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# PISOT SUBSTITUTIONS AND THE HAUSDORFF DIMENSION OF BOUNDARIES OF ATOMIC SURFACES

By

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**Abstract.** The atomic surface $X_\sigma$ from an unimodular Pisot substitution $\sigma$ usually has the fractal boundary and it generates a self-affine tiling. In this paper, we study the boundary $\partial X_\sigma$ as the graph directed self-affine fractal and estimate the Hausdorff dimension of the boundary.

## 0. Introduction

The several properties of self-affine tiles and their boundaries are studied for instance in the articles [26], [15], [3], [16], [9], [17], [18], [4], [27], [1], [24]. In this paper, we treat the sets which have the fractal boundary called atomic surfaces or self-affine tiles based on substitutions.

Let $\sigma$ be a primitive unimodular Pisot substitution on the free monoid $A^* = \bigcup_{n=0}^{\infty}\{1, 2, \dots, d\}^n$, that is,

**(1)** there exists an $n$ such that $i$ occurs in $\sigma^n(j)$ for any pair of letters $(i, j)$ (*primitive*);

**(2)** the characteristic polynomial of $L_\sigma$ is irreducible over $\boldsymbol{Q}$ and eigenvalues $\lambda_i$, $1 \le i \le d$ of $L_\sigma$ satisfy the followings:

$$\lambda_1 > 1 > |\lambda_i|, \quad i = 2, \dots, d \quad \text{(Pisot condition)};$$

**(3)** $\det L_\sigma = \pm 1$ (unimodular condition).

Let $\omega = (\omega_1, \omega_2, \dots)$ be the fixed point of the substitution $\sigma$ and $\pi : \boldsymbol{R}^d \to \mathscr{P}$ be the projection along the eigenvector with respect to the largest eigenvalue $\lambda_1$ of $L_\sigma$ to the contractive invariant plane $\mathscr{P}$ of $L_\sigma$. Let us define the set $X_\sigma$ by

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$$X_\sigma := \text{the closure of } \left\{\pi \sum_{k=1}^{n} \boldsymbol{e}_{\omega_k} \,|\, n = 1, 2, \dots\right\}$$

where $\boldsymbol{e}_i$, $i = 1, 2, \dots, d$ are the canonical basis of $\boldsymbol{R}^d$. The domain $X_\sigma$ called the *atomic surface* usually has a fractal boundary. This domain and its boundary are not only interesting from the viewpoint of the fractal geometry, but also ergodic theory, number theory and quasi-crystal theory (see [22], [10], [11], [19], [23], [7]). In this paper, we mainly study the boundary $\partial X_\sigma$ as the fractals which have graph self-affine in Theorem 2.6 (c.f. [5], [25]) and estimate the Hausdorff dimension of atomic surfaces as follows.

THEOREM 1. *Let $\sigma$ be a primitive unimodular Pisot substitution with $d$ letters and let $X_\sigma$ be the atomic surface based on the substitution $\sigma$. Then the Hausdorff dimension of the boundary $\partial X_\sigma$ is estimated by*

$$\dim_H \partial X_\sigma \le \frac{\log \gamma_1 - \log \lambda_1 - (d-1) \log|\lambda_d|}{-\log|\lambda_d|}$$

*where $\gamma_1$ is the largest eigenvalue of the graph matrix $M_\sigma$.*

Moreover, if the linear map $L_\sigma|_{\mathscr{P}}$ restricted to the contractive invariant plane $\mathscr{P}$ is a similitude, then the Hausdorff dimension of $\partial X_\sigma$ is given by

$$\dim_H \partial X_\sigma = \frac{(d-1) \log \gamma_1}{\log \lambda_1}.$$

## 1. Atomic Surfaces and Their Basic Properties

In this section, we give a survey of the property of the atomic surface which is discussed in [6], [2], [12]. Let $\mathscr{A}$ be an alphabet of $d$ letters $\{1, 2, \dots, d\}$. We denote $\mathscr{A}^* = \bigcup_{n=0}^{\infty} \mathscr{A}^n$ the free monoid of $\mathscr{A}$. The substitution $\sigma$ is a map from $\mathscr{A}$ to $\mathscr{A}^*$ such that $\sigma(i)$ is a non-empty word for any letter $i$. The substitution $\sigma$ naturally extends to an endomorphism of the free monoid $\mathscr{A}^*$ by the rule $\sigma(UV) = \sigma(U)\sigma(V)$. Denote $\sigma(i) = W^{(i)}$, where $W^{(i)}$ is a finite word of the length $l_i$, and we write $W^{(i)} = W_1^{(i)} \cdots W_{l_i}^{(i)}$. Denote by $P_k^{(i)}$ *the prefix* of the length $k-1$ of $W_k^{(i)}$ (for $k = 1$, this is the empty word), and $S_k^{(i)}$ *the suffix* of the length $l_i - k$, so that $\sigma(i) = P_k^{(i)} W_k^{(i)} S_k^{(i)}$. For the simplicity, we assume that $W_1^{(1)} = 1$. Under this assumption, the infinite sequence $\omega$ given by

$$\omega = \lim_{n\to\infty} \sigma^n(1)$$

is the fixed point of the substitution $\sigma$. There is a natural homomorphism $f : \mathscr{A}^* \to \boldsymbol{Z}^d$ obtained by the abeliarization of the free monoid $\mathscr{A}^*$, and we obtain a linear transformation $L_\sigma$ satisfying the commutative diagram:

$$\begin{array}{ccc} \mathscr{A}^* & \xrightarrow{\sigma} & \mathscr{A}^* \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \boldsymbol{Z}^d & \xrightarrow{L_\sigma} & \boldsymbol{Z}^d. \end{array}$$

From now on, we assume that the substitution $\sigma$ is *primitive*, that is, there exists an $n$ such that $i$ occurs in $\sigma^n(j)$ for any pair of letters $(i, j)$. It is equivalent to say that the matrix $L_\sigma$ of $\sigma$ is primitive. By Perron-Frobenius theorem, $L_\sigma$ has the largest eigenvalue $\lambda_1$ that is positive, simple and strictly bigger in modulus than the other eigenvalues. We denote $\boldsymbol{u}_\lambda$ and $\boldsymbol{v}_\lambda$ positive eigenvectors associated with $\lambda_1$ for $L_\sigma$ and the transpose of $L_\sigma$ respectively. Moreover, we assume that the substitution $\sigma$ satisfies *irreducible Pisot* and *unimodular condition*, that is,

**(1)** the characteristic polynomial of $L_\sigma$ is irreducible over $\boldsymbol{Q}$ and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_d$ of $L_\sigma$ satisfy

$$\lambda_1 > 1 > |\lambda_i|, \quad i = 2, \dots, d \quad \text{(Pisot condition);}$$

**(2)** the determinant of $L_\sigma$ is equal to $\pm 1$ (unimodular condition).

Let $\mathscr{P}$ be the plane orthogonal to $\boldsymbol{v}_\lambda$. It is clear that $\mathscr{P}$ is invariant by the linear transformation $L_\sigma$. Moreover, the linear transformation $L_\sigma$ is contractive on $\mathscr{P}$, that is, there exists a constant $0 < \lambda_0 < 1$ such that

$$d_{\mathscr{P}}(L_\sigma \boldsymbol{x}, L_\sigma \boldsymbol{y}) \le \lambda_0 d_{\mathscr{P}}(\boldsymbol{x}, \boldsymbol{y}) \quad \text{for } \boldsymbol{x}, \boldsymbol{y} \in \mathscr{P}$$

where $d_{\mathscr{P}}(\cdot\,,\cdot)$ is the restricted Euclid distance on $\mathscr{P}$. Let $\pi : \boldsymbol{R}^d \to \mathscr{P}$ be the projection along the eigenvector $\boldsymbol{u}_\lambda$.

Definition 1.1. *Let us denote the fixed point* $\omega = \lim_{n\to\infty} \sigma^n(1)$ *of* $\sigma$ *by*

$$\omega = s_1 s_2 \cdots s_n \cdots,$$

*and let us define the set* $X$ *and* $X_i$, $i = 1, 2, \dots, d$ *by*

$$X := \text{the closure of } \left\{\pi \sum_{j=1}^{k} \boldsymbol{e}_{s_j} \,|\, k = 1, 2, \dots\right\},$$

$$X_i := \text{the closure of } \left\{\pi \sum_{j=1}^{k} \boldsymbol{e}_{s_j} \,|\, s_k = i \text{ for some } k\right\}.$$

*The set* $X$ *is called the atomic surface associated with the substitution* $\sigma$.

With the notations above, we know the following theorem.

Theorem 1.2 ([2]). *Let $\sigma$ be a primitive unimodular Pisot substitution, and $X$ and $X_i$, $i = 1, 2, \dots d$ be the atomic surfaces of $\sigma$. Then $X_i$'s satisfy the following relations: for each $i = 1, \dots, d$,*

$$X_i = \sum_{j=1}^{d} \sum_{\substack{S_k^{(j)}: \\ W_k^{(j)}=i, \\ \sigma(j)=P_k^{(j)} W_k^{(j)} S_k^{(j)}}} (L_\sigma X_j - \pi f(S_k^{(j)})) \quad (\textit{non-overlap})$$

*where $\sum_{j=1}^{l} A_j$ (non-overlap) means that the Lebesgue measure $|A_j \cap A_k|$ of $A_j \cap A_k$ is equal to zero for each $1 \le j < k \le l$.*

In [2], we can see implicitly the set equation of $X_i$, $i = 1, 2, \dots, d$ holds. However, we will give an explicit proof here. For this purpose, we prepare some lemmas and propositions.

Lemma 1.3. *The set $X$ is bounded. More precisely, we can estimate*

$$\text{diam.} X \le \frac{2}{1 - \lambda_0} \cdot l \cdot m,$$

*where $L_\sigma = (l_{ij})$, $l = \max_{1 \le j \le d} \sum_{i=1}^{d} l_{ij}$, and $m = \max_{1 \le j \le d} d_{\mathscr{P}}(\mathbf{0}, \pi(f(j)))$.*

Proof. For any $k > 0$ there exists $n$ such that $l^{(n)} \le k < l^{(n+1)}$, where $l^{(n)} = |\sigma^n(1)|$ is the length of the word $\sigma^n(1)$. Therefore, there exists $j$ such that

$$s_1 \cdots s_k = \sigma^n(W_1^{(1)}) \cdots \sigma^n(W_{j-1}^{(1)}) t_1 \cdots t_{k'},$$

$$t_1 \cdots t_{k'} \prec \sigma^n(W_j^{(1)})$$

where $u_1 \cdots u_k \prec v_1 \cdots v_j$ means

$$v_1 \cdots v_j = u_1 \cdots u_k v_{k+1} \cdots v_j.$$

Therefore, we know

$$f(s_1 s_2 \cdots s_k) = f(\sigma^n(W_1^{(1)})) + \cdots + f(\sigma^n(W_{j-1}^{(1)})) + f(t_1 \cdots t_{k'}).$$

On the other hand, we know that

$$d_{\mathscr{P}}(\mathbf{0}, \pi f(\sigma^n(j))) \le \lambda_0^n d_{\mathscr{P}}(\mathbf{0}, \pi f(j))$$

where $\lambda_0 = \max_{2 \le i \le d}(|\lambda_i|)$. Therefore, we have

$$d_{\mathscr{P}}(\mathbf{0}, \pi f(s_1 \cdots s_k)) \le l \cdot \max_{1 \le j \le d} d_{\mathscr{P}}(\mathbf{0}, \pi f(j)) \lambda_0^n + d_{\mathscr{P}}(\mathbf{0}, \pi f(t_1 \cdots t_{k'}))$$

where $l = \max_{1 \le j \le d} \sum_{i=1}^{d} l_{ij}$ and $L_\sigma = (l_{ij})_{1 \le i,j \le d}$. Continue the procedure, then we get

$$\mathrm{diam}.X \le \frac{2}{1-\lambda_0} \cdot l \cdot \max_{1 \le j \le d} d_{\mathscr{P}}(\mathbf{0}, \pi f(j)). \qquad \square$$

Lemma 1.4. *The following set equation holds: for each* $i \in \{1, 2, \dots, d\}$

$$X_i = \bigcup_{j=1}^{d} \bigcup_{\substack{S_k^{(j)}: \\ W_k^{(j)} = i \\ \sigma(j) = P_k^{(j)} W_k^{(j)} S_k^{(j)}}} (L_\sigma X_j - \pi f(S_k^{(j)})).$$

Proof. It is enough to show that

$$L_\sigma^{-1} Y_i = \bigcup_{j=1}^{d} \bigcup_{\substack{S_k^{(j)}: \\ W_k^{(j)} = i, \\ \sigma(j) = P_k^{(j)} W_k^{(j)} S_k^{(j)}}} (Y_j - L_\sigma^{-1}(\pi f(S_k^{(j)})))$$

where $Y_i = \{\pi f(s_1 \cdots s_k) \mid s_k = i \text{ for some } k\}$. For any $k$ satisfying $s_k = i$, there exist $m$ and $t$ such that

$$s_1 s_2 \cdots s_k = \sigma(s_1 \cdots s_{m-1}) P_t^{(s_m)} W_t^{(s_m)},$$
$$W_t^{(s_m)} = i.$$

Therefore, we have

$$f(s_1 s_2 \cdots s_k) = f(\sigma(s_1 s_2 \cdots s_m)) - f(S_t^{(s_m)}).$$

Thus, the set equation holds. $\square$

Lemma 1.5. *Let $A$ be a $d \times d$ integer matrix and assume that the characteristic polynomial of $A$ is irreducible, then the eigenvector $\boldsymbol{u} = {}^t(1, u_1, \dots, u_{d-1})$ of the eigenvalue $\lambda$ of $A$ is $\boldsymbol{Q}$-basis of the field $\boldsymbol{Q}(\lambda)$, that is,*

**(1)** $\boldsymbol{Q} \cdot 1 + \boldsymbol{Q} \cdot u_1 + \cdots + \boldsymbol{Q} \cdot u_{d-1} = \boldsymbol{Q}(\lambda)$;

**(2)** $\{1, u_1, \dots, u_{d-1}\}$ *is $\boldsymbol{Q}$-independent.*

Proof. Let us denote the simple extension of $\boldsymbol{Q}$ adjoining $\lambda$ by $\boldsymbol{Q}(\lambda)$, then from the irreducibility of the characteristic polynomial of $A$, we see that $\{1, \lambda, \lambda^2, \dots, \lambda^{d-1}\}$ is the basis of $\boldsymbol{Q}(\lambda)$, that is,

**(1)** $\boldsymbol{Q} + \boldsymbol{Q}\lambda + \cdots + \boldsymbol{Q}\lambda^{d-1} = \boldsymbol{Q}(\lambda)$;

**(2)** $\{1, \lambda, \ldots, \lambda^{d-1}\}$ is $\boldsymbol{Q}$-independent.

On the other hand, from the definition:

$$A\,{}^t[1, u_1, \ldots, u_{d-1}] = \lambda\,{}^t[1, u_1, \ldots, u_{d-1}],$$

we see

$$\lambda = a_{11} + a_{12}u_1 + \cdots + a_{1d}u_{d-1} \tag{1.1}$$

and moreover from the fact that

$$A^k\,{}^t[1, u_1, \ldots, u_{d-1}] = \lambda^k\,{}^t[1, u_1, \ldots, u_{d-1}],$$

we have

$$\lambda^k = a_{11}^{(k)} + a_{12}^{(k)}u_1 + \cdots + a_{1d}^{(k)}u_d \tag{1.2}$$

and we see

$$\lambda^k \in \boldsymbol{Q} + \boldsymbol{Q}u_1 + \cdots + \boldsymbol{Q}u_{d-1}.$$

Therefore, we know that

$$(\boldsymbol{Q}(\lambda) =)\boldsymbol{Q} + \boldsymbol{Q}\lambda + \cdots + \boldsymbol{Q}\lambda^{d-1} \subset \boldsymbol{Q} + \boldsymbol{Q}u_1 + \cdots + \boldsymbol{Q}u_{d-1}.$$

Other direction

$$\boldsymbol{Q} + \boldsymbol{Q}\lambda + \cdots + \boldsymbol{Q}\lambda^{d-1} \supset \boldsymbol{Q} + \boldsymbol{Q}u_1 + \cdots + \boldsymbol{Q}u_{d-1}$$

is easy from the fact that

$$(A - \lambda E)\,{}^t[1, u_1, \ldots, u_{d-1}] = \mathbf{0}.$$

In fact, $\{1, u_1, \ldots, u_{d-1}\}$ is the solution of the linear equation $(A - \lambda E)\,{}^t[x_1, \ldots, x_d] = \mathbf{0}$, which is the equation with $\boldsymbol{Q}(\lambda)$-coefficient, therefore, we see $u_i \in \boldsymbol{Q}(\lambda)$. And, we have

$$\boldsymbol{Q} + \boldsymbol{Q}\lambda + \cdots + \boldsymbol{Q}\lambda^{d-1} = \boldsymbol{Q} \cdot 1 + \boldsymbol{Q} \cdot u_1 + \cdots + \boldsymbol{Q} \cdot u_{d-1},$$

that is, $\{1, u_1, \ldots, u_{d-1}\}$ is the basis of $\boldsymbol{Q}(\lambda)$. And so, we see $\{1, u_1, \ldots, u_{d-1}\}$ is $\boldsymbol{Q}$-linearly independent. □

As the corollary of Lemma 1.5, we have the following.

COROLLARY 1.6. *The closure of* $\pi\boldsymbol{Z}^d = \mathscr{P}$.

Proposition 1.7. *For the atomic surface $X$ associated with the substitution $\sigma$ we know the following properties*:

**(1)** $\bigcup_{z \in \{\sum_{i=2}^{d} n_i \pi(e_1 - e_i) \,|\, n_i \in Z\}} (X + z) = \mathscr{P}$;

**(2)** $\mathring{X} \neq \emptyset$.

Proof. For each $n$ let us consider the set of points $l_n = \{\sum_{j=1}^{k} e_{s_j} \,|\, 1 \le k \le l^{(n)}\}$. We define $Y_n = \pi l_n$ and let us consider the lattice $L_0 := \{\sum_{i=2}^{d} n_i(e_1 - e_i) \,|\, n_i \in Z\}$ on $\mathscr{P}_0 := \{x \in Z^d \,|\, \langle x, {}^t(\overbrace{1, 1, \dots, 1}^{\#d})\rangle = 0\}$ where $\langle x, y\rangle$ is the inner product of vectors $x$ and $y$.

Now define the set of the lattice points by

$$l_n + L_0 = \bigcup_{z \in L_0} (l_n + z).$$

The projection of $l_n + L_0$ by $\pi$ is denoted by $\bigcup_{z \in L_0} (Y_n + \pi z)$. On the other hand, for any substitution we can see easily the following relation:

$$l_n + L_0 = \{x \in Z^d \,|\, \langle x, {}^t(\overbrace{1, 1, \dots, 1}^{\#d})\rangle \ge 0\}.$$

Using the fact that

$$Y_n \subset Y_{n+1},$$

$$\text{the closure of } \bigcup Y_n = X,$$

we know from the boundedness of $X$ and Corollary 1.6,

$$\bigcup_{z \in L_0} (X + \pi z) = \mathscr{P}. \tag{1.3}$$

Using (1.3) and from Baire category theorem, we have $\mathring{X} = \mathring{\overline{Y}} \neq \emptyset$. From Theorem 1.2 and primitivity, we see that

$$\mathring{X}_i \neq \emptyset \quad \text{for all } i \in \{1, 2, \dots, d\}. \qquad \square$$

In order to know that $X_i$ are disjoint each other up to a set of measure 0 (about the sets of measure 0), we would prepare several lemmas. The next result can be found in [2], originally in [21].

Lemma 1.8. *Let $M$ be a primitive matrix with the largest eigenvalue $\lambda$. Suppose that $\mathbf{v}$ is a positive vector such that $M\mathbf{v} \ge \lambda \mathbf{v}$. Then the inequality is an equality and $\mathbf{v}$ is the eigenvector with respect to $\lambda$.*

Hereafter, we will note $|K|$ the measure of the set $K$.

LEMMA 1.9. *The vector of volumes* ${}^t(|X_i|)_{1\le i\le d}$ *satisfies the following inequality*:

$$L_\sigma^{-1}\,{}^t(|X_1|,\dots,|X_d|) \ge \lambda_1\,{}^t(|X_1|,\dots,|X_d|).$$

PROOF. From the form of $X_i$ in the equation of Lemma 1.4, we see

$$|L_\sigma^{-1}X_i| \le \sum\nolimits_{j=1}^{d}(L_\sigma)_{ij}|X_j|.$$

Since the determinant of $L_\sigma^{-1}$ restricted to $\mathscr{P}$ is $\lambda_1$, we know that $|L_\sigma^{-1}X_i| = \lambda_1|X_i|$. Hence we arrive at the conclusion. □

From the Lemma 1.8, Lemma 1.9 and the fact that $|X_j| > 0$, we obtain the proof of Theorem 1.2.

REMARK. We don't know whether

$$X = \sum\nolimits_{j=1}^{d} X_i \quad \text{(non-overlap)}$$

and we see in [2] that $X = \bigcup_{j=1}^{d} X_j$ is non-overlap if $\sigma$ satisfies the coincidence condition.

COROLLARY 1.10. *The relation that* $X =$ *the closure of* $\overset{\circ}{X}$ *holds.*

PROOF. Moreover by rewriting Theorem 1.2, for any $n > 0$ we have

$$X = \sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{\substack{S_{n,k}^{(j)}:\\ W_{n,k}^{(j)}=i,\\ \sigma^n(j)=P_{n,k}^{(j)}W_{n,k}^{(j)}S_{n,k}^{(j)}}} (L_\sigma^n X_j - \pi f(S_{n,k}^{(j)})).$$

For any $\boldsymbol{x} \in X$ and $\delta > 0$, let $B_x(\delta)$ be the ball with the center $\boldsymbol{x}$ and the radius $\delta$ on $\mathscr{P}$, then by the above rewritten formula, there exist $n$ and $S_{n,k}^{(j)}$ such that

$$B_x(\delta) \supset L_\sigma^n X_j - \pi f(S_{n,k}^{(j)}) \quad \text{and} \quad L_\sigma^n \overset{\circ}{X}_j \ne \emptyset.$$

This means that the relation that $X =$ the closure of $\overset{\circ}{X}$ holds. □

## 2. Structure of Boundary and Mauldin-Williams Graph

We say that the point $(\boldsymbol{x}, i^*) \in \boldsymbol{Z}^d \times \{1, 2, \dots, d\}$ is an element of the stepped surface $\mathscr{P}$ if $\langle \boldsymbol{x}, \boldsymbol{v}_\lambda\rangle \ge 0$ and $\langle \boldsymbol{x} - \boldsymbol{e}_i, \boldsymbol{v}_\lambda\rangle < 0$. Put all of the elements of the stepped surface $\mathscr{P}$ by $\boldsymbol{S}$.

Lemma 2.1. *If a pair* $(\boldsymbol{x}, i^*) \neq (\boldsymbol{y}, j^*)$ *are the elements of* $\boldsymbol{S}$, *then the element* $(\boldsymbol{z}, k^*)$ *given by*

$$(\boldsymbol{z}, k^*) := \begin{cases} (\boldsymbol{x} - \boldsymbol{y}, i^*) & \text{if } \langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{v}_\lambda \rangle \geq 0 \\ (\boldsymbol{y} - \boldsymbol{x}, j^*) & \text{if } \langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{v}_\lambda \rangle > 0 \end{cases}$$

*is also an element of* $\boldsymbol{S}$.

The proof is easy.

Let us define the map $\varphi : \boldsymbol{S} \times \boldsymbol{S} \to \boldsymbol{S} \times \boldsymbol{S}$ as follows:

$$\varphi((\boldsymbol{x}, i^*), (\boldsymbol{y}, j^*)) = ((\boldsymbol{0}, l^*), (\boldsymbol{z}, k^*))$$

where $(\boldsymbol{z}, k^*)$ is given as Lemma 2.1 and $l^*$ is given by

$$l^* = \begin{cases} j^* & \text{if } \langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{v}_\lambda \rangle \geq 0 \\ i^* & \text{if } \langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{v}_\lambda \rangle > 0 \end{cases}$$

$$\left( \boldsymbol{z} = \begin{cases} \boldsymbol{x} - \boldsymbol{y} & \text{if } \langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{v}_\lambda \rangle \geq 0 \\ \boldsymbol{y} - \boldsymbol{x} & \text{if } \langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{v}_\lambda \rangle > 0 \end{cases} \right).$$

Lemma 2.2. *Let us define the operator* $\sigma^*$ *on* $\boldsymbol{S}$ *by*

$$\sigma^* : (\boldsymbol{x}, i^*) := \sum_{j \in \{1, \dots, d\}} \sum_{\substack{S_k^{(j)}: \\ W_k^{(j)} = i, \\ \sigma(j) = P_k^{(j)} W_k^{(j)} S_k^{(j)}}} (L_\sigma^{-1} \boldsymbol{x} + L_\sigma^{-1} f(S_k^{(j)}), j^*).$$

*Then all of the elements in* $\sigma^*(\boldsymbol{x}, i^*)$ *are also the elements of* $\boldsymbol{S}$.

The proof can be found in [2].

Let us consider the set $\boldsymbol{V}_0$ of the pair of elements such that

$$\boldsymbol{V}_0 = \{((\boldsymbol{0}, i^*), (\boldsymbol{x}, j^*)) \mid (\boldsymbol{x}, j^*) \in \boldsymbol{S}, \|\pi \boldsymbol{x}\| < 2D\}$$

where $\|\boldsymbol{x}\|$ be the length of the vector $\boldsymbol{x}$ and $D$ be the diameter of $X$ estimated in Lemma 1.3. Then, we see that the cardinarity of $\boldsymbol{V}_0$ is finite. Let us define the set of the pair $\boldsymbol{V}^{(i)}$ such that

$$\boldsymbol{V}^{(i)} := \{\varphi((\boldsymbol{x}, j^*), (\boldsymbol{y}, k^*)) \mid (\boldsymbol{x}, j^*), (\boldsymbol{y}, k^*) \in \sigma^*(\boldsymbol{0}, i^*), \|\pi(\boldsymbol{x} - \boldsymbol{y})\| < 2D\},$$

and $\boldsymbol{V}_0^{(0)} := \bigcup_{i=1,2,\dots,d} \boldsymbol{V}^{(i)}$, then $\boldsymbol{V}_0^{(0)} \subset \boldsymbol{V}_0$.

Let us define the arrow from the point $((\boldsymbol{0}, i^*), (\boldsymbol{w}, j^*)) \in \boldsymbol{V}_0$ by the following manner: for each pair $((\boldsymbol{0}, i^*), (\boldsymbol{w}, j^*))$ let us pick up the pair such that $(\boldsymbol{x}, k^*) \in$

$\sigma^*(\mathbf{0}, i^*)$, $(\boldsymbol{y}, l^*) \in \sigma^*(\boldsymbol{w}, j^*)$ and if $\|\pi(\boldsymbol{x} - \boldsymbol{y})\| < 2D$, we give the arrow from $((\mathbf{0}, i^*), (\boldsymbol{w}, j^*))$ to $\varphi((\boldsymbol{x}, k^*), (\boldsymbol{y}, l^*))$.

Let us define the graph $G_0 = (\boldsymbol{V}_1, E, i, t)$ by the following manner:

**1st step:** let us consider the arrows starting from the vertex $u \in \boldsymbol{V}_0^{(0)}$. If we can not find the arrow from $u$, then the vertex $u$ is cancelled; If we can find the arrow $e$ from $u$ to $v$ we denote $i(e) = u$, $t(e) = v$ and if moreover the vertex is new, that is, $v \in \boldsymbol{V}_0 \backslash \boldsymbol{V}_0^{(0)}$, then we call $v$ the first generation of $u$. We denote the set of the first generation from $\boldsymbol{V}_0^{(0)}$ by $\boldsymbol{V}_0^{(1)}$.

**2nd step:** let us consider the arrow starting from the vertex of the first generation $v \in \boldsymbol{V}_0^{(1)}$. If we cannot find any arrows from $v$, then we cancell the vertex $v \in \boldsymbol{V}_0^{(1)}$ and the arrow $e$ such that $t(e) = v$; if we can find the arrow $e'$ from $v$ to $w$ and the terminal $t(e')$ is new, that is, $\omega = t(e') \in \boldsymbol{V}_0 \backslash (\boldsymbol{V}_0^{(0)} \cup \boldsymbol{V}_0^{(1)})$, then we call the terminal $\omega$ the 2nd generator of $u$ and denote $\boldsymbol{V}_0^{(2)}$.

**kth step:** if we can not find any arrows from the vertex $v_k$, we cancelled the vertex $v_k$ and the arrow $e$ such that $t(e) = v_k$. And by the cancellation of the arrow $e$ if $v_{k-1} = i(e)$ has no arrow $e'$ such that $v_{k-1} = i(e')$ then the vertex $v_{k-1}$ and the arrow $e''$ such that $t(e'') = v_{k-1}$ are also cancelled and so on. From the finiteness of the cardinarity of $\boldsymbol{V}_0$, we can stop this procedure. We denote the final step by $q$.

Now we get the graph with vertices $\boldsymbol{V}_1 = \bigcup_{j=1}^q \boldsymbol{V}_0^{(j)}$ and each vertex $u$ has the arrow $e$ such that $u = i(e)$.

We denote the graph by $G_B = (\boldsymbol{V}_1, E, i, t)$ and call the graph of the boundary of the atomic surface. For the simplicity, we denote the vertex $((\mathbf{0}, i^*), (\boldsymbol{x}, j^*))$ by $(i, j, \boldsymbol{x})$.

The existence of the arrow from $(i, p, \boldsymbol{x}_0)$ to $(j, q, \boldsymbol{x}_1)$ means that on the notation:

$$\sigma^*(\mathbf{0}, i^*) = \sum_{l \in \{1, \dots, d\}} \sum_{\substack{S_k^{(l)}: \\ \sigma(l) = P_k^{(l)} \cdot i \cdot S_k^{(l)}}} (-L_\sigma^{-1}(f(S_k^{(l)})), l^*) \tag{2.4}$$

$$\sigma^*(\boldsymbol{x}_0, p^*) = \sum_{m \in \{1, \dots, d\}} \sum_{\substack{S_{k'}^{(m)}: \\ \sigma(m) = P_{k'}^{(m)} \cdot p \cdot S_{k'}^{(m)}}} (-L_\sigma^{-1}(f(S_{k'}^{(m)})), m^*) + L_\sigma^{-1}(\boldsymbol{x}_0), \tag{2.5}$$

there exist $l$, $k$, $m$ and $k'$ such that $(j, q, \boldsymbol{x}_1)$ is given explicitly by

$$\boldsymbol{x}_1 = \begin{cases} L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \boldsymbol{x}_0) \\ \qquad \text{if } \langle L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \boldsymbol{x}_0), \boldsymbol{v}_\lambda \rangle \geq 0 \\ -L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \boldsymbol{x}_0) \\ \qquad \text{if } \langle L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \boldsymbol{x}_0), \boldsymbol{v}_\lambda \rangle < 0 \end{cases} \tag{2.6}$$

$$(j, q) = \begin{cases} (l, m) & \text{if } \langle L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \boldsymbol{x}_0), \boldsymbol{v}_\lambda \rangle \geq 0 \\ (m, l) & \text{if } \langle L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \boldsymbol{x}_0), \boldsymbol{v}_\lambda \rangle < 0 \end{cases}. \tag{2.7}$$

Proposition 2.3. *For each vertex $(i, j, \boldsymbol{x}) \in \boldsymbol{V}_1$ we know $X_i \cap (X_j + \pi \boldsymbol{x}) \neq \emptyset$ and $|X_i \cap (X_j + \pi \boldsymbol{x})| = 0$.*

Proof. Suppose that $X_i \cap (X_j + \pi \boldsymbol{x}) = \emptyset$, then from the compactness of $X_i$ and $X_j$ we see

$$d_{\mathscr{P}}(X_i, (X_j + \pi \boldsymbol{x})) > 0,$$

where $d_{\mathscr{P}}(A, B) := \inf\{d_{\mathscr{P}}(x, y) \mid x \in A, y \in B\}$, and so we have

$$d_{\mathscr{P}}(L_\sigma^{-1} X_i, L_\sigma^{-1}(X_j + \pi \boldsymbol{x})) \geq \lambda_0^{-1} d_{\mathscr{P}}(X_i, (X_j + \pi \boldsymbol{x})).$$

From the set equation given by Theorem 1.2 and the relation (2.4) and (2.5), we know that

$$L_\sigma^{-1} X_i \supset X_l - L_\sigma^{-1} \pi f(S_k^{(l)}) \quad \text{for } (l, k) \text{ satisfying } W_k^{(l)} = i$$

$$L_\sigma^{-1}(X_p + \boldsymbol{x}_0) \supset X_m - L_\sigma^{-1} \pi f(S_{k'}^{(m)}) + L_\sigma^{-1}(\boldsymbol{x}_0) \quad \text{for } (m, k') \text{ satisfying } W_{k'}^{(m)} = p.$$

Moreover, from the fact that the vertex $(i_1, j_1, \boldsymbol{x}_1)$ from $(i, j, \boldsymbol{x}) \in V_1$ is given by (2.6) and (2.7), in particular $(i_1, j_1)$ is chosen as $(l, m)$ or $(m, l)$ on the notation (2.4), (2.5). Therefore we see

$$d_{\mathscr{P}}(X_{i_1}, X_{j_1} + \pi \boldsymbol{x}_1) \geq d_{\mathscr{P}}(L_\sigma^{-1} X_i, L_\sigma^{-1}(X_j + \pi \boldsymbol{x})),$$

that is,

$$d_{\mathscr{P}}(X_{i_1}, X_{j_1} + \pi \boldsymbol{x}_1) \geq \lambda_0^{-1} d_{\mathscr{P}}(X_i, X_j + \pi \boldsymbol{x}).$$

Continuing this procedure, we have

$$d_{\mathscr{P}}(X_{i_n}, X_{j_n} + \pi \boldsymbol{x}_n) \geq \lambda_0^{-n} d_{\mathscr{P}}(X_i, X_j + \pi \boldsymbol{x}).$$

On the other hand, from the definition of $\boldsymbol{V}_0$ and Lemma 1.3, we know

$$d_{\mathscr{P}}(X_p, X_q + \pi \boldsymbol{x}) < 3D \quad \text{for all } (p, q, \boldsymbol{x}) \in \boldsymbol{V}_0.$$

Therefore, we see that

$$d_{\mathscr{P}}(X_i, X_j + \pi\boldsymbol{x}) = 0.$$

This contradicts to $d_{\mathscr{P}}(X_i, X_j + \pi\boldsymbol{x}) > 0$. From the definition of $(i, j, \boldsymbol{x}) \in \boldsymbol{V}_1$, there exist $k$, $n$, $(\boldsymbol{y}, l^*)$ and $(\boldsymbol{w}, m^*) \in {}^*\sigma^n(\boldsymbol{0}, k^*)$ such that

$$(i, j, \boldsymbol{x}) = \varphi((\boldsymbol{y}, l^*), (\boldsymbol{w}, m^*))$$

where we denote ${}^*\sigma^n$ instead of $(\sigma^*)^n$. Therefore, from the non-overlapping property in Theorem 1.2, we have

$$|X_i \cap (X_j + \pi\boldsymbol{x})| = 0. \qquad \square$$

PROPOSITION 2.4. *For each vertices $(i, j, \boldsymbol{x}) \in \boldsymbol{V}_1$, we see*

$$\partial X_i \supset X_i \cap (X_j + \pi\boldsymbol{x}).$$

PROOF. Assume that

$$\partial X_i \not\supset X_i \cap (X_j + \pi\boldsymbol{x}).$$

Then, we see that

$$(X_j + \pi\boldsymbol{x}) \cap \mathring{X}_i \neq \emptyset.$$

Therefore, there exist $a \in \mathring{X}_i$ and an open ball $B_\delta(a)$ with the center $a$ and the radius $\delta$ such that

$$a \in X_j + \pi\boldsymbol{x} \quad \text{and} \quad B_\delta(a) \subset \mathring{X}_i.$$

Since the closure of $\mathring{X}_j$ is equal to $X_j$, we know $B_\delta(a) \cap (\mathring{X}_j + \pi\boldsymbol{x}) \neq \emptyset$, and thus there exists $B_{\delta'}(b)$ such that

$$B_{\delta'}(b) \subset B_\delta(a) \cap (\mathring{X}_j + \pi\boldsymbol{x}).$$

Therefore,

$$|B_\delta(a) \cap (\mathring{X}_j + \pi\boldsymbol{x})| > 0.$$

From Proposition 2.3 this contradicts to

$$|X_i \cap (X_j + \pi\boldsymbol{x})| = 0. \qquad \square$$

PROPOSITION 2.5. *For each $j \in \{1, \ldots, d\}$, there exist $n$ and $W_0$ such that $\sigma^n(j) = Y \cdot 1 \cdot W_0$ and satisfying the following form*:

$$\partial(X_j - \pi L_\sigma^{-n}(f(W_0))) = \sum_{\substack{k,W:\\ \sigma^n(k)=Y'\cdot 1\cdot W \text{ if } k\neq j\\ \text{or } \sigma^n(k)=Y''\cdot 1\cdot W \text{ and } W\neq W_0 \text{ if } k=j}} ((X_j - \pi L_\sigma^{-n} f(W_0)) \cap (X_k - \pi L_\sigma^{-n}(f(W)))) \tag{2.8}$$

*and*

$$\varphi((j, f(W_0)), (k, f(W))) \in \boldsymbol{V}_1 \quad \text{if} \quad (X_j - \pi L_\sigma^{-n} f(W_0)) \cap (X_k - \pi L_\sigma^{-n}(f(W))) \neq \emptyset.$$

*In particular, we have*

$$\partial X_j = \sum_{\substack{k,W:\\ \sigma^n(k)=Y'\cdot 1\cdot W \text{ if } k\neq j\\ \text{or } \sigma^n(k)=Y''\cdot 1\cdot W \text{ and } W\neq W_0 \text{ if } k=j}} (X_j \cap (X_k - \pi(L_\sigma^{-n}(f(W) - f(W_0))))). \tag{2.9}$$

Proof. From Theorem 1.2, we know

$$L_\sigma^{-n} X_1 = \sum_{j=1}^{d} \sum_{\substack{W:\\ \sigma^n(j)=Y\cdot 1\cdot W}} (-\pi L_\sigma^{-n}(f(W)) + X_j).$$

For the fixed $j$ and the sufficient large $n$, we can find a ball $V$ contained $L_\sigma^{-n} X_1$ and $W_0$ such that the ball $V$ contains $X_j - \pi L_\sigma^{-n} f(W_0)$ and $W_0$ satisfies $\sigma^n(j) = Y \cdot 1 \cdot W_0$. Therefore, we see that

$$\partial(X_j - \pi L_\sigma^{-n}(f(W_0))) = \sum_{\substack{k,W:\\ \sigma^n(k)=Y'\cdot 1\cdot W \text{ if } k\neq j\\ \text{or } \sigma^n(k)=Y''\cdot 1\cdot W \text{ and } W\neq W_0 \text{ if } k=j}} (X_j - \pi L_\sigma^{-n} f(W_0)) \cap (X_k - \pi L_\sigma^{-n}(f(W))). \tag{2.10}$$

In the formula (2.8), if $(X_j - \pi L_\sigma^{-n} f(W_0)) \cap (X_k - \pi L_\sigma^{-n}(f(W))) \neq \emptyset$, then

$$\varphi((j, L_\sigma^{-n} f(W_0)), (k, L_\sigma^{-n} f(W))) \in \boldsymbol{V}_1. \qquad \square$$

For each arrow $e_{u,v} \in E$ let us define the transformation $T_{u,v} : \mathscr{P} \to \mathscr{P}$ by

$$T_{u,v}\boldsymbol{x} = L_\sigma \boldsymbol{x} + \pi \boldsymbol{f}_{u,v} \tag{2.11}$$

where $u = (i, p, \boldsymbol{x}_0)$ and $v = (j, q, \boldsymbol{x}_1)$ given by (2.6) and (2.7), and $\pi \boldsymbol{f}_{u,v}$ is given by

$$\pi \boldsymbol{f}_{u,v} = \begin{cases} -\pi f(S_{k'}^{(m)}) + \boldsymbol{x}_0 \\ \qquad \text{if } \langle L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \boldsymbol{x}_0), \boldsymbol{v}_\lambda \rangle \geq 0 \\ \pi f(S_k^{(l)}) \\ \qquad \text{if } \langle L_\sigma^{-1}(f(S_k^{(l)}) - f(S_{k'}^{(m)}) + \boldsymbol{x}_0), \boldsymbol{v}_\lambda \rangle < 0 \end{cases}.$$

Then the transformation $T_{u,v}$ on $\mathscr{P}$ is a contractive map. Therefore, we have the list of compact sets $(\mathscr{K}_u)_{u \in \boldsymbol{V}_1}$ uniquely satisfying $\mathscr{K}_u = \bigcup T_{u,v}(K_v)$ (see [20]). On the other hand, for each vertex $(i, p, \boldsymbol{x}_0)$ from Proposition 2.3, we know $X_i \cap (X_p + \pi \boldsymbol{x}_0) \neq \varnothing$ and each $X_i$ and $X_p + \boldsymbol{x}_0$ are decomposed by Theorem 1.2,

$$X_i = \sum_{l=1}^{d} \sum_{\substack{S_k^{(l)}: \\ W_k^{(l)} = i, \\ \sigma(l) = P_k^{(l)} W_k^{(l)} S_k^{(l)}}} (L_\sigma X_l - \pi f(S_k^{(l)})),$$

$$X_p + \pi \boldsymbol{x}_0 = \sum_{m=1}^{d} \sum_{\substack{S_{k'}^{(m)}: \\ W_{k'}^{(m)} = p, \\ \sigma(m) = P_{k'}^{(m)} W_{k'}^{(m)} S_{k'}^{(m)}}} (L_\sigma X_m - \pi(f(S_{k'}^{(m)}) - \boldsymbol{x}_0)).$$

Therefore, we have

$$X_i \cap (X_p + \pi \boldsymbol{x}_0) = \sum_{\substack{S_k^{(l)}, S_{k'}^{(m)}: \\ (W_k^{(l)}, W_{k'}^{(m)}) = (i,p)}} (L_\sigma(X_l) - \pi f(S_k^{(l)})) \cap (L_\sigma(X_m) - \pi(f(S_{k'}^{(m)})) + \boldsymbol{x}_0).$$

Using (2.6), (2.7) and $\pi \boldsymbol{f}_{u,v}$ we have

$$\begin{aligned} X_i \cap (X_p + \pi \boldsymbol{x}_0) &= \bigcup_{\substack{v: v = (j,q,\boldsymbol{x}_1) \in \boldsymbol{V}_1, \\ e \in E_{u,v}}} L_\sigma(X_j \cap (X_q + \pi L(\boldsymbol{x}_1))) + \pi \boldsymbol{f}_{u,v} \\ &= \bigcup_{\substack{v: v = (j,q,\boldsymbol{x}_1) \in \boldsymbol{V}_1, \\ e \in E_{u,v}}} T_{u,v}(X_j \cap (X_q + \pi \boldsymbol{x}_1)). \end{aligned}$$ □

Therefore, we have the following theorem.

THEOREM 2.6. *Let $G_B = (\boldsymbol{V}_1, E, i, t)$ be the graph from the substitution $\sigma$ and let $T_{u,v} : \mathscr{P} \to \mathscr{P}$ be the transformation given by* (2.11). *Then, the list of compact sets $(\mathscr{K}_u)_{u \in \boldsymbol{V}_1}$ satisfying*

$$\mathscr{K}_u = \bigcup_{\substack{v \in V_1, \\ e \in E_{u,v}}} T_{u,v}(\mathscr{K}_v)$$

*is given by*

$$\mathscr{K}_u = X_i \cap (X_j + \pi \boldsymbol{x})$$

*where* $u = (i, j, \boldsymbol{x}) \in V_1$.

## 3. Hausdorff Dimension of Boundaries

In this section, we discuss the Hausdorff dimension of the boundary of atomic surfaces.

THEOREM 3.1. *Let $\sigma$ be a primitive unimodular Pisot substitution with $d$ letters. Let $X$ be the atomic surface with respect to $\sigma$. Then the Hausdorff dimension of $\partial X$ is estimated by*

$$\dim_H \partial X \leq \dim_B \partial X \leq \frac{\log \gamma_1 - \log \lambda_1 - (d-1) \log|\lambda_d|}{-\log|\lambda_d|}$$

*where $\dim_B \partial X$ is the Box dimension of $\partial X$ and $\gamma_1$ is the largest eigenvalue of the matrix of the graph $G_B$.*

PROOF. By Proposition 2.5, the boundary $\partial X$ is constructed by the sets $(X_i \cap (X_j + \pi \boldsymbol{x}))$, $(i, j, \boldsymbol{x}) \in V_1$. For any $\varepsilon > 0$, each set $X_i \cap (X_j + \pi \boldsymbol{x})$ can be covered by $c(\gamma_1 + \varepsilon)^n$ pieces parallelograms $L_\sigma^n(\pi \mathscr{U})$ from the unit square $\mathscr{U}$ and the parallelogram $L_\sigma^n(\pi \mathscr{U})$ is covered at most $c'\left(\frac{|\lambda_2|}{|\lambda_d|} \cdot \frac{|\lambda_3|}{|\lambda_d|} \cdot \dots \cdot \frac{|\lambda_d|}{|\lambda_d|}\right)^n$ pieces of the cube whose length of the edge is $|\lambda_d|^n$. Therefore, the Box dimension of $X_i \cap (X_j + \pi \boldsymbol{x})$ can be estimated by

$$\begin{aligned} \dim_B(X_i \cap (X_j + \pi \boldsymbol{x})) &\leq \lim_{n \to \infty} \frac{\log c(\gamma_1 + \varepsilon)^n + \log c'(\lambda_1 |\lambda_d|^{d-1})^{-n}}{-\log|\lambda_d^n|} \\ &= \frac{\log(\gamma_1 + \varepsilon) - \log \lambda_1 - (d-1) \log|\lambda_d|}{-\log|\lambda_d|} \end{aligned}$$

for any $\varepsilon > 0$. Therefore, by Proposition 2.5, we see

$$\dim_H \partial X \leq \dim_B \partial X \leq \frac{\log \gamma_1 - \log \lambda_1 - (d-1) \log|\lambda_d|}{-\log|\lambda_d|}. \qquad \square$$

If we know the explicit values $\gamma_1$, $\lambda_1$ and $\lambda_d$, we see probably that $\dim_H \partial X < d - 1$. But we have no idea to say

$$\frac{\log \gamma_1 - \log \lambda_1 - (d-1)\log|\lambda_d|}{-\log|\lambda_d|} < d-1.$$

Therefore, we give the next theorem (c.f. [14]).

THEOREM 3.2. *Under the same assumption for* $\sigma$ *as in Theorem* 3.1, *we have*

$$\dim_H \partial X < d-1.$$

PROOF. From the set equations in Theorem 1.2 of $\{X_i\}_{i=1,2,\dots,d}$ and $\mathring{X}_1 \neq \emptyset$, for the sufficient large $n_0$ there exist $j_0 \in \{1,\dots,d\}$ and $k_0$ such that

$$\sigma^{n_0}(j_0) = P^{(j_0)}_{n_0,k_0} \cdot 1 \cdot S^{(j_0)}_{n_0,k_0},$$
$$L^{(n_0)}_\sigma X_{j_0} - \pi f(S^{(j_0)}_{n_0,k_0}) \subset \mathring{X}_1.$$

This means

$$\begin{cases} \partial X_1 \subset \displaystyle\sum_{j=1}^{d} \sum_{\substack{(j,S^{(j)}_{n,k}) \neq (j_0,S^{(j_0)}_{n_0,k_0}),\\ \sigma^{n_0}(j)=P^{(j)}_{n_0,k}\cdot 1\cdot S^{(j)}_{n_0,k}}} (L^{n_0}_\sigma(\partial X_j) - \pi f(S^{(j)}_{n_0,k})) \\ \partial X_i \subset \displaystyle\sum_{j=1}^{d} \sum_{\substack{S^{(j)}_{n,k}:\\ \sigma^{n_0}(j)=P^{(j)}_{n,k}\cdot i\cdot S^{(j)}_{n,k}}} (L^{n_0}_\sigma(\partial X_j) - \pi f(S^{(j)}_{n,k})) \end{cases}. \tag{3.12}$$

From the above properties, we say that we can cover $\partial X_1$ by

at most $L^{n_0}_\sigma(1,1)$ pieces of $L^{n_0}_\sigma(\partial X_1)$

$\cdots$

at most $L^{n_0}_\sigma(j_0,1) - 1$ pieces of $L^{n_0}_\sigma(\partial X_{j_0})$

$\cdots$

at most $L^{n_0}_\sigma(d,1)$ pieces of $L^{n_0}_\sigma(\partial X_d)$

and on the definition of the matrix

$$D = \begin{bmatrix} L^{n_0}_\sigma(1,1) & \cdots & L^{n_0}_\sigma(1,d) \\ \cdots & \cdots & \cdots \\ L^{n_0}_\sigma(j_0,1) - 1 & \cdots & L^{n_0}_\sigma(j_0,d) \\ \cdots & \cdots & \cdots \\ L^{n_0}_\sigma(d,1) & \cdots & L^{n_0}_\sigma(d,d) \end{bmatrix},$$

we see that $D < L_\sigma^{n_0}$ and $D$ is primitive for sufficient large $n_0$. Therefore, we know that the largest eigenvalue $\mu$ of $D$ is strictly smaller than $\lambda_1^{n_0}$. The boundary $\partial X_1$ can be covered by at most $c'^p$-pieces of paralleologram $\pi L_\sigma^{pn_0}(\mathscr{U})$ for any $\mu < \nu < \lambda_1^{n_0}$. By analogous discussion in Theorem 3.1, we see that the boundary $\partial X_1$ is covered by at most $c'^p(\lambda_1|\lambda_d|^{d-1})^{-pn_0}$ pieces of cubes with the length $|\lambda_d|^{pn_0}$. Therefore, the $\alpha$-dimensional Hausdorff measure $\mathscr{H}^\alpha(\partial X_1)$ can be estimated by

$$\mathscr{H}^\alpha(\partial X_1) \le \lim_{p\to\infty} \nu^p \frac{1}{(\lambda_1|\lambda_d|^{d-1})^{pn_0}}(|\lambda_d|^{pn_0})^\alpha.$$

Let us assume that $\nu = \lambda_1^{n_0 - x}$ for some $0 < x < 1$. Then the Hausdorff measure is estimated by

$$\mathscr{H}^\alpha(\partial X_1) \le \lim_{p\to\infty}(\lambda_1^{(x-1)}|\lambda_d^{\alpha-(d-1)}|)^{pn_0},$$

we can choose $\alpha_0 > 0$ such that

$$\alpha_0 < d-1 \quad \text{and} \quad \lambda_1^{(x-1)}\lambda_d^{\alpha_0-(d-1)} < 1,$$

and so we know that $\mathscr{H}^{\alpha_0}(\partial X_1) = 0$. Therefore we have

$$\dim_H(\partial X_1) \le \alpha_0 < d-1.$$

By analogous discussion, we see

$$\dim_H(\partial X_i) < d-1$$

and so we get

$$\dim_H(\partial X) < d-1. \qquad \square$$

From now on, we will assume that the linear transformation $L_\sigma$ on $\mathscr{P}$ is a similitude. In two cases (i) $d = 2$ (ii) $d = 3$ and $L_\sigma$ is the complex Pisot matrix, we know that the linear transformation is the similitude on $\mathscr{P}$.

Let the list $\{X_1 \dots, X_d\}$ of compact sets be the atomic surfaces, then we had known the sets satisfy the equation in Theorem 1.2. Therefore, we can get the graph $G_\sigma = \{\boldsymbol{V}, E, i, t\}$ which is constructed by $\boldsymbol{V} = \{1,\dots,d\}$, $e_{ij} \in E$ if there exists $j \in \{1,\dots,d\}$ such that $\sigma(i) = P_k^{(j)} \cdot i \cdot S_k^{(j)}$. And for each $e_{ij} \in E$ let us define the contracting transformation $T_{ij} : \mathscr{P} \to \mathscr{P}$ by

$$T_{ij}(\boldsymbol{x}) = L_\sigma \boldsymbol{x} - \pi f(S_k^{(j)})$$

which is the similitude with some contractive constant $0 < s < 1$. Then we see that $\{\boldsymbol{V}, E, i, t, \{T_{ij}\}\}$ is a Mauldin-Williams graph and that $\{X_i \,|\, i = 1, 2, \dots d\}$ is the graph construction set. Moreover, the graph satisfies the locally finite condition, that is, there exists a constant $H > 0$ such that for any $1 > r > 0$ and any $\boldsymbol{x} \in \mathscr{P}$

$$\#\left\{ (i_1 i_2 \cdots i_l) \middle| \begin{array}{l} e_{i_j, i_{j+1}} \in E, \ \ 1 \leq j \leq l-1, \\ tr \leq t^l \leq r, \\ T_{i_1 i_2} \circ T_{i_2 i_3} \circ \cdots T_{i_{l-1} i_l}(X_{i_l}) \cap B_x(r) \neq \emptyset \end{array} \right\} < H, \qquad (*)$$

since the sets $\overset{\circ}{X}_j$, $j = 1, 2, \ldots, d$ satisfy the open set condition.

Therefore, we have the following lemma.

LEMMA 3.3. *Let* $G_B = (V_1, E, i, t, \{T_{u,v}\})$ *be a Mauldin-William graph in Theorem* 2.6. *Then the graph satisfies the locally finite condition.*

PROOF. From the locally finite condition of $G_B = \{V, E, i, t, \{T_{ij}\}\}$, we see that

$$\#\left\{ (u_1, u_2, \ldots, u_n) \middle| \begin{array}{l} e_{u_i, u_{i+1}} \in E, \ \ tr < t^n < r, \\ T_{u_1 u_2} T_{u_2 u_3} \cdots T_{u_{n-1} u_n}(X_p \cap (X_q + \pi \boldsymbol{y})) \cap B_x(r) \neq \emptyset \end{array} \right\}$$
$$< C_H^2 = \frac{H(H-1)}{2}.$$

Using Lemma 3.3 and Theorem 1 in [20], we have the following theorem.

THEOREM 3.4. *Let* $\sigma$ *be the primitive unimodular Pisot substitution. Let us assume that the linear trasnformation* $L_\sigma$ *on the invariant surface* $\mathscr{P}$ *is a similitude. Then the Hausdorff dimension of* $\partial X$ *is given by*

$$\dim_H \partial X = \frac{(d-1) \log \gamma_1}{\log \lambda_1}$$

*where* $\gamma_1$ *is the largest eigenvalue of the matrix of the graph* $G_B$.

## 4. Examples

In this section, we propose some examples of atomic surfaces.

EXAMPLE 4.1. *Let* $\sigma$ *be the following substitution*:

$$\sigma : \begin{array}{l} 1 \to 112 \\ 2 \to 21 \end{array}.$$

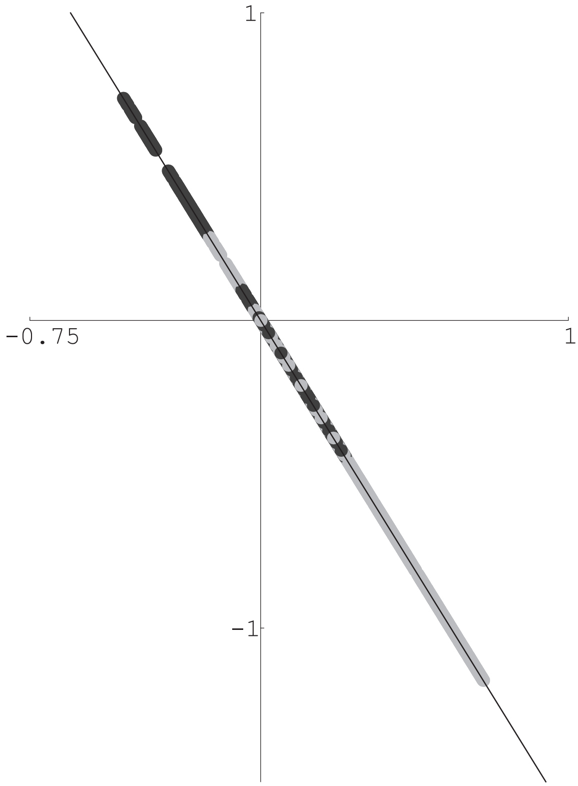


Figure 1: the atomic surface $X = \bigcup_{i=1,2} X_i$ in Example 4.1.

This substitution is a simple example which is not invertible. Therefore, the atomic surface is not an interval (see [6]). In this example, the graph $G_B$ of the boundary of the atomic surface is given by the following form (see Figure 2):

The matrix $M_\sigma$ of the graph $G_B$ is given by

$$M_\sigma = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the characteristic polynomial of $M_\sigma$ is given by

$$x^2(x^2 - 2x - 1)(x - 1)^2$$

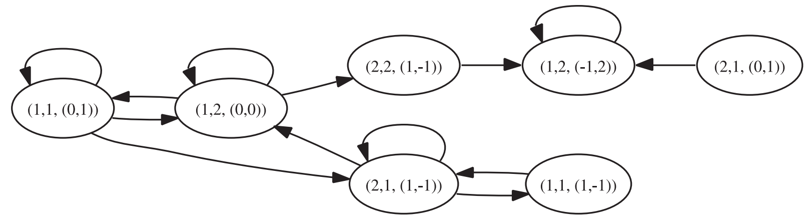


Figure 2: the graph $G_B$ from the substitution: $1 \mapsto 112$, $2 \mapsto 21$.

where the largest eigenvalue of $M_\sigma$ comes from $x^2 - 2x - 1$. And so by using Theorem 3.4, the Hausdorff dimension of the boundary of the atomic surface is given by

$$\dim_H \partial X = \frac{\log \gamma_1}{\log \lambda_1} = \frac{\log 2.41421}{\log 2.61803} = 0.915785\ldots$$

where $\gamma_1$ and $\lambda_1$ are the largest eigenvalues of the graph matrix $M_\sigma$ and $L_\sigma$ respectively.

EXAMPLE 4.2. *Let us consider the substitution called Rauzy substitution* [22]:

$$\sigma : \begin{array}{l} 1 \to 12 \\ 2 \to 13 \\ 3 \to 1. \end{array}$$

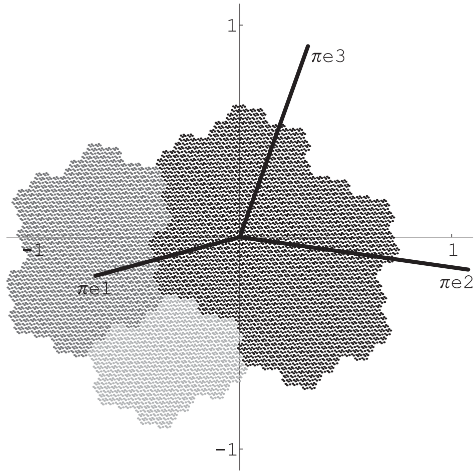


Figure 3: the atomic surface $X = \bigcup_{i=1,2,3} X_i$ in Example 4.2.

The Hausdorff dimension had been calculated in [10]. In our method, the graph $G_B$ of the boundary of the atomic surface is given by the following form (see Figure 4):

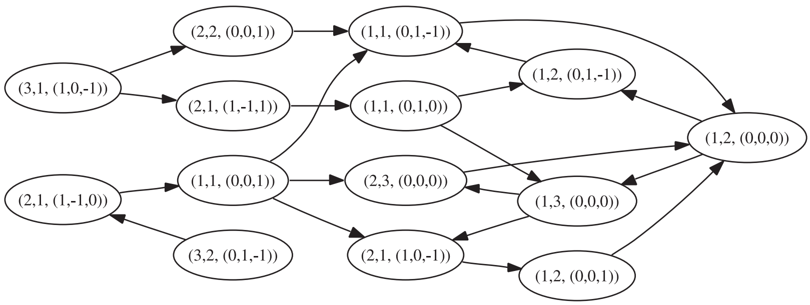


Figure 4: the graph $G_B$ from Rauzy substitution: $1 \mapsto 12,\ 2 \mapsto 13,\ \mapsto 1$.

The matrix $M_\sigma$ of the graph $G_B$ is given by

$$M_\sigma = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the characteristic polynomial of $M_\sigma$ is given by

$$x^3(x^4 - 2x - 1).$$

Therefore, the Hausdorff dimension of $\partial X_\sigma$ is caluculated by

$$\dim_H \partial X = \frac{2 \log \gamma_1}{\log \lambda_1} = \frac{2 \log 1.39534}{\log 1.83929} = 1.09337\ldots$$

where $\gamma_1$ and $\lambda_1$ are the largest eigenvalues of the graph matrix $M_\sigma$ and $L_\sigma$ respectively.

Example 4.3. *Let us consider the following substitution*:

$$\sigma : \begin{matrix} 1 \to 12 \\ 2 \to 31 \\ 3 \to 1 \end{matrix}.$$

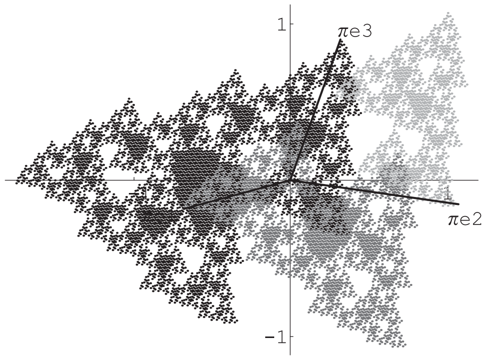


Figure 5: the atomic surface $X = \bigcup_{i=1,2,3} X_i$ in Example 4.3.

The matrix $L_\sigma$ of the substitution is same as one of Rauzy substitution. But the shape of the atomic surface is perfectly different. The graph $G_B$ of the boundary of the atomic surface is given by the following form (see Figure 6): The characteristic polynomial of $M_\sigma$ is given by

$$(x^6 - x^5 - x^4 - x^2 + x - 1)(x^2 + x + 1)^2 x^{15}(x - 1)^2.$$

Therefore, the Hausdorff dimension of $\partial X_\sigma$ is caluculated by

$$\dim_H \partial X = \frac{2 \log \gamma_1}{\log \lambda_1} = \frac{2 \log 1.72629}{\log 1.83929} = 1.7919\ldots$$

where $\gamma_1$ and $\lambda_1$ are the largest eigenvalues of the graph matrix $M_\sigma$ and $L_\sigma$ respectively.

EXAMPLE 4.4. *Let us consider the substitution*:

$$\sigma : \begin{array}{l} 1 \to 112 \\ 2 \to 13 \\ 3 \to 1. \end{array}$$

This substitution is an example of a class of Pisot substitutions:

$$\sigma_{k_1,k_2} : \begin{array}{l} 1 \to \overbrace{11\cdots 1}^{\#k_1}2 \\ 2 \to \overbrace{11\cdots 1}^{\#k_2}3 \\ 3 \to 1 \end{array}$$

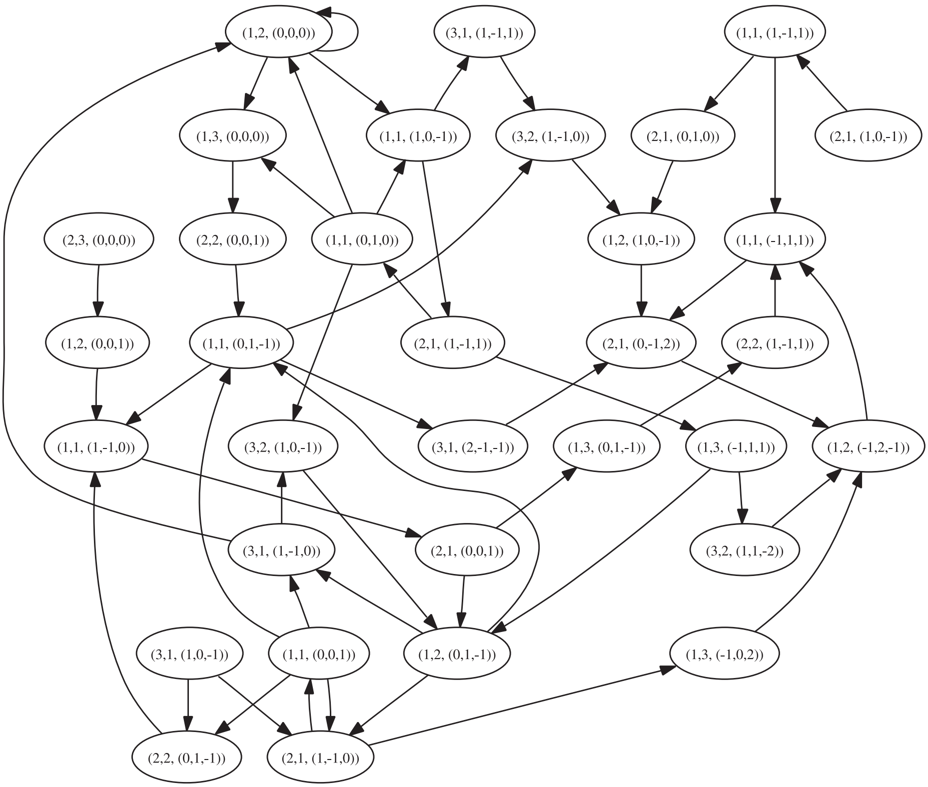


Figure 6: the graph $G_B$ from the substitution: $1 \mapsto 12,\ 1 \mapsto 31,\ 1 \mapsto 1$.

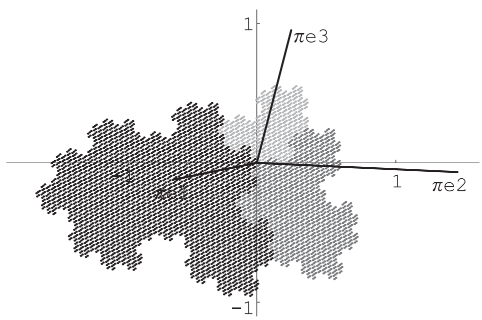


Figure 7: the atomic surface $X = \bigcup_{i=1,2,3} X_i$ in Example 4.4.

which is related to Pisot $\beta$-expansions (see [13]). The graph $G_B$ of the boundary of the atomic surface is given by the following form (see Figure 8):

The matrix $M_\sigma$ of the graph $G_B$ is given by

$$M_\sigma = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the characteristic polynomial of $M_\sigma$ is given by

$$x^5(x^4 - x^2 - 3x - 1).$$

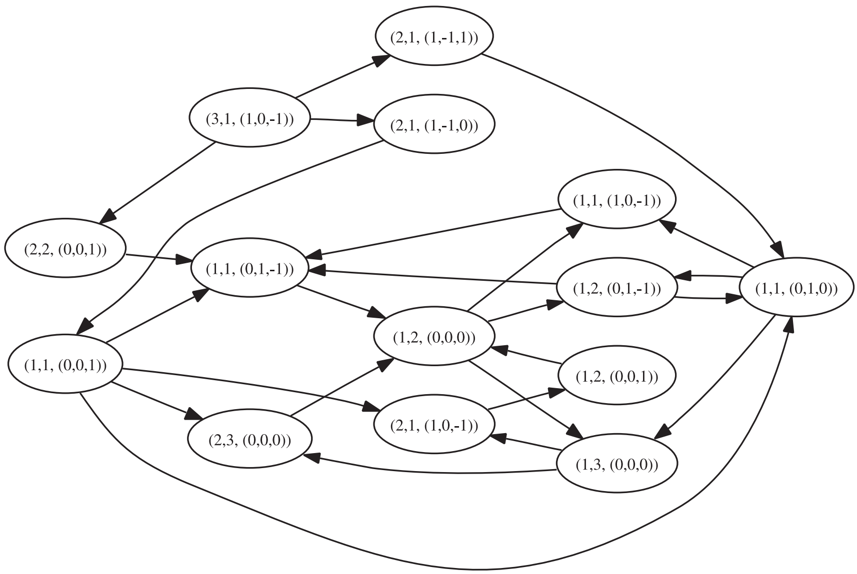


Figure 8: the graph $G_B$ from the $\beta$-substitution: $1 \mapsto 112,\ 2 \mapsto 13,\ 3 \mapsto 1$.

Therefore, the Hausdorff dimension of $\partial X_\sigma$ is caluculated by

$$\dim_H \partial X = \frac{2 \log \gamma_1}{\log \lambda_1} = \frac{2 \log 1.74553}{\log 2.54682} = 1.19177\ldots$$

where $\gamma_1$ and $\lambda_1$ are the largest eigenvalues of the graph matrix $M_\sigma$ and $L_\sigma$ respectively.

Example 4.5. *Let us consider the substitution*:

$$\sigma : \begin{array}{l} 1 \to 13 \\ 2 \to 1 \\ 3 \to 32 \end{array} .$$

*This substitution is coming from Example* 4 *in* [8] $(L_\sigma = M^2)$.

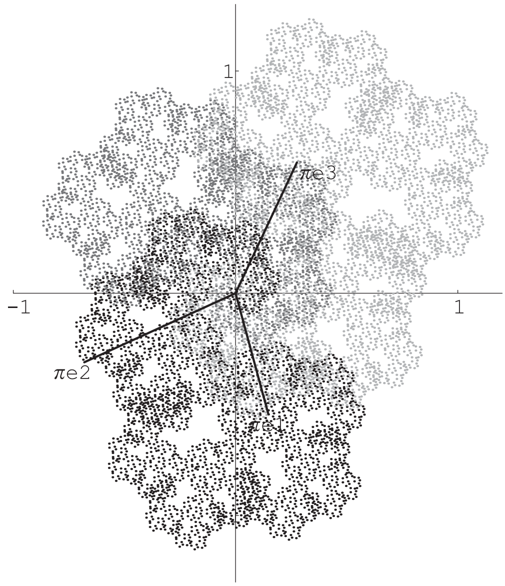


Figure 9: the atomic surface $X = \bigcup_{i=1,2,3} X_i$ in Example 4.5.

This example is that the atomic surface is not simply connected. The characteristic polynomial of $M_\sigma$ is given by

$$x^{32}(x^{13} - x^{12} - x^{10} + x^9 - 2x^8 - 4x^7 - 2x^5 - 4x^4 + x^3 - 4x^2 - 1)$$
$$\times (x^5 - 2x^3 + x - 1)(x^4 + x^3 + x^2 + x + 1)(x - 1)$$

and the largest eigenvalue of $M_\sigma$ is coming from the polynomial $(x^{13} - x^{12} - x^{10} + x^9 - 2x^8 - 4x^7 - 2x^5 - 4x^4 + x^3 - 4x^2 - 1)$. Therefore, the Hausdorff dimension of $\partial X_\sigma$ is caluculated by

$$\dim_H \partial X = \frac{2 \log \gamma_1}{\log \lambda_1} = \frac{2 \log 1.72864}{\log 1.75478} = 1.94643\ldots$$

where $\gamma_1$ and $\lambda_1$ are the largest eigenvalues of the graph matrix $M_\sigma$ and $L_\sigma$ respectively.

Example 4.6. *Let us consider the substitution*:

$$\sigma : \begin{array}{l} 1 \to 12123 \\ 2 \to 1 \\ 3 \to 12. \end{array}$$

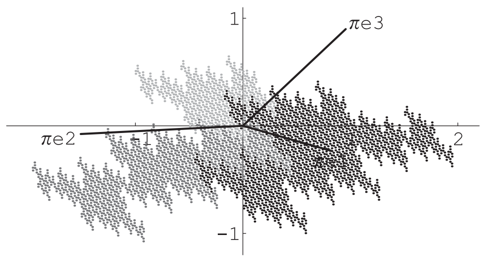


Figure 10: the atomic surface $X = \bigcup_{i=1,2,3} X_i$ in Example 4.6.

This substitution is coming from $\sigma_1 \circ \sigma_2$ for $\sigma_m$ Example 1 in [10].

This is an example such that the boundary of the atomic surface is not double point free. The graph $G_B$ of the boundary of the atomic surface is given the following form (see Figure 11);

The matrix $M_\sigma$ of the graph $G_B$ is given by

$$M_\sigma = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}.$$

The characteristic polynomial of $M_\sigma$ is given by

$$x^{13}(x^3 - 3x^2 + 2x - 1)(x - 1).$$

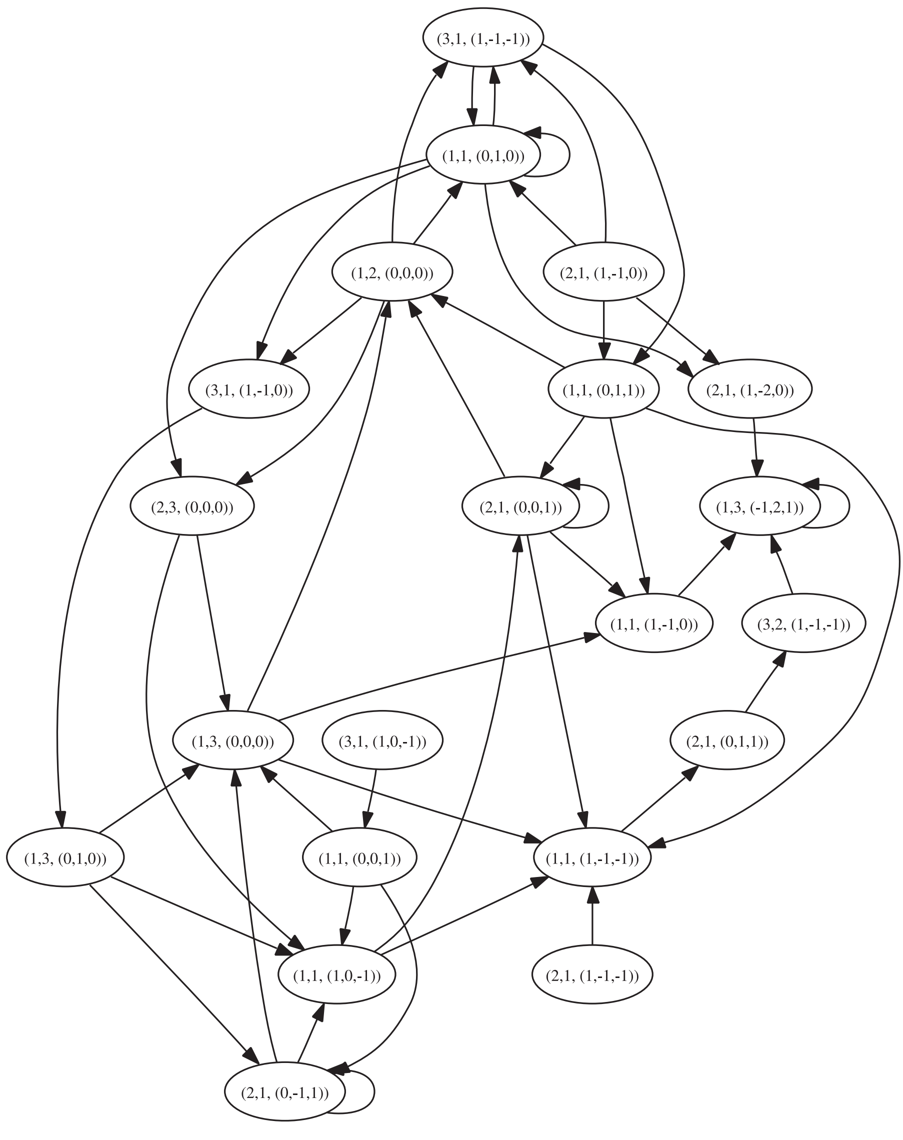


Figure 11: the graph $G_B$ from the substitution: $1 \mapsto 12123,\ 2 \mapsto 1,\ 3 \mapsto 12.$

Therefore, the Hausdorff dimension of $\partial X_\sigma$ is caluculated by

$$\dim_H \partial X = \frac{2 \log \gamma_1}{\log \lambda_1} = \frac{2 \log 2.32472}{\log 3.0796} = 1.5$$

where $\gamma_1$ and $\lambda_1$ are the largest eigenvalues of the graph matrix $M_\sigma$ and $L_\sigma$ respectively.

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