UPPER BOUND FOR SUM OF DIVISORS FUNCTION AND THE RIEMANN HYPOTHESIS

By

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Abstract. Let $\sigma(n)$ denote the sum of divisors function. We prove that if $(2, m) = 1$ and $2m > 3^9$ then 1^0 $\sigma(2m) < \frac{39}{40} e^{\gamma} 2m \log \log 2m$, and for all odd integers $m > \frac{3^9}{2}$, we have 2^0 $\sigma(m) < e^{\gamma} m \log \log m$. Moreover, we show that if $\sigma(2m) < \frac{3}{4} e^{\gamma} 2m \log \log 2m$, for $m > m_0$ and $(2, m) = 1$, then the inequality $\sigma(2^{\alpha} m) < e^{\gamma} 2^{\alpha} m \log \log 2^{\alpha} m$ is true for all integers $\alpha \geq 2$ and $m > m_0$. Robin criterion implies that the Riemann hypothesis is true for these cases.

1. Introduction

The Riemann zeta function $\zeta(s)$ for $s = \sigma + it$ is defined by Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

which converges for $\sigma > 1$, and it has analytic continuation to the complex plane with one singularity, a simple pole with residue equal to 1, at $s = 1$.

In 1859 Riemann [10] stated conjecture which concerns the complex zeros of the Riemann zeta function. Namely, the Riemann hypothesis states that the nonreal zeros of the Riemann zeta function $\zeta(s)$ all lie on the line $\sigma = \frac{1}{2}$. The connection of the Riemann hypothesis with prime numbers has been considered by Gauss.

Let

$$\pi(x) = \sum_{1 < p \leq x} 1,$$

then it is well-known that the Riemann hypothesis is equivalent to the assertion that for each $\varepsilon > 0$ there is a positive constant $C = C(\varepsilon)$ such that

$$|\pi(x) - Li(x)| \leq C(\varepsilon)x^{1/2+\varepsilon},$$

where

$$Li(x) = \int_2^x \frac{dt}{\log t}.$$

The Riemann hypothesis is a special case of questions concerning generalizations of L-functions and their connections with many important and difficult problems in number theory, algebraic geometry, topology, representation theory and modern physics (see; Berry and Keating [1], Katz and Sarnak [6], Murty [8]). In 1984 Robin [11] proved the following criterion connected with sum divisors function.

ROBIN'S CRITERION.   *The Riemann hypothesis is true if and only if*

(*) $\sigma(n) < e^\gamma n \log \log n$

for all positive integers $n \geq 5041$, where

$$\sigma(n) = \sum_{d|n} d, \quad and \quad \gamma \approx 0.57721 \ is \ Euler's \ constant.$$

In 2002 Lagarias [7] proved the following criterion.

LAGARIAS' CRITERION.   *Let $H_n = \sum_{j=1}^n \frac{1}{j}$. The Riemann hypothesis is true if and only if*

(L) $\sigma(n) \leq H_n + \exp(H_n) \log(H_n),$

for each positive integer $n \geq 1$, and with equality in (L) *only for $n = 1$.*

In this paper Lagarias noted that for all positive integers $n \geq 3$ we have

(L$_1$) $e^\gamma n \log \log n \leq \exp(H_n) \log(H_n),$

and therefore Lagarias' citerion is an extension Robin's criterion.

Many others criterions and interesting results connected with the Riemann hypothesis are described by Conrey in his elegant article [2].

We also note that it is well known that the Riemann hypothesis is related to estimates of error terms associated with the Farey sequence of reduced fractions in the unit interval. Interesting and important results in this direction has been given by Yoshimoto ([13], [14], [15]) and Kanemitsu and Yoshiomoto ([4], [5]).

2. Basic Lemmas

In the proofs of our results we use two following Lemmas:

LEMMA 1 (Rosser-Schoenfeld's inequality [12], Cf. [9], p. 169). *Let $\varphi(n)$ be the Euler's totient function. Then for all positive integers $n \geq 3$ the following inequality is true*

$$(**) \qquad \frac{n}{\varphi(n)} \leq e^{\gamma}\left(\log\log n + \frac{2.5}{e^{\gamma}\log\log n}\right)$$

except, when $n = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 = 223092870$, and in this case the constant $c = 2.5$ must be replaced by the constant $c_1 = 2.50637 < 2.51$.

LEMMA 2. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_j are primes and $\alpha_j \geq 1$ for $j = 1, 2, \ldots, k$ and let σ and φ be the sum divisors function and Euler's totient function, respectively. Then we have*

$$(***) \qquad \frac{\sigma(n)}{n} = \prod_{j=1}^{k}\left(1 - \frac{1}{p_j^{1+\alpha_j}}\right)\frac{n}{\varphi(n)}.$$

PROOF. Since $n = \prod_{j=1}^{k} p_j^{\alpha_j}$ then we have

$$(2.1) \qquad \sigma(n) = \prod_{j=1}^{k}\frac{p_j^{\alpha_j+1}-1}{p_j-1} = \frac{n\prod_{j=1}^{k}\left(1-\frac{1}{p_j^{\alpha_j+1}}\right)}{\prod_{j=1}^{k}\left(1-\frac{1}{p_j}\right)}.$$

On the other hand we have

$$(2.2) \qquad \varphi(n) = n\prod_{j=1}^{k}\left(1-\frac{1}{p_j}\right).$$

From (2.1) and (2.2) we obtain

$$\frac{\sigma(n)}{n} = \prod_{j=1}^{k}\left(1 - \frac{1}{p_j^{\alpha_j+1}}\right)\frac{n}{\varphi(n)},$$

and the proof of Lemma 2 is complete. ∎

3. The Results

First we prove of the following theorem.

THEOREM 1. *Let* $n = 2m$, $(2, m) = 1$ *and* $m = \prod_{j=1}^{k} p_j^{\alpha_j}$. *Then for all odd positive integers* $m > \frac{3^9}{2}$ *we have*

$$(3.1) \qquad\qquad \sigma(2m) < \frac{39}{40} e^{\gamma} 2m \log \log 2m$$

and

$$(3.2) \qquad\qquad \sigma(m) < e^{\gamma} m \log \log m.$$

PROOF. First we note that by (**) of Lemma 1 it follows that for all positive integers $n \geq 3$ we have

$$(3.3) \qquad\qquad \frac{n}{\varphi(n)} < e^{\gamma} \log \log n \left(1 + \frac{2.51}{e^{\gamma} (\log \log n)^2} \right).$$

Now, we remark that for all positive integers $n \geq 3^9 = 19683$ we have

$$(3.4) \qquad (\log \log n)^2 \geq (\log(9 \log 3))^2 > (\log 9.81)^2 > (2.28)^2 > 5.19.$$

Since $e^{\gamma} > e^{0.57} > \sqrt{e} > 1.64$ then

$$(3.5) \qquad\qquad e^{\gamma} (\log \log n)^2 > (1.64) x (5.19) = 8.5116 > 8.51$$

By (3.3), (3.4) and (3.5) it follows that for all $n \geq 3^9$ we have

$$(3.6) \qquad\qquad \frac{n}{\varphi(n)} < e^{\gamma} \log \log n \left(1 + \frac{2.51}{8.51} \right) < 1.3 e^{\gamma} \log \log n.$$

Let $n = 2m$, where $(2, m) = 1$ and let $m > \frac{3^9}{2} > 9841$. Then by (***) of Lemma 2 and (3.6) it follows that

$$(3.7) \qquad \frac{\sigma(2m)}{2m} = \left(1 - \frac{1}{2^2} \right) I(m) \frac{2m}{\varphi(2m)} < \frac{3}{4} \times \frac{13}{10} e^{\gamma} \log \log 2m = \frac{39}{40} e^{\gamma} \log \log 2m,$$

because $I(m) = \prod_{j=1}^{k} \left(1 - \frac{1}{p_j^{1+\alpha_j}} \right) < 1$.

Hence, the inequality (3.7) states that (3.1) is true for all even positive integers $n = 2m > 3^9$, such that $(2, m) = 1$.

From (3.7) and $\sigma(2m) = \sigma(2)\sigma(m) = 3\sigma(m)$, when $(2, m) = 1$ for $m > \frac{3^9}{2}$, we obtain

$$(3.8) \qquad \sigma(m) = \frac{1}{3}\sigma(2m) < \frac{1}{3} \times \frac{39}{40} e^{\gamma} 2m \log \log 2m = \frac{13}{20} e^{\gamma} m \log \log 2m.$$

It is easy to see that for all odd positive integers $m > \frac{3^9}{2}$ we have

$$(3.9) \qquad \frac{13}{20} \log \log 2m < \log \log m.$$

From (3.8) and (3.9) follows that (3.2) is true for all odd positive integers $m > \frac{3^9}{2} > 9841$ and the proof of the Theorem 1 is complete.   ■

THEOREM 2.   *If for odd positive integers $m > m_0$, we have*

$$(3.10) \qquad \sigma(2m) < \frac{3}{4} e^{\gamma} 2m \log \log 2m,$$

then for all integers $n = 2^{\alpha}m$, $(2, m) = 1$, $m > m_0$ and each integer $\alpha \geq 2$, we have

$$(3.11) \qquad \sigma(2^{\alpha}m) < e^{\gamma} 2^{\alpha}m \log \log 2^{\alpha}m.$$

PROOF.   By well-known property of the sum divisors function it follows that

$$(3.12) \qquad \sigma(2^{\alpha}m) = \sigma(2^{\alpha})\sigma(m) = (2^{\alpha+1} - 1)\sigma(m),$$

and

$$(3.13) \qquad \sigma(2m) = \sigma(2)\sigma(m) = 3\sigma(m).$$

From (3.12) and (3.13) we obtain

$$(3.14) \qquad \sigma(2^{\alpha}m) = \frac{2^{\alpha+1} - 1}{3}\sigma(2m).$$

By the assumption (3.10) and (3.14) it follows that

$$\sigma(2^{\alpha}m) < \frac{2^{\alpha+1} - 1}{3}\frac{3}{4} e^{\gamma} 2m \log \log 2m < e^{\gamma} 2^{\alpha}m \log \log 2^{\alpha}m,$$

and the proof of the Theorem 2 is complete.   ■

Now, we remark that if m is greater than m in the assumption of the Theorem 1, then we obtain better upper bound than (3.1) for $\sigma(2m)$.

Namely, we have of the following theorem.

THEOREM 3. *If* $(2, m) = 1$ *and* $n = 2m$ *then for all odd positive integers* $m > \frac{1}{2}e^{e^7}$ *we have*

(3.15) $$\sigma(n) = \sigma(2m) < \frac{39}{50}e^{\gamma}2m \log \log 2m.$$

PROOF.   Since $2m > e^{e^7}$ then $\log \log 2m > 7$ and since $e^{\gamma} > 1.64$ then we get

(3.16) $$e^{\gamma}(\log \log 2m)^2 > 80.36$$

By (3.16) it follows that

(3.17) $$\frac{2.51}{e^{\gamma}(\log \log 2m)^2} < \frac{2.51}{80.36} < 0.032.$$

From (3.17) and (3.3) we obtain

(3.18) $$\frac{n}{\varphi(n)} = \frac{2m}{\varphi(2m)} < \frac{129}{125}e^{\gamma} \log \log 2m.$$

By (3.18) and Lemma 2 for $n = 2m > e^{e^7}$, with $(2, m) = 1$ it follows that

$$\frac{\sigma(2m)}{2m} < \frac{3}{4}\frac{2m}{\varphi(2m)} < \frac{387}{500}e^{\gamma} \log \log 2m < \frac{39}{50}e^{\gamma} \log \log 2m,$$

and the proof of the Theorem 3 is complete.   ■

Applying this result we can obtain corresponding upper bound for $\sigma(n)$, if $n = 2^{\alpha}m$, $m > \frac{1}{2}e^{e^7}$, $(2, m) = 1$ and when $\alpha = 2$ or 3.

THEOREM 4. *If* $m > \frac{1}{2}e^{e^7}$ *and* $(m, 2) = 1$ *then we have*

$1^0.$ $\sigma(2^2m) < \frac{91}{100}e^{\gamma}2^2m \log \log 2^2m$,
$2^0.$ $\sigma(2^3m) < \frac{39}{40}e^{\gamma}2^3m \log \log 2^3m$.

PROOF.   By the assumption that $(m, 2) = 1$ and the multiplicative property of the sum divisors function it follows that

(3.19) $$\sigma(2m) = \sigma(2)\sigma(m) = (2^2 - 1)\sigma(m) = 3\sigma(m),$$

(3.20) $$\sigma(2^2m) = \sigma(2^2)\sigma(m) = (2^3 - 1)\sigma(m) = 7\sigma(m),$$

(3.21) $$\sigma(2^3m) = \sigma(2^3)\sigma(m) = (2^4 - 1)\sigma(m) = 15\sigma(m),$$

From (3.19) and (3.20) and Theorem 3 we obtain

$$\sigma(2^2 m) = \frac{7}{3}\sigma(2m) < \frac{91}{50}e^{\gamma}2m \log\log 2m < \frac{91}{100}e^{\gamma}2^2 m \log\log 2^2 m,$$

and 1^0 is proved.

In similar way from (3.20), (3.21) and Theorem 3 we get

$$\sigma(2^3 m) = \frac{15}{7}\sigma(2^2 m) < \frac{15}{7}\frac{91}{100}e^{\gamma}2^2 m \log\log 2^2 m < \frac{39}{40}e^{\gamma}2^3 m \log\log 2^3 m,$$

and the proof of the Theorem 4 is complete. ∎

Now, we prove the following theorem.

THEOREM 5. *Let* $n = 2m$, $(2, m) = 1$, $m = \prod_{j=1}^{k} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$.
If for odd integers $m > \frac{1}{2}e^{e^9}$ *the inequality*

$$(3.22) \qquad I(m) = \prod_{j=1}^{k}\left(1 - \frac{1}{p_j^{1+\alpha_j}}\right) < \frac{50}{51},$$

is satisfied, then for all integers $2m > e^{e^9}$ *we have*

$$(3.23) \qquad \sigma(2m) < \frac{3}{4}e^{\gamma}2m \log\log 2m.$$

PROOF. Since $2m > e^{e^9}$ then we have

$$(3.24) \qquad \frac{2.51}{e^{\gamma}(\log\log 2m)^2} < 0.02 = \frac{1}{50}.$$

By (3.24) and (2.3) it follows that

$$(3.23) \qquad \frac{2m}{\varphi(2m)} < \frac{51}{50}e^{\gamma}\log\log 2m.$$

From Lemma 2 we obtain,

$$(3.24) \qquad \frac{\sigma(2m)}{2m} = \left(1 - \frac{1}{2^2}\right)I(m)\frac{2m}{\varphi(2m)} = \frac{3}{4}I(m)\frac{2m}{\varphi(2m)}.$$

Hence, (3.24), (3.23) and (3.20) implies that for $m > \frac{1}{2}e^{e^9}$, we get

$$\frac{\sigma(2m)}{2m} < \frac{3}{4}e^{\gamma}\log\log 2m,$$

and the proof of the Theorem 5 is complete. ∎

4. Remarks

REMARK 1.   If $n = 2^\alpha m$, $\alpha \geq 2$, $(2, m) = 1$, $m > \frac{1}{2} e^{e^9}$ and $I(m) < \frac{50}{51}$, then

$$(4.1) \qquad\qquad \sigma(2^\alpha m) < e^\gamma 2^\alpha m \log \log 2^\alpha m.$$

inequality (4.1) follows immediately from the Theorem 5 and Theorem 2.

REMARK 2.   Robin criterion implies that for the complete proof of the Riemann hypothesis it suffices checked by computer the inequality (*) for integers $n \in (5040, e^{e^9})$ and proved that $I(m) < \frac{50}{51}$ for odd integers $m > \frac{1}{2} e^{e^9}$.

REMARK 3.   The inequality (L) has been checked by computer for all integers $n \in [1, 5040]$. (see, [7])

REMARK 4.   Gronwall in 1913 (see [3], Thm. 323, sect. 18.3 and 22.9) proved the following result.

(G) The divisor sum function $\sigma(n)$ satisfies

$$\limsup_{n \to \infty} \frac{\sigma(n)}{n \log \log n} = e^\gamma,$$

where γ is Euler's constant.

REMARK 5.   Robin proved ([11]) that for all positive integers $n \geq 3$, we have

$$(R) \qquad \sigma(n) < e^\gamma n \log \log n + 0.6482 \frac{n}{\log \log n} = e^\gamma n \log \log n \left(1 + \frac{0.6482}{e^\gamma (\log \log n)^2} \right).$$

From the inequality (R) follows that for integers $n > 3^9$, we have

$$(R_1) \qquad\qquad \sigma(n) < 1.076 e^\gamma n \log \log n.$$

Hence, upper bounds given in the Theorem 1 are better than (R_1) for $n = 2m > 3^9$, $(2, m) = 1$ and odd $m > \frac{3^9}{2}$.

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