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REMARKS ON THE BORDISM INTERSECTION MAP*

By

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Abstract. In this paper we give a characterization of the kernel of the bordism intersection map and we present some related results as the following. The set of bordism classes of C^{∞} maps $f : M \to N$ such that rank $df(x) \leq p$ for all x is contained in $J_{p,m-p}(N)$, where M is a smooth closed manifold of dimension m, N is a smooth closed manifold, df is the differential of f, $J_{p,m-p}(N)$ is the image of the homomorphism $\ell_* : \mathfrak{N}_m(N^{(p)}) \to \mathfrak{N}_m(N)$ induced by the inclusion, $0 \leq p \leq m$, and $N^{(p)}$ is the p-skeleton of N.

1. Introduction

Let $f : M \to N$ and $g : K \to N$ be differentiable maps, where M and K are smooth closed manifolds of dimensions m and k, respectively, and N is an n-dimensional smooth closed manifold. Let us consider a C^{∞} map $\varphi : M \times K \to N \times N$ homotopic to $f \times g$ and transversal to the diagonal $\triangle \subset N \times N$ and the $(m+k-n)$-dimensional manifold $V \subset M \times K$ obtained by $V = \varphi^{-1}(\triangle)$. We call V the intersection manifold, and we define the intersection map $h : V \to N$ by the composite $h = \pi_1 \circ \varphi \circ i$, where i is the inclusion map from V into $M \times K$ and π_1 is the projection of $N \times N$ onto the first factor.

Then we define the *bordism intersection product* $I_{m,k} : \mathfrak{N}_m(N) \times \mathfrak{N}_k(N) \to \mathfrak{N}_{m+k-n}(N)$ by $I_{m,k}([M,f],[K,g]) = [V,h]$, where $\mathfrak{N}_i(N)$ denotes the i-dimensional unoriented bordism group of N([5]). It is known that this is well-defined.

The map $I_{m,k}$ induces on $\mathfrak{N}_*(N)$ a product which, with the disjoint union, makes $\mathfrak{N}_*(N)$ a commutative ring. This product corresponds to a product in the cobordism ring $\mathfrak{N}^*(N)$, up to duality, and was studied by Quillen ([5]).

In this paper we consider $g : K \to N$ fixed and then we give a characterization

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of the kernel of the map $I_g : \mathfrak{N}_m(N) \to \mathfrak{N}_{m+k-n}(N)$ obtained from $I_{m,k}$. We also pay attention to the image of I_g.

For any map $f : M \to N$ we denote by U_f the Poincaré dual (P.D.) of $f_*(\mu_M)$, where μ_M is the fundamental class of M. In what follows we shall consider the map $\cdot u_g : H_i(N, \mathbf{Z}_2) \to H_{i+k-n}(N, \mathbf{Z}_2)$ defined by $\alpha \cdot u_g = P.D.(\beta \smile U_g)$, where β is the dual class of α, and w_i shall denote the i-th Stiefel-Whitney class. We also denote by $\pi_M = \pi_1 \circ i$ and $\pi_K = \pi_2 \circ i$, where π_1 and π_2 are the projection maps from $M \times K$ onto M and K respectively.

Let $J_{p,m-p}(X)$ be the image of the map $\ell_* : \mathfrak{N}_m(X^{(p)}) \to \mathfrak{N}_m(X)$, $0 \le p \le m$, induced by the inclusion in X of the p-skeleton $X^{(p)}$ of a finite CW-complex X. Since $\mathfrak{N}(\varnothing) = 0$, let us agree that $J_{p,m-p}(X) = 0$ for $p < 0$. It is known that $J_{p,m-p}(X)$ does not depend on a particular cell decomposition of X ([3]).

THEOREM 1.1. *The kernel of I_g coincides with $J_{n-k-1,m+k-n+1}(N)$ if one of the following conditions holds.*

a1) $g^*(w_i(N)) = w_i(K)$ *in the range* $0 \le i \le m+k-n$ *and* $\smile U_g : H^i(N, \mathbf{Z}_2) \to H^{i+n-k}(N, \mathbf{Z}_2)$ *is onto in the same range.*

a2) $\cdot u_g : H_i(N, \mathbf{Z}_2) \to H_{k-n+i}(N, \mathbf{Z}_2)$ *is a monomorphism for* $n-k-1 < i \le m$.

As an immediate consequence we observe that if K and N are manifolds of the same dimension n, and if $g : K \to N$ satisfies $g_*(\mu_K) = \mu_N$, then I_g is a monomorphism.

The following gives examples in which the conditions in Theorem 1.1 are satisfied. Let us consider $K = P^{n-4}$, $N = P^n$ and the inclusion map $g : P^{n-4} \to P^n$, where P^m denotes the m-dimensional real projective space. Then $g_*(\mu_{P^{n-4}}) \in H_{n-4}(P^n, \mathbf{Z}_2)$ is such that $U_g = P.D.^{-1}(g_*(\mu_{P^{n-4}}))$ generates $H^4(P^n, \mathbf{Z}_2)$ and $g^*(w_i(P^n)) = w_i(P^{n-4})$ for $0 \le i \le 2$. If the dimension of the manifold M is equal to 6 and $n \ge 5$, then we have that $\smile U_g : H^i(P^n, \mathbf{Z}_2) \to H^{i+4}(P^n, \mathbf{Z}_2)$ is an epimorphism for $i = 1, 2$. Thus, condition a1) is satisfied in this case. By considering $K = P^{n-2}$ and $N = P^n$, the above reasoning shows that condition a2) is satisfied, whenever the dimension of the manifold M is equal to 3 and $n \ge 3$.

The following example shows that the conditions in Theorem 1.1 are only sufficient ones. Consider the embedding $g : S^1 \to T^2 = S^1 \times S^1$ defined by $g(x) = (x, e)$. In this case, $\smile U_g : H^0(T^2, \mathbf{Z}_2) \to H^1(T^2, \mathbf{Z}_2)$ is not surjective, but for $I_g : \mathfrak{N}_2(T^2) \to \mathfrak{N}_1(T^2)$ we have $\ker I_g = J_{0,2}(T^2)$. We observe that $\cdot u_g : H_1(T^2, \mathbf{Z}_2) \to H_0(T^2, \mathbf{Z}_2)$ is not injective.

THEOREM 1.2. *If* $\cdot u_g : H_i(N, \mathbf{Z}_2) \to H_{k-n+i}(N, \mathbf{Z}_2)$ *is an epimorphism for* $n - k - 1 < i \leq m$, *then* I_g *is an epimorphism.*

The following gives an example in which the condition in Theorem 1.2 is satisfied. Let us consider $K = P^{n-4}$, $N = P^n$ and the inclusion map $g : P^{n-4} \to P^n$. If the dimension of the manifold M is m and $m \leq n$, then we have that $\cdot u_g : H_i(N, \mathbf{Z}_2) \to H_{i-4}(N, \mathbf{Z}_2)$ is an epimorphism for $3 < i \leq m$.

Let us consider now the following problem. Let V and K be submanifolds of M and N respectively, of the same codimension. Let v_K and v_V denote the normal bundles to K and V in M and N respectively. Given a C^∞ map $f_V : V \to K$ with $f_V^* v_K = v_V$, under what conditions does there exist an extension $f : M \to N$ of f_V such that f is transversal to K and $f^{-1}(K) = V$?

There are particular cases where it is possible to obtain such an extension using obstruction theory. We shall deal with this problem in a forthcoming paper.

Let $forg : \mathfrak{N}_{m+k-n}(N) \to \mathfrak{N}_{m+k-n}$ be the forgetful map and let us take $I_g' : \mathfrak{N}_m(N) \to \mathfrak{N}_{m+k-n}$ as the composite $I_g' = forg \circ I_g$.

REMARKS. 1) If $g : K \to N$ is the inclusion map and if I_g' is onto, then given an $(m + k - n)$-manifold V, there exists an $(M', f') \in \mathfrak{N}_m(N)$ such that f' is transversal to K and $f'^{-1}(K)$ is cobordant to V.

2) Let V and K be submanifolds of M and N respectively, of the same codimension. Given a C^∞ map $f_V : V \to K$ with $f_V^* v_K = v_V$, if $[V]$ does not belong to the image of I_g', where $g : K \to N$ is the inclusion map, then there does not exist an extension $f : M \to N$ of f_V such that f is transversal to K and $f^{-1}(K) = V$.

As *forg* restricted to $J_{0,m+k-n}(N)$ is surjective, we can say that I_g' is surjective, if $J_{0,m+k-n}(N)$ is contained in the image of I_g.

THEOREM 1.3. $J_{0,m+k-n}(N)$ *is contained in the image of* I_g *if the map* $\smile U_g : H^k(N, \mathbf{Z}_2) \to H^n(N, \mathbf{Z}_2)$ *is an epimorphism.*

We give now an example where $\smile U_g : H^k(N, \mathbf{Z}_2) \to H^n(N, \mathbf{Z}_2)$ is an epimorphism. Let N be a smooth connected n-dimensional manifold, which is the total space of a fiber bundle over a smooth closed connected k-dimensional manifold K and with fiber a smooth closed connected $(n - k)$-dimensional manifold F. Suppose that there exists a section $g : K \to N$. Since the class in $H_{n-k}(N, \mathbf{Z}_2)$ given by the inclusion $F \hookrightarrow N$ intersects the section g, $\smile U_g : H^k(N, \mathbf{Z}_2) \to H^n(N, \mathbf{Z}_2)$ is an epimorphism.

2. Whitney Numbers of Intersection Maps

Given smooth maps $f : M \to N$ and $g : K \to N$, where M and K are smooth closed manifolds of dimensions m and k, respectively, and N is an n-dimensional smooth closed manifold, we consider the intersection manifold V and the intersection map $h : V \to N$. We observe that h is homotopic to both $f \circ \pi_M$ and $g \circ \pi_K$, where $\pi_M = \pi_1 \circ i$, $\pi_K = \pi_2 \circ i$, and π_1 and π_2 are the projection maps from $M \times K$ onto M and K respectively. We remark that whenever f is transversal to g we can take $\varphi = f \times g$. In this case h coincides with $f \circ \pi_M$ and with $g \circ \pi_K$.

The following lemma is proved in [1].

Lemma 2.1. *Let (V, h) be obtained by the intersection of the maps $f : M \to N$ and $g : K \to N$. Then $f^*(U_g) = U_{\pi_M}$ and $g^*(U_f) = U_{\pi_K}$.*

Remark. If $\alpha \in H^{m+k-n}(N, \mathbf{Z}_2)$ is any class, then $\langle \alpha, h_*(\mu_V) \rangle = \langle \alpha, (f \circ \pi_M)_* \mu_V \rangle = \langle f^*(\alpha), \pi_{M*} \mu_V \rangle = \langle f^*(\alpha), f^*(U_g) \frown \mu_M \rangle = \langle \alpha \smile U_g, f_*(\mu_M) \rangle = \langle \alpha, (U_f \smile U_g) \frown \mu_N \rangle$.

Since the intersection of the homology classes $f_*(\mu_M)$ and $g_*(\mu_K)$, denoted by $f_*(\mu_M) \cdot g_*(\mu_K)$, is given by $P.D.(U_f \smile U_g)$, we conclude that $h_*(\mu_V) = f_*(\mu_M) \cdot g_*(\mu_K)$.

Let $f : M \to N$ be a map between closed manifolds and let $\alpha \in H^i(N, \mathbf{Z}_2)$ be any cohomology class. For every partition $\{i_1 \le i_2 \le \cdots \le i_r\}$ of $m - i$, the number $\langle w_{i_1}(M) \cdots w_{i_r}(M) \cdot f^*(\alpha), \mu_M \rangle \in \mathbf{Z}_2$ is defined and is called the *Whitney number* of f associated to α, where $w_i(M)$ is the i-th Stiefel-Whitney class of M.

Let us consider the tangent vector bundles TN, TM and TK as well as the respective vector bundles induced by h, π_M and π_K. We observe that $TV \oplus h^*(TN)$ and $\pi_M^*(TM) \oplus \pi_K^*(TK)$ are equivalent vector bundles over V. Therefore $w(V)h^*(w(N)) = \pi_M^*(w(M))\pi_K^*(w(K))$, where w denotes the total Stiefel-Whitney class.

Theorem 2.2. *Let $\alpha \in H^i(N, \mathbf{Z}_2)$ be any cohomology class and $\{i_1 \le \cdots \le i_s\}$*

be a partition of $m+k-n-i$. *If* $g^*(w_i(N)) = w_i(K)$, $0 \le i \le m+k-n$, *then* $\langle w_{i_1}(V)\cdots w_{i_s}(V)\cdot h^*(\alpha), \mu_V\rangle = \langle w_{i_1}(M)\cdots w_{i_s}(M)\cdot f^*(\alpha \smile U_g), \mu_M\rangle$.

Proof. We recall that h is homotopic to $g\circ\pi_K$. Then using the hypothesis and the remark above, we conclude that $w_i(V) = \pi_M^*(w_i(M))$ for $0 \le i \le m+k-n$.

Consequently,

$$\begin{aligned}\langle w_{i_1}(V)\cdots w_{i_s}(V)\cdot h^*(\alpha), \mu_V\rangle &= \langle \pi_M^*(w_{i_1}(M)\cdots w_{i_s}(M))\cdot \pi_M^* f^*(\alpha), \mu_V\rangle\\ &= \langle w_{i_1}(M)\cdots w_{i_s}(M)\cdot f^*(\alpha), U_{\pi_M}\frown \mu_M\rangle\\ &= \langle w_{i_1}(M)\cdots w_{i_s}(M)\cdot f^*(\alpha\smile U_g), \mu_M\rangle. \qquad \square\end{aligned}$$

Remark. If $g^*(w_i(N)) = w_i(K)$ and $f^*(w_i(N)) = w_i(M)$, $0\le i\le m+k-n$, then using Theorem 2.2 we have that: $\langle w_{i_1}(V)\cdots w_{i_s}(V)\cdot h^*(\alpha), \mu_V\rangle = \langle w_{i_1}(N)\cdots w_{i_s}(N)\cdot \alpha\cdot U_f\cdot U_g, \mu_N\rangle$.

Let X be a finite CW-complex and let us consider $\ell_* : \mathfrak{N}_m(X^{(p)}) \to \mathfrak{N}_m(X)$ induced by the inclusion of the p-skeleton $X^{(p)}$ of X in X.

If $J_{p,m-p}(X)$ is the image of ℓ_*, $0\le p\le m$, then we have the filtration $\mathfrak{N}_m(X) = J_{m,0}(X)\supset J_{m-1,1}(X)\supset\cdots\supset J_{0,m}(X)\supset 0$.

The unoriented bordism spectral sequence associated to this filtration is such that $E^2_{p,m-p} = H_p(X,\mathbf{Z}_2)\otimes\mathfrak{N}_{m-p}$ and this sequence is trivial. So we have $J_{p,m-p}(X)/J_{p-1,m-p+1}(X) = H_p(X,\mathbf{Z}_2)\otimes\mathfrak{N}_{m-p}$ ([3]).

Let $\{c_{m,i}\}$ be an additive homogeneous basis for $H_m(X,\mathbf{Z}_2)$. Since the homomorphism $\mu : \mathfrak{N}_m(X)\to H_m(X,\mathbf{Z}_2)$ defined by $\mu([M,f]) = f_*(\mu_M)$ is an epimorphism, for each $c_{m,i}$ we can select a singular manifold $(M_i^m, f_{m,i})$ such that $f_{m,i_*}(\mu_{M_i^m}) = c_{m,i}$. The set $\{[M_i^m, f_{m,i}]\}$ is a homogeneous $\mathfrak{N}$-module basis for $\mathfrak{N}_*(X)$. Let us consider the $\mathfrak{N}$-module isomorphism $\psi : H_*(X,\mathbf{Z}_2)\otimes\mathfrak{N}\to\mathfrak{N}_*(X)$ defined by $\psi(c_{m,i}\otimes 1) = [M_i^m, f_{m,i}]$.

We can see $J_{p,m-p}(X)$ as the image of $\sum_{j=0}^p H_j(X,\mathbf{Z}_2)\otimes\mathfrak{N}_{m-j}$ by the m-th component of ψ. Then a general element of $J_{p,m-p}(X)$ can be expressed as $\sum_{j=0}^p\sum_{i=1}^{k_j}[M_i^j\times Q_i^{m-j}, \bar f_{j,i}]$, where $\bar f_{j,i}$ is defined by the composite $\bar f_{j,i} = f_{j,i}\circ\pi_1$, with $\pi_1 : M_i^j\times Q_i^{m-j}\to M_i^j$ the projection to the first factor, $f_{j,i}$ a map from M_j^i into X chosen above, and Q_i^{m-j} a closed manifold of dimension $m-j$, given by $\mathfrak{N}$-module structure of $\mathfrak{N}_*(X)$.

It follows from the proof of (17.1) Theorem in [3] that:

Theorem 2.3. *$J_{p,m-p}(X)$ is the set made up of classes $[M,f]$ in $\mathfrak{N}_m(X)$ such that for all $\alpha \in H^j(X,\mathbf{Z}_2)$ and partition $\{i_1 \leq \cdots \leq i_s\}$ of $m-j$ with $j > p$, the corresponding Whitney number of f, $\langle w_{i_1}(M)\cdots w_{i_s}(M)\cdot f^*(\alpha), \mu_M\rangle$, associated to α is zero.*

3. Proof of the Theorems

The map $I_g : \mathfrak{N}_m(N) \to \mathfrak{N}_{m+k-n}(N)$ is obtained from $I_{m,k}$ by considering $g : K \to N$ fixed.

The kernel of I_g contains $J_{n-k-1,m+k-n+1}(N)$, because for $p+q<n$, $I_{m,k}$ restricted to $J_{p,m-p}(N) \times J_{q,k-q}(N)$ is a trivial map.

It is not always true that kernel of I_g coincides with $J_{n-k-1,m+k-n+1}(N)$, as the following example shows.

Example. Consider the embedding $g = i \times Id : S^p \times S^1 \to S^{p+1} \times S^1$, $p \geq 2$, where $i : S^p \to S^{p+1}$ is the inclusion map. Then we see that $I_g : \mathfrak{N}_{p+1}(S^{p+1} \times S^1) \to \mathfrak{N}_p(S^{p+1} \times S^1)$ vanishes and satisfies $I_g([S^{p+1}, f]) = [S^p, f \circ i]) = 0$, where $f : S^{p+1} \times \{point\} \to S^{p+1} \times S^1$ is the inclusion, while $[S^{p+1}, f]$ does not belong to $J_{0,p+1}(S^{p+1} \times S^1)$.

Let $I_g^p : J_{p,m-p}(N) \to J_{p+k-n,m-p}(N)$ be the map I_g restricted to $J_{p,m-p}(N)$. Then we have that

$$I_g^p\left(\sum_{j=0}^{p}\sum_{i=1}^{k_j}[M_i^j \times Q_i^{m-j}, \bar{f}_{j,i}]\right) = \sum_{j=0}^{p}\sum_{i=1}^{k_j}[V_i^{k-n+j} \times Q_i^{m-j}, \bar{h}_{k-n+j,i}],$$

where $[V_i^{k-n+j}, h_{k-n+j,i}] = I_g([M_i^j, f_{j,i}])$.

We observe that $I_g^m = I_g$, since $J_{m,0}(N) = \mathfrak{N}_m(N)$ and $J_{m+k-n,0}(N) = \mathfrak{N}_{m+k-n}(N)$.

Let us now consider the natural projection $\pi^i : J_{i,m-i}(N) \to J_{i,m-i}(N)/J_{i-1,m-i+1}(N) = E^2_{i,m-i} = H_i(N,\mathbf{Z}_2) \otimes \mathfrak{N}_{m-i}$ and the map $\cdot u_g : H_i(N,\mathbf{Z}_2) \to H_{i+k-n}(N,\mathbf{Z}_2)$ defined by $\alpha \cdot u_g = P.D.(\beta \smile U_g)$, where $u_g = g_*(\mu_K)$ and $\beta = P.D.^{-1}(\alpha)$. We can see $\cdot u_g$ as $g_* g_!$, where $g_! : H_i(N,\mathbf{Z}_2) \to H_{i+k-n}(K,\mathbf{Z}_2)$ is the homology transfer homomorphism. In the same way the map $\smile U_g : H^{n-i}(N,\mathbf{Z}_2) \to H^{2n-k-i}(N,\mathbf{Z}_2)$ is equal to $g^! g^*$, where $g^!$ is the cohomology transfer homomorphism.

With these notations we have the following commutative diagrams for $0 \leq i \leq m$.

$$
\begin{tikzcd}
H^{n-i}(K,\mathbf{Z}_2) \arrow[rr, "g!"] \arrow[ddd, "P.D."'] & & H^{2n-k-i}(N,\mathbf{Z}_2) \arrow[ddd, "P.D."] \\
& H^{n-i}(N,\mathbf{Z}_2) \arrow[ul, "g^*"] \arrow[ur, "\smile U_g"'] \arrow[d, "P.D."] & \\
& H_i(N,\mathbf{Z}_2) \arrow[dl, "g_!"'] \arrow[dr, "\cdot u_g"] & \\
H_{k-n+i}(K,\mathbf{Z}_2) \arrow[rr, "g_*"'] & & H_{k-n+i}(N,\mathbf{Z}_2)
\end{tikzcd}
\tag{3.1}
$$

$$
\begin{tikzcd}
J_{i,m-i}(N) \arrow[r, "I_g^i"] \arrow[d, "\pi^i"] & J_{k-n+i,m-i}(N) \arrow[d, "\pi^{k-n+i}"] \\
H_i(N,\mathbf{Z}_2) \otimes \mathfrak{N}_{m-i} \arrow[r, "\cdot u_g \otimes Id"] & H_{k-n+i}(N,\mathbf{Z}_2) \otimes \mathfrak{N}_{m-i}
\end{tikzcd}
\tag{3.2}
$$

Proof of Theorem 1.1. Let us suppose that for $0 \leq i \leq m+k-n$, $g^*(w_i(N)) = w_i(K)$ and that $\smile U_g : H^i(N,\mathbf{Z}_2) \to H^{i+n-k}(N,\mathbf{Z}_2)$ is onto. Let $\gamma \in H^j(N,\mathbf{Z}_2)$, $m \geq j > n-k-1$, be any class and let $\alpha \in H^{j-n+k}(N,\mathbf{Z}_2)$ be such that $\alpha \smile U_g = \gamma$.

Let us consider a partition $\{i_1 \leq \cdots \leq i_s\}$ of $m-j$ and let $[M,f]$ be a class in the kernel of I_g. Then we have $\langle w_{i_1}(M)\cdots w_{i_s}(M)\cdot f^*(\gamma), \mu_M\rangle = \langle w_{i_1}(M)\cdots w_{i_s}(M)\cdot f^*(\alpha \smile U_g), \mu_M\rangle$, which by Theorem 2.2 is equal to $\langle w_{i_1}(V)\cdots w_{i_s}(V)\cdot h^*(\alpha), \mu_V\rangle$. Since $[V,h] = I_g([M,f]) = 0$, we get $\langle w_{i_1}(M)\cdots w_{i_s}(M)\cdot f^*(\gamma), \mu_M\rangle = 0$.

It follows from Theorem 2.3 that $[M,f] \in J_{n-k-1,m+k-n+1}(N)$ and we conclude that $\ker I_g = J_{n-k-1,m+k-n+1}(N)$ as stated.

We suppose next that $\cdot u_g : H_i(N,\mathbf{Z}_2) \to H_{k-n+i}(N,\mathbf{Z}_2)$ is a monomorphism for $n-k-1 < i \leq m$. Let us show that $\ker I_g^i = J_{n-k-1,m+k-n+1}(N)$ for $n-k-1 \leq i \leq m$ by induction on i.

As the first step we observe that $J_{-1,m+k-n+1}(N) = 0$ and hence that $\ker I_g^{n-k-1} = J_{n-k-1,m+k-n+1}(N)$ holds.

Then suppose that $\ker I_g^i = J_{n-k-1,m+k-n+1}(N)$ for $n-k-1 \leq i < m$. By recalling that a general element β of $J_{i+1,m-i-1}(N)$ can be expressed as $\beta = \sum_{j=0}^{i+1}\sum_{l=1}^{k_j}[M_l^j \times Q_l^{m-j}, \bar{f}_{j,l}]$, we see that if such an element belongs to $\ker I_g^{i+1}$, then it follows from diagram (3.2) that $(\cdot u_g \otimes Id)(\pi^{i+1}(\beta)) = \pi^{k-n+i+1}(I_g^{i+1}(\beta)) = 0$, or equivalently, $(\cdot u_g \otimes Id)(\pi^{i+1}(\sum_{j=0}^{i}\sum_{l=1}^{k_j}[M_l^j \times Q_l^{m-j}, \bar{f}_{j,l}] + \sum_{l=1}^{k_{i+1}}[M_l^{i+1} \times Q_l^{m-i-1}, \bar{f}_{i+1,l}])) = (\cdot u_g \otimes Id)(\sum_{l=1}^{k_{i+1}}[M_l^{i+1} \times Q_l^{m-i-1}, \bar{f}_{i+1,l}]) = 0$. Since $\cdot u_g \otimes Id$ is a monomorphism, we have $\sum_{l=1}^{k_{i+1}}[M_l^{i+1} \times Q_l^{m-i-1}, \bar{f}_{i+1,l}] = 0$.

Since I_g^i is the restriction of I_g^{i+1} to $J_{i,m-i}(N)$, we have $0 = I_g^{i+1}(\beta) = I_g^{i+1}(\sum_{j=0}^{i}\sum_{l=1}^{k_j}[M_l^j \times Q_l^{m-j}, \bar{f}_{j,l}]) = I_g^i(\sum_{j=0}^{i}\sum_{l=1}^{k_j}[M_l^j \times Q_l^{m-j}, \bar{f}_{j,l}])$ and by the induction hypothesis we see that β is in $J_{n-k-1,m+k-n+1}(N)$. $\square$

Proof of Theorem 1.2. Let us suppose that $\cdot u_g : H_i(N, \mathbf{Z}_2) \to H_{k-n+i}(N, \mathbf{Z}_2)$ is an epimorphism for $n-k-1 < i \le m$. To show that I_g is an epimorphism let us show that $I_g^i : J_{i,m-i}(N) \to J_{i+k-n,m-i}(N)$ is an epimorphism for $n-k-1 \le i \le m$ by induction on i.

Let us observe that $J_{-1,m+k-n+1}(N) = 0$ and hence that I_g^{n-k-1} is an epimorphism. Let us suppose that I_g^{i-1}, $n-k-1 < i \le m$, is an epimorphism. If y is in $J_{i+k-n,m-i}(N)$ then $\pi^{i+k-n}(y) = y + J_{i+k-n-1,m-i+1}(N)$ in $J_{i+k-n,m-i}(N)/J_{i+k-n-1,m-i+1}(N) = H_{i+k-n}(N, \mathbf{Z}_2) \otimes \mathfrak{N}_{m-i}$. Since $\cdot u_g \otimes Id : H_i(N, \mathbf{Z}_2) \otimes \mathfrak{N}_{m-i} \to H_{k-n+i}(N, \mathbf{Z}_2) \otimes \mathfrak{N}_{m-i}$ is an epimorphism for $n-k-1 < i \le m$, there exists an $l \in J_{i,m-i}(N)$ such that $(\cdot u_g \otimes Id)(\pi^i(l)) = y + J_{i+k-n-1,m-i+1}(N) = y + I_g^{i-1}(J_{i-1,m-i+1}(N))$, the last equality following from the induction hypothesis. We have $\pi^{i+k-n}(I_g^i(l)) = (\cdot u_g \otimes Id)(\pi^i(l))$, due to diagram (3.2). On the other hand, we have $\pi^{i+k-n}(I_g^i(l)) = I_g^i(l) + I_g^{i-1}(J_{i-1,m-i+1}(N))$. Then $I_g^i(l) - y \in I_g^{i-1}(J_{i-1,m-i+1}(N))$ and $I_g^i(l) - y = I_g^{i-1}(x)$ for some $x \in J_{i-1,m-i+1}(N)$. Since I_g^{i-1} is the restriction of I_g^i to $J_{i-1,m-i+1}(N)$, we have that $y = I_g^i(l-x)$. Therefore, I_g^i is an epimorphism. $\square$

Proof of Theorem 1.3. If $\smile U_g$ is an epimorphism, then so is $\cdot u_g \otimes Id : H_{n-k}(N, \mathbf{Z}_2) \otimes \mathfrak{N}_{m+k-n} \to H_0(N, \mathbf{Z}_2) \otimes \mathfrak{N}_{m+k-n}$.

Considering diagram (3.2) for $i = n-k$, we see that $J_{0,m+k-n}(N)$ is contained in the image of I_g. $\square$

4. Related Results

We present now some related results.

Theorem 4.1. *The set of bordism classes of C^∞ maps $f : M \to N$ such that* rank $df(x) \le p$ *for all x is contained in $J_{p,m-p}(N)$, where M and N are smooth closed manifolds of dimension m and n, respectively.*

Proof. For every class $\alpha \in H_{n-j}(N, \mathbf{Z}_2)$ there exists a singular manifold (K, g') such that $g'_*(\mu_K) = \alpha$. By using l vector fields $X_1, X_2, \dots, X_l$ in N which generate $T_y(N)$ for each $y \in N$, we can construct a submersion, that is, a C^∞-map $G : V \times K \to N$ such that $G(0, x) = g'(x)$ for all $x \in K$ and the differential dG is surjective at every point, where V is a sufficiently small neighborhood of

$0 \in \mathbf{R}^l$. Then $G \times f : V \times K \times M \to N \times N$ is transversal to the diagonal $\triangle_N$ of $N \times N$. Applying [4, Chap. 3, Theorem 2.7], we obtain a C^∞ map $g : K \to N$ homotopic to g' and transversal to f. Then for every pair (x, y) with $f(x) = g(y)$ we have $T_{g(y)}N = df(x)T_xM + dg(y)T_yK$.

Since rank $df(x) \le p$ for all x, we see that $n = \dim(df(x)T_xM + dg(y)T_yK) \le p + n - j$, which is an absurd if $j > p$.

We conclude that $g(K) \subset N - f(M)$ if $j > p$, and so the map $H_{n-j}(N - f(M), \mathbf{Z}_2) \to H_{n-j}(N, \mathbf{Z}_2)$ induced by the inclusion of $N - f(M)$ in N is onto.

Let us consider the following commutative diagram:

$$\begin{array}{ccccccc}
\check{H}^j(N, f(M), \mathbf{Z}_2) & \longrightarrow & \check{H}^j(N, \mathbf{Z}_2) & \xrightarrow{k^*} & \check{H}^j(f(M), \mathbf{Z}_2) & \xrightarrow{\delta} & \check{H}^{j+1}(N, f(M), \mathbf{Z}_2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n-j}(N - f(M), \mathbf{Z}_2) & \longrightarrow & H_{n-j}(N, \mathbf{Z}_2) & \longrightarrow & H_{n-j}(N, N - f(M), \mathbf{Z}_2) & \longrightarrow & H_{n-j-1}(N - f(M), \mathbf{Z}_2),
\end{array}$$

where the top horizontal line is the exact Cěch cohomology sequence of the pair $(N, f(M))$, the bottom horizontal line is the exact homology sequence of the pair $(N, N - f(M))$, and the vertical arrows are either Poincaré duality or Alexander duality and are isomorphisms.

It follows that $k^* = 0$ for $j > p$. Recalling that for manifolds the Cěch cohomology agrees with the usual cohomology, we have that $f^* : H^j(N, \mathbf{Z}_2) \to H^j(M, \mathbf{Z}_2)$ is a trivial map for $j > p$.

The result follows from Theorem 2.3. □

In fact, by using a result of [2], we can prove the following.

THEOREM 4.2. *The set of bordism classes of* C^r *maps* $f : M \to N$ *with* $r \ge \max\{1, (m - p)/(s + 1)\}$, *s and p being nonnegative integers such that* rank $df(x) \le p$ *for all x is contained in* $J_{p+s, m-p-s}(N)$, *where M and N are smooth closed manifolds of dimensions m and n, respectively.*

PROOF. Under the hypothesis we have from [2] that $\dim f(M) \le p + s$. Therefore, $f^* : H^j(N, \mathbf{Z}_2) \to H^j(M, \mathbf{Z}_2)$ is a trivial map for $j > p + s$. Consequently, the set of such bordism classes is contained in $J_{p+s, m-p-s}(N)$. □

As a last remark, we observe that: *Given a codimension one submanifold K of an n-dimensional manifold N with inclusion map* $g : K \to N$, *if* $g_*(\mu_K) = 0$, *then* $I_g : \mathfrak{N}_m(N) \to \mathfrak{N}_{m-1}(N)$ *is the trivial map.*

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