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CONSTRUCT BI-FROBENIUS ALGEBRAS VIA QUIVERS*

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Abstract. The aim of this note is to construct explicitly a class of bi-Frobenius algebras via quivers. In particular, this kind of bi-Frobenius algebras are not Hopf algebras, and a necessary and sufficient condition for such algebras being symmetric is given.

1. Introduction

Typical Frobenius algebras are finite group algebras. In general, a finite-dimensional Hopf algebra is a Frobenius algebra (see Larson and Sweedler [LS], or 2.1.3 in Montgomery [M]). Relations of the Frobenius algebras with the Yang-Baxter equations and with the topological quantum field theory can be founded in [Kad] and [A], respectively. As a natural generalization of finite-dimensional Hopf algebras, the concept of a bi-Frobenius algebra was introduced by Doi and Takeuchi [DT] (see also [Kop]). Roughly speaking, this is a Frobenius algebra as well as a Frobenius coalgebra together with an antipode. Except for an example given in 2.5 in [DT], there are few explicit constructions of bi-Frobenius algebras which are not finite-dimensional Hopf algebras. The aim of this note is to provide such an explicit construction via quivers.

Motivations of our construction is the quiver method in the representation theory of algebras, see Ringel [R], coalgebra structure on quivers considered by Chin and Montgomery [CM], and constructing Hopf quiver and quiver quantum groups by Cibils and Rosso [C], [CR], and E. Green and Solberg [GS], etc.

We start from the algebra KZ_n/J^d, where KZ_n is the path algebra of the basic cycle with n vertices and J is the ideal generated by arrows with $d \geq 2$ an integer. This is an augmented Frobenius algebra, and it is a symmetric if and only if $d \equiv 1 \pmod{n}$ (see Theorem 2.3 below). Endowed with a suitable

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Frobenius coalgebra structure, this Frobenius algebra becomes a bi-Frobenius algebra, which is not a Hopf algebra (see Theorem 3.3).

The authors thank the referee for pointing out the reference [D]. By Lemma 1.2 in [D], one can prove the both Frobeniusness of KZ_n/J^d by showing that it has a bijective bi-Frobenius antipode. So we leave the proof of the Frobeniusness of algebra (see Lemma 2.1(ii)) and of coalgebra (see Lemma 3.2) to Theorem 3.3.

Throughout let K be a field. All algebras and coalgebras are over K. The notations Hom and $\otimes$ are over K.

2. Quivers and Frobenius Algebras

A quiver Q is an oriented graph given by the set Q_0 of vertices and the set Q_1 of arrows. Let KQ be the path algebra of a quiver Q (see e.g. [R]). We write the conjunction of paths from right to left.

A finite-dimensional algebra A is said to be elementary if $A/\mathrm{rad}\, A \cong k^n$ as algebras for some positive integer n, where $\mathrm{rad}\, A$ is the Jacobson radical. By Gabriel's theorem an elementary algebra is isomorphic to KQ/I, where Q is a finite quiver, and I is an admissible ideal of KQ (i.e., $J^N \subseteq I \subseteq J^2$ for some positive integer N, J is the ideal of KQ generated by the arrows). Such a quiver Q is uniquely determined by A, which is called the Gabriel quiver of A.

Let A be a finite-dimensional algebra, and $A^* = \mathrm{Hom}(A, K)$. Then A^* has a natural A-A-bimodule structure given by $(af)(b) = f(ba)$, $(fa)(b) = f(ab)$, $\forall f \in A^*$, $a, b \in A$. We say that A is a Frobenius algebra provided that ${}_AA \cong {}_AA^*$ as left A-modules, or equivalently $A_A \cong A^*_A$ as right A-modules; and that A is symmetric provided that ${}_AA_A \cong {}_AA^*_A$ as A-A-bimodules.

Let A be a Frobenius algebra with $\Phi : {}_AA \cong {}_AA^*$. Then $\phi := \Phi(1_A)$ is a cyclic generator of ${}_AA^*$. Also ϕ is a cyclic generator of A^*_A, and $a \mapsto \phi a$ is an isomorphism $A_A \cong A^*_A$. We will call the pair (A, ϕ) is a Frobenius algebra if ϕ is needed to be specified. Let (x_i, f_i), $x_i \in A$, $f_i \in A^*$, be a dual basis (i.e., for each $a \in A$, $\sum_i f_i(a)x_i = a$), and $y_i \in A$ with $f_i = \phi y_i$. Then we have

$$\sum_i x_i\phi(y_ia) = a = \sum_i \phi(ax_i)y_i, \quad \forall a \in A.$$

We refer to ϕ as a Frobenius homomorphism, (x_i, y_i) as a dual basis, $\sum_i x_i \otimes y_i$ as a Frobenius element, and (ϕ, x_i, y_i) as a Frobenius coordinate, see [Kad] or [KS]. The Nakayama automorphism is the unique algebra isomorphism $N : A \to A$, determined by

$$\phi a = N(a)\phi$$

for all $a \in A$. Then we have

$$N(a) = \sum_i x_i \phi(a y_i), \quad \forall a \in A.$$

It is well-known that a Frobenius algebra is symmetric if and only if its Nakayama automorphism is inner (see e.g. [Y]).

Let C be a finite-dimensional coalgebra with comultiplication $\triangle$, C^* its dual algebra. Then C is a C^*-C^*-bimodule via

$$fc = \sum c_1 f(c_2), \quad cf = \sum f(c_1) c_2 \quad \forall f \in C^*,\, c \in C.$$

By definition a pair (C, t) with $t \in C$ is called a Frobenius coalgebra if $C = tC^*$, or equivalently $C = C^*t$.

Let Z_n denote the basic cycle with n vertices. The set of vertices is denoted by $\{e_i \mid i \in Z/nZ\}$, and the set of arrows by $\{a_i = i \to i+1 \mid i \in \boldsymbol{Z}/n\boldsymbol{Z}\}$. Set $\gamma_i^m := a_{i+m} \cdots a_{i+1} a_i$, the path of length m starting at the vertex e_i. Taking the indices modulo n. Note that $\gamma_i^0 = e_i$ and $\gamma_i^1 = a_i$.

The following fact seems to be well-known. For use later, we write out a direct proof.

Lemma 2.1. (i) *Assume that $\boldsymbol{K}Z_n/I$ is a Frobenius algebra, where I is an admissible ideal. Then I must be of the form $I = J^d$ for some positive integer $d \geq 2$.*

(ii) *The algebra $\boldsymbol{K}Z_n/J^d$ is a Frobenius algebra, which is augmented (i.e. there is an algebra homomorphism $\varepsilon : A \to \boldsymbol{K}$).*

Proof. (i) Note that an admissible ideal I of $\boldsymbol{K}Z_n$ must be generated by some paths. While $\boldsymbol{K}Z_n/I$ is a self-injective algebra, it follows from a direct calculation that $I = J^d$ for some positive integer $d \geq 2$.

(ii) Write $A = \boldsymbol{K}Z_n/J^d$. We have remarked in the introduction that the proof of the Frobeniusness is left to Theorem 3.3. But for use later, we write out a left A-module isomorphism $\Phi : A \to A^*$. Note that $\{\gamma_i^m \mid i \in \boldsymbol{Z}/n\boldsymbol{Z}, 0 \leq m \leq d-1\}$ is a basis of A. Let $\{(\gamma_i^m)^* \mid i \in \boldsymbol{Z}/n\boldsymbol{Z}, 0 \leq m \leq d-1\}$ denote the dual basis of A^*, and $\Phi : A \to A^*$ the linear map determined by

$$\Phi(\gamma_i^m) = (\gamma_{i+m}^{d-1-m})^*, \quad \forall i \in \boldsymbol{Z}/n\boldsymbol{Z},\ 0 \leq m \leq d-1. \tag{1}$$

Actually, Φ is a left A-module isomorphism and hence A is a Frobenius algebra.

Define $\varepsilon : A \to \boldsymbol{K}$ to be the linear map determined by

$$\varepsilon(\gamma_i^m) = \bar{\delta}_{i,0} \delta_{m,0} \quad \forall i \in \boldsymbol{Z}/n\boldsymbol{Z},\ 0 \leq m \leq d-1 \tag{2}$$

where $\delta_{i,j}$ is the usual Kronecker symbol, and $\bar{\delta}_{i,j}$ is the one modulo n for $i, j \in \boldsymbol{Z}/n\boldsymbol{Z}$. Clearly ε is an algebra map. ■

The following facts can be obtained by direct calculations. We omit the details.

LEMMA 2.2. *For the Frobenius algebra $A = \boldsymbol{K}Z_n/J^d$ with isomorphism Φ given as above, we have*

(i) *The Frobenius homomorphism is $\phi = \sum_{u=0}^{n-1}(\gamma_u^{d-1})^*$.*
(ii) *The Frobenius element is $\sum_{i=0}^{n-1}\sum_{m=0}^{d-1}\gamma_i^m \otimes \gamma_{i+m}^{d-1-m}$.*
(iii) *The space of left and right integrals are $\boldsymbol{K}\gamma_{1-d}^{d-1}$ and $\boldsymbol{K}\gamma_0^{d-1}$, respectively. Hence kZ_n/J^d is unimodular if and only if $d \equiv 1 \pmod{n}$.*
(iv) *The right modular function $\alpha : A \to \boldsymbol{K}$ is given by $\alpha(\gamma_i^m) = \bar{\delta}_{i,d-1}\delta_{m,0}$.*
(v) *The Nakayama automorphism $N : A \to A$ is given by $N(\gamma_i^m) = \gamma_{i-(d-1)}^m$. Thus the order of N is exactly $n/(d-1, n)$, where $(d-1, n)$ is the greatest common divisor.*

THEOREM 2.3. *The Frobenius algebra $\boldsymbol{K}Z_n/J^d$ is symmetric if and only if $d \equiv 1 \pmod{n}$.*

PROOF. If $\boldsymbol{K}Z_n/J^d$ is symmetric, then $\boldsymbol{K}Z_n/J^d$ is unimodular, and hence $d \equiv 1 \pmod{n}$, by Lemma 2.2(iii). If $d \equiv 1 \pmod{n}$, then $N(\gamma_i^m) = \gamma_i^m$ by Lemma 2.2(v), and hence $\phi a = a\phi$ for $a \in \boldsymbol{K}Z_n/J^d$. It follows that $\boldsymbol{K}Z_n/J^d$ is symmetric. ■

REMARK 2.4. By [CHYZ], the Frobenius algebras and the symmetric algebras constructed above are all possible connected monomial Frobenius algebras and monomial symmetric algebras, respectively.

3. A Class of Bi-Frobenius Algebras

DEFINITION 3.1 ([DT]). Let A be a finite-dimensional algebra and coalgebra with $t \in A$ and $\phi \in A^*$. Suppose that

(i) the counit ε is an algebra map and 1_A is a group-like element;
(ii) (A, ϕ) is a Frobenius algebra, and (A, t) is a Frobenius coalgebra with comultiplication $\triangle$;
(iii) The linear map $\psi : A \to A$, given by

$$\psi(a) = \sum \phi(t_1 a) t_2 \tag{3}$$

for all $a \in A$, is an anti-algebra map as well as an anti-coalgebra map, where $\triangle(t) = \sum t_1 \otimes t_2$.

Then the quadruple (A, ϕ, t, ψ) is called a bi-Frobenius algebra, and the map ψ is called the antipode of A.

Now we attach a Frobenius coalgebra structure to $\boldsymbol{K}Z_n/J^d$.

LEMMA 3.2. *The quadruple* $(\boldsymbol{K}Z_n/J^d, t, \triangle, \varepsilon)$ *is a cocommutative Frobenius coalgebra, where*

$$t = \gamma_0^{d-1} \tag{4}$$

and

$$\triangle(\gamma_i^m) = \sum_{p+q=i,\, l+s=m} \gamma_p^l \otimes \gamma_q^s, \quad \forall i, p, q \in \boldsymbol{Z}/n\boldsymbol{Z},\ 0 \le m, l, s \le d-1, \tag{5}$$

and the counit ε *is defined as in* (2).

PROOF. By a routine verification one sees that $(\boldsymbol{K}Z_n/J^d, \triangle, \varepsilon)$ is a coalgebra. The Frobeniusness is proved in Theorem 3.3 below. ■

THEOREM 3.3. *The quadruple* $(\boldsymbol{K}Z_n/J^d, \phi, t, \psi)$ *is a bi-Frobenius algebra but not a Hopf algebra, with* t, ϕ, ψ *defined as above.*

PROOF. It is easy to check the identity $1 = \sum_{i=0}^{n-1} \gamma_i^0$ is a group-like element. By Lemma 1.2 in [D], it remains to check that ψ is an anti-algebra and anti-coalgebra automorphism.

Since $\psi(a) = \sum \phi(t_1 a) t_2$, we have

$$\begin{aligned}
\psi(\gamma_i^m) &= \sum_{j=0}^{n-1} \sum_{l=0}^{d-1} \phi(\gamma_j^l \gamma_i^m) \gamma_{n-j}^{d-1-l} = \sum_{j=0}^{n-1} \sum_{l=0}^{d-1} \bar{\delta}_{j,i+m} \phi(\gamma_i^{l+m}) \gamma_{n-j}^{d-1-l} \\
&= \sum_{j=0}^{n-1} \sum_{l=0}^{d-1} \bar{\delta}_{j,i+m} \left(\sum_{u=0}^{n-1} (\gamma_u^{d-1})^* \right) (\gamma_i^{l+m}) \gamma_{n-j}^{d-1-l} \\
&= \bar{\delta}_{j,i+m} \delta_{d-1,l+m} \gamma_{n-j}^{d-1-l} \\
&= \gamma_{-i-m}^m.
\end{aligned}$$

It is clear that ψ is bijective.

Obviously, $\psi(\sum_{i=0}^{n-1}\gamma_i^0) = \sum_{i=0}^{n-1}\gamma_i^0$, and $\varepsilon \circ \psi = \varepsilon$. Note that

$$\psi(\gamma_i^m\gamma_j^l) = \bar{\delta}_{i,j+l}\psi(\gamma_j^{m+l}) = \bar{\delta}_{i,j+l}\gamma_{-j-m-l}^{m+l}.$$

On the other hand,

$$\psi(\gamma_j^l)\psi(\gamma_i^m) = \gamma_{-j-l}^l\gamma_{-i-m}^m = \bar{\delta}_{-j-l,-i}\gamma_{-i-m}^{l+m} = \bar{\delta}_{j+l,i}\gamma_{-j-l-m}^{l+m}.$$

So ψ is an anti-algebra automorphism.

Let τ be the twist map. Then we have

$$\tau \circ \triangle(\psi(\gamma_i^m)) = \tau \circ \triangle(\gamma_{-i-m}^m) = \sum_{\substack{p+q=-i-m\\ l+s=m}} \gamma_p^l \otimes \gamma_q^s$$

and

$$\begin{aligned}(\psi\otimes\psi)(\triangle(\gamma_i^m)) &= (\psi\otimes\psi)\left(\sum_{\substack{u+v=i\\ t+r=m}} \gamma_u^t \otimes \gamma_v^r\right)\\ &= \sum_{\substack{u+v=i\\ t+r=m}} \gamma_{-u-t}^t \otimes \gamma_{-v-r}^r\\ &= \sum_{\substack{-u-t-v-r=-i-m\\ t+r=m}} \gamma_{-u-t}^t \otimes \gamma_{-v-r}^r.\end{aligned}$$

This means $\tau \circ \triangle\psi = (\psi\otimes\psi)\triangle$, i.e., ψ is an anti-coalgebra automorphism.

Since $\triangle$ is not an algebra map, it follows that it is not a Hopf algebra. ■

REMARKS 3.4.

(i) The order of ψ is 2 since the coalgebra is cocommutative.
(ii) There may be other comultiplication on KZ_n/J^d. In fact we can endow a non-cocommutative comultiplication to KZ_2/J^2 as follows

$$\triangle(e_0) = e_0\otimes e_0 + e_1\otimes e_1$$

$$\triangle(e_1) = e_1\otimes e_0 + e_0\otimes e_1$$

$$\triangle(a_0) = e_0\otimes a_0 + e_1\otimes a_1 + a_0\otimes e_0 - a_1\otimes e_1$$

$$\triangle(a_1) = e_1\otimes a_0 + e_0\otimes a_1 + a_1\otimes e_0 - a_0\otimes e_1$$

This is exactly Sweedler's 4-dimensional Hopf algebra

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx + xg = 0, \triangle(g) = g \otimes g, \triangle(x) = g \otimes x + x \otimes 1 \rangle,$$

via

$$1 = e_0 + e_1, \quad g = e_0 - e_1, \quad x = a_1 - a_0.$$

(iii) Actually, if $d \nmid n$, the algebra $\boldsymbol{K} Z_n / J^d$ can never become a Hopf algebra, with any comultiplication, see [CHYZ] for detail.

(iv) Dually, we may obtain a class of bi-Frobenius algebras starting from path coalgebras.

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