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DIOPHANTINE PHENOMENA IN COMMUTING VECTOR FIELDS AND DIFFEOMORPHISMS

By

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Abstract. This paper studies the simultaneous Diophantine phenomena of commuting systems of vector fields and diffeomorphisms. We consider how these conditions are related with the Diophantine phenomena of every element of the system.

1. Introduction

In [4] J. Moser studied a commuting system of smooth circle maps ϕ_ν $(\nu = 1, \dots, d)$ with the rotation numbers $\alpha_\nu = \lim_{k\to\infty}((\phi_\nu)^k - I)/k$, $((\phi_\nu)^k = \phi_\nu \circ \cdots \circ \phi_\nu)$ satisfying a simultaneous Diophantine condition, i.e., a simultaneous approximation of a set of numbers by rationals. He showed that there exists a set of α_ν $(\nu = 1, \dots, d)$ satisfying a simultaneous Diophantine condition, whereas $\sum_{\nu=1}^d \alpha_\nu p_\nu$ is a Liouville number for any $(p_1, \dots, p_d) \in \boldsymbol{Z}^d \backslash 0$. (cf. Theorem 2 of [4]). This implies that even in the Diophantine case one cannot reduce the simultaneous linearization problem under certain regularity to the case of a single map.

In this paper we show that the same phenomenon occurs for a commuting maps in $\boldsymbol{C}^n$ fixing the origin. (cf. Theorem 5.1). On the contrary, in the case of vector fields we will show that a simultaneous Diophantine condition is equivalent to a Diophantine condition for some element in the system. More precisely, there exists an element in the system having the same Diophantine property and the resonance as those for the system. In case no Diophantine condition appears we will show that both for maps and for vector fields a si-

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multaneous Poincaré condition is equivalent to the one for some element in the system.

2. Simultaneous Siegel and Bruno Condition

Let $x = (x_1, \dots, x_n)$ be the variable in $\boldsymbol{R}^n$. We denote the differentials by $\partial_x = (\partial_{x_1}, \dots, \partial_{x_n})$, $\partial_{x_j} = \partial/\partial x_j$, $(j = 1, \dots, n)$. Let $d \geq 1$ be an integer. We consider a commuting system of analytic vector fields $\mathscr{X} := \{\mathscr{X}_\mu; \mu = 1, \dots, d\}$, where $\mathscr{X}_\mu := \sum_{j=1}^n X_j^\mu(x)\partial_{x_j}$ $(\mu = 1, \dots, d)$, and $[\mathscr{X}_\nu, \mathscr{X}_\mu] = 0$ $(\nu, \mu = 1, \dots, n)$. Define $X^\mu := (X_1^\mu, \dots, X_n^\mu)$ and $\Lambda^\mu = \nabla_x X^\mu(0)$. Note that $x\Lambda^\mu$ is the linear part of X^μ. We assume that $\mathscr{X}$ is singular at the origin. Hence we can write

$$X^\mu(x) := X^\mu = (X_1^\mu(x), \dots, X_n^\mu(x)) = x\Lambda^\mu + R^\mu(x), \quad 1 \leq \mu \leq d, \tag{2.1}$$

where $R^\mu(x)$ is analytic in x in some neighborhood of the origin such that

$$R^\mu(0) = \partial_x R^\mu(0) = 0, \quad 1 \leq \mu \leq d. \tag{2.2}$$

We assume that the diagonal entries of a (real) Jordan normal form of Λ^μ is given by $(\Lambda_1^\mu, \dots, \Lambda_{n_1}^\mu, \xi_{n_1+1}^\mu, \dots, \xi_{n_1+n_2}^\mu)$, where $\xi_j^\mu \in \boldsymbol{R}$, $(2n_1 + n_2 = n)$, and Λ_j^μ is given by

$$\Lambda_j^\mu = \begin{pmatrix} \xi_j^\mu & -\eta_j^\mu \\ \eta_j^\mu & \xi_j^\mu \end{pmatrix}, \quad \xi_j^\mu, \eta_j^\mu \in \boldsymbol{R}, \tag{2.3}$$

where $\eta_j^\mu \neq 0$ for some μ, $1 \leq \mu \leq d$.

We define λ_j^μ $(j = 1, \dots, n, \mu = 1, \dots, d)$ by $\lambda_{2j-1}^\mu = \xi_j^\mu + i\eta_j^\mu$, $\lambda_{2j}^\mu = \xi_j^\mu - i\eta_j^\mu$ for $j = 1, \dots, n_1$ and $\lambda_j^\mu = \xi_{j-n_1}^\mu$ for $j = 2n_1 + 1, \dots, n$. Then we set $\lambda^\mu = (\lambda_1^\mu, \dots, \lambda_n^\mu)$, $(\mu = 1, \dots, d)$. For a multiinteger $\alpha = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{Z}_+^n$ we set $\langle \lambda^\nu, \alpha \rangle = \sum_{j=1}^n \lambda_j^\nu \alpha_j$ and define

$$\omega(\alpha) = \min_{1 \leq j \leq n} \sum_{\nu=1}^d |\langle \alpha, \lambda^\nu \rangle - \lambda_j^\nu|. \tag{2.4}$$

DEFINITION 2.1. *We say that $\mathscr{X} := \{\mathscr{X}_\nu; \nu = 1, \dots, d\}$ is non simultaneously resonant if $\omega(\alpha) \neq 0$ for all $\alpha \in \boldsymbol{Z}_+^n$, $|\alpha| \geq 2$. The set of $\alpha \in \boldsymbol{Z}_+^n$, $|\alpha| \geq 2$ such that $\omega(\alpha) = 0$ is called a simultaneous resonance of $\mathscr{X}$.*

DEFINITION 2.2. *Let ω_k $(k = 2, 3, \dots)$ be given by*

$$\omega_k = \inf\{\omega(\alpha); \omega(\alpha) \neq 0, \alpha \in \boldsymbol{Z}_+^n, 2 \leq |\alpha| < 2^k\}. \tag{2.5}$$

We say that the system $\mathscr{X}$ satisfies a simultaneous Siegel condition, a simultaneous Bruno type condition and a simultaneous Bruno condition respectively if,

$$\omega_k \geq C(1+2^k)^{-\tau},$$

$$\omega_k \geq \exp(-C2^k/(k+1)^{1+\tau}),$$

for some constants $C > 0$ and $\tau > 0$ independent of k, and

$$-\sum_{k=2}^{\infty} \ln \omega_k/2^k < \infty.$$

In the case $d = 1$ we say that the vector field $\mathscr{X} = \mathscr{X}_1$ satisfies a Siegel condition, a Bruno type condition and a Bruno condition, respectively if the corresponding simultaneous condition is verified.

THEOREM 2.3. *Let $\mathscr{X} = \{\mathscr{X}_\nu; \nu = 1, \ldots, d\}$ be a commuting system of vector fields. Then $\mathscr{X}$ satisfies one of a simultaneous Siegel condition, a simultaneous Bruno type condition and a simultaneous Bruno condition if and only if there exist real numbers c_ν $(\nu = 1, \ldots, d)$ such that* (i) *the vector field $\mathscr{X}_0 = \sum_{\nu=1}^{d} c_\nu \mathscr{X}_\nu$ satisfies a Siegel condition, a Bruno type condition and a Bruno condition, respectively*; (ii) *the resonance of $\mathscr{X}_0$ coincides with the simultaneous resonance of the system $\mathscr{X}$.*

The proof of this theorem is given in Section 3.

REMARK 2.4. *By the same argument Theorem 2.3 holds for a commuting system of holomorphic vector fields if we replace the condition $c_\nu \in \boldsymbol{R}$ $(\nu = 1, \ldots, d)$ with the one $c_\nu \in \boldsymbol{C}$ $(\nu = 1, \ldots, d)$.*

3. Proof of Theorem 2.3

PROOF OF THEOREM 2.3. We will show the necessity of (i) and (ii). We note that the commutativity of $\mathscr{X}_\nu$ implies that the linear parts of $\mathscr{X}_\nu$ are pairwise commuting. Without loss of generality we may assume that the linear part A_1 of $\mathscr{X}_1$ is put in a Jordan normal form.

Let $c_1, \ldots, c_d$ be real numbers. By the commutativity, the real parts of eigenvalues of the linear part of $\mathscr{X}_0 := \sum_{\nu=1}^{d} c_\nu \mathscr{X}_\nu$ are given by $\sum_{\nu=1}^{d} c_\nu \xi_j^\nu$ $(j = 1, \ldots, n_1 + n_2)$. For $c = (c_1, \ldots, c_d) \in \boldsymbol{R}_+^d$ and $\alpha \in \boldsymbol{Z}_+^n$ we define

$$\Omega(\alpha, c) = \min_{1 \leq j \leq n} \left| \sum_{\nu=1}^{d} c_\nu (\langle \alpha, \lambda^\nu \rangle - \lambda_j^\nu) \right|. \tag{3.1}$$

Let $\omega(\alpha)$ and ω_k be given by (2.4) and Definition 2.2, respectively. Then we define

$$A_k = \{c = (c, \dots, c_d) \in \boldsymbol{R}_+^d; \exists \alpha \in \boldsymbol{Z}_+^n, 2 \le |\alpha| < 2^k \text{ such that } \omega(\alpha) \neq 0, \Omega(\alpha, c) < 2^{-nk-k}\omega_k\}. \tag{3.2}$$

We want to show that the Lebesgue measure of the set $A := \overline{\lim}_{k\to\infty} A_k$ is equal to zero. Without loss of generality we may assume that A_k is contained in some bounded ball B. By assumption and Definition 2.2 the length of the vector

$$\omega_k^{-1}(\langle \alpha, \lambda^1 \rangle - \lambda_j^1, \dots, \langle \alpha, \lambda^d \rangle - \lambda_j^d), \quad j = 1, \dots, n$$

is bounded from the below by some constant $K_1 >$ independent of j and k. It follows that the Lebesgue measure of the set of $c = (c_1, \dots, c_d)$ in the ball B such that $\Omega(\alpha, c) < 2^{-nk-k}\omega_k$ is bounded by $K_2 2^{-nk-k}$ for some $K_2 > 0$ independent of k. Because the number of α such that $2 \le |\alpha| < 2^k$ is bounded by $K_3 2^{nk}$ for some $K_3 > 0$ independent of k, the Lebesgue measure of A_k is bounded by $K2^{-k}$ for some $K > 0$ independent of k. Hence the Lebesgue measure of A is bounded by $K \lim_{k\to\infty} \sum_{v=k}^{\infty} 2^{-v} = 0$ for some $K > 0$ independent of k.

Therefore, if $c \notin A$ there exists $k_0 \ge 1$ such that

$$\Omega(\alpha, c) > \omega_k 2^{-nk-k}, \quad \forall k \ge k_0.$$

This proves that $\mathscr{X}_0$ satisfies a Siegel, a Bruno type and a Bruno condition, respectively.

In order to show (ii) we note that if α is not in a simultaneous resonance set of $\mathscr{X}$, the set of $c \in \boldsymbol{R}^n$ such that $\sum_{v=1}^{d} c_v(\langle \alpha, \lambda^v \rangle - \lambda_j^v) = 0$ is a hyperplane for each j. The Lebesgue measure of the sum of these hyperplanes is zero. By adding A to the sum of these hyperplanes we can choose $c \notin A$ such that the resonance of $\mathscr{X}_0$ is equal to the simultaneous resonance of $\mathscr{X}$.

We will prove the sufficiency. We define $\tilde{\omega}(\alpha)$ by

$$\tilde{\omega}(\alpha) = \min_j \left| \left\langle \alpha, \sum_v c_v \lambda^v \right\rangle - \sum_v c_v \lambda_j^v \right|.$$

We also define $\tilde{\omega}_k$ by (2.5) with $\omega(\alpha)$ replaced by $\tilde{\omega}(\alpha)$. We can easily show that $\tilde{\omega}(\alpha) \le M\omega(\alpha)$ for some $M > 0$ independent of α. If follows from the assumption (ii) that $\tilde{\omega}_k \le M\omega_k$. This implies that if $\mathscr{X}_0$ satisfies a Siegel condition (or Bruno type condition) the system $\mathscr{X}$ also satisfies a simultaneous Siegel and Bruno type condition, respectively. Now, let us assume that $\mathscr{X}_0$ satisfies a Bruno condition. Because $\ln \tilde{\omega}_k < \ln M + \ln \omega_k$, it follows that $-\sum_k \ln \tilde{\omega}_k / 2^k >$

$-\sum_k(\ln M+\ln \omega_k)/2^k$. Hence $\mathscr{X}$ satisfies a simultaneous Bruno condition. This ends the proof.

4. Note on the non Diophantine Case

We know that for commuting diffeomorphisms, a simultaneous Diophantine condition does not necessarily imply a Diophantine condition for any element of the system. In this section we will show that no such phenomena occur both for commuting diffeomorphisms and for vector fields if we assume much stronger condition than a Diophantine condition, namely a simultaneous Poincaré condition. Although the following results hold for commuting vector fields as well as for (local) commuting diffeomorphisms we state only in the case of diffeomorphisms for the sake of simplicity. The precise statements in the case of vector fields are left to the reader.

Let us start with (seemingly) weaker definition of a Poincaré condition for a system. Let $\mathscr{X}:=\{\Phi_\mu(x);\mu=1,\dots,d\}$ be a commuting system of diffeomorphisms near the origin of $\boldsymbol{C}^n$, $[\Phi_\mu,\Phi_\nu]=0$ $(\forall\nu,\forall\mu)$ such that

$$\Phi_\mu(x)=\Lambda^\mu x+\phi_\mu(x),\phi_\mu(x)=O(|x|^2),\quad \Lambda^\mu\in GL(n,\mathbf{C}). \tag{4.1}$$

By the commutativity, Λ^μ $(\mu=1,\dots,d)$ commute with each other. Hence every Λ^μ $(\mu=1,\dots,d)$ has the same Jordan block structure with diagonal entries given by

$$(\lambda_1^\mu,\dots,\lambda_n^\mu),\quad \lambda_j^\mu\neq 0, \tag{4.2}$$

where we denote with multiplicity. We set $\xi_j=(\log|\lambda_j^1|,\dots,\log|\lambda_j^d|)$ for $j=1,\dots,n$ and define

$$\Gamma=\left\{\sum_{j=1}^n c_j\xi_j\in\boldsymbol{R}^d;c_j\geq 0,c_1^2+\cdots+c_n^2\neq 0\right\}.$$

DEFINITION 4.1. *The maps Φ_μ $(\mu=1,\dots,d)$ satisfy a simultaneous Poincaré condition if Γ does not contain the origin.*

REMARK 4.2. *In the case $d=1$, the above definition is equivalent to the usual Poincaré condition for a single map, $|\lambda_j^1|>1$ $(j=1,\dots,n)$ or $|\lambda_j^1|<1$ $(j=1,\dots,n)$.*

We set $\lambda^\mu=(\lambda_1^\mu,\dots,\lambda_n^\mu)$ and $(\lambda^\mu)^\alpha=(\lambda_1^\mu)^{\alpha_1}\cdots(\lambda_n^\mu)^{\alpha_n}$.

We say that $\mathscr{X}$ is simultaneously nonresonant if

$$\min_{|\alpha|=m, \alpha\in \mathbf{Z}_+^n, 1\le j\le n} \sum_{\mu=1}^{d} |(\lambda^\mu)^\alpha - \lambda_j^\mu| \neq 0 \quad \text{for } m = 2, 3, \ldots. \tag{4.3}$$

The set of multiintegers α which does not satisfy (4.3) is called a simultaneous resonance of $\mathscr{X}$. In the case $d = 1$, it coincides with the resonance of a single map.

THEOREM 4.3. *The system $\mathscr{X}$ satisfies a simultaneous Poincaré condition if and only if there exist $t_\mu \in \mathbf{Z}$ $(\mu = 1, \ldots, d)$ such that* (i) *$\Phi_0 := \prod_{\mu=1}^d \Phi_\mu^{t_\mu}$ satisfies a Poincaré condition*; (ii) *the resonance of Φ_0 coincides with a simultaneous resonance of $\mathscr{X}$.*

PROOF. Let Γ' be the dual cone of Γ

$$\Gamma' = \left\{(c_1, \ldots, c_d) \in \mathbf{R}^d; \sum_j c_j t_j > 0, \forall (t_1, \ldots, t_d) \in \Gamma\right\}. \tag{4.4}$$

The eigenvalues of the linear part of Φ_0 is given by $\prod_{\mu=1}^d (\lambda_j^\mu)^{t_\mu}$, $j = 1, \ldots, n$. Hence Φ_0 satisfies a Poincaré condition iff, by replacing t_μ with $-t_\mu$, if necessary,

$$\sum_{\mu=1}^d t_\mu \log|\lambda_j^\mu| > 0 \quad \text{for } j = 1, \ldots, n.$$

It follows that $\Gamma' \cap \mathbf{Z}^d \neq \emptyset$, where Γ' is a dual cone of Γ. Because Γ' is an open cone this is equivalent to $\Gamma' \neq \emptyset$. Because $0 \notin \Gamma$ if and only if $\Gamma' \neq \emptyset$, $\mathscr{X}$ satisfies a simultaneous Poincaré condition. This proves the sufficiency part and the necessity of (i).

In order to show the necessity of (ii) we want to show that we can choose $(t_1, \ldots, t_d)$ so that the resonance of Φ_0 is equal to the simultaneous resonance of $\mathscr{X}$. We first note that Φ_0 is resonant for a simultaneous resonance of $\mathscr{X}$. Suppose that α is not a simultaneous resonance of $\mathscr{X}$. By definition α is not a resonance of Φ_0 if,

$$\prod_{j=1}^n \prod_{\mu=1}^d (\lambda_j^\mu)^{t_\mu \alpha_j} - \prod_{\mu=1}^d (\lambda_\ell^\mu)^{t_\mu} \neq 0, \quad \ell = 1, \ldots, n. \tag{4.5}$$

By taking a logarithm of both sides of (4.5) we have

$$\sum_{\mu=1}^d t_\mu \left(\log \prod_{j=1}^n (\lambda_j^\mu)^{\alpha_j} - \log \lambda_\ell^\mu\right) \neq 0, \quad \ell = 1, \ldots, n. \tag{4.6}$$

The condition (4.6) holds for all except a finite number of α's in view of a Poincaré condition. On the other hand, for each α and ℓ, $1 \le \ell \le n$ the simultaneous nonresonant condition implies the existence of μ such that $\log \prod_{j=1}^{n} (\lambda_j^{\mu})^{\alpha_j} - \log \lambda_\ell^{\mu} \neq 0$. Because (4.6) means that $(t_1, \dots, t_d)$ does not lie on a sum of hyperplanes and because Γ' is an open cone there exists $t_\mu \in \Gamma' \cap \boldsymbol{Z}^d$ satisfying (4.6). Hence α is not a resonance of Φ_0. $\square$

Finally we give expressions of a simultaneous Poincaré condition.

REMARK 4.4. *The simultaneous Poincaré condition is equivalent to each of the following conditions.*

(a) *There exist $c_1 > 0$ and $c_2 > 0$ such that*

$$\prod_{\nu, |(\lambda^\nu)^\alpha| \ge 1} |(\lambda^\nu)^\alpha| \ge e^{c_1 |\alpha|} \prod_{\nu, |(\lambda^\nu)^\alpha| < 1} |(\lambda^\nu)^\alpha|, \quad \forall \alpha \in \mathbf{Z}_+^n,\ |\alpha| > c_2. \tag{4.7}$$

(b) *There exist $c_1 > 0$ and $c_2 > 0$ such that for each $\alpha \in \mathbf{Z}_+^d$, $|\alpha| > c_1$ we can choose $\nu = \nu(\alpha)$, $1 \le \nu \le d$ such that either $|(\lambda^\nu)^\alpha| > e^{c_2 |\alpha|}$ or $|(\lambda^\nu)^\alpha| < e^{-c_2|\alpha|}$ holds.*

(c) *The dual cone Γ' of Γ is nonempty.*

Because the proof of these facts are elementary we omit the proof.

5. Diffeomorphisms

In this section we will show that the simultaneous Diophantine condition for a commuting system of diffeomorphisms in $\boldsymbol{C}^n$ does not imply the Diophantine condition for any element of the system. We consider the commuting system of diffeomorphisms as in (4.1). We continue to use the same notations as in Section 4. We assume that we are in a Siegel domain. Namely we assume that the eigenvalues satisfy

$$|\lambda_j^\nu| = 1, \quad (\nu = 1, \dots, d; j = 1, \dots, n). \tag{5.1}$$

We say that the system $\{\Phi_\nu\}_{\nu=1}^d$ satisfies a simultaneous Diophantine condition if there exist $c > 0$ and a real number τ such that

$$\min_{1 \le k \le n} \sum_{\nu=1}^{d} \left| \prod_{j=1}^{n} (\lambda_j^\nu)^{\alpha_j} - \lambda_k^\nu \right| \ge c|\alpha|^{-\tau}, \quad \forall |\alpha| \ge 2,\ \alpha \in \boldsymbol{Z}_+^n. \tag{5.2}$$

If we set $\lambda_j^\nu = \exp(2\pi i \theta_j^\nu)$, $0 \le \theta_j^\nu \le 1$ and $\theta^\nu = (\theta_1^\nu, \dots, \theta_n^\nu)$, $\langle \alpha, \theta^\nu \rangle = \sum_{j=1}^n \alpha_j \theta_j^\nu$, (5.2) is equivalent to the following condition

$$(5.3) \qquad \min_{1\le k\le n}\sum_{\nu=1}^{d}\|\langle\alpha,\theta^{\nu}\rangle-\theta_k^{\nu}\|\ge c|\alpha|^{-\tau},\quad \forall|\alpha|\ge 2,\ \alpha\in \boldsymbol{Z}_+^n,$$

where $\|t\| = \inf_{p\in Z}|t-p|$. Let $p_\nu \in \boldsymbol{Z}$ $(\nu = 1,\dots,d)$ and set $\delta_j = \sum_{\nu=1}^{d}\theta_j^{\nu}p_\nu$ and $\delta = (\delta_1,\dots,\delta_n)$. We say that the vector δ satisfies a Liouville condition if, for every $\lambda > 0$ the inequality

$$(5.4) \qquad 0 < \min_{1\le k\le n}\|\langle\alpha,\delta\rangle-\delta_k\| < |\alpha|^{-\lambda},$$

holds for infinitely many $\alpha\in \boldsymbol{Z}_+^n$. We note that δ gives the eigenvalues of the map $\Phi \equiv \Phi_1^{p_1}\cdots\Phi_d^{p_d}$.

In the following, without loss of generality we assume that $\Phi_1 = Identity$. Then we have

THEOREM 5.1. *Suppose that $d > n \ge 2$. Then there exists a set of linearly independent vectors $\theta_j = (\theta_j^1,\dots,\theta_j^d)$ $(j = 1,\dots,n)$ with the density of continuum which satisfies a simultaneous Diophantine condition* (5.3), *whereas for any $p = (p_1,\dots,p_d)\in \boldsymbol{Z}^d\backslash 0$ the $\delta = (\delta_1,\dots,\delta_n)$, $\delta_j = \sum_{\nu=1}^{d}\theta_j^{\nu}p_\nu$ satisfies a Liouville condition* (5.4).

In order to prove Theorem 5.1 we need a result in [4]. We state it for the reader's convenience. (For the detail, see [4]). Let $E^n\subset \boldsymbol{R}^d$ be a real subspace in $\boldsymbol{R}^d$. With the norm $|\cdot|$ in (5.3) we define

$$\mathrm{dist}(x,E^n) = \min_{y\in E^n}|x-y|,\quad x\in \boldsymbol{R}^d.$$

DEFINITION 5.2. *We define $\mu := \mu(E^n)$ as the supremum of the numbers λ for which*

$$\mathrm{dist}(j,E^n) < |j|^{-\lambda},\quad j\in \boldsymbol{Z}^d$$

possesses infinitely many solutions. Here $\mu = \infty$ is admitted.

Clearly, the definition is independent of the norm. Note that, if $\boldsymbol{Z}^d\cap E^n = \{0\}$ and $\tau > \mu$ then there exists a positive constant c such that

$$(5.5) \qquad \mathrm{dist}(j,E^n)\ge c|j|^{-\tau},\quad \text{for all } j\in \boldsymbol{Z}^d\backslash\{0\}.$$

A subspace E^n satisfying $\boldsymbol{Z}^d\cap E^n = \{0\}$ and (5.5) is called a Diophantine subspace with respect to $\boldsymbol{Z}^d$. The following theorem is given in Moser [Theorem 2.1, 4]. (See also [5]).

THEOREM 5.3. *For almost all E^n in the Grassmann manifold $G_n(\boldsymbol{R}^d)$ one has $\mu(E^n) = n/(d-n)$.*

PROOF OF THEOREM 5.1. We use the argument in Moser [4]. Suppose that there exists a subspace E^n in $\boldsymbol{R}^d$ generated by the linearly independent vectors $\theta_j = (\theta_j^1, \dots, \theta_j^d)$, $(j = 1, \dots, n)$ such that $\mu(E^n) = n/(d-n)$. Let τ be such that $\tau > n/(d-n)$. Then we have (5.5). We consider the left-hand side of (5.3)

$$\min_{1\le k\le n} \sum_{v=1}^{d} \|\langle\alpha, \theta^v\rangle - \theta_k^v\| = \min_{1\le k\le n} \sum_{v=1}^{d} \inf_{p_v \in \boldsymbol{Z}} |\langle\alpha, \theta^v\rangle - \theta_k^v - p_v|. \tag{5.6}$$

We set

$$y = y_k = (\langle\alpha, \theta^v\rangle - \theta_k^v)_{v\downarrow 1,\dots,d} \in E^n, \quad k = 1, \dots, n.$$

Let $j = (p_v)_{v\downarrow 1,\dots,d} \in \boldsymbol{Z}^d$ be a multiinteger for which the infimum in the right-hand side of (5.6) is taken. Then the right-hand side of (5.6) is bounded from the below by $c_1 \min_{1\le k\le n}|j - y_k|$ for some positive constant c_1 independent of j and k. By the inequality $|j - y_k| \ge \operatorname{dist}(j, E^n)$ for $k = 1, \dots, n$ and (5.5) we can estimate the right-hand side of (5.6) from the below in the following way

$$\ge c_1 \min_{1\le k\le n} |j - y_k| \ge c_1 \operatorname{dist}(j, E^n) \ge c_2 |j|^{-\tau}, \tag{5.7}$$

for some positive constant c_2 independent of j. Because the infimum in (5.6) is taken for j such that $|j - y_k| \le M|y_k|$ for some constant M independent of k, we obtain, by the condition $|\alpha| \ge 2$

$$|j| \le (1 + M)|y_k| \le c'(1 + |\alpha|) \le c''|\alpha|$$

for some positive constants c' and c''. It follows that the right-hand side of (5.7) is bounded from the below by $c|\alpha|^{-\tau}$ for some positive constant c independent of α. This proves (5.3).

We want to show that there exists E^n satisfying $\mu(E^n) = n/(d-n)$ and the Liouville property (5.4) for any $p = (p_1, \dots, p_d) \in \boldsymbol{Z}^d \backslash 0$. Let $g, h \in \boldsymbol{Z}^d$ be the given two linearly independent vectors and define a $\Gamma_2 = \boldsymbol{Z}^d \cap (\operatorname{span}\{g, h\})^\perp$. The set Γ_2 is a sublattice of $\boldsymbol{Z}^d$ of codimension 2. We choose a basis $\gamma_3, \dots, \gamma_d$ of Γ_2 and, extend it by γ_1, γ_2 to a basis of $\boldsymbol{Z}^d$. With $\Gamma_1 = \operatorname{span}_{\boldsymbol{Z}}\{\gamma_1, \gamma_2\}$ we have a splitting $\boldsymbol{Z}^d = \Gamma_1 \oplus \Gamma_2$. We define a rational projection $P : \boldsymbol{Z}^d \to \Gamma_1$ by $P(v_1 + v_2) = v_1$ for $v_i \in \Gamma_i$. The dimension of Γ_1 is called a rank of P.

Let ξ_j, η $(j = 1, \dots, n-1)$ be a basis of E^n, and define $\zeta_j = \lambda\xi_j + \eta$ for $j = 1, \dots, n-1$, where λ is a real parameter. Suppose that we can choose $\eta \in E^n$ so that $P\eta \neq 0$. We define

$$f_{ghj}(\lambda) = \frac{\langle g, \zeta_j \rangle}{\langle h, \zeta_j \rangle}. \tag{5.8}$$

We want to show that $f_{ghj}(\lambda)$ is not constant for some j, $1 \le j \le n-1$. Suppose that this is not true. Then the differentiation vanishes, $f'_{ghj}(\lambda) = 0$ for $j = 1, \dots, n-1$. By simple calculations we have

$$\frac{\langle g, \xi_j \rangle}{\langle h, \xi_j \rangle} = \frac{\langle g, \eta \rangle}{\langle h, \eta \rangle}, \quad j = 1, \dots, n-1. \tag{5.9}$$

Then the slopes of the vectors $P\xi_j$ and $P\eta$ in the plane coincides. It follows that there exist numbers α_j and β_j such that the linear combination $\omega_j = \alpha_j \xi_j + \beta_j \eta$ satisfies $P\omega_j = 0$ and $\omega_j \ne 0$ for $j = 1, \dots, n-1$.

We will show that ω_j $(j = 1, \dots, n-1)$ are linearly independent. Assume that $\sum_j c_j \omega_j = 0$. It follows that $\sum_j c_j \alpha_j \xi_j + (\sum_k c_k \beta_k)\eta = 0$. Hence we have $c_j \alpha_j = 0$ for $j = 1, \dots, n-1$. Suppose that $\beta_j \ne 0$. Because $P\eta \ne 0$ and $P\omega_j = 0$ we have $0 \ne \beta_j P\eta = -\alpha_j P\xi_j$. Thus we have $\alpha_j \ne 0$. It follows that $c_j = 0$. In case $\beta_j = 0$, it follows from the definition of ω_j that $0 \ne \omega_j = \alpha_j \xi_j$. Hence we have $\alpha_j \ne 0$. Because $c_j \alpha_j = 0$, it follows that $c_j = 0$. Hence we have $c_j = 0$ for $j = 1, \dots, n-1$. It follows that, if F is a rational subspace of codimension 2 defined by Γ_2 we have

$$\dim(E^n \cap F) \ge n - 1. \tag{5.10}$$

Next we consider the case where there does not exist an $\eta \in E^n$ such that $P\eta \ne 0$. It follows from the definition of P that $\dim(E^n \cap F) = n \ge n-1$. Hence we obtain (5.10).

We want to show that the Lebesgue measure of the subspace E^n satisfying (5.10) in a Grassmann manifold is zero. We consider only the special subspace E^n given by

$$E^n : x_{\nu+n} - \sum_{\mu=1}^{n} c_{\nu\mu} x_\mu = 0, \quad \nu = 1, 2, \dots, d-n. \tag{5.11}$$

We define the matrix A by $A = (c_{\nu\mu})_{\nu\mu}$. Let $s = (s_1, \dots, s_d) \in \boldsymbol{Z}^d$ and $t = (t_1, \dots, t_d) \in \boldsymbol{Z}^d$ span $F^\perp$. By (5.10) the $d-n$ vectors which define E^n and s, t are not linearly independent. It follows that every $d-n+2$ cofactor matrix of

$$\begin{pmatrix} A, & -I_{d-n} \\ & t_1, \dots, t_d \\ & s_1, \dots, s_d \end{pmatrix}$$

vanishes. On the other hand, since t and s are linearly independent the determinant is a polynomial of $c_{\nu\mu}$ of degree $d-n$ which does not vanish identically. It follows that the $n(d-n)$ dimensional Lebesgue measure of E^n in the Grassmann manifold is zero.

Since the set of subspace E^n in a Grassmann manifold satisfying a Diophantine condition $\mu(E^n) = n/(d-n)$ has positive measure we can take E^n so that (5.10) does not hold. Therefore $f_{ghj}(\lambda)$ is not constant for some j, $1 \leq j \leq n-1$. In the following, we take such a j and we fix it. Because the Liouville numbers form a residual set $\mathscr{L}$ on $\boldsymbol{R}$, i.e., countable intersection of open dense sets, the same property holds for $\cap f_{ghj}^{-1}(\mathscr{L})$, where the intersection is taken over all linearly independent vectors $g, h \in \boldsymbol{Z}^d$. Hence the set is residual, dense in $\boldsymbol{R}$ and of the cardinality of continuum. For every λ in this set the vector $\zeta_j = \lambda\xi_j + \eta$ satisfies that $\langle g, \zeta_j\rangle / \langle h, \zeta_j\rangle$ is a Liouville number. We set $h = (1, 0, \ldots, 0)$ and define θ_j by $\theta_j = \zeta_j/\zeta_j^1$, where ζ_j^1 is the first component of ζ_j. If we take a basis of E^n containing θ_j we have (5.4). $\square$

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