

## REAL HYPERSURFACES OF A NONFLAT COMPLEX SPACE FORM IN TERMS OF THE RICCI TENSOR

By

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**Abstract.** We know the fact that there are no real hypersurfaces with parallel Ricci tensors in a nonflat complex space form (cf. [5]). In this paper we investigate real hypersurfaces in a nonflat complex space form using some conditions of the Ricci tensor  $S$  which are weaker than  $\nabla S = 0$ . We characterize Hopf hypersurfaces of a nonflat complex space form.

### 0 Introduction

A Kähler manifold of constant holomorphic sectional curvature  $c$  is called a complex space form, which is denoted by  $M_n(c)$ . A complete and simply connected complex space forms are isometric to a complex projective space  $CP_n$ , a complex Euclidean space  $C^n$  or a complex hyperbolic space  $CH_n$  as  $c > 0$ ,  $c = 0$  or  $c < 0$ .

Let  $M$  be a real hypersurface of  $M_n(c)$ . Then  $M$  has an almost contact metric structure  $(\phi, \xi, \eta, g)$  induced from the complex structure  $J$  and the Kähler metric of  $M_n(c)$  (for details see §1). The structure vector  $\xi$  is said to be principal if  $A\xi = \alpha\xi$  is satisfied, where  $A$  is the shape operator of  $M$  and  $\alpha = \eta(A\xi)$ . A real hypersurface is said to be a Hopf hypersurface if the structure vector  $\xi$  of  $M$  is principal.

Typical examples of real hypersurfaces in  $CP_n$  are homogeneous ones which are orbits under subgroups of  $PU(n+1)$ . The complete classification of them was obtained by Takagi [10] as follows:

**THEOREM T [10].** *Let  $M$  be a homogeneous real hypersurface of  $CP_n$ . Then  $M$  is a tube of radius  $r$  over one of the following Kähler submanifolds:*

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- (A<sub>1</sub>) a hyperplane  $CP_{n-1}$ , where  $0 < r < \frac{\pi}{2}$ ,
- (A<sub>2</sub>) a totally geodesic  $CP_k$  ( $1 \leq k \leq n-2$ ), where  $0 < r < \frac{\pi}{2}$ ,
- (B) a complex quadric  $Q_{n-1}$ , where  $0 < r < \frac{\pi}{4}$ ,
- (C)  $CP_1 \times CP_{(n-1)/2}$ , where  $0 < r < \frac{\pi}{4}$  and  $n \geq 5$  is odd,
- (D) a complex Grassmann  $G_{2,5}C$ , where  $0 < r < \frac{\pi}{4}$  and  $n = 9$ ,
- (E) a Hermitian symmetric space  $SO(10)/U(5)$ , where  $0 < r < \frac{\pi}{4}$  and  $n = 15$ .

Also Berndt [1] classified all Hopf real hypersurfaces in  $CH_n$  with constant principal curvatures as follows:

**THEOREM B** [1]. *Let  $M$  be a real hypersurface of  $CH_n$ . Then  $M$  has constant principal curvatures and  $\xi$  is principal if and only if  $M$  is locally congruent to one of the following:*

- (A<sub>0</sub>) a self-tube, that is, a horosphere,
- (A<sub>1</sub>) a geodesic hypersphere, or a tube over a hyperplane  $CH_{n-1}$ ,
- (A<sub>2</sub>) a tube over a totally geodesic  $CH_k$  ( $1 \leq k \leq n-2$ ),
- (B) a tube over a totally real hyperbolic space  $RH_n$ .

Let  $\nabla$  and  $S$  be the Levi-Civita connection and the Ricci tensor of  $M$ , respectively. There are many studies about Ricci tensors of real hypersurfaces (cf. [2], [3], [4], [5], [6], [7], [8], [9]). Very important fact is that there are no real hypersurfaces with parallel Ricci tensors  $S$  (that is,  $\nabla_X S = 0$  for each vector field  $X$  tangent to  $M$ ) in  $M_n(c)$ ,  $c \neq 0$ ,  $n \geq 3$  (cf. [5]). Especially, there exist no Einstein real hypersurfaces  $M$  in  $M_n(c)$ ,  $c \neq 0$ ,  $n \geq 3$ . So, it is natural to investigate real hypersurfaces  $M$  by using some conditions (on the derivatives of  $S$ ) which are weaker than  $\nabla S = 0$ .

Recently, the first author, Hwang and Kim proved the following theorem:

**THEOREM 0.1.** *Let  $M$  be a real hypersurface in a nonflat complex space form. If the Ricci tensor  $S$  of  $M$  satisfies  $\nabla_\xi S = 0$ ,  $\nabla_{\phi \nabla_\xi \xi} S = 0$  and  $S\xi = g(S\xi, \xi)\xi$ , then  $M$  is locally congruent to one of the homogeneous real hypersurfaces of Theorem T and Theorem B.*

In this paper we pay particular attention to the fact that for each Hopf hypersurface  $M$  in  $M_n(c)$ ,  $c \neq 0$  the characteristic vector  $\xi$  of  $M$  is an eigenvector of the Ricci tensor  $S$  of  $M$ . So it is natural to consider a problem that if the vector  $\xi$

is an eigenvector of the Ricci tensor  $S$  of a real hypersurface  $M$  in  $M_n(c)$ ,  $c \neq 0$ , is  $M$  a Hopf hypersurface?

The purpose of this paper is to establish the following theorem which gives a partial answer to this problem:

**THEOREM 4.1.** *Let  $M$  be a real hypersurface in  $M_n(c)$ ,  $c > 0$ . If it satisfies  $\nabla_{\phi\nabla_\xi\xi}S = 0$  and at the same time satisfies  $S\xi = \sigma\xi$  for some constant  $\sigma$ , then  $M$  is a Hopf hypersurface.*

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### 1 Preliminaries

Let  $M$  be a real hypersurface immersed in a complex space form  $(M_n(c), G)$  with almost complex structure  $J$  and the Kähler metric  $G$  of constant holomorphic sectional curvature  $c$ , and let  $C$  be a unit normal vector field on  $M$ . The Riemannian connection  $\tilde{\nabla}$  in  $M_n(c)$  and  $\nabla$  in  $M$  are related by the following formulas for any vector fields  $X$  and  $Y$  on  $M$ :

$$\tilde{\nabla}_Y X = \nabla_Y X + g(AY, X)C, \tag{1.1}$$

$$\tilde{\nabla}_X C = -AX, \tag{1.2}$$

where  $g$  denotes the Riemannian metric on  $M$  induced from that  $G$  of  $M_n(c)$  and  $A$  is the shape operator of  $M$  in  $M_n(c)$ . An eigenvector  $X$  of the shape operator  $A$  is called a principal curvature vector. Also an eigenvalue  $\lambda$  of  $A$  is called a principal curvature. It is known that  $M$  has an almost contact metric structure induced from the almost complex structure  $J$  on  $M_n(c)$ , that is, we define a tensor field  $\phi$  of type  $(1, 1)$ , a vector field  $\xi$ , an 1-form  $\eta$  on  $M$  by  $g(\phi X, Y) = G(JX, Y)$  and  $g(\xi, X) = \eta(X) = G(JX, C)$ . Then we have

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\xi, \xi) = 1, \quad \phi\xi = 0. \tag{1.3}$$

From (1.1) we see that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{1.4}$$

$$\nabla_X \xi = \phi AX. \tag{1.5}$$

Since the ambient space is of constant holomorphic sectional curvature  $c$ , equations of the Gauss and Codazzi are respectively given by

$$R(X, Y)Z = \frac{c}{4} \{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \quad (1.6)$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4} \{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \quad (1.7)$$

for any vector fields  $X$ ,  $Y$  and  $Z$  on  $M$ , where  $R$  denotes the Riemannian curvature tensor of  $M$ . We shall denote the Ricci tensor of type  $(1, 1)$  by  $S$ . Then it follows from (1.6) that

$$SX = \frac{c}{4} \{(2n + 1)X - 3\eta(X)\xi\} + hAX - A^2X, \quad (1.8)$$

where  $h = \text{trace } A$ . Further, using (1.5), we obtain

$$(\nabla_X S)Y = -\frac{3}{4}c\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\} + (Xh)AY \\ + (hI - A)(\nabla_X A)Y - (\nabla_X A)AY, \quad (1.9)$$

where  $I$  is the identity map.

To write our formulas in convention forms, we denote  $\alpha = \eta(A\xi)$ ,  $\beta = \eta(A^2\xi)$ ,  $\mu^2 = \beta - \alpha^2$  and  $\nabla f$  by the gradient vector field of a function  $f$  on  $M$ . In the following, we use the same terminology and notation as above unless otherwise stated.

If we put  $U = \nabla_\xi \xi$ , then  $U$  is orthogonal to the structure vector field  $\xi$ . Then it is, using (1.3) and (1.5), seen that

$$\phi U = -A\xi + \alpha\xi, \quad (1.10)$$

which shows that  $g(U, U) = \beta - \alpha^2$ . By the definition of  $U$ , (1.3) and (1.5) it is verified that

$$g(\nabla_X \xi, U) = g(A^2\xi, X) - \alpha g(A\xi, X). \quad (1.11)$$

Now, differentiating (1.10) covariantly along  $M$  and using (1.4) and (1.5), we find

$$\eta(X)g(AU + \nabla\alpha, Y) + g(\phi X, \nabla_Y U) \\ = g((\nabla_Y A)X, \xi) - g(A\phi AX, Y) + \alpha g(A\phi X, Y), \quad (1.12)$$

which enables us to obtain

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha \quad (1.13)$$

because of (1.7). From (1.12) we also have

$$\nabla_{\xi}U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha, \tag{1.14}$$

where we have used (1.3), (1.5) and (1.11).

If  $A\xi - g(A\xi, \xi)\xi \neq 0$ , then we can put

$$A\xi = \alpha\xi + \mu W, \tag{1.15}$$

where  $W$  is a unit vector field orthogonal to  $\xi$ . Then from (1.10) it is seen that  $U = \mu\phi W$  and hence  $g(U, U) = \mu^2$ , and  $W$  is also orthogonal to  $U$ . Thus, we see, making use of (1.5), that

$$\mu g(\nabla_X W, \xi) = g(AU, X). \tag{1.16}$$

### 2 Real Hypersurfaces Satisfying $S\xi = g(S\xi, \xi)\xi$

Let  $M$  be a real hypersurface of a nonflat complex space form  $M_n(c)$ . If it satisfies

$$S\xi = g(S\xi, \xi)\xi, \tag{2.1}$$

then we have by (1.8)

$$A^2\xi = hA\xi + (\beta - h\alpha)\xi, \tag{2.2}$$

where we have put  $g(S\xi, \xi) = \sigma$ ,

$$\beta - h\alpha = \frac{c}{2}(n - 1) - \sigma. \tag{2.3}$$

In what follows we assume that  $\mu \neq 0$  on  $M$ , that is,  $\xi$  is not a principal curvature vector field and we put  $\Omega = \{p \in M \mid \mu(p) \neq 0\}$ . Then  $\Omega$  is an open subset of  $M$ , and from now on we discuss our arguments on  $\Omega$ .

From (1.15) and (2.2), we see that

$$AW = \mu\xi + (h - \alpha)W \tag{2.4}$$

and hence

$$A^2W = hAW + (\beta - h\alpha)W \tag{2.5}$$

because of  $\mu \neq 0$ .

Now, differentiating (2.4) covariantly along  $\Omega$ , we find

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X\xi + X(h - \alpha)W + (h - \alpha)\nabla_X W. \tag{2.6}$$

By taking the inner product with  $W$  in the last equation, we obtain

$$g((\nabla_X A)W, W) = -2g(AX, U) + Xh - X\alpha \quad (2.7)$$

since  $W$  is a unit vector field orthogonal to  $\xi$ . We also have by applying  $\xi$  to (2.6)

$$\mu g((\nabla_X A)W, \xi) = (h - 2\alpha)g(AU, X) + \mu(X\mu), \quad (2.8)$$

where we have used (1.16), which together with the Codazzi equation (1.7) gives

$$\mu(\nabla_W A)\xi = (h - 2\alpha)AU - \frac{c}{2}U + \mu\nabla\mu, \quad (2.9)$$

$$\mu(\nabla_\xi A)W = (h - 2\alpha)AU - \frac{c}{4}U + \mu\nabla\mu. \quad (2.10)$$

Replacing  $X$  by  $\xi$  in (2.6) and taking account of (2.10), we find

$$\begin{aligned} & (h - 2\alpha)AU - \frac{c}{4}U + \mu\nabla\mu + \mu\{A\nabla_\xi W - (h - \alpha)\nabla_\xi W\} \\ &= \mu(\xi\mu)\xi + \mu^2U + \mu(\xi h - \xi\alpha)W. \end{aligned} \quad (2.11)$$

By the way, from  $\phi U = -\mu W$  we have

$$g(AU, X)\xi - \phi\nabla_X U = (X\mu)W + \mu\nabla_X W.$$

Replacing  $X$  by  $\xi$  in this and using (1.10) and (1.14), we get

$$\mu\nabla_\xi W = 3AU - \alpha U + \nabla\alpha - (\xi\alpha)\xi - (\xi\mu)W, \quad (2.12)$$

which implies

$$W\alpha = \xi\mu. \quad (2.13)$$

From the last equations, it follows that

$$\begin{aligned} & 3A^2U - 2hAU + A\nabla\alpha + \frac{1}{2}\nabla\beta - h\nabla\alpha + \left(\alpha h - \beta - \frac{c}{4}\right)U \\ &= 2\mu(W\alpha)\xi + \mu(\xi h)W - (h - 2\alpha)(\xi\alpha)\xi, \end{aligned} \quad (2.14)$$

which enables us to obtain

$$\xi\beta = 2\alpha(\xi\alpha) + 2\mu(W\alpha). \quad (2.15)$$

Differentiating (2.2) covariantly and making use of (1.5), we get

$$\begin{aligned} & (\nabla_X A)A\xi + A(\nabla_X A)\xi + A^2\phi AX - hA\phi AX \\ &= (Xh)A\xi + h(\nabla_X A)\xi + X(\beta - h\alpha)\xi + (\beta - h\alpha)\phi AX, \end{aligned} \quad (2.16)$$

which together with (1.7) implies that

$$\begin{aligned} & \frac{c}{4} \{u(Y)\eta(X) - u(X)\eta(Y)\} + \frac{c}{2}(h - \alpha)g(\phi Y, X) - g(A^2\phi AX, Y) \\ & + g(A^2\phi AY, X) + 2hg(\phi AX, AY) - (\beta - h\alpha)\{g(\phi AY, X) - g(\phi AX, Y)\} \\ & = g(AY, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi) + (Yh)g(A\xi, X) - (Xh)g(A\xi, Y) \\ & + Y(\beta - h\alpha)\eta(X) - X(\beta - h\alpha)\eta(Y), \end{aligned} \tag{2.17}$$

where we have defined an 1-form  $u$  by  $u(X) = g(U, X)$  for any vector field  $X$ . If we replace  $X$  by  $\mu W$  to the both sides of (2.17) and take account of (1.13), (2.4), (2.5), (2.8) and (2.9), then we obtain

$$\begin{aligned} & (3\alpha - 2h)A^2U + 2\left(h^2 + \beta - 2h\alpha + \frac{c}{4}\right)AU + (h - \alpha)\left(\beta - h\alpha - \frac{c}{2}\right)U \\ & = \mu A\nabla\mu + (\alpha h - \beta)\nabla\alpha - \frac{1}{2}(h - \alpha)\nabla\beta + \mu^2\nabla h \\ & - \mu(Wh)A\xi - \mu W(\beta - h\alpha)\xi. \end{aligned} \tag{2.18}$$

Using (1.15), the equation (2.16) can be written as

$$\begin{aligned} & A(\nabla_X A)\xi + (\alpha - h)(\nabla_X A)\xi + \mu(\nabla_X A)W \\ & = (Xh)A\xi + X(\beta - h\alpha)\xi + (\beta - h\alpha)\phi AX + hA\phi AX - A^2\phi AX. \end{aligned}$$

Thus, replacing  $X$  by  $\alpha\xi + \mu W$  in this and making use of (1.5), (1.13), (1.15) and (2.7)–(2.9), we find

$$\begin{aligned} & 2hA^2U + 2\left(\alpha h - \beta - h^2 - \frac{c}{4}\right)AU + \left(h^2\alpha - h\beta + \frac{c}{2}h - \frac{3}{4}c\alpha\right)U \\ & = g(A\xi, \nabla h)A\xi - \frac{1}{2}A\nabla\beta + \frac{1}{2}(h - 2\alpha)\nabla\beta + \beta\nabla\alpha \\ & - \mu^2\nabla h + g(A\xi, \nabla(\beta - h\alpha))\xi. \end{aligned} \tag{2.19}$$

### 3 Real Hypersurfaces Satisfying $\nabla_{\phi\nabla_\xi S} S = 0$ and $S\xi = g(S\xi, \xi)\xi$

We continue now, our arguments under the same hypothesis  $S\xi = g(S\xi, \xi)\xi$  as in section 2. Furthermore, suppose that  $\nabla_{\phi\nabla_\xi S} S = 0$ , that is,  $\nabla_W S = 0$  since we now suppose that  $\mu \neq 0$ .

Then, by replacing  $X$  by  $W$ , we have from (1.9)

$$\begin{aligned}
& -\frac{3}{4}c(h-\alpha)(u(Y)\xi + \eta(Y)U) + \mu(Wh)AY + \mu h(\nabla_W A)Y \\
& = \mu A(\nabla_W A)Y - \mu(\nabla_W A)AY,
\end{aligned} \tag{3.1}$$

where we have used (1.5) and (2.4). If we replace  $Y$  by  $W$  and make use of (2.7) and (2.9), then we find

$$(Wh)AW = hAU - \frac{c}{2}U - 2A^2U + \frac{1}{2}\nabla\beta - \alpha\nabla h + A\nabla h - A\nabla\alpha \tag{3.2}$$

because of  $\mu \neq 0$ .

In the following we assume that  $\sigma$  is constant on  $M$  and then  $\beta - h\alpha =$  constant. In this case we notice here that the following fact:

**REMARK 3.1.**  $h - \alpha \neq 0$  on  $\Omega$ .

In fact, if not, then we have  $h = \alpha$  and hence  $\beta - \alpha^2 =$  constant, because  $\sigma =$  constant. Thus (3.2) implies  $Wh = W\alpha = 0$  and hence

$$2A^2U = \alpha AU - \frac{c}{2}U. \tag{3.3}$$

Further, (2.14) and (2.18) turns out respectively to

$$2A^2U - 2\alpha AU + \left(\alpha^2 - \beta - \frac{c}{4}\right)U = -A\nabla\alpha + (\xi\alpha)A\xi, \tag{3.4}$$

$$\alpha A^2U + 2\left(\beta - \alpha^2 + \frac{c}{4}\right)AU = 0. \tag{3.5}$$

It is, using (3.3)–(3.5), verified that  $\alpha \neq 0$  on this set.

Combining (3.3) with (3.5), we see that

$$\alpha AU = 2\left(\alpha^2 - \beta - \frac{c}{4}\right)U \tag{3.6}$$

and thus  $AU = vU$  because of  $\alpha \neq 0$ , where we have put

$$\alpha v = 2\left(\alpha^2 - \beta - \frac{c}{4}\right). \tag{3.7}$$

From this and (3.3), we obtain



$$v^2 + \beta - \alpha^2 + \frac{c}{2} = 0. \tag{3.8}$$

Therefore  $v = \text{constant} \neq 0$  because of (3.3). Hence it is, using (3.7), seen that  $\alpha = \text{constant}$  and thus

$$3v^2 - 2\alpha v + \alpha^2 - \beta - \frac{c}{4} = 0,$$

which together with (3.7) and (3.8), produces a contradiction. Consequently  $h - \alpha \neq 0$  on  $\Omega$  is proved. In what follows we assume that  $h - \alpha \neq 0$  is satisfied everywhere.

Differentiating (2.1) covariantly, we find

$$(\nabla_X S)\xi + S\nabla_X \xi = \sigma\nabla_X \xi$$

because  $\sigma = \text{constant}$  is assumed, which together with hypothesis  $\nabla_W S = 0$  yields

$$S\nabla_W \xi = \sigma\nabla_W \xi. \tag{3.9}$$

By the way we have  $\mu\nabla_W \xi = (h - \alpha)U$  with the aid of (1.5) and (2.4), (3.9) implies  $SU = \sigma U$  because of Remark 3.1. Hence (1.8) leads to

$$A^2U = hAU + \left(\beta - h\alpha + \frac{3}{4}c\right)U. \tag{3.10}$$

From (2.3) we have

$$\nabla\beta = \alpha\nabla h + h\nabla\alpha. \tag{3.11}$$

Thus (2.15) is reduced to

$$2\mu(W\alpha) = (h - 2\alpha)(\xi\alpha) + \alpha(\xi h). \tag{3.12}$$

Using (1.15), (3.10) and (3.12), the equation (2.14) turns out to be

$$hAU + 2(\beta - h\alpha + c)U = (\xi h)A\xi - A\nabla\alpha + h\nabla\alpha - \frac{1}{2}\nabla\beta. \tag{3.13}$$

From (2.19) and (2.10), we also find

$$\begin{aligned} &\left(2\beta - 2h\alpha + \frac{c}{2}\right)AU + \left\{h(h\alpha - \beta) + \frac{c}{4}(3\alpha - 8h)\right\}U + g(A\xi, \nabla h)A\xi \\ &= \frac{1}{2}A\nabla\beta - \beta\nabla\alpha + \left(\alpha - \frac{1}{2}h\right)\nabla\beta + \mu^2\nabla h. \end{aligned} \tag{3.14}$$

Because of (3.2) and (3.10), we see that

$$(Wh)AW = -hAU - 2(\beta - h\alpha + c)U + A\nabla h - A\nabla\alpha + \frac{1}{2}\nabla\beta - \alpha\nabla h,$$

which together with (3.10) and (3.11) gives

$$A\nabla h = (Wh)AW + (\xi h)A\xi. \quad (3.15)$$

Making use of (3.13) and (3.15), we have from (3.14)

$$\begin{aligned} & (4\beta - 4h\alpha + h^2 + c)AU + \left(\frac{3}{2}c\alpha - 2ch\right)U \\ &= \alpha(Wh)AW - \{(\alpha - h)(\xi h) + 2\mu(Wh)\}A\xi \\ & \quad + \left(2\alpha h - 2\beta - \frac{1}{2}h^2\right)\nabla\alpha + \left(2\beta - \frac{3}{2}h\alpha\right)\nabla h. \end{aligned} \quad (3.16)$$

If we use (2.2), (2.5) and (3.10), then above equation implies

$$\begin{aligned} & \frac{3}{4}c \left\{ (4\beta - 4h\alpha + h^2 + c)AU + \left(\frac{3}{2}c\alpha - 2ch\right)U \right\} \\ &= \left(2\alpha h - 2\beta - \frac{1}{2}h^2\right) \{A^2\nabla\alpha - hA\nabla\alpha - (\beta - h\alpha)\nabla\alpha\} \\ & \quad + \left(2\beta - \frac{3}{2}h\alpha\right) \{A^2\nabla h - hA\nabla h - (\beta - h\alpha)\nabla h\}, \end{aligned}$$

which together with (3.15) yields

$$\begin{aligned} & \frac{3}{4}c \left\{ (4\beta - 4h\alpha + h^2 + c)AU + \frac{c}{2}(3\alpha - 4h)U \right\} \\ &= \left(2\alpha h - 2\beta - \frac{1}{2}h^2\right) \{A^2\nabla\alpha - hA\nabla\alpha - (\beta - h\alpha)\nabla\alpha\} \\ & \quad + \left(2\beta - \frac{3}{2}h\alpha\right)(\beta - h\alpha) \{(Wh)W + (\xi h)\xi - \nabla h\}. \end{aligned} \quad (3.17)$$

On the other hand, we have from (3.13)

$$\begin{aligned} & A^2\nabla\alpha - hA\nabla\alpha + (h^2 + 2\beta - 2h\alpha + 2c)AU + h\left(\beta - h\alpha + \frac{3}{4}c\right)U \\ &= (\xi h)A^2\xi - \frac{1}{2}A\nabla\beta, \end{aligned}$$

where we have used (3.10), or using (3.11) and (3.14),

$$\begin{aligned} & A^2\nabla\alpha - hA\nabla\alpha + (\beta - h\alpha)\nabla\alpha \\ &= \left(4h\alpha - 4\beta - h^2 - \frac{5}{2}c\right)AU + \frac{c}{4}(5h - 3\alpha)U \\ &\quad - \frac{1}{2}h^2\nabla\alpha + \left(\beta - \frac{1}{2}h\alpha\right)\nabla h + (\xi h)A^2\xi - g(A\xi, \nabla h)A\xi. \end{aligned} \quad (3.18)$$

If we take the inner product  $\xi$  with this and make use of (1.15) and (2.2), then we obtain

$$\mu\alpha(Wh) = \left(2h\alpha - 2\beta - \frac{1}{2}h^2\right)(\xi\alpha) + \left(2\beta - \frac{1}{2}h\alpha - \alpha^2\right)(\xi h). \quad (3.19)$$

Substituting (3.18) into (3.17) and taking account of (3.16), we find

$$\begin{aligned} & \frac{3}{2}c\left\{cAU + \frac{c}{2}(3\alpha - 4h)U + (h - \alpha)\left(2\alpha h - 2\beta - \frac{1}{2}h^2\right)U\right\} \\ &= h(h - \alpha)(\beta - h\alpha)\{\nabla h - (\xi h)\xi - (Wh)W\}. \end{aligned} \quad (3.20)$$

Applying  $A$  to both sides of this and using (3.10) and (3.15), we have

$$\left\{\frac{c}{2}(3\alpha - 2h) + (h - \alpha)\left(2\alpha h - 2\beta - \frac{1}{2}h^2\right)\right\}AU + c\left(\beta - h\alpha + \frac{3}{4}c\right)U = 0. \quad (3.21)$$

**LEMMA 3.1.** *Let  $M$  be a real hypersurface of  $M_n(c)$  ( $c \neq 0$ ). If it satisfies  $\nabla_W S = 0$  and  $S\xi = \sigma\xi$  for some constant  $\sigma$ , then we have*

$$AU = \lambda U \quad (3.22)$$

on  $\Omega$ , where  $\mu^2\lambda = g(AU, U)$ .

**PROOF.** Let  $\Omega_0$  be a set of points in  $M$  such that  $\|AU - \lambda U\| \neq 0$  on  $\Omega$  and suppose that  $\Omega_0$  be nonempty. If  $\beta - h\alpha + \frac{3}{4}c \neq 0$ , then we have from (3.21)

$$\frac{c}{2}(3\alpha - 2h) + (h - \alpha)\left(2\alpha h - 2\beta - \frac{1}{2}h^2\right) \neq 0$$

and hence (3.22) is valid. Thus it is, using (3.21), seen that

$$\beta - h\alpha + \frac{3}{4}c = 0 \quad (3.23)$$

and therefore  $h(h^2 - \alpha h - c) = 0$  on  $\Omega_0$ . So we have

$$h^2 - \alpha h - c = 0 \quad (3.24)$$

on  $\Omega_0$ . In fact, if not, then we have  $h = 0$ . Thus (3.10) and (3.23) are respectively to

$$AU = 0, \quad \beta + \frac{3}{4}c = 0.$$

Hence (3.13) becomes  $2(\beta + c)U + A\nabla\alpha = 0$ . But, by (3.14) we have  $\nabla\alpha = \alpha U$ . Combining the last two equations, we obtain  $\beta + c = 0$ , a contradiction. Thus (3.24) is accomplished.

Differentiating (3.24), and using (3.23), we find

$$2h\nabla h = \alpha\nabla h + h\nabla\alpha = \nabla\beta. \quad (3.25)$$

From this and (3.15) we obtain

$$A\nabla\beta = 2h\{(Wh)AW + (\xi h)A\xi\}. \quad (3.26)$$

If we take account of (3.23)–(3.26), then (3.14) turns out to be

$$\begin{aligned} -cAU + \frac{c}{4}(3\alpha - 5h)U &= (h - \alpha)(\xi h)A\xi - \mu(Wh)A\xi + h(Wh)AW \\ &\quad + (\mu^2 + \alpha h - c)\nabla h - \beta\nabla\alpha. \end{aligned} \quad (3.27)$$

On the other hand, we have from (3.13)

$$h^2AU + \frac{c}{2}hU = (\alpha - h)(\xi h)A\xi + (\alpha - 2h)(Wh)AW + c\nabla h$$

because of (3.24)–(3.26). Comparing with the last two equations, it follows that

$$\begin{aligned} (h^2 - c)AU + \frac{3}{4}c(\alpha - h)U \\ = (\alpha - h)(Wh)AW - \mu(Wh)A\xi + (\beta - \alpha^2 + \alpha h)\nabla h - \beta\nabla\alpha. \end{aligned}$$

Applying this by  $hA$  and making use of (2.2), (2.5) and (3.23), we find

$$\begin{aligned} &\left\{ h^2(h^2 - c) + \frac{3}{4}ch(\alpha - h) \right\} AU \\ &= h(\alpha - h)(Wh) \left\{ hAW - \frac{3}{4}cW \right\} - \mu h(Wh) \left( hA\xi - \frac{3}{4}c\xi \right) \\ &\quad + h(\beta - \alpha^2 + \alpha h)A\nabla h - \beta hA\nabla\alpha, \end{aligned}$$

which together with (3.15) and (3.23)–(3.25) implies that

$$\begin{aligned} & \left\{ (\alpha h + c)\alpha h - \frac{3}{4}c^2 \right\} AU \\ &= h(\alpha - h)(Wh) \left( hAW - \frac{3}{4}cW \right) - \mu h(Wh) \left( hA\xi - \frac{3}{4}c\xi \right) \\ & \quad + \frac{3}{4}c(h - \alpha)\{(Wh)AW + (\xi h)A\xi\}. \end{aligned} \tag{3.28}$$

Furthermore, using (2.2) and (2.5), we have from (3.28)

$$\left\{ (\alpha h + c)\alpha h - \frac{3}{4}c^2 \right\} AU = 0$$

because  $U$  is orthogonal to  $\xi$  and  $W$ . Hence we have

$$(\alpha^3 + c\alpha)h + c\alpha^2 - \frac{3}{4}c^2 = 0$$

on  $\Omega_0$ . Since  $c \neq 0$ , it follows that

$$h = \frac{\frac{3}{4}c^2 - c\alpha^2}{\alpha(\alpha^2 + c)}. \tag{3.29}$$

From this and (3.24) we have  $12\alpha^4 + 52c\alpha^2 - 9c^2 = 0$  on  $\Omega_0$ . So we see that  $\nabla\alpha = 0$  and hence  $\nabla h = 0$  because of (3.29). Thus (3.27) becomes  $AU = \frac{1}{4}(3\alpha - 5h)U$  on  $\Omega_0$ . Therefore  $\Omega_0$  is void. This completes the proof. ■

**LEMMA 3.2.** *Under the same assumptions as those stated in Lemma 3.1, we have  $\xi\alpha = 0$ ,  $W\alpha = 0$ ,  $\xi h = 0$  and  $Wh = 0$  on  $\Omega$ .*

**PROOF.** As in the proof of Lemma 3.1, it is sufficient to show that the following two cases:

Case 1.  $\beta - h\alpha + \frac{3}{4}c = 0$  and  $h^2 - h\alpha - c = 0$ ,

Case 2.  $\frac{c}{2}(3\alpha - 2h) + (h - \alpha)(2\alpha h - 2\beta - \frac{1}{2}h^2) \neq 0$ .

Case 1: By taking the inner product with  $\xi$  in (3.14), we obtain

$$\mu(h - \alpha)(Wh) = \left( 2\alpha^2 - 3h\alpha + \frac{7}{4}c \right) (\xi h) + \left( h\alpha - \frac{3}{4}c \right) (\xi\alpha). \tag{3.30}$$

From (3.19) we have

$$\mu\alpha(Wh) = -\frac{1}{2}(h\alpha - 2c)(\xi\alpha) + \frac{1}{2}(3h\alpha - 2\alpha^2 - 3c)(\xi h). \quad (3.31)$$

Using (3.24), (3.30) and (3.31), we are led to

$$\{(\xi h)^2 + (\xi\alpha)^2\}(25h\alpha + 14c - 3\alpha^2) = 0. \quad (3.32)$$

So, on the set of points satisfying  $25h\alpha + 14c - 3\alpha^2 \neq 0$ ,

$$\xi h = \xi\alpha = 0.$$

On account of Remark 3.1 and (3.30), we deduce that

$$Wh = 0.$$

Further, from (3.12), we get  $W\alpha = 0$  since  $\mu \neq 0$ .

If  $2h\alpha + 14c - 3\alpha^2 \equiv 0$ , then  $\alpha \neq 0$  since  $c \neq 0$ . So, we have

$$h = \frac{3\alpha^2 - 14c}{25\alpha}. \quad (3.33)$$

Combining this with (3.24), we see that

$$(3\alpha^2 - 14c)^2 - 25\alpha^2(3\alpha^2 - 14c) - 625c\alpha^2 \equiv 0.$$

Therefore we have  $\nabla\alpha = 0$ . So we have  $\nabla h = 0$  by (3.33).

Case 2: Putting  $\beta - h\alpha + \frac{3}{4}c = c'$ , (3.21) is reduced to

$$\left\{ \frac{c}{2}(3\alpha - 2h) + (h - \alpha) \left( \frac{3}{2}c - 2c' - \frac{1}{2}h^2 \right) \right\} AU + cc'U = 0.$$

From this we have

$$AU = \lambda U, \quad \lambda = \frac{-2cc'}{c(3\alpha - 2h) + (h - \alpha)(3c - 4c' - h^2)}.$$

Therefore we are led to the following equation by (3.10):

$$(4c' + h^2)\{(4c' + h^2)\alpha^2 - 2h(4c' + h^2)\alpha + h^2(4c' + h^2) - c^2\} = 0. \quad (3.34)$$

If  $4c' + h^2 \equiv 0$ , then  $h = \text{constant}$ . So, using (3.19), we are led to  $\xi\alpha = 0$  since  $c \neq 0$ . Furthermore, from (3.12), we have  $W\alpha = 0$ .

If  $4c' + h^2 \neq 0$ , then from (3.34) we have

$$(4c' + h^2)\alpha^2 - 2h(4c' + h^2)\alpha + h^2(4c' + h^2) - c^2 = 0. \quad (3.35)$$

Differentiating both sides of (3.35), we obtain

$$(\alpha - h)(4c' + h^2)\nabla\alpha + \{h\alpha^2 - (4c' + h^2)\alpha + 2h(2c' + h^2)\}\nabla h = 0. \quad (3.36)$$

By taking the inner products with  $\xi$  in (3.14), we obtain

$$\begin{aligned} \mu\alpha(Wh) - \mu h(W\alpha) &= \left(-\alpha^2 + h\alpha + 2c' - \frac{3}{2}c\right)(\xi h) \\ &\quad + \left(h\alpha - 2c' + \frac{3}{2}c - h^2\right)(\xi\alpha). \end{aligned} \quad (3.37)$$

By our assumption (3.19) is reduced to

$$\mu\alpha(Wh) = \left(\frac{3}{2}h\alpha - \alpha^2 + 2c' - \frac{3}{2}c\right)(\xi h) - \left(\frac{1}{2}h^2 + 2c' - \frac{3}{2}c\right)(\xi\alpha). \quad (3.38)$$

Using (3.36) and (3.37), we obtain

$$\begin{aligned} &2\mu(h^2 + 2c')(\alpha - h)(Wh) \\ &= \left\{-2h(h^2 + 4c')\alpha + (h^2 + 4c')\left(h^2 + h\alpha + 2c' - \frac{3}{2}c\right) - c^2\right\}(\xi h) \\ &\quad + (h^2 + 4c')\left(h\alpha - 2c' + \frac{3}{2}c - h^2\right)(\xi\alpha). \end{aligned} \quad (3.39)$$

Making use of (3.35), we have from (3.38) and (3.39)

$$\begin{aligned} &\left[-2(h^2 + 2c')\alpha^3 + 2h(3h^2 + 7c')\alpha^2 + \left\{-4h^4 - \left(8c' + \frac{3}{2}c\right)h^2 + c^2\right\}\alpha\right. \\ &\quad \left.+ (3c - 4c')h(h^2 + 2c')\right](\xi h) - \left\{h^2\left(2h^2 - \frac{3}{2}c + 8c'\right)\alpha\right. \\ &\quad \left.+ h(c^2 - 10c'h^2 - 2h^4 + 3ch^2 - 8c'^2 + 6cc')\right\}(\xi\alpha) = 0. \end{aligned} \quad (3.40)$$

From (3.36) we have

$$(\alpha - h)(h^2 + 4c')(\xi\alpha) + \{h\alpha^2 - (4c' + 3h^2)\alpha + 2h(2c' + h^2)\}(\xi h) = 0. \quad (3.41)$$

From (3.40) and (3.41) we obtain

$$\begin{aligned}
& \{(\xi h)^2 + (\xi \alpha)^2\} \\
& \times \left[ (\alpha - h)(h^2 + 4c') \left\{ -2(h^2 + 2c')\alpha^3 + 2h(3h^2 + 7c')\alpha^2 \right. \right. \\
& \quad \left. \left. - 4h^4\alpha - \left(8c' + \frac{3}{2}c\right)h^2\alpha + c^2\alpha + (3c - 4c')h(h^2 + 2c') \right\} \right. \\
& \quad \left. + \{h\alpha^2 - (4c' + 3h^2)\alpha + 2h(2c' + h^2)\} \left\{ h^2 \left(2h^2 - \frac{3}{2}c + 8c'\right)\alpha \right. \right. \\
& \quad \left. \left. + h(c^2 - 10c'h^2 - 2h^4 + 3ch^2 - 8c'^2 + 6c'c) \right\} \right] = 0. \tag{3.42}
\end{aligned}$$

If  $(\xi h)^2 + (\xi \alpha)^2 \neq 0$ , then from (3.42) we have

$$\begin{aligned}
& (-12h^2c' - 2h^4 - 16c'^2)\alpha^4 \\
& + \left( -\frac{3}{2}h^3c + 72hc'^2 + 58h^3c' + 10h^5 \right)\alpha^3 \\
& + \left( 2h^2c^2 + 3h^4c + \frac{9}{2}h^2c + 4c'^2c^2 - 88c'h^4 \right. \\
& \quad \left. - 6h^4 - 14h^6 - 24c'h^2 - 128c'^2h^2 + 6c'ch^2 \right)\alpha^2 \\
& + \left( -18c'ch - 8c'^2h + 6h^5 + 62c'h^5 - 3c^2h - 2h^3c^2 \right. \\
& \quad \left. + 24c'^2h + 10h^7 + 88c'h^3 - 9ch^3 - \frac{3}{2}ch^5 + 30c'h^3 \right)\alpha \\
& + 6c'ch^4 + 4c'^2c^2h^2 - 4h^8 - 24c'h^6 - 32c'h^4 + 2c^2h^4 + 3ch^6 = 0. \tag{3.43}
\end{aligned}$$

Using Sylvester's elimination method to (3.35) and (3.43), we deduce that

$$\begin{aligned}
& (-24cc' - 7c^2 + 16c'^2)h^{20} + (-576c'^2c + 72c'c + 384c'^3 - 48c'^2 \\
& \quad + 21c^2 + 36c^3 - 120c'c^2)h^{18} + f(h) = 0, \tag{3.44}
\end{aligned}$$

where  $f(h)$  is the polynomial of  $h$  of degree  $\leq 16$ . (We use a computer to calculate this.)

We can check that the coefficients of  $h^{20}$  and  $h^{18}$  does not vanish simultaneously since  $c \neq 0$ . (We use a computer to check this.)

By the above argument, we know that (3.44) is a non-trivial algebraic equation of  $h$ . So, we arrive at  $h = \text{constant}$ . From (3.41), we have  $\xi\alpha = 0$ . These



are contradictions. So, we have  $\xi\alpha = \xi h = 0$ . Furthermore, using (3.12) and (3.39), we arrive at  $W\alpha = Wh = 0$ . We have thus proved the lemma. ■

**4 Proof of the Theorem**

We continue our discussion under the same assumption of §3. First, we prove the following two lemmas:

LEMMA 4.1. *Let  $\lambda$  be a principal curvature corresponding to  $U$ . Then  $\lambda$  does not vanish identically on  $\Omega = \{p \in M \mid \mu(p) \neq 0\}$ .*

PROOF. From Lemma 3.1 and (3.10) the following equation holds on  $\Omega$ :

$$\lambda^2 = \lambda h + \beta - h\alpha + \frac{3}{4}c. \tag{4.1}$$

By Lemma 3.2, (3.15) becomes

$$A\nabla h = 0, \quad \lambda(Uh) = 0. \tag{4.2}$$

Because of Lemma 3.1 and Lemma 3.2, (3.13) and (3.16) are reduced respectively to

$$\{h\lambda + 2(\beta - h\alpha + c)\}U = -A\nabla\alpha + \frac{1}{2}(h\nabla\alpha - \alpha\nabla h), \tag{4.3}$$

$$\theta U = \left(2\alpha h - 2\beta - \frac{1}{2}h^2\right)\nabla\alpha + \left(2\beta - \frac{3}{2}h\alpha\right)\nabla h, \tag{4.4}$$

where we define  $\theta$  by  $\theta = (4\beta - 4h\alpha + h^2 + c)\lambda + \frac{3}{2}c\alpha - 2ch$ .

From (3.11) and Lemma 3.2, we have  $\xi\beta = 0$ . Therefore it is seen, using Lemma 3.2, that

$$\xi\theta = 0.$$

From this and Lemma 3.1, we see, making use of (4.4), that

$$\theta \, du(\xi, X) = 0 \tag{4.5}$$

for any vector fields  $X$  on  $\Omega$ , where  $u$  is defined by  $u(X) = g(U, X)$ , and exterior derivation  $du$  of  $u$  is given by

$$du(\xi, X) = \frac{1}{2}\{\xi u(X) - Xu(\xi) - u([\xi, X])\}.$$

On the other hand, using (1.15) and  $AU = \lambda U$ , the equation (1.14) turns out to be

$$\nabla_{\xi} U = \mu(\alpha - 3\lambda)W - \mu^2\xi + \phi\nabla\alpha,$$

which together with (1.11) and (2.2) implies that

$$du(\xi, X) = (h - 3\lambda)\mu w(X) + g(\phi\nabla\alpha, X), \quad (4.6)$$

where  $w(X) = g(W, X)$ .

If  $\lambda = 0$ , then by (3.1) we have

$$\beta - h\alpha = -\frac{3}{4}c. \quad (4.7)$$

Thus (4.3) and (4.4) becomes respectively

$$cU = -2A\nabla\alpha + h\nabla\alpha - \alpha\nabla h, \quad (4.8)$$

$$(3c\alpha - 4ch)U = (3c - h^2)\nabla\alpha - (3c - h\alpha)\nabla h. \quad (4.9)$$

Because of Lemma 3.1 and (4.2), we see, using (4.9), that

$$(3c - h^2)A\nabla\alpha = 0. \quad (4.10)$$

If the set of points satisfying  $A\nabla\alpha \neq 0$  is not empty, then on that set we have

$$h = \text{constant}$$

because of (4.10). So, from (4.9), we are led to

$$\nabla\alpha = 0.$$

This is a contradiction. So, we obtain

$$A\nabla\alpha = 0 \quad \text{on } \Omega. \quad (4.11)$$

Thus (4.7) becomes

$$cU = h\nabla\alpha - \alpha\nabla h.$$

So, we have

$$du(\xi, X) = 0$$

because of Lemma 3.2. Therefore (4.6) means that

$$\phi\nabla\alpha = \mu(h - 3\lambda)W.$$

Since  $\xi\alpha = 0$ , it follows that

$$\nabla\alpha = hU. \quad (4.12)$$

So, from (4.8), we have

$$\alpha\nabla h = (h^2 - c)U. \quad (4.13)$$

Combining last two equations with (3.2) and (3.11), we obtain

$$A\nabla\beta = 0, \quad A\nabla\mu = 0.$$

Thus (2.18) with  $AU = 0$  and (4.7) implies

$$\begin{aligned} -\frac{5}{4}c(h - \alpha)U &= \frac{3}{4}c\nabla\alpha - \frac{1}{2}(h - \alpha)\{\alpha\nabla h + h\nabla\alpha\} \\ &\quad + \left(h\alpha - \frac{3}{4}c - \alpha^2\right)\nabla h. \end{aligned} \quad (4.14)$$

Substituting (4.12) and (4.13) in the right-hand side of (4.14), we are led to

$$(h - \alpha)^2 = c. \quad (4.15)$$

Combining this with (4.12) and (4.13), we have

$$\alpha(h - \alpha) = 0.$$

Since  $h - \alpha \neq 0$ , we have

$$\alpha = 0. \quad (4.16)$$

So, (4.12) implies that  $h = 0$ . These are contradictions. We have thus proved the lemma.  $\blacksquare$

LEMMA 4.2.  $\theta = 0$  on  $\Omega$ .

PROOF. If not, then from (4.5) we have

$$du(\xi, X) = 0.$$

By (4.6), we obtain

$$\nabla\alpha = (h - 3\lambda)U. \quad (4.17)$$

Hence (4.3) is reduced to

$$\alpha\nabla h = \{h^2 - 7\lambda h + 6\lambda^2 - 4(\beta - h\alpha + c)\}U. \quad (4.18)$$

Applying  $A$  to both sides of (4.18), we have

$$4(\beta - h\alpha) = h^2 - 7h\lambda + 6\lambda^2 - 4c \quad (4.19)$$

since  $A\nabla h = 0$  and  $\lambda \neq 0$  on  $\Omega$ .

Combining (4.19) with (4.1), we are led to

$$2\lambda^2 - 3\lambda h + h^2 - c = 0. \quad (4.20)$$

Differentiating both sides of (4.20), we obtain

$$(4\lambda - 3h)\nabla\lambda + (2h - 3\lambda)\nabla h = 0. \quad (4.21)$$

On the other hand, from (4.1) we have

$$(2\lambda - h)\nabla\lambda = \lambda\nabla h. \quad (4.22)$$

Combining (4.22) with (4.21), we are led to

$$(h - \lambda)^2\nabla\lambda = 0.$$

Furthermore, we have

$$\nabla\lambda = 0$$

since  $h \neq \lambda$  by (4.20) and  $c \neq 0$ . So, from (4.22) we obtain

$$\nabla h = 0 \quad (4.23)$$

since  $\lambda \neq 0$  by Lemma 4.1. Thus (4.4) becomes

$$(4\beta - 4h\alpha + h^2 + c)\lambda + \frac{3}{2}c\alpha - 2ch = (h - 3\lambda)\left(2\alpha h - 2\beta - \frac{1}{2}h^2\right). \quad (4.24)$$

Differentiating both sides of (4.24), we have

$$\nabla\alpha = 0 \quad (4.25)$$

since  $c \neq 0$ .

From (4.4), (4.23) and (4.25), we are led to

$$\theta = 0.$$

This is a contradiction. We have thus proved the lemma. ■

Finally, we prove

**THEOREM 4.1.** *Let  $M$  be a real hypersurface in  $M_n(c)$ ,  $c > 0$ . If it satisfies  $\nabla_{\phi\nabla_\xi}\mathcal{S} = 0$  and at the same time satisfies  $S\xi = \sigma\xi$  for some constant  $\sigma$ , then  $M$  is a Hopf hypersurface.*

PROOF. By Lemma 4.2 and (4.1), we have

$$\lambda(4\lambda^2 - 4h\lambda + h^2 - 2c) = \frac{c}{2}(4h - 3\alpha), \quad (4.26)$$

$$\left(2\alpha h - 2\beta - \frac{1}{2}h^2\right)\nabla\alpha + \left(2\beta - \frac{3}{2}h\alpha\right)\nabla h = 0. \quad (4.27)$$

Applying  $A$  to both sides of (4.27) and using (4.2), we obtain

$$\left(2\alpha h - 2\beta - \frac{1}{2}h^2\right)A\nabla\alpha = 0.$$

Now, suppose that  $A\nabla\alpha \neq 0$ , then we have

$$2\alpha h - 2\beta - \frac{1}{2}h^2 = 0.$$

From this and our assumption  $\sigma = \text{constant}$ , we have

$$\nabla h = 0. \quad (4.28)$$

Differentiating both sides of (4.1), we obtain

$$(h - 2\lambda)\nabla\lambda = 0. \quad (4.29)$$

From (4.28) and (4.29), we are led to

$$\nabla\lambda = 0. \quad (4.30)$$

Thus from (4.26) we see that

$$\nabla\alpha = 0.$$

This contradicts to  $A\nabla\alpha = 0$ . So, we have

$$A\nabla\alpha = 0, \quad U\alpha = 0 \quad (4.31)$$

since  $\lambda \neq 0$ .

Using (4.2) and (4.31) and applying  $U$  to both sides of (4.3), we have

$$h\lambda + 2(\beta - h\alpha + c) = 0. \quad (4.32)$$

From (4.1) and (4.32), we obtain

$$\lambda^2 = \frac{1}{2}h\lambda - \frac{1}{4}c. \quad (4.33)$$

Substituting (4.33) to both sides of (4.26), we are led to

$$\alpha = h + 2\lambda \quad (4.34)$$

since  $c \neq 0$ .

Combining (4.34) with (4.32), we have

$$g(U, U) = \beta - \alpha^2 = -7\lambda^2 - \frac{9}{4}c < 0.$$

This is a contradiction. The theorem is now proved by all the above arguments. ■

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