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ON FIRST ORDER LINEAR PDE SYSTEMS ALL OF WHOSE SOLUTIONS ARE HARMONIC FUNCTIONS

Dedicated to the memory of Gianfranco Cimmino

By

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Abstract. We study the first order linear system $u_{\bar{z}} + \bar{v}_w = 0$, $u_{\bar{w}} - \bar{v}_z = 0$ in a domain $\Omega \subset \mathbf{C}^2$ (first considered by G. Cimmino, [3]). We prove a Morera type theorem, emphasizing the analogy to the Cauchy-Riemann system, and a representation formula yielding a result on removable singularities of solutions to (2). We derive (by a Hilbert space technique outlined in [5]) compatibility relations among the free terms and boundary data in the boundary value problem $u_{\bar{z}} + \bar{v}_w = f$, $u_{\bar{w}} - \bar{v}_z = g$ in Ω, and $u = \varphi$, $v = \psi$ on $\partial\Omega$. If $F = (u, v) : \Omega \to \mathbf{C}^2$ is a solution to (2) such that $\sup_{\varepsilon>0} \int_{\partial\Omega_\varepsilon} |F(z,w)|^p \, d\sigma_\varepsilon(z,w) < \infty$ for some $p \geq 2$ then we show that F admits nontangential limits at almost every $(\zeta, \omega) \in \partial\Omega$.

1. A Morera Type Theorem

The systems of first order linear partial differential equations all of whose solutions are harmonic functions bear, as demonstrated by G. Cimmino (cf. [3]), many similarities to the ordinary Cauchy-Riemann system. Interesting examples occur however only in higher dimensions [first order linear homogeneous systems with two unknown functions in two real variables, possessing the required property, are equivalent (up to a linear transformation of the dependent variables) to the Cauchy-Riemann equations, while there are no such systems in dimension three, [3], p. 91–94]. Let us consider (together with G. Cimmino, cf. *op. cit.*) the following system of first order linear homogeneous equations

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$$
\begin{cases} X_x - Y_y + Z_\xi - T_\eta = 0 \\ X_y + Y_x - Z_\eta - T_\xi = 0 \\ X_\xi - Y_\eta - Z_x + T_y = 0 \\ X_\eta + Y_\xi + Z_y + T_x = 0, \end{cases} \tag{1}
$$

with the (real valued) unknown functions $X(x, y, \xi, \eta)$, $Y(x, y, \xi, \eta)$, $Z(x, y, \xi, \eta)$ and $T(x, y, \xi, \eta)$. Each C^2 solution (X, Y, Z, T) to (1) is harmonic. This is most easily seen by setting $z = x + iy$, $w = \xi + i\eta$ and $f = X + iY$, $g = Z + iT$ ($i = \sqrt{-1}$) and rewriting (1) as

$$
\frac{\partial f}{\partial \bar{z}} + \frac{\partial \bar{g}}{\partial w} = 0, \quad \frac{\partial f}{\partial \overline{w}} - \frac{\partial \bar{g}}{\partial z} = 0. \tag{2}
$$

Indeed, if $\Omega \subset \mathbf{C}^2$ is an open set and $f, g \in C^2(\Omega)$ satisfy (2) then (differentiating the first equation in (2) with respect to z, the second with respect to $\overline{w}$, and summing up the two resulting equations)

$$
\Delta f = 2(f_{z\bar{z}} + f_{w\overline{w}}) = 0
$$

in Ω. Similarly $\Delta g = 0$ in Ω. The differential operator

$$
Q = \begin{pmatrix} \partial/\partial \bar{z} & \partial/\partial w \\ \partial/\partial \overline{w} & -\partial/\partial z \end{pmatrix}
$$

is referred to as the *Cimmino operator* and $QF = 0$ is the *Cimmino system* (where $F = (f, \bar{g})$). We may tentatively define weak solutions to the Cimmino system as follows. Let $\Omega \subset \mathbf{C}^2$ be a bounded domain. A pair of functions $f, g \in L^2(\Omega)$ is a *weak solution* to (2) if

$$
\int_\Omega (f\varphi_{\bar{z}} + \bar{g}\varphi_w)\, dz \wedge d\bar{z} \wedge dw \wedge d\overline{w} = 0, \tag{3}
$$

$$
\int_\Omega (f\varphi_{\overline{w}} - \bar{g}\varphi_z)\, dz \wedge d\bar{z} \wedge dw \wedge d\overline{w} = 0, \tag{4}
$$

for any $\varphi \in C_0^\infty(\Omega)$. Nevertheless, if $\psi \in C_0^\infty(\Omega)$ and we set $\varphi = \psi_z$ in (3), respectively $\varphi = \psi_w$ in (4), and add up the resulting equations we obtain $\int_\Omega f \Delta\varphi = 0$, i.e. f is C^∞ (and similarly $g \in C^\infty$). More generally, we have

LEMMA 1. *The Cimmino operator is hypoelliptic.*

PROOF. If $f \in L^1_{\mathrm{loc}}(\Omega)$ let T_f be the distribution associated to f. Given two distributions $u, v \in C_0^\infty(\Omega)'$ such that $u_{\bar{z}} + v_w = T_f$ and $u_{\overline{w}} - v_z = T_g$, for some $f, g \in C^\infty(\Omega)$, one has

$$(\Delta u)(\varphi) = 2(u_{z\bar{z}} + u_{w\bar{w}})(\varphi) = -2u_{\bar{z}}(\varphi_{\bar{z}}) - 2u_{\bar{w}}(\varphi_{\bar{w}})$$
$$= 2(v_w - T_f)(\varphi_{\bar{z}}) - 2(v_z + T_g)(\varphi_{\bar{w}})$$
$$= -2\int_\Omega (f\bar{\varphi}_z + g\bar{\varphi}_w)\, dV = 2\int_\Omega (f_z + g_w)\bar{\varphi}\, dV$$

for any $\varphi \in C_0^\infty(\Omega)$, that is $\Delta u = 2T_{f_z+g_w}$ (in distribution sense) and $f_z + g_w \in C^\infty(\Omega)$, hence u (and similarly v) is C^∞. Q.e.d.

The following analog to the fundamental Cauchy theorem (cf. e.g. Theorem 1.5 in [8], p. 42) holds

Proposition 1. *Let $f, g \in C^1(\Omega)$ be a solution to the Cimmino system. Then*

$$\int_{\partial D} (f\, dz \wedge dw \wedge d\bar{w} - \bar{g}\, dz \wedge d\bar{z} \wedge d\bar{w}) = 0, \tag{5}$$

$$\int_{\partial D} (f\, dz \wedge d\bar{z} \wedge dw + \bar{g}\, d\bar{z} \wedge dw \wedge d\bar{w}) = 0, \tag{6}$$

for any domain $D \subset \mathbf{C}^2$ with $\bar{D} \subset \Omega$ on which the Stokes theorem holds.

Compare to (9) in [3], p. 95. Indeed, let us consider the (complex valued) differential 1-form (of class C^1)

$$\omega = f\, dz \wedge dw \wedge d\bar{w} - \bar{g}\, dz \wedge d\bar{z} \wedge d\bar{w}.$$

Then ω is closed

$$d\omega = df \wedge dz \wedge dw \wedge d\bar{w} - d\bar{g} \wedge dz \wedge d\bar{z} \wedge d\bar{w}$$
$$= -(f_{\bar{z}} + \bar{g}_{\bar{w}})\, dz \wedge d\bar{z} \wedge dw \wedge d\bar{w} = 0$$

(by the first equation of (2)) and one may apply the Stokes theorem $\int_{\partial D} \omega = \int_D d\omega = 0$. Similarly $f\, dz \wedge d\bar{z} \wedge dw + \bar{g}\, d\bar{z} \wedge dw \wedge d\bar{w}$ is a closed 1-form. Q.e.d.

The following converse to Proposition 1 (an analog to the classical Morera theorem, cf. e.g. Theorem 1.10 in [8], p. 56) is claimed in [3]

Theorem 1. *Let $\Omega \subset \mathbf{C}^2$ be a domain and $f, g : \Omega \to \mathbf{C}$ two locally Hölder continuous functions. Assume that for any $x_0 \in \Omega$ there is $R > 0$ such that $B(x_0, R) \subset \Omega$ and f, g satisfy (5)–(6) on any cube D with $\bar{D} \subset B(x_0, R)$. Then f, g are harmonic in Ω and a solution to (2).*

The locally Hölder continuous assumption is employed to solve the Dirichlet problem for the Poisson equation (cf. e.g. Theorem 4.3 in [6], p. 56). Then f, g may be recast in terms of second order derivatives of the solution (similar to our (7)–(8) below). Therefore, the differential forms appearing in the integral identities at hand (cf. (15) in [3], p. 97) are but C^0 and the Stokes theorem cannot be applied. This difficulty is circumnavigated by explicit integration on the boundary of a cube (rather than passing to a volume integral, which is prevented by the lack of differentiability) and the use of a mean value theorem (to get harmonicity). G. Cimmino's ideas may be used to generalize Theorem 1 above, as follows

Theorem 2. *Let $\Omega \subset \mathbf{C}^2$ be a domain and $f, g : \Omega \to \mathbf{C}$ continuous functions satisfying* (5)–(6) *for any ball $D = B(x_0, R)$ such that $\bar{D} \subset \Omega$. Then f, g are harmonic in Ω and a solution to* (2).

The main ingredient is to use the mollifications of f and g (whose regularity allows us to give an elegant proof based on the Stokes theorem).

Proof of Theorem 2. Let $x_0 = (z_0, w_0) \in \Omega$ and let us consider a ball $B = B(x_0, 2R) \subset \Omega$ such that $0 < R < \frac{1}{6}\,\mathrm{dist}(x_0, \partial\Omega)$. Also, let us set

$$\tilde{f}(x) = \begin{cases} f(x), & x \in B, \\ 0, & x \in \mathbf{C}^2 \backslash B. \end{cases}$$

Let $f_\varepsilon = J_\varepsilon * \tilde{f}$ $(\varepsilon > 0)$ be the mollification of $\tilde{f}$. As $\tilde{f} \in L^1_{\mathrm{loc}}(\bar{B})$ it follows (cf. e.g. Lemma 2.18 in [1], p. 29–30) that $f_\varepsilon \in C^\infty(\mathbf{C}^2)$ and $\tilde{f} \in C^0(B)$ yields $\lim_{\varepsilon \to 0^+} f_\varepsilon(x) = f(x)$ uniformly for $x \in A$, for any $A \subset\subset B$. Let $F_\varepsilon, G_\varepsilon \in C^\infty(B)$ be solutions to the Poisson equations $\Delta F = f_\varepsilon$ and $\Delta G = g_\varepsilon$. Moreover, let $\varphi_\varepsilon, \psi_\varepsilon \in C^\infty(B)$ be the functions given by

$$\varphi_\varepsilon = 2\left(\frac{\partial F_\varepsilon}{\partial \bar{z}} + \frac{\partial \overline{G}_\varepsilon}{\partial w}\right), \quad \psi_\varepsilon = 2\left(\frac{\partial F_\varepsilon}{\partial \overline{w}} - \frac{\partial \overline{G}_\varepsilon}{\partial z}\right).$$

Note that

$$\frac{\partial \varphi_\varepsilon}{\partial z} + \frac{\partial \psi_\varepsilon}{\partial w} = \Delta F_\varepsilon = f_\varepsilon \tag{7}$$

and, similarly

$$\frac{\partial \bar{\varphi}_\varepsilon}{\partial w} - \frac{\partial \overline{\psi}_\varepsilon}{\partial z} = \Delta G_\varepsilon = g_\varepsilon \tag{8}$$

in B.

Lemma 2. *Let* $\varphi, \psi \in C^\infty(B)$ *such that* $f := \varphi_z + \psi_w$ *and* $g := \bar{\varphi}_w - \bar{\psi}_z$ *satisfy* (5)–(6) *for* $D = B(x_0, R)$. *Then* φ *and* ψ *are harmonic in D. Consequently*, f *and* g *are harmonic in D and* (f, g) *is a solution to* (2) *in D.*

Proof. The assumptions (5)–(6) may be written

$$\int_{\partial D} \{(\varphi_z + \psi_w)\, dz \wedge dw \wedge d\overline{w} - (\varphi_{\overline{w}} - \psi_{\bar{z}})\, dz \wedge d\bar{z} \wedge d\overline{w}\} = 0, \tag{9}$$

$$\int_{\partial D} \{(\varphi_z + \psi_w)\, dz \wedge d\bar{z} \wedge dw + (\varphi_{\overline{w}} - \psi_{\bar{z}})\, d\bar{z} \wedge dw \wedge d\overline{w}\} = 0. \tag{10}$$

Yet (by the Stokes theorem)

$$\begin{aligned}&\int_{\partial D} (\psi_w\, dz \wedge dw \wedge d\overline{w} + \psi_{\bar{z}}\, dz \wedge d\bar{z} \wedge d\overline{w})\\ &\qquad = \int_D (d\psi_w \wedge dz \wedge dw \wedge d\overline{w} + d\psi_{\bar{z}} \wedge dz \wedge d\bar{z} \wedge d\overline{w})\\ &\qquad = \int_D (-\psi_{w\bar{z}} + \psi_{\bar{z}w})\, dz \wedge d\bar{z} \wedge dw \wedge d\overline{w} = 0\end{aligned}$$

hence (by (9))

$$\begin{aligned}0 &= \int_{\partial D} (\varphi_z\, dz \wedge dw \wedge d\overline{w} - \varphi_{\overline{w}}\, dz \wedge d\bar{z} \wedge d\overline{w})\\ &= -\int_D (\varphi_{z\bar{z}} + \varphi_{w\overline{w}})\, dz \wedge d\bar{z} \wedge dw \wedge d\overline{w} = -\frac{1}{2}\int_D (\Delta\varphi)\, dz \wedge d\bar{z} \wedge dw \wedge d\overline{w}\end{aligned}$$

that is $\Delta\varphi = 0$ in D. Similarly (again by the Stokes theorem)

$$\int_{\partial D} (\varphi_z\, dz \wedge d\bar{z} \wedge dw + \varphi_{\overline{w}}\, d\bar{z} \wedge dw \wedge d\overline{w}) = 0$$

and (10) yields $\Delta\psi = 0$ in D. Then (by the very definition of f, g) $\Delta f = 0$ and $\Delta g = 0$ in D. Finally f, g satisfy (2). For instance

$$f_{\bar{z}} + \bar{g}_w = (\varphi_z + \psi_w)_{\bar{z}} + (\varphi_{\overline{w}} - \psi_{\bar{z}})_w = \Delta\varphi = 0.$$

Lemma 2 is proved. Next, we need

Lemma 3. *The mollifications* f_ε, g_ε *satisfy* (5)–(6) *for* $D = B(x_0, R)$.

Let us end the proof of Theorem 2. By Lemma 3 the functions φ_ε and ψ_ε satisfy the hypothesis of Lemma 2. Then Lemma 2 yields $\Delta f_\varepsilon = 0$, $\Delta g_\varepsilon = 0$ in D

and $(f_\varepsilon, g_\varepsilon)$ is a solution to (2) in D. Thus f, g are harmonic in D and, as uniform limits of sequences of harmonic functions on relatively compact subdomains of D, satisfy (2) in D, and therefore in Ω. Q.e.d.

It remains to prove Lemma 3. Let dm be the Lebesgue measure on $\mathbf{R}^4$. We may conduct the following calculation

$$\int_{\partial D} f_\varepsilon(z,w)\, dz \wedge dw \wedge d\overline{w}$$

$$= \int_{\partial D} \left\{ \int_{\mathbf{R}^4} J_\varepsilon(z-\zeta, w-\omega) \tilde{f}(\zeta,\omega)\, dm(\zeta,\omega) \right\} dz \wedge dw \wedge d\overline{w}$$

(by a change of variables and Fubini's theorem)

$$= \int_{\mathbf{R}^4} J_\varepsilon(\zeta,\omega) \left\{ \int_{\partial D} f(z-\zeta, w-\omega)\, dz \wedge dw \wedge d\overline{w} \right\} dm(\zeta,\omega)$$

(as $\tilde{f}$ vanishes outside B we may assume w.l.o.g. that $(\zeta,\omega) \in B$ hence $\partial B((z_0-\zeta, w_0-\omega), R) \subset \Omega$)

$$= \int_{\mathbf{R}^4} J_\varepsilon(\zeta,\omega) \left\{ \int_{\partial B((z_0-\zeta, w_0-\omega), R)} f(z,w)\, dz \wedge dw \wedge d\overline{w} \right\} dm(\zeta,\omega)$$

(by (5) with $D = B((z_0-\zeta, w_0-\omega), R)$)

$$= \int_{\mathbf{R}^4} J_\varepsilon(\zeta,\omega) \left\{ \int_{\partial B((z_0-\zeta, w_0-\omega), R)} g(z,w)\, dz \wedge d\bar{z} \wedge d\overline{w} \right\} dm(\zeta,\omega)$$

$$= \int_{\partial D} g_\varepsilon(z,w)\, dz \wedge d\bar{z} \wedge d\overline{w}.$$

Similarly, f_ε and g_ε satisfy (6). Lemma 3 is proved.

As well known, an application of Morera's theorem is to establish the so called (first) Weierstrass theorem (cf. e.g. Theorem 2.3 in [8], p. 63). The similar application holds for solutions to (2) (though the regularity requirements and proof are much simplified by the results in harmonic function theory)

Corollary 1. *Let* $\Omega \subset \mathbf{C}^2$ *be a domain and* $f_n, g_n : \Omega \to \mathbf{C}$, $n \geq 1$, *a sequence of solutions to* (2). *Assume that the series* $\sum_{n=1}^\infty f_n(z,w)$ *and* $\sum_{n=1}^\infty g_n(z,w)$ *converge respectively to* $f(z,w)$ *and* $g(z,w)$, *uniformly in* $(z,w) \in \overline{D}$, *for any domain* $D \subset\subset \Omega$. *Then* i) f, g *is a solution to* (2), ii) *for any multi-index* α *the series* $\sum_{n=1}^\infty D^\alpha f_n(z,w)$ *and* $\sum_{n=1}^\infty D^\alpha g_n(z,w)$ *converge respectively to* $D^\alpha f(z,w)$ *and* $D^\alpha g(z,w)$, *uniformly in* $(z,w) \in \overline{D}$, *for any domain* $D \subset\subset \Omega$.

Indeed, let us consider a domain $D \subset\subset \Omega$. As f, g are continuous in Ω we may consider the integral in the left hand member of (5), which may be computed by termwise integration of the relevant series, hence (by Proposition 1)

$$\int_{\partial D} (f\, dz \wedge dw \wedge d\overline{w} - \bar{g}\, dz \wedge d\bar{z} \wedge d\overline{w})$$
$$= \sum_{n=1}^{\infty} \int_{\partial D} (f_n\, dz \wedge dw \wedge d\overline{w} - \overline{g_n}\, dz \wedge d\bar{z} \wedge d\overline{w}) = 0,$$

that is (7) holds. Similarly, one may prove (8). Then (by Theorem 2) f, g is a solution to (2). Of course, the following direct proof may be adopted, as well. All solutions to (2) are harmonic and the limit of a uniformly convergent (on closed subdomains $\overline{D} \subset \Omega$) sequence of harmonic functions is known to be harmonic and moreover its derivative of any order is the uniform limit of the termwise derivative of the given sequence. Hence $f_{\bar{z}} + \bar{g}_w = 0$ follows from $\sum_{n=1}^{\infty}(\partial f_n/\partial \bar{z} + \partial \bar{g}_n/\partial w) = 0$.

2. Representation of Solutions

2.1. Representation by harmonic function techniques. Let $\Omega \subset \mathbf{C}^2$ be a domain and $f, g : \Omega \to \mathbf{C}$ a solution to (2). Let

$$\Gamma(x - y) = -\frac{1}{4\pi^2}|x - y|^{-2}$$

be the fundamental solution to the Laplace operator in $\mathbf{R}^4$ and $D \subset \mathbf{C}^2$ a bounded domain such that $\overline{D} \subset \Omega$ and the Green formula holds for D. As f is harmonic in Ω

$$f(y) = \int_{\partial D} \left\{ f(x)\frac{\partial}{\partial \nu}\Gamma(x - y) - \Gamma(x - y)\frac{\partial f}{\partial \nu}(x) \right\} ds_x \tag{11}$$

for any $y \in D$, where ν is the unit outward normal to ∂D. As a consequence of (11) (and the similar representation of g) we may establish the following representation formulae for the solutions to (2)

Theorem 3. *Suppose there is an open set $U \subseteq \mathbf{C}^2$ such that $U \cap \partial D = \{(z, \xi + i\eta) \in \mathbf{C}^2 : (z, \xi) \in A, \eta = a\}$, for some bounded domain $A \subset \mathbf{R}^3$ and some $a > 0$, and moreover $f = 0$ and $g = 0$ in $\partial D \backslash U$. Then*

$$(12)\qquad f(\zeta,\omega) = \frac{1}{2\pi^2 i}\int_A \{\bar{g}(\psi(u))(\bar{z}(u)-\bar{\zeta}) - f(\psi(u))(\overline{w}(u)-\overline{\omega})\}\frac{du}{|(z(u)-\zeta, w(u)-\omega)|^4},$$

$$(13)\qquad g(\zeta,\omega) = -\frac{1}{2\pi^2 i}\int_A \{\bar{f}(\psi(u))(\bar{z}(u)-\bar{\zeta}) + g(\psi(u))(\overline{w}(u)-\overline{\omega})\}\frac{du}{|(z(u)-\zeta, w(u)-\omega)|^4},$$

for any $(\zeta,\omega)\in D$, *where* $\psi(u) = (z(u),w(u)) = (u^1+iu^2, u^3+ia)$, $u\in A$, *is the parametrization of* $U\cap\partial D$.

PROOF. Set $x=(z,w)$ and $y=(\zeta,\omega)$. By the last equation in (1)

$$\int_{\partial D}\Gamma(x-y)\frac{\partial X}{\partial \nu}(x)\,ds_x = \int_{U\cap\partial D}\Gamma(x-y)X_{x^4}(x)\,ds_x = -\int_{U\cap\partial D}\Gamma(x-y)(Y_{x^3}(x)+Z_{x^2}(x)+T_{x^1}(x))\,ds_x$$

(the variables in $\mathbf{R}^4$ are relabeled $x^1=x$, $x^2=y$, $x^3=\xi$ and $x^4=\eta$). Note that

$$\frac{\partial}{\partial u^\alpha}Y(\psi(u)) = Y_{x^\alpha}(\psi(u)),\quad 1\le\alpha\le 3.$$

As $f(\cdot,a)=0$ and $g(\cdot,a)=0$ outside A, we may integrate by parts and use

$$D_i\Gamma(x-y) = \frac{1}{2\pi^2}(x^i-y^i)|x-y|^{-4}\quad 1\le i\le 4,$$

so that to obtain (by (11))

$$(14)\quad X(y) = \frac{1}{2\pi^2}\int_A |\psi(u)-y|^{-4}\{(a-y^4)X(\psi(u)) - (u^3-y^3)Y(\psi(u)) - (u^2-y^2)Z(\psi(u)) - (u^1-y^1)T(\psi(u))\}\,du.$$

Similarly

$$(15)\quad Y(y) = \frac{1}{2\pi^2}\int_A |\psi(u)-y|^{-4}\{(a-y^4)Y(\psi(u)) + (u^3-y^3)X(\psi(u)) - (u^1-y^1)Z(\psi(u)) + (u^2-y^2)T(\psi(u))\}\,du,$$

(16) $$Z(y) = \frac{1}{2\pi^2}\int_A |\psi(u) - y|^{-4}\{(a - y^4)Z(\psi(u))$$
$$+ (u^2 - y^2)X(\psi(u)) + (u^1 - y^1)Y(\psi(u)) - (u^3 - y^3)T(\psi(u))\}\,du,$$

(17) $$T(y) = \frac{1}{2\pi^2}\int_A |\psi(u) - y|^{-4}\{(a - y^4)T(\psi(u))$$
$$+ (u^1 - y^1)X(\psi(u)) - (u^2 - y^2)Y(\psi(u)) + (u^3 - y^3)Z(\psi(u))\}\,du.$$

Now we may add (14) (respectively (16)) to (15) (respectively (17)) multiplied by i so that to obtain (12)–(13). Q.e.d.

As an application of the representation formulae (12)–(13) we obtain the following results ("removing the singularities" of solutions to (2))

Theorem 4. *Let $A \subset \mathbf{R}^3$ be a bounded domain and f, g two continuous functions in $\{(z, \xi + i\eta) \in \mathbf{C}^2 : (z, \xi) \in \bar{A},\ \eta = a\}$. Then the functions f_a, g_a given by*

(18) $$f_a(\zeta, \omega) = \frac{1}{2\pi^2 i}\int_A \{\bar{g}(\psi_a(u))(\bar{z}_a(u) - \bar{\zeta})$$
$$- f(\psi_a(u))(\overline{w}_a(u) - \overline{\omega})\}\frac{du}{|\psi_a(u) - (\zeta, \omega)|^4},$$

(19) $$g_a(\zeta, \omega) = -\frac{1}{2\pi^2 i}\int_A \{\bar{f}(\psi_a(u))(\bar{z}_a(u) - \bar{\zeta})$$
$$+ g(\psi_a(u))(\overline{w}_a(u) - \overline{\omega})\}\frac{du}{|\psi_a(u) - (\zeta, \omega)|^4},$$

for any $y = (\zeta, \omega) \in H_a = \{(z, \xi + i\eta) \in \mathbf{C}^2 : (z, \xi) \in A, \eta > a\}$, are a solution (f_a, g_a) to (2) in H_a. Here $\psi_a(u) = (z_a(u), w_a(u)) = (u^1 + iu^2, u^3 + ia)$, $u \in A$.

Proof. The function $F : \bar{A} \times H_a \to \mathbf{C}$ given by

$$F(u, \zeta, \omega) = \frac{\bar{g}(\psi(u))(\bar{z}(u) - \bar{\zeta}) - f(\psi(u))(\overline{w}(u) - \overline{\omega})}{|\psi(u) - (\zeta, \omega)|^4}$$

(where $\psi = \psi_a$) is continuous on $\bar{A}$ and differentiable in H_a. As $\bar{A}$ is compact we may differentiate under the integral sign in $\int_A F(u, \zeta, \omega)\,du$ so that to obtain

$$\frac{\partial f_a}{\partial \bar{\zeta}} = -\frac{1}{2\pi^2 i}\int_A \left\{ \frac{\bar{g}(\psi(u))}{|\psi(u) - (\zeta,\omega)|^4} - \frac{2[\bar{g}(\psi(u))|z(u)-\zeta|^2 - f(\psi(u))(z(u)-\zeta)(\overline{w}(u)-\bar{\omega})]}{|\psi(u)-(\zeta,\omega)|^6} \right\} du$$

and (with similar arguments)

$$\frac{\partial \bar{g}_a}{\partial \omega} = -\frac{1}{2\pi^2 i}\int_A \left\{ \frac{\bar{g}(\psi(u))}{|\psi(u) - (\zeta,\omega)|^4} - \frac{2[f(\psi(u))(\overline{w}(u)-\bar{\omega})(z(u)-\zeta) + \bar{g}(\psi(u))|w(u)-\omega|^2]}{|\psi(u)-(\zeta,\omega)|^6} \right\} du$$

hence $\partial f_a/\partial\bar{\zeta} + \partial \bar{g}_a/\partial\omega = 0$ in $(\zeta,\omega) \in H_a$. The proof of $\partial f_a/\partial\bar{\omega} - \partial\bar{g}/\partial\zeta = 0$ is similar to the above. Q.e.d.

Combining Theorems 3 and 4 we obtain

Corollary 2. *Let $D \subset \mathbf{C}^2$ be a bounded domain such that the Green lemma holds on D. Assume there is an open set $U \subset \mathbf{C}^2$ such that $U \cap \partial D = \{(z, \xi + i\eta) \in \mathbf{C}^2 : (z,\xi) \in A, \eta = a\}$ and $D \cap H_a \neq \emptyset$, for some bounded domain $A \subset \mathbf{R}^3$ and some $a > 0$. Let $S \subset D \cap H_a$ be a closed subset. Let $\Omega \subseteq \mathbf{C}^2$ be a neighborhood of $\bar{D}$ and $f, g : \Omega \backslash S \to \mathbf{C}$ a solution to (2) in $\Omega\backslash S$ such that $f = 0$ and $g = 0$ on $\partial D \backslash U$. Then (f,g) extends to a solution to (2) in Ω.*

Historical remark. G. Cimmino realized (cf. [3], p. 97–99) the importance of (11) and attempted to derive a representation formula for a solution (X, Y, Z, T) to (1) in a domain $\Omega \subset \mathbf{R}^4$. There (cf. *op. cit.*) it is claimed that, given a bounded domain $D \subset\subset \Omega$ such that ∂D is smooth and a point $(\zeta,\omega) \in D$, one has

$$X_0 = \int_A \begin{vmatrix} \Gamma_x X + \Gamma_y Y - \Gamma_\xi Z + \Gamma_\eta T & x_{u^1} & x_{u^2} & x_{u^3} \\ \Gamma_y X - \Gamma_x Y + \Gamma_\eta Z + \Gamma_\xi T & y_{u^1} & y_{u^2} & y_{u^3} \\ \Gamma_\xi X + \Gamma_\eta Y + \Gamma_x Z - \Gamma_y T & \xi_{u^1} & \xi_{u^2} & \xi_{u^3} \\ \Gamma_\eta X - \Gamma_\xi Y - \Gamma_y Z - \Gamma_x T & \eta_{u^1} & \eta_{u^2} & \eta_{u^3} \end{vmatrix} du^1 du^2 du^3 \tag{20}$$

(together with similar formulae for Y_0, Z_0, T_0) where $X_0 = X(\zeta,\omega)$ and Γ is short for $\Gamma(z-\zeta, w-\omega)$. Also

$$\psi(u) = (x(u), y(u), \xi(u), \eta(u)), \quad u \in A \subset \mathbf{R}^3,$$

is a parametrization of ∂D. No proof is given. Clearly, for (20) to follow from (11) one needs either ∂D to be covered by a single chart, or that $\{X^j\} = \{X, Y, Z, T\}$ vanish on ∂D outside the given coordinate patch. To integrate by parts in

$$\int_A \Gamma(z(u) - \zeta, w(u) - \omega) \frac{\partial X}{\partial v}(\psi(u)) \sqrt{\det[g_{\alpha\beta}(u)]}\, du$$

(as in the proof of Theorem 3) where $g_{\alpha\beta} = \sum_{j=1}^4 \psi^j_{u^\alpha} \psi^j_{u^\beta}$ one must compute (assuming for instance that $\det[\psi^\alpha_{u^\beta}] \neq 0$ in the neighborhood of a point) X^j_η, $1 \leq j \leq 4$, from

$$\frac{\partial}{\partial u^\alpha}(X^j \circ \psi)(u) = X^j_{x^i}(\psi(u)) \frac{\partial \psi^i}{\partial u^\alpha}$$

and the four equations (1), which seems of little hope. Of course, as ∂D is a real hypersurface in $\mathbf{R}^4$ there are local coordinates (u^α) on ∂D and local coordinates (x^i) on $\mathbf{R}^4$ at a point $p \in \partial D$ such that ∂D is given, in a neighborhood of p, by the equations $x^\alpha = u^\alpha$, $x^4 = 0$. However the choice of (x^i) involves a transformation of local coordinates on the ambient space and the system (1) is not invariant.

2.2. Cauchy-Pompeiu type integral formulae. Let $\Omega \subset \mathbf{C}^2$ be a domain and $f, g \in C^1(\Omega)$. Let $a \in \Omega$ and let $D_i \subset \mathbf{C}$, $i = 1, 2$, be two simply connected domains such that $\bar{D}_i$ is compact, ∂D_i is piecewise smooth, $\bar{D}_1 \times \bar{D}_2 \subset \Omega$, and $a_i \in D_i$, $i = 1, 2$. By the classical Cauchy-Pompeiu formula

$$2\pi i f(\zeta, w) = \int_{\partial D} \frac{f(z, w)\, dz}{z - \zeta} - \text{p.v.} \int_D \frac{f_{\bar{z}}(z, w)\, dz \wedge d\bar{z}}{z - \zeta}, \tag{21}$$

for any $\zeta \in D = D_1$ and $w \in \mathbf{C}$ such that $(z, w) \in \Omega$ for any $z \in \bar{D}$. As well known, the principal value is

$$\text{p.v.} \int_D \frac{f_{\bar{z}}(z, w)\, dz \wedge d\bar{z}}{z - \zeta} = \lim_{\varepsilon \to 0^+} \int_{D \setminus B(\zeta, \varepsilon)} \frac{f_{\bar{z}}(z, w)\, dz \wedge d\bar{z}}{z - \zeta}$$

and the convergence is uniform in $w \in D$, as shown by the following elementary

Lemma 4. *Let $F : \Omega \to \mathbf{C}$ be a continuos function and set*

$$\varphi_\varepsilon(w) = \int_{D \setminus B(\zeta, \varepsilon)} \frac{F(z, w)\, dz \wedge d\bar{z}}{z - \zeta}.$$

Then the limit $\lim_{\varepsilon \to 0^+} \varphi_\varepsilon(w)$ exists and is uniform in $w \in D$.

Proof. For any $\gamma > 0$

$$|\varphi_{\varepsilon+\gamma}(w) - \varphi_{\varepsilon}(w)| = \left| \int_{\varepsilon \le |z-\zeta| < \varepsilon+\gamma} \frac{F(z,w)\, dz \wedge d\bar{z}}{z-\zeta} \right| \le 2\pi C\gamma$$

where $C = 2 \sup_{(z,w) \in \bar{D}_1 \times \bar{D}_2} |F(z,w)|$, i.e. $\varphi_{\varepsilon}(w)$ is uniformly Cauchy as $\varepsilon \to 0^+$. Q.e.d.

Given $\omega \in D_2$ let us divide (21) by $\overline{w} - \overline{\omega}$ and integrate over $D_2 \backslash B(\omega, \delta)$, for sufficiently small $\delta > 0$. By Lemma 4

$$\int_{D_2 \backslash B(\omega,\delta)} \left[2\pi i f(\zeta, w) - \int_{\partial D} \frac{f(z,w)\, dz}{z-\zeta} \right] \frac{dw \wedge d\overline{w}}{\overline{w} - \overline{\omega}} \tag{22}$$

$$= -\lim_{\varepsilon \to 0^+} \int_{D_2 \backslash B(\omega,\delta)} \left[\int_{D \backslash B(\zeta,\varepsilon)} \frac{f_{\bar{z}}(z,w)\, dz \wedge d\bar{z}}{z-\zeta} \right] \frac{dw \wedge d\overline{w}}{\overline{w} - \overline{\omega}}.$$

The limit in the right hand side of (22) is uniform in $\delta \ge \delta_0$ (where $\delta_0 = \frac{1}{2}\,\mathrm{dist}(\omega, \partial D_2)$). Indeed, let us set

$$F_\delta(z) = \int_{D_2 \backslash B(\omega,\delta)} \frac{f_{\bar{z}}(z,w)\, dw \wedge d\overline{w}}{\overline{w} - \overline{\omega}}.$$

By Fubini's theorem the integral in the right hand side of (22) is

$$h_\varepsilon(\delta) = \int_{D \backslash B(\zeta,\varepsilon)} \frac{F_\delta(z)\, dz \wedge d\bar{z}}{z-\zeta}$$

hence for any $\gamma > 0$

$$|h_{\varepsilon+\gamma}(\delta) - h_\varepsilon(\delta)| = \left| \int_{\varepsilon \le |z-\zeta| < \varepsilon+\gamma} \frac{F_\delta(z)\, dz \wedge d\bar{z}}{z-\zeta} \right|$$

$$\le 2\pi C\gamma \int_{D_2 \backslash B(\omega,\delta)} \frac{d\xi \wedge d\eta}{|\xi + i\eta - \omega|}$$

where $C = 2 \sup_{(z,w) \in \bar{D}_1 \times \bar{D}_2} |f_{\bar{z}}(z,w)|$. As $\xi + i\eta \in D_2 \backslash B(\omega,\delta)$ one has $1/|\xi + i\eta - \omega| \le 1/\delta$ hence the last integral is $\le (|D_2| - \pi\delta^2)/\delta$ which is strictly decreasing. Here $|A|$ is the Lebesgue measure of A. We may conclude that $|h_{\varepsilon+\gamma}(\delta) - h_\varepsilon(\delta)| \le 2\pi C\gamma(|D_2| - \delta_0^2)/\delta_0$, i.e. $h_\varepsilon(\delta)$ is uniformly Cauchy as $\varepsilon \to 0^+$. By Lemma 4 the limit $\lim_{\delta \to 0^+} F_\delta(z)$ exists. Let us take $\delta \to 0^+$ in (22) and use uniformity to interchange limits. We obtain

$$\int_{D_2}\left[2\pi i f(\zeta,w)-\int_{\partial D}\frac{f(z,w)\,dz}{z-\zeta}\right]\frac{dw\wedge d\overline{w}}{\overline{w}-\overline{\omega}} \tag{23}$$

$$=-\int_D\left[\iint_{D_2}\frac{f_{\bar z}(z,w)\,dw\wedge d\overline{w}}{\overline{w}-\overline{\omega}}\right]\frac{dz\wedge d\bar z}{z-\zeta}$$

where all double integrals are meant in the sense of principal value. Similarly we obtain

$$\int_D\left[2\pi i\bar g(z,\omega)+\int_{\partial D_2}\frac{\bar g(z,w)\,d\overline{w}}{\overline{w}-\overline{\omega}}\right]\frac{dz\wedge d\bar z}{z-\zeta}$$

$$=-\int_D\left[\iint_{D_2}\frac{\bar g_w(z,w)\,dw\wedge d\overline{w}}{\overline{w}-\overline{\omega}}\right]\frac{dz\wedge d\bar z}{z-\zeta}.$$

Summing up the last two identities we obtain (24) in

THEOREM 5. *Let $\Omega\subset\mathbf{C}^2$ be a domain and $f,g\in C^1(\Omega)$. Let $(\zeta,\omega)\in\Omega$ and let $D_i\subset\mathbf{C}$ $(i=1,2)$ be two simply connected domains such that $\overline{D}_i$ is compact, ∂D_i is piecewise smooth, $\zeta\in D_1$, $\omega\in D_2$, and $\overline{D}_1\times\overline{D}_2\subset\Omega$. Then*

$$\int_{D_2}\left[2\pi i f(\zeta,w)-\int_{\partial D_1}\frac{f(z,w)\,dz}{z-\zeta}\right]\frac{dw\wedge d\overline{w}}{\overline{w}-\overline{\omega}} \tag{24}$$

$$+\int_{D_1}\left[2\pi i\bar g(z,\omega)+\int_{\partial D_2}\frac{\bar g(z,w)\,d\overline{w}}{\overline{w}-\overline{\omega}}\right]\frac{dz\wedge d\bar z}{z-\zeta}$$

$$=-\int_{D_1}\left[\iint_{D_2}\frac{(f_{\bar z}+\bar g_w)\,dw\wedge d\overline{w}}{\overline{w}-\overline{\omega}}\right]\frac{dz\wedge d\bar z}{z-\zeta},$$

$$\int_{D_1}\left[2\pi i f(z,\omega)-\int_{\partial D_2}\frac{f(z,w)\,dw}{w-\omega}\right]\frac{dz\wedge d\bar z}{\bar z-\bar\zeta} \tag{25}$$

$$-\int_{D_2}\left[2\pi i\bar g(\zeta,w)+\int_{\partial D_1}\frac{\bar g(z,w)\,d\bar z}{\bar z-\bar\zeta}\right]\frac{dw\wedge d\overline{w}}{w-\omega}$$

$$=-\int_{D_1}\left[\iint_{D_2}\frac{(f_{\overline{w}}-\bar g_z)\,dw\wedge d\overline{w}}{w-\omega}\right]\frac{dz\wedge d\bar z}{\bar z-\bar\zeta}.$$

The proof of (25) is similar. If f, g is a solution to (2) then (by (24)–(25)) we obtain the following identities (similar to the Cauchy integral formula for a holomorphic function)

$$
\int_{D_2}\left[f(\zeta,w)-\frac{1}{2\pi i}\int_{\partial D_1}\frac{f(z,w)\,dz}{z-\zeta}\right]\frac{dw\wedge d\overline{w}}{\overline{w}-\overline{\omega}} + \int_{D_1}\left[\bar{g}(z,\omega)+\frac{1}{2\pi i}\int_{\partial D_2}\frac{\bar{g}(z,w)\,d\overline{w}}{\overline{w}-\overline{\omega}}\right]\frac{dz\wedge d\bar{z}}{z-\zeta}=0, \tag{26}
$$

$$
\int_{D_1}\left[f(z,\omega)-\frac{1}{2\pi i}\int_{\partial D_2}\frac{f(z,w)\,dw}{w-\omega}\right]\frac{dz\wedge d\bar{z}}{\bar{z}-\bar{\zeta}} - \int_{D_2}\left[\bar{g}(\zeta,w)+\frac{1}{2\pi i}\int_{\partial D_1}\frac{\bar{g}(z,w)\,d\bar{z}}{\bar{z}-\bar{\zeta}}\right]\frac{dw\wedge d\overline{w}}{w-\omega}=0. \tag{27}
$$

No applications of (26)–(27) are known as yet.

3. Inhomogeneous Systems

Systems similar to (1) (all of whose solutions are harmonic functions) appear as (subspaces of) perps of ranges of certain linear operators of Hilbert spaces associated to a boundary value problem for a given PDE system. The phenomenon has been discovered by G. Cimmino (cf. [2] and [5]) in an attempt to formulate compatibility conditions for the boundary data (and free terms), in a given boundary value problem. Given a linear operator $L:\mathscr{X}\to\mathscr{Y}$ of Hilbert spaces, the basic idea is that whenever a solution $u\in\mathscr{D}(L)$ to the equation $Lu=f$ exists, f must satisfy compatibility conditions of the form $\langle f,g\rangle_{\mathscr{Y}}=0$, for any $g\in\mathscr{Z}$, where $\mathscr{Z}\subseteq\mathscr{R}(L)^{\perp}$ is some subspace which may be described[1] explicitly. Of course, when $\mathscr{R}(L)$ is closed in $\mathscr{Y}$ and $\mathscr{Z}$ is dense in $\mathscr{R}(L)^{\perp}$ the compatibility relations are also sufficient for solving $Lu=f$. G. Cimmino uses (cf. [5]) this tautology to write compatibility conditions for the problem

$$
\begin{cases} X_x - Y_y + Z_\xi - T_\eta = a \\ X_y + Y_x - Z_\eta - T_\xi = b \\ X_\xi - Y_\eta - Z_x + T_y = c \\ X_\eta + Y_\xi + Z_y + T_x = d \end{cases} \quad \text{in } \Omega, \tag{28}
$$

$$
X=\alpha,\quad Y=\beta,\quad Z=\gamma,\quad T=\delta \quad \text{on } \partial\Omega, \tag{29}
$$

where $\Omega\subset\mathbf{R}^4$ is a domain. There (cf. *op. cit.*) it is suggested that the compatibility conditions may be obtained when either strong or weak solutions to (28)–(29) are assumed to exist. However, neither the required regularity con-

[1] A heuristic description (for linear operators of *finite dimensional* spaces) of the general method of identifying subspaces $\mathscr{Z}\subseteq\mathscr{R}(L)^{\perp}$ (spaces of solutions to a homogeneous system associated with the given system) is given in [4].

ditions are specified, nor proofs are given. We solve the problem (along the guidelines traced by G. Cimmino, cf. *op. cit.*) when (28)–(29) admits suitable strong solutions and obtain

Theorem 6. *Let $\Omega \subset \mathbf{C}^2$ be a bounded domain on which Green's formula holds and $f, g \in L^2(\Omega)$, $\varphi, \psi \in L^2(\partial\Omega)$. If there is a solution $u, v \in C^1(\Omega) \cap C^0(\overline{\Omega})$ to the boundary value problem*

$$u_{\bar z} + \bar v_w = f, \quad u_{\bar w} - \bar v_z = g \quad \text{in } \Omega,$$

$$u = \varphi, \quad v = \psi \quad \text{on } \partial\Omega,$$

then (f, g, φ, ψ) satisfies the compatibility relations

$$\text{(30)} \qquad \operatorname{Re}\left\{2\int_\Omega (f\bar h + g\bar k)\, dV - \int_{\partial\Omega} \{\varphi[(n_1 + in_2)\bar h + (n_3 + in_4)\bar k] \right.$$
$$\left. + \psi[(n_3 + in_4)h - (n_1 + in_2)k]\}\, d\sigma\right\} = 0,$$

for any solution $h, k \in C^1(\Omega) \cap C^0(\overline{\Omega})$ to

$$h_z + k_w = 0, \quad h_{\bar w} - k_{\bar z} = 0 \quad \text{in } \Omega,$$

where (n_1, n_2, n_3, n_4) is the outward unit normal on $\partial\Omega$.

A similar result may be obtained for the boundary value problem for the inhomogeneous Cauchy-Riemann system (cf. [5], p. 62–63, and our Theorem 7). It is an open problem whether the compatibility relations (30) are sufficient for solving the boundary value problem with the data (f, g, φ, ψ).

Proof of Theorem 6. On $\mathscr{Y} = L^2(\Omega)^2 \times L^2(\partial\Omega)^2$ we consider the scalar product

$$(f, g, \varphi, \psi) \cdot (h, k, \lambda, \mu) = \operatorname{Re}\int_\Omega (f\bar h + g\bar k)\, dV + \operatorname{Re}\int_{\partial\Omega} (\varphi\bar\lambda + \psi\bar\mu)\, d\sigma$$

(making $\mathscr{Y}$ into a Hilbert space). Let $\mathscr{X} = L^2(\Omega)^2$ and $L : \mathscr{X} \to \mathscr{Y}$ be the operator given by

$$L(u, v) = (u_{\bar z} + \bar v_w, u_{\bar w} - \bar v_z, u|_{\partial\Omega}, v|_{\partial\Omega}),$$

with the domain $\mathscr{D}(L) = [C^1(\Omega) \cap C^0(\overline{\Omega})]^2$ (clearly $C^0(\overline{\Omega}) \subset L^2(\Omega)$ as Ω is bounded, and then $\mathscr{D}(L)$ is dense in $\mathscr{X}$). If a solution $(u, v) \in \mathscr{D}(L)$ to $L(u, v) =$

(f, g, φ, ψ) exists then one may produce a subspace $\mathscr{Z} \subset \mathscr{R}(L)^{\perp}$ such that the orthogonality condition $(f, g, \varphi, \psi) \cdot (h, k, \lambda, \mu) = 0$, for any $(h, k, \lambda, \mu) \in \mathscr{Z}$, implies (30). Indeed, we may set $\mathscr{Z} = \mathscr{R}(L)^{\perp} \cap \{[C^1(\Omega) \cap C^0(\overline{\Omega})]^2 \times L^2(\partial\Omega)^2\}$ and then $(h, k, \lambda, \mu) \in \mathscr{Z}$ if

$$\text{(31)} \qquad \operatorname{Re}\left\{\int_{\Omega} [(u_{\bar z} + \bar v_w)\bar h + (u_{\bar w} - \bar v_z)\bar k]\, dV + \int_{\partial\Omega} (u\bar\lambda + v\bar\mu)\, d\sigma\right\} = 0,$$

for any $(u, v) \in \mathscr{D}(L)$. The first integral in (31) may be calculated as

$$\begin{aligned}
&\int_{\Omega} [(u_{\bar z} + \bar v_w)\bar h + (u_{\bar w} - \bar v_z)\bar k]\, dV \\
&\quad = \int_{\Omega} [(u\bar h)_{\bar z} - u\bar h_{\bar z} + (\bar v, \bar h)_w - \bar v\bar h_w + (u\bar k)_{\bar w} - u\bar k_{\bar w} - (\bar v\bar k)_z + \bar v\bar k_z]\, dV \\
&\quad = \int_{\Omega} \left[\operatorname{div}\left(u\bar h \frac{\partial}{\partial \bar z}\right) - u\bar h_{\bar z} + \operatorname{div}\left(\bar v\bar h \frac{\partial}{\partial w}\right) - \bar v\bar h_w \right. \\
&\qquad\qquad \left. + \operatorname{div}\left(u\bar k \frac{\partial}{\partial \bar w}\right) - u\bar k_{\bar w} - \operatorname{div}\left(\bar v\bar k \frac{\partial}{\partial z}\right) + \bar v\bar k_z\right] dV \\
&\quad = \frac{1}{2}\int_{\partial\Omega} [u\bar h(n_1 + in_2) + \bar v\bar h(n_3 - in_4) + u\bar k(n_3 + in_4) - \bar v\bar k(n_1 - in_2)]\, d\sigma \\
&\qquad - \int_{\Omega} [u(\bar h_{\bar z} + \bar k_{\bar w}) + \bar v(\bar h_w - \bar k_z)]\, dV
\end{aligned}$$

(by Green's formula). Therefore (31) may be written

$$\text{(32)} \qquad \begin{aligned}
&\operatorname{Re}\int_{\Omega} \{u(\bar h_{\bar z} + \bar k_{\bar w}) + \bar v(\bar h_w - \bar k_z)\}\, dV \\
&\qquad - \operatorname{Re}\int_{\partial\Omega} \left\{u\left[\bar\lambda + \frac{1}{2}(n_1 + in_2)\bar h + \frac{1}{2}(n_3 + in_4)\bar k\right]\right. \\
&\qquad \left. + v\left[\bar\mu + \frac{1}{2}(n_3 + in_4)h - \frac{1}{2}(n_1 + in_2)k\right]\right\} d\sigma = 0.
\end{aligned}$$

In particular (32) holds for any $u, v \in C_0^\infty(\Omega)$

$$\text{(33)} \qquad \operatorname{Re}\int_{\Omega} \{u(\bar h_{\bar z} + \bar k_{\bar w}) + \bar v(\bar h_w - \bar k_z)\}\, dV = 0,$$

which implies that $h_z + k_w = 0$, $h_{\bar w} - k_{\bar z} = 0$ in Ω, by an elementary argument. Indeed, let $u_\nu, v_\nu \in C_0^\infty(\Omega)$ such that $u_\nu \to h_z + k_w$ and $v_\nu \to h_{\bar w} - k_{\bar z}$ in $L^2(\Omega)$ as $\nu \to \infty$. Then

$$\int_\Omega u_\nu(\bar{h}_{\bar{z}} + \bar{k}_{\bar{w}})\, dV = \int_\Omega (u_\nu - h_z - k_w)(\bar{h}_{\bar{z}} + \bar{k}_{\bar{w}})\, dV + \|h_z + k_w\|^2,$$

$$\left|\int_\Omega (u_\nu - h_z - k_w)(\bar{h}_{\bar{z}} + \bar{k}_{\bar{w}})\, dV\right| \le \|u_\nu - h_z - k_w\|\, \|h_z + k_w\| \to 0,$$

for $\nu \to \infty$, hence (by (33)) $\|h_z + k_w\|^2 + \|h_{\bar{w}} - k_{\bar{z}}\|^2 = 0$. Q.e.d.

Then (32) yields

$$\lambda + \frac{1}{2}(n_1 - in_2)h + \frac{1}{2}(n_3 - in_4)k = 0, \tag{34}$$

$$\mu + \frac{1}{2}(n_3 - in_4)\bar{h} - \frac{1}{2}(n_1 - in_2)\bar{k} = 0 \tag{35}$$

on $\partial\Omega$. Let $u, v \in C^1(\Omega) \cap C^0(\overline{\Omega})$ be a solution to $L(u, v) = (f, g, \varphi, \psi)$. We may substitute from (34)–(35) into

$$\operatorname{Re} \int_\Omega (f\bar{h} + g\bar{k})\, dV + \operatorname{Re} \int_{\partial\Omega} (\varphi\bar{\lambda} + \psi\bar{\mu})\, d\sigma = 0$$

so that to obtain (30). Q.e.d.

The differential operator

$$Q^* = \begin{pmatrix} \partial/\partial z & \partial/\partial w \\ \partial/\partial \overline{w} & -\partial/\partial \overline{z} \end{pmatrix}$$

is referred to as the *adjoint Cimmino operator*. We shall need

Lemma 5. *The solutions to $Q^*F = 0$ are harmonic functions. More generally, the adjoint Cimmino operator is hypoelliptic.*

The proof is similar to that of Lemma 1. It is tempting to look for a characterization of the full $\mathscr{R}(L)^\perp$ (similar to that of $\mathscr{L}$). The proof of Theorem 6 requires that given $(h, k, \lambda, \mu) \in \mathscr{R}(L)^\perp$ the functions h, k be smooth. This is indeed so as a consequence of Lemma 5. Precisely, we have

Proposition 2. *Let $P : \mathscr{Y} \to L^2(\Omega)^2$ be the natural projection. Then $P\mathscr{R}(L)^\perp \subseteq \mathscr{H}^2(\Omega)^2$, where $\mathscr{H}^2(\Omega)$ is the Bergman space of all harmonic L^2 functions on Ω.*

Proof. We set $\langle f, g\rangle = \int_\Omega f\bar{g}\,dV$. Then $\mathscr{R}(L)^\perp$ consists of all (h,k,λ,μ) in $\mathscr{Y}_0$ such that

$$\mathrm{Re}\left\{\langle u_{\bar z} + \bar v_w, h\rangle + \langle u_{\bar w} - \bar v_z, k\rangle + \int_{\partial\Omega} (u\bar\lambda + v\bar\mu)\,d\sigma\right\} = 0,$$

for any $u, v \in C^1(\Omega)\cap C^0(\overline{\Omega})$. In particular for $u = \varphi$ and $v = \psi$, $\varphi, \psi \in C_0^\infty(\Omega)$

$$\begin{aligned}0 &= \mathrm{Re}\{T_h(\varphi_{\bar z} + \bar\psi_w) + T_k(\varphi_{\bar w} - \bar\psi_z)\}\\ &= -\mathrm{Re}\{(\partial T_h/\partial z + \partial T_k/\partial w)(\varphi) + (\partial T_h/\partial\bar w - \partial T_k/\partial\bar z)(\psi)\}\end{aligned}$$

hence (for $\psi = 0$) $\partial T_h/\partial z + \partial T_k/\partial w = 0$ (in distribution sense) and similarly $\partial T_h/\partial\bar w - \partial T_k/\partial\bar z = 0$. Then (by Lemma 5) $h, k \in \mathscr{H}^2(\Omega)$. Q.e.d.

However, the proof of Theorem 6 also requires continuity of h, k up to the boundary (so that one may apply Green's formula). One may restrict the domain of L to be $\mathscr{D}(L) = C^1(\overline{\Omega})$ so that $\mathscr{R}(L) \subseteq C^0(\overline{\Omega})^2 \times C^0(\partial\Omega)^2 =: \mathscr{Y}_0$ (a pre-Hilbert subspace of $\mathscr{Y}$). Let $\mathscr{R}(L)^\perp = \{y \in \mathscr{Y}_0 : L(x)\cdot y = 0,\ \text{for any } x \in \mathscr{D}(L)\}$. Then (by Green's formula)

Proposition 3. *For any* $(h,k,\lambda,\mu) \in \mathscr{R}(L)^\perp$ *one has* $h, k \in \mathscr{H}^2(\Omega)\cap C^0(\overline{\Omega})$ *and* $Q^*(h,k)^t = 0$ *and* λ, μ *are given by* (34)–(35). *In particular, if* $(f, g, \varphi, \psi) \in \mathscr{Y}_0$ *satisfies the compatibility condition* (30) *then* $(f, g, \varphi, \psi) \in [\mathscr{R}(L)^\perp]^\perp$.

Proposition 3 is of limited use as $\mathscr{Y}_0$ is not complete (and $\mathscr{R}(L)^\perp$ may fail to be closed). We end this section by proving a result similar to that in Theorem 6 for the inhomogeneous Cauchy-Riemann system.

Theorem 7. *Let* $\Omega \subset \mathbf{C}$ *be a bounded domain such that Green's formula holds on* Ω, *and* $f \in L^2(\Omega)$, $\varphi \in L^2(\partial\Omega)$. *If the Dirichlet problem*

$$u_{\bar z} = f \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega,$$

admits a solution $u \in C^1(\Omega)\cap C^0(\overline{\Omega})$ *then* (f, φ) *satisfies the compatibility relation*

$$2\,\mathrm{Re}\int_\Omega f\bar g\,dV - \mathrm{Re}\int_{\partial\Omega} \varphi(n_1 + in_2)\bar g\,d\sigma = 0$$

for any antiholomorphic function $g : \Omega \to \mathbf{C}$ *which is continuous up to the boundary, where* (n_1, n_2) *is the outward unit normal on* $\partial\Omega$.

Proof. Let $\mathscr{Y} = L^2(\Omega) \times L^2(\partial\Omega)$ with the scalar product

$$(f, \varphi) \cdot (g, \psi) = \operatorname{Re} \int_\Omega f \bar{g} \, dV + \operatorname{Re} \int_{\partial\Omega} \varphi \bar{\psi} \, d\sigma.$$

Clearly $\mathscr{Y}$ is complete. Let $L : L^2(\Omega) \to \mathscr{Y}$ be the operator given by

$$Lu = (u_{\bar{z}}, u|_{\partial\Omega})$$

with the domain $\mathscr{D}(L) = C^1(\Omega) \cap C^0(\overline{\Omega})$. Moreover, we set $\mathscr{Z} = \mathscr{R}(L)^\perp \cap \{[C^1(\Omega) \cap C^0(\overline{\Omega})] \times L^2(\partial\Omega)\}$ so that $(g, \psi) \in \mathscr{Z}$ if

$$\operatorname{Re} \int_\Omega u_{\bar{z}} \bar{g} \, dV + \operatorname{Re} \int_{\partial\Omega} u \bar{\psi} \, d\sigma = 0,$$

for any $u \in \mathscr{D}(L)$. By Green's formula

$$\int_\Omega u_{\bar{z}} \bar{g} \, dV = \int_\Omega \{(u\bar{g})_{\bar{z}} - u\bar{g}_{\bar{z}}\} \, dV$$

$$= \int_\Omega \left\{ \operatorname{div}\left(u\bar{g} \frac{\partial}{\partial \bar{z}} \right) - u\bar{g}_{\bar{z}} \right\} dV = \frac{1}{2} \int_{\partial\Omega} u\bar{g}(n_1 + in_2) \, d\sigma - \int_\Omega u\bar{g}_{\bar{z}} \, dV$$

hence

$$\operatorname{Re} \int_\Omega u\bar{g}_{\bar{z}} \, dV - \operatorname{Re} \int_{\partial\Omega} u \left\{ \bar{\psi} + \frac{1}{2}(n_1 + in_2)\bar{g} \right\} d\sigma = 0, \tag{36}$$

which is easily seen to yield $g_z = 0$ in Ω and $\psi + \frac{1}{2}(n_1 - in_2)g = 0$ on $\partial\Omega$. Indeed, let $u_\nu \in C_0^\infty(\Omega)$ such that $u_\nu \to g_z$ in $L^2(\Omega)$, as $\nu \to \infty$. Then

$$\int_\Omega u_\nu \bar{g}_{\bar{z}} \, dV = \int_\Omega (u_\nu - g_z)\bar{g}_{\bar{z}} \, dV + \|g_z\|^2,$$

$$\left| \int_\Omega (u_\nu - g_z)\bar{g}_{\bar{z}} \, dV \right| \le \|u_\nu - g_z\| \, \|g_z\| \to 0, \quad \nu \to \infty,$$

hence $\|g_z\|^2 = 0$. Q.e.d.

4. Nontangential Limits for Solutions to the Cimmino System

Let $\Omega \subset \mathbf{C}^2$ be a bounded domain with smooth (C^2) boundary. For every $0 < p < \infty$ let $S^p(\Omega)$ be the class of solutions $(f, g) : \Omega \to \mathbf{C}^2$ to the system (2) such that

$$\sup_{\varepsilon > 0} \int_{\partial\Omega_\varepsilon} (|f(z, w)|^2 + |g(z, w)|^2)^{p/2} \, d\sigma_\varepsilon(z, w) < \infty$$

for any fixed family of approximating domains Ω_ε, i.e. if φ is a C^2 defining function for Ω $(\Omega = \{\varphi < 0\})$ then

$$\Omega_\varepsilon = \{(z, w) \in \Omega : \varphi(z, w) < -\varepsilon\} \quad (\varepsilon > 0).$$

By analogy with the boundary behavior of holomorphic functions it is a natural problem whether nontangential limits

$$\lim_{\mathscr{A}_\alpha(\zeta,\omega) \ni (z,w) \to (\zeta,\omega)} F(z, w)$$

exist. Here the approach region is

$$\mathscr{A}_\alpha(\zeta, \omega) = \{(z, w) \in \Omega : |(z - \zeta, w - \omega) \cdot \bar{v}_{(\zeta,\omega)}| < (1 + \alpha)\delta_{(\zeta,\omega)}(z, w),$$
$$|z - \zeta|^2 < \alpha\delta_{(\zeta,\omega)}(z, w)\} \quad (\alpha > 0, (\zeta, \omega) \in \partial\Omega)$$

and $\delta_{(\zeta,\omega)}(z, w)$ is the minimum among $\delta(z, w) = \operatorname{dist}((z, w), \partial\Omega)$ and $\operatorname{dist}((z, w), T_{(\zeta,\omega)}(\partial\Omega))$. Also $v_{(\zeta,\omega)} \in \mathbf{C}^2$ is the complex unit normal at (ζ, ω) (pointing outward Ω). While we leave this problem open we rely once again on the theory of harmonic functions to derive the (more modest) Theorem 8 below. Let $\Gamma_\alpha(\zeta, \omega)$ be the cone of aperture α and vertex (ζ, ω) i.e.

$$\Gamma_\alpha(\zeta, \omega) = \{(z, w) \in \Omega : |(z - \zeta, w - \omega)| < (1 + \alpha)\delta(z, w)\}.$$

We may state

THEOREM 8. *Assume that $F = (f, g) : \Omega \to \mathbf{C}^2$ belongs to $S^p(\Omega)$ for some $p \geq 1$. Then the function $|F|^p$ is subharmonic on Ω and is harmonic if and only if F is a constant map. In particular, if $p \geq 2$ then F admits nontangential limits*

$$\lim_{\Gamma_\alpha(\zeta,\omega) \ni (z,w) \to (\zeta,\omega)} F(z, w) \tag{37}$$

at almost every boundary point $(\zeta, \omega) \in \partial\Omega$.

PROOF. Let $F = (f, g) \in S^p(\Omega)$. As both f, g are harmonic in Ω

$$\Delta|F|^p = 2\{(|F|^p)_{z\bar{z}} + (|F|^p)_{w\bar{w}}\}$$
$$= p|F|^{p-2}\{|\partial f|^2 + |\bar{\partial} f|^2 + |\partial g|^2 + |\bar{\partial} g|^2\}$$
$$+ \frac{p(p-2)}{2}|F|^{p-4}\{|f_{\bar{z}}\bar{f} + f\bar{f}_{\bar{z}} + g_{\bar{z}}\bar{g} + g\bar{g}_{\bar{z}}|^2$$
$$+ |f_{\bar{w}}\bar{f} + f\bar{f}_{\bar{w}} + g_{\bar{w}}\bar{g} + g\bar{g}_{\bar{w}}|^2\},$$

where

$$|\partial f|^2 = |f_z|^2 + |f_w|^2, \quad |\bar{\partial} f|^2 = |f_{\bar{z}}|^2 + |f_{\bar{w}}|^2.$$

Therefore, to conclude that $\Delta|F|^p \geq 0$ in Ω based on the above calculation one should assume that $p \geq 4$. However, the following elementary result circumvents this difficulty.

Lemma 6. *Let $G : \mathbf{R}^n \to \mathbf{R}$ be a continuous convex function. Let $\Omega \subseteq \mathbf{R}^m$ be a domain and let $u_j : \Omega \to \mathbf{R}$, $1 \leq j \leq n$, be harmonic functions. Then $G(u_1, \dots, u_n)$ is subharmonic in Ω. If additionally* i) *G is of class C^2 and strictly convex in $\mathbf{R}^n \setminus \{0\}$,* ii) *$G(\xi) \geq 0$, for any $\xi \in \mathbf{R}^n$, and $G(0) = 0$, and* iii) *$G(u_1, \dots, u_n)$ is harmonic in Ω, then each u_j is constant.*

Indeed the first statement in Theorem 8 follows from Lemma 6 for $G(x) = |x|^p$, $x \in \mathbf{R}^4$ $(p \geq 1)$. The remaining part of the proof is standard. Indeed, if additionally $p \geq 2$ then $u(z, w) = |F(z, w)|^{p/2}$ is subharmonic in Ω and

$$\sup_{\varepsilon > 0} \int_{\partial\Omega_\varepsilon} u(z, w)^2 \, d\sigma_\varepsilon(z, w) < \infty,$$

hence (cf. e.g. [7], p. 8–9) there is a harmonic function h which is the Poisson integral of a function $f \in L^2(\partial\Omega)$ such that $u(z, w) \leq h(z, w)$. Then we may apply Theorem 3 in [7], p. 11, to the function h hence there is $C_\alpha > 0$ such that for any $(\zeta, \omega) \in \partial\Omega$

$$u(z, w) \leq C_\alpha \sum_{k=1}^{\infty} \frac{\int_{B((\zeta,\omega), 2^k\eta)} |f(x)| \, d\sigma(x)}{2^k |B((\zeta, \omega), 2^k\eta)|}$$

for any $(z, w) \in \{\Gamma_\alpha(\zeta, \omega) : |(z - \zeta, w - \omega)| = \eta\}$. Here $B((\zeta, \omega), \rho) = \{(z, w) \in \partial\Omega : |(z - \zeta, w - \omega)| < \rho\}$. Consequently, one may argue as in the proof of Theorem 4 in [7], p. 12, to conclude that the nontangential limit (37) exists. Q.e.d.

Proof of Lemma 6. Let $u = (u_1, \dots, u_n)$ and $v = G \circ u$. Also, let $x \in \Omega$ and $r > 0$ such that $\overline{B(x, r)} \subset \Omega$ and let us set

$$M_r(u)(x) = \frac{1}{|\partial D(x, r)|} \int_{\partial D(x, r)} u(x) \, d\sigma(x).$$

As each u_j is harmonic, $u(x) = M_r(u)(x)$ hence (by the Jensen inequality)

$$v(x) = G(M_r(u)(x)) \leq M_r(v)(x)$$

hence (as v is continuous) v is subharmonic in Ω. Let $\Omega_0 = \{x \in \Omega : u(x) \neq 0\}$. Then

$$\Delta v = \sum_{j=1}^{n} \frac{\partial G}{\partial \xi_j}(u)\Delta u_j + \sum_{\alpha=1}^{n} \langle (D^2 G)(u)\partial_\alpha u, \partial_\alpha u \rangle$$

where $D^2 G$ is the Hessian of G and $\partial_\alpha u = \partial u / \partial x_\alpha$. As $(D^2 G)(u(x))$ is positive definite for any $x \in \Omega_0$, $\Delta u = 0$ and $\Delta v = 0$ yield $\partial_\alpha u = 0$ in Ω_0. Let $\Omega_j = \{x \in \Omega : u_j(x) \neq 0\}$. Then (as $\Omega_j \subseteq \Omega_0$) $\partial_\alpha u_j^2 = 2u_j \partial_\alpha u_j = 0$ in Ω. Q.e.d.

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