proof in Theorem 3.7, however we write as detailed as possible. Let $O(6, C) = O((V^C)^6) = \{\beta \in \mathrm{Iso}_C((V^C)^6) \mid (\beta X, \beta Y) = (X, Y)\}$. We consider the restriction $\beta = \alpha|_{(V^C)^6}$ of $\alpha \in (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$ to $(V^C)^6$, then we have $\beta \in O(6, C)$. Hence we define a homomorphism $p : (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1} \to O(6, C) = O((V^C)^6)$ by

$$p(\alpha) = \alpha|_{(V^C)^6}.$$

Moreover since the mapping p is continuous and the group $(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$ is connected (Proposition 3.15), the mapping p induces a homomorphism

$$p : (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1} \to SO(6, C) = SO((V^C)^6).$$

It is easy to obtain that $\mathrm{Ker}\, p = \{1, \sigma\} \cong \boldsymbol{Z}_2$. Indeed, Let $\alpha \in \mathrm{Ker}\, p$. Then, since $\alpha \in (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1} \subset (F_4^C)_{E_1, E_2, E_3} \cong Spin(8, C)$ (Theorem 3.1), we can set $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. In addition, from $\alpha F_1(e_k) = F_1(e_k)$, $k = 0, 1$ and $\alpha|_{(V^C)^6} = 1$, we have $\alpha_1 x = x$ for all $x \in \mathfrak{C}^C$, that is, $\alpha_1 = 1$. Hence we have that

$$\alpha = (1, 1, 1) \quad \text{or} \quad \alpha = (1, -1, -1) = \sigma,$$

that is, $\mathrm{Ker}\, p \subset \{1, \sigma\}$ and vice versa. Thus we obtain $\mathrm{Ker}\, p = \{1, \sigma\}$. We shall prove that p is surjection. From Lemma 3.11, we have that

$\dim_C((\mathfrak{f}_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1}) = 15 = \dim_C(\mathfrak{so}(6,C))$, and in addition to this, since $SO(6,C)$ is connected and Ker p is discrete, p is surjection. Thus we have the isomorphism

$$(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1}/\boldsymbol{Z}_2 \cong SO(6,C).$$

Therefore the group $(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1}$ is isomorphic $Spin(6,C)$ as the universal covering group of $SO(6,C)$, that is, $(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1} \cong Spin(6,C)$. □

Here, we make a summary of the results as the low dimensional spinor groups which were constructed in this section. It is as follows:

$$\begin{array}{l}
(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1} \cong Spin(6,C) \\
\qquad\quad \cup \\
(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1,2} \cong Spin(5,C) \\
\qquad\quad \cup \\
(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,\dots,3} \cong Spin(4,C) \\
\qquad\quad \cup \\
(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,\dots,4} \cong Spin(3,C) \\
\qquad\quad \cup \\
(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,\dots,5} \cong Spin(2,C) \cong U(1,\boldsymbol{C}^C).
\end{array}$$

In the last of this subsection, we prove the following important lemma.

LEMMA 3.17. *The group* $(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1} \cong Spin(6,C)$ *is the subgroup of the group* $(F_4^C)^{\sigma_4'} = \{\alpha \in F_4^C \mid \sigma_4'\alpha = \alpha\sigma_4'\}$: $Spin(6,C) \cong (F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1} \subset (F_4^C)^{\sigma_4'}$.

PROOF. We consider the following complex eigenspaces of σ_4' in $\mathfrak{J}^C$:

$$\begin{aligned}
(\mathfrak{J}^C)_{\sigma_4'} &= \{X \in \mathfrak{J}^C \mid \sigma_4' X = X\}(\subset (\mathfrak{J}^C)_\sigma) \\
&= \{X = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1(x_1) \mid \xi_k \in C, x_1 \in \boldsymbol{C}^C\}, \\
(\mathfrak{J}^C)_{-\sigma_4'} &= \{X \in \mathfrak{J}^C \mid \sigma_4' X = -X\}(= \{F_1(x_1^\perp)\} \oplus (\mathfrak{J}^C)_{-\sigma}) \\
&= \{X = F_1(x_1^\perp) + F_2(x_2) + F_3(x_3)) \mid x_1^\perp \in (\boldsymbol{C}^C)^\perp \text{ in } \mathfrak{C}^C, x_k \in \mathfrak{C}^C\},
\end{aligned}$$

where $(\mathfrak{J}^C)_\sigma$, $(\mathfrak{J}^C)_{-\sigma}$ are the complex eigenspaces of the C-linear transformation σ in $\mathfrak{J}^C$.

Now, let $\alpha \in (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$, and set $X = X_1 + X_2 \in (\mathfrak{J}^C)_{\sigma'_4} \oplus (\mathfrak{J}^C)_{-\sigma'_4} = \mathfrak{J}^C$. Then, noting that $\alpha \in (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1} \subset (F_4^C)_{E_1} = (F_4^C)^\sigma$, we have that

$$\begin{aligned}
\sigma'_4\alpha X &= \sigma'_4\alpha(X_1 + X_2) = \sigma'_4\alpha X_1 + \sigma'_4\alpha X_2 \\
&= \sigma'_4 X_1 + \sigma'_4\alpha(F_1(x_1^\perp) + F_2(x_2) + F_3(x_3)) \\
&= X_1 + \sigma'_4\alpha F_1(x_1^\perp) - (F_2(x'_2) + F_3(x'_3)) \\
&= X_1 + \sigma'_4 F_1(x_1^\perp) - (F_2(x'_2) + F_3(x'_3)) \\
&= X_1 - F_1(x_1^\perp) - F_2(x'_2) - F_3(x'_3),
\end{aligned}$$

on the other hand, we have that

$$\begin{aligned}
\alpha\sigma'_4 X &= \alpha\sigma'_4(X_1 + X_2) = \alpha(X_1 - X_2) = X_1 - \alpha X_2 \\
&= X_1 - \alpha(F_1(x_1^\perp) + F_2(x_2) + F_3(x_3)) \\
&= X_1 - \alpha F_1(x_1^\perp) - (F_2(x'_2) + F_3(x'_3)) \\
&= X_1 - F_1(x_1^\perp) - F_2(x'_2) - F_3(x'_3).
\end{aligned}$$

Hence from the result of computation above, we see $\sigma'_4\alpha = \alpha\sigma'_4$, that is, $Spin(6, C) \cong (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1} \subset (F_4^C)^{\sigma'_4}$. □

3.2. The Groups $(E_7^C)^{\sigma'_4}$ and $(E_7^C)^{\sigma'_4, \mathfrak{so}(6, C)}$

The aim of this subsection is to show the connectedness of the group $(E_7^C)^{\sigma'_4, \mathfrak{so}(6, C)}$ after determining the structure of the group $(E_7^C)^{\sigma'_4}$.

Now, we define subgroups $(E_7^C)^{\sigma'_4}$ and $(E_7^C)^{\sigma'_4, \mathfrak{so}(6, C)}$ of E_7^C respectively by

$$\begin{aligned}
(E_7^C)^{\sigma'_4} &= \{\alpha \in E_7^C \mid \sigma'_4\alpha = \alpha\sigma'_4\}, \\
(E_7^C)^{\sigma'_4, \mathfrak{so}(6, C)} &= \{\alpha \in (E_7^C)^{\sigma'_4} \mid \Phi_D\alpha = \alpha\Phi_D \text{ for all } D \in \mathfrak{so}(6, C)\},
\end{aligned}$$

where $\Phi_D = (D, 0, 0, 0) \in \mathfrak{e}_7^C$, $D \in \mathfrak{so}(6, C) \cong (\mathfrak{f}_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$. Hereafter, we often denote D above by D_6.

LEMMA 3.18. *We have the following*

(1) *The Lie algebra* $(\mathfrak{e}_7^C)^{\sigma_4'}$ *of the group* $(E_7^C)^{\sigma_4'}$ *is given by*

$$(\mathfrak{e}_7^C)^{\sigma_4'} = \{\Phi \in \mathfrak{e}_7^C \mid \sigma_4'\Phi = \Phi\sigma_4'\}$$

$$= \left\{ \Phi(\phi, A, B, v) \in \mathfrak{e}_7^C \;\middle|\; \begin{array}{l} \phi = \left(\begin{array}{c|c} D_2 & 0 \\ \hline 0 & D_6 \end{array}\right) \begin{array}{l} +\tilde{A}_1(a) \\ +(\tau_1 E_1 + \tau_2 E_2 + \tau_3 E_3 + F_1(t_1))^\sim, \end{array} \\ \quad D_2 \in \mathfrak{so}(2, C),\ D_6 \in \mathfrak{so}(6, C),\ a \in \boldsymbol{C}^C,\ \tau_k \in C, \\ \quad \tau_1 + \tau_2 + \tau_3 = 0,\ t_1 \in \boldsymbol{C}^C, \\ A = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix},\ B = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & y_1 \\ 0 & \bar{y}_1 & \eta_3 \end{pmatrix}, \\ \quad \xi_k \in C,\ x_1 \in \boldsymbol{C}^C,\ \eta_k \in C,\ y_1 \in \boldsymbol{C}^C, \\ v \in C \end{array} \right\}.$$

In particular, $\dim_C((\mathfrak{e}_7^C)^{\sigma_4'}) = ((1 + 15) + 2 + (2 + 2)) + (3 + 2) \times 2 + 1 = 33.$

(2) *The Lie algebra* $(\mathfrak{e}_7^C)^{\sigma_4', \mathfrak{so}(6, C)}$ *of the group* $(E_7^C)^{\sigma_4', \mathfrak{so}(6, C)}$ *is given by*

$$(\mathfrak{e}_7^C)^{\sigma_4', \mathfrak{so}(6, C)}$$

$$= \{\Phi \in (\mathfrak{e}_7^C)^{\sigma_4'} \mid [\Phi, \Phi_D] = 0 \ \text{for all}\ D \in \mathfrak{so}(6, C)\}$$

$$= \left\{ \Phi(\phi, A, B, v) \in \mathfrak{e}_7^C \;\middle|\; \begin{array}{l} \phi = \left(\begin{array}{c|c} D_2 & 0 \\ \hline 0 & 0 \end{array}\right) \begin{array}{l} +\tilde{A}_1(a) \\ +(\tau_1 E_1 + \tau_2 E_2 + \tau_3 E_3 + F_1(t_1))^\sim, \end{array} \\ \quad D_2 \in \mathfrak{so}(2, C),\ a \in \boldsymbol{C}^C,\ \tau_k \in C, \\ \quad \tau_1 + \tau_2 + \tau_3 = 0,\ t_1 \in \boldsymbol{C}^C, \\ A = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix},\ B = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & y_1 \\ 0 & \bar{y}_1 & \eta_3 \end{pmatrix}, \\ \quad \xi_k \in C, x_1 \in \boldsymbol{C}^C,\ \eta_k \in C,\ y_1 \in \boldsymbol{C}^C, \\ v \in C \end{array} \right\}.$$

In particular, $\dim_C((\mathfrak{e}_7^C)^{\sigma_4', \mathfrak{so}(6, C)}) = (1 + 2 + (2 + 2)) + (3 + 2) \times 2 + 1 = 18.$

PROOF. By doing simple computation, this lemma is proved easily. □

First, making some preparations, we shall determine the structure of the group $(E_7^C)^{\sigma_4'}$. Hereafter, we often use the following notations in $\mathfrak{P}^C$:

$$\dot{X} = (X, 0, 0, 0), \quad \underset{\cdot}{Y} = (0, Y, 0, 0), \quad \dot{\xi} = (0, 0, \xi, 0), \quad \underset{\cdot}{\eta} = (0, 0, 0, \eta),$$

$$\tilde{E}_1 = (0, E_1, 0, 1), \quad \tilde{E}_{-1} = (0, -E_1, 0, 1), \quad E_2 \dotplus E_3 = (E_2 + E_3, 0, 0, 0),$$

$$E_2 \dot{-} E_3 = (E_2 - E_3, 0, 0, 0), \quad \dot{F}_1(e_k) = (F_1(e_k), 0, 0, 0), \quad k = 0, \dots, 7.$$

We define C-linear transformations κ, μ of $\mathfrak{P}^C$ by

$$\kappa(X, Y, \xi, \eta) = (-\kappa_1 X, \kappa_1 Y, -\xi, \eta),$$

$$\mu(X, Y, \xi, \eta) = (2E_1 \times Y + \eta E_1, 2E_1 \times X + \xi E_1, (E_1, Y), (E_1, X)),$$

respectively, where $\kappa_1 X = (E_1, X)E_1 - 4E_1 \times (E_1 \times X)$, $X \in \mathfrak{J}^C$. The explicit forms of κ, μ are as follows:

$$\kappa(X, Y, \xi, \eta) = \kappa(\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}, \xi, \eta)$$

$$= (\begin{pmatrix} -\xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & -\eta_2 & -y_1 \\ 0 & -\bar{y}_1 & -\eta_3 \end{pmatrix}, -\xi, \eta),$$

$$\mu(X, Y, \xi, \eta) = (\begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta_3 & -y_1 \\ 0 & -\bar{y}_1 & \eta_2 \end{pmatrix}, \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi_3 & -x_1 \\ 0 & -\bar{x}_1 & \xi_2 \end{pmatrix}, \eta_1, \xi_1).$$

By doing simple computation, we can easily confirm that $\kappa\sigma_4' = \sigma_4'\kappa$, $\mu\sigma_4' = \sigma_4'\mu$.

We define a group $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, E_2 \dotplus E_3, E_2 \dot{-} E_3, \dot{F}_1(e_k), k=0,1}$ by

$$((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, E_2 \dotplus E_3, E_2 \dot{-} E_3, \dot{F}_1(e_k), k=0,1} = \left\{ \alpha \in E_7^C \;\middle|\; \begin{array}{l} \kappa\alpha = \alpha\kappa, \ \mu\alpha = \alpha\mu, \\ \alpha\tilde{E}_1 = \tilde{E}_1, \ \alpha\tilde{E}_{-1} = \tilde{E}_{-1} \\ \alpha(E_2 \dotplus E_3) = E_2 \dotplus E_3, \\ \alpha(E_2 \dot{-} E_3) = E_2 \dot{-} E_3, \\ \alpha\dot{F}_1(e_k) = \dot{F}_1(e_k), \ k = 0, 1 \end{array} \right\}.$$

PROPOSITION 3.19. *The group* $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, E_2 \dotplus E_3, E_2 \dot{-} E_3, \dot{F}_1(e_k), k=0,1}$ *is isomorphic to* $Spin(6, C)$: $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, E_2 \dotplus E_3, E_2 \dot{-} E_3, \dot{F}_1(e_k), k=0,1} \cong Spin(6, C)$.

Proof. Let $\alpha \in ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, E_2 \dot{+} E_3, E_2 \dot{-} E_3, \dot{F}_1(e_k), k=0,1}$. Then from $\alpha(0, E_1, 0, 1) = (0, E_1, 0, 1)$ and $\alpha(0, -E_1, 0, 1) = (0, -E_1, 0, 1)$, we have that $\alpha(0, E_1, 0, 0) = (0, E_1, 0, 0)$ and $\alpha(0, 0, 0, 1) = (0, 0, 0, 1)$. Hence we see that $\alpha(E_1, 0, 0, 0) = (E_1, 0, 0, 0)$ and $\alpha(0, 0, 1, 0) = (0, 0, 1, 0)$. Indeed, it follows that

$$\alpha(E_1, 0, 0, 0) = \alpha\mu(0, 0, 0, 1) = \mu\alpha(0, 0, 0, 1) = \mu(0, 0, 0, 1) = (E_1, 0, 0, 0),$$

$$\alpha(0, 0, 1, 0) = \alpha\mu(0, E_1, 0, 0) = \mu\alpha(0, E_1, 0, 0) = \mu(0, E_1, 0, 0) = (0, 0, 1, 0).$$

Thus from $\alpha\dot{1} = \dot{1}$ and $\alpha\underset{\cdot}{1} = \underset{\cdot}{1}$, we see $\alpha \in E_6^C$, moreover from $\alpha E_i = E_i$, $i = 1, 2, 3$, that is, $\alpha E = E$, we see $\alpha \in F_4^C$. Note that suppose $\alpha \in F_4^C$, α satisfies $\kappa\alpha = \alpha\kappa$, $\alpha\mu = \mu\alpha$, automatically. Hence we have $\alpha \in (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$, and vice versa. Thus we have

$$((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, E_2 \dot{+} E_3, E_2 \dot{-} E_3, \dot{F}_1(e_k), k=0,1} = (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}.$$

Therefore, from Theorem 3.16 we have the required isomorphism

$$((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, E_2 \dot{+} E_3, E_2 \dot{-} E_3, \dot{F}_1(e_k), k=0,1} \cong Spin(6, C). \qquad \square$$

In order to construct one more $Spin(6, C)$ in E_7^C, after this we shall construct $Spin(3, C)$, $Spin(4, C)$ and $Spin(5, C)$ stepwisely.

First, we shall construct one more $Spin(3, C) \subset F_4^C$ which is different from $Spin(3, C)$ constructed in Theorem 3.7.

Now, we consider a group $(F_4^C)_{E_1, F_1(e_k), k=2,\ldots,7}$:

$$(F_4^C)_{E_1, F_1(e_k), k=2,\ldots,7} = \{\alpha \in F_4^C \mid \alpha E_1 = E_1, \alpha F_1(e_k) = F_1(e_k), k = 2, \ldots, 7\},$$

moreover define a 3-dimensional C-vector subspace $(V_-^C)^3$ of $\mathfrak{J}^C$ by

$$(V_-^C)^3 = \{X \in \mathfrak{J}^C \mid E_1 \circ X = 0, \operatorname{tr}(X) = 0, (F_1(e_k), X) = 0, k = 2, \ldots, 7\}$$
$$= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\xi \end{pmatrix} \middle| \, \xi \in C, x \in \boldsymbol{C}^C \right\}$$

with the norm $(X, X) = 2(\xi^2 + \bar{x}x)$. Obviously, the group $(F_4^C)_{E_1, F_1(e_k), k=2,\ldots,7}$ acts on $(V_-^C)^3$.

Lemma 3.20. *The Lie algebra* $(\mathfrak{f}_4^C)_{E_1, F_1(e_k), k=2,\ldots,7}$ *of the group* $(F_4^C)_{E_1, F_1(e_k), k=2,\ldots,7}$ *is given by*

$$(\tilde{\mathfrak{f}}_4^C)_{E_1, F_1(e_k), k=2,\dots,7} = \left\{ \delta \in \tilde{\mathfrak{f}}_4^C \,\middle|\, \begin{array}{l} \delta E_1 = 0, \\ \delta F_1(e_k) = 0,\ k = 2, \dots, 7 \end{array} \right\}$$

$$= \left\{ \delta = \left(\begin{array}{c|c} D_2 & 0 \\ \hline 0 & 0 \end{array} \right) + \tilde{A}_1(a) \,\middle|\, D_2 \in \mathfrak{so}(2, C),\ a \in \boldsymbol{C}^C \right\}.$$

In particular, $\dim_C((\tilde{\mathfrak{f}}_4^C)_{E_1, F_1(e_k), k=2,\dots,7}) = 3$.

Proof. By doing simple computation, this lemma is proved easily. □

Proposition 3.21. *The homogeneous space* $(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7}/U(1, \boldsymbol{C}^C)$ *is homeomorphic to the complex sphere* $(S_-^C)^2$: $(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7}/U(1, \boldsymbol{C}^C) \simeq (S_-^C)^2$.

In particular, the group $(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7}$ *is connected.*

Proof. We define a 2-dimensional complex sphere $(S_-^C)^2$ by

$$(S_-^C)^2 = \{X \in (V_-^C)^3 \mid (X, X) = 2\}$$

$$= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\xi \end{pmatrix} \,\middle|\, \xi^2 + x\bar{x} = 1, \xi \in C, x \in \boldsymbol{C}^C \right\}.$$

Then the group $(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7}$ acts on $(S_-^C)^2$, obviously. We shall show that this action is transitive. In order to prove this, it is sufficient to show that any element $X \in (S_-^C)^2$ can be transformed to $E_2 - E_3 \in (S_-^C)^2$. Here, we prepare some element of $(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7}$. For $a \in \boldsymbol{C}^C$ such that $a\bar{a} \neq 0$, we define a C-linear transformation $\alpha(a)$ of $\mathfrak{J}^C$, $\alpha(a)X(\xi, x) =: Y(\eta, y)$, by

$$\begin{cases} \eta_1 = \xi_1 \\ \eta_2 = \dfrac{\xi_2 + \xi_3}{2} + \dfrac{\xi_2 - \xi_3}{2} \cos 2v + \dfrac{(a, x_1)}{v} \sin 2v \\ \eta_3 = \dfrac{\xi_2 + \xi_3}{2} - \dfrac{\xi_2 - \xi_3}{2} \cos 2v - \dfrac{(a, x_1)}{v} \sin 2v, \end{cases}$$

$$\begin{cases} y_1 = x_1 - \dfrac{(\xi_2 - \xi_3)a}{2v} \sin 2v - \dfrac{2(a, x_1)a}{v^2} \sin^2 v \\ y_2 = x_2 \cos v - \dfrac{\overline{x_3 a}}{v} \sin v \\ y_3 = x_3 \cos v + \dfrac{\overline{a x_2}}{v} \sin v, \end{cases}$$

where $v \in C$, $v^2 = a\bar{a}$.

Then we see $\alpha(a) = \exp \tilde{A}(a) \in ((F_4^C)_{E_1, F_1(e_k), k=2,\dots,7})_0$ (Lemma 3.20).

Now, let $X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\xi \end{pmatrix} \in (S_-^C)^2$. We choose $a \in \boldsymbol{C}^C$ such that $(a, x) = 0$ and $a\bar{a} = (\pi/4)^2$. Operate $\alpha(a) \in ((F_4^C)_{E_1, F_1(e_k), k=2,\dots,7})_0$ on X, then we have that

$$\alpha(a)X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x' \\ 0 & \bar{x}' & 0 \end{pmatrix} =: X', \quad x'\bar{x}' = 1.$$

Moreover, using this x' above, operate $\alpha((\pi/4)x')$ on X', then we have

$$\alpha\left(\frac{\pi}{4}x'\right)X' = E_2 - E_3.$$

This shows the transitivity of this action to $(S_-^C)^2$ by the group $(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7}$. The isotropy subgroup of $(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7}$ at $E_2 - E_3$ is the group

$$\begin{aligned}(F_4^C)_{E_1, E_2-E_3, F_1(e_k), k=2,\dots,7} &= (F_4^C)_{E_1, E_2+E_3, E_2-E_3, F_1(e_k), k=2,\dots,7} \\ &= (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=2,\dots,7} \cong U(1, \boldsymbol{C}^C) \quad \text{(Theorem 3.3)}.\end{aligned}$$

Thus we have the required homeomorphism

$$(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7}/U(1, \boldsymbol{C}^C) \simeq (S_-^C)^2.$$

Therefore we see that the group $(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7}$ is connected. □

THEOREM 3.22. *The group $(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7}$ is isomorphic to $Spin(3, C)$: $(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7} \cong Spin(3, C)$.*

PROOF. Let $O(3, C) = O((V_-^C)^3) = \{\beta \in \mathrm{Iso}_C((V_-^C)^3) \,|\, (\beta X, \beta Y) = (X, Y)\}$. We consider the restriction $\beta = \alpha|_{(V_-^C)^3}$ of $\alpha \in (F_4^C)_{E_1, F_1(e_k), k=2,\dots,7}$ to $(V_-^C)^3$, then we have $\beta \in O(3, C)$. Hence we define a homomorphism $p : (F_4^C)_{E_1, F_1(e_k), k=2,\dots,7} \to O(3, C) = O((V_-^C)^3)$ by

$$p(\alpha) = \alpha|_{(V_-^C)^3}.$$

Moreover since the mapping p is continuous and the group $(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7}$ is connected (Proposition 3.21), the mapping p induces a homomorphism

$$p : (F_4^C)_{E_1, F_1(e_k), k=2,\dots,7} \to SO(3, C) = SO((V_-^C)^3).$$

It is not difficult to obtain that $\mathrm{Ker}\, p = \{1, \sigma\} \cong \boldsymbol{Z}_2$. Indeed, let $\alpha \in \mathrm{Ker}\, p = \{\alpha \in (F_4^C)_{E_1, F_1(e_k), k=2,\dots,7} \mid p(\alpha) = 1\}$, that is, $\alpha \in \{\alpha \in (F_4^C)_{E_1, F_1(e_k), k=2,\dots,7} \mid \alpha|_{(V_-^C)^3} = 1\}$. It follows from $\alpha E_1 = E_1$, $\alpha E = E$ that $\alpha(E_2 + E_3) = E_2 + E_3$, moreover since $\alpha|_{(V_-^C)^3} = 1$, we also see $\alpha(E_2 - E_3) = E_2 - E_3$. Hence, since we have $\alpha E_1 = E_1$, $\alpha E_2 = E_2$, $\alpha E_3 = E_3$, we see $\alpha \in (F_4^C)_{E_1, E_2, E_3} \cong Spin(8, C)$, and so set $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_k \in SO(8, C)$. Thus again from $\alpha F_1(e_k) = F_1(e_k), k=2,\dots,7$ and $\alpha|_{(V_-^C)^3} = 1$, we have $\alpha_1 = 1$. Hence from the Principle of triality on $SO(8, C)$ (Theorem 2.3), we see

$$\alpha = (1, 1, 1) = 1 \quad \text{or} \quad \alpha = (1, -1, -1) = \sigma,$$

that is, $\mathrm{Ker}\, p \subset \{1, \sigma\}$ and vice versa. Thus we see $\mathrm{Ker}\, p = \{1, \sigma\} \cong \boldsymbol{Z}_2$. Finally, we shall show that p is surjection. Since $SO(3, C)$ is connected, $\mathrm{Ker}\, p$ is discrete and $\dim_C((\mathfrak{f}_4^C)_{E_1, F_1(e_k), k=2,\dots,7}) = 3 = \dim_C(\mathfrak{so}(3, C))$ (Lemma 3.20), p is surjection. Thus we have that

$$(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7} / \boldsymbol{Z}_2 \cong SO(3, C).$$

Therefore the group $(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7}$ is isomorphic to $Spin(3, C)$ as a universal covering group of $SO(3, C)$, that is, $(F_4^C)_{E_1, F_1(e_k), k=2,\dots,7} \cong Spin(3, C)$. □

Next, we shall construct $Spin(4, C)$ in E_6^C.

Now, we consider subgroups $((E_6^C)^\sigma)_{E_1}, ((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$ of E_6^C:

$$((E_6^C)^\sigma)_{E_1} = \{\alpha \in E_6^C \mid \sigma\alpha = \alpha\sigma, \alpha E_1 = E_1\} (\cong Spin(10, C)),$$

$$((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7} = \{\alpha \in ((E_6^C)^\sigma)_{E_1} \mid \alpha F_1(e_k) = F_1(e_k), k = 2, \dots, 7\},$$

respectively, where as for $((E_6^C)^\sigma)_{E_1} \cong Spin(10, C)$, see [6, Proposition 3.6.4] in detail, and the C-linear transformation σ defined in Section 2 induces the involutive automorphism $\tilde{\sigma}$ on E_6^C, moreover define a 4-dimensional C-vector subspace $(V_-^C)^4$ of $\mathfrak{J}^C$ by

$$(V_-^C)^4 = \{X \in \mathfrak{J}^C \mid 4E_1 \times (E_1 \times X) = X, F_1(e_k) \times X = 0, k = 2, \dots, 7\}$$

$$= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix} \middle| \, \xi_i \in C, x_1 \in \boldsymbol{C}^C \subset \mathfrak{C}^C \right\}$$

with the norm $(-E_1, X, X) = -\xi_2\xi_3 + \bar{x}_1 x_1$. The group $((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$ acts on $(V_-^C)^4$, obviously.

Lemma 3.23. *The Lie algebra* $((\mathfrak{e}_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$ *of the group* $((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$ *is given by*

$$((\mathfrak{e}_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7} = \left\{ \phi \in \mathfrak{e}_6^C \,\middle|\, \begin{array}{l} \sigma\phi = \phi\sigma, \\ \phi E_1 = 0,\ \phi F_1(e_k) = 0,\ k = 2, \dots, 7 \end{array} \right\}$$

$$= \left\{ \phi = \left(\begin{array}{c|c} D_2 & 0 \\ \hline 0 & 0 \end{array} \right) \begin{array}{l} +\tilde{A}_1(a) \\ +\tau(E_2 - E_3)^\sim \\ +F_1(t)^\sim \end{array} \,\middle|\, \begin{array}{l} D_2 \in \mathfrak{so}(2, C), \\ a, t \in \boldsymbol{C}^C,\ \tau \in C \end{array} \right\}.$$

In particular, $\dim_C(((\mathfrak{e}_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}) = 6$.

Proof. By doing simple computation, this lemma is proved easily. □

Proposition 3.24. *The homogeneous space* $((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}/Spin(3, C)$ *is homeomorphic to the complex sphere* $(S_-^C)^3$: $((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}/Spin(3, C) \simeq (S_-^C)^3$.

In particular, the group $((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$ *is connected.*

Proof. We define a 3-dimensional complex sphere $(S_-^C)^3$ by

$$(S_-^C)^3 = \{X \in (V_-^C)^4 \,|\, (-E_1, X, X) = 1\}$$

$$= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix} \,\middle|\, -\xi_2\xi_3 + \bar{x}_1 x_1 = 1, \xi_k \in C, x_1 \in \boldsymbol{C}^C \subset \mathfrak{C}^C \right\}.$$

The group $((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$ acts on $(S_-^C)^3$. Indeed, for $\alpha \in ((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7} \subset ((E_6^C)^\sigma)_{E_1}$, it follows from [6, Lemma 3.6.2] that ${}^t\alpha^{-1} \in ((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$. Hence, for $X \in (S_-^C)^3$ we have that

$$4E_1 \times (E_1 \times \alpha X) = 4\,{}^t\alpha^{-1}E_1 \times (\alpha E_1 \times \alpha X) = 4\,{}^t\alpha^{-1}E_1 \times {}^t\alpha^{-1}(E_1 \times X)$$

$$= \alpha(4E_1 \times (E_1 \times X)) = \alpha X,$$

and

$$F_1(e_k) \times \alpha X = \alpha F_1(e_k) \times \alpha X = {}^t\alpha^{-1}(F_1(e_k) \times X) = 0,$$

that is, $\alpha X \in (V_-^C)^4$. Moreover, it is clear that $(-E_1, \alpha X, \alpha X) = 1$. Thus we see $\alpha X \in (S_-^C)^3$. We shall show that this action is transitive. In order to prove this, it is sufficient to show that any element $X \in (S_-^C)^3$ can be transformed to $i(E_2+E_3) \in (S_-^C)^3$. Then we prepare some elements of $((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$.

For $t \in \boldsymbol{C}^C \subset \mathfrak{C}^C$ such that $t\bar{t} \neq 0$, we define a C-linear transformation $\beta_1(t)$ of $\mathfrak{J}^C$, $\beta_1(t)X(\xi, x) = Y(\eta, y)$, by

$$\begin{cases} \eta_1 = \xi_1 \\ \eta_2 = \dfrac{\xi_2 - \xi_3}{2} + \dfrac{\xi_2 + \xi_3}{2} \cosh v + \dfrac{(t, x_1)}{v} \sinh v \\ \eta_3 = -\dfrac{\xi_2 - \xi_3}{2} + \dfrac{\xi_2 + \xi_3}{2} \cosh v + \dfrac{(t, x_1)}{v} \sinh v, \end{cases}$$

$$\begin{cases} y_1 = x_1 + \dfrac{(\xi_2 + \xi_3)t}{2v} \sinh v + \dfrac{2(t, x_1)t}{v^2} \sinh^2 \dfrac{v}{2} \\ y_2 = x_2 \cosh \dfrac{v}{2} + \dfrac{\overline{x_3 t}}{v} \sinh \dfrac{v}{2} \\ y_3 = x_3 \cosh \dfrac{v}{2} + \dfrac{\overline{t x_2}}{v} \sinh \dfrac{v}{2}, \end{cases}$$

where $v \in C$, $v^2 = t\bar{t}$, moreover define a C-linear transformation $\alpha_{23}(c)$ of $\mathfrak{J}^C$ by

$$\alpha_{23}(c)\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & e^{c/2}x_3 & e^{-c/2}\bar{x}_2 \\ e^{c/2}\bar{x}_3 & e^c\xi_2 & x_1 \\ e^{-c/2}x_2 & \bar{x}_1 & e^{-c}\xi_3 \end{pmatrix},$$

where $c \in C$. Then since we can express $\beta_1(t) = \exp F_1(t)^\sim$ and $\alpha_{23}(c) = \exp c(E_2 - E_3)^\sim$ for $F_1(t)^\sim$, $c(E_2 - E_3)^\sim \in ((\mathfrak{e}_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$ (Lemma 3.23), we see that $\beta_1(t)$, $\alpha_{23}(c) \in (((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7})_0$.

Now, let

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix} \in (S_-^C)^3.$$

Operate some $\alpha_0 \in (((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7})_0$ on X, and so X can be transformed to $\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x_1' \\ 0 & \bar{x}_1' & -\xi \end{pmatrix} \in (S_-^C)^2$, that is,

$$\alpha_0 X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x_1' \\ 0 & \bar{x}_1' & -\xi \end{pmatrix} \in (S_-^C)^2.$$

Indeed, we can confirm the existence of α_0 above as follows.

Case (i) where $x_1\bar{x}_1 \neq 0$.

We choose some $t_0 = i(\pi/2)(e_1x_1/(x_1\bar{x}_1)^{1/2}) \in \boldsymbol{C}^C$. Then since it is easy to verify that

$$\begin{aligned}(t_0, x_1) &= \left(i\left(\frac{\pi}{2}\right)\frac{e_1x_1}{\sqrt{x_1\bar{x}_1}}, x_1\right) = i\left(\frac{\pi}{2}\right)\frac{1}{\sqrt{x_1\bar{x}_1}}(e_1x_1, x_1) \\ &= i\left(\frac{\pi}{2}\right)\frac{1}{\sqrt{x_1\bar{x}_1}}(e_1, 1)(x_1, x_1) = 0, \\ v^2 = t_0\bar{t}_0 &= i\left(\frac{\pi}{2}\right)\frac{e_1x_1}{\sqrt{x_1\bar{x}_1}}\overline{i\left(\frac{\pi}{2}\right)\frac{e_1x_1}{\sqrt{x_1\bar{x}_1}}} = -\left(\frac{\pi}{2}\right)^2\frac{(e_1x_1)(\overline{e_1x_1})}{\sqrt{x_1\bar{x}_1}^2} \\ &= -\left(\frac{\pi}{2}\right)^2\frac{x_1\bar{x}_1}{x_1\bar{x}_1} = -\left(\frac{\pi}{2}\right)^2,\end{aligned}$$

operate $\alpha_0 := \beta_1(t_0)$ on X, and so we easily see that the η_2-part and the η_3-part of α_0X are $(\xi_2 - \xi_3)/2$ and $-(\xi_2 - \xi_3)/2$, respectively. Hence we can confirm the form of α_0X as above.

Case (ii) where $x_1\bar{x}_1 = 0$.

Together with the condition of $(S_-^C)^3$, we have $\xi_2\xi_3 = 1$. Then set $\xi_2 = e^{r_2+i\theta_2}$, $r_2, \theta_2 \in \boldsymbol{R}$. Operate $\alpha_{23}(-r_2 - i\theta_2)$ on X, and so we easily see that the η_2-part and the η_3-part of $\alpha_{23}(-r_2 - i\theta_2)X$ are equal to 1, that is,

$$\alpha_{23}(-r_2 - i\theta_2)X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & x_1 \\ 0 & \bar{x}_1 & 1 \end{pmatrix} =: X_1.$$

Moreover, operate $\alpha_{23}(i\pi/2) \in (((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}))_0$ on X_1, then we have that

$$\alpha_{23}\left(i\frac{\pi}{2}\right)X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & x_1 \\ 0 & \bar{x}_1 & -i \end{pmatrix} \in (S_-^C)^2.$$

Hence this case is reduced to Case (i).

Since $Spin(3, C) \cong (F_4^C)_{E_1, F_1(e_k), k=2,\dots,7} (\subset ((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7})$ acts transitively on $(S_-^C)^2$ (Proposition 3.21), there exists $\alpha \in Spin(3, C)$ such that

$$\alpha X = E_2 - E_3.$$

Again, operate $\alpha_{23}(i\pi/2)$ on $E_2 - E_3$, then we have that

$$\alpha_{23}\left(i\frac{\pi}{2}\right)(E_2 - E_3) = i(E_2 + E_3).$$

This shows the transitivity of this action to $(S_-^C)^3$ by the group $((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$. The isotropy subgroup of the group $((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$ at $i(E_2+E_3)$ is $Spin(3,C) \cong (F_4^C)_{E_1, F_1(e_k), k=2,\dots,7} = ((E_6^C)^\sigma)_{E_1, E_2+E_3, F_1(e_k), k=2,\dots,7}$ (Theorem 3.22, Section 2). Thus we have the required homeomorphism

$$((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}/Spin(3,C) \simeq (S_-^C)^3.$$

Therefore we see that the group $((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$ is connected. □

Theorem 3.25. *The group* $((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$ *is isomorphic to* $Spin(4,C)$: $((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7} \cong Spin(4,C)$.

Proof. Let $O(4,C) = O((V_-^C)^4) = \{\beta \in \mathrm{Iso}_C((V_-^C)^4) \,|\, (E_1, \beta X, \beta Y) = (E_1, X, Y)\}$. We consider the restriction $\beta = \alpha|_{(V_-^C)^4}$ of $\alpha \in ((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$ to $(V_-^C)^4$, then we have $\beta \in O(4,C)$. Hence we define a homomorphism $p : ((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7} \to O(4,C) = O((V_-^C)^4)$ by

$$p(\alpha) = \alpha|_{(V_-^C)^4}.$$

Moreover since the mapping p is continuous and the group $((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$ is connected (Proposition 3.24), the mapping p induces a homomorphism

$$p : ((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7} \to SO(4,C) = SO((V_-^C)^4).$$

It is not difficult to obtain that $\mathrm{Ker}\, p = \{1, \sigma\} \cong \boldsymbol{Z}_2$. Indeed, let $\alpha \in \mathrm{Ker}\, p$. For $E_2+E_3, E_2-E_3 \in (V_-^C)^4$, since $\alpha(E_2+E_3) = E_2+E_3$ and $\alpha(E_2-E_3) = E_2-E_3$, that is, $\alpha E_2 = E_2$, $\alpha E_3 = E_3$, we have that $\alpha \in ((E_6^C)^\sigma)_{E_1, E_2, E_3, F_1(e_k), k=2,\dots,7} \cong (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=2,\dots,7} \cong U(1, \boldsymbol{C}^C)$ (Theorem 3.3). Hence there exists $\theta \in U(1, \boldsymbol{C}^C)$ such that $\alpha = \phi(\theta)$, where ϕ is defined in Theorem 3.3, then it follows from $\alpha F_1(1) = F_1(1)$, $F_1(1) \in (V_-^C)^4$ that we have $(\bar{\theta})^2 = 1$, that is, $\theta = 1$ or $\theta = -1$. Thus we have that

$$\alpha = \phi(1) = 1 \quad \text{or} \quad \alpha = \phi(-1) = \sigma,$$

that is, $\mathrm{Ker}\, p \subset \{1, \sigma\}$ and vice versa. Hence we obtain that $\mathrm{Ker}\, p = \{1, \sigma\}$. Finally, we shall show that p is surjection. Since $SO(4,C)$ is connected, $\mathrm{Ker}\, p$ is discrete and $\dim_C(((\mathfrak{e}_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}) = 6 = \dim_C(\mathfrak{so}(4,C))$ (Lemma 3.23), p is surjection. Thus we have that

$$((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}/\boldsymbol{Z}_2 \cong SO(4,C).$$

Therefore the group $((E_6^C)^{\sigma})_{E_1, F_1(e_k), k=2,\dots,7}$ is isomorphic to $Spin(4, C)$ as the universal double covering group of $SO(4, C)$, that is, $((E_6^C)^{\sigma})_{E_1, F_1(e_k), k=2,\dots,7} \cong Spin(4, C)$. □

We define a group $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, \dot{F}_1(e_k), k=2,\dots,7}$ by

$$((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, \dot{F}_1(e_k), k=2,\dots,7} = \left\{ \alpha \in E_7^C \;\middle|\; \begin{array}{l} \kappa\alpha = \alpha\kappa,\ \mu\alpha = \alpha\mu, \\ \alpha\tilde{E}_1 = \tilde{E}_1,\ \alpha\tilde{E}_{-1} = \tilde{E}_{-1} \\ \alpha\dot{F}_1(e_k) = \dot{F}_1(e_k),\ k = 2, \dots, 7 \end{array} \right\}.$$

Then we have the following proposition.

Proposition 3.26. *The group* $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, \dot{F}_1(e_k), k=2,\dots,7}$ *consists with the group* $((E_6^C)^{\sigma})_{E_1, F_1(e_k), k=2,\dots,7}$: $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, \dot{F}_1(e_k), k=2,\dots,7} = ((E_6^C)^{\sigma})_{E_1, F_1(e_k), k=2,\dots,7}$ $\cong Spin(4, C)$.

Proof. Let $\alpha \in ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, F_1(e_k), k=2,\dots,7}$. From $\alpha\tilde{E}_1 = \tilde{E}_1$ and $\alpha\tilde{E}_{-1} = \tilde{E}_{-1}$, we have $\alpha\underset{\cdot}{1} = \underset{\cdot}{1}$, and so as in the proof of Proposition 3.19, we have $\alpha\dot{1} = \dot{1}$. Thus we see $\alpha \in (E_7^C)_{\dot{1}, \underset{\cdot}{1}} = E_6^C$. Moreover, since we can confirm $\alpha\dot{E}_1 = \dot{E}_1$ from the condition above, we have $\alpha \in (E_6^C)_{E_1}$, and from $\kappa\alpha = \alpha\kappa$, together with $-\sigma = \exp(\pi i\kappa)$, we have $(-\sigma)\alpha = \alpha(-\sigma)$, that is, $\sigma\alpha = \alpha\sigma$. Thus we have $\alpha \in ((E_6^C)^{\sigma})_{E_1, F_1(e_k), k=2,\dots,7}$.

Conversely, let $\beta \in ((E_6^C)^{\sigma})_{E_1, F_1(e_k), k=2,\dots,7}$. It is clear that $\beta\tilde{E}_1 = \tilde{E}_1$ and $\beta\tilde{E}_{-1} = \tilde{E}_{-1}$. For C-linear transformation κ_1 of $\mathfrak{J}^C : \kappa_1 X = (E_1, X)E_1 - 4E_1 \times (E_1 \times X)$, we have $\kappa_1\beta = \beta\kappa_1$. Indeed, note that suppose $\beta E_1 = E_1$, we have ${}^t\beta^{-1}E_1 = E_1$ (see [6, Lemma 3.6.2]).

$$\begin{aligned}
\kappa_1\beta X &= (E_1, \beta X)E_1 - 4E_1 \times (E_1 \times \beta X) \\
&= ({}^t\beta^{-1}E_1, \beta E_1)\beta E_1 - 4\,{}^t\beta^{-1}E_1 \times (\beta E_1 \times \beta X) \\
&= ({}^t\beta\,{}^t\beta^{-1}E_1, E_1)\beta E_1 - 4\,{}^t\beta^{-1}E_1 \times {}^t\beta^{-1}(E_1 \times X) \\
&= (E_1, \beta X)\beta E_1 - 4\beta(E_1 \times (E_1 \times \beta X)) \\
&= \beta((E_1, X)E_1 - 4E_1 \times (E_1 \times X)) \\
&= \beta\kappa_1 X,
\end{aligned}$$

that is, $\kappa_1\beta = \beta\kappa_1$. Similarly, we can show $\kappa_1\,{}^t\beta^{-1} = {}^t\beta^{-1}\kappa_1$. Hence we have that

$$\kappa\beta(X, Y, \xi, \eta) = \kappa(\beta X, {}^t\beta^{-1}, \xi, \eta) = (-\kappa_1\beta X, \kappa_1{}^t\beta^{-1}Y, -\xi, \eta)$$
$$= (-\beta\kappa_1 X, {}^t\beta^{-1}\kappa_1 Y, -\xi, \eta) = \beta(-\kappa_1 X, \kappa_1 Y, -\xi, \eta)$$
$$= \beta\kappa(X, Y, \xi, \eta),$$

that is, $\kappa\beta = \beta\kappa$. Additionally, we can show that $\mu\beta = \beta\mu$. Indeed, for $(X, Y, \xi, \eta) \in \mathfrak{P}^C$, we do simple computation as follows:

$$\begin{aligned}
\mu\beta(X, Y, \xi, \eta) &= \mu(\beta X, {}^t\beta^{-1}Y, \xi, \eta) \\
&= \Phi(0, E_1, E_1, 0)(\beta X, {}^t\beta^{-1}Y, \xi, \eta) \\
&= (2E_1 \times {}^t\beta^{-1}Y + \eta E_1, 2E_1 \times \beta X + \xi E_1, (E_1, {}^t\beta^{-1}Y), (E_1, \beta X)) \\
&= (2{}^t\beta^{-1}E_1 \times {}^t\beta^{-1}Y + \eta\beta E_1, 2\beta E_1 \times \beta X + \xi{}^t\beta^{-1}E_1, \\
&\qquad (\beta E_1, {}^t\beta^{-1}Y), ({}^t\beta^{-1}E_1, \beta X)) \\
&= (2\beta(E_1 \times Y) + \eta\beta E_1, 2{}^t\beta^{-1}(E_1 \times X) + \xi{}^t\beta^{-1}E_1, \\
&\qquad (\beta^{-1}\beta E_1, Y), ({}^t\beta^{-1}\beta^{-1}E_1, X)) \\
&= (\beta(2E_1 \times Y + \eta E_1), {}^t\beta^{-1}(2E_1 \times X + \xi E_1), (E_1, Y), (E_1, X)) \\
&= \beta(2E_1 \times Y + \eta E_1, 2E_1 \times X + \xi E_1, (E_1, Y), (E_1, X)) \\
&= \beta\Phi(0, E_1, E_1, 0)(X, Y, \xi, \eta) \\
&= \beta\mu(X, Y, \xi, \eta),
\end{aligned}$$

that is, $\mu\beta = \beta\mu$. Hence we have $\beta \in ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, \dot{F}_1(e_k), k=2,\dots,7}$. The proof of this proposition is completed. □

Continuously, we shall construct $Spin(5, C)$ in E_7^C.

Now, we consider a group $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7}$:

$$((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7} = \left\{ \alpha \in E_7^C \;\middle|\; \begin{array}{l} \kappa\alpha = \alpha\kappa, \mu\alpha = \alpha\mu, \\ \alpha\tilde{E}_1 = \tilde{E}_1, \\ \alpha\dot{F}_1(e_k) = \dot{F}_1(e_k), k = 2, \dots, 7 \end{array} \right\},$$

moreover define a 5-dimensional C-vector subspace $(V_-^C)^5$ of $\mathfrak{P}^C$ by

$$(V_-^C)^5 = \left\{ P \in \mathfrak{P}^C \,\middle|\, \begin{array}{l} \kappa P = P,\ P \times \tilde{E}_1 = 0, \\ P \times \dot{F}_1(e_k) = 0,\ k = 2, \ldots, 7 \end{array} \right\}$$

$$= \{(X, -\eta E_1, 0, \eta) \,|\, 4E_1 \times (E_1 \times X) = X, X \times F_1(e_k) = 0, k = 2, \ldots, 7, \eta \in C\}$$

$$= \left\{ \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} -\eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \eta \right) \,\middle|\, x_1 \in \boldsymbol{C}^C, \xi_2, \xi_3, \eta \in C \right\}$$

with the norm $(P, P)_\mu = (1/2)\{\mu P, P\} = -\xi_2\xi_3 + x_1\bar{x}_1 - \eta^2$, here the alternative inner product $\{P, Q\}$ is defined as follows: $\{P, Q\} = (X, W) - (Z, Y) + \xi\omega - \zeta\eta$ for $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega)$. Obviously, the group $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, F_1(e_k), k=2,\ldots,7}$ acts on $(V_-^C)^5$.

Lemma 3.27. *The Lie algebra* $((\mathfrak{e}_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\ldots,7}$ *of the group* $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\ldots,7}$ *is given by*

$$((\mathfrak{e}_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\ldots,7} = \left\{ \Phi(\phi, A, B, v) \in \mathfrak{e}_7^C \,\middle|\, \begin{array}{l} \kappa\Phi = \Phi\kappa,\ \mu\Phi = \Phi\mu, \\ \Phi\tilde{E}_1 = 0, \\ \Phi\dot{F}_1(e_k) = 0,\ k = 2, \ldots, 7 \end{array} \right\}$$

$$= \left\{ \Phi(\phi, A, B, 0) \in \mathfrak{e}_7^C \,\middle|\, \begin{array}{l} \phi \in (\mathfrak{e}_6^C)^\sigma, \\ \quad \phi E_1 = \phi F_1(e_k) = 0, \\ A = \varepsilon_2 E_2 + \varepsilon_3 E_3 + F_1(a), \\ \quad \varepsilon_k \in C,\ a \in \boldsymbol{C}^C, \\ B = -2E_1 \times A \\ \quad = -\varepsilon_3 E_2 - \varepsilon_2 E_3 + F_1(a) \end{array} \right\}.$$

In particular, $\dim_C(((\mathfrak{e}_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\ldots,7}) = 6 + (1 + 1 + 2) = 10.$

Proof. Suppose $\kappa\Phi = \Phi\kappa$ for $\Phi \in \mathfrak{e}_7^C$, from $-\sigma = \exp(\pi i\kappa)$ we see that $(-\sigma)\Phi = \Phi(-\sigma)$, that is, $\sigma\Phi = \Phi\sigma$. Hence, we have $\phi \in (\mathfrak{e}_6^C)^\sigma$. Moreover, from $\mu\Phi = \Phi\mu$, the condition $\Phi\tilde{E}_1 = 0$ is equivalent to the condition $\Phi(E_1, 0, 1, 0) = 0$. Using these facts, by doing simple computation, we have the explicit form of the Lie algebra $((\mathfrak{e}_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\ldots,7}$ above. □

Lemma 3.28. *For* $0 \neq a \in C$, *we define a mapping* $\alpha_i(a) : \mathfrak{P}^C \to \mathfrak{P}^C$, $i = 1, 2, 3$ *by*

$$\alpha_i(a) = \begin{pmatrix} 1+(\cos|a|-1)p_i & -2\tau a\dfrac{\sin|a|}{|a|}E_i & 0 & a\dfrac{\sin|a|}{|a|}E_i \\ 2a\dfrac{\sin|a|}{|a|}E_i & 1+(\cos|a|-1)p_i & -\tau a\dfrac{\sin|a|}{|a|}E_i & 0 \\ 0 & a\dfrac{\sin|a|}{|a|}E_i & \cos|a| & 0 \\ -\tau a\dfrac{\sin|a|}{|a|}E_i & 0 & 0 & \cos|a| \end{pmatrix},$$

then we have $\alpha_i(a) \in E_7 \subset E_7^C$, *where* $p_i : \mathfrak{J}^C \to \mathfrak{J}^C$ *is the* C*-linear mapping defined by*

$$p_i\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \delta_{i3}x_3 & \delta_{i2}\bar{x}_2 \\ \delta_{i3}\bar{x}_3 & \xi_2 & \delta_{i1}x_1 \\ \delta_{i2}x_2 & \delta_{i1}\bar{x}_1 & \xi_3 \end{pmatrix},$$

where δ_{ij} *is the Kronecker delta symble. The mappings* $\alpha_1(a_1)$, $\alpha_2(a_2)$, $\alpha(a_3)$, $(a_i \in C)$ *are commutative for each other.*

Proof. For $\Phi_i(a) = \Phi(0, aE_i, -\tau aE_i, 0) \in \mathfrak{e}_7$, it follows from $\alpha_i(a) = \exp \Phi_i(a)$ that $\alpha_i(a) \in E_7 \subset E_7^C$. The relation formula $[\Phi_i(a_i), \Phi_j(a_j)] = 0$ implies that $\alpha_i(a_i)$ and $\alpha_j(a_j)$ are commutative (As for the Lie algebra $\mathfrak{e}_7$ of the compact Lie group E_7, see [9, Theorem 4.3.4] in detail). □

Proposition 3.29. *The homogeneous space* $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1,\dot{F}_1(e_k),k=2,\ldots,7}/Spin(4,C)$ *is homeomorphic to the complex sphere* $(S_-^C)^4$: $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1,\dot{F}_1(e_k),k=2,\ldots,7}/Spin(4,C) \simeq (S_-^C)^4$.

In particular, the group $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1,\dot{F}_1(e_k),k=2,\ldots,7}$ *is connected.*

Proof. We define a 4-dimensional complex sphere $(S_-^C)^4$ by

$$\begin{aligned}(S_-^C)^4 &= \{P \in (V_-^C)^5 \mid (P,P)_\mu = 1\} \\ &= \left\{ \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} -\eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \eta \right) \,\middle|\, -\xi_2\xi_3 + x_1\bar{x}_1 - \eta^2 = 1 \right\}.\end{aligned}$$

The group $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1,\dot{F}_1(e_k),k=2,\ldots,7}$ acts on $(S_-^C)^4$. Indeed, for $\alpha \in ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1,\dot{F}_1(e_k),k=2,\ldots,7}$ and $P \in (S_-^C)^4$, from the following relational formulas:

$$\kappa\alpha P = \alpha\kappa P = \alpha P,$$

$$\alpha P \times \tilde{E}_1 = \alpha P \times \alpha\tilde{E}_1 = \alpha(P \times \tilde{E}_1)^t\alpha^{-1} = 0,$$

$$\alpha P \times \dot{F}_1(e_k) = \alpha P \times \alpha\dot{F}_1(e_k) = \alpha(P \times \dot{F}_1(e_k))^t\alpha^{-1} = 0,$$

$$(\alpha P, \alpha P)_\mu = \frac{1}{2}\{\mu\alpha P, \alpha P\} = \frac{1}{2}\{\alpha\mu P, \alpha P\} = \frac{1}{2}\{\mu P, P\} = 1,$$

we have $\alpha P \in (S_-^C)^4$. We shall show that this action is transitive. In order to prove this, it is sufficient to show that any element $P \in (S_-^C)^4$ can be transformed to $\tilde{E}_{-1} = (0, -E_1, 0, 1)$.

Now, for a given

$$P = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} -\eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \eta \right) \in (S_-^C)^4,$$

we choose $a \in \boldsymbol{R}$, $0 \le a \le \pi/2$ such that $\tan 2a = 2\,\mathrm{Re}(\eta)/\mathrm{Re}(\xi_2 + \xi_3)$ (if $\mathrm{Re}(\xi_2 + \xi_3) = 0$, let $a = \pi/4$). Operate $\alpha_{23}(a) := \alpha_2(a)\alpha_3(a) = \exp(\Phi(0, a(E_2 + E_3), -a(E_2 + E_3), 0)) \in (((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7})_0$ on P (Lemmas 3.27, 3.28), then the part $(1/2)\,\mathrm{Re}(\xi_2 + \xi_3)\sin 2a - \mathrm{Re}(\eta)\cos 2a$ of η-term in $\alpha_{23}(a)P$ is equal to 0, that is,

$$(1/2)\,\mathrm{Re}(\xi_2 + \xi_3)\sin 2a - \mathrm{Re}(\eta)\cos 2a = 0.$$

Moreover, we choose $b \in \boldsymbol{R}$, $0 \le b \le \pi/2$ such that $\tan 2b = 2\,\mathrm{Im}(\eta)/\mathrm{Im}(\xi_2 + \xi_3)$ (if $\mathrm{Im}(\xi_2 + \xi_3) = 0$, let $b = \pi/4$), then η-term of $\alpha_{23}(b)\alpha_{23}(a)P$ is equal to 0.

Hence we have that

$$\alpha_{23}(b)\alpha_{23}(a)P =: P' \in (S_-^C)^3.$$

Since $Spin(4, C) \cong ((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7} (\subset ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7})$ acts transitively on $(S_-^C)^3$, there exists $\beta \in Spin(4, C)$ such that

$$\beta P' = (i(E_2 + E_3), 0, 0, 0) =: P''.$$

Again, operate $\alpha_{23}(-\pi/4)$ on P'', then we have that

$$\alpha_{23}\left(-\frac{\pi}{4}\right)P'' = (0, -iE_1, 0, i)(= i\tilde{E}_{-1}).$$

This shows the transitivity of this action to $(S_-^C)^4$ by the group $(E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7}$. The isotropy subgroup of the group $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7}$

at $i\tilde{E}_{-1}$ is $Spin(4, C)$ (Theorem 3.25, Proposition 3.26). Thus we have the required homeomorphism

$$((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7}/Spin(4, C) \simeq (S_-^C)^4.$$

Therefore we see that the group $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7}$ is connected. □

Theorem 3.30. *The group* $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7}$ *is isomorphic to* $Spin(5, C)$: $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7} \cong Spin(5, C)$.

Proof. Let $O(5, C) = O((V_-^C)^5) = \{\beta \in \mathrm{Iso}_C((V_-^C)^5) \mid (\alpha P, \alpha P)_\mu = (P, P)_\mu\}$. We consider the restriction $\beta = \alpha|_{(V_-^C)^5}$ of $\alpha \in ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7}$ to $(V_-^C)^5$, then we have $\beta \in O(5, C)$. Hence we define a homomorphism $p : ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7} \to O(5, C) = O((V_-^C)^5)$ by

$$p(\alpha) = \alpha|_{(V_-^C)^5}.$$

Moreover since the mapping p is continuous and the group $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7}$ is connected (Proposition 3.29), the mapping p induces a homomorphism

$$p : ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7} \to SO(5, C) = SO((V_-^C)^5).$$

It is not difficult to obtain that $\mathrm{Ker}\, p = \{1, \sigma\} \cong \boldsymbol{Z}_2$. Indeed, let $\alpha \in \mathrm{Ker}\, p$. For $\tilde{E}_{-1} = (0, -E_1, 0, 1) \in (V^C)^5$, since $\alpha\tilde{E}_{-1} = \tilde{E}_{-1}$, together with $\alpha\tilde{E}_1 = \tilde{E}_1$, we have that $\alpha\dot{E}_1 = \dot{E}_1$ and $\alpha\underset{\cdot}{1} = \underset{\cdot}{1}$. Hence we have that $\alpha \in ((E_7^C)^{\kappa,\mu})_{\dot{E}_1, \underset{\cdot}{1}, \dot{F}_1(e_k), k=2,\dots,7} \cong ((E_6^C)^\sigma)_{E_1, F_1(e_k), k=2,\dots,7}$. Moreover, for $E_2 \dotplus E_3, E_2 \dot{-} E_3 \in (V_-^C)^5$, since $\alpha(E_2 \dotplus E_3) = E_2 \dotplus E_3$ and $\alpha(E_2 \dot{-} E_3) = E_2 \dot{-} E_3$, we have that $\alpha \in ((E_6^C)^\sigma)_{E_1, E_2, E_3, F_1(e_k), k=2,\dots,7} = (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=2,\dots,7} \cong U(1, \boldsymbol{C}^C)$. Hence there exists $\theta \in U(1, \boldsymbol{C}^C)$ such that $\alpha = \phi(\theta)$, where ϕ is defined in Theorem 3.3, and so since $\alpha F_1(1) = F_1(1)$, $F_1(1) \in (V_-^C)^5$, we have $(\bar{\theta})^2 = 1$, that is, $\theta = 1$ or $\theta = -1$. Thus we have that

$$\alpha = \phi(1) = 1 \quad \text{or} \quad \alpha = \phi(-1) = \sigma,$$

that is, $\mathrm{Ker}\, p \subset \{1, \sigma\}$ and vice versa. Hence we obtain that $\mathrm{Ker}\, p = \{1, \sigma\}$. Finally, we shall show that p is surjection. Since $SO(5, C)$ is connected, $\mathrm{Ker}\, p$ is discrete and $\dim_C(((\mathfrak{e}_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7}) = 10 = \dim_C(\mathfrak{so}(5, C))$ (Lemma 3.27), p is surjection. Thus we have that

$$((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7}/\boldsymbol{Z}_2 \cong SO(5, C).$$

Therefore the group $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7}$ is isomorphic to $Spin(5, C)$ as the universal double covering group of $SO(5, C)$, that is, $((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{F}_1(e_k), k=2,\dots,7} \cong Spin(5, C)$. □

Continuously, we shall construct $Spin(6, C)$ in E_7^C.

Now, we consider a group $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7}$:

$$((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7} = \left\{ \alpha \in E_7^C \,\middle|\, \begin{array}{l} \kappa\alpha = \alpha\kappa,\ \mu\alpha = \alpha\mu, \\ \alpha\dot{F}_1(e_k) = \dot{F}_1(e_k),\ k = 2, \dots, 7 \end{array} \right\},$$

moreover, define a 6-dimensional C-vector subspace $(V_-^C)^6$ of $\mathfrak{P}^C$ by

$$\begin{aligned} (V_-^C)^6 &= \left\{ P \in \mathfrak{P}^C \,\middle|\, \begin{array}{l} \kappa P = P, \\ P \times \dot{F}_1(e_k) = 0,\ k = 2, \dots, 7 \end{array} \right\} \\ &= \left\{ \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \eta \right) \,\middle|\, x_1 \in \boldsymbol{C}^C, \xi_2, \xi_3, \eta_1, \eta \in C \right\} \end{aligned}$$

with the norm $(P, P)_\mu = (1/2)\{\mu P, P\} = -\xi_2\xi_3 + x_1\bar{x}_1 + \eta_1\eta$.

Lemma 3.31. *The Lie algebra* $((\mathfrak{e}_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7}$ *of the group* $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7}$ *is given by*

$$\begin{aligned} &((\mathfrak{e}_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7} \\ &= \left\{ \Phi(\phi, A, B, \nu) \in \mathfrak{e}_7^C \,\middle|\, \begin{array}{l} \kappa\Phi = \Phi\kappa,\ \mu\Phi = \Phi\mu, \\ \Phi\dot{F}_1(e_k) = 0,\ k = 2, \dots, 7 \end{array} \right\} \\ &= \left\{ \Phi(\phi, A, B, \nu) \in \mathfrak{e}_7^C \,\middle|\, \begin{array}{l} \phi = \left(\begin{array}{c|c} D_2 & 0 \\ \hline 0 & 0 \end{array} \right) \begin{array}{l} + \tilde{A}_1(a) \\ + (\tau_1 E_1 + \tau_2 E_2 + \tau_3 E_3 + F_1(t_1))^{\sim}, \end{array} \\ \quad D_2 \in \mathfrak{so}(2, C),\ a \in \boldsymbol{C}^C,\ \tau_k \in C, \\ \quad \tau_1 + \tau_2 + \tau_3 = 0,\ t_1 \in \boldsymbol{C}^C, \\ A = \varepsilon_2 E_2 + \varepsilon_3 E_3 + F_1(a),\ \varepsilon_k \in C,\ a \in \boldsymbol{C}^C, \\ B = \upsilon_2 E_2 + \upsilon_3 E_3 + F_1(b),\ \upsilon_k \in C,\ b \in \boldsymbol{C}^C, \\ \nu = -(3/2)\tau_1 \end{array} \right\}. \end{aligned}$$

In particular, $\dim_C(((\mathfrak{e}_7^C)^{\kappa,\mu})_{\tilde{E}_1,\dot{F}_1(e_k),k=2,\dots,7}) = (1 + 2 + (2 + 2)) + 4 + 4 = 15.$

Proof. From [7, Section 4.6], we see that the explicit form of $(\mathfrak{e}_7^C)^{\kappa,\mu}$ is given by

$$(\mathfrak{e}_7^C)^{\kappa,\mu} = \{\Phi(\phi, A, B, \nu) \in \mathfrak{e}_7^C \mid \kappa\Phi = \Phi\kappa, \mu\Phi = \Phi\mu\}$$

$$= \left\{ \Phi(\phi, A, B, \nu) \in \mathfrak{e}_7^C \;\middle|\; \begin{array}{l} \phi \in (\mathfrak{e}_6^C)^\sigma, \\ A = \varepsilon_2 E_2 + \varepsilon_3 E_3 + F_1(a), \\ B = \upsilon_2 E_2 + \upsilon_3 E_3 + F_1(b), \\ \quad \varepsilon_k, \upsilon_k \in C,\ a, b \in \boldsymbol{C}^C, \\ \nu = -(3/2)(\phi E_1, E_1) \end{array} \right\}.$$

Since the result of direct computation of $\Phi\dot{F}_1(e_k)$ is as follows:

$$\Phi(\phi, A, B, \nu)\dot{F}_1(e_k) = (\phi F_1(e_k) - \frac{\nu}{3}F_1(e_k), 2A \times F_1(e_k), 0, (B, F_1(e_k))),$$

for $\Phi\dot{F}_1(e_k) = 0$ we have that

$$\begin{cases} \phi F_1(e_k) - \dfrac{\nu}{3}F_1(e_k) = 0 \cdots (1) \\ 2A \times F_1(e_k) = 0 \cdots (2) \\ (B, F_1(e_k)) = 0 \cdots (3). \end{cases}$$

From the conditions (2) and (3), it is easy to verify that $x, y \in \boldsymbol{C}^C$. As for the condition (1), by doing direct computation, we obtain that

$$\begin{aligned}
&\phi F_1(e_k) - \frac{\nu}{3}F_1(e_k) \\
&\quad = (\delta + \tilde{T})F_1(e_k) - \frac{\nu}{3}F_1(e_k) \quad (\delta \in \mathfrak{f}_4^C, T \in (\mathfrak{J}^C)_\sigma, \mathrm{tr}(T) = 0) \\
&\quad = \delta F_1(e_k) + \tilde{T}F_1(e_k) - \frac{\nu}{3}F_1(e_k) \\
&\quad = (D + \tilde{A}_1(a_1))F_1(e_k) + \tilde{T}F_1(e_k) - \frac{\nu}{3}F_1(e_k) \quad (D \in \mathfrak{so}(8, C), a_1 \in \mathfrak{C}^C) \\
&\quad = F_1(De_k) + (a_1, e_k)(E_2 - E_3) \\
&\qquad\qquad + \left(\frac{1}{2}\tau_2 F_1(e_k) + \frac{1}{2}\tau_3 F_1(e_k) + (t_1, e_k)(E_2 + E_3)\right) - \frac{\nu}{3}F_1(e_k) \\
&\quad = \{(a_1, e_k) + (t_1, e_k)\}E_2 + \{(a_1, e_k) - (t_1, e_k)\}E_3 \\
&\qquad\qquad + F_1(De_k + \frac{1}{2}(\tau_2 + \tau_3)e_k - \frac{\nu}{3}e_k),
\end{aligned}$$

where $T = \tau_1 E_1 + \tau_2 E_2 + \tau_3 E_3 + F_1(t_1)$, $\tau_k \in C$, $t_1 \in \mathfrak{C}^C$.

Hence, from the condition (1), we see that

$$\begin{cases} (a_1, e_k) + (t_1, e_k) = (a_1, e_k) - (t_1, e_k) = 0 \cdots (4) \\ De_k - \dfrac{1}{2}\tau_1 e_k - \dfrac{v}{3} e_k = 0 \cdots (5)(\tau_1 + \tau_2 + \tau_3 = 0), \end{cases}$$

moreover, for the condition (5), together with $v = (-3/2)\tau_1$, we have $De_k = 0$, $k = 2, \dots, 7$. Thus we have $D \in \mathfrak{so}(2, C) \subset \mathfrak{so}(8, C)$. From the condition (4), we have $a_1, t_1 \in \boldsymbol{C}^C$. Therefore we have the required the explicit form of the Lie algebra $((\mathfrak{e}_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7}$. □

Lemma 3.32. *For $v \in C$, we define a mapping $\beta(v) : \mathfrak{P}^C \to \mathfrak{P}^C$ by*

$$\beta(v)(X, Y, \xi, \eta) = (\begin{pmatrix} e^{2v}\xi_1 & e^v x_3 & e^v \bar{x}_2 \\ e^v \bar{x}_3 & \xi_2 & x_1 \\ e^v x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} e^{-2v}\eta_1 & e^{-v} y_3 & e^{-v} \bar{y}_2 \\ e^{-v} \bar{y}_3 & \eta_2 & y_1 \\ e^{-v} y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}, e^{-2v}\xi, e^{-v}\eta).$$

Then we have $\beta(v) \in (((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7})_0$.

Proof. From Lemma 3.31, for $v \in C$ we see that $\Phi((2/3)v(2E_1 - (E_2 + E_3))^{\sim}, 0, 0, -2v) \in ((\mathfrak{e}_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7}$. Hence we have that

$$\beta(v) = \exp(\Phi((2/3)v(2E_1 - (E_2 + E_3))^{\sim}, 0, 0, -2v)) \in (((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7})_0.$$

□

Proposition 3.33. *The homogeneous space $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7}/Spin(5, C)$ is homeomorphic to the complex sphere $(S_-^C)^5$: $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7}/Spin(5, C) \simeq (S_-^C)^5$.*

In particular, the group $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7}$ is connected.

Proof. We define a 5-dimensional complex sphere $(S_-^C)^5$ by

$$(S_-^C)^5 = \{P \in (V_-^C)^6 \mid (P, P)_\mu = 1\}$$

$$= \left\{ (\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \eta) \,\middle|\, -\xi_2\xi_3 + x_1\bar{x}_1 + \eta_1\eta = 1 \right\}.$$

As in the proof of Proposition 3.29, it is easy to verify that the group $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7}$ acts on $(S_-^C)^5$, and so we shall show that this action is transitive. In order to prove this, it is sufficient to show that any $P \in (S_-^C)^5$ can be transformed to $\tilde{E}_1 \in (S_-^C)^5$.

Now, for a given

$$P = (\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \eta) \in (S^C)^5,$$

first we shall show that there exists some $\alpha \in ((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7}$ such that $\alpha P \in (S_-^C)^4$.

Case (i) where $\eta_1 \neq 0,\ \eta \neq 0$.

We choose $v \in C$ such that $-e^{-2v}\eta_1 = e^{2v}\eta$, and operate $\beta(v)$ of Lemma 3.32 on P, then we have $\beta(v)P \in (S_-^C)^4$.

Case (ii) where $\eta_1 = 0,\ \eta \neq 0,\ \xi_2 \neq 0$.

Operate $\alpha = \exp \Phi(0, E_3, 0, 0) \in (((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7})_0$ on P (Lemma 3.31), then we have that

$$\alpha P = (\xi_2 E_2 + (\xi_3 + \eta)E_3 + F_1(x_1), \xi_2 E_1, 0, \eta).$$

Hence this case is reduced to Case (i).

Case (iii) where $\eta_1 = 0,\ \eta \neq 0,\ \xi_3 \neq 0$.

As in Case (ii), operate $\alpha = \exp \Phi(0, E_2, 0, 0) \in (((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7})_0$ on P (Lemma 3.31), then we have that

$$\alpha P = ((\xi_2 + \eta)E_2 + \xi_3 E_3 + F_1(x_1), \xi_3 E_1, 0, \eta).$$

Hence this case is also reduced to Case (i).

Case (iv) where $\eta_1 = \xi_2 = \xi_3 = 0,\ \eta \neq 0$.

For some $t \in \boldsymbol{R}$, operate $\alpha = \exp \Phi(0, tF_1(x_1), 0, 0) \in (((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7})_0$ on $P = (F_1(x), 0, 0, \eta)$ (Lemma 3.31), then we have that

$$\begin{aligned} \alpha P &= ((1 + t\eta)F_1(x_1), -(2t + t^2\eta)(x_1, x_1)E_1, 0, \eta) \quad ((x_1, x_1) = 1) \\ &= ((1 + t\eta)F_1(x_1), -(2t + t^2\eta)E_1, 0, \eta). \end{aligned}$$

Hence this case is also reduced to Case (i) for some $t \in \boldsymbol{R}$.

Case (v) where $\eta_1 \neq 0,\ \eta = 0,\ \xi_2 \neq 0$.

Operate $\alpha = \exp \Phi(0, 0, E_2, 0) \in (((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7})_0$ on P (Lemma 3.31), then we have that

$$\alpha P = (\xi_2 E_2 + \xi_3 E_3 + F_1(x_1), \eta_1 E_1, 0, \xi_2).$$

Hence this case is also reduced to Case (i).

Case (vi) where $\eta_1 \neq 0,\ \eta = 0,\ \xi_3 \neq 0$.

As in Case (v), operate $\alpha = \exp \Phi(0, 0, E_3, 0) \in (((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7})_0$ on P (Lemma 3.31), then we have that

$$\alpha P = (\xi_2 E_2 + \xi_3 E_3 + F_1(x_1), \eta_1 E_1, 0, \xi_3).$$

Hence this case is also reduced to Case (i).

Case (vii) where $\eta_1 \neq 0$, $\eta = 0$, $\xi_2 = \xi_3 = 0$.

For some $t \in \boldsymbol{R}$, operate $\alpha = \exp \Phi(0, 0, tF_1(x_1), 0) \in (((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7})_0$ on $P = (F_1(x_1), \eta_1 E_1, 0, 0)$ (Lemma 3.31), then we have that

$$\begin{aligned} \alpha P &= ((1 - t\eta_1)F_1(x_1), \eta_1 E_1, 0, (2t - t^2\eta_1(x_1, x_1)))((x_1, x_1) = 1) \\ &= ((1 - t\eta_1)F_1(x_1), \eta_1 E_1, 0, (2t - t^2\eta_1)). \end{aligned}$$

Hence this case is also reduced to Case (i) for some $t \in \boldsymbol{R}$.

Case (viii) where $\eta_1 = \eta = 0$.

Then we see that $P \in (S_-^C)^3 \subset (S_-^C)^4$.

From above since

$$Spin(5, C) \cong ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \dot{1}, \dot{F}_1(e_k), k=2,\dots,7} (\subset ((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7})$$

acts transitively on $(S_-^C)^4$ (Proposition 3.29), there exists $\delta \in Spin(5, C)$ such that

$$\delta(\alpha P) = (0, -iE_1, 0, i)(= i\tilde{E}_{-1}).$$

Again, operate $\beta(-i\pi/4)$ of Lemma 3.32 on $\delta(\alpha P)$, then we have that

$$\beta\left(-i\frac{\pi}{4}\right)(\delta(\alpha P)) = (0, E_1, 0, 1)(= \tilde{E}_1).$$

This shows the transitivity of this action to $(S_-^C)^5$ by the group $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7}$. The isotropy subgroup of the group $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7}$ at $\tilde{E}_1$ is $Spin(5, C)$ (Theorem 3.30).

Thus we have the required homeomorphism

$$((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7}/Spin(5, C) \simeq (S_-^C)^5.$$

Therefore we see that the group $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7}$ is connected. □

THEOREM 3.34. *The group* $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7}$ *is isomorphic to* $Spin(6, C)$: $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7} \cong Spin(6, C)$.

PROOF. Let $O(6, C) = O((V_-^C)^6) = \{\beta \in \mathrm{Iso}_C((V_-^C)^6) \mid (\alpha P, \alpha P)_\mu = (P, P)_\mu\}$. We consider the restriction $\beta = \alpha|_{(V_-^C)^6}$ of $\alpha \in ((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7}$ to $(V_-^C)^6$, then we have $\beta \in O(6, C)$. Hence we define a homomorphism $p : ((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7} \to O(6, C) = O((V_-^C)^6)$ by

$$p(\alpha) = \alpha|_{(V_-^C)^6}.$$

Moreover since the mapping p is continuous and the group $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\ldots,7}$ is connected (Proposition 3.33), the mapping p induces a homomorphism

$$p : ((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\ldots,7} \to SO(6,C) = SO((V_-^C)^6).$$

It is not difficult to obtain that Ker $p = \{1,\sigma\} \cong \boldsymbol{Z}_2$. Indeed, let $\alpha \in$ Ker p. For $\tilde{E}_1 = (0, E_1, 0, 1)$, $\tilde{E}_{-1} = (0, -E_1, 0, 1) \in (V_-^C)^6$, since $\alpha\tilde{E}_1 = \tilde{E}_1$ and $\alpha\tilde{E}_{-1} = \tilde{E}_{-1}$, we have that $\alpha\dot{E}_1 = \dot{E}_1$ and $\alpha\underset{\cdot}{1} = \underset{\cdot}{1}$. Hence we have that $\alpha \in ((E_7^C)^{\kappa,\mu})_{\dot{E}_1,\underset{\cdot}{1},\dot{F}_1(e_k),k=2,\ldots,7} \cong ((E_6^C)^{\sigma})_{E_1,F_1(e_k),k=2,\ldots,7}$. Moreover, for $E_2 \dotplus E_3$, $E_2 \dot{-} E_3 \in (V_-^C)^6$, since $\alpha(E_2 \dotplus E_3) = E_2 \dotplus E_3$ and $\alpha(E_2 \dot{-} E_3) = E_2 \dot{-} E_3$, we have that $\alpha \in ((E_6^C)^{\sigma})_{E_1,E_2,E_3,F_1(e_k),k=2,\ldots,7} \cong (F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=2,\ldots,7} \cong U(1,\boldsymbol{C}^C)$. Hence there exists $\theta \in U(1,\boldsymbol{C}^C)$ such that $\alpha = \phi(\theta)$, where ϕ is defined in Theorem 3.3, and so since $\alpha F_1(1) = F_1(1)$, $F_1(1) \in (V_-^C)^6$, we have $(\bar{\theta})^2 = 1$, that is, $\theta = 1$ or $\theta = -1$. Thus since we see

$$\alpha = \phi(1) = 1 \quad \text{or} \quad \alpha = \phi(-1) = \sigma,$$

we have that Ker $p \subset \{1,\sigma\}$ and vice versa. Hence we obtain that Ker $p = \{1,\sigma\}$. Finally, we shall show that p is surjection. Since $SO(6,C)$ is connected, Ker p is discrete and $\dim_C(((\mathfrak{e}_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\ldots,7}) = 15 = \dim_C(\mathfrak{so}(6,C))$ (Lemma 3.31), p is surjection. Thus we have that

$$((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\ldots,7}/\boldsymbol{Z}_2 \cong SO(6,C).$$

Therefore the group $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\ldots,7}$ is isomorphic to $Spin(6,C)$ as the universal double covering group of $SO(6,C)$, that is, $((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\ldots,7} \cong Spin(6,C)$. □

Here, as in previous section, we make a summary of the results as the low dimensional spinor groups which were constructed in this section.

$$\begin{array}{l}
((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\ldots,7} \cong Spin(6,C) \\
\qquad\quad \cup \\
((E_7^C)^{\kappa,\mu})_{\tilde{E}_1,\dot{F}_1(e_k),k=2,\ldots,7} \cong Spin(5,C) \\
\qquad\quad \cup \\
((E_6^C)^{\sigma})_{E_1,F_1(e_k),k=2,\ldots,7} \cong Spin(4,C) \\
\qquad\quad \cup \\
(F_4^C)_{E_1,F_1(e_k),k=2,\ldots,7} \cong Spin(3,C) \\
\qquad\quad \cup \\
(F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=2,\ldots,7} \cong Spin(2,C) \cong U(1,\boldsymbol{C}^C)
\end{array}$$

Together with the results of previous section, we have had two sequences as for the low dimensional spinor groups.

After this, by using two $Spin(6, C)$, we determine the structure of the groups $((E_7^C)^{\kappa,\mu})^{\sigma_4'}$, $(E_7^C)^{\sigma_4'}$, and we shall prove the connectedness of the group $(E_7^C)^{\sigma_4', \mathfrak{so}(6,C)}$.

First, we determine the structure of the group $((E_7^C)^{\kappa,\mu})^{\sigma_4'}$.

Lemma 3.35. *The Lie algebra* $((\mathfrak{e}_7^C)^{\kappa,\mu})^{\sigma_4'}$ *of the group* $((E_7^C)^{\kappa,\mu})^{\sigma_4'}$ *is given by*

$$((\mathfrak{e}_7^C)^{\kappa,\mu})^{\sigma_4'}$$
$$= \left\{ \Phi(\phi, A, B, \nu) \in \mathfrak{e}_7^C \;\middle|\; \begin{array}{l} \kappa\Phi = \Phi\kappa,\ \mu\Phi = \Phi\mu, \\ \sigma_4'\Phi = \Phi\sigma_4' \end{array} \right\}$$
$$= \left\{ \Phi(\phi, A, B, \nu) \in \mathfrak{e}_7^C \;\middle|\; \begin{array}{l} \phi = \left(\begin{array}{c|c} D_2 & 0 \\ \hline 0 & D_6 \end{array}\right) \begin{array}{l} +\tilde{A}_1(a) \\ +(\tau_1 E_1 + \tau_2 E_2 + \tau_3 E_3 + F_1(t_1))^{\sim}, \end{array} \\ \qquad D_2 \in \mathfrak{so}(2, C),\ D_6 \in \mathfrak{so}(6, C),\ a \in \boldsymbol{C}^C,\ \tau_k \in C, \\ \qquad \tau_1 + \tau_2 + \tau_3 = 0,\ t_1 \in \boldsymbol{C}^C, \\ A = \varepsilon_2 E_2 + \varepsilon_3 E_3 + F_1(a),\ \varepsilon_k \in C,\ a \in \boldsymbol{C}^C, \\ B = \upsilon_2 E_2 + \upsilon_3 E_3 + F_1(b),\ \upsilon_k \in C,\ b \in \boldsymbol{C}^C, \\ \nu = -(3/2)\tau_1 \end{array} \right\}.$$

In particular, $\dim_C(((\mathfrak{e}_7^C)^{\kappa,\mu})^{\sigma_4'}) = ((1 + 15) + 2 + (2 + 2)) + (2 + 2) \times 2 = 30.$

Proof. By doing simple computation, we can obtain the result above. □

Proposition 3.36. *The group* $((E_7^C)^{\kappa,\mu})^{\sigma_4'}$ *is isomorphic to* $(Spin(6, C) \times Spin(6, C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(1, 1), (\sigma, \sigma)\}$: $((E_7^C)^{\kappa,\mu})^{\sigma_4'} \cong (Spin(6, C) \times Spin(6, C))/\boldsymbol{Z}_2$.

Proof. Let

$$Spin(6, C) \cong (F_4^C)_{E_1, E_2, E_3, \dot{F}_1(e_k), k=0,1} \cong ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, E_2 \dot{+} E_3, E_2 \dot{-} E_3, \dot{F}_1(e_k), k=0,1}$$

(Theorem 3.16, Proposition 3.19) and one more $Spin(6, C) \cong ((E_7^C)^{\kappa,\mu})_{F_1(e_k), k=2,\dots,7}$ (Theorem 3.34). Then we define a mapping $\varphi_{\kappa,\mu,\sigma_4'} : Spin(6, C) \times Spin(6, C) \to ((E_7^C)^{\kappa,\mu})^{\sigma_4'}$ by

$$\varphi_{\kappa,\mu,\sigma_4'}(\beta_1, \beta_2) = \beta_1 \beta_2.$$

First, we have to prove that the mapping $\varphi_{\kappa,\mu,\sigma_4'}$ is well-defined. It follows from Lemma 3.17 and Proposition 3.19 that $Spin(6,C) \cong (F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1} \subset (F_4^C)^{\sigma_4'} \subset ((E_7^C)^{\kappa,\mu})^{\sigma_4'}$, and since $Spin(6,C) \cong ((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7}$ and $((E_7^C)^{\kappa,\mu})^{\sigma_4'}$ are connected, in order to prove $Spin(6,C) \cong ((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7} \subset (E_7^C)^{\kappa,\mu})^{\sigma_4'}$, it is sufficient to show that the Lie algebra $\mathfrak{spin}(6,C) \cong ((\mathfrak{e}_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7}$ is the subalgebra of the Lie algebra $((\mathfrak{e}_7^C)^{\kappa,\mu})^{\sigma_4'}$. However, from Lemmas 3.31, 3.35, it is clear. Hence the mapping $\varphi_{\kappa,\mu,\sigma_4'}$ is well-defined.

Next, we shall show that the mapping $\varphi_{\kappa,\mu,\sigma_4'}$ is a homomorphism. Since $Spin(6,C) \cong (F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1}$ and $Spin(6,C) \cong ((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7}$ are connected, in order to prove that the mapping $\varphi_{\kappa,\mu,\sigma_4'}$ is a homomorphism, it is sufficient to show that Φ_1 commutes with Φ_2, that is, $[\Phi_1,\Phi_2] = 0$ for $\Phi_1 \in \mathfrak{spin}(6,C) \cong (\mathfrak{f}_4^C)_{E_1,E_2,E_3,\dot{F}_1(e_k),k=0,1}$ and $\Phi_2 \in \mathfrak{spin}(6,C) \cong ((\mathfrak{e}_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7}$. However, it is also clear from Lemmas 3.14, 3.31.

We determine the $\operatorname{Ker}\varphi_{\kappa,\mu,\sigma_4'}$. From the definition of kernel, we have that

$$\begin{aligned}\operatorname{Ker}\varphi_{\kappa,\mu,\sigma_4'} &= \{(\beta_1,\beta_2) \in Spin(6,C) \times Spin(6,C) \,|\, \varphi_{\kappa,\mu,\sigma_4'}(\beta_1,\beta_2) = 1\} \\ &= \{(\beta_1,\beta_2) \in Spin(6,C) \times Spin(6,C) \,|\, \beta_1 = \beta_2^{-1}\}.\end{aligned}$$

Then, from the condition $\beta_1 = \beta_2^{-1}$, we see $\beta_1 \dot{F}_1(e_k) = \beta_2^{-1}\dot{F}_1(e_k) = \dot{F}_1(e_k)$, $k = 2,\dots,7$, that is, $\beta_1 \dot{F}_1(e_k) = \dot{F}_1(e_k)$. Moreover, since $\beta_1 \in Spin(6,C) \cong (F_4^C)_{E_1,E_2,E_3,\dot{F}_1(e_k),k=0,1}$, we see that $\beta_1 \dot{F}_1(x) = \dot{F}_1(x)$ for all $x \in \mathfrak{C}^C$. Here, from $\beta_1 \in (F_4^C)_{E_1,E_2,E_3} \cong Spin(8,C)$, β_1 can be expressed by $\beta_1 = (\delta_1,\delta_2,\delta_3) \in SO(8,C)^{\times 3}$ such that $(\delta_1 x)(\delta_2 y) = \overline{\delta_3(\overline{xy})}$, $x, y \in \mathfrak{C}^C$, and so we have that $\delta_1 x = x$ for all $x \in \mathfrak{C}^C$. Hence we have $\delta_1 = 1$, and so we see that

$$\beta_1 = (1,1,1) = 1 \quad \text{or} \quad \beta_1 = (1,-1,-1) = \sigma.$$

Hence it follows from the condition $\beta_1 = \beta_2^{-1}$ that $\beta_2 = 1$ or $\beta_2 = \sigma$, that is, $\operatorname{Ker}\varphi_{\kappa,\mu,\sigma_4'} \subset \{(1,1),(\sigma,\sigma)\}$ and vice versa. Thus we obtain that $\operatorname{Ker}\varphi_{\kappa,\mu,\sigma_4'} = \{(1,1),(\sigma,\sigma)\} \cong \boldsymbol{Z}_2$. Finally, we shall show that $\varphi_{\kappa,\mu,\sigma_4'}$ is surjection. Since $\operatorname{Ker}\varphi_{\kappa,\mu,\sigma_4'}$ is discrete, the group $((E_7^C)^{\kappa,\mu})^{\sigma_4'}$ is connected because of $(E_7^C)^{\kappa,\mu} \cong Spin(12,C)$ (see [7, Proposition 4.6.10]) and $\dim_C(((\mathfrak{e}_7^C)^{\kappa,\mu})^{\sigma_4'}) = 30 = \dim_C(\mathfrak{so}(6,C) \oplus \mathfrak{so}(6,C))$ (Lemma 3.35), $\varphi_{\kappa,\mu,\sigma_4'}$ is surjection.

Therefore we have the required isomorphism

$$((E_7^C)^{\kappa,\mu})^{\sigma_4'} \cong (Spin(6,C) \times Spin(6,C))/\boldsymbol{Z}_2. \qquad \square$$

We determine the structure of the group $(E_7^C)^{\sigma_4'}$ as one of aims of this subsection.

Lemma 3.37. *The group* $(E_7^C)^{\sigma_4'}$ *contains a subgroup*

$$\psi(SL(2,C)) = \{\psi(A) \in E_7^C \mid A \in SL(2,C)\}$$

which is isomorphic to the special linear group $SL(2,C) = \{A \in M(2,C) \mid \det A = 1\}$. *Here, for* $A \in SL(2,C)$, *a mapping* $\psi(A) : \mathfrak{P}^C \to \mathfrak{P}^C$ *is defined by*

$$\psi(A)(\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}, \xi, \eta)$$

$$=: (\begin{pmatrix} \xi_1' & x_3' & \bar{x}_2' \\ \bar{x}_3' & \xi_2' & x_1' \\ x_2' & \bar{x}_1' & \xi_3' \end{pmatrix}, \begin{pmatrix} \eta_1' & y_3' & \bar{y}_2' \\ \bar{y}_3' & \eta_2' & y_1' \\ y_2' & \bar{y}_1' & \eta_3' \end{pmatrix}, \xi', \eta'),$$

where

$$\begin{pmatrix} \xi_1' \\ \eta' \end{pmatrix} = A\begin{pmatrix} \xi_1 \\ \eta \end{pmatrix}, \quad \begin{pmatrix} \xi' \\ \eta_1' \end{pmatrix} = A\begin{pmatrix} \xi \\ \eta_1 \end{pmatrix}, \quad \begin{pmatrix} \eta_2' \\ \xi_3' \end{pmatrix} = A\begin{pmatrix} \eta_2 \\ \xi_3 \end{pmatrix}, \quad \begin{pmatrix} \eta_3' \\ \xi_2' \end{pmatrix} = A\begin{pmatrix} \eta_3 \\ \xi_2 \end{pmatrix},$$

$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \tau A\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} x_3' \\ y_3' \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}.$$

Proof. The action of $\Phi(\phi(v), aE_1, bE_1, v) \in (\mathfrak{e}_7^C)^{\sigma_4'}$ (Lemma 3.18) ($\phi(v) = (2/3)v(2E_1 - (E_2 + E_3))^\sim, a, b, v \in C$) on $\mathfrak{P}^C$ is as follows:

$$\Phi(\phi(v), aE_1, bE_1, v)(X, Y, \xi, \eta) =: (X', Y', \xi', \eta'),$$

where

$$\begin{pmatrix} \xi_1' \\ \eta' \end{pmatrix} = \begin{pmatrix} v & a \\ b & -v \end{pmatrix}\begin{pmatrix} \xi_1 \\ \eta \end{pmatrix}, \qquad \begin{pmatrix} \xi' \\ \eta_1' \end{pmatrix} = \begin{pmatrix} v & a \\ b & -v \end{pmatrix}\begin{pmatrix} \xi \\ \eta_1 \end{pmatrix},$$

$$\begin{pmatrix} \eta_2' \\ \xi_3' \end{pmatrix} = \begin{pmatrix} v & a \\ b & -v \end{pmatrix}\begin{pmatrix} \eta_2 \\ \xi_3 \end{pmatrix}, \qquad \begin{pmatrix} \eta_3' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} v & a \\ b & -v \end{pmatrix}\begin{pmatrix} \eta_3 \\ \xi_2 \end{pmatrix},$$

$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \begin{pmatrix} \tau v & \tau a \\ \tau b & -\tau v \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} x_3' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, for $A = \exp\begin{pmatrix} v & a \\ b & -v \end{pmatrix} \in SL(2,C)(\begin{pmatrix} v & a \\ b & -v \end{pmatrix} \in \mathfrak{sl}(2,C))$, we have that

$$\exp(\Phi(\phi(v), aE_1, bE_1, v)) = \psi(A) \in \psi(SL(2,C)) \subset (E_7^C)^{\sigma_4'}. \qquad \square$$

Theorem 3.38. *We have that* $(E_7^C)^{\sigma'_4} \cong (SL(2,C) \times Spin(6,C) \times Spin(6,C))/\boldsymbol{Z}_4$, $\boldsymbol{Z}_4 = \{(E,1,1), (E,\sigma,\sigma), (-E,\sigma'_4,-\sigma'_4), (-E,\sigma\sigma'_4,-\sigma\sigma'_4)\}$.

Proof. Let $SL(2,C) = \{A \in M(2,C) \mid \det A = 1\}$ and two $Spin(6,C)$ as in Proposition 3.36. Then we define a mapping $\varphi_{E_7^C,\sigma'_4} : SL(2,C) \times Spin(6,C) \times Spin(6,C) \to (E_7^C)^{\sigma'_4}$ by

$$\varphi_{E_7^C,\sigma'_4}(A,\beta_1,\beta_2) = \psi(A)\beta_1\beta_2.$$

From Lemma 3.37 and Proposition 3.36, it is clear that the mapping $\varphi_{E_7^C,\sigma'_4}$ is well-defined. It is to verify that $\varphi_{E_7^C,\sigma'_4}$ is a homomorphism. Indeed, note that $\beta_1,\beta_2 \in Spin(12,C) \cong (E_7^C)^{\kappa,\mu}$. From [7, Theorem 4.6.13], we see that $\psi(A)$ commutes with β_1, β_2, respectively. Moreover, as in Proposition 3.36, β_1 commutes with β_2. Hence since $\psi(A)$, β_1, β_2 commute each other, $\varphi_{E_7^C,\sigma'_4}$ is a homomorphism. We shall show that $\varphi_{E_7^C,\sigma'_4}$ is surjection. For $\alpha \in (E_7^C)^{\sigma'_4} \subset (E_7^C)^{\sigma}$, there exist $A \in SL(2,C)$ and $\beta \in Spin(12,C)$ such that $\alpha = \varphi(A,\beta)$ (see [7, Theorem 4.6.13]). Moreover, from the condition $\sigma'_4\alpha = \alpha\sigma'_4$, that is, $\varphi(A, \sigma'_4\beta{\sigma'_4}^{-1}) = \varphi(A,\beta)$, we have that

$$\begin{cases} A = A \\ \sigma'_4\beta{\sigma'_4}^{-1} = \beta \end{cases} \quad \text{or} \quad \begin{cases} A = -A \\ \sigma'_4\beta{\sigma'_4}^{-1} = -\sigma\beta. \end{cases}$$

Then the latter case is impossible because of $A \neq 0$. As for the former case, from Proposition 3.36, there exist $\beta_1 \in Spin(6,C)$ and $\beta_2 \in Spin(6,C)$ such that $\beta = \varphi_{\kappa,\mu,\sigma'_4}(\beta_1,\beta_2)$. Thus, $\varphi_{E_7^C,\sigma'_4}$ is surjection. Finally, we determine the $\operatorname{Ker} \varphi_{E_7^C,\sigma'_4}$. From $\operatorname{Ker} \varphi = \{(E,1), (-E,-\sigma)\}$ (see [7, Theorem 4.6.13]) , we have that

$$\begin{aligned} \operatorname{Ker} \varphi_{E_7^C,\sigma'_4} = {} & \{(A,\beta_1,\beta_2) \in SL(2,C) \times Spin(6,C) \times Spin(6,C) \mid A = E, \beta_1\beta_2 = 1\} \\ & \cup \{(A,\beta_1,\beta_2) \in SL(2,C) \times Spin(6,C) \\ & \qquad \times Spin(6,C) \mid A = -E, \beta_1\beta_2 = -\sigma\}. \end{aligned}$$

So, we obtain the following results.

Case (i) where $A = E$, $\beta_1\beta_2 = 1$.

From $\operatorname{Ker} \varphi_{\kappa,\mu,\sigma'_4} = \{(1,1), (\sigma,\sigma)\}$ (Proposition 3.36), we have that

$$\begin{cases} A = E \\ \beta_1 = 1 \\ \beta_2 = 1 \end{cases} \quad \text{or} \quad \begin{cases} A = E \\ \beta_1 = \sigma \\ \beta_2 = \sigma. \end{cases}$$

Case (ii) where $A = -E$, $\beta_1\beta_2 = -\sigma$.

Since $\beta_1\beta_2 = -\sigma \in ((E_7^C)^{\kappa,\mu})^{\sigma'_4}$, there exist $\beta_1 \in Spin(6, C)$ and $\beta_2 \in Spin(6, C)$ such that $-\sigma = \beta_1\beta_2$ (Proposition 3.36). Here, we easily see that

$$\sigma'_4 \in Spin(6, C) \cong (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1} \cong ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, E_2 \dot{+} E_3, E_2 \dot{-} E_3, \dot{F}_1(e_k), k=0,1},$$

$$-\sigma'_4 \in Spin(6, C) \cong ((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7},$$

and $\sigma'_4(-\sigma'_4) = -\sigma$, and so together with $\operatorname{Ker} \varphi_{\kappa,\mu,\sigma'_4} = \{(1,1), (\sigma,\sigma)\}$, we have that

$$\begin{cases} \beta_1 = \sigma'_4 \\ \beta_2 = -\sigma'_4 \end{cases} \quad \text{or} \quad \begin{cases} \beta_1 = \sigma\sigma'_4 \\ \beta_2 = \sigma(-\sigma'_4). \end{cases}$$

Hence we see

$$\operatorname{Ker} \varphi_{E_7^C, \sigma'_4} \subset \{(E,1,1), (E,\sigma,\sigma), (-E,\sigma'_4,-\sigma'_4), (-E,\sigma\sigma'_4,-\sigma\sigma'_4)\},$$

and vice versa. Thus we obtain that

$$\operatorname{Ker} \varphi_{E_7^C, \sigma'_4} = \{(E,1,1), (E,\sigma,\sigma), (-E,\sigma'_4,-\sigma'_4), (-E,\sigma\sigma'_4,-\sigma\sigma'_4)\} \cong \boldsymbol{Z}_4.$$

Therefore we have the required isomorphism

$$(E_7^C)^{\sigma'_4} \cong (SL(2, C) \times Spin(6, C) \times Spin(6, C))/\boldsymbol{Z}_4. \qquad \square$$

Now, we also determine the structure of the group $(E_7^C)^{\sigma'_4, \mathfrak{so}(6,C)}$, and prove the connectedness of its group as another aim of this subsection.

THEOREM 3.39. *We have that* $(E_7^C)^{\sigma'_4, \mathfrak{so}(6,C)} \cong SL(2, C) \times Spin(6, C)$. *In particular, the group* $(E_7^C)^{\sigma'_4, \mathfrak{so}(6,C)}$ *is connected.*

PROOF. Let $SL(2, C)$ and $Spin(6, C) \cong ((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k), k=2,\dots,7}$ as in Theorem 3.38. Note that

$$\mathfrak{so}(6, C) = \{\Phi_D = (D, 0, 0, 0) \in \mathfrak{e}_7^C \mid D \in \mathfrak{so}(6, C) \cong (\mathfrak{f}_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}\}.$$

Then we define a mapping $\varphi_{E_7^C, \sigma'_4, \mathfrak{so}(6,C)} : SL(2, C) \times Spin(6, C) \to (E_7^C)^{\sigma'_4, \mathfrak{so}(6,C)}$ by

$$\varphi_{E_7^C, \sigma'_4, \mathfrak{so}(6,C)}(A, \beta_2) = \psi(A)\beta_2,$$

where note that the mapping $\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}$ is the restricted mapping of the mapping $\varphi_{E_7^C,\sigma_4'}$ in Theorem 3.38. We have to prove that $\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}$ is well-defined. In order to prove this, since $\psi(SL(2,C))$ and $Spin(6,C)$ are connected, it is sufficient to show that for $\Phi(\phi(v),aE_1,bE_1,v) \in \psi_*(\mathfrak{sl}(2,C))$, $\Phi_2 \in \mathfrak{spin}(6,C) \cong ((\mathfrak{e}_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7}$, the following formulas hold:

$$[\Phi_D, \Phi(\phi(v),aE_1,bE_1,v)] = 0, \quad [\Phi_D,\Phi_2] = 0,$$

where $\Phi(\phi(v),aE_1,bE_1,v) \in \psi_*(\mathfrak{sl}(2,C))$, $\Phi_2 \in \mathfrak{spin}(6,C) \cong ((\mathfrak{e}_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7}$, here a mapping ψ_* is the differential mapping of the mapping ψ in Lemma 3.37. However, it is clear that $[\Phi_D, \Phi(\phi(v),aE_1,bE_1,v)] = 0$, moreover from Lemma 3.32, it is easy to verify that $[\Phi_D,\Phi_2] = 0$. Hence $\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}$ is well-defined. Since the mapping $\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}$ is the restricted mapping $\varphi_{E_7^C,\sigma_4'}$, it is clear that the mapping $\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}$ is a homomorphism. We shall show that the mapping $\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}$ is injection. Since $\dim_C(\varphi_{\mathfrak{e}_7^C,\sigma_4',\mathfrak{so}(6,C)}) = 18 = 3 + 15 = \dim_C(\mathfrak{sl}(2,C) \oplus \mathfrak{spin}(6,C))$ (Lemma 3.18 (2)), the differential mapping ${\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}}_*$ of $\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}$ is injection. Hence we see that $\mathrm{Ker}\,{\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}}_* = \{0\}$, that is, $\mathrm{Ker}\,\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}$ is discrete. Hence, $\mathrm{Ker}\,\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}$ is contained in the center $z(SL(2,C) \times Spin(6,C)) = \{(E,1),(E,\sigma),(E,-\sigma_4'),(E,-\sigma\sigma_4'),(-E,1),(-E,\sigma),(-E,\sigma_4'),(-E,-\sigma\sigma_4')\}$. Note that in general because of the center $z(Spin(6,C)) = \boldsymbol{Z}_4$, we see that $z(Spin(6,C)) = \{1,\sigma,-\sigma\sigma_4',-\sigma_4'\}$ (cf. in the proof of Theorem 3.38). However, since the mapping $\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}$ maps the elements of $z(SL(2,C) \times Spin(6,C))$ to 1, σ, $-\sigma_4'$, $-\sigma\sigma_4'$, -1, $-\sigma$, σ_4', $\sigma\sigma_4'$, respectively, we have that $\mathrm{Ker}\,\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)} = \{(E,1)\}$, that is, the mapping $\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}$ is injection. Finally, We shall show that the mapping $\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}$ is surjection. For $\alpha \in (E_7^C)^{\sigma_4',\mathfrak{so}(6,C)} \subset (E_7^C)^{\sigma_4'}$, there exist $A \in SL(2,C)$ and $\beta_1 \in Spin(6,C) \cong (F_4^C)_{E_1,E_2,E_3,\dot{F}_1(e_k),k=0,1} \cong ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1,\tilde{E}_{-1},E_2\dot{+}E_3,E_2\dot{-}E_3,\dot{F}_1(e_k),k=0,1}$ and $\beta_2 \in Spin(6,C) \cong ((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7}$ such that $\alpha = \psi(A)\beta_1\beta_2$ (Theorem 3.38). Moreover, from the condition $\Phi_D\alpha = \alpha\Phi_D$, together with $\Phi_D\psi(A) = \psi(A)\Phi_D$ (Lemma 3.37), $\Phi_D\beta_2 = \beta_2\Phi_D$ (Lemma 3.31), we have $\Phi_D\beta_1 = \beta_1\Phi_D$ for all $D \in \mathfrak{so}(6,C)$. Hence β_1 is contained in the center $z(Spin(6,C)) = z((F_4^C)_{E_1,E_2,E_3,\dot{F}_1(e_k),k=0,1}) = \{1,\sigma,\sigma_4',\sigma\sigma_4'\} \cong \boldsymbol{Z}_4$. However, we see that

$$\sigma = (1)\sigma = \psi(E)\sigma \in \psi(SL(2,C))\ Spin(6,C),$$

$$\sigma_4' = (-1)(-\sigma_4') = \psi(-E)(-\sigma_4') \in \psi(SL(2,C))\ Spin(6,C),$$

$$\sigma\sigma_4' = (-1)(-\sigma\sigma_4') = \psi(-E)(-\sigma\sigma_4') \in \psi(SL(2,C))\ Spin(6,C),$$

that is, $\sigma,\sigma_4',\sigma\sigma_4' \in \psi(SL(2,C))\ Spin(6,C)$, where $Spin(6,C) \cong ((E_7^C)^{\kappa,\mu})_{\dot{F}_1(e_k),k=2,\dots,7}$.

Consequently, we have $\beta_1 = 1$. Hence, $\varphi_{E_7^C,\sigma_4',\mathfrak{so}(6,C)}$ is surjection.

Thus we have the required isomorphism

$$(E_7^C)^{\sigma_4', \mathfrak{so}(6,C)} \cong SL(2,C) \times Spin(6,C).$$

Therefore we see that the group $(E_7^C)^{\sigma_4', \mathfrak{so}(6,C)}$ is connected. □

3.3. Connectedness of the Group $(E_8^C)^{\sigma_4', \mathfrak{so}(6,C)}$

We define a subgroup $(E_8^C)^{\sigma_4', \mathfrak{so}(6,C)}$ of the group $(E_8^C)^{\sigma_4'}$ by

$$(E_8^C)^{\sigma_4', \mathfrak{so}(6,C)} = \left\{ \alpha \in E_8^C \,\middle|\, \begin{array}{l} \sigma_4'\alpha = \alpha\sigma_4', \\ \Theta(R_D)\alpha = \alpha\Theta(R_D) \text{ for all } D \in \mathfrak{so}(6,C) \end{array} \right\},$$

where $R_D = (\Phi_D, 0, 0, 0, 0, 0) \in \mathfrak{e}_8^C$ and $\Theta(R_D)$ means $\mathrm{ad}(R_D)$. Hereafter for $R \in \mathfrak{e}_8^C$, we denote $\mathrm{ad}(R)$ by $\Theta(R)$, moreover in $\mathfrak{e}_8^C$, we often use the following notations:

$$\Phi = (\Phi, 0, 0, 0, 0, 0), \quad P^- = (0, P, 0, 0, 0, 0), \quad Q_- = (0, 0, Q, 0, 0, 0),$$
$$\tilde{r} = (0, 0, 0, r, 0, 0), \quad s^- = (0, 0, 0, 0, s, 0), \quad t_- = (0, 0, 0, 0, 0, t).$$

In order to prove the connectedness of the group $(E_8^C)^{\sigma_4', \mathfrak{so}(6,C)}$, we use the method used in [5]. However, we write this method in detail again.

First, we consider a subgroup $((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-}$ of the group $(E_8^C)^{\sigma_4', \mathfrak{so}(6,C)}$:

$$((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-} = \{\alpha \in (E_8^C)^{\sigma_4', \mathfrak{so}(6,C)} \,|\, \alpha 1_- = 1_-\}.$$

Lemma 3.40. *We have the following*

(1) *The Lie algebra* $((\mathfrak{e}_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-}$ *of the group* $((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-}$ *is given by*

$$((\mathfrak{e}_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-}$$
$$= \left\{ R \in \mathfrak{e}_8^C \,\middle|\, \begin{array}{l} \sigma_4' R = R, \\ [R, R_D] = 0 \text{ for all } D \in \mathfrak{so}(6,C),\ [R, 1_-] = 0 \end{array} \right\}$$
$$= \left\{ (\Phi, 0, Q, 0, 0, t) \in \mathfrak{e}_8^C \,\middle|\, \begin{array}{l} \Phi \in (\mathfrak{e}_7^C)^{\sigma_4', \mathfrak{so}(6,C)}, \\ Q = (Z, W, \zeta, \omega), \\ \quad Z = \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & \zeta_2 & z \\ 0 & \bar{z} & \zeta_3 \end{pmatrix}, W = \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & w \\ 0 & \overline{w} & \omega_3 \end{pmatrix}, \\ \quad \zeta_k, \omega_k, \zeta, \omega \in C,\ z, w \in \boldsymbol{C}^C, \\ t \in C \end{array} \right\},$$

where as for the explicit form of the Lie algebra $(\mathfrak{e}_7^C)^{\sigma_4', \mathfrak{so}(6,C)}$, *see Lemma* 3.18 (2).

In particular,

$$\dim_C(((\mathfrak{e}_8^C)^{\sigma'_4, \mathfrak{so}(6,C)})_{1_-}) = 18 + ((3+2) \times 2 + 1 \times 2) + 1 = 31.$$

(2) *The Lie algebra* $(\mathfrak{e}_8^C)^{\sigma'_4, \mathfrak{so}(6,C)}$ *of the group* $(E_8^C)^{\sigma'_4, \mathfrak{so}(6,C)}$ *is given by*

$$(\mathfrak{e}_8^C)^{\sigma'_4, \mathfrak{so}(6,C)}$$

$$= \left\{ R \in \mathfrak{e}_8^C \;\middle|\; \begin{array}{l} \sigma'_4 R = R, \\ [R, R_D] = 0 \ \ \textit{for all} \ \ D \in \mathfrak{so}(6,C) \end{array} \right\}$$

$$= \left\{ (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C \;\middle|\; \begin{array}{l} \Phi \in (\mathfrak{e}_7^C)^{\sigma'_4, \mathfrak{so}(6,C)}, \\ P = (X, Y, \xi, \eta), \\ \quad X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & x \\ 0 & \bar{x} & \xi_3 \end{pmatrix}, \ Y = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & y \\ 0 & \bar{y} & \eta_3 \end{pmatrix}, \\ \quad \xi_k, \eta_k, \xi, \eta \in C, \ x, y \in \boldsymbol{C}^C, \\ Q = (Z, W, \zeta, \omega) \ \textit{is same form as} \ P, \\ r, s, t \in C \end{array} \right\}.$$

In particular,

$$\dim_C((\mathfrak{e}_8^C)^{\sigma'_4, \mathfrak{so}(6,C)}) = 18 + ((3+2) \times 2 + 1 \times 2) \times 2 + 3 = 45.$$

Proof. (1) For $R = (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C$, from the condition $\sigma'_4 R = R$, we have that

$$\Phi \in (\mathfrak{e}_7^C)^{\sigma'_4},$$

$$P = (X, Y, \xi, \eta), \quad \begin{array}{l} X = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1(x), \xi_k, \xi \in C, x \in \boldsymbol{C}^C, \\ Y = \eta_1 E_1 + \eta_2 E_2 + \eta_3 E_3 + F_1(y), \eta_k, \eta \in C, y \in \boldsymbol{C}^C, \end{array}$$

$$Q = (Z, W, \zeta, \omega), \quad \begin{array}{l} Z = \zeta_1 E_1 + \zeta_2 E_2 + \zeta_3 E_3 + F_1(z), \zeta_k, \zeta \in C, z \in \boldsymbol{C}^C, \\ W = \omega_1 E_1 + \omega_2 E_2 + \omega_3 E_3 + F_1(w), \omega_k, \omega \in C, w \in \boldsymbol{C}^C, \end{array}$$

$$r, s, t \in C.$$

Moreover, from the condition $[R, R_D] = 0$, we have that

$$\Phi \in (\mathfrak{e}_7^C)^{\sigma'_4, \mathfrak{so}(6,C)}, \quad P, Q \text{ are same form above, and so are } r, s, t.$$

Finally, from the condition $[R, 1_-] = 0$, we have that $P = 0$ and $s = r = 0$.

Hence we have the required explicit form of the Lie algebra $((\mathfrak{e}_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-}$.
(2) By an argument similar to (1) above, we have the required result. □

In Proposition 3.41 below, note that the subspace $(\mathfrak{P}^C)_{\sigma_4'}$ of $\mathfrak{P}^C$ is defined by

$$\begin{aligned}(\mathfrak{P}^C)_{\sigma_4'} &= \{P \in \mathfrak{P}^C \mid \sigma_4' P = P\} \\ &= \{(X, Y, \xi, \eta) \in \mathfrak{P}^C \mid X, Y \in (\mathfrak{J}^C)_{\sigma_4'}, \xi, \eta \in C\},\end{aligned}$$

where $(\mathfrak{J}^C)_{\sigma_4'} = \{X \in \mathfrak{J}^C \mid \sigma_4' X = X\}$.

Proposition 3.41. *The group* $((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-}$ *is a semi-direct product of groups* $\exp(\Theta(((\mathfrak{P}^C)_{\sigma_4'})_- \oplus C_-))$ *and* $(E_7^C)^{\sigma_4', \mathfrak{so}(6,C)}$:

$$((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-} = \exp(\Theta(((\mathfrak{P}^C)_{\sigma_4'})_- \oplus C_-)) \rtimes (E_7^C)^{\sigma_4', \mathfrak{so}(6,C)}.$$

In particular, the group $((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-}$ *is connected.*

Proof. Let $((\mathfrak{P}^C)_{\sigma_4'})_- \oplus C_- = \{(0,0,Q,0,0,t) \mid Q \in (\mathfrak{P}^C)_{\sigma_4'}, t \in C\}$ be a Lie subalgebra of the Lie algebra $((\mathfrak{e}_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-}$ (Lemma 3.40 (1)). Since it follows from $[Q_-, t_-] = 0$ that $\Theta(Q_-)$ commutes with $\Theta(t_-)$, we have $\exp(\Theta(Q_- + t_-)) = \exp(\Theta(Q_-))\exp(\Theta(t_-))$, and so we also see that $\exp(\Theta(((\mathfrak{P}^C)_{\sigma_4'})_- \oplus C_-))$ is the connected subgroup of the group $((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-}$.

Now, let $\alpha \in ((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-}$ and set

$$\alpha\tilde{1} = (\Phi, P, Q, r, s, t), \quad \alpha 1^- = (\Phi_1, P_1, Q_1, r_1, s_1, t_1).$$

Then, from the relation formulas $[\alpha\tilde{1}, 1_-] = \alpha[\tilde{1}, 1_-] = -2\alpha 1_- = -21_-$, $[\alpha 1^-, 1_-] = \alpha[1^-, 1_-] = \alpha\tilde{1}$, we have that

$$P = 0, \quad s = 0, \quad r = 1, \quad \Phi = 0, \quad P_1 = -Q, \quad s_1 = 1, \quad r_1 = -\frac{t}{2}.$$

Moreover, from $[\alpha\tilde{1}, \alpha 1^-] = \alpha[\tilde{1}, 1^-] = 2\alpha 1^-$, we have that

$$\Phi_1 = \frac{1}{2} Q \times Q, \quad Q_1 = -\frac{t}{2} Q - \frac{1}{3}\Phi_1 Q, \quad t_1 = -\frac{t^2}{4} - \frac{1}{16}\{Q, Q_1\}.$$

Hence we see that α is of the form

$$\alpha = \begin{pmatrix} * & * & * & 0 & \frac{1}{2}Q \times Q & 0 \\ * & * & * & 0 & -Q & 0 \\ * & * & * & Q & -\frac{t}{2}Q - \frac{1}{6}(Q \times Q)Q & 0 \\ * & * & * & 1 & -\frac{t}{2} & 0 \\ * & * & * & 0 & 1 & 0 \\ * & * & * & t & -\frac{t^2}{4} + \frac{1}{96}\{Q, (Q \times Q)Q\} & 1 \end{pmatrix}.$$

On the other hand, we have that

$$\begin{aligned} \delta 1^- &= \exp\left(\Theta\left(\left(\frac{t}{2}\right)_-\right)\right) \exp(\Theta(Q_-))1^- \\ &= \begin{pmatrix} \frac{1}{2}Q \times Q \\ -Q \\ -\frac{t}{2}Q - \frac{1}{6}(Q \times Q)Q \\ -\frac{t}{2} \\ 1 \\ -\frac{t^2}{4} + \frac{1}{96}\{Q, (Q \times Q)Q\} \end{pmatrix} = \alpha 1^-, \end{aligned}$$

and also that

$$\delta \tilde{1} = \alpha \tilde{1}, \quad \delta 1_- = \alpha 1_-.$$

Hence we see that $\delta^{-1}\alpha \in ((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{\tilde{1}, 1^-, 1_-} = (E_7^C)^{\sigma_4', \mathfrak{so}(6,C)}$.

Thus we have that

$$((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-} = \exp(\Theta((((\mathfrak{P}^C)_{\sigma_4'})_- \oplus C_-))(E_7^C)^{\sigma_4', \mathfrak{so}(6,C)}.$$

Furthermore, for $\beta \in (E_7^C)^{\sigma_4', \mathfrak{so}(6,C)}$, it is easy to verify that

$$\beta(\exp(\Theta(Q_-)))\beta^{-1} = \exp(\Theta(\beta Q_-)), \quad \beta(\exp(\Theta(t_-)))\beta^{-1} = \exp(\Theta(t_-)).$$

Indeed, for $(\Phi', P', Q', r', s', t') \in \mathfrak{e}_8^C$, by doing simple computation, we have that

$$\begin{aligned}
\beta\Theta(Q_-)\beta^{-1}(\Phi', P', Q', r', s', t') &= \beta[Q_-, \beta^{-1}(\Phi', P', Q', r', s', t')] \\
&= [\beta Q_-, \beta\beta^{-1}(\Phi', P', Q', r', s', t')](\beta \in E_7^C \subset E_8^C) \\
&= [\beta Q_-, (\Phi', P', Q', r', s', t')] \\
&= \Theta(\beta Q_-)(\Phi', P', Q', r', s', t'),
\end{aligned}$$

that is, $\beta\Theta(Q_-)\beta^{-1} = \Theta(\beta Q_-)$. Hence we obtain that

$$\begin{aligned}
\beta(\exp(\Theta(Q_-)))\beta^{-1} &= \beta\left(\sum_{n=0}^{\infty}\frac{1}{n!}\Theta(Q_-)^n\right)\beta^{-1} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}(\beta\Theta(Q_-)\beta^{-1})^n(\beta\Theta(Q_-)\beta^{-1} = \Theta(\beta Q_-)) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}(\Theta(\beta Q_-))^n \\
&= \exp(\Theta(\beta Q_-)).
\end{aligned}$$

By the argument similar to above, we have that $\beta(\exp(\Theta(t_-)))\beta^{-1} = \exp(\Theta(t_-))$.

This shows that $\exp(\Theta(((\mathfrak{P}^C)_{\sigma_4'})_- \oplus C_-)) = \exp(\Theta(((\mathfrak{P}^C)_{\sigma_4'})_-)) \exp(\Theta(C_-))$ is a normal subgroup of the group $((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-}$.

Moreover, we have a split exact sequence

$$1 \to \exp(\Theta(((\mathfrak{P}^C)_{\sigma_4'})_- \oplus C_-)) \to ((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-} \to (E_7^C)^{\sigma_4', \mathfrak{so}(6,C)} \to 1.$$

Hence the group $((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-}$ is a semi-direct product of $\exp(\Theta(((\mathfrak{P}^C)_{\sigma_4'})_- \oplus C_-))$ and $(E_7^C)^{\sigma_4', \mathfrak{so}(6,C)}$:

$$((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-} = \exp(\Theta(((\mathfrak{P}^C)_{\sigma_4'})_- \oplus C_-)) \rtimes (E_7^C)^{\sigma_4', \mathfrak{so}(6,C)}.$$

Therefore since $\exp(\Theta(((\mathfrak{P}^C)_{\sigma_4'})_- \oplus C_-))$ is connected and $(E_7^C)^{\sigma_4', \mathfrak{so}(6,C)}$ is connected (Theorem 3.39), we have that the group $((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_{1_-}$ is connected. □

For $R \in \mathfrak{e}_8^C$, we define a C-linear mapping $R \times R : \mathfrak{e}_8^C \to \mathfrak{e}_8^C$ by

$$(R \times R)R_1 = [R, [R, R_1]] + \frac{1}{30} B_8(R, R_1)R, \quad R_1 \in \mathfrak{e}_8^C,$$

where B_8 is the Killing form of the Lie algebra $\mathfrak{e}_8^C$ (As for the Killing form B_8, see [9, Theorem 5.3.2]), and using this mapping we define a space $\mathfrak{W}^C$ by

$$\mathfrak{W}^C = \{R \in \mathfrak{e}_8^C \mid R \times R = 0, R \neq 0\},$$

moreover, define a subspace $(\mathfrak{W}^C)_{\sigma_4', \mathfrak{so}(6,C)}$ of $\mathfrak{W}^C$ by

$$(\mathfrak{W}^C)_{\sigma_4', \mathfrak{so}(6,C)} = \{R \in \mathfrak{W}^C \mid \sigma_4' R = R, [R_D, R] = 0 \text{ for all } D \in \mathfrak{so}(6, C)\}.$$

Lemma 3.42. *For $R = (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C$ satisfying $\sigma_4' R = R$ and $[R_D, R] = 0$ for all $D \in \mathfrak{so}(6, C)$, $R \neq 0$, R belongs to $(\mathfrak{W}^C)_{\sigma_4', \mathfrak{so}(6,C)}$ if and only if R satisfies the following conditions*:

(1) $2s\Phi - P \times P = 0$ (2) $2t\Phi + Q \times Q = 0$
(3) $2r\Phi + P \times Q = 0$ (4) $\Phi P - 3rP - 3sQ = 0$
(5) $\Phi Q + 3rQ - 3tP = 0$ (6) $\{P, Q\} - 16(st + r^2) = 0$
(7) $2(\Phi P \times Q_1 + 2P \times \Phi Q_1 - rP \times Q_1 - sQ \times Q_1) - \{P, Q_1\}\Phi = 0$
(8) $2(\Phi Q \times P_1 + 2Q \times \Phi P_1 + rQ \times P_1 - tP \times P_1) - \{Q, P_1\}\Phi = 0$
(9) $8((P \times Q_1)Q - stQ_1 - r^2 Q_1 - \Phi^2 Q_1 + 2r\Phi Q_1) + 5\{P, Q_1\}Q - 2\{Q, Q_1\}P = 0$
(10) $8((Q \times P_1)P + stP_1 + r^2 P_1 + \Phi^2 P_1 + 2r\Phi P_1) + 5\{Q, P_1\}P - 2\{P, Q_1\}Q = 0$
(11) $18((\operatorname{ad}\Phi)^2\Phi_1 + Q \times \Phi_1 P - P \times \Phi_1 Q) + B_7(\Phi, \Phi_1)\Phi = 0$
(12) $18(\Phi_1 \Phi P - 2\Phi\Phi_1 P - r\Phi_1 P - s\Phi_1 Q) + B_7(\Phi, \Phi_1)P = 0$
(13) $18(\Phi_1 \Phi Q - 2\Phi\Phi_1 Q + r\Phi_1 Q - t\Phi_1 P) + B_7(\Phi, \Phi_1)Q = 0$,

(*where B_7 is the Killing form of the Lie algebra $\mathfrak{e}_7^C$*) *for all $\Phi_1 \in \mathfrak{e}_7^C$, $P_1, Q_1 \in \mathfrak{P}^C$.*

Proof. For $R = (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C$ satisfying $\sigma_4' R = R$ and $[R_D, R] = 0$ for all $D \in \mathfrak{so}(6, C)$, $R \neq 0$, by doing simple computation of $(R \times R)R_1 = 0$ for all $R_1 = (\Phi_1, P_1, Q_1, r_1, s_1, t_1) \in \mathfrak{e}_8^C$, we have the required relational formulas above. □

Proposition 3.43. *The group $((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_0$ acts on $(\mathfrak{W}^C)_{\sigma_4', \mathfrak{so}(6,C)}$ transitively.*

PROOF. Since $\alpha \in (E_8^C)^{\sigma_4', \mathfrak{so}(6,C)}$ leaves invariant the Killing form B_8 of $\mathfrak{e}_8^C : B_8(\alpha R, \alpha R') = B_8(R, R')$, $R, R' \in \mathfrak{e}_8^C$, we have $\alpha R \in (\mathfrak{W}^C)_{\sigma_4', \mathfrak{so}(6,C)}$ for $R \in (\mathfrak{W}^C)_{\sigma_4', \mathfrak{so}(6,C)}$. Indeed, since we see that

$$\begin{aligned}
(\alpha R \times \alpha R)R_1 &= [\alpha R, [\alpha R, \alpha R_1]] + \frac{1}{30} B_8(\alpha R, R_1)\alpha R \\
&= \alpha[[R, [R, \alpha^{-1} R_1]] + \frac{1}{30} B_8(R, \alpha^{-1} R_1)\alpha R \\
&= \alpha((R \times R)\alpha^{-1} R_1 \\
&= 0, \\
[R_D, \alpha R] &= \alpha[\alpha^{-1} R_D, R] = \alpha[R_D, R] \\
&= 0,
\end{aligned}$$

this shows that the group $(E_8^C)^{\sigma_4', \mathfrak{so}(6,C)}$ acts on $(\mathfrak{W}^C)_{\sigma_4', \mathfrak{so}(6,C)}$. We shall show that this action is transitive. First, for $R_1 \in \mathfrak{e}_8^C$, it follows from

$$\begin{aligned}
(1_- \times 1_-)R_1 &= [1_-, [1_-, (\Phi_1, P_1, Q_1, r_1, s_1, t_1)]] + \frac{1}{30} B_8(1_-, R_1)1_- \\
&= [1_-, (0, 0, P_1, -s_1, 0, 2r_1)] + 2s_1 1_- \\
&= (0, 0, 0, 0, -2s_1) + 2s_1 1_- \\
&= 0, \\
[R_D, 1_-] &= 0,
\end{aligned}$$

and $\sigma_4' 1_- = 1_-$ that we confirm $1_- \in (\mathfrak{W}^C)_{\sigma_4', \mathfrak{so}(6,C)}$. Then, in order to prove the transitivity of this action, it is sufficient to show that any element $R \in (\mathfrak{W}^C)_{\sigma_4', \mathfrak{so}(6,C)}$ can be transformed to $1_- \in (\mathfrak{W}^C)_{\sigma_4', \mathfrak{so}(6,C)}$ by some $\alpha \in (E_8^C)^{\sigma_4', \mathfrak{so}(6,C)}$. Indeed, we have the following.

Case (i) where $R = (\Phi, P, Q, r, s, t), t \neq 0$.

From Lemma 3.42 (2), (5) and (6), we have that

$$\Phi = -\frac{1}{2t} Q \times Q, \quad P = \frac{r}{t} Q - \frac{1}{6t^2}(Q \times Q)Q, \quad s = -\frac{r^2}{t} + \frac{1}{96t^3}\{Q, (Q \times Q)Q\}.$$

Now, for $\Theta = \Theta(0, P_1, 0, r_1, s_1, 0) \in \Theta((\mathfrak{e}_8^C)^{\sigma_4', \mathfrak{so}(6,C)})$ (Lemma 3.40 (2)), we compute $\Theta^n 1_-$:

$\Theta^n 1_-$

$$= \begin{pmatrix} ((-2)^{n-1}+(-1)^n)r_1^{n-2}P_1\times P_1 \\ \left((-2)^{n-1}-\dfrac{1+(-1)^{n-1}}{2}\right)r_1^{n-2}s_1P_1+\left(\dfrac{1-(-2)^n}{6}+\dfrac{(-1)^n}{2}\right)r_1^{n-3}(P_1\times P_1)P_1 \\ ((-2)^n+(-1)^{n-1})r_1^{n-1}P_1 \\ (-2)^{n-1}r_1^{n-1}s_1 \\ -((-2)^{n-2}+2^{n-2})r_1^{n-2}s_1^2+\dfrac{2^{n-2}+(-2)^{n-2}-(-1)^{n-1}}{24}r_1^{n-4}\{P_1,(P_1\times P_1)P_1\} \\ (-2)^n r_1^n \end{pmatrix}.$$

Then, by doing simple computation, we have that

$\exp(\Phi(0,P_1,0,r_1,s_1,0))1_-$

$= (\exp\,\Theta)1_- = \left(\sum_{n=0}^{\infty}\frac{1}{n!}\Theta^n\right)1_-$

$$= \begin{pmatrix} -\dfrac{1}{2r_1^2}(e^{-2r_1}-2e^{-r_1}+1)P_1\times P_1 \\ \dfrac{s_1}{2r_1^2}(-e^{-2r_1}-e^{r_1}+e^{-r_1}+1)P_1+\dfrac{1}{6r_1^3}(-e^{-2r_1}+e^{r_1}+3e^{-r_1}-3)(P_1\times P_1)P_1 \\ \dfrac{1}{r_1}(e^{-2r_1}-e^{-r_1})P_1 \\ \dfrac{s_1}{2r_1}(1-e^{-2r_1}) \\ -\dfrac{s_1^2}{4r_1^2}(e^{-2r_1}+e^{2r_1}-2)+\dfrac{1}{96r_1^4}(e^{2r_1}+e^{-2r_1}-4e^{r_1}-4e^{-r_1}+6)\{P_1,(P_1\times P_1)P_1\} \\ e^{-2r_1} \end{pmatrix}.$$

Note that if $r_1 = 0$,

$$\frac{f(r_1)}{r_1^k} \text{ means } \lim_{r_1\to 0}\frac{f(r_1)}{r_1^k}.$$

Here we set

$$Q = \frac{1}{r_1}(e^{-2r_1}-e^{-r_1})P_1, \quad r = \frac{s_1}{2r_1}(1-e^{-2r_1}), \quad t = e^{-2r_1}.$$

Then we have that

$$(\exp \Theta)1_- = \begin{pmatrix} -\dfrac{1}{2t} Q \times Q \\ \dfrac{r}{t} Q - \dfrac{1}{6t^2}(Q \times Q)Q \\ Q \\ r \\ -\dfrac{r^2}{t} + \dfrac{1}{96t^3}\{Q, (Q \times Q)Q\} \\ t \end{pmatrix} = \begin{pmatrix} \Phi \\ P \\ Q \\ r \\ s \\ t \end{pmatrix} =: R.$$

Thus R is transformed to 1_- by $(\exp \Theta)^{-1} \in ((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_0$.

Case (ii) where $R = (\Phi, P, Q, r, s, 0)$, $s \neq 0$.

First, we denote $\exp(\Theta(0,0,0,0,\pi/2,-\pi/2)) \in ((E_8^C)^{\sigma_4', \mathfrak{so}(6,C)})_0$ by λ' (Lemma 3.40 (2)). Here, operate λ' on R, then we have that

$$\lambda' R = \lambda'(\Phi, P, Q, r, s, 0) = (\Phi, Q, -P, -r, 0, -s), \quad -s \neq 0.$$

Hence this case is reduced to Case (i).

Case (iii) where $R = (\Phi, P, Q, r, 0, 0)$, $r \neq 0$.

From Lemma 3.42 (2), (5) and (6), we have that

$$Q \times Q = 0, \quad \Phi Q = -3rQ, \quad \{P, Q\} = 16r^2.$$

Then, for $\Theta = \Theta(0, Q, 0, 0, 0, 0) \in \Theta((\mathfrak{e}_8^C)^{\sigma_4', \mathfrak{so}(6,C)})$ (Lemma 3.40 (2)), we see that

$$(\exp \Theta)R = (\Phi, P + 2rQ, Q, r, -4r^2, 0), \quad -4r^2 \neq 0.$$

Hence this case is reduced to Case (ii).

Case (iv) where $R = (\Phi, P, Q, 0, 0, 0)$, $Q \neq 0$.

We choose $P_1 \in (\mathfrak{P}^C)_{\sigma_4'}$ such that $\{P_1, Q\} \neq 0$. Then, for $\Theta = \Theta(0, P_1, 0, 0, 0, 0) \in \Theta((\mathfrak{e}_8^C)^{\sigma_4', \mathfrak{so}(6,C)})$ (Lemma 3.40 (2)), we have that

$$(\exp \Theta)R = \Big(\Phi + P_1 \times Q, P - \Phi P_1 + \frac{1}{2}(P_1 \times Q)P_1, Q, -\frac{1}{8}\{P_1, Q\},$$

$$\frac{1}{4}\{P_1, P\} + \frac{1}{8}\{P_1, -\Phi P_1\} + \frac{1}{24}\{P_1, (P_1 \times Q)P_1\}, 0\Big),$$

$$-\frac{1}{8}\{P_1, Q\} \neq 0.$$

Hence this case is reduced to Case (iii).

Case (v) where $R = (\Phi, P, 0, 0, 0, 0)$, $P \neq 0$.

We choose $Q_1 \in (\mathfrak{P}^C)_{\sigma'_4}$ such that $\{P, Q_1\} \neq 0$. Then, for $\Theta = \Theta(0, 0, Q_1, 0, 0, 0) \in \Theta((\mathfrak{e}_8^C)^{\sigma'_4, \mathfrak{so}(6, C)})$ (Lemma 3.40 (2)), we have

$$(\exp \Theta)R = \left(\Phi - P \times Q_1, P, -\Phi Q_1 - \frac{1}{2}(P \times Q_1)Q_1, \frac{1}{8}\{P, Q_1\},\right.$$
$$\left. 0, -\frac{1}{8}\{Q_1, -\Phi Q_1\} - \frac{1}{24}\{Q_1, -(P \times Q_1)Q_1\}\right), \quad \frac{1}{8}\{P, Q_1\} \neq 0.$$

Hence this case is reduced to Case (iii).

Case (vi) where $R = (\Phi, 0, 0, 0, 0, 0)$, $\Phi \neq 0$. From Lemma 3.42 (10), we have $\Phi^2 = 0$. We choose $P_1 \in (\mathfrak{P}^C)_{\sigma'_4}$ such that $\Phi P_1 \neq 0$. Then, for $\Theta = \Theta(0, P_1, 0, 0, 0, 0) \in \Theta((\mathfrak{e}_8^C)^{\sigma'_4, \mathfrak{so}(6, C)})$ (Lemma 3.40 (2)), we have that

$$(\exp \Theta)R = \left(\Phi, -\Phi P_1, 0, 0, \frac{1}{8}\{\Phi P_1, P_1\}, 0\right).$$

Hence this case is also reduced to Case (v).

Thus the proof of this proposition is completed. □

Now, we shall prove the theorem as the aim of this section.

Theorem 3.44. *The homogeneous space* $(E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)}/((E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)})_{1_-}$ *is diffeomorphic to the space* $(\mathfrak{W}^C)_{\sigma'_4, \mathfrak{so}(6, C)}$: $(E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)}/((E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)})_{1_-} \simeq (\mathfrak{W}^C)_{\sigma'_4, \mathfrak{so}(6, C)}$.

In particular, the group $(E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)}$ *is connected.*

Proof. Since the group $(E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)}$ acts on the space $(\mathfrak{W}^C)_{\sigma'_4, \mathfrak{so}(6, C)}$ transitively (Proposition 3.43), the former half of this theorem is proved.

The latter half can be shown as follows. Since $((E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)})_{1_-}$ and $(\mathfrak{W}^C)_{\sigma'_4}$ are connected (Propositions 3.41, 3.43), we have that $(E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)}$ is also connected. □

3.4. Construction of $Spin(10, C)$ in E_8^C

We define a subgroup $(E_8)^{\sigma'_4, \mathfrak{so}(6)}$ of E_8 by

$$(E_8)^{\sigma'_4, \mathfrak{so}(6)} = \left\{ \alpha \in E_8 \,\middle|\, \begin{array}{l} \sigma'_4\alpha = \alpha\sigma'_4, \\ \Theta(R_D)\alpha = \alpha\Theta(R_D) \text{ for all } D \in \mathfrak{so}(6) \end{array} \right\}.$$

Then we have the following lemma.

LEMMA 3.45. *The Lie algebra* $(\mathfrak{e}_8)^{\sigma_4', \mathfrak{so}(6)}$ *of the group* $(E_8)^{\sigma_4', \mathfrak{so}(6)}$ *is given by*

$$(\mathfrak{e}_8)^{\sigma_4', \mathfrak{so}(6)} = \left\{ R \in \mathfrak{e}_8 \;\middle|\; \begin{array}{l} \sigma_4' R = R, \\ [R, R_D] = 0 \ \ for \ \ all \ \ D \in \mathfrak{so}(6) \end{array} \right\}$$

$$= \left\{ (\Phi, P, -\tau\lambda P, r, s, -\tau s) \in \mathfrak{e}_8 \;\middle|\; \begin{array}{l} \Phi \in (\mathfrak{e}_7)^{\sigma_4', \mathfrak{so}(6)}, \\ P = (X, Y, \xi, \eta), \\ \quad X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & x \\ 0 & \bar{x} & \xi_3 \end{pmatrix}, \ Y = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & y \\ 0 & \bar{y} & \eta_3 \end{pmatrix}, \\ \quad \xi_k, \eta_k, \xi, \eta \in C, \ x, y \in \boldsymbol{C}^C, \\ r \in i\boldsymbol{R}, \ s \in C, \end{array} \right\},$$

where as mentioned in Lemma 3.28, *as for the Lie algebra* $\mathfrak{e}_7$ *of the compact Lie group* E_7, *see* [9, Theorem 4.3.4] *in detail, and so the Lie algebra* $(\mathfrak{e}_7)^{\sigma_4', \mathfrak{so}(6)}$ *above is defined as follows*:

$$(\mathfrak{e}_7)^{\sigma_4', \mathfrak{so}(6)} = \left\{ \Phi(\phi, A, -\tau A, v) \;\middle|\; \begin{array}{l} \sigma_4' \Phi = \Phi \sigma_4', \\ [\Phi, \Phi_D] = 0 \ \ for \ \ all \ \ D \in \mathfrak{so}(6) \end{array} \right\}$$

$$= \left\{ \Phi(\phi, A, -\tau A, v) \in \mathfrak{e}_7 \;\middle|\; \begin{array}{l} \phi = \left(\begin{array}{c|c} D_2 & 0 \\ \hline 0 & 0 \end{array} \right) \begin{array}{l} + \tilde{A}_1(a) \\ + i(\tau_1 E_1 + \tau_2 E_2 + \tau_3 E_3 + F_1(t_1))^{\sim} \end{array} \\ \quad D_2 \in \mathfrak{so}(2), \ a \in \boldsymbol{C}, \ \tau_k \in \boldsymbol{R}, \\ \quad \tau_1 + \tau_2 + \tau_3 = 0, \ t_1 \in \boldsymbol{C}, \\ A = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \ \xi_k \in C, \ x_1 \in \boldsymbol{C}^C, \\ v \in i\boldsymbol{R} \end{array} \right\}.$$

In particular,

$$\dim_C((\mathfrak{e}_8^C)^{\sigma_4', \mathfrak{so}(6, C)}) = 18 + ((3 + 2) \times 2 + 1 \times 2) \times 2 + 3 = 45.$$

PROOF. By the argument similar to Lemma 3.40 (2), this lemma is proved. □

PROPOSITION 3.46. *The Lie algebras* $(\mathfrak{e}_8)^{\sigma_4', \mathfrak{so}(6)}$ *and* $(\mathfrak{e}_8^C)^{\sigma_4', \mathfrak{so}(6, C)}$ *are isomorphic to the Lie algebras* $\mathfrak{so}(10)$ *and* $\mathfrak{so}(10, C)$, *respectively*: $(\mathfrak{e}_8)^{\sigma_4', \mathfrak{so}(6)} \cong \mathfrak{so}(10)$ *and* $(\mathfrak{e}_8^C)^{\sigma_4', \mathfrak{so}(6, C)} \cong \mathfrak{so}(10, C)$.

PROOF. We provide the correspondence φ_* between the Lie algebra $\mathfrak{so}(10) = \{D \in M(10, \boldsymbol{R}) \mid {}^t D = -D\}$ and the Lie algebra $(\mathfrak{e}_8)^{\sigma_4', \mathfrak{so}(6)}$ explicitly as follows:

$$\varphi_* : \mathfrak{so}(10) \to (\mathfrak{e}_8)^{\sigma_4', \mathfrak{so}(6)},$$
$$G_{ij} \mapsto R_{ij}, \quad 0 \le i < j \le 9,$$

where the elements G_{ij} are $\boldsymbol{R}$-basis in $\mathfrak{so}(10)$. As its example, the explicit form of G_{26} as matrix is of form with $(3, 7)$-component $= 1$, $(7, 3)$-component $= -1$, other components $= 0$, moreover the explicit forms of C-basis R_{ij} in $(\mathfrak{e}_8)^{\sigma_4', \mathfrak{so}(6)}$ are as follows.

$$R_{01} = (\Phi(-i(E_2 - E_3)^{\sim}, 0, 0, 0), 0, 0, 0, 0, 0)$$

$$R_{02} = \left(\Phi\left(0, -\frac{i}{2}(E_2 - E_3), -\frac{i}{2}(E_2 - E_3), 0\right), 0, 0, 0, 0, 0\right)$$

$$R_{12} = \left(\Phi\left(0, \frac{1}{2}(E_2 + E_3), -\frac{1}{2}(E_2 + E_3), 0\right), 0, 0, 0, 0, 0\right)$$

$$R_{03} = \left(\Phi\left(0, -\frac{1}{2}(E_2 - E_3), \frac{1}{2}(E_2 - E_3), 0\right), 0, 0, 0, 0, 0\right)$$

$$R_{13} = \left(\Phi\left(0, -\frac{i}{2}(E_2 + E_3), -\frac{i}{2}(E_2 + E_3), 0\right), 0, 0, 0, 0, 0\right)$$

$$R_{23} = (\Phi(-i(E_1 \vee E_1), 0, 0, i), 0, 0, 0, 0, 0)$$

$$R_{04} = (0, (-(E_2 - E_3), 0, 0, 0), (0, -(E_2 - E_3), 0, 0), 0, 0, 0)$$

$$R_{14} = (0, (-i(E_2 + E_3), 0, 0, 0), (0, i(E_2 + E_3), 0, 0), 0, 0, 0)$$

$$R_{24} = (0, (0, iE_1, 0, -i), (iE_1, 0, -i, 0), 0, 0, 0)$$

$$R_{34} = (0, (0, E_1, 0, 1), (-E_1, 0, -1, 0), 0, 0, 0)$$

$$R_{05} = (0, (-i(E_2 - E_3), 0, 0, 0), (0, i(E_2 - E_3), 0, 0), 0, 0, 0)$$

$$R_{15} = (0, (E_2 + E_3, 0, 0, 0), (0, E_2 + E_3, 0, 0), 0, 0, 0)$$

$$R_{25} = (0, (0, -E_1, 0, 1), (E_1, 0, -1, 0), 0, 0, 0)$$

$$R_{35} = (0, (0, iE_1, 0, i), (iE_1, 0, i, 0), 0, 0, 0)$$

$$R_{45} = \left(\Phi\left(i(E_1 \vee E_1), 0, 0, \frac{i}{2}\right), 0, 0, -\frac{i}{2}, 0, 0\right)$$

$$R_{06} = (0, (0, -(E_2 - E_3), 0, 0), (E_2 - E_3, 0, 0, 0), 0, 0, 0)$$

$$R_{16} = (0, (0, i(E_2 + E_3), 0, 0), (i(E_2 + E_3), 0, 0, 0), 0, 0, 0)$$

$$R_{26} = (0, (iE_1, 0, -i, 0), (0, -iE_1, 0, i), 0, 0, 0)$$

$$R_{36} = (0, (-E_1, 0, -1, 0), (0, -E_1, 0, -1), 0, 0, 0)$$

$$R_{46} = \left(\Phi\left(0, \frac{1}{2}E_1, -\frac{1}{2}E_1, 0\right), 0, 0, 0, -\frac{1}{2}, \frac{1}{2}\right)$$

$$R_{56} = \left(\Phi\left(0, -\frac{i}{2}E_1, -\frac{i}{2}E_1, 0\right), 0, 0, 0, -\frac{i}{2}, -\frac{i}{2}\right)$$

$$R_{07} = (0, (0, -i(E_2 - E_3), 0, 0), (-i(E_2 - E_3), 0, 0, 0), 0, 0, 0)$$

$$R_{17} = (0, (0, -(E_2 + E_3), 0, 0), (E_2 + E_3, 0, 0, 0), 0, 0, 0)$$

$$R_{27} = (0, (-E_1, 0, 1, 0), (0, -E_1, 0, 1), 0, 0, 0)$$

$$R_{37} = (0, (-iE_1, 0, -i, 0), (0, iE_1, 0, i), 0, 0, 0)$$

$$R_{47} = \left(\Phi\left(0, \frac{i}{2}E_1, \frac{i}{2}E_1, 0\right), 0, 0, 0, -\frac{i}{2}, -\frac{i}{2}\right)$$

$$R_{57} = \left(\Phi\left(0, \frac{1}{2}E_1, -\frac{1}{2}E_1, 0\right), 0, 0, 0, \frac{1}{2}, -\frac{1}{2}\right)$$

$$R_{67} = \left(\Phi\left(-i(E_1 \vee E_1), 0, 0, -\frac{i}{2}\right), 0, 0, -\frac{i}{2}, 0, 0\right)$$

$$R_{08} = (\Phi(\tilde{A}_1(i), 0, 0, 0), 0, 0, 0, 0, 0)$$

$$R_{18} = (\Phi(-F_1(1)^{\sim}, 0, 0, 0), 0, 0, 0, 0, 0)$$

$$R_{28} = \left(\Phi\left(0, -\frac{1}{2}F_1(1), -\frac{1}{2}F_1(1), 0\right), 0, 0, 0, 0, 0\right)$$

$$R_{38} = \left(\Phi\left(0, \frac{i}{2}F_1(1), -\frac{i}{2}F_1(1), 0\right), 0, 0, 0, 0, 0\right)$$

$$R_{48} = (0, (iF_1(1), 0, 0, 0), (0, iF_1(1), 0, 0), 0, 0, 0)$$

$$R_{58} = (0, (-F_1(1), 0, 0, 0), (0, F_1(1), 0, 0), 0, 0, 0)$$

$$R_{68} = (0, (0, iF_1(1), 0, 0), (-iF_1(1), 0, 0, 0), 0, 0, 0)$$

$$R_{78} = (0, (0, -F_1(1), 0, 0), (-F_1(1), 0, 0, 0), 0, 0, 0)$$

$$R_{09} = (\Phi(i\tilde{A}_1(e_1), 0, 0, 0), 0, 0, 0, 0, 0)$$

$$R_{19} = (\Phi(-F_1(e_1)^{\sim}, 0, 0, 0), 0, 0, 0, 0, 0)$$

$$R_{29} = \left(\Phi\left(0, -\frac{1}{2}F_1(e_1), -\frac{1}{2}F_1(e_1), 0\right), 0, 0, 0, 0, 0\right)$$

$$R_{39} = \left(\Phi\left(0, \frac{i}{2}F_1(e_1), -\frac{i}{2}F_1(e_1), 0\right), 0, 0, 0, 0, 0\right)$$

$$R_{49} = (0, (iF_1(e_1), 0, 0, 0), (0, iF_1(e_1), 0, 0), 0, 0, 0)$$

$$R_{59} = (0, (-F_1(e_1), 0, 0, 0), (0, F_1(e_1), 0, 0), 0, 0, 0)$$

$$R_{69} = (0, (0, -iF_1(e_1), 0, 0), (iF_1(e_1), 0, 0, 0), 0, 0, 0)$$

$$R_{79} = (0, (0, -F_1(e_1), 0, 0), (-F_1(e_1), 0, 0, 0), 0, 0, 0)$$

$$R_{89} = (\Phi(-[\tilde{A}_1(1), \tilde{A}_1(e_1)], 0, 0, 0), 0, 0, 0, 0, 0)(-[\tilde{A}_1(1), \tilde{A}_1(e_1)] \in \mathfrak{D}_4)$$

Then we can confirm that φ_* is a Lie-homomorphism, that is:

$$\varphi_*([G_{ij}, G_{kl}]) = [\varphi_*(G_{ij}), \varphi_*(G_{kl})].$$

In order to prove these, we need to check ${}_{45}C_2 = 990$ times if honestly doing. Then, we give only five examples. As the first example, we shall show that $\varphi_*([G_{01}, G_{02}]) = [\varphi_*(G_{01}), \varphi_*(G_{02})]$. Indeed, $\varphi_*([G_{01}, G_{02}]) = \varphi_*(-G_{12}) = -R_{12}$.

On the other hand,

$$\begin{aligned}
[\varphi_*(G_{01}), \varphi_*(G_{02})] &= [R_{01}, R_{02}] \\
&= [(\Phi(-i(E_2 - E_3)^{\sim}, 0, 0, 0), 0, 0, 0, 0, 0), \\
&\qquad \left(\Phi\left(0, -\frac{i}{2}(E_2 - E_3), -\frac{i}{2}(E_2 - E_3), 0\right), 0, 0, 0, 0, 0\right)] \\
&= \Bigg([\Phi(-i(E_2 - E_3)^{\sim}, 0, 0, 0), \\
&\qquad \left(\Phi\left(0, -\frac{i}{2}(E_2 - E_3), -\frac{i}{2}(E_2 - E_3), 0\right)\right)], 0, 0, 0, 0, 0\Bigg) \\
&= \left(\Phi\left(0, -\frac{1}{2}(E_2 + E_3), \frac{1}{2}(E_2 + E_3), 0\right), 0, 0, 0, 0, 0\right) \\
&= -R_{12}.
\end{aligned}$$

As the second example, we shall show that $\varphi_*([G_{04}, G_{45}]) = [\varphi_*(G_{04}), \varphi_*(G_{45})]$. Indeed, $\varphi_*([G_{04}, G_{45}]) = \varphi_*(G_{05}) = R_{05}$.

On the other hand,

$$
\begin{aligned}
[\varphi_*(G_{04}), \varphi_*(G_{45})] &= [R_{04}, R_{45}] \\
&= [(0, (-(E_2 - E_3), 0, 0, 0), (0, -(E_2 - E_3), 0, 0), 0, 0, 0), \\
&\quad \left(\Phi\left(i(E_1 \vee E_1), 0, 0, \frac{i}{2}\right), 0, 0, -\frac{i}{2}, 0, 0\right)] \\
&= \left(0, -\left(\Phi\left(i(E_1 \vee E_1), 0, 0, \frac{i}{2}\right)(-(E_2 - E_3), 0, 0, 0)\right.\right. \\
&\quad + \frac{i}{2}(-(E_2 - E_3), 0, 0, 0), -\left(\Phi\left(i(E_1 \vee E_1), 0, 0, \frac{i}{2}\right)\right. \\
&\quad \left.\times (0, -(E_2 - E_3), 0, 0) - \frac{i}{2}(0, -(E_2 - E_3), 0, 0), 0, 0, 0\right) \\
&= (0, (-i(E_2 - E_3), 0, 0, 0), (0, i(E_2 - E_3), 0, 0), 0, 0, 0) \\
&= R_{05}.
\end{aligned}
$$

As the third example, we shall show that $\varphi_*([G_{57}, G_{67}]) = [\varphi_*(G_{57}), \varphi_*(G_{67})]$. Indeed, $\varphi_*([G_{57}, G_{67}]) = \varphi_*(-G_{56}) = -R_{56}$.

On the other hand,

$$
\begin{aligned}
[\varphi_*(G_{57}), \varphi_*(G_{67})] &= [R_{57}, R_{67}] \\
&= [\left(\Phi\left(0, \frac{1}{2}E_1, -\frac{1}{2}E_1, 0\right), 0, 0, 0, \frac{1}{2}, -\frac{1}{2}\right), \\
&\quad \left(\Phi\left(-i(E_1 \vee E_1), 0, 0, -\frac{i}{2}\right), 0, 0, -\frac{i}{2}, 0, 0\right)] \\
&= \left([\Phi\left(0, \frac{1}{2}E_1, -\frac{1}{2}E_1, 0\right), \Phi\left(-i(E_1 \vee E_1), 0, 0, -\frac{i}{2}\right)],\right. \\
&\quad \left. 0, 0, 0, -2\left(-\frac{i}{2}\right)\left(\frac{1}{2}\right), 2\left(-\frac{i}{2}\right)\left(-\frac{1}{2}\right)\right) \\
&= \left(\Phi\left(0, \frac{i}{2}E_1, \frac{i}{2}E_1, 0\right), 0, 0, 0, \frac{i}{2}, \frac{i}{2}\right) \\
&= -R_{56}.
\end{aligned}
$$

As the forth example, we shall show that $\varphi_*([G_{36}, G_{68}]) = [\varphi_*(G_{36}), \varphi_*(G_{68})]$. Indeed, $\varphi_*([G_{36}, G_{68}]) = \varphi_*(G_{38}) = R_{38}$.

On the other hand,

$$\begin{aligned}
[\varphi_*(G_{36}), \varphi_*(G_{68})] &= [R_{36}, R_{68}] \\
&= [(0, (-E_1, 0, -1, 0), (0, -E_1, 0, -1), 0, 0, 0), \\
&\quad (0, iF_1(1), 0, 0), (-iF_1(1), 0, 0, 0), 0, 0, 0)] \\
&= ((-E_1, 0, -1, 0) \times (-iF_1(1), 0, 0, 0)) \\
&\quad - (0, iF_1(1), 0, 0) \times (0, -E_1, 0, -1), 0, 0, 0, 0, 0) \\
&= \left(\Phi\left(0, \frac{i}{4}F_1(1), -\frac{i}{4}F_1(1), 0\right) \right. \\
&\quad \left. - \Phi\left(0, -\frac{i}{4}F_1(1), -\frac{i}{4}F_1(1), 0\right), 0, 0, 0, 0, 0 \right) \\
&= \left(\Phi\left(0, \frac{i}{2}F_1(1), -\frac{i}{2}F_1(1), 0\right), 0, 0, 0, 0, 0 \right) \\
&= R_{38}.
\end{aligned}$$

Finally, as the fifth example, we shall show that $\varphi_*([G_{28}, G_{89}]) = [\varphi_*(G_{28}), \varphi_*(G_{89})]$. Indeed, $\varphi_*([G_{28}, G_{89}]) = \varphi_*(G_{29}) = R_{29}$.

On the other hand,

$$\begin{aligned}
[\varphi_*(G_{28}), \varphi_*(G_{89})] &= [R_{28}, R_{89}] \\
&= \left[\left(\Phi\left(0, -\frac{i}{2}F_1(1), -\frac{i}{2}F_1(1), 0\right), 0, 0, 0, 0, 0 \right), \right. \\
&\quad (\Phi(-[\tilde{A}_1(1), \tilde{A}_1(e_1)], 0, 0, 0), 0, 0, 0, 0, 0)] \\
&= \left(\left[\Phi\left(0, -\frac{i}{2}F_1(1), -\frac{i}{2}F_1(1), 0\right), \right. \right. \\
&\quad \left. \Phi(-[\tilde{A}_1(1), \tilde{A}_1(e_1)], 0, 0, 0)], 0, 0, 0, 0, 0 \right) \\
&= \left(\Phi\left(0, [\tilde{A}_1(1), \tilde{A}_1(e_1)]\left(-\frac{1}{2}F_1(1)\right), \right. \right. \\
&\quad \left. \left. [\tilde{A}_1(1), \tilde{A}_1(e_1)]\left(-\frac{1}{2}F_1(1)\right), 0\right), 0, 0, 0, 0, 0 \right) \\
&= \left(\Phi\left(0, -\frac{1}{2}F_1(e_1), -\frac{1}{2}F_1(e_1), 0\right), 0, 0, 0, 0, 0 \right) \\
&= R_{29}.
\end{aligned}$$

Since we have $\dim(\mathfrak{so}(10)) = 45 = \dim((\mathfrak{e}_8)^{\sigma'_4, \mathfrak{so}(6)})$ from Lemma 3.45, we see that φ_* is an isomorphism. Thus we have the required isomorphism $(\mathfrak{e}_8)^{\sigma'_4, \mathfrak{so}(6)} \cong \mathfrak{so}(10)$ as a Lie algebra. The other case is the complexification of the case above (cf. [3]). □

Now, we shall prove the theorem as the aim of this subsection.

THEOREM 3.47. *The group* $(E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)}$ *is isomorphic to* $Spin(10, C)$: $(E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)} \cong Spin(10, C)$.

PROOF. The group $(E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)}$ is connected (Theorem 3.44) and its type is $\mathfrak{so}(10, C)$ (Proposition 3.46). Hence the group $(E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)}$ is isomorphic to either one of the following groups:

$$Spin(10, C), \quad SO(10, C), \quad Spin(10, C)/\boldsymbol{Z}_4.$$

Their centers of groups above are $\boldsymbol{Z}_4$, $\boldsymbol{Z}_2$, $\{1\}$, respectively. However, we see that the center of $(E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)}$ has the elements 1, σ, σ'_4, $\sigma\sigma'_4$, and so its center is $\boldsymbol{Z}_4$. Hence the group $(E_8^C)^{\sigma'_4, \mathfrak{so}(6, C)}$ have to be isomorphic to $Spin(10, C)$. □

3.5. The Structure of the Group $(E_8^C)^{\sigma'_4}$

By using the results of previous subsection, the aim of this subsection is to determine the structure of the group $(E_8^C)^{\sigma'_4}$.

LEMMA 3.48. *The Lie algebra* $(\mathfrak{e}_8^C)^{\sigma'_4}$ *of the group* $(E_8^C)^{\sigma'_4}$ *is given by*

$$(\mathfrak{e}_8^C)^{\sigma'_4} = \{R \in \mathfrak{e}_8^C \mid \sigma'_4 R = R\}$$

$$= \left\{ (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C \;\middle|\; \begin{array}{l} \Phi \in (\mathfrak{e}_7^C)^{\sigma'_4}, \\ P = (X, Y, \xi, \eta), \\ \quad X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & x \\ 0 & \bar{x} & \xi_3 \end{pmatrix}, \; Y = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & y \\ 0 & \bar{y} & \eta_3 \end{pmatrix}, \\ \quad \xi_k, \eta_k, \xi, \eta \in C, \; x, y \in \boldsymbol{C}^C, \\ Q = (Z, W, \zeta, \omega) \text{ is same form as } P, \\ r, s, t \in C \end{array} \right\},$$

as for the explicit form of the Lie algebra $(\mathfrak{e}_7^C)^{\sigma'_4}$, *see Lemma* 3.18 (1).

In particular,

$$\dim_C((\mathfrak{e}_8^C)^{\sigma'_4}) = 33 + ((3+2)\times 2 + 1\times 2)\times 2 + 3 = 60.$$

Proof. By the argument similar to Lemma 3.40, we have the required result. □

Now, we shall prove the following theorem as the aim of this subsection.

Theorem 3.49. *We have that* $(E_8^C)^{\sigma'_4} \cong (Spin(6,C)\times Spin(10,C))/\boldsymbol{Z}_4$, $\boldsymbol{Z}_4 = \{(1,1),(\sigma'_4,\sigma\sigma'_4),(\sigma,\sigma),(\sigma\sigma'_4,\sigma'_4)\}$.

Proof. Let

$$Spin(6,C) \cong (F_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1} \cong ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1,\tilde{E}_{-1},E_2\dot{+}E_3,E_2\dot{-}E_3,\dot{F}_1(e_k),k=0,1}$$
$$\subset (E_7^C)^{\sigma'_4} \subset (E_8^C)^{\sigma'_4}$$

(Theorems 3.16, 3.38) and $Spin(10,C) \cong (E_8^C)^{\sigma'_4,\mathfrak{so}(6,C)} \subset (E_8^C)^{\sigma'_4}$ (Theorem 3.47). Then we define a mapping $\varphi_{E_8^C,\sigma'_4} : Spin(6,C)\times Spin(10,C) \to (E_8^C)^{\sigma'_4}$ by

$$\varphi_{E_8^C,\sigma'_4}(\alpha,\beta) = \alpha\beta.$$

It is clear that $\varphi_{E_8^C,\sigma'_4}$ is well-defined. Since $[R_D, R_{10}] = 0$ for $R_D \in \mathfrak{spin}(6,C) = \mathfrak{so}(6,C) \cong (\mathfrak{f}_4^C)_{E_1,E_2,E_3,F_1(e_k),k=0,1}$, $R_{10} \in \mathfrak{spin}(10,C) = \mathfrak{so}(10,C) \cong (\mathfrak{e}_8^C)^{\sigma'_4,\mathfrak{so}(6,C)}$ (Lemmas 3.14, 3.45) and $Spin(6,C)$, $Spin(10,C)$ are connected, we see that $\alpha\beta = \beta\alpha$. Hence $\varphi_{E_8^C,\sigma'_4}$ is a homomorphism. Moreover, we obtain that $\operatorname{Ker}\varphi_{E_8^C,\sigma'_4} \cong \boldsymbol{Z}_4$. Indeed, since we see that $\dim_C(\mathfrak{spin}(6,C)\oplus\mathfrak{spin}(10,C)) = 15+45 = 60 = \dim_C((\mathfrak{e}_8)^{\sigma'_4})$ (Lemma 3.48) and from $z(\varphi_{E_8^C,\sigma'_4 *}) = \{0\}$ (the mapping $\varphi_{E_8^C,\sigma'_4 *}$ is the differential mapping of $\varphi_{E_8^C,\sigma'_4}$) we have that $\operatorname{Ker}\varphi_{E_8^C,\sigma'_4}$ is discrete. Hence $\operatorname{Ker}\varphi_{E_8^C,\sigma'_4}$ is contained in the center $z(Spin(6,C)\times Spin(10,C)) = z(Spin(6,C))\times z(Spin(10,C)) = \{1,\sigma,\sigma'_4,\sigma\sigma'_4\}\times\{1,\sigma,\sigma'_4,\sigma\sigma'_4\}$. Note that in general because of the center $z(Spin(10,C)) = \boldsymbol{Z}_4$, we see $z(Spin(10,C)) = \{1,\sigma,\sigma'_4,\sigma\sigma'_4\}$ and $z(Spin(6,C)) = \{1,\sigma,\sigma'_4,\sigma\sigma'_4\}$ as in the proof of Theorem 3.39. Then, among them, the mapping $\varphi_{E_8^C,\sigma'_4}$ maps only $(1,1)$, $(\sigma'_4,\sigma\sigma'_4)$, (σ,σ), $(\sigma\sigma'_4,\sigma'_4)$ to the identity 1. Hence we have that $\operatorname{Ker}\varphi_{E_8^C,\sigma'_4} \subset \{(1,1),((\sigma'_4,\sigma\sigma'_4),(\sigma,\sigma),(\sigma\sigma'_4,\sigma'_4)\}$, and vice versa. Thus we see that

$$\operatorname{Ker}\varphi_{E_8^C,\sigma'_4} = \{(1,1),((\sigma'_4,\sigma\sigma'_4),(\sigma,\sigma),(\sigma\sigma'_4,\sigma'_4)\} \cong \boldsymbol{Z}_4.$$

Since $(E_8^C)^{\sigma'_4}$ is connected and $\operatorname{Ker}\varphi_{E_8^C,\sigma'_4}$ is discrete, again together with $\dim_C(\mathfrak{so}(6,C)\oplus\mathfrak{so}(10,C)) = 15+45 = 60 = \dim_C((\mathfrak{e}_8)^{\sigma'_4})$, $\varphi_{E_8^C,\sigma'_4}$ is surjection.

Therefore we have the required isomorphism

$$(E_8^C)^{\sigma'_4} \cong (Spin(6, C) \times Spin(10, C))/\boldsymbol{Z}_4. \qquad \square$$

3.6. Main Theorem

By using results above, we shall determine the structure of the group $(E_8)^{\sigma'_4}$, which is the main theorem.

THEOREM 3.50. *We have that* $(E_8)^{\sigma'_4} \cong (Spin(6) \times Spin(10))/\boldsymbol{Z}_4$, $\boldsymbol{Z}_4 = \{(1,1), (\sigma'_4, \sigma\sigma'_4), (\sigma, \sigma), (\sigma\sigma'_4, \sigma'_4)\}$.

PROOF. For $\delta \in (E_8)^{\sigma'_4} = ((E_8^C)^{\tau\lambda_\omega})^{\sigma'_4} = ((E_8^C)^{\sigma'_4})^{\tau\lambda_\omega} \subset (E_8^C)^{\sigma'_4}$, there exist $\alpha \in Spin(6, C) \cong (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1} \cong ((E_7^C)^{\kappa,\mu})_{\tilde{E}_1, \tilde{E}_{-1}, E_2 \dot{+} E_3, E_2 \dot{-} E_3, \dot{F}_1(e_k), k=0,1} \subset (E_7^C)^{\sigma'_4} \subset (E_8^C)^{\sigma'_4}$ and $\beta \in Spin(10, C) \cong (E_8^C)^{\sigma'_4, \mathfrak{so}(6,C)} \subset (E_8^C)^{\sigma'_4}$ such that $\delta = \varphi(\alpha, \beta) = \alpha\beta$ (Theorem 3.49). From the condition $\tau\lambda_\omega\delta\lambda_\omega\tau = \delta$, that is, $\tau\lambda_\omega\varphi(\alpha,\beta)\lambda_\omega\tau = \varphi(\alpha,\beta)$, we have $\varphi(\tau\lambda_\omega\alpha\lambda_\omega\tau, \tau\beta\tau) = \varphi(\alpha,\beta)$. Hence, we have that

$$\text{(i)} \begin{cases} \tau\alpha\tau = \alpha \\ \tau\lambda_\omega\beta\lambda_\omega\tau = \beta, \end{cases} \qquad \text{(ii)} \begin{cases} \tau\alpha\tau = \sigma\sigma'_4\alpha \\ \tau\lambda_\omega\beta\lambda_\omega\tau = \sigma'_4\beta, \end{cases}$$

$$\text{(iii)} \begin{cases} \tau\alpha\tau = \sigma\alpha \\ \tau\lambda_\omega\beta\lambda_\omega\tau = \sigma\beta, \end{cases} \qquad \text{(iv)} \begin{cases} \tau\alpha\tau = \sigma'_4\alpha \\ \tau\lambda_\omega\beta\lambda_\omega\tau = \sigma\sigma'_4\beta. \end{cases}$$

Case (i). From the condition $\tau\alpha\tau = \alpha$, we have $\alpha \in Spin(6) \cong (F_4)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$. Indeed, first since $Spin(6, C) \cong (F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$ is simply connected, the group $(F_4)_{E_1, E_2, E_3, F_1(e_k), k=0,1} = ((F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1})^\tau$ is connected. Since $(F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$ acts on $(V^C)^6$, the group $(F_4)_{E_1, E_2, E_3, F_1(e_k), k=0,1} = ((F_4^C)_{E_1, E_2, E_3, F_1(e_k), k=0,1})^\tau$ acts on

$$\begin{aligned} V^6 &= \{X \in (V^C)^6 \,|\, \tau X = X\} \\ &= \{X = F_1(t) \,|\, t = t_2e_2 + t_3e_3 + t_4e_4 + t_5e_5 + t_6e_6 + t_7e_7, t_k \in \boldsymbol{R}\} \end{aligned}$$

with the norm $(X, X) = 2t\bar{t}$. We can define a homomorphism $\pi : (F_4)_{E_1, E_2, E_3, F_1(e_k), k=0,1} \to SO(6) = SO(V^6)$ by $\pi(\alpha) = \alpha|_{V^6}$. Then it is easy to obtain that $\operatorname{Ker} \pi = \{1, \sigma\} \cong \boldsymbol{Z}_2$. Since, by doing simple computation as in Lemma 3.14 we see $\dim((\mathfrak{f}_4)_{E_1, E_2, E_3, F_1(e_k), k=0,1}) = 15$, we have that $\dim((\mathfrak{f}_4)_{E_1, E_2, E_3, F_1(e_k), k=0,1}) = 15 = \dim(\mathfrak{so}(6))$, moreover $SO(6)$ is connected. Hence the mapping π is surjection. Thus we have that $(F_4)_{E_1, E_2, E_3, F_1(e_k), k=0,1}/\boldsymbol{Z}_2 \cong SO(6)$. Therefore the group $(F_4)_{E_1, E_2, E_3, F_1(e_k), k=0,1}$ is isomorphic to $Spin(6)$ as the universal covering group $SO(6)$, that is, $(F_4)_{E_1, E_2, E_3, F_1(e_k), k=0,1} \cong Spin(6)$.

Next, from the condition $\tau\lambda_\omega\beta\lambda_\omega\tau = \beta$, we see that the group $\{\alpha \in Spin(10, C) \,|\, \tau\lambda_\omega\beta\lambda_\omega\tau = \beta\} = (Spin(10, C))^{\tau\lambda_\omega}$ (which is connected) $= ((E_8^C)^{\sigma_4', \mathfrak{so}(6, C)})^{\tau\lambda_\omega} = (E_8)^{\sigma_4', \mathfrak{so}(6)}$ (Theorem 3.47), and so its type is $\mathfrak{so}(10)$ (Proposition 3.46) (Note that $(E_8^C)^{\tau\lambda_\omega} = E_8$ and the C-linear transformation $\tau\lambda_\omega$ induces the involutive automorphism of the group $(E_8^C)^{\sigma_4', \mathfrak{so}(6, C)}$. Hence we see that the group $(E_8)^{\sigma_4', \mathfrak{so}(6)}$ is isomorphic to either one of

$$Spin(10),\ SO(10),\ Spin(10)/\boldsymbol{Z}_4.$$

Their center are $\boldsymbol{Z}_4$, $\boldsymbol{Z}_2$, $\{1\}$, respectively. However, since the center of $(E_8)^{\sigma_4', \mathfrak{so}(6)}$ has the elements 1, $\sigma\sigma_4'$, σ, σ_4', the group $(E_8)^{\sigma_4', \mathfrak{so}(6)}$ has to be isomorphic to $Spin(10)$ and its center is $\{1, \sigma\sigma_4', \sigma, \sigma_4'\} \cong \boldsymbol{Z}_4$. Hence the group of Case (i) is isomorphic to the group $(Spin(6) \times Spin(10))/\boldsymbol{Z}_4$.

Case (ii). This case is impossible. Indeed, for $\alpha \in Spin(6, C) \subset Spin(8, C)$ we can set $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_1 \in SO(6, C) \subset SO(8, C)$, $\alpha_2, \alpha_3 \in SO(8, C)$ satisfying $(\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})}$, $x, y \in \mathfrak{C}^C$, and similarly for $\sigma_4' \in Spin(8)$, set $\sigma_4' = (\sigma_1', \sigma_2', \sigma_3')$, $\sigma_k \in SO(8) \subset SO(8, C)$ satisfying $(\sigma_1' x)(\sigma_2' y) = \overline{\sigma_3'(\overline{xy})}$, $x, y \in \mathfrak{C}$. Note that as a matrix, σ_1', σ_2' and σ_3' are expressed as follows:

$$\sigma_1' = \mathrm{diag}(1, 1, -1, -1, -1, -1, -1, -1),$$

$$\sigma_2' = \mathrm{diag}(-J, -J, -J, -J), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\sigma_3' = \mathrm{diag}(J, -J, -J, -J).$$

Then, from the condition $\tau\alpha\tau = \sigma\sigma_4'\alpha$, we have $(\tau\alpha_1, \tau\alpha_2, \tau\alpha_3) = (\sigma_1'\alpha_1, -\sigma_2'\alpha_2, -\sigma_3'\alpha_3)$, that is,

$$\tau\alpha_1 = \sigma_1'\alpha_1, \quad \tau\alpha_2 = -\sigma_2'\alpha_2, \quad \tau\alpha_3 = -\sigma_3'\alpha_3.$$

Here, as a matrix, α_1 is expressed as follows:

$$\alpha_1 = \left(\begin{array}{cc|c} 1 & 0 & \multirow{2}{*}{0} \\ 0 & 1 & \\ \hline \multicolumn{2}{c|}{0} & A \end{array}\right), \quad A \in SO(6, C).$$

Then, from $\tau\alpha_1 = \sigma'_1\alpha_1$, we have

$$\alpha_1 = \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & 0 \\ \hline & 0 & A' \end{array}\right), \quad A' = iB \in SO(6, C),\ i^2 = -1.$$

As for α_2, from $\tau\alpha_2 = -\sigma'_2\alpha_2$, we have $\alpha_2 = 0$. Indeed, from the explicit form of σ_2, it is sufficient to confirm this in the case 2×2-matrix. From

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} = \begin{pmatrix} \tau a & \tau b \\ \tau c & \tau d \end{pmatrix},$$

we have that $\tau a = b$, $\tau b = -a$, $\tau c = d$, $\tau d = -c$, that is, $a = b = c = d = 0$. Hence we see $\alpha_2 = 0$. This is contrary to $\alpha_2 \in SO(8, C)$.

Case (iii). This case is also impossible. Indeed, from the condition $\tau\alpha\tau = \sigma\alpha$, we have $(\tau\alpha_1, \tau\alpha_2, \tau\alpha_3) = (\alpha_1, -\alpha_2, -\alpha_3)$, that is,

$$\tau\alpha_1 = \alpha_1, \quad \tau\alpha_2 = -\alpha_2, \quad \tau\alpha_3 = -\alpha_3.$$

From $\tau\alpha_1 = \alpha_1$, we have $\alpha_1 \in SO(6) \subset SO(8)$. Hence, by the Principal of triality on $SO(8)$ (Theorem 2.3) we see that $\alpha_k \in SO(8)$, $k = 2, 3$, that is, $\tau\alpha_k = \alpha_k$, $k = 2, 3$. However, from $\tau\alpha_k = -\alpha_k$, $k = 2, 3$, we have $\alpha_k = \tau\alpha_k = -\alpha_k$, that is, $\alpha_k = 0$. This is contrary to $\alpha_k \in SO(8)$.

Case (iv). This case is also impossible. Indeed, from the condition $\tau\alpha\tau = \sigma'_4\alpha$, we have $(\tau\alpha_1, \tau\alpha_2, \tau\alpha_3) = (\sigma'_1\alpha_1, \sigma'_2\alpha_2, \sigma'_3\alpha_3)$, that is,

$$\tau\alpha_1 = \sigma'_1\alpha_1, \quad \tau\alpha_2 = \sigma'_2\alpha_2, \quad \tau\alpha_3 = \sigma'_3\alpha_3.$$

As in the Case (ii), we have $\alpha_2 = 0$. This is contrary to $\alpha_2 \in SO(8, C)$.

Therefore we have the required isomorphism

$$(E_8)^{\sigma'_4} \cong (Spin(6) \times Spin(10))/\boldsymbol{Z}_4, \quad \boldsymbol{Z}_4 = \{(1,1), (\sigma'_4, \sigma\sigma'_4), (\sigma, \sigma), (\sigma\sigma'_4, \sigma'_4)\}. \quad \square$$

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