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SIMPLE COMPONENTS OF $Q[Sp_4(F_q)]$

By

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Abstract. The character table of $G = Sp_4(F_q)$, q odd, was calculated by B. Srinivasan in 1968 [22]. The rational and the real Schur indices of each complex irreducible character of G were calculated by A. Przygocki in 1982 [21]. We calculate the Hasse invariants of each simple component of the group algebra $Q[G]$ of G over Q.

Introduction

Let F_q be a finite field with q elements of characteristic p. In this paper we shall calculate the Hasse invariants of each simple component of the group algebra $Q[Sp_4(F_q)]$, q odd. Our main interest is to seek the distribution of the invariants. We will see that the results are similar to those obtained by G. J. Janusz for $SL_2(F_q)$ ([9]). In this connection, we should mention that A. Przygocki has already determined the rational and the real Schur indices of each complex irreducible character of $Sp_4(F_q)$, q odd, ([21]) and R. Gow has shown that each complex irreducible character of $Sp_4(F_{2^f})$ has the rational Schur index 1 ([6]).

It may be needed to explain why we treat such a special finite group $Sp_4(F_q)$. In the following discussion, if χ is a complex irreducible character of a finite group, then $m_Q(\chi)$ denotes the Schur index of χ with respect to Q and, for a rational prime $r, m_{Q_r}(\chi)$ denotes the r-local Schur index of χ.

Let $\tilde{G}$ be a connected, reductive linear algebraic group, defined over F_q, and let G be the group of F_q-rational points of $\tilde{G}$. Let $\tilde{Z}$ be the centre of $\tilde{G}$. Let χ be a complex irreducible character of G. Then the following theorems hold:

THEOREM 1 ([14, 15, 18]). *Assume (for the sake of simplicity) that p is good for $\tilde{G}$, and that $(\lambda^G, \chi)_G = 1$ for some linear character λ of a Sylow p-subgroup of*

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G or the degree of χ *is coprime to* p. *Then* $m_Q(\chi) \leqq 2$. *If* $\tilde{Z}$ *is connected or if* q *is an even power of* $p \neq 2$, *then we have* $m_{Q_r}(\chi) = 1$ *for each prime number* $r \neq p$. *If* $\tilde{Z}$ *is connected and* $\tilde{G}$ *is split over* F_q, *then we have* $m_Q(\chi) = 1$.

THEOREM 2 ([1, 4, 6, 16, 17, 18, 28]). *If* $G = GL_n(F_q), SO_5(F_q), CSp_4(F_q), G_2(F_q)$ ($\tilde{Z}$ *is connected and* $\tilde{G}$ *is split over* F_q) *or* ${}^3D_4(F_q)$ ($\tilde{Z}$ *is trivial*), *then* $m_Q(\chi) = 1$. *If* $G = U_n(F_q)$ ($\tilde{Z}$ *is connected but* $\tilde{G}$ *is not split over* F_q), *then* $m_Q(\chi) \leqq 2$ *and, for each prime number* $r \neq p$, *we have* $m_{Q_r}(\chi) = 1$.

THEOREM 3 ([10, 12, 19]). *If* χ *is a unipotent character of* G, *then, for each prime number* $r \neq p$, *we have* $m_{Q_r}(\chi) = 1$.

We should remark that M. Geck has shown that the cuspidal unipotent characters of $E_7(F_q)$ have the p-local Schur indices 2 provided that q is an even power of p such that $p \equiv 1 \pmod 4$ and that p is sufficiently large so the G. Lusztig's results in [11] can be used ([5]) (the condition that $p \gg 0$ can be removed [20]); thus $E_8(F_q)$ also has unipotent characters having the same rationality when q is an even power of p such that $p \equiv 1 \pmod 4$.

We note that the characters χ which satisfy the condition in Theorem 1 occupy "almost all" the complex irreducible characters of G, so that Theorems 1, 2, 3 suggest that when $\tilde{Z}$ is connected or q is an even power of $p \neq 2$, the distribution of the invariants will be comparatively simple. On the other hand, when $\tilde{Z}$ is not connected (e.g. $\tilde{G}$ is a non-adjoint semi-simple algebraic group) and q is an odd power of $p \neq 2$ the known results are considerable complicated.

Let, for example, $G = SL_n(F_q)$ (see [6, 7, 9, 13, 23, 28]). Then we have $m_Q(\chi) \leqq 2$, and, if $p = 2$, or n is odd, or $\mathrm{ord}_2\, n > \mathrm{ord}_2(p-1)$, we have $m_Q(\chi) = 1$; if $1 \leqq \mathrm{ord}_2\, n \leqq \mathrm{ord}_2(p-1)$ and q is an even power of p, we have $m_{Q_r}(\chi) = 1$ for each prime number $r \neq p$; if $1 \leqq \mathrm{ord}_2\, n \leqq \mathrm{ord}_2(p-1)$ and q is an odd power of p, it often happens that $m_{Q_r}(\chi) = 2$ for some prime numbers $r \neq p$. Thus it would be natural to wish to know the distribution of the invariants for other groups, such as $Sp_4(F_q)$, as an example when $\tilde{Z}$ is not connected.

Let $G = Sp_4(F_q)$, q odd. Then, as is well known, the character table of G was first calculated by B. Srinivasan in [22]. Later Hiromichi Yamada reconstructed the character table of G in [25] (unpublished) along the same line as in the paper [3] of H. Enomoto. Gow has obtained some results about the rationality-properties of characters of $Sp_{2n}(F_q)$ ([6, 7, 8]). In some cases of our arguments below, we can follow Przygocki's arguments in [21].

Our first task was to calculate the value fields $Q(\chi)$ $Q(\chi(g), g \in G)$. But, in Srinivasan's character table in [22], some character-values are omitted. I asked Professor Ken-ichi Shinoda about these omitted values. Then Professor H. Yamada sent me his preprint [25] and permitted me to use it; in [25] all character-values are typed; he also taught me that the omitted values in [22] can be obtained from informations in [22]. I wish to thank these two professors for their kindness. Professor Toshihiko Yamada has published many works on the rationality-properties of characters of finite groups; in particular, I employed in several places of my proofs his index formulas [27, Chap. 4] which have been very useful. Finally, I wish to thank the referee for his (her) kind advice.

Notation

F_q is a finite field with q elements of characteristic $p \neq 2$ and $\bar{F}_q$ is an algebraic closure of F_q. $\tilde{G}$ is the group of all non-singular matrices X of degree 4 with entries in $\bar{F}_q$ such that $XJ{}^tX = J$ where

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(${}^tX$ is the transpose of X), and G is the subgroup of $\tilde{G}$ consisting of the matrices in $\tilde{G}$ with entries in F_q. The Frobenius endomorphism of the algebraic group $\tilde{G}$ will be given by $a \to a^{(q)}$ (if $a = [a_{ij}]$, then $a^{(q)} = [a_{ij}^q]$). The centre $\tilde{Z}$ of $\tilde{G}$ is $\{1, -1\}$, which is also the centre of G.

If K is a field, then $K^\times$ denotes the multiplicative group of K.

As in [22], let κ be a fixed element of order $q^4 - 1$ in $\bar{F}_q^\times$, and let $\theta = \kappa^{q^2+1}$, $\zeta = \kappa^{q^2-1}$, $\eta = \theta^{q-1}$, $\gamma = \theta^{q+1}$ and $v = \gamma^{(q-1)/(p-1)}$. Fixing an isomorphism Θ of the cyclic group $\langle\kappa\rangle$ into $C^\times$, we put $\tilde{\theta} = \Theta(\theta)$, $\tilde{\zeta} = \Theta(\zeta)$, $\tilde{\eta} = \Theta(\eta)$, $\tilde{\gamma} = \Theta(\gamma)$ and $\tilde{v} = \Theta(v)$. For an integer k, let $\alpha_k = \tilde{\gamma}^k + \tilde{\gamma}^{-k}$ and $\beta_k = \tilde{\eta}^k + \tilde{\eta}^{-k}$. For a positived integer n, ζ_n is a certain primitive n-th root of unity in some algebraically closed field of characteristic 0.

If α, β are class functions on a finite group H over an algebraically closed field of characteristic 0, then $(\alpha, \beta)_H = (1/|H|) \sum_{h \in H} \alpha(h)\beta(h^{-1})$. If f is a function on a set S and if T is a subset of S, then $f|T$ denotes the restriction of f to T.

As to the notation of the complex irreducible characters of G, we follow that of Srinivasan in [22]; in particular, the following notation will be used as parameter-sets of some of the complex irreducible characters of G: $R_1 =$

$\{1, 2, \ldots, (1/4)(q^2-1)\}$, R_2 is a set of $(1/4)(q-1)^2$ distinct positive integers i such that $\theta^i, \theta^{-i}, \theta^{qi}, \theta^{-qi}$ are all distinct, $T_1 = \{1, 2, \ldots, (1/2)(q-3)\}$ and $T_2 = \{1, 2, \ldots, (1/2)(q-1)\}$.

Let K be a field of characteristic 0, and let L be an algebraically closed extension of K. Let χ be a (generalized) character of a finite group H over L. Then we set $K(\chi) = K(\chi(h), h \in H)$. If χ is absolutely irreducible, then $A(\chi, H)$ denotes the simple component of the group algebra $K[H]$ of H over K associated with χ and $m_K(\chi)$ denotes the Schur index of χ with respect to K. In this case $A(\chi, K)$ is isomorphic over K to $A(\chi, K(\chi))$ (see, e.g., [27, Proposition 1.5, p. 8] and $m_K(\chi)$ is equal to the index of $A(\chi, K)$.

Let K be a field. If A is a finite-dimensional central simple algebra over K, then $[A]$ denotes the class of A in the Brauer group of K; for two such algebras A, B over K, we write $A \sim B$ if $[A] = [B]$.

Let K be a finite algebraic extension of Q. Then, for a place v of K (see Weil [24, pp. 43–44]), K_v is the completion of K at v. If v is a place of K lying above a finite place r of Q (we write $w|r$), then Q_r is the topological closure of Q in K_v and may be idetified with the r-adic rational field. If A is a finite-dimensional central simple algebra over K, then, for a place v of K, A_v denotes the simple algebra $A \otimes_K K_v$ over K_v, and $h_v(A)$ or $h(A_v)$ denotes the Hasse invariant of A_v ($h_v(A) \in Q/Z$).

Let K be a field and let L be a finite Galois extension of K. Then $\mathrm{Gal}(L/K)$ denotes the Galois group of L over K. Put $H = \mathrm{Gal}(L/K)$, and let $\beta : H \times H \to L$ be a factor set (i.e. 2-cocycle) of H with values in L. Then $(\beta, L/k)$ denotes the crossed product algebra over K corresponding to β: $(\beta, L/K)$ is a left vactor-space over L with a basis $\{u_\sigma, \sigma \in H\}$ such that the multiplication law is given by the following formula:

$$\left(\sum_{\sigma \in H} x_\sigma u_\sigma\right)\left(\sum_{\tau \in H} y_\tau u_\tau\right) = \sum_{v \in H}\left(\sum_{\substack{\sigma, \tau \in H \\ \sigma\tau = v}} x_\sigma \sigma(y_\tau)\beta(\sigma, \tau) u_v\right) \quad (x_\sigma, y_\tau \in L).$$

Assume that L is a cyclic extension of K of degree n, and let $H = \langle\sigma\rangle$. Then, for $a \in K^\times$, $(a, L/K, \sigma)$ or (a, L, σ) denotes the cyclic algebra over K corresponding to a (with respect to σ); if we set $\beta(\sigma^i, \sigma^j) = a^{[(i+j)/n]-[i/n]-[j/n]}$ ([*] is the Gauss symbol), then we have $(a, L, \sigma) = (\beta, L/K)$.

Let a, b be rational integers with $a \neq 0$. Then (a, b) is the greatest commomn divisor of a, b. We wrire $a|b$ (resp. $a \nmid b$) if a divides b (resp. if a does not divide b). Let r be a prime number such that $r|a$. Then we write $r^e \| a$ (e is a positive integer) if $r^e | a$ but $r^{e+1} \nmid a$; in this case we write $\mathrm{ord}_r\, a = e$.

1. Preliminaries

In the following, K is a field of characteristic 0, C is an algebraically closed extension of K, H is a finite group and χ is an absolutely irreducible character of H over C.

PROPOSITION A (E. Witt; see [27, Proposition 3.8, p. 29]). *Assume that $K(\chi) = K$. Let M be a subgroup of H and let ξ be an absolutely irreducible character of M over C such that $K(\xi) = K$ and $(\chi|M, \xi)_M = n \neq 0$. Then, for each prime number r such that $(r, n) = 1$, the r-parts of $[A(\chi, K)]$ and $[A(\xi, K)]$ are the same.*

COROLLARY B. *Let the notation and the assumption be as in Proposition A. Assume that $m_K(\chi)$ and $m_K(\xi)$ are coprime to n. Then we have $[A(\chi, K)] = [A(\xi, K)]$. In particular, (T. Yamada) if $\chi = \xi^H$, then $[A(\chi, K)] = [A(\xi, K)]$.*

Let L be a finite Galois extension of K of the form $K(\varepsilon)$ for some root of unity ε. Then, if β is a factor set of $\mathrm{Gal}(L/K)$ such that, for any $\sigma, \tau \in \mathrm{Gal}(L/K)$, $\beta(\sigma, \tau)$ is a root of unity in L, the crossed product algebra $(\beta, L/K)$ will be called a cyclotomic algebra over K.

In the following two propositions and the remark, N is a normal subgroup of H. If ψ is a character of N, then, for $h \in H$, ψ^h is the character of N defined by $\psi^h(x) = \psi(hxh^{-1})$, $x \in N$.

PROPOSITION C (T. Yamada [27, Proposition 3.4, p. 23]). *Suppose that χ is induced by an absolutely irreducible character ψ of N over C and that $K(\chi) = K$. Set $F = \{f \in H \mid \psi^f = \psi^{\tau(f)}$ for some $\tau(f) \in \mathrm{Gal}(K(\psi)/K)\}$. Let $Nf_1, Nf_2, \dots, Nf_t$ $(f_1 = 1)$ be all the distinct cosets of N in F, and set $\tau_i = \tau(f_i)$, $1 \leqq i \leqq t$. Then $F/N \simeq \{\tau_1, \tau_2, \dots, \tau_t\} = \mathrm{Gal}(K(\psi)/K)$ and $K(\psi^F) = K$.*

PROPOSITION D (T. Yamada [27, Proposition 3.5, p. 24]). *Let the notation and the assumption be as in Proposition C. For $1 \leqq i, j \leqq t$, let $f_i f_j = n_{ij} f_{v(i,j)}$, $n_{ij} \in N$, $v(i, j) \in \{1, 2, \dots, t\}$. Suppose that ψ is a linear character of N, and put $\beta(\tau_i, \tau_j) = \psi(n_{ij})$, $l \leqq i, j \leqq t$. Then β is a factor set of $\mathrm{Gal}(K(\psi)/K)$ consisting of roots of unity in $K(\psi)$ and the algebra $A(\psi^F, K)$ over K is isomorphic over K to the cyclotomic algebra $(\beta, K(\psi)/K)$ over K.*

REMARK E. Let the notation and the assumption be as in Propositions C, D. Suppose that F/N is a cyclic group of order t. Let f be an element of F

such that $F/N = \langle Nf \rangle$, and put $\tau = \tau(f)$. Then $(\beta, K(\psi)/K)$ is a cyclic algebra $(\psi(f^t), K(\psi)/K, \tau)$ (the verification is easy).

LEMMA F (see, e.g., [26, Lemma 7]). *Suppose that K is a finite algebraic extension of Q. Let L be a finite Galois extension of K, let β be a factor set of* $\mathrm{Gal}(L/K)$ *such that for $\sigma, \tau \in \mathrm{Gal}(L/K)$, $\beta(\sigma, \tau)$ is a unit in L, and let $A = (\beta, L/K)$. Then, if v is a finite place of K that is unramified in L, we have* $h_v(A) \equiv 0 \pmod 1$.

THEOREM G (Hasse's sum formula; see, e.g., [24, Theorem 2, p. 255]). *If A is a finite-dimensional central simple algebra over a finite algebraic extension K of Q, then we have* $\sum_v h_v(A) \equiv 0 \pmod 1$, *where the sum is taken over all the places v of K.*

LEMMA H ([22, Lemma 3.1]). *Let $\tilde{H}$ be a subgroup of $\tilde{G}$. If there is a matrix y in $\tilde{G}$ such that $y^{-1}ay = a^{(q)}$ for all $a \in \tilde{H}$, then there is a matrix z in $\tilde{G}$ such that $z^{-1}\tilde{H}z \subset G$.*

LEMMA I ([22, Lemma 1.1]). *Let S be the set of non-zero elements of F_q which are squares in F_q and let S' be the set of elements of F_q which are not squares in F_q. Put $s = (-1)^{(q-1)/2}$ (recall that q is odd). Then there are complex additive characters $\varepsilon, \varepsilon'$ of F_q such that*

$$\sum_{x \in S} \varepsilon(x) = \sum_{x \in S'} \varepsilon'(x) = -\frac{s}{2}(s + \sqrt{sq}),$$

$$\sum_{x \in S'} \varepsilon(x) = \sum_{x \in S} \varepsilon'(x) = -\frac{s}{2}(s - \sqrt{sq}).$$

THEOREM J (The Brauer-Speiser Theorem; see, e.g., [27, Corollary 1.8, p. 9]). *If χ is a complex irreducible character of a finite group whose values are real, then we have $m_Q(\chi) \leqq 2$.*

THEOREM K (R. Gow [7, Theorem 2.9]). *For any complex irreducible character χ of $Sp_{2n}(F_q)$, we have $m_Q(\chi) \leqq 2$.*

PROPOSITION L (G. J. Janusz [9, Proposition 1]). *Let n be an integer $\geqq 3$, $K = Q(\zeta_n + \zeta_n^{-1})$,* $\mathrm{Gal}(Q(\zeta_n)/K) = \langle \iota \rangle$, *where $\zeta_n^\iota = \zeta_n^{-1}$, and $A = (-1, Q(\zeta_n)/K, \iota)$. Let v be a place of K. Then, if v is infinite (i.e. real), we have $h_v(A) \equiv 1/2$*

(mod 1), *and, if* v *is finite, we have* $h_v(A) \equiv 0$ (mod 1) *except in the following cases*: *If* n *is of the form* r^m *or* $2r^m$, *where* r *is an odd prime number of the form* $4k-1$ *and* $m \geqq 1$, *and if* $v|r$, *we have* $h_v(A) = 1/2$ (mod 1). *If* $n = 4$ *and* $v = 2$, *we have* $h_2(A) = 1/2$ (mod 1).

PROPOSITION N (Janusz [9, Proposition 3]). *Let* τ *be an automorphism of* $Q(\zeta_p)$ *having order either* $p-1$ *or* $(p-1)/2$. *Let* K *be the subfield* $Q(\zeta_p)^{\langle\tau\rangle}$ *of* $Q(\zeta_p)$ *fixed by* $\langle\tau\rangle$ *and let* $A = (-1, Q(\zeta_p)/K, \tau)$. *Then*: (i) *If* τ *has order* $p-1$, $K = Q$, $h_\infty(A) \equiv h_p(A) \equiv 1/2$ (mod 1) *and* $h_r(A) \equiv 0$ (mod 1) *for each finite place* $r \neq p$ *of* Q. (ii) *If* τ *has order* $(p-1)/2$ *and* $p \equiv 1$ (mod 4), *then* $K = Q(\sqrt{p})$, $h_v(A) \equiv 1/2$ (mod 1) *for two real places* v *of* K *and* $h_v(A) \equiv 0$ (mod 1) *for any finite place* v *of* K. (iii) *If* τ *has order* $(p-1)/2$ *and* $p \equiv -1$ (mod 4), *then* $K = Q(\sqrt{-p})$ *and* $A \sim K$.

2. The Hasse Invariants of $A(\chi_1(j), Q)$

Let $\chi = \chi_1(j)$ $(j \in R_1)$, $K = Q(\chi)$ and $A = A(\chi, K)$. In this section we calculate the invariants of A.

We have $K = Q(\tilde{\zeta}^{ij} + \tilde{\zeta}^{-ij} + \tilde{\zeta}^{qij} + \tilde{\zeta}^{-qij}, i \in R_1) = Q(\tilde{\zeta}^j)^{\langle\tau\rangle}$, where τ is the automorphism of $Q(\tilde{\zeta}^j)$ given by $(\tilde{\zeta}^j)^\tau = (\tilde{\zeta}^j)^{-q}$ (see below). In the first part of the following arguments, we follow those in Przygocki [21, (3.1)]: Let

$$\tilde{a} = \begin{pmatrix} \zeta & & & 0 \\ & \zeta^{-1} & & \\ & & \zeta^q & \\ 0 & & & \zeta^{-q} \end{pmatrix}, \quad \tilde{x} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and let $\tilde{H} = \langle \tilde{a}, \tilde{x} \rangle$. Then $\tilde{x}^{-1} y \tilde{x} = y^{(q)}$ for all $y \in \tilde{H}$, so that, by Lemma H, there is an element z of $\tilde{G}$ such that $z^{-1}\tilde{H}z \subset G$. Fixing one such element z, put $a = z^{-1}\tilde{a}z$, $x = z^{-1}\tilde{x}z$ and $H = z^{-1}\tilde{H}z$. Then $a^{q^2+1} = x^8 = 1$ and $xax^{-1} = a^{-q}$. Let $N = \langle a \rangle$ (a normal subgroup of H), and let ψ be the linear character of N defined by $\psi(a) = \tilde{\zeta}^j$. Then ψ^H is an irreducible character of H and $Q(\psi^H) = K$ (cf. [22, Table on p. 496]). Since χ and ψ^H are real characters, by the Brauer-Speiser theorem (Theorem J), we have $m_Q(\chi) \leqq 2$ and $m_Q(\psi^H) \leqq 2$. And $(\chi|H, \psi^H)_H = (\chi|N, \psi)_N = 2q^2 - 5$, odd. So $(\chi|H, \psi^H)_H$ is coprime to $m_Q(\chi)$ and $m_Q(\psi^H)$. Therefore, by Corollary B, we have $[A] = [A(\psi^H, K)]$. Thus it suffices to calculate the invariants of $A(\psi^H, K)$.

Set $F = \{f \in H \mid \psi^f = \psi^{\tau(f)}$ for some $\tau(f) \in \mathrm{Gal}(K(\psi)/K)\}$. Since $\psi^x =$

$\psi^{-q} = \psi^{\tau}$, x belongs to F and $\tau = \tau(x)$. So $F = H$. By Proposition C, we see that $F/N \simeq \langle\tau\rangle = \mathrm{Gal}(K(\psi)/K) \simeq Z/4Z$ (thus $K = Q(\psi)^{\langle\tau\rangle} = Q(\tilde{\zeta}^j)^{\langle\tau\rangle}$). By Proposition D and Remark E, we see that $A(\psi^F, K) \simeq C$, where $C = (\psi(x^4), K(\psi)/K, \tau) = ((-1)^j, Q(\tilde{\zeta}^j), \tau)$.

If j is even, then $C \sim K$. Suppose therefore j is odd. Let $L = K(\psi) = Q(\tilde{\zeta}^j)$. Let w be any infinite place of L, and let v be the place of K lying below v. Then $\mathrm{Gal}(L_w/K_v)$ can be canonically viewed as the subgroup $\langle\tau^2\rangle$ of $\mathrm{Gal}(L/K)$, and we see that $C_v \sim (-1, L_w/K_v, \tau^2)$. L_w is isomorphic to C and K_v is isomorphic to R, so $(-1, L_w/K_v, \tau^2)$ is isomorphic (as rings) to the quaternion algebra over R. Thus $h(C_v) \equiv 1/2 \pmod 1$.

Let v be a finite place of K. The determinetion of the invariant of C_v is rather formally and is essestially achieved by Janusz in [9]. Since C is a cyclotomic algebra over K, by Lemma F, we have $h(C_v) \equiv 0 \pmod 1$ whenever v is unramified in L. Let $n = (q^2+1)/(j, q^2+1)$, the order of $\tilde{\zeta}^j$. L is a cyclic extension of K of degree 4 and K is real, so the real field $Q(\zeta_n + \zeta_n^{-1})$ is the unique internediate subfield of L containing K. Thus v is ramified in L only if n is of the form $2r^m$, where r is an odd prime number and $m \geqq 1$, and $v|r$. In this case, r is totally ramified in L, so v is unique. Since $[L : Q] = (r-1)r^{m-1}$ and $[L : K] = 4$, we must have $r \equiv 1 \pmod 4$, and K has just $((r-1)/4)r^{m-1}$ real places. Thus Hasse's sum formula (Theorem G) forces that $h_v(C) \equiv 0$ or $1/2 \pmod 1$ according as $(r-1)/4$ is even or odd respectively. We note that we see easily that when q is an even power of p we have $h_{v'}(A) \equiv 0 \pmod 1$ for each finite place v' of K.

Thus we get

PROPOSITION 1 (cf. Przygocki [21, (3.1)]). *Let* $\chi = \chi_1(j)$ $(j \in R_1)$, $K = Q(\chi)$ *and* $A = A(\chi, K)$. *Then* $K = Q(\tilde{\zeta}^{ij} + \tilde{\zeta}^{-ij} + \tilde{\zeta}^{qij} + \tilde{\zeta}^{-qij}, i \in R_1) = Q(\tilde{\zeta}^j)^{\langle\tau\rangle}$, *where* τ *is the automorphism of* $Q(\tilde{\zeta}^j)$ *given by* $(\tilde{\zeta}^j)^{\tau} = (\tilde{\zeta}^j)^{-q}$. *If* j *is even,* $A \sim K$. *Suppose that* j *is odd and let* v *be a place of* K. *Then, if* v *is infinite* (*real*), *we have* $h_v(A) \equiv 1/2 \pmod 1$, *and if* v *is finite, we have* $h_v(A) \equiv 0 \pmod 1$ *except in the following case*: *If* $(q^2+1)/(j, q^2+1)$ *is of the form* $2r^m$, *where* r *is an odd prime number of the form* $8k+5$ *and* $m \geqq 1$, *and if* $v|r$, *then* $h_v(A) \equiv 1/2 \pmod 1$. *In particular, if* q *is an even power of* p, *then* $h_v(A) \equiv 0 \pmod 1$ *for all finite places* v *of* K.

EXAMPLE. Let $q = p = 3$. Then $R_1 = \{1, 2\}$. We have $Q(\chi_1(1)) = Q(\chi_1(2)) = Q$, $A(\chi_1(2), Q) \sim Q$, $m_R(\chi_1(1)) = m_{Q_5}(\chi_1(1)) = 2$ and $m_{Q_r}(\chi_1(1)) = 1$ for each prime number $r \neq 5$.

REMARK. For $j,k \in R_1$, $\chi_1(k)$ is algebraically conjugate to $\chi_1(j)$ is and only if there is an integer m such that $\tilde{\zeta}^k = (\tilde{\zeta}^j)^m$ and $(m,(q^2+1)/(j,q^2+1)) = 1$.

3. Hasse Invariants of $A(-\chi_2(j),Q)$

Let $\chi = -\chi_2(j)$ $(j \in R_2)$. In [21, (3.2)], Przygocki states that $m_Q(\chi) = 1$ if j is even and $m_R(\chi) = 2$ if j is odd. As we shall see below, his statement is correct, but his argument is valid only when $(q-1)/2 \nmid j$ and $(q+1)/2 \nmid j$, since his calculation on p. 294, line 26, is not right: in fact, we have

$$(\chi,\delta^H) = \begin{cases} 2q^2+1 \text{ (odd)} & \text{if } (q-1)/2 \nmid j \text{ and } (q+1)/2 \nmid j, \\ 2(q^2+1) \text{ (even)} & \text{if } (q-1)/2 \mid j \text{ and } (q+1)/2 \nmid j, \\ 2q^2 \text{ (even)} & \text{if } (q-1)/2 \nmid j \text{ and } (q+1)/2 \mid j \end{cases}$$

(the case where $(q-1)/2 \mid j$ and $(q+1)/2 \mid j$ does not happen). We have $Q(\chi) = Q(\alpha_j, \beta_j, \tilde{\theta}^{ij} + \tilde{\theta}^{-ij} + \tilde{\theta}^{qij} + \tilde{\theta}^{-qij}, j \in R_2)$.

3.1. The case $(q-1)/2 \nmid j$ and $(q+1)/2 \nmid j$: In this case, we can follow the argument of Przygocki in [21, (3.2)]. Let

$$\tilde{a} = \begin{pmatrix} \theta & & & 0 \\ & \theta^{-1} & & \\ & & \theta^q & \\ 0 & & & \theta^{-q} \end{pmatrix}, \quad \tilde{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\tilde{y} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and let $\tilde{H} = \langle \tilde{a}, \tilde{x}, \tilde{y} \rangle$. Then $\tilde{x}^{-1}c\tilde{x} = c^{(q)}$ for all $c \in \tilde{H}$. Let z be an element of $\tilde{G}$ such that $z^{-1}\tilde{H}z \subset G$, and put $a = z^{-1}\tilde{a}z$, $x = z^{-1}\tilde{x}z$, $y = z^{-1}\tilde{y}z$ and $H = z^{-1}\tilde{H}z$ (see [22, p. 496]). Let $N = \langle a \rangle$ (a normal subgroup of H), and let ψ be the linear character of N defined by $\psi(a) = \tilde{\theta}^j$. Then $\psi^x = \psi^q$, $\psi^y = \psi^{-q}$ and $\psi^{xy} = \psi^{-1}$. It follows that ψ^H is an irreducible character of H and $Q(\psi^H) = Q(\chi)$, real. Since $(\chi|H,\psi^H)_H = (\chi|N,\psi)_N = 2q^2+1$, odd, we have $[A(\chi,K)] = [A(\psi^H,K)]$ with $K = Q(\chi) = Q(\psi^H)$.

Let $L = K(\psi) = Q(\tilde{\theta}^j)$, and let σ and τ be the automorphisms of L over K given by $(\tilde{\theta}^j)^\sigma = (\tilde{\theta}^j)^q$ and $(\tilde{\theta}^j)^\tau = (\tilde{\theta}^j)^{-q}$ respectively. Then we see that

$F = \{f \in H \mid \psi^f = \psi^{\tau(f)}$ for some $\tau(f) \in \mathrm{Gal}(L/K)\} = H$, $\sigma = \tau(x)$, $\tau = \tau(y)$ and $\mathrm{Gal}(L/K) = \langle \sigma, \tau \rangle \simeq F/N = H/N$ (Proposition C). $R = \{1, x, y, xy\}$ is a set of complete system of representatives for F/N, and the factor set β of $\mathrm{Gal}(L/K)$ with respect to R (Proposition D) is given by $\beta(\sigma, \sigma) = (-1)^j$, $\beta(\sigma, \tau) = \beta(\tau, \sigma) = \beta(\tau, \tau) = 1$. Let σ' and τ' be respectively the restriction of σ to $L^{\langle \tau \rangle}$ and the restriction of τ to $L^{\langle \sigma \rangle}$. Then we have:

$$\begin{aligned}
A(\psi^H, K) &\simeq (\beta, L/K) = \sum_{s=0}^{1}\sum_{t=0}^{1} L u_{\sigma^s \tau^t} \\
&= \sum_{s=0}^{1}\sum_{t=0}^{1} L^{\langle \tau \rangle} L^{\langle \sigma \rangle} u_{\sigma^s} u_{\tau^t} = \sum_{s=0}^{1}\sum_{t=0}^{1} L^{\langle \tau \rangle} u_{\sigma^s} L^{\langle \sigma \rangle} u_{\tau^t} \\
&= \left(\sum_{s=0}^{1} L^{\langle \tau \rangle} u_{\sigma^s}\right) \cdot \left(\sum_{t=0}^{1} L^{\langle \sigma \rangle} u_{\tau^t}\right) = \left(\sum_{s=0}^{1} L^{\langle \tau \rangle} (u_{\sigma})^s\right) \cdot \left(\sum_{t=0}^{1} L^{\langle \sigma \rangle} (u_{\tau})^t\right) \\
&\simeq \left(\sum_{s=0}^{1} L^{\langle \tau \rangle} (v_{\sigma'})^s\right) \otimes_K \left(\sum_{t=0}^{1} L^{\langle \tau \rangle} (v_{\tau'})^t\right) \\
&= ((-1)^j, L^{\langle \tau \rangle}/K, \sigma') \otimes_K (1, L^{\langle \sigma \rangle}/K, \tau') \sim ((-1)^j, L^{\langle \tau \rangle}/K, \sigma').
\end{aligned}$$

Thus, if j is even, $A(\psi^H, K) \sim K$.

Suppose that j is odd, and let $C = (-1, L^{\langle \tau \rangle}/K, \sigma')$. Let $n = (q^2 - 1)/(j, q^2 - 1)$, the order of $\tilde{\theta}^j$, and let $n = n_2 n_{2'}$, where n_2 is the 2-part of n and $n_{2'}$ is the odd part of n. Then we see that $L^{\langle \tau \rangle} = K(\zeta_{n_2} + \zeta_{n_2}^{-q})$ if $\mathrm{ord}_2(q+1) = 1$ and $L^{\langle \tau \rangle} = K(\zeta_{n_2/2})$ if $\mathrm{ord}_2(q-1) = 1$. Since $\mathrm{ord}_2 n_2 \geqq 3$, $L^{\langle \tau \rangle}$ is not a real field.

Put $M = L^{\langle \tau \rangle}$. Let w be a finite place of M and let v be the place of K lying below w. Then $K_v \simeq R$ and $M_w = M \cdot K_v \simeq C$, and $\mathrm{Gal}(M_w/K_v) \simeq \langle \sigma' \rangle$. So, if we let $\mathrm{Gal}(M_w/K_v) = \langle \iota \rangle$, then $C_v \simeq (-1, M \cdot K_v/K_v, \iota) \simeq (-1, C/R, \eta)$ (an isomorphism of rings) with $\langle \eta \rangle = \mathrm{Gal}(C/R)$. The last algebra is the quaternion algebra over R. Hence $h(C_v) \equiv 1/2 \pmod 1$.

Put $d = (q+1)_{2'}/(j, q+1)$, where $(q+1)_{2'}$ is the odd part of $q+1$. Then we see that $M = K(\zeta_d)$ if $d > 1$. Thus, if $d > 1$, since M is also contained in $K(\zeta_{n_2})$, any finite place of K is unramified in M, and if $d = 1$, each finite place v of K such that $v \nmid 2$ is unramified in M. Thus, by Lemma F, we see that, if $d > 1$, we have $h_v(C) \equiv 0 \pmod 1$ for any finite place v of K, and if $d = 1$, we have $h_v(C) \equiv 0 \pmod 1$ for each finite place v of K such that $v \nmid 2$. But when $d = 1$ and v is a finite place of K lying above 2, we can prove, by rather long considerations, that $h_v(C) \equiv 0 \pmod 1$.

We give here a sketch of these considerations. Let σ'' and τ'' be respectively the restrictions of σ and τ to $Q(\zeta_{n_2})$. Let $P = Q(\zeta_{n_2})^{\langle\tau''\rangle}$ and $S = Q(\zeta_{n_2})^{\langle\sigma'',\tau''\rangle}$. Then we see that $M = K \cdot P$ and $K \cap P = S$ and that $C \sim D \otimes_S K$, where $D = (-1, P/S, \sigma''')$ (σ''' is the restriction of σ'' to P). Let v' be the place of S lying below v, and let $f = [K_v : S_{v'}]$. Then $h_v(C) \equiv f \cdot h_{v'}(D) \pmod 1$. Since $[D]^2 = [S]$, we have $h_{v'}(D) \equiv 0$ or $1/2 \pmod 1$. Put $c = [Q_2(\zeta_n) : K_v]$. Then we see that $c = 2$ or 4, and that, if $c = 4$, then $2 \mid [Q_2(\zeta_{n_{2'}}) : Q_2]/c$. Finally, we have $f = 4 \cdot [Q_2(\zeta_{n_{2'}}) : Q_2]/c$, even. Thus $h_v(C) \equiv 0 \pmod 1$.

3.2. The case $(q-1)/2 \mid j$ and $(q+1)/2 \nmid j$: Let $\sigma(j)$ be the character of G defined in [22, (3.13), pp. 502–3]. Then we have $(\chi, \sigma(j))_G = 2q^2 - 2q + 1$, odd, and $Q(\sigma(j)) = Q(\beta_j) \subset Q(\chi)$. In this subsection we use this $\sigma(j)$ in order to investigate the rationality of χ.

Let δ' be an element of $\bar{F}_q$ such that $\delta'^2 = \gamma$; we have $\delta'^q = -\delta'$. Let

$$\tilde{w} = \begin{pmatrix} \eta & & & 0 \\ & \eta^{-1} & & \\ & & \eta^{-1} & \\ 0 & & & \eta \end{pmatrix}, \qquad \tilde{d}_\beta = \begin{pmatrix} 1 & 0 & 0 & \delta'\beta \\ 0 & 1 & 0 & 0 \\ 0 & \delta'\beta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\beta \in F_q),$$

$$\tilde{v} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and

$$\tilde{b} = \begin{pmatrix} \theta^{-((q-1)/(p-1))} & & & 0 \\ & \theta^{(q-1)/(p-1)} & & \\ & & \theta^{-((q-1)/(p-1))q} & \\ 0 & & & \theta^{((q-1)/(p-1))q} \end{pmatrix}$$

(cf. [22, p. 500]), and let $\tilde{H} = \langle \tilde{w}, \tilde{d}_\beta (\beta \in F_q), \tilde{v}, \tilde{b} \rangle$. Then $\tilde{v}$ transforms each element y of H to $y^{(q)}$. Let z be an element of $\tilde{G}$ such that $z^{-1}\tilde{H}z \subset G$, and put $w = z^{-1}\tilde{w}z$, $d_\beta = z^{-1}\tilde{d}_\beta z$ $(\beta \in F_q)$, $v = z^{-1}\tilde{v}z$, $b = z^{-1}\tilde{b}z$ and $H = z^{-1}\tilde{H}z$. Then we have: $w^{q+1} = v^4 = d_\beta^p = 1$ $(\beta \in F_q)$, $v^2 = -1$, $vwv^{-1} = w^{-1} = w^q$, $vd_\beta v^{-1} = d_{-\beta} = d_\beta^{-1}$ $(\beta \in F_q)$, $bwb^{-1} = w$, $b^{p-1} = w^{-1}$, $bd_\beta b^{-1} = d_{\nu^{-1}\beta}$ $(\beta \in F_q)$ and $vbv^{-1} = b^q$.

Let $N = \langle w, d_\beta(\beta \in F_q)\rangle = \langle w\rangle \times \{d_\beta \mid \beta \in F_q\}$. Then N is a normal subgroup of H, $(H:N) = 2(p-1)$ and $R = \{b^i, vb^i, i = 0, 1, \dots, p-2\}$ is a set of complete system of representatives for H/N. Let ε be an additive character of F_q as in Lemma I, and let ψ be the linear character of N defined by $\psi(w^k d_\beta) = (\tilde{\eta}^j)^k \varepsilon(\beta)$. Then $\psi^G = \sigma(j)$ ([22, pp. 502–3]). We have $\psi^{b^i}(w^k d_\beta) = (\tilde{\eta}^j)^k \varepsilon(\beta)^{g^i}$ and $\psi^{vb^i}(w^k b_\beta) = \psi^{b^i}(w^k d_\beta)^{-1}$, where g is an integer such that $g \bmod pZ = v^{-1}$ in $F_p = Z/pZ$. From this we see that $\psi^H(w^k d_\beta) = (p-1)\beta_{jk}$ if $\beta \in \mathrm{Ker}(\varepsilon)$, and $= -\beta_{jk}$ otherwise. Then it follows that $(\psi^H, \psi^H)_H = 1$, so ψ^H is an irreducible character of H. We have $Q(\psi^H) = Q(\beta_j) \subset Q(\chi)$, $(\chi|H, \psi^H)_H = (\chi, \sigma(j))_G = 2q^2 - 2q + 1$, odd, and χ and ψ^H are real. Therefore we have $[A(\chi, Q(\chi))] = [A(\psi^H, Q(\chi))]$.

Let $K = Q(\psi^H)$, $L = K(\psi) = Q(\tilde{\eta}^j, \zeta_p)$ and $B = A(\psi^H, K)$. Then $\mathrm{Gal}(L/K) = \langle\omega\rangle \times \langle\phi\rangle$, where $(\tilde{\eta}^j)^\omega = \tilde{\eta}^j$, $\zeta_p^\omega = \zeta_p^g$, $(\tilde{\eta}^j)^\phi = (\tilde{\eta}^j)^{-1}$ and $\zeta_p^\phi = \zeta_p$. We see that $\psi^b = \psi^\omega$ and $\psi^{vb^{(p-1)/2}} = \psi^\phi$, so $F = \{f \in H \mid \psi^f = \psi^{\tau(f)}$ for some $\tau(f) \in \mathrm{Gal}(L/K)\} = H$, hence $B \simeq (\beta, L/K)$, where β is the factor set of $\mathrm{Gal}(L/K)$ with respect to R.

Let u' be any infinite place of L, and let u be the place of K lying below u'. Then $L_{u'} \simeq C$ and $K_u \simeq R$ and $\mathrm{Gal}(L_{u'}/K_u)$ is canonically isomorphic to the subgroup $\langle\omega^{(p-1)/2}\phi\rangle$ of $\mathrm{Gal}(L/K)$. Let $\mathrm{Gal}(L_{u'}/K_u) = \langle\iota\rangle$, and let β_u be the restriction of β to $\langle\iota\rangle$. Then $\beta_u(\iota, \iota) = (-1)^j$, so $B_u \sim (\beta_u, L_{u'}/K_u)$ (see, e.g. [24, Chap. IX, §3, Corollary to Proposition 7, p. 174]) $= ((-1)^j, L_{u'}/K_u, \iota)$. Thus $h_u(B) \equiv 0 \pmod 1$ if j is even, and $h_u(B) \equiv 1/2 \pmod 1$ if j is odd.

Let u be a finite place of K. Let $n = (q+1)/(j, q+1)$, the order of $\tilde{\eta}^j$. Then $L = Q(\zeta_n, \zeta_p) = Q(\zeta_{np})$. We see that u is unramified in L except in the following cases: (a) If $u|p$, then u is ramified in L. (b) If n is of the form r^m or $2r^m$, where r is an odd prime number and $m \geqq 1$, and if $u|r$, then u is ramified in L. (c) If n is of the form 2^m with $m \geqq 2$ and if $u|2$, then u is ramified in L.

Suppose that $u|p$. Then, by using the index formula of T. Yamada ([27, Theorem 4.4, p. 43]), we see that the index m_p of B_u is equal to $(p-1)/((q-s)/n, p-1)$, where $s = p^{f/2}$ with $f = [Q_p(\zeta_n) : Q_p]$ (even), and, by relatively elementary arguments, we can conclude that $m_p = 1$.

Suppose that n is of the form r^m or $2r^m$, where r is an odd prime number and $m \geqq 1$, and that $u|r$. Then r is totally ramified in K, so that u is unique. We see that j is even, so that, by sum formula, we must have $h_u(B) \equiv 0 \pmod 1$.

Suppose that n is of the form 2^m with $m \geqq 2$ and that $u|2$. Then 2 is totally ramified in K, so that u is unique. Thus, if j is even, we must have $h_u(B) \equiv 0 \pmod 1$. Suppose that j is odd. Since $[K:Q] = 2^{m-2}$, K has just 2^{m-2} real places. Therefore we must have $h_u(B) \equiv 0 \pmod 1$ if $m > 2$, and $h_u(B) \equiv 1/2$

(mod 1) if $m=2$. If $m=2$, then we see that $Q(\chi)=Q$, so we have $m_{Q_2}(\chi)=m_{Q_2}(\psi^H)=2$.

3.3. The case $(q-1)/2\nmid j$ and $(q+1)/2\,|\,j$: In this case we have $(\chi,\sigma(j))_G=2q(q-1)$, even. But, if $\rho(j)$ is the character of G which is defined in [22, pp. 502–3], we have $(\chi,\rho(j))_G=2q^2+2q+3$, odd, and $Q(\rho(j))=Q(\alpha_j)\subset Q(\chi)$. So in this subsection, we use $\rho(j)$ in order to investigate the rationality of χ.

Let

$$u=\begin{pmatrix}\gamma & & & 0\\ & \gamma^{-1} & & \\ & & \gamma^{-1} & \\ 0 & & & \gamma\end{pmatrix},\qquad b_\beta=\begin{pmatrix}1&0&0&\beta\\0&1&0&0\\0&\beta&1&0\\0&0&0&1\end{pmatrix}\quad(\beta\in F_q),$$

$$v=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\qquad s=\begin{pmatrix}1 & & & 0\\ & 1 & & \\ & & \nu^{-1} & \\ 0 & & & \nu\end{pmatrix},$$

and let $H=\langle u,b_\beta(\beta\in F_q),v,s\rangle$ (cf. [22, p. 499]). Then $v^2=s^{p-1}=u^{q-1}=1$, $vuv^{-1}=u^{-1}$, $vb_\beta v=b_\beta$ $(\beta\in F_q)$, $sus^{-1}=u$, $sb_\beta s^{-1}=b_{\nu^{-1}\beta}$ $(\beta\in F_q)$ and $ub_\beta u^{-1}=b_\beta$ $(\beta\in F_q)$. Let $N=\langle u,b_\beta(\beta\in F_q)\rangle=\langle u\rangle\times\{b_\beta\,|\,\beta\in F_q\}$. Then N is a normal subgroup of H, $(H:N)=2(p-1)$ and $R=\{s^i,vs^i,i=0,1,\dots,p-2\}$ is a set of complete system of representatives for H/N. Let ψ be the linear character of N defined by $\psi(u^kb_\beta)=(\tilde{\gamma}^j)^k\varepsilon(\beta)$. Then $\psi^G=\rho(j)$ ([22, p. 502]). We have $\psi^{s^i}(u^kb_\beta)=\psi(u^kb_{\nu^{-i}\beta})=(\tilde{\gamma}^j)^k\varepsilon(\beta)^{g^i}$ and $\psi^{vs^i}(u^kb_\beta)=\psi(u^{-k}b_{\nu^{-i}\beta})=(\tilde{\gamma}^j)^{-k}\varepsilon(\beta)^{g^i}$. It follows that $\psi^H(u^kb_\beta)=(p-1)\beta_{jk}$ if $\beta\in\mathrm{Ker}(\varepsilon)$, and $=0$ otherwise, and we see that ψ^H is an irreducible character of H and $Q(\psi^H)=Q(\alpha_j)\subset Q(\chi)$. Since ψ^H and χ are real and $(\chi|H,\psi^H)_H=(\chi,\rho(j))_G=2q^2+2q+3$, odd, we have $[A(\chi,Q(\chi))]=[A(\psi^H,Q(\chi))]$.

Let $K=Q(\psi^H)$, $L=K(\psi)=Q(\tilde{\gamma}^j,\zeta_p)$ and $B=A(\psi^H,K)$. Then we have $\mathrm{Gal}(L/K)=\langle\omega\rangle\times\langle\phi\rangle$, where $(\tilde{\gamma}^j)^\omega=\tilde{\gamma}^j$, $\zeta_p^\omega=\zeta_p^g$, $(\tilde{\gamma}^j)^\phi=(\tilde{\gamma}^j)^{-1}$ and $\zeta_p^\phi=\zeta_p$. We have $\psi^s=\psi^\omega$ and $\psi^v=\psi^\phi$. Therefore $B\simeq(\beta,L/K)$, where β is the factor set of $\mathrm{Gal}(L/K)$ with respect to R.

As in 3.2, we see that, for an infinite place w of K, we have $h_w(B)\equiv 0$

(mod 1) if j is even and $h_w(B) \equiv 1/2$ (mod 1) if j is odd. When w is a finite place of K the argument goes similarly as in 3.2.

We get

Proposition 2 (cf. [21, (3.2)]). *Let* $\chi = -\chi_2(j)$ $(j \in R_2)$. *Then* $K = Q(\chi) = Q(\alpha_j, \beta_j, \tilde\theta^{ij} + \tilde\theta^{-ij} + \tilde\theta^{qij} + \tilde\theta^{-qij}, i \in R_2)$. *Let* $A = A(\chi, Q)$. *Then, if* j *is even,* $A \sim K$. *Suppose that* j *is odd. Then* $h_v(A) \equiv 1/2$ (mod 1) *for all infinite places* v *of* K, *and if* v *is a finite place of* K, *we have* $h_v(A) \equiv 0$ (mod 1) *except in the following cases*: (a) *If* $(q-1)/2 \mid j$ *and* $(q+1)/2 \nmid j$ *and* $(q+1)/(j, q+1) = 4$, *then* $K = Q$ *and* $h_2(A) \equiv 1/2$ (mod 1). (b) *If* $(q-1)/2 \nmid j$ *and* $(q+1)/2 \mid j$ *and* $(q-1)/(j, q-1) = 4$, *then* $K = Q$ *and* $h_2(A) \equiv 1/2$ (mod 1).

4. The Hasse Invariants of $A(\chi_3(k, \ell), Q)$

Let $\chi = \chi_3(k, \ell)$ $(k, \ell \in T_1, k \neq \ell)$. Then $Q(\chi) = Q((-1)^{\ell}\alpha_{ik} + (-1)^{j}\alpha_{i\ell}$ $(i \in T_1)$, $\alpha_{ik} + \alpha_{i\ell}$ $(i \in T_1)$, $\alpha_{ik}\alpha_{i\ell}$ $(i \in T_1)$, $\alpha_{ik}\alpha_{j\ell} + \alpha_{i\ell}\alpha_{jk}$ $(i, j \in T_1, i \neq j)$. Generally, this differs from the calculation in [21, p. 295, line 12] that $Q(\chi) = Q(\lambda(k) \times \lambda(\ell)) = Q(\alpha_k, \alpha_\ell)$, so the assertion in [21, (3.3)] is not correct.

4.1. The case $(q-1)/2 \nmid k + \ell$: Let

$$a = \begin{pmatrix} \gamma & & & 0 \\ & \gamma^{-1} & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \gamma & \\ 0 & & & \gamma^{-1} \end{pmatrix},$$

$$x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and let $H = \langle a, b, x, y, z \rangle$ (cf. [22, p. 496]). Let $N = \langle a, b \rangle$ (a normal subgroup of H), and let ψ be the linear character of N defined by $\psi(a^i b^j) = (\tilde\gamma^k)^i (\tilde\gamma^\ell)^j$. Then we see that ψ^H is an irreducible character of H and $Q(\psi^H) = Q(\chi)$, real. We have

$$(\chi|H, \psi^H)_H = \begin{cases} 2q^2 + 8q + 29 \text{ (odd)} & \text{if } 2|k, \ell \text{ or } 2\nmid k, \ell, \\ 2q^2 + 8q + 25 \text{ (odd)} & \text{if } 2|k \text{ and } 2\nmid \ell, \\ 2q^2 + 8q + 25 \text{ (odd)} & \text{if } 2\nmid k \text{ and } 2|\ell. \end{cases}$$

Therefore we have $[A(\chi, K)] = [A(\psi^H, K)]$ with $K = Q(\chi)$.

Set $F = \{f \in H \mid \psi^f = \psi^{\tau(f)}$ for some $\tau(f) \in \mathrm{Gal}(K(\psi)/K)\}$. Then, by Proposition C, we have $K(\psi^F) = K$ and $F/K \simeq \mathrm{Gal}(K(\psi)/K)$, an abelian group. Since $\psi^H = (\psi^F)^H$, by Corollary B, we have $[A(\psi^H, K)] = [A(\psi^F, K)]$. In our case, the group F is not uniquely determined. For $h \in H$, let $\bar{h} = Nh \in H/N$. Then the abelian subgroup of H/N are $\{\bar{1}\}, \langle\bar{x}\rangle, \langle\bar{y}\rangle, \langle\bar{z}\rangle, \langle\overline{yz}\rangle, \langle\bar{y}, \bar{z}\rangle, \langle\bar{x}, \overline{yz}\rangle, \langle\overline{xyz}\rangle, \langle\overline{xy}\rangle = \langle\overline{xz}\rangle$. We see easily that the cases $F/N = \{\bar{1}\}, \langle\bar{x}\rangle, \langle\bar{y}\rangle, \langle\bar{z}\rangle, \langle\overline{xyz}\rangle$ cannot happen.

(1) The case $F/N = \langle\overline{yz}\rangle$: This case happens when, for instance, $q = 11$, $k = 1$ and $\ell = 2$. Put $\tau = \tau(yz)$. Then $\mathrm{Gal}(K(\psi)/K) = \langle\tau\rangle \simeq Z/2Z$, where $(\tilde{y}^k)^\tau = (\tilde{y}^k)^{-1}$ and $(\tilde{y}^\ell)^\tau = (\tilde{y}^\ell)^{-1}$. So $K(\psi) = Q(\tilde{y}^k, \tilde{y}^\ell) = Q(\tilde{y}^m)$ with $m = (k, \ell)$ and $K = Q(\alpha_m)$. We see that $A(\psi^F, K) = ((-1)^{k+\ell}, Q(\tilde{y}^m)/Q(\alpha_m), \tau)$. Thus, if $k + \ell$ is even, $A(\psi^F, K) \sim K$, and, if $k + \ell$ is odd, the invariants of $A(\psi^F, K)$ can be determined by using Proposition L. We note that the case $(q-1)/(m, q-1) = 4$ cannot happen.

(2) The case $F/N = \langle\bar{y}, \bar{z}\rangle$: This case happens when, for instance, $q = 13$, $k = 4$ and $\ell = 3$.

Put $\tau = \tau(y)$ and $\upsilon = \tau(z)$. Then $\mathrm{Gal}(K(\psi)/K) = \langle\tau\rangle \times \langle\upsilon\rangle \simeq Z/2Z \simeq Z/2Z$, where $(\tilde{y}^k)^\tau = (\tilde{y}^k)^{-1}$, $(\tilde{y}^\ell)^\tau = \tilde{y}^\ell$, $(\tilde{y}^k)^\upsilon = \tilde{y}^k$ and $(\tilde{y}^\ell)^\upsilon = (\tilde{y}^\ell)^{-1}$. We have $K(\psi) = Q(\tilde{y}^k, \tilde{y}^\ell)$ and $K = Q(\alpha_k, \alpha_\ell)$. (We note that $((q-1)/(k, q-1), (q-1)/(\ell, q-1)) \leqq 2$.) $R = \{1, y, z, yz\}$ is a set of complete system of representatives for F/N, and the factor set β of $G(K(\psi)/K)$ with respect to R is given by $\beta(\tau, \upsilon) = \beta(\upsilon, \tau) = 1$, $\beta(\tau, \tau) = (-1)^k$ and $\beta(\upsilon, \upsilon) = (-1)^\ell$. Let τ'' be the restriction of τ to $Q(\tilde{y}^k)$ and let υ'' be the restriction of υ to $Q(\tilde{y}^\ell)$. Then we see that $A(\psi^F, K) \sim (B_1 \otimes_{Q(\alpha_k)} K) \otimes_K (B_2 \otimes_{Q(\alpha_\ell)} K)$, where $B_1 = ((-1)^k, Q(\tilde{y}^k)/Q(\alpha_k), \tau'')$ and $B_2 = ((-1)^\ell, Q(\tilde{y}^\ell)/Q(\alpha_\ell), \upsilon'')$. Thus the calculation of the invariants of $A(\psi^F, K)$ is easy (by using Proposition L).

(3) The case $F/N = \langle\bar{x}, \overline{yz}\rangle$: This case happens when, for instance, $q = 31$, $k = 2$ and $\ell = 8$.

Let $\tau = \tau(x)$ and $\upsilon = \tau(yz)$. Then $\mathrm{Gal}(K(\psi)/K) = \langle\tau, \upsilon\rangle \simeq Z/2Z \times Z/2Z$, where $(\tilde{y}^k)^\tau = \tilde{y}^\ell$, $(\tilde{y}^\ell)^\tau = \tilde{y}^k$, $(\tilde{y}^k)^\upsilon = (\tilde{y}^k)^{-1}$ and $(\tilde{y}^\ell)^\upsilon = (\tilde{y}^\ell)^{-1}$. So we have $K(\psi) = Q(\tilde{y}^k, \tilde{y}^\ell) = Q(\tilde{y}^k) = Q(\tilde{y}^\ell)$ and $K = Q(\alpha_k + \alpha_\ell, \alpha_k\alpha_\ell)$. Since $\tilde{y}^k$ and $\tilde{y}^\ell$ are conjugate over Q, they have the same order, that is, $(q-1)/(k, q-1) = (q-1)/(\ell, q-1)$, so that $k + \ell$ is even. $R = \{1, x, yz, xyz\}$ is a set of complete system of representatives for F/N, and the factor set of $\mathrm{Gal}(K(\psi)/K)$ with respect to R is 1. Therefore $A(\psi^F, K) \sim K$.

(4) The case $F/N = \langle\overline{xy}\rangle = \langle\overline{xz}\rangle$: This case happens when, for instance, $q = 11$, $k = 1$ and $\ell = 3$.

Let $\tau = \tau(xy)$. Then $\mathrm{Gal}(K(\psi)/K) = \langle\tau\rangle \simeq Z/4Z$, where $(\tilde{y}^k)^\tau = (\tilde{y}^\ell)^{-1}$ and

$(\tilde{\gamma}^{\ell})^{\tau} = \tilde{\gamma}^{k}$. So $K(\psi) = Q(\tilde{\gamma}^{k}) = Q(\tilde{\gamma}^{\ell})$ and $K = Q(\alpha_k + \alpha_\ell, \alpha_k\alpha_\ell)$. Since $\tilde{\gamma}^{k}$ and $\tilde{\gamma}^{\ell}$ are conjugate over Q, $k+\ell$ is even. We see that $A(\psi^F, K) = (\psi(xy)^4, K(\psi)/K, \tau) = ((-1)^{k+\ell}, K(\psi)/K, \tau) \sim K$.

4.2. The case $(q-1)/2 \mid k+\ell$, i.e., $k+\ell = (q-1)/2$. If H and ψ are as in 4.1, then we have $(\chi|H, \psi^H)_H = 2(q^2+4q+15)$ if k,ℓ are even or k,ℓ are odd, and $(\chi|H, \psi^H)_H = 2(q^2+4q+13)$ otherwise. So we cannot use this ψ. Instead, let H, N be as in 3.3, and let ψ be the linear character of N defined by $\psi(u^i b_\beta) = (\tilde{\gamma}^{k-\ell})^i \varepsilon(\beta)$. Then we have $(\chi|H, \psi^H)_H = 2q^2+6q+11$, odd, and $Q(\psi^H) \subset Q(\chi)$. So in this subsection we use this ψ^H. Then we have $[A(\chi, Q(\chi))] = [A(\psi^H, Q(\chi))]$. And the arguments go as in 3.3.

We get:

Proposition 3. *Let* $\chi = \chi_3(k,\ell)$ $(k,\ell \in T_1, k \neq \ell)$. *Then* $K = Q(\chi) = Q((-1)^{\ell}\alpha_{ik} + (-1)^{k}\alpha_{i\ell}$ $(i \in T_1)$, $\alpha_{ik} + \alpha_{i\ell}$ $(i \in T_1)$, $\alpha_{ik}\alpha_{i\ell}$ $(i \in T_1)$, $\alpha_{ik}\alpha_{j\ell} + \alpha_{i\ell}\alpha_{jk}$ $(i, j \in T_1, i \neq j)$. *Let* $A = A(\chi, Q)$. *Then we have the following*:

(I) *Assume that* $(q-1)/2 \nmid k+\ell$. *Put* $\Pi = \mathrm{Gal}(Q(\tilde{\gamma}^{k}, \tilde{\gamma}^{\ell})/K)$.

(i) *Assume that* $\Pi = \langle \iota \rangle$, *where* $(\tilde{\gamma}^{k})^{\iota} = (\tilde{\gamma}^{k})^{-1}$ *and* $(\tilde{\gamma}^{\ell})^{\iota} = (\tilde{\gamma}^{\ell})^{-1}$. *Put* $m = (k,\ell)$. *Then, if* $k+\ell$ *is even,* $A \sim K$. *Suppose that* $k+\ell$ *is odd. Then, if* w *is any infinite place of* K, *we have* $h_w(A) \equiv 1/2 \pmod 1$, *and if* w *is a finite place of* K, *we have* $h_w(A) \equiv 0 \pmod 1$ *except in the following case*: *If* $(q-1)/(m, q-1)$ *is of the form* r^c *or* $2r^c$, *where* r *is an odd prime number of the form* $4u-1$ *and* $c \geqq 1$, *and if* $w|r$, *then* $h_w(A) \equiv 1/2 \pmod 1$.

(ii) *Assume that* $\Pi = \langle \tau, \upsilon \rangle$, *where* $(\tilde{\gamma}^{k})^{\tau} = (\tilde{\gamma}^{k})^{-1}$, $(\tilde{\gamma}^{\ell})^{\tau} = \tilde{\gamma}^{\ell}$, $(\tilde{\gamma}^{k})^{\upsilon} = \tilde{\gamma}^{k}$ *and* $(\tilde{\gamma}^{\ell})^{\upsilon} = (\tilde{\gamma}^{\ell})^{-1}$. *Then*: (a) *If* k,ℓ *are even,* $A \sim K$. (b) *Assume that* k *is even and* ℓ *is odd. Then, if* w *is any infinite place of* K, *we have* $h_w(A) \equiv 1/2 \pmod 1$, *and if* w *is a finite place of* K, *we have* $h_w(A) \equiv 0 \pmod 1$ *except in the following cases*: *Put* $n = (q-1)/(\ell, q-1)$. *If* n *is of the form* $2r^c$, *where* r *is an odd prime number of the form* $4u-1$ *and* $c \geqq 1$, *if* $[Q_r(\alpha_k) : Q_r]$ *is odd and if* $w|r$, *then* $h_w(A) \equiv 1/2 \pmod 1$. *If* $n = 4$, *if* $[Q_2(\alpha_k) : Q_2]$ *is odd and if* $w|2$, *then* $h_w(A) \equiv 1/2 \pmod 1$. (c) *Assume that* k *is odd and* ℓ *is even. Then, if* w *is any infinite place of* K, *we have* $h_w(A) \equiv 1/2 \pmod 1$, *and if* w *is a finite place of* K, *we have* $h_w(A) \equiv 0 \pmod 1$ *except in the following cases*: *Put* $m = (q-1)/(k, q-1)$. *If* m is *of the form* $2r^c$, *where* r *is an odd prime number of the form* $4u-1$ *and* $c \geqq 1$, *if* $[Q_r(\alpha_\ell) : Q_r]$ *is odd and if* $w|r$, *then* $h_w(A) \equiv 1/2 \pmod 1$. *If* $m = 4$, *if* $[Q_2(\alpha_\ell) : Q_2]$ *is odd and if* $w|2$, *then* $h_w(A) \equiv 1/2 \pmod 1$. (d) *Assume that* k,ℓ *are odd. Then, if* w *is any infinite place of* K, *we have* $h_w(A) \equiv 0 \pmod 1$, *and if* w *is a finite place of* K, *we have* $h_w(A) \equiv 0 \pmod 1$ *except in the following cases*: *Put* $m = (q-1)/(k, q-1)$ *and* $n = (q-1)/(\ell, q-1)$. *If* m *is of* *the form* $2r^c$, *where r is an odd prime number of the form* $4u-1$ *and* $c \geqq 1$, *if* $[Q_r(\alpha_\ell) : Q_r]$ *is odd and if* $w|r$, *then* $h_w(A) \equiv 1/2 \pmod 1$. *If n is of the form* $2r^c$, *where r is an odd prime number of the form* $4u-1$ *and* $c \geqq 1$, *if* $[Q_r(\alpha_k) : Q_r]$ *is odd and if* $w|r$, *then* $h_w(A) \equiv 1/2 \pmod 1$. (*The case that* $m = 4$ *or* $n = 4$ *does not happen.*)

(iii) *Assume that* $\Pi = \langle \tau, \upsilon \rangle$, *where* $(\tilde{\gamma}^k)^\tau = \tilde{\gamma}^\ell$, $(\tilde{\gamma}^\ell)^\tau = \tilde{\gamma}^k$, $(\tilde{\gamma}^k)^\upsilon = (\tilde{\gamma}^k)^{-1}$ *and* $(\tilde{\gamma}^\ell)^\upsilon = (\tilde{\gamma}^\ell)^{-1}$. *Then* $K = Q(\alpha_k + \alpha_\ell, \alpha_k\alpha_\ell)$ *and* $A \sim K$.

(iv) *Assume that* $\pi = \langle \tau \rangle \simeq Z/4Z$, *where* $(\tilde{\gamma}^k)^\tau = (\tilde{\gamma}^\ell)^{-1}$ *and* $(\tilde{\gamma}^\ell)^\tau = \tilde{\gamma}^k$. *Then* $K = Q(\alpha_k + \alpha_\ell, \alpha_k\alpha_\ell)$ *and* $A \sim K$.

(II) *Assume that* $(q-1)/2 \,|\, k+\ell$, *i.e.*, $k+\ell = (q-1)/2$. *Then, if* $k+\ell$ *is even*, $A \sim K$. *Suppose that* $k+\ell$ *is odd. Then, if w is any infinite place of* K, *we have* $h_w(A) \equiv 1/2 \pmod 1$, *and if w is a finite place of* K, *we have* $h_w(A) \equiv 0 \pmod 1$ *except in the following case*: *If* $(q-1)/(k-\ell, q-1)$ *is of the form* $2r^c$, *where r is an odd prime number of the form* $4u-1$ *and* $c \geqq 1$, *and if* $w|r$, *then* $h_w(A) \equiv 1/2 \pmod 1$.

5. The Hasse Invariants of $A(\chi_4(k,\ell), Q)$

Let $\chi = \chi_4(k,\ell)$ $(k, \ell \in T_2, k \neq \ell)$. Then $Q(\chi) = Q((-1)^\ell \beta_{ik} + (-1)^k \beta_{i\ell}\ (i \in T_2),$ $\beta_{ik} + \beta_{i\ell}\ (i \in T_2),\ \beta_{ik}\beta_{i\ell}\ (i \in T_2),\ \beta_{ik}\beta_{j\ell} + \beta_{jk}\beta_{i\ell}\ (i, j \in T_2, i \neq j))$. We note that, generally, $Q(\chi) \neq Q(\lambda'(k) \times \lambda'(\ell)) = Q(\beta_k, \beta_\ell)$. So the assertion in [21, (3.4)] is not correct.

5.1. The case $(q+1)/2 \nmid k+\ell$: Let

$$\tilde{a} = \begin{pmatrix} \eta & & & 0 \\ & \eta^{-1} & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \eta & \\ 0 & & & \eta^{-1} \end{pmatrix},$$

$$\tilde{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \tilde{y} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$v = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and let $\tilde{H} = \langle \tilde{a}, \tilde{b}, \tilde{x}, \tilde{y}, \tilde{z} \rangle$. Then v transforms each element c of H to $c^{(q)}$. Let d be an element of $\tilde{G}$ such that $d^{-1}\tilde{H}d \subset G$, and put $a = d^{-1}\tilde{a}d$, $b = d^{-1}\tilde{b}d$, $x = d^{-1}\tilde{x}d$, $y = d^{-1}\tilde{y}d$, $z = d^{-1}\tilde{z}d$ and $H = d^{-1}\tilde{H}d$. Let $N = \langle a, b \rangle$ (a normal subgroup of H), and let ψ be the linear character of N defined by $\psi(a^i b^j) = (\tilde{\eta}^k)^i (\tilde{\eta}^\ell)^j$. Then ψ^H is an irreducible character of H and $Q(\psi^H) = Q(\chi)$. We have

$$(\chi | H, \psi^H)_H = \begin{cases} 2q^2 - 8q + 17 & \text{if } k + \ell \text{ is even,} \\ 2q^2 - 8q + 13 & \text{if } k + \ell \text{ is odd.} \end{cases}$$

Since χ and ψ^H are real, we have $[A(\chi, Q(\chi))] = [A(\psi^H, Q(\chi))]$. The arguments go as in 4.1. We omit the details.

5.2. The case $(q+1)/2 \,|\, k+\ell$, i.e., $k + \ell = (q+1)/2$: We have $(\chi, \sigma(k-\ell))_G = 2q^2 - 6q + 9$ (odd) and $Q(\sigma(k-\ell)) = Q(\beta_{2k}) \subset Q(\chi)$ (we have $Q(\chi) = Q(\beta_k)$ if $k+\ell$ is odd and $Q(\chi) = Q(\beta_{2k})$ if $k+\ell$ is even). Let H, N be as in 3.2, and let ψ be the linear character of N defined by $\psi(w^i d_\beta) = (\tilde{\eta}^{k-\ell})^i \varepsilon(\beta)$. Then ψ^H is an irreducible character of H and $Q(\psi^H) = Q(\sigma(k-\ell))$. Since $\psi^G = \sigma(k-\ell)$, we have $(\chi | H, \psi^H)_H = 2q^2 - 6q + 9$, odd. Thus, since χ and ψ^H are real, we have $[A(\chi, Q(\chi))] = [A(\psi^H, Q(\chi))]$. We omit the detailed calculation.

We get

PROPOSITION 4. *Let* $\chi = \chi_4(k, \ell)$ $(k, \ell \in T_2, k \neq \ell)$. *Then* $Q(\chi) = Q((-1)^\ell \beta_{ik} + (-1)^k \beta_{i\ell}$ $(i \in T_2)$, $\beta_{ik} + \beta_{i\ell}$ $(i \in T_2)$, $\beta_{ik}\beta_{i\ell}$ $(i \in T_2)$, $\beta_{ik}\beta_{j\ell} + \beta_{jk}\beta_{i\ell}$ $(i, j \in T_2, i \neq j))$. *Put* $K = Q(\chi)$ *and* $A = A(\chi, Q)$. *Then we have the following*:

(I) *Assume that* $(q+1)/2 \nmid k+\ell$. *Put* $L = Q(\tilde{\eta}^k, \tilde{\eta}^\ell)$, *and* $\Pi = \mathrm{Gal}(L/K)$. *Then*:

(i) *Assume that* $\Pi = \langle \iota \rangle$, *where* $(\tilde{\eta}^k)^\iota = (\tilde{\eta}^k)^{-1}$ *and* $(\tilde{\eta}^\ell)^\iota = (\tilde{\eta}^\ell)^{-1}$. *Put* $m = (k, \ell)$. *Then* $L = Q(\tilde{\eta}^m)$ *and* $K = Q(\beta_m)$. *If* $k+\ell$ *is even,* $A \sim K$. *Suppose that* $k+\ell$ *is odd. Then* $h_u(A) \equiv 1/2 \pmod 1$ *for any infinite place* u *of* K, *and if* u *is a finite place of* K, *we have* $h_u(A) \equiv 0 \pmod 1$ *except in the following case: If* $(q+1)/(m, q+1)$ *is of the form* r^c *or* $2r^c$, *where* r *is an odd prime number of the form* $4s-1$ *and* $c \geqq 1$, *and if* $u|r$, *then* $h_u(A) \equiv 1/2 \pmod 1$.

(ii) *Assume that* $\Pi = \langle \tau, v \rangle \simeq Z/2Z \times Z/2Z$, *where* $(\tilde{\eta}^k)^\tau = (\tilde{\eta}^k)^{-1}$, $(\tilde{\eta}^\ell)^\tau = \tilde{\eta}^\ell$, $(\tilde{\eta}^k)^v = \tilde{\eta}^k$ *and* $(\tilde{\eta}^\ell)^v = (\tilde{\eta}^\ell)^{-1}$. *Put* $m = (q+1)/(k, q+1)$ *and* $n = (q+1)/(\ell, q+1)$. *Then* $(m, n) \leqq 2$ *and* $K = Q(\beta_k, \beta_\ell)$. (a) *If* k, ℓ *are even,* $A \sim K$. (b) *Suppose that* k *is even and* ℓ *is odd. Then* $h_u(A) \equiv 1/2 \pmod 1$ *for any infinite place* u *of* K, *and if* u *is a finite place of* K, *we have* $h_u(A) \equiv 0 \pmod 1$ *except in the following cases: If* n *is of the form* $2r^c$, *where* r *is an odd prime number of the* *form* $4s-1$ *and* $c \geqq 1$, *if* $[Q_r(\beta_k):Q_r]$ *is odd and if* $u|r$, *then* $h_u(A) \equiv 1/2 \pmod 1$. *If* $n=4$, *if* $[Q_2(\beta_k):Q_2]$ *is odd and if* $u|2$, *then* $h_u(A) \equiv 1/2 \pmod 1$. (c) *Suppose that* k *is odd and* ℓ *is even. Then* $h_u(A) \equiv 1/2 \pmod 1$ *for any infinite place* u *of* K, *and if* u *is a finite place of* K, *we have* $h_u(A) \equiv 0 \pmod 1$ *except in the following cases: If* m *is of the form* $2r^c$, *where* r *is an odd prime number of the form* $4s-1$ *and* $c \geqq 1$, *if* $[Q_r(\beta_\ell):Q_r]$ *is odd and if* $u|r$, *we have* $h_u(A) \equiv 1/2 \pmod 1$. *If* $m=4$, *if* $[Q_2(\beta_\ell):Q_2]$ *is odd and if* $u|2$, *then* $h_u(A) \equiv 1/2 \pmod 1$. (d) *Suppose that* k,ℓ *are odd. Then we have* $h_u(A) \equiv 0 \pmod 1$ *for any infinite place* u *of* K, *and if* u *is a finite place of* K, *we have* $h_u(A) \equiv 0 \pmod 1$ *except in the following cases: If* m *is of the form* $2r^c$, *where* r *is an odd prime number of the form* $4s-1$ *and* $c \geqq 1$, *if* $[Q_r(\beta_\ell):Q_r]$ *is odd and if* $u|r$, *then* $h_u(A) \equiv 1/2 \pmod 1$. *If* n *is of the form* $2r^c$, *where* r *is an odd prime number of the form* $4s-1$ *and* $c \geqq 1$, *if* $[Q_r(\beta_k):Q_r]$ *is odd and if* $u|r$, *then* $h_u(A) \equiv 1/2 \pmod 1$.

(iii) *Assume that* $\Pi = \langle \tau, v \rangle \simeq Z/2Z \times Z/2Z$, *where* $(\tilde{\eta}^k)^\tau = \tilde{\eta}^\ell$, $(\tilde{\eta}^\ell)^\tau = \tilde{\eta}^k$, $(\tilde{\eta}^k)^v = (\tilde{\eta}^k)^{-1}$ *and* $(\tilde{\eta}^\ell)^v = (\tilde{\eta}^\ell)^{-1}$. *Then* $L = Q(\tilde{\eta}^k) = Q(\tilde{\eta}^\ell)$, $K = Q(\beta_k + \beta_\ell, \beta_k\beta_\ell)$ *and* $A \sim K$.

(iv) *Assume that* $\Pi = \langle \tau \rangle \simeq Z/4Z$, *where* $(\tilde{\eta}^k)^\tau = (\tilde{\eta}^\ell)^{-1}$ *and* $(\tilde{\eta}^\ell)^\tau = \tilde{\eta}^k$. *Then* $L = Q(\tilde{\eta}^k) = Q(\tilde{\eta}^\ell)$, $K = Q(\beta_k + \beta_\ell, \beta_k\beta_\ell)$ *and* $A \sim K$.

(II) *Assume that* $(q+1)/2 \,|\, k+\ell$, *i.e.,* $k+\ell = (q+1)/2$. *Then, if* $k-\ell$ *is even,* $A \sim K$. *Suppose that* $k-\ell$ *is odd. Then* $h_u(A) \equiv 1/2 \pmod 1$ *for any infinite place* u *of* K, *and if* u *is a finite place of* K, *we have* $h_u(A) \equiv 0 \pmod 1$ *except in the following case: If* $(q+1)/(k-\ell, q+1)$ *is of the form* $2r^c$, *where* r *is an odd prime number of the form* $4s-1$ *and* $c \geqq 1$, *and if* u *is the unique place of* K *that lies above* r, *then* $h_u(A) \equiv 1/2 \pmod 1$.

6. The Hasse Invariants of $A(-\chi_5(k,\ell), Q)$

Let $\chi = -\chi_5(k,\ell)$ $(k \in T_2, \ell \in T_1)$. Then $Q(\chi) = Q(\beta_k, \alpha_\ell) = Q(-\lambda'(k) \times \lambda(\ell))$, where $-\lambda'(k)$ and $\lambda(\ell)$ are irreducible characters of $SL_2(F_q)$ whose character-values are listed up on p. 504 of [22]. In this case the assertion in [21, (3.5)] is correct. Let ψ_3, ψ_3' be characters of G which are constructed in [22, pp. 494–5]. Then, by [22, p. 505], we have $-\chi = (\lambda'(k) \times \lambda(\ell))^G + \tilde{\psi}_3$, where $\tilde{\psi}_3 = \psi_3$ (resp. $\tilde{\psi}_3 = \psi_3'$) if $k+\ell$ is even (resp. odd). We have $(\chi, \tilde{\psi}_3)_G = 3$, hence, since $(\chi,\chi)_G = 1$, we must have $(\chi, (-\lambda'(k) \times \lambda(\ell))^G)_G = 3$, odd. So, since χ and $-\lambda'(k) \times \lambda(\ell)$ are real, we have $[A(\chi, Q(\chi))] = [A(-\lambda'(k) \times \lambda(\ell), Q(\chi))]$. The local Schur indices of the character $-\lambda'(k) \times \lambda(\ell)$ have been calculated in [21, Proposition (2.5)]. Here we shall give a more direct treatment.

Let

$$\tilde{a} = \begin{pmatrix} \eta & & & 0 \\ & \eta^{-1} & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \gamma & \\ 0 & & & \gamma^{-1} \end{pmatrix},$$

$$\tilde{y} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and let $\tilde{H} = \langle \tilde{a}, \tilde{b}, \tilde{y}, \tilde{z} \rangle$. The group $SL_2(\bar{F}_q) \times SL_2(\bar{F}_q)$ can be embedded into $\tilde{G}$ diagonally, and $\tilde{H}$ is a subgroup of $SL_2(\bar{F}_q) \times SL_2(\bar{F}_q)$. $\tilde{y}$ transforms each element w of $\tilde{H}$ to $w^{(q)}$, so that, by a result similar to Lemma H, there is an element d of $SL_2(\bar{F}_q) \times SL_2(\bar{F}_q)$ such that $d^{-1}\tilde{H}d \subset SL_2(F_q) \times SL_2(F_q)$. Fixing one such element d, we put $a = d^{-1}\tilde{a}d$, $b = d^{-1}\tilde{b}d$, $y = d^{-1}\tilde{y}d$, $z = d^{-1}\tilde{z}d$ and $H = d^{-1}\tilde{H}d$. Let $N = \langle a, b \rangle$ (a normal subgroup of H), and let ψ be the linear character of N defined by $\psi(a^i b^j) = (\tilde{\eta}^k)^i (\tilde{\gamma}^\ell)^j$. Then $((-\lambda'(k) \times \lambda(\ell)) \,|\, N, \psi)_N = 3$. We see that ψ^H is an irreducible character of H and $Q(\psi^H) = Q(\chi)$. Therefore we have $[A(-\lambda'(k) \times \lambda(\ell), Q(\chi))] = [A(\psi^H, Q(\chi))]$. The calculation of the invariants of $A(\psi^H, Q(\psi^H))$ is standard.

We get

PROPOSITION 5 ([21, (3.5), (2.4)]). *Let $\chi = -\chi_5(k, \ell)$ $(k \in T_2, \ell \in T_1)$. Then $Q(\chi) = Q(\beta_k, \alpha_\ell)$. Put $K = Q(\chi)$ and $A = A(\chi, Q)$. Then we have the following:*

(a) *If k, ℓ are even, $A \sim K$.*

(b) *Suppose that k is even and ℓ is odd. Then $h_v(A) \equiv 1/2 \pmod 1$ for any infinite place v of K, and if v is a finite place of K, we have $h_v(A) \equiv 0 \pmod 1$ except in the following cases: Put $n = (q-1)/(\ell, q-1)$. If n is of the form $2r^c$, where r is an odd prime number of the form $4u-1$ and $c \geqq 1$, if $[Q_r(\beta_k) : Q_r]$ is odd and if $v | r$, then $h_v(A) \equiv 1/2 \pmod 1$. If $n = 4$, if $[Q_2(\beta_k) : Q_2]$ is odd and if $v | 2$, then $h_v(A) \equiv 1/2 \pmod 1$.*

(c) *Suppose that k is odd and ℓ is even. Then $h_v(A) \equiv 1/2 \pmod 1$ for any infinite place v of K, and if v is a finite place of K, we have $h_v(A) \equiv 0 \pmod 1$ except in the following cases: Put $m = (q+1)/(k, q+1)$. If m is of the form $2r^c$, where r is an odd prime number of the form $4u-1$ and $c \geqq 1$, if $[Q_r(\alpha_\ell) : Q_r]$ is* *odd and if* $v|r$, *then* $h_v(A) \equiv 1/2 \pmod 1$. *If* $m = 4$, *if* $[Q_2(\alpha_\ell) : Q_2]$ *is odd and if* $v|2$, *then* $h_v(A) \equiv 1/2 \pmod 1$.

(d) *Suppose that* k, ℓ *are odd. Then* $h_v(A) \equiv 0 \pmod 1$ *for any infinite place* v *of* K, *and if* v *is a finite place of* K, *we have* $h_v(A) \equiv 0 \pmod 1$ *except in the following cases: Put* $m = (q+1)/(k, q+1)$ *and* $n = (q-1)/(\ell, q-1)$. *If* m *is of the form* $2r^c$, *where* r *is an odd prime number of the form* $4u-1$ *and* $c \geqq 1$, *if* $[Q_r(\alpha_\ell) : Q_r]$ *is odd and if* $v|r$, *then* $h_v(A) \equiv 1/2 \pmod 1$. *If* $m = 4$, *if* $[Q_2(\alpha_\ell) : Q_2]$ *is odd and if* $v|2$, *then* $h_v(A) \equiv 1/2 \pmod 1$. *If* n *is of the form* $2s^d$, *where* s *is an odd prime number of the form* $4t-1$ *and* $d \geqq 1$, *if* $[Q_s(\beta_k) : Q_s]$ *is odd and if* $v|s$, *then* $h_v(A) \equiv 1/2 \pmod 1$. *If* $n = 4$, *if* $[Q_2(\beta_k) : Q_2]$ *is odd and if* $v|2$, *then* $h_v(A) \equiv 1/2 \pmod 1$.

7. The Hasse Invariants of $A(\chi_i(k), Q)$, $i = 6, 7, 8, 9$

In this case Przygocki has obtained the following result:

PROPOSITION 6 ([21, (3.6)]). *We have* $Q(-\chi_6(k)) = Q(\chi_7(k)) = Q(\beta_k)$ $(k \in T_2)$ *and* $Q(\chi_8(k)) = Q(\chi_9(k)) = Q(\alpha_k)$ $(k \in T_1)$. *And, for* $\chi = -\chi_6(k), \chi_7(k), \chi_8(k)$ *and* $\chi_9(k)$, $A(\chi, Q) \sim Q(\chi)$.

8. The Hasse Invariants of $A(-\xi_1(k), Q)$

Let $\chi = -\xi_1(k)$ $(k \in T_2)$. Then $Q(\chi) = Q(\beta_k)$. Let

$$\tilde{h} = \begin{pmatrix} \eta & & & 0 \\ & \eta^{-1} & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}, \quad \tilde{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\tilde{H} = \langle \tilde{h}, \tilde{x} \rangle$. Then $\tilde{x}$ transforms each element c of $\tilde{H}$ to $c^{(q)}$. Let z be an element of G such that $z^{-1}\tilde{H}z \subset G$, and let $h = z^{-1}\tilde{h}z$, $x = z^{-1}\tilde{x}z$ and $H = z^{-1}\tilde{H}z$. Let $N = \langle h \rangle$ (a normal subgroup of H), and let ψ be the linear character of N defined by $\psi(h) = \tilde{\eta}^k$. Then, as Przygocki observed in [21, (3.7)], we have $(\chi|N, \psi)_N = q^2 - 2q + 2$ (odd). We see that ψ^H is an irreducible character of H and $Q(\psi^H) = Q(\chi)$ (real). Therefore we have $[A(\chi, Q(\chi))] = [A(\psi^H, Q(\chi))]$. We see that $A(\psi^H, Q(\chi)) \simeq ((-1)^k, Q(\tilde{\eta}^k)/Q(\beta_k), \tau)$, where τ is the automorphism of $Q(\tilde{\eta}^k)$ given by $(\tilde{\eta}^k)^\tau = (\tilde{\eta}^k)^{-1}$. Thus, by Proposition L, we get:

PROPOSITION 7 (cf. [21, (3.7)]). *Let* $\chi = -\xi_1(k)$ $(k \in T_2)$. *Then* $Q(\chi) = Q(\beta_k)$. *Put* $K = Q(\chi)$ *and* $A = A(\chi, Q)$. *Then, if k is even,* $A \sim K$. *Suppose that k is odd. Then* $h_v(A) \equiv 1/2 \pmod 1$ *for any infinite place v of K, and if v is a finite place of K, we have* $h_v(A) \equiv 0 \pmod 1$ *except in the following cases: Put* $n = (q+1)/(k, q+1)$. *If n is of the form* $2r^c$, *where r is an odd prime number of the form* $4u-1$ *and* $c \geqq 1$, *and if* $v|r$, *then* $h_v(A) \equiv 1/2 \pmod 1$. *If* $n = 4$, *then* $h_2(A) \equiv 1/2 \pmod 1$.

9. The Hasse Invariants of $A(-\xi_1'(k), Q)$

Let $\chi = -\xi_1'(k)$ $(k \in T_2)$. Then $Q(\chi) = Q(\beta_k)$. Let H, N and ψ be as in §8. Then we have $(\chi|N, \psi)_N = q^3 - 2q^2 + 4q - 2$ if k is even (this differs from [21, p. 296]), and $= q(q^2 - 2q + 2)$ if k is odd. So $[A(\chi, Q(\chi))] = [A(\psi^H, Q(\chi))]$. Thus the same statements as in Proposition 7 holds for χ.

10. The Hasse Invariants of $A(\xi_3(k), Q)$

Let $\chi = \xi_3(k)$ $(k \in T_1)$. Then $Q(\chi) = Q(\alpha_k)$. Let

$$b = \begin{pmatrix} \gamma & & & 0 \\ & \gamma^{-1} & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $H = \langle b, y \rangle$. Let $N = \langle b \rangle$ (a normal subgroup of H), and let ψ be the linear character of N defined by $\psi(b) = \tilde{\gamma}^k$. Then, as Przygocki observed in [21, p. 296], we have $(\chi|N, \psi)_N = q^2 + 2q + 4$ (odd). We see that ψ^H is an irreducible character of H and $Q(\psi^H) = Q(\chi)$ (real). Therefore we have $[A(\chi, Q(\chi))] = [A(\psi^H, Q(\chi))]$. We see that $A(\psi^H, Q(\chi)) \simeq ((-1)^k, Q(\tilde{\gamma}^k)/Q(\alpha_k), \tau)$, where τ is the automorphism of $Q(\tilde{\gamma}^k)$ given by $(\tilde{\gamma}^k)^\tau = (\tilde{\gamma}^k)^{-1}$. Thus, by Proposition L, we get

PROPOSITION 8 (cf. [21, (3.7)]). *Let* $\chi = \xi_3(k)$ $(k \in T_1)$. *Then* $Q(\chi) = Q(\alpha_k)$. *Put* $K = Q(\chi)$ *and* $A = A(\chi, Q)$. *Then, if k is even,* $A \sim K$. *Suppose that k is odd. Then we have* $h_v(A) \equiv 1/2 \pmod 1$ *for any infinite place v of K, and if v is a finite place of K, we have* $h_v(A) \equiv 0 \pmod 1$ *except in the following cases: Put* $n = (q-1)/(k, q-1)$. *If n is of the form* $2r^c$, *where r is an odd prime number of the form* $4u-1$ *and* $c \geqq 1$, *and if* $v|r$, *then* $h_v(A) \equiv 1/2 \pmod 1$. *If* $n = 4$, *then* $h_2(A) \equiv 1/2 \pmod 1$.

11. The Hasse Invariants of $A(\xi_3'(k), Q)$

Let $\chi = \xi_3'(k)$ $(k \in T_1)$. Then $Q(\chi) = Q(\alpha_k)$. Let H, N and ψ be as in §10. Then we have $(\chi|N, \psi)_N = q^3 + 2q^2 + 6q + 2$ if k is even, and $= q(q^2 + 2q + 4)$ if k is odd (this differs from [21, p. 296]). Thus the same statement as in Proposition 8 holds for χ.

12. The Hasse Invariants of $A(-\xi_{21}(k), Q)$

Let $\chi = -\xi_{21}(k)$ $(k \in T_2)$. Then $Q(\chi) = Q(\sqrt{sq}, \beta_k)$, where $s = (-1)^{(q-1)/2}$. Let

$$\tilde{u} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{a} = \begin{pmatrix} \eta & & & 0 \\ & \eta^{-1} & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix},$$

$$\tilde{k}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\beta \in F_q), \qquad \tilde{t} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \xi^{-1} & \\ 0 & & & \xi \end{pmatrix} \quad (\xi^2 = \nu)$$

and

$$\tilde{t}' = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \nu^{-1} & \\ 0 & & & \nu \end{pmatrix}.$$

Put $\tilde{H} = \langle \tilde{u}, \tilde{a}, \tilde{k}_\beta\ (\beta \in F_q), \tilde{t} \rangle$ if q is square (i.e. q is an even power of p), and $\tilde{H}' = \langle \tilde{u}, \tilde{a}, \tilde{k}_\beta\ (\beta \in F_q), \tilde{t}' \rangle$ if q is non-square. Then, if q is square (resp. non-square), $\tilde{u}$ transforms each element c of $\tilde{H}$ (resp. $\tilde{H}'$) to $c^{(q)}$, so that there is an element z of $\tilde{G}$ such that $z^{-1}\tilde{H}z \subset G$ (resp. $z^{-1}\tilde{H}'z \subset G$). Fixing one such element z, we put $u = z^{-1}\tilde{u}z$, $a = z^{-1}\tilde{a}z$, $k_\beta = z^{-1}\tilde{k}_\beta z$ $(\beta \in F_q)$, $t = z^{-1}\tilde{t}z$ (resp. $t' = z^{-1}\tilde{t}'z$) and $H = z^{-1}\tilde{H}z$ (resp. $H' = z^{-1}\tilde{H}'z$). Let $N = \langle a, k_\beta\ (\beta \in F_q) \rangle = \langle a \rangle \times \{k_\beta \mid \beta \in F_q\}$. Then N is a normal subgroup of H (resp. H'). Let ψ and ψ'

be the linear characters of N defined by $\psi(a^i k_\beta) = (\tilde{\eta}^k)^i \varepsilon(\beta)$ and $\psi'(a^i k_\beta) = (\tilde{\eta}^k)^i \varepsilon'(\beta)$ (see Lemma I). Then we have

$$(\chi|N, \psi)_N = (q^2+1)/2 \ (\text{odd}) \quad \text{if } q \equiv 1 \pmod 4 \text{ and } 2|k,$$

$$(\chi|N, \psi)_N = (q^2-2q+3)/2 \ (\text{odd}) \quad \text{if } q \equiv -1 \pmod 4 \text{ and } 2|k,$$

$$(\chi|N, \psi')_N = (q^2-2q+3)/2 \ (\text{odd}) \quad \text{if } q \equiv 1 \pmod 4 \text{ and } 2\nmid k,$$

$$(\chi|N, \psi')_N = (q^2+1)/2 \ (\text{odd}) \quad \text{if } q \equiv -1 \pmod 4 \text{ and } 2\nmid k.$$

12.1. Assume that q is square. Then $Q(\chi) = Q(\beta_k)$. Put $K = Q(\chi)$. Let $Z = \langle -1 \rangle$ (the centre of Z). For $i = 0, 1$, let ψ_i (resp. ψ_i') be the linear character of NZ defined by $\psi_i|N = \psi$ and $\psi_i(-1) = (-1)^i$ (resp. $\psi_i'|N = \psi'$ and $\psi_i'(-1) = (-1)^i$). $R_0 = \{t^i, ut^i, 0 \leqq i \leqq 2(p-1)-1\}$ (resp. $R = \{t^i, ut^i, 0 \leqq i \leqq p-2\}$) is a set of complete system of representatives for H/N (resp. H/NZ). For $i = 0, 1$, ψ_i^H (resp. $\psi_i'^H$) is an irreducible character of H and $\psi^H = \psi_0^H + \psi_1^H$ (resp. $\psi'^H = \psi_0'^H + \psi_1'^H$).

Suppose that k is even. Then $(\chi|H, \psi^H) = (\chi|H, \psi_0^H)_H + (\chi|H, \psi_1^H)_H = (q^2+1)/2$. Since $\chi(-1) = \chi(1)$, $\psi_0^H(-1) = \psi_0^H(1)$ and $\psi_1^H(-1) = -\psi_1^H(1)$, by Schur's lemma, we must have $(\chi|H, \psi_0^H)_H = (q^2+1)/2$ (odd) and $(\chi|H, \psi_1^H)_H = 0$. We have $Q(\psi_0^H) = K$ (real). Therefore we have $[A(\chi, K)] = [A(\psi_0^H, K)]$. Put $B = A(\psi_0^H, K)$.

Let $L = K(\psi_0) = Q(\tilde{\eta}^k, \zeta_p)$. Then $\mathrm{Gal}(L/K) = \langle \omega \rangle \times \langle \phi \rangle$, where $(\tilde{\eta}^k)^\omega = \tilde{\eta}^k$, $\zeta_p^\omega = \zeta_p^g$ ($g \bmod pZ = v^{-1}$), $(\tilde{\eta}^k)^\phi = (\tilde{\eta}^k)^{-1}$ and $\zeta_p^\phi = \zeta_p$. Set $F = \{f \in H \mid \psi_0^f = \psi_0^{\tau(f)}$ for some $\tau(f) \in \mathrm{Gal}(L/K)\}$. Then we see that $t, u \in F$ and $\omega = \tau(t)$, $\phi = \tau(u)$. So $F = H$. Let β be the factor set of $\mathrm{Gal}(L/K)$ with respect to R. Then $B \simeq (\beta, L/K)$. We have $\beta(\phi, \phi) = \psi_0(u^2) = (-1)^k = 1$, $\beta(\omega, \phi) = \beta(\phi, \omega) = \psi_0(1) = 1$ and $\beta(\omega^{p-2}, \omega) = \psi_0(t^{p-1}) = \psi_0(u^2(-1)) = (-1)^k(-1)^0 = 1$. Let ω' (resp. ϕ') be the restriction of ω (resp. ϕ) to $Q(\zeta_p, \beta_k)$ (resp. $Q(\tilde{\eta}^k)$). Then $(\beta, L/K) \simeq (1, L^{\langle\omega\rangle}, \phi') \otimes_K (1, L^{\langle\phi\rangle}, \omega') \sim K$.

Suppose that k is odd. Then we have $(\chi|H, \psi_1'^H)_H = (q^2-2q+3)/2$ (odd) and $Q(\psi_1'^H) = K$, so $[A(\chi, K)] = [A(\psi_1'^H, K)]$. Put $B' = A(\psi_1'^H, K)$. Then we see that $B' \simeq B_1 \otimes_K B_2$, where $B_1 = (-1, L^{\langle\omega\rangle}, \phi')$ and $B_2 = (1, L^{\langle\phi\rangle}, \omega')$ (cf. $K(\psi_1') = L$). So $B_2 \sim K$, and $B' \sim (-1, Q(\tilde{\eta}^k)/Q(\beta_k), \phi')$. By Proposition L, we have $h_v(B') \equiv 1/2 \pmod 1$ for any infinite place v of K. Put $n = (q+1)/(k, q+1)$. Then, since q is square, $\mathrm{ord}_2\, n = 1$, so that the case $n = 4$ cannot happen. Moreover, since q is square, for any odd prime divisor r of n, the congruence relation $x^2 \equiv -1 \pmod r$ has an integral solution (e.g.

$q \equiv -1 \pmod r$), so that the Legendre symbol $(-1/r) = 1$, hence $r \equiv 1 \pmod 4$. Therefore, by Proposition L, we see that $h_v(B') \equiv 0 \pmod 1$ for any finite place v of K.

12.2. Assume that q is an odd power of $p \equiv 1 \pmod 4$. Then $Q(\chi) = Q(\sqrt{p}, \beta_k)$. Put $K = Q(\chi)$.

Let ψ_i, ψ_i' $(i = 0, 1)$ be as in 12.1. Then we have the irreducible decompositions $\psi^{H'} = \psi_0^{H'} + \psi_1^{H'}$, $\psi'^{H'} = \psi_0'^{H'} + \psi_1'^{H'}$ and $Q(\psi_i^{H'}) = Q(\psi_i'^{H'}) = K$.

Suppose that k is even. Then, since $\chi(-1) = \chi(1)$, we must have $(\chi|H', \psi_0^{H'})_{H'} = (q^2+1)/2$ (odd), so $[A(\chi, K)] = [A(\psi_0^{H'}, K)]$. Let $L = K(\psi_0) = Q(\tilde{\eta}^k, \zeta_p)$. Then we have $\mathrm{Gal}(L/K) = \langle v \rangle \times \langle \phi \rangle$, where $(\tilde{\eta}^k)^v = \tilde{\eta}^k$, $\zeta_p^v = \zeta_p^{g^2}$, $(\tilde{\eta}^k)^\phi = (\tilde{\eta}^k)^{-1}$ and $\zeta_p^\phi = \zeta_p$. We see that $F = \{f \in H' \mid \psi_0^f = \psi_0^{\tau(f)}$ for some $\tau(f) \in \mathrm{Gal}(L/K)\} = H'$, and $v = \tau(t')$ and $\phi = \tau(u)$. Let v' be the restriction of v to $Q(\beta_k, \zeta_p)$ and let ϕ' be the restriction of ϕ to $Q(\tilde{\eta}^k, \sqrt{p})$. Then we see that $B \sim (1, L^{\langle v \rangle}, \phi') \otimes_K (1, L^{\langle \phi \rangle}, v') \sim K$.

Suppose that k is odd. Then, since $\chi(-1) = -\chi(1)$, we must have $(\chi|H', \psi_1'^{H'})_{H'} = (q^2 - 2q + 3)/2$ (odd), so $[A(\chi, K)] = [A(\psi_1'^{H'}, K)]$. Let L, v, ϕ, v', ϕ' be as above. Then we see that $B' = A(\psi_1'^{H'}, K) = (-1, L^{\langle v \rangle}, \phi') \otimes_K (1, L^{\langle \phi \rangle}, v') \sim (-1, L^{\langle v \rangle}, \phi') \sim (-1, Q(\tilde{\eta}^k)/Q(\beta_k), \phi'') \otimes_{Q(\beta_k)} K$, where ϕ'' is the restriction of ϕ' to $Q(\tilde{\eta}^k)$. The invariants of $(-1, Q(\tilde{\eta}^k), \phi'')$ can be determined by using Proposition L.

Put $n = (q+1)/(k, q+1)$. We note that, since $(p-1, n) = 2$ and $p \equiv 1 \pmod 4$, the case $n = 4$ cannot happen. If v is any infinite place of K and v' is the place of $Q(\beta_k)$ that lies below v, then $K_v = Q(\beta_k)_{v'} \simeq R$, so we have $h_v(B') \equiv 1/2 \pmod 1$. Suppose that n is of the form $2r^m$, where r is an odd prime number of the form $4s - 1$ and $m \geqq 1$, let w be a place of K that lies above r and let v be the place of $Q(\beta_k)$ that lies below w. Put $f = [K_w : Q(\beta_k)_v]$. Then we have $h_w(B') \equiv f \times 1/2 \pmod 1$. We show that $f = 2$, which would imply that $h_w(B') \equiv 0 \pmod 1$.

In fact, since $K_w = Q_r(\beta_k, \sqrt{p}) = Q_r(\beta_k)(\sqrt{p})$ and $Q(\beta_k)_v = Q_r(\beta_k)$, $f = [Q_r(\beta_k)(\sqrt{p}) : Q_r(\beta_k)] \leqq 2$. Suppose that $f = 1$. Then $\sqrt{p}$ must belong to $Q_r(\beta_k) = Q_r(\zeta_{r^m} + \zeta_{r^m}^{-1})$ $(\subset Q_r(\zeta_{r^m}))$. Since $Q_r(\zeta_p)/Q_r$ is unramified and $Q_r(\zeta_{r^m})/Q_r$ is totally ramified, we have $Q_r(\zeta_p) \cap Q_r(\zeta_{r^m}) = Q_r$. So, since $\sqrt{p} \in Q(\zeta_p) \cap Q_r(\zeta_{r^m})$, $\sqrt{p}$ belongs to Q_r, hence $\sqrt{p} \in Z_r$, where Z_r is the integer ring of Q_r. Since $Z_r/rZ_r = Z/rZ$, this implies that there must be a rational integer x, coprime to r, such that $x^2 \equiv p \pmod r$. Then $p^{(r-1)/2} \equiv (x^2)^{(r-1)/2} = x^{r-1} \equiv 1 \pmod r$, so that the order e of $p \bmod rZ$ in $F_r^\times = (Z/rZ)^\times$ divides $(r-1)/2$. Since $r = 4s - 1$, $(r-1)/2 = 2s - 1$, odd, so e must be odd. Put $q = p^d$ with d an odd positive integer. Then, since r is a divisor of $q+1=p^d+1$, we have $p^{2d}\equiv 1 \pmod r$. Then, since e is equal to the smallest positive integer h such that $p^h\equiv 1 \pmod r$, e must divide $2d$, hence, since e is odd, e must divide d. Hence $p^d\equiv 1 \pmod r$. Hence r divides $(q+1,q-1)=2$, a contradiction, since r is odd. Therefore we must have $f=2$.

12.3. Assume that q is an odd power of $p\equiv -1 \pmod 4$. Then $K=Q(\chi)=Q(\sqrt{-p},\beta_k)$. We show that $A(\chi,K)\sim K$.

Suppose that k is even. Then, since $\chi(-1)=-\chi(1)$, we must have $(\chi|H',\psi_1^{H'})_{H'}=(q^2-2q+3)/2$ (odd), and $Q(\psi_1^{H'})=K$. Put $B=A(\psi_1^{H'},K)$. Let $L=K(\psi_1)=Q(\tilde{\eta}^k,\zeta_p)$. Then $\mathrm{Gal}(L/K)=\langle v\rangle\times\langle\phi\rangle$, where $(\tilde{\eta}^k)^v=\tilde{\eta}^k$, $\zeta_p^v=\zeta_p^{g^2}$, $(\tilde{\eta}^k)^\phi=(\tilde{\eta}^k)^{-1}$ and $\zeta_p^\phi=\zeta_p$. We have $F=\{f\in H'\mid \psi_1^f=\psi_1^{\tau(f)}$ for some $\tau(f)\in \mathrm{Gal}(L/K)\}=H'$ and $v=\tau(t')$ and $\phi=\tau(u)$. We see that $B\simeq(1,L^{\langle v\rangle},\phi')\otimes_K(-1,L^{\langle\phi\rangle},v')\sim(-1,L^{\langle\phi\rangle},v')\sim(-1,Q(\zeta_p)/Q(\sqrt{-p}),v'')\otimes_{Q(\sqrt{-p})}K$. Here ϕ' (resp. v') is the restriction of ϕ (resp. v) to $Q(\tilde{\eta}^k,\sqrt{-p})$ (resp. $Q(\beta_k,\zeta_p)$), and v'' is the restriction of v' to $Q(\zeta_p)$. But, by Proposition M, (iii), we see that $(-1,Q(\zeta_p)/Q(\sqrt{-p}),v'')\sim Q(\sqrt{-p})$. Thus $m_K(\psi_1^{H'})=1$. By Theorem K, we have $m_K(\chi)\leqq 2$. Therefore, since $(\chi|H',\psi_1^{H'})_{H'}$ is odd, by Corollary B, we conclude that $A(\chi,k)\sim B\sim K$.

Finally, suppose that k is odd. Then, since $\chi(-1)=\chi(1)$, we must have $(\chi|H',\psi_0'^{H'})_{H'}=(q^2+1)/2$ (odd), and $Q(\psi_0'^{H'})=K$. Put $B'=A(\psi_0'^{H'},K)$. Then we see that $B'\simeq B_1'\otimes_K B_2'$, where $B_1'=(-1,L^{\langle v\rangle},\phi')$ and $B_2'=(-1,L^{\langle\phi\rangle},v')$. Here the notation is the same as that in the case when k is even. By Proposition M, (iii), we see that $B_2'\sim(-1,Q(\zeta_p)/Q(\sqrt{-p}),v'')\otimes_{Q(\sqrt{-p})}K\sim K$. So $B'\sim B_1'\sim D\otimes_{Q(\beta_k)}K$, where $D=(-1,Q(\tilde{\eta}^k)/Q(\beta_k),\phi'')$ (ϕ'' is the restriction of ϕ' to $Q(\tilde{\eta}^k)$). The invariants of D can be determined by Proposition D. If w is an infinite place of K, then $K_w\simeq C$, so $h_w(B')\equiv 0 \pmod 1$. Put $n=(q+1)/(k,q+1)$. Then, since $\mathrm{ord}_2(q+1)=\mathrm{ord}_2(p+1)\geqq 2$ and k is odd, $\mathrm{ord}_2 n\geqq 2$, so that, for a finite place v of $Q(\beta_k)$, we have $h_v(D)\equiv 1/2 \pmod 1$ only when $n=4$ and $v=2$. Suppose therefore that is the case, and let w be any place of K that lies above 2. Put $f=[K_w:Q_2]=[Q_2(\sqrt{-p}):Q_2]$. Since $4=n=(q+1)/(k,q+1)$ and k is odd, $\mathrm{ord}_2(q+1)=\mathrm{ord}_2(p+1)=2$, so $-p\not\equiv 1 \pmod 8$. This implies that $-p$ is not a square in Q_2. Therefore $f=2$, and $h_w(B_1')\equiv 2\times 1/2\equiv 0 \pmod 1$. Thus $B'\sim B_1'\sim K$ and $m_K(\psi_0'^{H'})=1$. Since $m_K(\chi)\leqq 2$, we see that $A(\chi,K)\sim A(\psi_0'^{H'},K)\sim K$.

By summarizing the results obtained above, we get:

PROPOSITION 9 (cf. [21, (3.8)]). *Let* $\chi=-\xi_{21}(k)$ $(k\in T_2)$. *Then* $K=Q(\chi)=$

$Q(\sqrt{sq}, \beta_k)$, *where* $s = (-1)^{(q-1)/2}$. *Put* $A = A(\chi, Q)$. *Then, if* k *is even,* $A \sim K$. *Suppose that* k *is odd. Then*: (a) *If* $q \equiv 1 \pmod 4$, *we have* $h_v(A) \equiv 1/2 \pmod 1$ *for any infinite place* v *of* K, *and* $h_v(A) \equiv 0 \pmod 1$ *for any finite place* v *of* K. (b) *If* $q \equiv -1 \pmod 4$, *then* $A \sim K$.

13. The Hasse Invariants of $A(-\xi_{22}(k), Q)$

Let $\chi = -\xi_{22}(k)$ $(k \in T_2)$. Then $Q(\chi) = Q(\sqrt{sq}, \beta_k)$, where $s = (-1)^{(q-1)/2}$. If q is non-square, then χ is the conjugate of $-\xi_{21}(k)$ under the automorphism $\sqrt{sq} \to -\sqrt{sq}$ of $Q(\chi)$ over $Q(\beta_k)$. Assume that q is square. Let H, N, ψ and ψ' be as in §12. Then we have $(\chi|N, \psi')_N = (q^2+1)/2$ (odd) if k is even, and $(\chi|N, \psi)_N = (q^2-2q+3)/2$ (odd) if k is odd. Therefore the rationality of χ is the same as that of $-\xi_{21}(k)$. Thus the same statement as in Proposition 9 holds for χ.

14. The Hasse Invariants of $A(\xi'_{21}(k), Q)$

Let $\chi = \xi'_{21}(k)$ $(k \in T_2)$. Then $Q(\chi) = Q(\sqrt{sq}, \beta_k)$, where $s = (-1)^{(q-1)/2}$. Let H, H', N, ψ and ψ' be as in §12. Then we have:

$$(\chi|N, \psi)_N = \begin{cases} (q^2-4q+5)/2 \text{ (odd)} & \text{if } q \equiv 1 \pmod 4 \text{ and } 2|k, \\ (q^2-2q+3)/2 \text{ (odd)} & \text{if } q \equiv -1 \pmod 4 \text{ and } 2|k, \end{cases}$$

$$(\chi|N, \psi')_N = \begin{cases} (q^2-2q+3)/2 \text{ (odd)} & \text{if } q \equiv 1 \pmod 4 \text{ and } 2\nmid k, \\ (q^2-4q+5)/2 \text{ (odd)} & \text{if } q \equiv -1 \pmod 4 \text{ and } 2\nmid k. \end{cases}$$

Thus, by arguments similar to those in §12, we get:

Proposition 10 (cf. [21, (3.9)]). *Let* $\chi = \xi'_{21}(k)$ $(k \in T_2)$. *Then* $K = Q(\chi) = Q(\sqrt{sq}, \beta_k)$, *where* $s = (-1)^{(q-1)/2}$. *Put* $A = A(\chi, Q)$. *Then*: (a) *Assume that* $q \equiv 1 \pmod 4$. *Then, if* k *is even,* $h_v(A) \equiv 1/2 \pmod 1$ *for any infinite place* v *of* K, *and* $h_v(A) \equiv 0 \pmod 1$ *for any finite place* v *of* K. *If* k *is odd, then* $A \sim K$. (b) *If* $q \equiv -1 \pmod 4$, *then* $A \sim K$.

15. The Hasse Invariants of $A(\xi'_{22}(k), Q)$

Let $\chi = \xi'_{22}(k)$ $(k \in T_2)$. Then $Q(\chi) = Q(\sqrt{sq}, \beta_k)$, where $s = (-1)^{(q-1)/2}$. If q is non-square, then χ is a conjugate of $\xi'_{21}(k)$. Assume that q is square, and let H, N, ψ and ψ' be as in §12. Then we have $(\chi|N, \psi)_N = (q^2-2q+3)/2$ (odd) if k is odd and $(\chi|N, \psi')_N = (q^2-4q+5)/2$ (odd) if k is even. Therefore the rationality of χ is the same as that of $\xi'_{21}(k)$. Thus the same statement as in Proposition 10 holds for χ.

16. The Hasse Invariants of $A(\xi_{41}(k), Q)$

Let $\chi = \xi_{41}(k)$ $(k \in T_1)$. Then $K = Q(\chi) = Q(\sqrt{sq}, \alpha_k)$, where $s = (-1)^{(q-1)/2}$. Let

$$a = \begin{pmatrix} \gamma & & & 0 \\ & \gamma^{-1} & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}, \quad h_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\beta \in F_q),$$

$$u = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and let $N = \langle a, h_\beta \ (\beta \in F_q)\rangle = \langle a \rangle \times \{h_\beta \mid \beta \in F_q\}$. Let ψ and ψ' be the linear characters of N defined by $\psi(a^i h_\beta) = (\tilde{\gamma}^k)^i \varepsilon(\beta)$ and $\psi'(a^i h_\beta) = (\tilde{\gamma}^k)^i \varepsilon'(\beta)$ respectively. Then we have:

$$(\chi|N, \psi)_N = \begin{cases} (q^2 + 2q + 3)/2 \ (\text{odd}) & \text{if } q \equiv 1 \pmod 4 \text{ and } 2|k, \\ (q^2 + 4q + 9)/2 \ (\text{odd}) & \text{if } q \equiv -1 \pmod 4 \text{ and } 2|k, \end{cases}$$

$$(\chi|N, \psi')_N = \begin{cases} (q^2 + 4q + 9)/2 \ (\text{odd}) & \text{if } q \equiv 1 \pmod 4 \text{ and } 2\nmid k, \\ (q^2 + 2q + 3)/2 \ (\text{odd}) & \text{if } q \equiv -1 \pmod 4 \text{ and } 2\nmid k. \end{cases}$$

In the case where q is square we set $H = \langle t, u, N\rangle$, where $t = \mathrm{diag}(1, 1, \xi^{-1}, \xi)$ with $\xi^2 = v$, and in the case where q is non-square we set $H' = \langle t', u, N\rangle$, where $t' = \mathrm{diag}(1, 1, v^{-1}, v)$. Then the arguments go similarly as in § 12. We get:

PROPOSITION 11 (cf. [21, (3.10)]). *Let $\chi = \xi_{41}(k)$ $(k \in T_1)$. Then $K = Q(\chi) = Q(\sqrt{sq}, \alpha_k)$, where $s = (-1)^{(q-1)/2}$. Let $A = A(\chi, Q)$. Then*: (a) *Assume that* $q \equiv 1 \pmod 4$. *Then, if k is even, $A \sim K$. If k is odd, we have $h_v(A) \equiv 1/2 \pmod 1$ for any infinite place of K, and $h_v(A) \equiv 0 \pmod 1$ for any finite place v of K.* (b) *If $q \equiv -1 \pmod 4$, then $A \sim K$.*

17. The Hasse Invariants of $A(\xi_{42}, Q)$

Let $\chi = \xi_{42}(k)$ $(k \in T_1)$. Then $Q(\chi) = Q(\sqrt{sq}, \alpha_k)$, where $s = (-1)^{(q-1)/2}$. If q is non-square, then χ is a conjugate of $\xi_{41}(k)$. Assume that q is square, and let H, N, ψ and ψ' be as in § 16. Then we have $(\chi|H, \psi)_N = (q^2 + 4q + 9)/2$ (odd) if

k is odd, and $(\chi|N,\psi')_N = (q^2+2q+3)/2$ (odd) if k is even. Therefore the rationality of χ is the same as that of $\xi_{41}(k)$. Thus the same statement as in Proposition 11 holds for χ.

18. The Hasse Invariants of $A(-\xi'_{41}(k), Q)$

Let $\chi = -\xi'_{41}(k)$ $(k \in T_1)$. Then $Q(\chi) = Q(\sqrt{sq}, \alpha_k)$, where $s = (-1)^{(q-1)/2}$. Let N, ψ, ψ', H, H' be as in §16. Then we have:

$$(\chi|N,\psi)_N = \begin{cases} (q^2+2q+7)/2 \text{ (odd)} & \text{if } q \equiv 1 \pmod 4 \text{ and } 2|k, \\ (q^2+2q+3)/2 \text{ (odd)} & \text{if } q \equiv -1 \pmod 4 \text{ and } 2|k, \end{cases}$$

$$(\chi|N,\psi')_N = \begin{cases} (q^2+1)/2 \text{ (odd)} & \text{if } q \equiv 1 \pmod 4 \text{ and } 2\nmid k, \\ (q^2+2q+7)/2 \text{ (odd)} & \text{if } q \equiv -1 \pmod 4 \text{ and } 2\nmid k \end{cases}$$

Thus, by a rather long consideration as in §12, we get:

Proposition 12 (cf. [21, (3.11)]). *Let* $\chi = -\xi'_{41}(k)$ $(k \in T_1)$. *Then* $K = Q(\chi) = Q(\sqrt{sq}, \alpha_k)$, *where* $s = (-1)^{(q-1)/2}$. *Let* $A = A(\chi, Q)$. *Then we have the following*: (a) *Assume that q is square. Then, if k is even, we have* $h_v(A) \equiv 1/2 \pmod 1$ *for any infinite place v of K, and if v is a finite place of K, we have* $h_v(A) \equiv 0 \pmod 1$ *except in the following case. Put* $n = (q-1)/(k, q-1)$ *and let* $q = p^{2^t u}$ *with* $(2, u) = 1$. *Then, if* $n \mid p^u - 1$ *or* $n \mid p^u + 1$ *and if* $v|p$, *we have* $h_v(A) \equiv 1/2 \pmod 1$. *If k is odd, then* $A \sim K$. (b) *Assume that q is an odd power of* $p \equiv 1 \pmod 4$. *Then, if k is even, we have* $h_v(A) \equiv 1/2 \pmod 1$ *for any infinite place v of K, and* $h_v(A) \equiv 0 \pmod 1$ *for any finite place v of K. If k is odd, then* $A \sim K$. (c) *If q is an odd power of* $p \equiv -1 \pmod 4$, *then* $A \sim K$.

19. The Hasse Invariants of $A(-\xi'_{42}(k), Q)$

Let $\chi = -\xi'_{42}(k)$ $(k \in T_1)$. Then $Q(\chi) = Q(\sqrt{sq}, \alpha_k)$, where $s = (-1)^{(q-1)/2}$. If q is non-square, then χ is a conjugate of $-\xi'_{41}(k)$. If q is square and if N, ψ and ψ' are as in §16, then we have $(\chi|N,\psi)_N = (q^2+1)/2$ (odd) if k is odd, and $(\chi|N,\psi')_N = (q^2+2q+7)/2$ (odd) if k is even. Therefore the rationality of χ is the same as that of $-\xi'_{42}(k)$. Thus the same statement as in Proposition 12 holds for χ.

20. The Hasse Invariants of $A(\Phi_i, Q)$ $(1 \leqq i \leqq 8)$

We have $Q(\Phi_i) = Q(\sqrt{sq})$, $1 \leqq i \leqq 8$, where $s = (-1)^{(q-1)/2}$. Let

$$h_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\beta \in F_q),$$

and let $N = \{h_\beta \mid \beta \in F_q\}$. Let ψ and ψ' be respectively the linear characters of N defined by $\psi(h_\beta) = \varepsilon(\beta)$ and $\psi'(h_\beta) = \varepsilon'(\beta)$. Then we have the following:

$$(-\Phi_1|N, \psi)_N = \begin{cases} q(q+1)/2 & \text{if } q \equiv 1 \pmod 4, \\ q(q-1)/2 & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(-\Phi_2|N, \psi')_N = \begin{cases} q(q+1)/2 & \text{if } q \equiv 1 \pmod 4, \\ q(q-1)/2 & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(-\Phi_3|N, \psi)_N = \begin{cases} q(q^2+1)/2 & \text{if } q \equiv 1 \pmod 4, \\ q(q-1)^2/2 \text{ (even)} & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(-\Phi_3|N, \psi')_N = q(q^2+1)/2 \quad \text{if } q \equiv -1 \pmod 4,$$

$$(-\Phi_4|N, \psi')_N = q(q^2+1)/2 \quad \text{if } q \equiv 1 \pmod 4,$$

$$(-\Phi_4|N, \psi)_N = q(q^2+1)/2 \quad \text{if } q \equiv -1 \pmod 4,$$

$$(\Phi_5|N, \psi)_N = \begin{cases} q(q+1)/2 & \text{if } q \equiv 1 \pmod 4, \\ q(q-1)/2 & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(\Phi_6|N, \psi')_N = \begin{cases} q(q+1)/2 & \text{if } q \equiv 1 \pmod 4, \\ q(q-1)/2 & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(\Phi_7|N, \psi)_N = \begin{cases} q(q+1)^2/2 \text{ (even)} & \text{if } q \equiv 1 \pmod 4, \\ q(q^2+1)/2 & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(\Phi_7|N, \psi')_N = q(q^2+1)/2 \quad \text{if } q \equiv 1 \pmod 4,$$

$$(\Phi_8|N, \psi)_N = q(q^2+1)/2 \quad \text{if } q \equiv 1 \pmod 4,$$

$$(\Phi_8|N, \psi')_N = q(q^2+1)/2 \quad \text{if } q \equiv -1 \pmod 4.$$

Put $K = Q(\sqrt{sq})$.

Let $\chi = -\Phi_1$. Assume that q is square. Then $K = Q$. Let $t = \mathrm{diag}(\xi^{-1}, \xi, \xi^{-1}, \xi)$ with $\xi^2 = \nu$, and let $H = \langle N, t \rangle$. For $i = 0, 1$, let ψ_i be the extension of ψ to NZ such that $\psi_i(-1) = (-1)^i$. Then we have the irreducible decomposition $\psi^H = \psi_0^H + \psi_1^H$ and $Q(\psi_i^H) = Q$ $(i = 0, 1)$. Since $\chi(-1) = -\chi(1)$, we must have $(\chi|H, \psi_1^H)_H = q(q+1)/2$, odd, so we have $[A(\chi, Q)] = [A(\psi_1^H, Q)]$. We see that $A(\psi_1^H, Q) \simeq (\psi_1(t^{p-1}), Q(\zeta_p), \tau) = (-1, Q(\zeta_p), \tau)$, where $\zeta_p^\tau = \zeta_p^g$ ($g \bmod pZ = v^{-1}$). Thus the invariants of $A(\psi_1^H, Q)$ can be determined by using Proposition M, (i).

Assume that q is an odd power of $p \equiv 1 \pmod 4$. Then $K = Q(\sqrt{p})$. Let $t' = \mathrm{diag}(v^{-1}, v, v^{-1}, v)$, and let $H' = \langle N, t' \rangle$. We have the irreducible decomposition $\psi^{H'} = \psi_0^{H'} + \psi_1^{H'}$ and $Q(\psi_i^{H'}) = K$ ($i = 0, 1$). Since $\chi(-1) = -\chi(1)$, we must have $(\chi|H', \psi_1^{H'})_{H'} = q(q+1)/2$, odd, so $[A(\chi, K)] = [A(\psi_1^{H'}, K)]$. We see that $[A(\psi_1^{H'}, K)] \simeq (-1, Q(\zeta_p), \tau^2)$, so the invariants of $A(\psi_1^{H'}, K)$ can be determined by using Proposition M, (ii).

Assume that q is an odd power of $p \equiv -1 \pmod 4$. Then $A(\chi, K) \sim A(\psi_0^{H'}, K) \simeq (1, Q(\zeta_p), \tau^2) \sim K$, since $m_K(\chi) \leqq 2$.

The characters Φ_i, $2 \leqq i \leqq 8$, can be treated similarly. Thus we get

PROPOSITION 13 (cf. [21, (3.12), (3.14), (3.15)]). *Let $\chi = -\Phi_1, -\Phi_2, -\Phi_3$ or $-\Phi_4$. Then $K = Q(\chi) = Q(\sqrt{sq})$, where $s = (-1)^{(q-1)/2}$. Put $A = A(\chi, K)$. Then, if q is square, we have $h_\infty(A) \equiv h_p(A) \equiv 1/2 \pmod 1$ and $h_r(A) \equiv 0 \pmod 1$ for each finite place $r \neq p$ of $K = Q$. If q is an odd power of $p \equiv 1 \pmod 4$, then $h_v(A) \equiv 1/2 \pmod 1$ for each real place v of $K = Q(\sqrt{p})$ and $h_v(A) \equiv 0 \pmod 1$ for any finite place v of K. If q is an odd power of $p \equiv -1 \pmod 4$, then $A \sim K = Q(\sqrt{-p})$.*

For $5 \leqq i \leqq 8$, $Q(\Phi_i) = Q(\sqrt{sq})$, where $s = (-1)^{(q-1)/2}$, and $A(\Phi_i, Q) \sim Q(\Phi_i)$.

Przygocki has observed

PROPOSITION 14 ([21, (3.16)]). $A(\Phi_9, Q) \sim Q$.

21. The Hasse Invariants of $A(\theta_i, Q)$ $(1 \leqq i \leqq 8)$

We have $Q(\theta_i) = Q(\sqrt{sq})$ $(1 \leqq i \leqq 8)$, where $s = (-1)^{(q-1)/2}$. Let N, ψ and ψ' be as in §20. Then we have the following:

$$(\theta_1|N, \psi)_N = \begin{cases} q^2(q+1)/2 & \text{if } q \equiv 1 \pmod 4, \\ q^2(q-1)/2 & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(\theta_2|N, \psi')_N = \begin{cases} q^2(q+1)/2 & \text{if } q \equiv 1 \pmod 4, \\ q^2(q-1)/2 & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(\theta_3|N, \psi)_N = \begin{cases} q & \text{if } q \equiv 1 \pmod 4, \\ 0 & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(\theta_3|N, \psi')_N = q \quad \text{if } q \equiv -1 \pmod 4,$$

$$(\theta_4|N,\psi')_N = \begin{cases} q & \text{if } q \equiv 1 \pmod 4, \\ 0 & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(\theta_4|N,\psi)_N = q \quad \text{if } q \equiv -1 \pmod 4,$$

$$(-\theta_5|N,\psi)_N = \begin{cases} q^2(q+1)/2 & \text{if } q \equiv 1 \pmod 4, \\ q^2(q-1)/2 & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(-\theta_6|N,\psi')_N = \begin{cases} q^2(q+1)/2 & \text{if } q \equiv 1 \pmod 4, \\ q^2(q-1)/2 & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(-\theta_7|N,\psi)_N = \begin{cases} q & \text{if } q \equiv 1 \pmod 4, \\ 0 & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(-\theta_7|N,\psi')_N = q \quad \text{if } q \equiv -1 \pmod 4,$$

$$(-\theta_8|N,\psi')_N = \begin{cases} q & \text{if } q \equiv 1 \pmod 4, \\ 0 & \text{if } q \equiv -1 \pmod 4, \end{cases}$$

$$(-\theta_8|N,\psi)_N = q \quad \text{if } q \equiv -1 \pmod 4.$$

Thus, by the arguments similar to those in §20, we get:

PROPOSITION 15 (cf. [21, (3.17), (3.18), (3.19), (3.20)]). *Suppose that* $\chi = \theta_1, \theta_2, \theta_3$ *or* θ_4. *Then* $Q(\chi) = Q(\sqrt{sq})$, *where* $s = (-1)^{(q-1)/2}$, *and* $A(\chi, Q) \sim Q(\chi)$. *Suppose that* $\chi = -\theta_5, -\theta_6, -\theta_7$ *or* $-\theta_8$. *Then* $K = Q(\chi) = Q(\sqrt{sq})$. *Put* $A = A(\chi, Q)$. *Then, if* q *is square, we have* $h_\infty(A) \equiv h_p(A) \equiv 1/2 \pmod 1$ *and* $h_r(A) \equiv 0 \pmod 1$ *for each finite place* $r \neq p$ *of* $K = Q$. *If* q *is an odd power of* $p \equiv 1 \pmod 4$, *then* $h_v(A) \equiv 1/2 \pmod 1$ *for each real place* v *of* $K = Q(\sqrt{p})$ *and* $h_v(A) \equiv 0 \pmod 1$ *for each finite place* v *of* K. *If* q *is an odd power of* $p \equiv -1 \pmod 4$, *then* $A \sim K = Q(\sqrt{-p})$.

22. The Hasse Invariants of $A(\theta_i, Q)$ $(9 \leqq i \leqq 13)$

PROPOSITION 16 (cf. [21, (3.21), (3.23)]). *For* $9 \leqq i \leqq 13$, $A(\theta_i, Q) \sim Q$.

REMARK. The characters above are the unipotent characters of G, so Proposition 16 is well known (Benson and Curtis [2], Lusztig [10, (7.6)]).

References

[1] Barry, M. J. J., Schur indices and ${}^3D_4(q^3)$, q odd, J. Algebra **117** (1988), 522–524.
[2] Benson, C. T., and Curtis, C. W., On the degrees and rationality of certain characters of finite Chevalley groups, Trans. Amer. Math. Soc. **165** (1972), 251–273.

[3] Enomoto, H., The characters of the finite symplectic group $Sp(4,q)$, $q=2^f$, Osaka J. Math. **9** (1972), 74–94.

[4] Enomoto, H., and Ohmori, Z., The Schur indices of the irreducible characters of $G_2(2^n)$, Proceed. Japan Acad. **69** (1993), Ser. A., No. 9, 350–352.

[5] Geck, M., The Schur indices of the cuspidal unipotent characters of the finite Chevalley groups $E_7(q)$, arXiv: math. RT/0306268 v1 18 Jun 2003.

[6] Gow, R., Schur indices of some groups of Lie type, J. Algebra **42** (1976), 102–120.

[7] Gow, R., On the Schur indices of characters of finite classical groups, J. London Math. Soc. **24** (1981), 135–147.

[8] Gow, R., Real representations of finite orthogonal and symplectic groups of odd characteristics, J. Algebra 96 (1985), 249–274.

[9] Janusz, G. J., Simple components of $Q[SL(2,q)]$, Commun. Algebra **1** (1974), 1–20.

[10] Lusztig, G., Coxeter orbits and eigenspaces of Frobenius, Invent. Math. **38** (1976), 101–159.

[11] Lusztig, G., A unipotent support for irreducible representations, Advances in Math. **94** (1992), 139–179.

[12] Lusztig, G., Rationality properties of unipotent representations, J. Algebra **258** (2002), 1–22.

[13] Ohmori, Z., On the Schur indices of $GL(n,q)$ and $SL(2n+1,q)$, J. Math. Soc. Japan **29** (1977), 693–707.

[14] Ohmori, Z., On the Schur indices of reductive groups II, Quart. J. Math. Oxford (2), **32** (1981), 443–452.

[15] Ohmori, Z., On the Schur indices of certain irreducible characters of reductive groups over finite fields, Osaka J. Math. **25** (1988), 149–159.

[16] Ohmori, Z., Schur indices of the irreducible characters of some finite Chevalley groups of rank 2, I, Tokyo J. Math. **8** (1985), 133–150.

[17] Ohmori, Z., On a Zelevinsky theorem and the Schur indices of the finite unitary groups, J. Math. Sci. Univ. Tokyo **4** (1997), 417–433.

[18] Ohmori, J., On the Schur indices of characters of finite reductive groups in bad characteristic cases, Osaka J. Math. **40** (2003), 1011–1019.

[19] Ohmori, J., On Feit's definition of the Schur index, To appear in Hokkaido Mathematical J.

[20] Ohmori, J., Rationality of the cuspidal unipotent representations of $E_7(q)$ in a crystalline cohomology group, Preprint.

[21] Przygocki, A., Schur indices of symplectic groups, Commun. Algebra **10** (1982), 279–310.

[22] Srinivasan, B., The characters of the finite symplectic group $Sp(4,q)$, Trans. Amer. Math. Soc. **131** (1968), 488–525.

[23] Turull, A., The Schur indices of the irreducible characters of the special linear groups, J. Algebra **235** (2001), 275–314.

[24] Weil, A., Basic Number Theory, 2nd ed., Springer-Verlag, 1973.

[25] Yamada, H., The characters of $Sp_4(q)$, q odd, Preprint.

[26] Yamada, T., The Schur subgroup of 2-adic field, J. Math. Soc. Japan **26** (1974), 168–179.

[27] Yamada, T., The Schur subgroup of the Brauer group, Lecture Notes in Mathematics 397, Springer-Verlag, 1974.

[28] Zelevinsky, A. V., Representations of finite classical groups—a Hopf algebra approach, Lecture Notes in Math. 869, Springer-Verlag, 1981.

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