

## NORMAL GENERATION OF LINE BUNDLES OF DEGREE $2g - 2h^1(L) - \text{Cliff}(X) - k$ ( $k = 2, 3, 4$ ) ON CURVES

By

Katsumi AKAHORI

**Abstract.** Let  $\text{Cliff}(X)$  be the Clifford index of a curve. We determine necessary conditions for very ample line bundles of degree  $\deg(L) = 2g - 2h^1(L) - \text{Cliff}(X) - k$  ( $k = 2, 3, 4$ ) being not normally generated.

### 1 Introduction

This is a continuation of the previous paper [2]. Let  $X$  be a smooth projective curve of genus  $g \geq 4$  over an algebraically closed field of characteristic 0. Let  $L$  be a very ample line bundle on  $X$ . One says that  $L$  is normally generated if  $X$  is projectively normal under the associated projective embedding.

The most remarkable result on the normal generation of a line bundle is the following one of Green and Lazarsfeld:

**PROPOSITION 1.1** ([6], Theorem 1). *Let  $L$  be a very ample line bundle on  $X$  with*

$$\deg(L) \geq 2g + 1 - 2h^1(L) - \text{Cliff}(X)$$

*(and hence  $h^1(L) \leq 1$ ). Then  $L$  is normally generated.*

Furthermore, in order to show that the inequality above is in general the best possible, they also determined necessary and sufficient conditions for very ample line bundles not being normally generated for the case of degree  $\deg(L) = 2g - 2h^1(L) - \text{Cliff}(X)$ . Their result is as follows:

**PROPOSITION 1.2** ([6], (2.1)). *Let  $c = \text{Cliff}(X)$  and  $L$  be a very ample line bundle with*

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$$\deg(L) = 2g - 2h^1(L) - \text{Cliff}(X).$$

Assume that  $g > \max\{(c+3)(c+2)/2, 10c+6\}$  and that  $X$  is neither hyperelliptic nor bielliptic. Then  $L$  fails to be normally generated if and only if either:

(I)  $X$  is  $(c+2)$ -gonal and  $L \cong K - g^1_{(c+2)} + D_4$  with  $h^1(L) = 0$  for some effective divisor  $D_4 \in X_4$ ;

(II)  $X$  is a double covering  $\pi: X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $f+2$  ( $c = 2f \geq 4$ ) and  $L \cong K - \pi^*(g^2_{(f+2)}) + D_4$  with  $h^1(L) = 1$  for some effective divisor  $D_4 \in X_4$ ;

(III)  $X$  is a double covering  $\pi: X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $f+2$  ( $c = 2f \geq 4$ ) and  $L \cong K - \pi^*(g^2_{(f+2)}) + D_6$  with  $h^1(L) = 0$  for some effective divisor  $D_6 \in X_6$ .

According to these propositions, we are provoked to discuss the following question: Which pairs  $(X, L)$  fail to be normally generated in the case of

$$\deg(L) = 2g - 2h^1(L) - \text{Cliff}(X) - k(k \geq 1)?$$

In the previous paper [2], we have determined necessary and sufficient conditions for very ample line bundles not being normally generated for the case of  $k = 1$ . The result is as follows:

**THEOREM 1.3** ([2], (1.3)). *Let  $c = \text{Cliff}(X)$  and  $L$  be a very ample line bundle with*

$$\deg(L) = 2g - 1 - 2h^1(L) - \text{Cliff}(X).$$

Assume that  $g > \max\{(c+4)(c+3)/2, 6c+8\}$  and that  $X$  is neither hyperelliptic nor bielliptic. Then  $L$  fails to be normally generated if and only if either:

(I)  $X$  is a double covering  $\pi: X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $f+2$  ( $c = 2f \geq 4$ ) and  $L \cong K - \pi^*(g^2_{(f+2)}) + D_5$  with  $h^1(L) = 0$  for some effective divisor  $D_5 \in X_5$ ;

(II)  $X$  is a  $(c+3)$ -gonal and  $L \cong K - g^1_{(c+3)} + D_4$  with  $h^1(L) = 0$  for some effective divisor  $D_4 \in X_4$ ;

(III)  $X$  is a triple covering  $\pi: X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $h$  ( $5+c = 3h \geq 6$ ) and  $L \cong K - \pi^*(g^2_h) + D_6$  with  $h^1(L) = 0$  for some effective divisor  $D_6 \in X_6$ ;

(IV)  $X$  is a triple covering  $\pi: X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $h$  ( $5+c = 3h \geq 12$ ) and  $L \cong K - \pi^*(g^2_h) + D_4$  with  $h^1(L) = 1$  for some effective divisor  $D_4 \in X_4$ .

Our purpose in this paper is to describe necessary conditions for very ample line bundles not being normally generated for the case of  $k = 2, 3, 4$ . Our results are as follows:

**THEOREM 1.4.** *Let  $c = \text{Cliff}(X)$  and  $L$  be a very ample line bundle with*

$$\text{deg}(L) = 2g - 2 - 2h^1(L) - \text{Cliff}(X).$$

*Assume that  $g > \max\{(c + 5)(c + 4)/2, 6c + 14\}$ . Then  $L$  is normally generated unless  $X$  and  $L$  are the following cases:*

(I)  *$X$  is a triple covering  $\pi : X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $h$  ( $c + 5 = 3h \geq 6$ ) and  $L \cong K - \pi^*(g_h^2) + D_5$  with  $h^1(L) = 0$  for some effective divisor  $D_5 \in X_5$ ;*

(II)  *$X$  is a  $(c + 4)$ -gonal curve and  $L \cong K - g_{(c+4)}^1 + D_4$  with  $h^1(L) = 0$  for some effective divisor  $D_4 \in X_4$ ;*

(III)  *$X$  is a 4-sheeted covering  $\pi : X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $f$  ( $c + 6 = 4f \geq 8$ ) and  $L \cong K - \pi^*(g_f^2) + D_6$  with  $h^1(L) = 0$  for some effective divisor  $D_6 \in X_6$ ;*

(IV)  *$X$  is a 4-sheeted covering  $\pi : X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $f$  ( $c + 6 = 4f \geq 16$ ) and  $L \cong K - \pi^*(g_f^2) + D_4$  with  $h^1(L) = 1$  for some effective divisor  $D_4 \in X_4$ ;*

(V)  *$X$  is a trigonal curve and  $L \cong K - 3g_3^1 + D_8$  with  $h^1(L) = 0$  for some effective divisor  $D_8 \in X_8$ ;*

(VI)  *$X$  is a triple covering  $\pi : X \rightarrow Y \subset \mathbf{P}^3$  of an elliptic curve  $Y$  and  $L \cong K - \pi^*g_4^3 + D_8$  with  $h^1(L) = 0$  for some effective divisor  $D_8 \in X_8$ ;*

(VII)  *$X$  is a double covering of a curve  $Y$  and  $\text{Cliff}(X)$  is even.*

The following result in the case of  $k = 3$  is very similar to the one in the case of  $k = 2$ .

**THEOREM 1.5.** *Let  $c = \text{Cliff}(X)$  and  $L$  be a very ample line bundle with*

$$\text{deg}(L) = 2g - 3 - 2h^1(L) - \text{Cliff}(X).$$

*Assume that  $g > \max\{(c + 6)(c + 5)/2, 6c + 20\}$ . Then  $L$  is normally generated unless  $X$  and  $L$  are the following cases:*

(I)  *$X$  is a 4-sheeted covering  $\pi : X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $h$  ( $c + 6 = 4h \geq 8$ ) and  $L \cong K - \pi^*(g_h^2) + D_5$  with  $h^1(L) = 0$  for some effective divisor  $D_5 \in X_5$ ;*

(II)  $X$  is a  $(c+5)$ -gonal curve and  $L \cong K - g_{(c+5)}^1 + D_4$  with  $h^1(L) = 0$  for some effective divisor  $D_4 \in X_4$ ;

(III)  $X$  is a 5-sheeted covering  $\pi : X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $f$  ( $c+7 = 5f \geq 10$ ) and  $L \cong K - \pi^*(g_f^2) + D_6$  with  $h^1(L) = 0$  for some effective divisor  $D_6 \in X_6$ ;

(IV)  $X$  is a 5-sheeted covering  $\pi : X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $f$  ( $c+7 = 5f \geq 20$ ) and  $L \cong K - \pi^*(g_f^2) + D_4$  with  $h^1(L) = 1$  for some effective divisor  $D_4 \in X_4$ ;

(V)  $X$  is a trigonal curve and  $L \cong K - 4g_3^1 + D_{10}$  with  $h^1(L) = 0$  for some effective divisor  $D_{10} \in X_{10}$ ;

(VI)  $X$  is a triple covering  $\pi : X \rightarrow Y \subset \mathbf{P}^4$  of an elliptic curve  $Y$  and  $L \cong K - \pi^*g_5^4 + D_{10}$  with  $h^1(L) = 0$  for some effective divisor  $D_{10} \in X_{10}$ ;

(VII)  $X$  is a double covering of a curve  $Y$  and  $\text{Cliff}(X)$  is even.

The result in the case of  $k = 4$  is more complicated than the preceding ones.

**THEOREM 1.6.** *Let  $c = \text{Cliff}(X)$  and  $L$  be a very ample line bundle with*

$$\deg(L) = 2g - 4 - h^1(L) - \text{Cliff}(X).$$

*Assume that  $g > \max\{(c+7)(c+6)/2, 6c+26\}$ . Then  $L$  is normally generated unless  $X$  and  $L$  are the following cases:*

(I)  $X$  is a 5-sheeted covering  $\pi : X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $h$  ( $c+7 = 5h \geq 10$ ) and  $L \cong K - \pi^*(g_h^2) + D_5$  with  $h^1(L) = 0$  for some effective divisor  $D_5 \in X_5$ ;

(II)  $X$  is a  $(c+6)$ -gonal curve and  $L \cong K - g_{(c+6)}^1 + D_4$  with  $h^1(L) = 0$  for some effective divisor  $D_4 \in X_4$ ;

(III)  $X$  is a 6-sheeted covering  $\pi : X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $f$  ( $c+8 = 6f \geq 12$ ) and  $L \cong K - \pi^*(g_f^2) + D_6$  with  $h^1(L) = 0$  for some effective divisor  $D_6 \in X_6$ ;

(IV)  $X$  is a 6-sheeted covering  $\pi : X \rightarrow Y \subset \mathbf{P}^2$  of a smooth plane curve  $Y$  of degree  $f$  ( $c+8 = 6f \geq 24$ ) and  $L \cong K - \pi^*(g_f^2) + D_4$  with  $h^1(L) = 1$  for some effective divisor  $D_4 \in X_4$ ;

(V)  $X$  is a trigonal curve and  $L \cong K - 5g_3^1 + D_{12}$  with  $h^1(L) = 0$  for some effective divisor  $D_{12} \in X_{12}$ ;

(VI)  $X$  is a triple covering  $\pi : X \rightarrow Y \subset \mathbf{P}^5$  of an elliptic curve  $Y$  and  $L \cong K - \pi^*g_6^5 + D_{12}$  with  $h^1(L) = 0$  for some effective divisor  $D_{12} \in X_{12}$ ;

(VII)  $X$  is a double covering of a curve  $Y$  and  $\text{Cliff}(X)$  is even;

(VIII)  $X$  is a 4-gonal curve and  $L \cong K - 3g_4^1 + D_8$  with  $h^1(L) = 0$  for some effective divisor  $D_8 \in X_8$ ;

(IX)  $X$  is a 4-sheeted covering  $\pi : X \rightarrow Y \subset \mathbf{P}^3$  of an elliptic curve  $Y$  and  $L \cong K - \pi^*g_4^3 + D_8$  with  $h^1(L) = 0$  for some effective divisor  $D_8 \in X_8$ ;

(X)  $X$  is a triple covering  $\pi : X \rightarrow Y \subset \mathbf{P}^2$  of a plane curve  $Y$  of degree  $f$  ( $c + 8 = 3f \geq 9$ ) and  $L \cong K - \pi^*(g_f^2) + D_6$  with  $h^1(L) = 0$  for some effective divisor  $D_6 \in X_6$ ;

(XI)  $X$  is a triple covering  $\pi : X \rightarrow Y \subset \mathbf{P}^2$  of a plane curve  $Y$  of degree  $f$  ( $c + 8 = 3f \geq 15$ ) and  $L \cong K - \pi^*(g_f^2) + D_4$  with  $h^1(L) = 1$  for some effective divisor  $D_4 \in X_4$ .

NOTATION

(1)  $X$  is a smooth projective curve of genus  $g \geq 4$  and  $L$  is a very ample line bundle on  $X$ . We denote by  $K$  the canonical bundle on  $X$ .

(2) For a line bundle  $A$  on a curve  $X$ , we denote by  $h^i(A)$  the dimension of an  $i$ -th cohomology group  $H^i(X, A)$ .

(3) The Clifford index of a line bundle  $A$  on  $X$  is defined by

$$\text{Cliff}(A) = \text{deg}(A) - 2(h^0(A) - 1).$$

The Clifford index of  $X$  is taken to be

$$\text{Cliff}(X) = \min\{\text{Cliff}(A) \mid h^0(A) \geq 2, h^1(A) \geq 2\}.$$

(4) We denote by  $g_d^r$  an  $r$ -dimensional linear series of degree  $d$  on a curve  $X$ .

**2 Preparatory Propositions**

Our main tool is the following result of [6].

PROPOSITION 2.1 ([6], Theorem 3 and Remark 1.3). *Let  $L$  be a very ample line bundle on  $X$  with  $\text{deg}(L) = 2g + 1 - k$ . Assume that  $2k + 4e + 1 \leq g$  and  $e \geq -1$ , and consider the embedding  $\phi_L : X \subseteq \mathbf{P}(H^0(L)) = \mathbf{P}^r$  defined by  $L$ . Then the following conditions (i), (ii), and (iii) are equivalent:*

(i)  $H^0(L) \otimes H^0(L) \rightarrow H^0(L^2)$  is not surjective,

(ii) there exists an integer  $1 \leq n \leq r - 2 - e - h^1(L)$ , and an effective divisor  $D$  on  $X$  of degree at least  $2n + 2$ , such that

(a)  $H^1(X, L^2(-D)) = 0$

(b)  $\phi_L(D)$  spans an  $n$ -plane  $\Lambda \subseteq \mathbf{P}^r$ , and  $H^0(\Lambda, \mathcal{O}(2)) \rightarrow H^0(\mathcal{O}_D)$  is not surjective,

(iii) *there exists an effective divisor  $D$  on  $X$  such that*

(a)  $H^1(X, L^2(-D)) = 0$

(b')  $H^0(\mathbf{P}^r, \mathcal{O}(2)) \rightarrow H^0(\mathcal{O}_D)$  *is not surjective.*

The following proposition can be derived from Castelnuovo's genus bound.

**PROPOSITION 2.2.** *Let  $X$  be a curve of genus  $g$ , and let  $B$  be a line bundle on  $X$  with  $r(B) = r$  and  $\deg(B) = 2r + e$  for some  $e \geq 0$ . If  $B$  is birationally very ample, then either*

$$r \geq g - 2e - 1 \quad \text{or} \quad g \leq (e + 3)(e + 2)/2.$$

Since we only deal with very ample line bundles, we shall prepare a criterion for a line bundle being not very ample on a triple covering of an elliptic curve.

**PROPOSITION 2.3.** *Let  $\pi : X \rightarrow Y$  be a triple covering of an elliptic curve  $Y$ . Let  $L$  be a line bundle of degree  $2g - 2 - d$  with  $\dim|K - L| = s \geq 1$  on  $X$ . If  $d < (g - 1)/2$  and  $\deg|K - L| = d > 3s + 3$ , then  $L$  is not very ample.*

**PROOF.** Since  $d < (g - 1)/2$ , it follows from ([7], (2.5)) that  $|K - L| = \pi^*(g_{s+1}^s) + D_{d-3s-3}$  for some  $D_{d-3s-3} \in X_{d-3s-3}$ . For  $p \in D_{d-3s-3}$ , we have  $|K - L + \pi^*(\pi(p)) - p| = \pi^*(g_{s+2}^{s+1}) + D_{d-3s-3} - p$ . Therefore we get  $\dim|L - \pi^*(\pi(p)) + p| \geq \dim|L| - 1$  and  $L$  is not very ample.  $\square$

### 3 The Proof of Our Theorems

We shall utilize the following proposition for the proof of our theorems.

**PROPOSITION 3.1.** *Let  $L$  be a line bundle and  $|L| = g_d^r$  be a linear series on a curve  $X$ . Assume that  $|L| = g_d^r$  is base-point-free,  $r \geq 2$ ,  $h^1(L) \geq 2$ , and  $\text{Cliff}(L) = \text{Cliff}(X) + k$  ( $k \geq 1$ ). Let  $\pi : X \rightarrow Y \subset \mathbf{P}^r$  be the morphism defined by  $|L| = g_d^r$ . Then we get  $\deg \pi \leq k + 2$ .*

*Furthermore, if  $\deg \pi \geq k + 1$ , then  $Y$  is a smooth curve in  $\mathbf{P}^r$ .*

**PROOF.** Let  $D = \pi^*\mathcal{O}_Y(y)$  for any  $y \in Y$ . If  $\deg \pi \geq k + 3$ , then we have  $\text{Cliff}(L - D) = \deg(L - D) - 2r(L - D) \leq \deg(L - D) - 2(r(L) - 1) = \text{Cliff}(L) - \deg D + 2 = \text{Cliff}(X) + k + 2 - \deg D < \text{Cliff}(X)$ . Since  $h^0(L - D) \geq 2$  and  $h^1(L - D) \geq 2$ , this is a contradiction. Therefore  $\deg \pi \leq k + 2$ .

Let  $L = \pi^*(L_0)$  for a line bundle  $L_0$  on  $Y$ . Assume that  $L_0$  is not very ample and  $\deg \pi \geq k + 1$ . Since  $|L_0|$  is base-point-free, we have  $h^0(L_0(-y_1 - y_2)) =$

$h^0(L_0) - 1$  for some points  $y_1, y_2 \in Y$ . Let  $D' = \pi^* \mathcal{O}_Y(y_1 + y_2)$ . Then  $h^0(L - D') = h^0(\pi^*(L_0(-y_1 - y_2))) \geq h^0(L_0(-y_1 - y_2)) = h^0(L_0) - 1 = r \geq 2$ . Since  $h^1(L - D') \geq h^1(L) \geq 2$ , we get  $\text{Cliff}(L - D') \geq \text{Cliff}(X) = \text{Cliff}(L) - k$ . On the other hand,  $\text{Cliff}(L - D') = \text{deg } L - \text{deg } D' - 2r(L - D') \leq \text{deg } L - 2(k + 1) - 2(r - 1) = \text{Cliff}(L) - 2k$ . It is a contradiction. Hence  $L_0$  is very ample.  $\square$

PROOF OF THEOREM 1.4. Assume that  $L$  is not normally generated. Since  $L$  is very ample and  $g > (c + 5)(c + 4)/2$ , it follows from (2.2) that

$$r(L) = g - i - c - 2 \geq g - 2(c + 2) - 1,$$

i.e. that  $c + 2 - i \geq -1$ . By applying (2.1) for  $k = 2i + c + 3$  and  $e = c + 2 - i$ , there exists an integer

$$1 \leq n \leq r(L) - 2 - e - h^1(L) = r(L) - c - 4 \tag{1}$$

and an effective divisor  $D$  on  $X$  of degree  $d \geq 2n + 2$  which spans an  $n$ -plane in  $\mathbf{P}(H^0(L))$ . Since  $n = r(L) - h^0(L - D)$ , we get  $h^0(L - D) = r(L) - n \geq c + 4 \geq 4$  and  $h^1(L - D) = i + d - n - 1 \geq i + n + 1 \geq 2$ . Therefore  $L - D$  contributes to  $\text{Cliff}(X)$ , and so  $\text{Cliff}(X) \leq \text{Cliff}(L - D)$ . On the other hand, thanks to  $\text{deg } D \geq 2n + 2$ , we have  $\text{Cliff}(L - D) \leq \text{Cliff}(L) = c + 2$ . Therefore we get  $\text{Cliff}(L - D) = c, c + 1, c + 2$ .

First, let  $\text{Cliff}(L - D) = c$ . Then we have  $d = 2n + 4$ , and hence  $h^1(L - D) = n + i + 3$ . Let

$$B = K - L + D.$$

Now we consider  $|B| = g_{2(n+i+2)+c}^{n+i+2}$ . Since  $\text{Cliff}(X) = \text{Cliff}(B)$ ,  $|B|$  is base-point-free, and so we can consider the morphism

$$\pi : X \rightarrow Y \subset \mathbf{P}^{n+i+2}.$$

By using (3.1), we get  $\text{deg } \pi \leq 2$ .

We assume that  $\text{deg } \pi = 1$ . By applying (2.2), thanks to the assumption we get  $n + i + 2 = r(B) \geq g - 2c - 1$ , i.e.

$$n \geq g - 2c - 3 - i = r(L) - c - 1.$$

This contradicts (1). Therefore  $\text{deg } \pi = 2$ . Since  $r(B) = n + i + 2 \geq 3$ , we get  $g(Y) \leq 1$  by ([8], p. 113), and we are in case (VII).

Secondly, let  $\text{Cliff}(L - D) = c + 1$ . Then we have  $d = 2n + 3$ , and hence  $h^1(L - D) = n + i + 2$ . Now we consider  $|B| = g_{2(n+i+1)+c+1}^{n+i+1}$ . If  $|B|$  has a base

point, then we revert to the case of  $\text{Cliff}(B) = \text{Cliff}(X)$ . Therefore we may assume that  $|B|$  is base-point-free. We consider the morphism

$$\pi : X \rightarrow Y \subset \mathbf{P}^{n+i+1}.$$

Let  $\deg \pi = 1$ . By applying (2.2), thanks to the assumption we get  $n + i + 1 = r(B) \geq g - 2(c + 1) - 1$ , i.e.

$$n \geq g - 2c - 4 - i = r(L) - c - 2.$$

This contradicts (1). Hence we have  $\deg \pi \geq 2$ . If  $r(B) \geq 3$ , then  $|B|$  must be simple by virtue of ([2], (3.1)). Therefore we get  $r(B) = 2$ . Furthermore,  $\deg \pi = 2, 3$  by virtue of ([2], (3.1)). If  $\deg \pi = 2$ , then  $c + 1 = 2f$  is even and  $\deg Y = 2 + f$  ( $f \geq 1$ ). By using Castelnuovo's genus bound ([4], p. 116), we get  $g(Y) \leq (f + 1)f/2 = (c + 3)(c + 1)/8$ . On the other hand, since  $g > (c + 5)(c + 4)/2 > (c + 3)(c + 2)/2$  for any  $c \geq 0$ ,  $X$  carries a  $g_{c+2}^1$  by virtue of [9]. Applying Castelnuovo's lemma ([4], p. 366, C-1), we get  $g \leq (c + 2) \times 0 + 2 \times (c + 3)(c + 1)/8 + \{(c + 2) - 1\}(2 - 1) = (c^2 + 8c + 7)/4$ . Since  $(c^2 + 8c + 7)/4 < \max\{(c + 5)(c + 4)/2, 6c + 14\}$  for any  $c \geq 0$ , this is a contradiction. Let  $\deg \pi = 3$  and  $5 + c = 3h$ . Then  $X$  is a triple covering of a smooth plane curve  $Y$  of degree  $h$ . Since  $r(B) = n + i + 1 = 2$ , we have  $(n, i) = (1, 0)$  and we are in case (I).

Lastly, let  $\text{Cliff}(L - D) = c + 2$ . Then we have  $d = 2n + 2$ , and hence  $h^1(L - D) = n + i + 1$ . Here we consider  $|B| = g_{2(n+i)+c+2}^{n+i}$  and may assume that  $|B|$  is base-point-free. If  $r(B) = 1$ , then  $(n, i) = (1, 0)$ ,  $|B| = g_{c+4}^1$ , and we are in case (II). Assuming that  $r(B) \geq 2$ , consider the morphism

$$\pi : X \rightarrow Y \subset \mathbf{P}^{n+i}.$$

Let  $\deg \pi = 1$ . By applying (2.2), thanks to the assumption we have  $n + i = r(B) \geq g - 2(c + 2) - 1$ , i.e.

$$n \geq g - 2c - 5 - i = r(L) - c - 3.$$

This contradicts (1). Therefore  $\deg \pi \geq 2$ . Furthermore, using (3.1) we have  $\deg \pi \leq 4$ .

Let  $r(B) = 2$ . If  $\deg \pi = 2$ , then we are in case (VII). Let  $\deg \pi = 3$  and  $c + 6 = 3f$  ( $f \geq 2$ ). Then we have  $\deg Y = f$ . By using Castelnuovo's genus bound ([4], p. 116), we get  $g(Y) \leq (f - 1)(f - 2)/2 = c(c + 3)/18$ . If  $c = 0$ , then  $X$  is a hyperelliptic curve and we are in case (VII). Therefore we assume that  $c > 0$ . Since  $g > (c + 5)(c + 4)/2 > (c + 3)(c + 2)/2$  for any  $c > 0$ ,  $X$  carries a  $g_{c+2}^1(c + 2 = 3(f - 1) - 1)$  by virtue of [9]. Applying Castelnuovo's lemma ([4],



p. 366, C-1), we get  $g \leq (c + 2) \times 0 + 3 \times c(c + 3)/18 + \{(c + 2) - 1\}(3 - 1) = (c^2 + 15c + 12)/6$ . Since  $(c^2 + 15c + 12)/6 < \max\{(c + 5)(c + 4)/2, 6c + 14\}$  for any  $c > 0$ , this is a contradiction. Let  $\deg \pi = 4$  and  $c + 6 = 4f$ . Since  $r(B) = n + i = 2$ , we obtain  $(n, i) = (2, 0), (1, 1)$ , and we are in case (III), (IV), respectively. Let  $(n, i) = (1, 1)$ . If  $f = 2$ , then  $X$  is a 4-gonal curve and  $\deg L = 2g - 6$ . Hence  $L$  is not very ample by ([7], (4.2)). If  $f = 3$ , then  $X$  is a 4-sheeted covering of an elliptic curve and  $\deg L = 2g - 10$ . Let  $D' = K - L$ . We have  $|D' + D| = \pi^*(q_1 + q_2 + q_3)$  for some  $q_1, q_2, q_3 \in Y$ . Let  $\pi^*(q_i) = p_{i1} + p_{i2} + p_{i3} + p_{i4}$  ( $i = 1, 2, 3$ ). Since  $\dim|D'| = 0$ , we may assume that  $D' = p_{11} + p_{12} + p_{13} + p_{21} + p_{22} + p_{23} + p_{31} + p_{32}$  or  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{21} + p_{22} + p_{23} + p_{31}$  or  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{21} + p_{22} + p_{31} + p_{32}$ . Then we have  $\dim|D' + p_{14} + p_{24}| \geq \dim|\pi^*(q_1 + q_2)| \geq \dim|q_1 + q_2| = 1$  or  $\dim|D' + p_{24} + p_{32}| \geq 1$  or  $\dim|D' + p_{23} + p_{24}| \geq 1$ , respectively. Thus  $\dim|D' + p + p'| \geq 1$  for a suitable choice  $p, p'$  from  $D$ . This implies that  $L$  is not very ample. Therefore  $f \geq 4$  in case (IV).

If  $r(B) \geq 3$ , then we consider only three cases by virtue of ([5], (2.3)).

(1) If  $X$  is a trigonal curve,  $\text{Cliff}(X) = 1$ , and  $|B| = g_9^3$ , then  $(n, i) = (1, 2), (2, 1), (3, 0)$ . If  $(n, i) = (1, 2), (2, 1)$ , then  $L$  is not very ample by virtue of ([7], (2.7)). If  $(n, i) = (3, 0)$ , then we are in case (V).

(2) If  $X$  is a triple covering of an elliptic curve,  $\text{Cliff}(X) = 4$ , and  $|B| = g_{12}^3$ , then  $(n, i) = (1, 2), (2, 1), (3, 0)$ . If  $(n, i) = (1, 2), (2, 1)$ , then  $L$  is not very ample by virtue of (2.3) and ([1], (4.1)). If  $(n, i) = (3, 0)$ , we are in case (VI).

(3) If  $\deg \pi = 2$ , then we are in case (VII). □

REMARK. If  $L$  is in case (I), (II), (III), (IV), then using (2.1) we can prove that  $L$  fails to be normally generated.

If  $L$  is in case (V), then  $L$  fails to be normally generated by ([10], Cor. 1).

PROOF OF THEOREM 1.5. Assume that  $L$  is not normally generated. Since  $L$  is very ample and  $g > (c + 6)(c + 5)/2$ , it follows from (2.2) that

$$r(L) = g - i - c - 3 \geq g - 2(c + 3) - 1,$$

i.e. that  $c + 3 - i \geq -1$ . By applying (2.1) for  $k = 2i + c + 4$  and  $e = c + 3 - i$ , there exists an integer

$$1 \leq n \leq r(L) - 2 - e - h^1(L) = r(L) - c - 5 \tag{2}$$

and an effective divisor  $D$  on  $X$  of degree  $d \geq 2n + 2$  which spans an  $n$ -plane in  $\mathbf{P}(H^0(L))$ . Since  $n = r(L) - h^0(L - D)$ , we get  $h^0(L - D) = r(L) - n \geq c + 5 \geq 5$

and  $h^1(L - D) = i + d - n - 1 \geq i + n + 1 \geq 2$ . Therefore  $L - D$  contributes to  $\text{Cliff}(X)$ , and so  $\text{Cliff}(X) \leq \text{Cliff}(L - D)$ . On the other hand, thanks to  $\deg D \geq 2n + 2$ , we have  $\text{Cliff}(L - D) \leq \text{Cliff}(L) = c + 3$ . Therefore we get  $\text{Cliff}(L - D) = c, c + 1, c + 2, c + 3$ .

First, let  $\text{Cliff}(L - D) = c$ . Then we have  $d = 2n + 5$ , and hence  $h^1(L - D) = n + i + 4$ . Let

$$B = K - L + D.$$

Now we consider  $|B| = g_{2(n+i+3)+c}^{n+i+3}$ . Since  $\text{Cliff}(X) = \text{Cliff}(B)$ ,  $|B|$  is base-point-free, and so we can consider the morphism

$$\pi : X \rightarrow Y \subset \mathbf{P}^{n+i+3}.$$

By using (3.1), we get  $\deg \pi \leq 2$ .

We assume that  $\deg \pi = 1$ . By applying (2.2), thanks to the assumption we get  $n + i + 3 = r(B) \geq g - 2c - 1$ , i.e.

$$n \geq g - 2c - 4 - i = r(L) - c - 1.$$

This contradicts (2). Therefore  $\deg \pi = 2$ . Since  $r(B) = n + i + 3 \geq 4$ , we get  $g(Y) \leq 1$  by ([8], p. 113), and we are in case (VII).

Secondly, let  $\text{Cliff}(L - D) = c + 1$ . Then we have  $d = 2n + 4$ , and hence  $h^1(L - D) = n + i + 3$ . Now we consider  $|B| = g_{2(n+i+2)+c+1}^{n+i+2}$ . If  $|B|$  has a base point, then we revert to the case of  $\text{Cliff}(B) = \text{Cliff}(X)$ . Therefore we may assume that  $|B|$  is base-point-free. We consider the morphism

$$\pi : X \rightarrow Y \subset \mathbf{P}^{n+i+2}.$$

Let  $\deg \pi = 1$ . By applying (2.2), thanks to the assumption we get  $n + i + 2 = r(B) \geq g - 2(c + 1) - 1$ , i.e.

$$n \geq g - 2c - 5 - i = r(L) - c - 2.$$

This contradicts (2). Hence we have  $\deg \pi \geq 2$ . On the other hand, since  $r(B) \geq 3$ ,  $|B|$  must be simple by virtue of ([2], (3.1)). It is a contradiction.

Thirdly, let  $\text{Cliff}(L - D) = c + 2$ . Then we have  $d = 2n + 3$ , and hence  $h^1(L - D) = n + i + 2$ . Here we consider  $|B| = g_{2(n+i+1)+c+2}^{n+i+1}$  and may assume that  $|B|$  is base-point-free. Since  $r(B) \geq 2$ , we consider the morphism

$$\pi : X \rightarrow Y \subset \mathbf{P}^{n+i+1}.$$

Let  $\deg \pi = 1$ . By applying (2.2), thanks to the assumption we have  $n + i + 1 = r(B) \geq g - 2(c + 2) - 1$ , i.e.

$$n \geq g - 2c - 6 - i = r(L) - c - 3.$$

This contradicts (2). Therefore  $\deg \pi \geq 2$ . Furthermore, using (3.1) we have  $\deg \pi \leq 4$ .

Let  $r(B) = 2$ . If  $\deg \pi = 2$ , then we are in case (VII). Let  $\deg \pi = 3$  and  $c + 6 = 3f$  ( $f \geq 2$ ). Then we have  $\deg Y = f$ . By the same way as we have done in the case of  $\text{Cliff}(L - D) = c + 2$  of the proof of (1.4), we get  $g \leq (c^2 + 15c + 12)/6$ . But  $(c^2 + 15c + 12)/6 < \max\{(c + 6)(c + 5)/2, 6c + 14\}$  for any  $c > 0$ . It is a contradiction. Let  $\deg \pi = 4$  and  $c + 6 = 4h$ . Since  $r(B) = n + i + 1 = 2$ , we obtain  $(n, i) = (1, 0)$  and we are in case (I).

If  $r(B) \geq 3$ , then we consider only three cases by virtue of ([5], (2.3)).

(1) If  $X$  is a trigonal curve,  $\text{Cliff}(X) = 1$ , and  $|B| = g_9^3$ , then  $(n, i) = (1, 1), (2, 0)$ . If  $(n, i) = (1, 1)$ , then  $L$  is not very ample by virtue of ([7], (2.7)). If  $(n, i) = (2, 0)$ , then we have  $L \cong K - 3g_3^1 + D_7$ . Since this is a special case of  $L \cong K - 4g_3^1 + D_{10}$ , we are in case (V).

(2) If  $X$  is a triple covering of an elliptic curve,  $\text{Cliff}(X) = 4$ , and  $|B| = g_{12}^3$ , then  $(n, i) = (1, 1), (2, 0)$ . If  $(n, i) = (1, 1)$ , then  $L$  is not very ample by virtue of and ([1], (4.1)). If  $(n, i) = (2, 0)$ , then we have  $L \cong K - \pi^*g_4^3 + D_7$ . Since this is a special case of  $L \cong K - \pi^*g_5^4 + D_{10}$ , we are in case (VI).

(3) If  $\deg \pi = 2$ , then we are in case (VII).

Lastly, let  $\text{Cliff}(L - D) = c + 3$ . Then we have  $d = 2n + 2$ , and hence  $h^1(L - D) = n + i + 1$ . Here we consider  $|B| = g_{2(n+i)+c+3}^{n+i}$  and may assume that  $|B|$  is base-point-free. If  $r(B) = 1$ , then  $(n, i) = (1, 0)$ ,  $|B| = g_{c+5}^1$ , and we are in case (II). Assuming that  $r(B) \geq 2$ , consider the morphism

$$\pi : X \rightarrow Y \subset \mathbf{P}^{n+i}.$$

Let  $\deg \pi = 1$ . By applying (2.2), thanks to the assumption we have  $n + i = r(B) \geq g - 2(c + 3) - 1$ , i.e.

$$n \geq g - 2c - 7 - i = r(L) - c - 4.$$

This contradicts (2). Therefore  $\deg \pi \geq 2$ . Furthermore, using (3.1) we have  $\deg \pi \leq 5$ .

Let  $r(B) = 2$ . If  $\deg \pi = 2$ , then we shall treat it with the case of  $r(B) \geq 3$ .

Let  $\deg \pi = 3$  and  $c + 7 = 3f$ . Then we have  $\deg Y = f$  ( $f \geq 3$ ). By using Castelnuovo's genus bound ([4], p. 116), we get  $g(Y) \leq (f - 1)(f - 2)/2 = (c + 1)(c + 4)/18$ . Since  $g > (c + 6)(c + 5)/2 > (c + 3)(c + 2)/2$  for any  $c \geq 0$ ,  $X$  carries a  $g_{c+2}^1(c + 2 = 3(f - 2) + 1)$  by virtue of [9]. Applying Castelnuovo's lemma ([4], p. 366, C-1), we get  $g \leq (c + 2) \times 0 + 3 \times (c + 1)(c + 4)/18 +$

$\{(c+2)-1\}(3-1) = (c^2 + 17c + 16)/6$ . Since  $(c^2 + 17c + 16)/6 < \max\{(c+6) \cdot (c+5)/2, 6c+20\}$  for any  $c \geq 0$ , this is a contradiction.

Let  $\deg \pi = 4$  and  $c+7 = 4f$  ( $f \geq 2$ ). By using Castelnuovo's genus bound ([4], p. 116), we get  $g(Y) \leq (f-1)(f-2)/2 = (c-1)(c+3)/32$ . Since  $g > (c+6)(c+5)/2 > (c+3)(c+2)/2$  for any  $c \geq 0$ ,  $X$  carries a  $g_{c+2}^1(c+2 = 4(f-1)-1)$  by virtue of [9]. Applying Castelnuovo's lemma ([4], p. 366, C-1), we get  $g \leq (c+2) \times 0 + 4 \times (c-1)(c+3)/32 + \{(c+2)-1\}(4-1) = (c^2 + 26c + 21)/8$ . Since  $(c^2 + 26c + 21)/8 < \max\{(c+6)(c+5)/2, 6c+20\}$  for any  $c \geq 0$ , this is a contradiction.

Let  $\deg \pi = 5$  and  $c+7 = 5h$ . Since  $r(B) = n+i = 2$ , we obtain  $(n, i) = (2, 0), (1, 1)$ , and we are in case (III), (IV), respectively. Let  $(n, i) = (1, 1)$ . If  $f = 2$ , then  $X$  is a 5-gonal curve and  $\deg L = 2g - 8$ . Hence  $L$  is not very ample by ([1], (4.3)). If  $f = 3$ , then  $X$  is a 5-sheeted covering of an elliptic curve and  $\deg L = 2g - 13$ . Let  $D' = K - L$ . We have  $|D' + D| = \pi^*(q_1 + q_2 + q_3)$  for some  $q_1, q_2, q_3 \in Y$ . Let  $\pi^*(q_i) = p_{i1} + p_{i2} + p_{i3} + p_{i4} + p_{i5}$  ( $i = 1, 2, 3$ ). Since  $\dim|D'| = 0$ , we may assume that  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{21} + p_{22} + p_{23} + p_{24} + p_{31} + p_{32} + p_{33}$  or  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{15} + p_{21} + p_{22} + p_{23} + p_{24} + p_{31} + p_{32}$  or  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{15} + p_{21} + p_{22} + p_{23} + p_{31} + p_{32} + p_{33}$ . Then we have  $\dim|D' + p_{15} + p_{25}| \geq \dim|\pi^*(q_1 + q_2)| \geq \dim|q_1 + q_2| = 1$  or  $\dim|D' + p_{25} + p_{33}| \geq 1$  or  $\dim|D' + p_{24} + p_{25}| \geq 1$ , respectively. Thus  $\dim|D' + p + p'| \geq 1$  for a suitable choice  $p, p'$  from  $D$ . This implies that  $L$  is not very ample. Hence  $f \geq 4$  in case (IV).

If  $r(B) \geq 3$ , then we consider only three cases by virtue of ([3], (1.4)).

(1) If  $X$  is a trigonal curve,  $\text{Cliff}(X) = 1$ , and  $|B| = g_{12}^4$ , then  $(n, i) = (1, 3), (2, 2), (3, 1), (4, 0)$ . If  $(n, i) = (1, 3), (2, 2), (3, 1)$ , then  $L$  is not very ample by virtue of ([7], (2.7)). If  $(n, i) = (4, 0)$ , then we are in case (V).

(2) If  $X$  is a triple covering of an elliptic curve,  $\text{Cliff}(X) = 4$ , and  $|B| = g_{15}^4$ , then  $(n, i) = (1, 3), (2, 2), (3, 1), (4, 0)$ . If  $(n, i) = (1, 3), (2, 2), (3, 1)$ , then  $L$  is not very ample by virtue of (2.3) and ([1], (4.1)). If  $(n, i) = (4, 0)$ , then we are in case (VI).

(3) If  $\deg \pi = 2$ , then  $c+3 = 2f$  is even. Let  $r = r(B) \geq 2$  and  $m = [(r-1+f)/(r-1)]$ . Then  $m \leq (r-1+f)/(r-1) < m+1$ . Hence we have  $f/m < r-1$ . By using Castelnuovo's genus bound ([4], p. 116) and the inequality above, we get  $g(Y) \leq m(m-1)(r-1)/2 + m\{(r-1) + f - m(r-1)\} = -m(m-1)(r-1)/2 + mf < -m(m-1)(f/m)/2 + mf = f(1+m)/2 \leq f(1+(1+f))/2 = (c+7)(c+3)/8$ . On the other hand, since  $g > (c+6)(c+5)/2 > (c+3)(c+2)/2$  for any  $c \geq 0$ ,  $X$  carries a  $g_{c+2}^1$  by virtue of [9]. Applying Castelnuovo's lemma ([4], p. 366, C-1), we get  $g \leq (c+2) \times 0 + 2 \times (c+7) \cdot$

$(c + 3)/8 + \{(c + 2) - 1\}(2 - 1) = (c^2 + 14c + 25)/4$ . Since  $(c^2 + 14c + 25)/4 < \max\{(c + 6)(c + 5)/2, 6c + 20\}$  for any  $c \geq 0$ , this is a contradiction.  $\square$

REMARK. If  $L$  is in case (I), (II), (III), (IV), then thanks to (2.1) we can prove that  $L$  fails to be normally generated.

If  $L$  is in case (V), then  $L$  fails to be normally generated by ([10], Cor. 1).

PROOF OF THEOREM 1.6. Assume that  $L$  is not normally generated. Since  $L$  is very ample and  $g > (c + 7)(c + 6)/2$ , it follows from (2.2) that

$$r(L) = g - i - c - 4 \geq g - 2(c + 4) - 1,$$

i.e. that  $c + 4 - i \geq -1$ . By applying (2.1) for  $k = 2i + c + 5$  and  $e = c + 4 - i$ , there exists an integer

$$1 \leq n \leq r(L) - 2 - e - h^1(L) = r(L) - c - 6 \tag{3}$$

and an effective divisor  $D$  on  $X$  of degree  $d \geq 2n + 2$  which spans an  $n$ -plane in  $\mathbf{P}(H^0(L))$ . Since  $n = r(L) - h^0(L - D)$ , we get  $h^0(L - D) = r(L) - n \geq c + 6 \geq 6$  and  $h^1(L - D) = i + d - n - 1 \geq i + n + 1 \geq 2$ . Therefore  $L - D$  contributes to  $\text{Cliff}(X)$ , and so  $\text{Cliff}(X) \leq \text{Cliff}(L - D)$ . On the other hand, thanks to  $\text{deg } D \geq 2n + 2$ , we have  $\text{Cliff}(L - D) \leq \text{Cliff}(L) = c + 4$ . Therefore we get  $\text{Cliff}(L - D) = c, c + 1, c + 2, c + 3, c + 4$ .

First, let  $\text{Cliff}(L - D) = c$ . Then we have  $d = 2n + 6$ , and hence  $h^1(L - D) = n + i + 5$ . Let

$$B = K - L + D.$$

Now we consider  $|B| = g_{2(n+i+4)+c}^{n+i+4}$ . Since  $\text{Cliff}(X) = \text{Cliff}(B)$ ,  $|B|$  is base-point-free, and so we can consider the morphism

$$\pi : X \rightarrow Y \subset \mathbf{P}^{n+i+4}.$$

By using (3.1), we get  $\text{deg } \pi \leq 2$ .

We assume that  $\text{deg } \pi = 1$ . By applying (2.2), thanks to the assumption we get  $n + i + 4 = r(B) \geq g - 2c - 1$ , i.e.

$$n \geq g - 2c - 5 - i = r(L) - c - 1.$$

This contradicts (3). Therefore  $\text{deg } \pi = 2$ . Since  $r(B) = n + i + 4 \geq 5$ , we get  $g(Y) \leq 1$  by ([8], p. 113), and we are in case (VII).

Secondly, let  $\text{Cliff}(L - D) = c + 1$ . Then we have  $d = 2n + 5$ , and hence  $h^1(L - D) = n + i + 4$ . Now we consider  $|B| = g_{2(n+i+3)+c+1}^{n+i+3}$ . If  $|B|$  has a base

point, then we revert to the case of  $\text{Cliff}(B) = \text{Cliff}(X)$ . Therefore we may assume that  $|B|$  is base-point-free. We consider the morphism

$$\pi : X \rightarrow Y \subset \mathbf{P}^{n+i+3}.$$

Let  $\deg \pi = 1$ . By applying (2.2), thanks to the assumption we get  $n + i + 3 = r(B) \geq g - 2(c + 1) - 1$ , i.e.

$$n \geq g - 2c - 6 - i = r(L) - c - 2.$$

This contradicts (3). Hence we have  $\deg \pi \geq 2$ . On the other hand, since  $r(B) \geq 4$ ,  $|B|$  must be simple by virtue of ([2], (3.1)). It is a contradiction.

Thirdly, let  $\text{Cliff}(L - D) = c + 2$ . Then we have  $d = 2n + 4$ , and hence  $h^1(L - D) = n + i + 3$ . Here we consider  $|B| = g_{2(n+i+2)+c+2}^{n+i+2}$  and may assume that  $|B|$  is base-point-free. Since  $r(B) \geq 3$ , we consider the morphism

$$\pi : X \rightarrow Y \subset \mathbf{P}^{n+i+2}.$$

Let  $\deg \pi = 1$ . By applying (2.2), thanks to the assumption we have  $n + i + 2 = r(B) \geq g - 2(c + 2) - 1$ , i.e.

$$n \geq g - 2c - 7 - i = r(L) - c - 3.$$

This contradicts (3). Therefore  $\deg \pi \geq 2$ . Furthermore, since  $r(B) \geq 3$ , we consider only three cases by virtue of ([5], (2.3)).

(1) If  $X$  is a trigonal curve,  $\text{Cliff}(X) = 1$ , and  $|B| = g_9^3$ , then  $(n, i) = (1, 0)$  and  $L \cong K - 3g_3^1 + D_6$ . Since this is a special case of  $L \cong K - 5g_3^1 + D_{12}$ , we are in case (V).

(2) If  $X$  is a triple covering of an elliptic curve,  $\text{Cliff}(X) = 4$ , and  $|B| = g_{12}^3$ , then  $(n, i) = (1, 0)$  and we have  $L \cong K - \pi^*g_4^3 + D_6$ . Since this is a special case of  $L \cong K - \pi^*g_6^5 + D_{12}$ , we are in case (VI).

(3) If  $\deg \pi = 2$ , then we are in case (VII).

Fourthly, let  $\text{Cliff}(L - D) = c + 3$ . Then we have  $d = 2n + 3$ , and hence  $h^1(L - D) = n + i + 2$ . Here we consider  $|B| = g_{2(n+i+1)+c+3}^{n+i+1}$  and may assume that  $|B|$  is base-point-free. Since  $r(B) \geq 2$ , we consider the morphism

$$\pi : X \rightarrow Y \subset \mathbf{P}^{n+i+1}.$$

Let  $\deg \pi = 1$ . By applying (2.2), thanks to the assumption we have  $n + i + 1 = r(B) \geq g - 2(c + 3) - 1$ , i.e.

$$n \geq g - 2c - 8 - i = r(L) - c - 4.$$

This contradicts (3). Therefore  $\deg \pi \geq 2$ . Furthermore, using (3.1) we have  $\deg \pi \leq 5$ .

Let  $r(B) = 2$ . If  $\deg \pi = 2$ , we shall treat it with  $r(B) \geq 3$ .

Let  $\deg \pi = 3$  and  $c + 7 = 3f$ . Then we have  $\deg Y = f$  ( $f \geq 3$ ). By using the same way as we have done in the case of  $\text{Cliff}(L - D) = c + 3$  of the proof of (1.5), we get  $g \leq (c^2 + 17c + 16)/6$ . But  $(c^2 + 17c + 16)/6 < \max\{(c + 7)(c + 6)/2, 6c + 26\}$  for any  $c \geq 0$ . It is a contradiction.

Let  $\deg \pi = 4$  and  $c + 7 = 4f$  ( $f \geq 2$ ). Since we can use the same way as we have done in the case of  $\text{Cliff}(L - D) = c + 3$  of the proof of (1.5), we get  $g \leq (c^2 + 26c + 21)/8$ . Since  $(c^2 + 26c + 21)/8 < \max\{(c + 7)(c + 6)/2, 6c + 26\}$  for any  $c \geq 0$ , this is a contradiction.

Let  $\deg \pi = 5$  and  $c + 7 = 5h$ . Since  $r(B) = n + i + 1 = 2$ , we obtain  $(n, i) = (1, 0)$ , and we are in case (I).

If  $r(B) \geq 3$ , then we consider only three cases by virtue of ([3], (1.4)).

(1) If  $X$  is a trigonal curve,  $\text{Cliff}(X) = 1$ , and  $|B| = g_{12}^4$ , then  $(n, i) = (1, 2), (2, 1), (3, 0)$ . If  $(n, i) = (1, 2), (2, 1)$ , then  $L$  is not very ample by virtue of ([7], (2.7)). If  $(n, i) = (3, 0)$ , then  $L \cong K - 4g_3^1 + D_9$ . Since this is a special case of  $L \cong K - 5g_3^1 + D_{12}$ , we are in case (V).

(2) If  $X$  is a triple covering of an elliptic curve,  $\text{Cliff}(X) = 4$ , and  $|B| = g_{15}^4$ , then  $(n, i) = (1, 2), (2, 1), (3, 0)$ . If  $(n, i) = (1, 2), (2, 1)$ , then  $L$  is not very ample by virtue of (2.3) and ([1], (4.1)). If  $(n, i) = (3, 0)$ , then  $L \cong K - \pi^*g_5^4 + D_9$ . Since this a special case of  $L \cong K - \pi^*g_6^5 + D_{12}$ , we are in case (VI).

(3) Let  $\deg \pi = 2$  and  $r(B) \geq 2$ . Then we can repeat the same way as we have done in the case of  $\text{Cliff}(L - D) = c + 3$  of the proof of (1.5). Hence we get  $g \leq (c^2 + 14c + 25)/4$ . Since  $(c^2 + 14c + 25)/4 < \max\{(c + 7)(c + 6)/2, 6c + 26\}$  for any  $c \geq 0$ , this is a contradiction.

Lastly, let  $\text{Cliff}(L - D) = c + 4$ . Then we have  $d = 2n + 2$ , and hence  $h^1(L - D) = n + i + 1$ . Here we consider  $|B| = g_{2(n+i)+c+4}^{n+i}$  and may assume that  $|B|$  is base-point-free. If  $r(B) = 1$ , then  $(n, i) = (1, 0)$ ,  $|B| = g_{c+6}^1$ , and we are in case (II). Assuming that  $r(B) \geq 2$ , consider the morphism

$$\pi : X \rightarrow Y \subset \mathbf{P}^{n+i}.$$

Let  $\deg \pi = 1$ . By applying (2.2), thanks to the assumption we have  $n + i = r(B) \geq g - 2(c + 4) - 1$ , i.e.

$$n \geq g - 2c - 9 - i = r(L) - c - 5.$$

This contradicts (3). Therefore  $\deg \pi \geq 2$ . Furthermore, using (3.1) we have  $\deg \pi \leq 6$ .

Let  $r(B) = 2$ . If  $\deg \pi = 2$ , then we are in case (VII).

Let  $\deg \pi = 3$  and  $c + 8 = 3f$ . Then we have  $\deg Y = f$  ( $f \geq 3$ ). Since  $r(B) = n + i = 2$ , we obtain  $(n, i) = (2, 0), (1, 1)$ , and we are in case (X), (XI), respectively. Let  $(n, i) = (1, 1)$ . If  $c = 1$ , then  $X$  is a trigonal curve. Hence  $L$  is not very ample by virtue of ([7], (2.7)). If  $c = 4$ , then  $X$  is a triple covering of an elliptic curve. Hence  $L$  is not very ample by virtue of ([1], (4.1)). Therefore  $f \geq 5$  in case (XI).

Let  $\pi$  be a simple 4-sheeted covering and  $c + 8 = 4f$  ( $f \geq 2$ ). By using Castelnuovo's genus bound ([4], p. 116), we get  $g(Y) \leq (f - 1)(f - 2)/2 = c(c + 4)/32$ . If  $c = 0$ , then  $X$  is a hyperelliptic curve and we are in case (VII). We assume  $c > 0$ . Since  $g > (c + 7)(c + 6)/2 > (c + 3)(c + 2)/2$  for any  $c > 0$ ,  $X$  carries a  $g_{c+2}^1(c + 2 = 4(f - 1) - 2)$  by virtue of [9]. Applying Castelnuovo's lemma ([4], p. 366, C-1), we get  $g \leq (c + 2) \times 0 + 4 \times c(c + 4)/32 + \{(c + 2) - 1\} \cdot (4 - 1) = (c^2 + 28c + 24)/8$ . Since  $(c^2 + 28c + 24)/16 < \max\{(c + 7)(c + 6)/2, 6c + 26\}$  for any  $c > 0$ , this is a contradiction.

Let  $\deg \pi = 5$  and  $c + 8 = 5(f \geq 2)$ . By using Castelnuovo's genus bound ([4], p. 116), we get  $g(Y) \leq (f - 1)(f - 2)/2 = (c + 3)(c - 2)/50$ . Since  $g > (c + 7)(c + 6)/2 > (c + 3)(c + 2)/2$  for any  $c \geq 0$ ,  $X$  carries a  $g_{c+2}^1(c + 2 = 5(f - 1) - 1)$  by virtue of [9]. Applying Castelnuovo's lemma ([4], p. 366, C-1), we get  $g \leq (c + 2) \times 0 + 5 \times (c + 3)(c - 2)/50 + \{(c + 2) - 1\}(5 - 1) = (c^2 + 41c + 34)/10$ . Since  $(c^2 + 41c + 34)/10 < \max\{(c + 7)(c + 6)/2, 6c + 26\}$  for any  $c \geq 0$ , this is a contradiction.

Let  $\deg \pi = 6$  and  $c + 8 = 6f$ . Since  $r(B) = n + i = 2$ , we obtain  $(n, i) = (2, 0), (1, 1)$ , and we are in case (III), (IV), respectively. Let  $(n, i) = (1, 1)$ . If  $f = 2$ , then  $X$  is a 6-gonal curve and  $\deg L = 2g - 10$ . Let  $D' = K - L$ . We have  $|D' + D| = \pi^*(q_1 + q_2)$  for some  $q_1, q_2 \in Y$ . Let  $\pi^*(q_i) = p_{i1} + p_{i2} + p_{i3} + p_{i4} + p_{i5} + p_{i6}$  ( $i = 1, 2$ ). Since  $\dim|D'| = 0$ , we may assume that  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{15} + p_{21} + p_{22} + p_{23}$  or  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{21} + p_{22} + p_{23} + p_{24}$ . Then we have  $\dim|D' + p_{16} + p_{24}| = 1$  or  $\dim|D' + p_{15} + p_{16}| = 1$ , respectively. Thus  $\dim|D' + p + p'| = 1$  for a suitable choice  $p, p'$  from  $D$ . This implies that  $L$  is not very ample. If  $f = 3$ , then  $X$  is a 6-sheeted covering of an elliptic curve and  $\deg L = 2g - 16$ . Let  $D' = K - L$ . We have  $|D' + D| = \pi^*(q_1 + q_2 + q_3)$  for some  $q_1, q_2, q_3 \in Y$ . Let  $\pi^*(q_i) = p_{i1} + p_{i2} + p_{i3} + p_{i4} + p_{i5} + p_{i6}$  ( $i = 1, 2, 3$ ). Since  $\dim|D'| = 0$ , we may assume that  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{15} + p_{21} + p_{22} + p_{23} + p_{24} + p_{25} + p_{31} + p_{32} + p_{33} + p_{34}$  or  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{15} + p_{16} + p_{21} + p_{22} + p_{23} + p_{24} + p_{25} + p_{31} + p_{32} + p_{33}$  or  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{15} + p_{16} + p_{21} + p_{22} + p_{23} + p_{24} + p_{31} + p_{32} + p_{33} + p_{34}$ . Then we have  $\dim|D' + p_{16} + p_{26}| \geq \dim|\pi^*(q_1 + q_2)| \geq \dim|q_1 + q_2| = 1$  or  $\dim|D' + p_{26} + p_{34}|$



$\geq 1$  or  $\dim|D' + p_{25} + p_{26}| \geq 1$ , respectively. Thus  $\dim|D' + p + p'| \geq 1$  for a suitable choice  $p, p'$  from  $D$ . This implies that  $L$  is not very ample. Hence  $f \geq 4$  in case (IV).

If  $r(B) \geq 3$ , then we consider only five cases by virtue of ([3], (1.5)).

(1) If  $X$  is a 4-gonal curve,  $\text{Cliff}(X) = 2$ , and  $|B| = g_{12}^3$ , then  $(n, i) = (1, 2), (2, 1), (3, 0)$ . Let  $(n, i) = (1, 2)$  and  $D' = K - L$ . We have  $|D' + D| = \pi^*(q_1 + q_2 + q_3)$  for some  $q_1, q_2, q_3 \in Y$ . Let  $\pi^*(q_i) = p_{i1} + p_{i2} + p_{i3} + p_{i4}$  ( $i = 1, 2, 3$ ). Since  $\dim|D'| = 1$ , we may assume that  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{21} + p_{22} + p_{23} + p_{31}$  or  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{21} + p_{22} + p_{31} + p_{32}$ . Then we have  $\dim|D' + p_{24} + p_{32}| = 2$  or  $\dim|D' + p_{23} + p_{24}| = 2$ , respectively. Thus  $\dim|D' + p + p'| = 2$  for a suitable choice  $p, p'$  from  $D$ . This implies that  $L$  is not very ample. If  $(n, i) = (2, 1)$ , then  $\deg L = 2g - 8$ . Hence  $L$  is not very ample by virtue of ([1], (4.3)). If  $(n, i) = (3, 0)$ , then we are in case (VIII).

(2) If  $X$  is a 4-sheeted covering of an elliptic curve,  $\text{Cliff}(X) = 6$ , and  $|B| = g_{16}^3$ , then  $(n, i) = (1, 2), (2, 1), (3, 0)$ . Let  $(n, i) = (1, 2)$  and  $D' = K - L$ . We have  $|D' + D| = \pi^*(q_1 + q_2 + q_3 + q_4)$  for some  $q_1, q_2, q_3, q_4 \in Y$ . Let  $\pi^*(q_i) = p_{i1} + p_{i2} + p_{i3} + p_{i4}$  ( $i = 1, 2, 3, 4$ ). Since  $\dim|D'| = 1$ , we may assume that  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{21} + p_{22} + p_{23} + p_{24} + p_{31} + p_{32} + p_{33} + p_{41}$  or  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{21} + p_{22} + p_{23} + p_{24} + p_{31} + p_{32} + p_{41} + p_{42}$ . Then we have  $\dim|D' + p_{34} + p_{42}| \geq \dim|\pi^*(q_1 + q_2 + q_3)| \geq \dim|q_1 + q_2 + q_3| \geq 2$  or  $\dim|D' + p_{33} + p_{34}| \geq 2$ , respectively. Thus  $\dim|D' + p + p'| \geq 2$  for a suitable choice  $p, p'$  from  $D$ . This implies that  $L$  is not very ample. If  $(n, i) = (2, 1)$ , then  $\dim|D'| = 0$ , and we may assume that  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{21} + p_{22} + p_{23} + p_{31} + p_{32} + p_{33}$  or  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{21} + p_{22} + p_{23} + p_{31} + p_{32} + p_{41}$  or  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{21} + p_{22} + p_{31} + p_{32} + p_{41} + p_{42}$  or  $D' = p_{11} + p_{12} + p_{13} + p_{14} + p_{21} + p_{22} + p_{23} + p_{31} + p_{32} + p_{33} + p_{41}$  or  $D' = p_{11} + p_{12} + p_{13} + p_{21} + p_{22} + p_{23} + p_{31} + p_{32} + p_{41} + p_{42}$ . Then we have  $\dim|D' + p_{24} + p_{41}| \geq \dim|\pi^*(q_1 + q_2)| \geq \dim|q_1 + q_2| \geq 1$  or  $\dim|D' + p_{24} + p_{33}| \geq 1$  or  $\dim|D' + p_{23} + p_{24}| \geq 1$  or  $\dim|D' + p_{14} + p_{24}| \geq 1$  or  $\dim|D' + p_{14} + p_{24}| \geq 1$ , respectively. Thus  $\dim|D' + p + p'| \geq 1$  for a suitable choice  $p, p'$  from  $D$ . This implies that  $L$  is not very ample. If  $(n, i) = (3, 0)$ , then we are in case (IX).

(3) If  $X$  is a trigonal curve,  $\text{Cliff}(X) = 1$ , and  $|B| = g_{15}^5$ , then  $(n, i) = (1, 4), (2, 3), (3, 2), (4, 1), (5, 0)$ . If  $(n, i) = (1, 4), (2, 3), (3, 2), (4, 1)$ , then  $L$  is not very ample by virtue of ([7], (2.7)). If  $(n, i) = (5, 0)$ , then we are in case (V).

(4) If  $X$  is a triple covering of an elliptic curve,  $\text{Cliff}(X) = 4$ , and  $|B| = g_{18}^5$ , then  $(n, i) = (1, 4), (2, 3), (3, 2), (4, 1), (5, 0)$ . If  $(n, i) = (1, 4), (2, 3), (3, 2), (4, 1)$ , then  $L$  is not very ample by virtue of (2.3) and ([1], (4.1)). If  $(n, i) = (5, 0)$ , then we are in case (VI).

(5) If  $\deg \pi = 2$ , then we are in case (VII).  $\square$

REMARK. If  $L$  is in case (I), (II), (III), (IV), (X), (XI), then  $L$  fails to be normally generated by virtue of (2.1).

If  $L$  is in case (V), then  $L$  fails to be normally generated by ([10], Cor. 1).

### References

- [ 1 ] Akahori, K., Remarks on normal generation of line bundles on algebraic curves, *Abh. Math. Sem. Univ. Hamburg* **74** (2004), 181–191.
- [ 2 ] Akahori, K., Normal generation of line bundles of degree  $2g - 1 - 2h^1(L) - \text{Cliff}(X)$  on curves, *Abh. Math. Sem. Univ. Hamburg* **75** (2005), 77–82.
- [ 3 ] Akahori, K., Remarks on the Clifford index of algebraic curves, preprint.
- [ 4 ] Arbarello, E., Cornalba, M., Griffiths, P. A., Harris, J., *Geometry of algebraic curves, Vol. I.* *Grundle. der math. Wiss.* **267**, Berlin-Heidelberg-New York 1985.
- [ 5 ] Ballico, E., and Keem, C., A remark on the Clifford index and higher order Clifford indices, *J. Korean Math. Soc.* **28** (1991), No. 1, 37–42.
- [ 6 ] Green, M., and Lazarsfeld, R., On the projective normality of complete linear series on an algebraic curve, *Invent. Math.* **83** (1986), 73–90.
- [ 7 ] Kato, T., Keem, C., and Ohbuchi, A., Normal generation of line bundles of high degrees on smooth algebraic curves, *Abh. Math. Sem. Univ. Hamburg* **69** (1999), 319–333.
- [ 8 ] Keem, C., Kim, S., and Martens, G., On a result of Farkas, *J. reine angew. Math.* **405** (1990), 112–116.
- [ 9 ] Martens, G., Funktionen von vorgegebener Ordnung auf komplexen Kurven, *J. reine angew. Math.* **320** (1980), 68–85.
- [10] Martens, G., and Schreyer, F.-O., Line bundle and syzygies of trigonal curves, *Abh. Math. Sem. Univ. Hamburg* **56** (1986), 169–189.

Department of Mathematics  
Kyushu University  
6-10-1 Hakozaki Higasi-ku  
Fukuoka, 812 Japan