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A WEIGHTED POINTWISE ESTIMATE FOR TWO DIMENSIONAL WAVE EQUATIONS AND ITS APPLICATION TO NONLINEAR SYSTEMS

Dedicated to professor Kunihiko Kajitani on his sixtieth birth day

By

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Abstract. This paper is devoted to some weighted pointwise estimate for solutions of two dimensional wave equations. It gives a simple proof to obtain the best order of the lower bound of the lifespan of classical solutions to nonlinear systems.

1. Introduction

We are first concerned with pointwise estimates of a classical solution of the following initial value problem for inhomogeneous wave equations in low space dimensions.

$$\begin{cases} \square u = H & \text{in } \mathbf{R}^n \times [0, \infty), \\ u|_{t=0} = f,\ u_t|_{t=0} = g, \end{cases} \tag{1.1}$$

where $\square$ is an usual D'Alembertian in $\mathbf{R}^n$ and f, g are given smooth functions of compact support in $\mathbf{R}^n$. $H = H(x,t)$ is a smooth function in $\mathbf{R}^n \times [0, \infty)$ whose support is admissible to the initial data. Our attention goes to pointwise estimates, so that we consider the case of $n = 2, 3$ only, in which a fundamental solution of $\square$ is positive.

Most of weighted L^∞ estimates of a solutions of (1.1) are global type. Actually, one can prove

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$$\|wu\|_{L^\infty(\mathbf{R}^n\times[0,\infty))} \le C_{f,g} + C\|w^pH\|_{L^\infty(\mathbf{R}^n\times[0,\infty))} \tag{1.2}$$

for a suitable weight w and some power p, where $C_{f,g}$ is a positive constant depending on the initial data. This inequality is often applied to nonlinear problem. For example, putting $H = |u|^{p-1}u$, or $|u|^p$ $(p \ge 2)$, we have a global existence of a classical solution of (1.1) with small initial data by (1.2) together with a contraction mapping argument, namely, a constructive method. See early works on this problem, F. John [6] and R. T. Glassey [4]. In this sense, the global estimate (1.2) is enough.

However, in order to investigate the lower bound of the lifespan of the solution when a blow-up occurs in the nonlinear problem, more precise estimate will be required rather than (1.2). To see this, we put a small parameter $\varepsilon > 0$ into the initial data, i.e.

$$f(x) = \varepsilon\varphi(x), \quad g(x) = \varepsilon\psi(x) \tag{1.3}$$

for arbitrarily fixed functions φ, ψ. Then we have a similar estimate to (1.2) such that

$$\|wu\|_{L^\infty(\mathbf{R}^n\times[0,T])} \le C_{\varphi,\psi}\varepsilon + C\|w^pH\|_{L^\infty(\mathbf{R}^n\times[0,T])}F(T,p), \tag{1.4}$$

where F is a function of p and a time T. But we have to improve this. Because a function space in which we will make a contraction has a bad order of ε such as

$$\|wu\|_{L^\infty(\mathbf{R}^n\times[0,T])} = O(\varepsilon). \tag{1.5}$$

Hence the local in time existence of a solution is guaranteed by

$$\varepsilon^{p-1}F(T,p) \ll 1 \tag{1.6}$$

in the contraction. Unfortunately, this condition does not make a optimal order of ε of the lifespan, sup T. For example, see R. Agemi & H. Takamura [2] for $n = 2$.

In three space dimensions, $n = 3$, one can improve (1.4) easily by making use of strong Huygens' principle. Actually, there is a space-time domain A in which we find $C_{\varphi,\psi} \equiv 0$ by compactness of the support of the initial data. Therefore we have new estimates

$$\begin{aligned} \|wu\|_{L^\infty(\bar{A})} &\le C\{\|w^pH\|_{L^\infty(\bar{A})}F(T,p) + \|w^pH\|_{L^\infty(\bar{B})}\}, \\ \|wu\|_{L^\infty(\bar{B})} &\le C_{\varphi,\psi}\varepsilon + C\|w^pH\|_{L^\infty(\bar{B})}G(T,p), \end{aligned} \tag{1.7}$$

where $B = \mathbf{R}^3 \times [0, T] \setminus A$ and F, G are functions of T, p. Then the good orders of ε in function spaces are obtained such as

$$\|wu\|_{L^\infty(\bar{A})} = O(\varepsilon^p), \quad \|wu\|_{L^\infty(\bar{B})} = O(\varepsilon). \tag{1.8}$$

Hence the local in time existence of a solution is guaranteed by

$$\varepsilon^{p(p-1)} F(T, p), \ \varepsilon^{p-1} G(T, p) \ll 1 \tag{1.9}$$

in the contraction. The first quantity is always bigger than the second one, and makes the optimal lifespan. See F. John [6] for a quadratic nonlinearity, or Zhou Yi [10] and R. Agemi & Y. Kurokawa & H. Takamura [1] for any power.

On the contrary, it is hard to prove a suitable weighted L^∞ estimate in two space dimensions like (1.7) by lack of the strong Huygens' principle. However, other clever proofs overcome the difficulity, which can be found in H. Lindblad [9] and Zhou Yi [11]. They made pointwise estimates and extended a local in time solution to the longest time by continuation principle, namely the contradiction argument. This method is sharper than constructive one. The lower and upper bounds of the lifespan coincide with each other and can be written by known quantity when ε goes to 0. But it works only for the sub-critical case of the nonlinear problem because a scaling argument is essential. For the critical case, the optimal lower bound of the lifespan has been obtained. We also remark that the pointwise estimate with some special function is required in two space dimensions.

The aim of the present paper is to prove some weighted pointwise estimate in two space dimensions by almost the same way as the three dimensional case. Moreover, as an application, we easily obtain the weighted L^∞ estimates like (1.7). This gives us the two dimensional version of R. Agemi & Y. Kurokawa & H. Takamura [1] in which the lifespan of a classical solution to the initial value problem for

$$\begin{cases} \Box u = |v|^p, \\ \Box v = |u|^q \end{cases} \tag{1.10}$$

is precisely estimated from below and above in three space dimensions. This system was first studied by D. Del Santo & V. Georgiev & E. Mitidieri [3].

This paper is organized as follows. In the next section, we state our main result. Section 3 is devoted to its proof which proceeds along with the method originally introduced by F. John [6]. In section 4, main result is applied to nonlinear systems. From section 5 to section 8, we get the lower bound of the lifespan in a very simple way.

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2. Main Result

We shall consider the linear problem (1.1) with a scaled data (1.3). A key assumption on the present problem is

$$\operatorname{supp} \varphi, \operatorname{supp} \psi \subset \{x \in \mathbf{R}^2; |x| \le k\}. \tag{2.1}$$

Without loss of generality, we may assume that $k > 1$. By finite propagation speed of the wave, the admissible support of H is

$$\operatorname{supp} H \subset \{(x,t) \in \mathbf{R}^2 \times [0,\infty); |x| \le t + k\}. \tag{2.2}$$

The weight function w is defined by

$$w(|x|,t) = \left(\frac{t+|x|+2k}{k}\right)^{1/2} N(t-|x|), \tag{2.3}$$

where $N(s)$ is a given function of s.

Our main result is the following theorem.

THEOREM 1. *Let u be a classical solution of the initial value problem* (1.1) *in* $\mathbf{R}^2 \times [0,T]$ *with a scaled data* (1.3) *under the assumptions on the compactness of support of the initial data,* (2.1), *and* H, (2.2). *Then, for* $0 < \varepsilon < 1$ *and* $p \in \mathbf{R}$, *there exist positive constants* $C_{\varphi,\psi}$ *depending on the initial data not on* ε, *and* C *independent of* k, ε *such that the following inequality holds for any* $(x,t) \in \mathbf{R}^2 \times [0,T]$.

$$\sqrt{\frac{t+|x|+2k}{k}}\sqrt{\frac{t-|x|+2k}{k}}|u(x,t)| \le C_{\varphi,\psi}\varepsilon + CkI(|x|,t), \tag{2.4}$$

where, when $p > 3$,

$$I(r,t) = \int_{-k}^{t-r}\left(1+\sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right)\frac{F_{(p-2)/2}(t-r,\beta)}{N(\beta)^p}\|w^p H\|(\beta)\,d\beta. \tag{2.5}$$

Here $F_P(b,a)$ *and* $\|\cdot\|$ *are defined by*

$$F_P(b,a) = \begin{cases} \left(\dfrac{a+2k}{k}\right)^{1-P} & \text{when } P > 1, \\ \log 2\dfrac{b+2k}{a+2k} & \text{when } P = 1, \\ \left(\dfrac{b+2k}{k}\right)^{1-P} & \text{when } P < 1 \end{cases} \tag{2.6}$$

and

$$\|u\|(t-|x|) = \sup_{t+|x|} \sup_{x/|x| \in S^1} |u(x,t)|. \tag{2.7}$$

For the case $p \le 3$, $F_{(p-2)/2}(t-r,\beta)$ *in* I *is replaced by* $G_p(t+r,t-r)$, *where*

$$G_p(\alpha,\beta) = \begin{cases} \left(\dfrac{\beta+2k}{k}\right)^{1/2} \log 2\dfrac{\alpha+2k}{\beta+2k} & \text{when } p = 3, \\ \left(\dfrac{\beta+2k}{k}\right)^{1/2} \left(\dfrac{\alpha+2k}{k}\right)^{(3-p)/2} & \text{when } p < 3. \end{cases} \tag{2.8}$$

REMARK 2.1. For $p > (3+\sqrt{17})/2$, a weighted L^∞ estimate in two space dimensions,

$$\|wu\|_{L^\infty(\mathbf{R}^2\times[0,\infty))} \le C_{\varphi,\psi}\varepsilon + Ck^2\|w^pH\|_{L^\infty(\mathbf{R}^2\times[0,\infty))} \tag{2.9}$$

has been obtained by R. T. Glassey [4] with a suitable weight w. See also Appendix of R. Agemi & H. Takamura [2] for a simplified proof. This estimate is enough for the global existence for nonlinear problem.

3. Proof of the Main Theorem

First, we shall follow some basic facts of a representation formula of a solution of (1.1) with (1.3) which has to satisfy the following integral equation.

$$u(x,t) = \varepsilon u^0(x,t) + L(H)(x,t), \tag{3.1}$$

where u^0 is a solution of $\square u^0 = 0$ with the same initial data to u/ε. Moreover, $L(H)$ is a solution of $\square L(H) = H$ with zero data. In two dimensional case, u^0 and $L(H)$ can be expressed by

$$u^0(x,t) = \frac{\partial}{\partial t}R(\varphi\,|\,x,t) + R(\psi\,|\,x,t), \quad L(H)(x,t) = \int_0^t R(H(\cdot,\tau)\,|\,x,t-\tau)\,d\tau, \tag{3.2}$$

where a function R is defined by

$$R(\psi \mid x,t) = \frac{1}{2\pi}\int_0^t \frac{\rho\, d\rho}{\sqrt{t^2-\rho^2}}\int_{|\omega|=1} \psi(x+\rho\omega)\, dS_\omega. \tag{3.3}$$

Applying this formula to $L(H)$ in our problem, we note that the weight function is spherically symmetric. So the method of the spherical mean developed by F. John [5] gives us the following useful formula.

$$|L(H)(x,t)| \le \frac{2}{\pi}\int_0^t d\tau \int_0^{t-\tau} \frac{\rho\, d\rho}{\sqrt{(t-\tau)^2-\rho^2}}\int_{|r-\rho|}^{r+\rho} \frac{\|w^pH\|(\tau-\lambda)\lambda w^{-p}(\lambda,\tau)\, d\lambda}{\sqrt{\lambda^2-(r-\rho)^2}\sqrt{(r+\rho)^2-\lambda^2}}, \tag{3.4}$$

where $r=|x|$. Then, it follows from inverting the order of (λ,ρ)-integral that

$$|L(H)(x,t)| \le \frac{2}{\pi}\left\{\int_0^t d\tau \int_{|t-\tau-r|}^{t-\tau+r} d\lambda \int_{|\lambda-r|}^{t-\tau} d\rho + \int_0^{(t-r)_+} d\tau \int_0^{t-\tau-r} d\lambda \int_{|\lambda-r|}^{\lambda+r} d\rho\right\}$$
$$\times \frac{\|w^pH\|(\tau-\lambda)\lambda w^{-p}(\lambda,\tau)\rho}{\sqrt{(t-\tau)^2-\rho^2}\sqrt{\rho^2-(\lambda-r)^2}\sqrt{(\lambda+r)^2-\rho^2}}. \tag{3.5}$$

After some estimates, one can find that ρ-integral will disappear by

$$\int_a^b \frac{\rho\, d\rho}{\sqrt{\rho^2-a^2}\sqrt{b^2-\rho^2}} = \frac{1}{2}B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\pi}{2}. \tag{3.6}$$

Introducing characteristic variables

$$\alpha = \tau+\lambda, \quad \beta = \tau-\lambda \tag{3.7}$$

and extending the domain of the integral, we have

$$|L(H)(x,t)| \le C(I_1(r,t)+I_2(r,t)), \tag{3.8}$$

where

$$I_1(r,t) = \int_{-k}^{t-r} \|w^pH\|(\beta)\, d\beta \int_{|t-r|}^{t+r} \frac{\lambda w(\lambda,\tau)^{-p}}{(\sqrt{r} \text{ or } \sqrt{\lambda})\sqrt{\alpha-(t-r)}}\, d\alpha \tag{3.9}$$

and

$$I_2(r,t) = \int_{-k}^{t-r} \|w^pH\|(\beta)\, d\beta \int_{\beta}^{(t-r)_+} \frac{\alpha w(\lambda,\tau)^{-p}}{(\sqrt{r} \text{ or } \sqrt{t-r-\beta})\sqrt{t-r-\alpha}}\, d\alpha. \tag{3.10}$$

All detailed proofs of inequalities above can be found in R. Agemi & H. Takamura [2]. Here we use a slightly modified form of I_2. The bottom of the α-integral is replaced by β instead of 0 which is made by different extension of the domain. The estimate for I_1 and I_2 will be divided into two cases. Here and hereafter, a constant C independent of ε and k may change from line to line.

Case $4r \geq t+r+2k$, *i.e.* $t+r+2k \geq 2(t-r+2k)$.

In this case, $\sqrt{r}$ must be taken in the definition of I_1, (3.9). Then it follows from the definition of w, (2.3), that

$$I_1(r,t) \leq \frac{k}{\sqrt{r}}\int_{-k}^{t-r} \|w^p H\|(\beta)N(\beta)^{-p}\,d\beta \int_{t-r}^{t+r} \frac{((\alpha+2k)/k)^{(2-p)/2}}{\sqrt{\alpha-(t-r)}}\,d\alpha. \tag{3.11}$$

With the help of integration by parts, the α-integral is dominated by

$$C\sqrt{r}\left(\frac{t+r+2k}{k}\right)^{(2-p)/2} + \frac{C}{\sqrt{k}}\int_{t-r}^{t+r}\left(\frac{\alpha+2k}{k}\right)^{(1-p)/2} d\alpha$$

$$\leq C\sqrt{k}\begin{cases} \left(\dfrac{t-r+2k}{k}\right)^{(3-p)/2} & \text{when } p>3, \\ \log 2\dfrac{t+r+2k}{t-r+2k} & \text{when } p=3, \\ \left(\dfrac{t+r+2k}{k}\right)^{(3-p)/2} & \text{when } p<3. \end{cases} \tag{3.12}$$

Hence, by simple inequality

$$\sqrt{r} \geq \frac{\sqrt{k}}{2}\left(\frac{t+r+2k}{k}\right)^{1/2}, \tag{3.13}$$

we obtain, when $p>3$,

$$\sqrt{\frac{t+r+2k}{k}}\sqrt{\frac{t-r+2k}{k}}I_1(r,t) \leq Ck\int_{-k}^{t-r}\frac{F_{(p-2)/2}(t-r,\beta)}{N(\beta)^p}\|w^p H\|(\beta)\,d\beta, \tag{3.14}$$

Because we have

$$\left(\frac{t-r+2k}{k}\right)^{(4-p)/2} \leq F_{(p-2)/2}(t-r,\beta). \tag{3.15}$$

It is clear that, when $p \leq 3$, $F_{(p-2)/2}(t-r,\beta)$ is replaced by $G_p(t+r,t-r)$.

Similarly to I_1, $\sqrt{r}$ is taken in the definition of I_2, (3.10). Here we have to assume that $t-r \geq 0$. Then we get

$$I_2(r,t) \leq \frac{k}{\sqrt{r}}\int_{-k}^{t-r} \|w^p H\|(\beta)N(\beta)^{-p}\,d\beta \int_{\beta}^{t-r} \frac{((\alpha+2k)/k)^{(2-p)/2}}{\sqrt{t-r-\alpha}}\,d\alpha. \tag{3.16}$$

In order to investigate the α-integral, we need the following inequality.

LEMMA 3.1. *For $P \in \mathbf{R}$, $b > 0$ and $b \geq a \geq -k$, it holds that*

$$J \equiv \int_a^b \sqrt{\frac{b+2k}{b-\alpha}} \left(\frac{\alpha+2k}{k}\right)^{-P} d\alpha \leq CkF_P(b,a), \tag{3.17}$$

where F_P is the one in (2.6).

PROOF. When $b \geq a \geq b/2 - k$, we have

$$\frac{J}{\sqrt{b+2k}} \leq C\left(\frac{b+2k}{k}\right)^{-P} \int_{b/2-k}^b \frac{d\alpha}{\sqrt{b-\alpha}} \tag{3.18}$$

which implies

$$J \leq Ck\left(\frac{b+2k}{k}\right)^{1-P} \leq CkF_P(b,a). \tag{3.19}$$

When $b/2 - k \geq a$, we have

$$\frac{J}{\sqrt{b+2k}} \leq \frac{1}{\sqrt{b/2+k}} \int_a^{b/2-k} \left(\frac{\alpha+2k}{k}\right)^{-P} d\alpha + C\left(\frac{b+2k}{k}\right)^{-P} \int_{b/2-k}^b \frac{d\alpha}{\sqrt{b-\alpha}} \tag{3.20}$$

which implies

$$J \leq Ck\left\{F_P(b/2-k,a) + \left(\frac{b+2k}{k}\right)^{1-P}\right\} \leq CkF_P(b,a). \tag{3.21}$$

Therefore the lemma follows.

Applying Lemma 3.1 to our case with $P = (p-2)/2$, $b = t - r$, $a = \beta$, we can find that the α-integral is dominated by

$$\frac{CkF_{(p-2)/2}(t-r,\beta)}{\sqrt{t-r+2k}}. \tag{3.22}$$

Hence it follows from (3.13) that

$$\sqrt{\frac{t+r+2k}{k}}\sqrt{\frac{t-r+2k}{k}} I_2(r,t) \leq Ck \int_{-k}^{t-r} \frac{F_{(p-2)/2}(t-r,\beta)}{N(\beta)^p} \|w^p H\|(\beta)\, d\beta. \tag{3.23}$$

Case 2. $4r \leq t + r + 2k$, *i.e.* $t + r + 2k \leq 2(t - r + 2k)$.

In this case, $\sqrt{\lambda}$ must be taken in (3.9). Then we have

$$I_1(r,t) \leq \sqrt{k} \int_{-k}^{t-r} \|w^p H\|(\beta) N(\beta)^{-p}\, d\beta \int_{t-r}^{t+r} \frac{((\alpha+2k)/k)^{(1-p)/2}}{\sqrt{\alpha-(t-r)}}\, d\alpha. \tag{3.24}$$

The α-integral is dominated by

$$C\left(\frac{t-r+2k}{k}\right)^{(1-p)/2}\int_{t-r}^{t+r}\frac{d\alpha}{\sqrt{\alpha-(t-r)}}\leq C\sqrt{k}\left(\frac{t-r+2k}{k}\right)^{(2-p)/2} \tag{3.25}$$

because $2r < t-r+2k$. This implies (3.14). Similarly, taking $\sqrt{t-r-\beta}$ in (3.10), we have

$$I_2(r,t)\leq k\int_{-k}^{t-r}\frac{\|w^pH\|(\beta)}{\sqrt{t-r-\beta}}N(\beta)^{-p}\,d\beta\int_{\beta}^{t-r}\frac{((\alpha+2k)/k)^{(2-p)/2}}{\sqrt{t-r-\alpha}}\,d\alpha. \tag{3.26}$$

The α-integral is estimated in completely the same way of previous case. Hence we have

$$\sqrt{\frac{t+r+2k}{k}}\sqrt{\frac{t-r+2k}{k}}I_2(r,t)\leq Ck\int_{-k}^{t-r}\frac{\|w^pH\|(\beta)}{\sqrt{t-r-\beta}}\frac{F_{(p-2)/2}(t-r,\beta)}{N(\beta)^p}\,d\beta. \tag{3.27}$$

We note that $F_{(p-2)/2}(t-r,\beta)\leq G_p(t+r,t-r)$ is always valid for $p\leq 3$.

As for u^0, we employ the well-known estimate.

Proposition 3.2. *Let u^0 be a solution of $\Box u^0=0$ in $\mathbf{R}^2\times[0,\infty)$ with an initial data $u|_{t=0}=\varphi$, $u_t|_{t=0}=\psi$ of compact support such as* (2.1). *Then there exists a positive constant $C_{\varphi,\psi}$ such that*

$$|u^0(x,t)|\leq\frac{C_{\varphi,\psi}k}{\sqrt{t+|x|+2k}\sqrt{t-|x|+2k}}. \tag{3.28}$$

Proof. For example, see R. T. Glassey [4].

Therefore the proof of Theorem 1 is finished.

4. Application to Nonlinear Systems

Now, as announced in Introduction, our attention goes to the following nonlinear systems.

$$\begin{cases}\Box u=|v|^p,\ \Box v=|u|^q, & \text{in } \mathbf{R}^2\times[0,\infty),\\ u(x,0)=\varepsilon f_1(x),\ (\partial u/\partial t)(x,0)=\varepsilon g_1(x),\\ v(x,0)=\varepsilon f_2(x),\ (\partial v/\partial t)(x,0)=\varepsilon g_2(x)\end{cases} \tag{4.1}$$

We shall investigate the lifespan defined by

$$T(\varepsilon)=\sup\{T\in(0,\infty]:\text{ There exists a unique solution }(u,v)$$

$$\in\{C^2(\mathbf{R}^2\times[0,T])\}^2\text{ of (4.1).}\}. \tag{4.2}$$

The full histories on (4.1) including the general space dimensions can be found in the introduction of R. Agemi & Y. Kurokawa & H. Takamura [1]. Therefore we omit all of them and shall prove the existence part of the following theorem.

THEOREM 2. *Let $p, q > 3$. Suppose that both $f_i \in C_0^3(\mathbf{R}^2)$ and $g_i \in C_0^2(\mathbf{R}^2)$ do not identically vanish for each $i = 1, 2$. Then there exists a positive constant ε_0 such that, for any ε with $0 < \varepsilon \le \varepsilon_0$, the lifespan $T(\varepsilon)$ of the classical solution (u, v) of* (4.1) *satisfies*

$$T(\varepsilon) = \infty \tag{4.3}$$

provided $F(p, q) < 0$,

$$\exp(c\varepsilon^{-\min\{p(pq-1), q(pq-1)\}}) \le T(\varepsilon) \le \exp(C\varepsilon^{-\min\{p(pq-1), q(pq-1)\}}). \tag{4.4}$$

provided $F(p, q) = 0$ with $p \neq q$,

$$\exp(c\varepsilon^{-p(p-1)}) \le T(\varepsilon) \le \exp(C\varepsilon^{-p(p-1)}) \tag{4.5}$$

provided $F(p, q) = 0$ with $p = q$, and

$$c\varepsilon^{-F(p,q)^{-1}} \le T(\varepsilon) \le C\varepsilon^{-F(p,q)^{-1}} \tag{4.6}$$

provided $F(p, q) > 0$, where c and C are positive constants independent of ε, and

$$F(p, q) \equiv \max\left\{\frac{p + 2 + q^{-1}}{pq - 1}, \frac{q + 2 + p^{-1}}{pq - 1}\right\} - \frac{1}{2}. \tag{4.7}$$

For the upper bound, the positivity on the initial data is required which makes the spherical mean of the solution to be positive.

REMARK 4.1. H. Kubo & M. Ohta [8] have proved the upper bound of Theorem 2.

REMARK 4.2. According to the short time existence of a classical solution of (4.1), we must have the same result when $2 \le p, q \le 3$. But the technical difficulty will appear. Actually we cannot apply Theorem 1 directly. The proof will be much complicated if we can prove by similar estimate to Theorem 1. The interesting result on (4.1) comes from the neighborhood of a cusp $p = q$ on the critical curve $\{F(p, q) = 0\}$. So, in this sense, the case $2 \le p, q \le 3$ is not essential in Theorem 2. We shall discuss this case in another paper.

Now we shall estimate the lower bound of the lifespan by the following proposition.

PROPOSITION 4.3. *Under the same assumption as Theorem* 2, *there exists a positive constant* ε_0 *such that* (4.1) *admits a unique solution* $(u,v) \in \{C^2(\mathbf{R}^2 \times [0,T])\}^2$, *as far as* T *satisfies*

$$T \leq \begin{cases} \infty & \text{if } F(p,q) < 0, \\ \exp(c\varepsilon^{-\min\{p(pq-1),q(pq-1)\}}) & \text{if } F(p,q) = 0 \text{ with } p \neq q, \\ \exp(c\varepsilon^{-p(p-1)}) & \text{if } F(p,q) = 0 \text{ with } p = q, \\ c\varepsilon^{-F(p,q)^{-1}} & \text{if } F(p,q) > 0 \end{cases} \tag{4.8}$$

for $0 < \varepsilon \leq \varepsilon_0$ *and some positive constant* c *independent of* ε.

PROOF. We note that our system (4.1) has a symmetry on p,q and u,v. Hence it suffice to investigate only the case

$$p \leq q \quad \text{and} \quad F(p,q) = \frac{q+2+1/p}{pq-1} - \frac{1}{2}. \tag{4.9}$$

Without loss of generality, we may assume that, for $k > 1$,

$$\operatorname{supp} f_i, \operatorname{supp} g_i \subset \{x \in \mathbf{R}^2; |x| \leq k\} \quad (i = 1,2). \tag{4.10}$$

Then it follows from the dependence domain of the solution of (4.1) that

$$\operatorname{supp} u, \operatorname{supp} v \subset \{(x,t) \in \mathbf{R}^2 \times [0,T]; |x| \leq t+k\}. \tag{4.11}$$

This fact is established as an easy application of the single case as well as the uniqueness of the solution. See Appendix of F. John [7] for example. Hence we can adapt $|u|^q$ and $|v|^p$ to H in Theorem 1.

We shall define the function space X by

$$\begin{aligned} X = \{(u,v) \in \{C^2(\mathbf{R}^2 \times [0,T])\}^2 : \operatorname{supp}(u,v) \subset \{|x| \leq t+k\}, \\ \|(u,v)\|_X < \infty\}. \end{aligned} \tag{4.12}$$

Here we put

$$\|(u,v)\|_X = \sum_{|\alpha| \leq 2} (\|\tilde{w}\nabla_x^\alpha u\|_{L^\infty(\mathbf{R}^2 \times [0,T])} + \|\tilde{w}\nabla_x^\alpha v\|_{L^\infty(\mathbf{R}^2 \times [0,T])}), \tag{4.13}$$

where $\tilde{w}$ stands for suitable weight functions which will be defined later. Remark that $\partial u/\partial t$ and $\partial v/\partial t$ are expressed by $\nabla_x u$ and $\nabla_x v$ in view of the representation formula of the solution. So, it is sufficient to consider the spatial derivatives only. A classical solution (u,v) of (4.1) will be constructed by the iteration argument in X as a classical solution of the corresponding systems of integral equations. More precisely, a solution will be obtained by convergence in a closed subspace in X of a sequence $\{(u_n, v_n)\}_{\mathbf{N}}$ defined by

$$\begin{cases} u_n = u_0 + L(|v_{n-1}|^p), & u_0 = \varepsilon u^0, \\ v_n = v_0 + L(|u_{n-1}|^q), & v_0 = \varepsilon v^0, \end{cases} \tag{4.14}$$

where all u^0, v^0 and a operator L appeared in the integral representations of u and v. See the beginning of the previous section.

In order to apply Theorem 1 to (4.1), we have to define suitable weights w_1, w_2 for u, v, respectively;

$$\begin{aligned} w(r,t) &= \left(\frac{t+r+2k}{k}\right)^{1/2}\left(\frac{t-r+2k}{k}\right)^{1/2}, \\ w_i(r,t) &= \left(\frac{t+r+2k}{k}\right)^{1/2} N_i(t-r) \quad (i=1,2), \end{aligned} \tag{4.15}$$

where

$$N_1(s) = \begin{cases} \left(\frac{s+2k}{k}\right)^{1/2} & \text{when } p > 4, \\ \left(\frac{s+2k}{k}\right)^{1/2}\left(\log\frac{s+3k}{k}\right)^{-1} & \text{when } p = 4, \\ \left(\frac{s+2k}{k}\right)^{(p-3)/2} & \text{when } 3 < p < 4, \end{cases} \tag{4.16}$$

$$N_2(s) = \begin{cases} \left(\frac{s+2k}{k}\right)^{1/2} & \text{when } \mu \geq 1/2 \text{ except for } p = q = 4, \\ \left(\frac{s+2k}{k}\right)^{1/2}\left(\log\frac{s+3k}{k}\right)^{-1} & \text{when } p = q = 4, \\ \left(\frac{s+2k}{k}\right)^{\mu}\left(\log\frac{s+3k}{k}\right)^{\nu} & \text{when } F(p,q) = 0 \text{ with } p \neq q, \\ \left(\frac{s+2k}{k}\right)^{\mu} & \text{otherwise.} \end{cases} \tag{4.17}$$

Here we set

$$\begin{aligned} \mu &= \frac{q(p-3)}{2p} + \frac{(q-p)(pq-5)}{2p(pq-1)}, \\ \nu &= \frac{q(p-1)}{p(pq-1)}. \end{aligned} \tag{4.18}$$

REMARK 4.4. Note that $\mu > 0$ when $3 < p \leq q$ and that μ satisfies

$$1 - p\mu = p(q-1)F(p,q). \tag{4.19}$$

This relation gives us the fact that, in (p,q)-plane, a curve $\{F(p,q) = 0\}$ is under a curve $\{\mu = 1/2\}$ and over a curve $\{\mu = 0\}$. Moreover, one can find that two curves, $\{\mu = 1/2\}$ and $\{\mu = 0\}$, cross a line $\{p = q\}$ at $p = q = 4$ and $p = q = 3$, respectively. Therefore we don't have to consider the case $\mu \leq 0$. We also note that ν satisfies

$$1 - p\nu = \frac{q-1}{pq-1} > 0. \tag{4.20}$$

The proof of Proposition 4.3 will be devided into the following four cases.

5. Proof for $p \geq 4$

In this case, due to Remark 4.4, we have

$$\mu \geq 1/2 \quad \text{and} \quad F(p,q) < 0. \tag{5.1}$$

For the sake of simplicity, we shall put

$$\|u_n\| = \|w_1 u_n\|_{L^\infty(\mathbf{R}^n \times [0,T])}, \quad \|v_n\| = \|w_2 v_n\|_{L^\infty(\mathbf{R}^n \times [0,T])}. \tag{5.2}$$

First we define a closed subspace Y of X by

$$Y = \{(u,v) \in X : \|u\| \leq 2C_{f_1,g_1}\varepsilon, \|v\| \leq 2C_{f_2,g_2}\varepsilon\}. \tag{5.3}$$

Then we presumably assume that

$$\begin{aligned} \|u_{n-1}\| &\leq 2C_{f_1,g_1}\varepsilon, \\ \|v_{n-1}\| &\leq 2C_{f_2,g_2}\varepsilon. \end{aligned} \tag{5.4}$$

When Theorem 1 is applied to u, we put $w = w_1$, $\varphi = f_1$, $\psi = g_1$ and $N = N_2$, $w^p H = (w_2|v|)^p$ in the β-integral. Similary, when Theorem 1 is applied to v, we put $w = w_2$, $\varphi = f_2$, $\psi = g_2$ and $N = N_1$, $p = q$, $w^q H = (w_1|u|)^q$ in the β-integral. Then it follows from

$$F_{(p-2)/2}(t-r,\beta) = \begin{cases} \left(\dfrac{\beta + 2k}{k}\right)^{(4-p)/2} & \text{when } p > 4, \\ \log 2\dfrac{t-r+2k}{\beta+2k} & \text{when } p = 4 \end{cases} \tag{5.5}$$

that

$$\begin{aligned} w_1(r,t)|u_n(x,t)| &\le C_{f_1,g_1}\varepsilon + Ck(2C_{f_2,g_2}\varepsilon)^p I_p(r,t), \\ w_2(r,t)|v_n(x,t)| &\le C_{f_2,g_2}\varepsilon + Ck(2C_{f_1,g_1}\varepsilon)^q I_q(r,t), \end{aligned} \quad \text{in } \mathbf{R}^2 \times [0,T] \tag{5.6}$$

where

$$I_p(r,t) = \int_{-k}^{t-r} \left(1 + \sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right)\left(\frac{\beta+2k}{k}\right)^{2-p+p\delta} d\beta \tag{5.7}$$

because of

$$\frac{4-p}{2} - \frac{p}{2} = 2 - p. \tag{5.8}$$

Here $\delta < (p-3)/p$ appears only in the case $p = 4$ and comes from the logarithmic term in the weight. Hence applying Lemma 3.1 with $P = p - 2 - p\delta > 1$ to I_p. We obtain $I_p(r,t), I_q(r,t) \le Ck$, so that

$$\begin{aligned} \|u_n\| &\le 2C_{f_1,g_1}\varepsilon, \\ \|v_n\| &\le 2C_{f_2,g_2}\varepsilon \end{aligned} \tag{5.9}$$

if the following inequalities hold.

$$\begin{aligned} Ck^2(2C_{f_2,g_2}\varepsilon)^p &\le C_{f_1,g_1}\varepsilon, \\ Ck^2(2C_{f_1,g_1}\varepsilon)^q &\le C_{f_2,g_2}\varepsilon. \end{aligned} \tag{5.10}$$

Therefore the boundedness of a sequence $\{(u_n, v_n)\}_{\mathbf{N}}$ in Y is obtained for any $T > 0$ if ε satisfies (5.10). Because we can take

$$\begin{aligned} \|u_0\| &\le C_{f_1,g_1}\varepsilon, \\ \|v_0\| &\le C_{f_2,g_2}\varepsilon. \end{aligned} \tag{5.11}$$

Next we shall estimate the differences under (5.10). The iteration frame (4.14) gives us

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|L(|v_n|^p - |v_{n-1}|^p)\| \\ &\le p\|L(|v_n - v_{n-1}|(|v_n| + |v_{n-1}|)^{p-1})\|. \end{aligned} \tag{5.12}$$

Hölder's inequality for the norm $\|\cdot\|$ and above estimates yield that

$$\begin{aligned} \|u_{n+1} - u_n\| &\le pCk^2\|(|v_n| + |v_{n-1}|)^{(p-1)/p}|v_n - v_{n-1}|^{1/p}\|^p \\ &\le pCk^2(\|v_n\| + \|v_{n-1}\|)^{p-1}\|v_n - v_{n-1}\|. \end{aligned} \tag{5.13}$$

Similarly we get

$$\|v_{n+1} - v_n\| \leq qCk^2(\|u_n\| + \|u_{n-1}\|)^{q-1}\|u_n - u_{n-1}\|. \tag{5.14}$$

Therefore a convergence of $\{(u_n, v_n)\}$ follows from

$$\begin{cases} \|u_{n+1} - u_n\| \leq 2^{-1}\|u_{n-1} - u_{n-2}\| \\ \|v_{n+1} - v_n\| \leq 2^{-1}\|v_{n-1} - v_{n-2}\|, \end{cases} \tag{5.15}$$

provided

$$2^{p+q-2}pqC^2k^4(2C_{f_2,g_2}\varepsilon)^{p-1}(2C_{f_1,g_1}\varepsilon)^{q-1} \leq 2^{-1}. \tag{5.16}$$

(5.16) is always valid for sufficiently small ε while T can be arbitrarily. Hence we get a C^0 solution (u, v) in Y.

In order to establish the existence of a classical solution, we require the convergence of sequences $\{(\partial_{x_i}u_n, \partial_{x_i}v_n)\}, \{(\partial_{x_i}\partial_{x_j}u_n, \partial_{x_i}\partial_{x_j}v_n)\}$ in Y. At the same step in [1], trivial mistakes have appeared in the form of the derivatives of non-linearlities. But they are not essential, meaning that one can find a correct proof very easily, for example along with the original proof of F. John [6]. A key role in estimating of $\{(\partial_{x_i}u_n, \partial_{x_i}v_n)\}$ is the following.

$$\begin{aligned}(|s|^p)'|_{s=u_{n+1}} - (|s|^p)'|_{s=u_n} &= \int_0^{u_{n+1}} (|s|^p)''\, ds - \int_0^{u_n} (|s|^p)''\, ds \\ &= \int_{u_n}^{u_{n+1}} (|s|^p)''\, ds. \end{aligned} \tag{5.17}$$

Similary to C^0 solution with this argument, one can verify the existence of C^1 and C^2 solutions. Hence here and hereafter we shall check only the existence of C^0 solution in each cases.

6. Proof for $3 < p < 4$ and $\mu \geq 1/2$

In this case we note that

$$F(p,q) < 0 \quad \text{and} \quad q > 4. \tag{6.1}$$

See Remark 4.4 again. The condition $\mu \geq 1/2$ makes the restriction $q \geq 4$, but $q = 4$ is valid only when $p = 4$ under this condition. The case $p = q = 4$ is already investigated in the previous case. We shall put

$$\|u_n\|_{S_{11}} = \|wu_n\|_{L^\infty(S_{11})}, \quad \|u_n\|_{S_{12}} = \|w_1u_n\|_{L^\infty(S_{12})}. \tag{6.2}$$

First we define a closed subspace Y of X by

$$Y = \{(u,v) \in X : \|u\|_{S_{11}} \le 2C_{f_1,g_1}\varepsilon, \|u\|_{S_{12}} \le 2M_1\varepsilon^p, \|v\| \le 2C_{f_2,g_2}\varepsilon\}. \tag{6.3}$$

Then we presumably assume that

$$\begin{aligned} &\|u_{n-1}\|_{S_{11}} \le 2C_{f_1,g_1}\varepsilon, \\ &\|u_{n-1}\|_{S_{12}} \le 2M_1\varepsilon^p, \\ &\|v_{n-1}\| \le 2C_{f_2,g_2}\varepsilon, \end{aligned} \tag{6.4}$$

where domains S_{11} and S_{12} are defined by

$$\begin{aligned} S_{11} &= \{(x,t) \in \mathbf{R}^2 \times [0,T]; -k < t - |x| \le K_1\varepsilon^{-L_1}k\}, \\ S_{12} &= \{(x,t) \in \mathbf{R}^2 \times [0,T]; t - |x| \ge K_1\varepsilon^{-L_1}k\}. \end{aligned} \tag{6.5}$$

Here we put

$$\begin{aligned} K_1 &\le (2^{p-(p-4)/2}Ck^2C^p_{f_2,g_2}C^{-1}_{f_1,g_1})^{2/(p-4)}, \\ L_1 &= \frac{2(p-1)}{4-p}, \\ M_1 &= 2C_{f_1,g_1}K_1^{(p-4)/2} + 2Ck^2(2C_{f_2,g_2})^p. \end{aligned} \tag{6.6}$$

K_1 will be fixed later.

Now, we assume that

$$K_1\varepsilon^{-L_1} \ge 2. \tag{6.7}$$

Then it follows, similarly to the previous case, from Theorem 1 with

$$F_{(p-2)/2}(t-r,\beta) = \left(\frac{t-r+2k}{k}\right)^{(4-p)/2} \tag{6.8}$$

that

$$\begin{aligned} w_1(r,t)|u_n(x,t)| &\le C_{f_1,g_1}\varepsilon\left(\frac{t-r+2k}{k}\right)^{(p-4)/2} + Ck(2C_{f_2,g_2}\varepsilon)^pJ_{11}(r,t), \\ w_2(r,t)|v_n(x,t)| &\le C_{f_2,g_2}\varepsilon + Ck\{(2C_{f_1,g_1}\varepsilon)^qJ_{21}(r,t) + (2M_1\varepsilon^p)^qJ_{22}(r,t)\}, \end{aligned} \tag{6.9}$$

where

$$J_{11}(r,t) = \int_{-k}^{t-r}\left(1 + \sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right)\left(\frac{\beta+2k}{k}\right)^{-p/2} d\beta \tag{6.10}$$

and

$$J_{21}(r,t) = \int_{-k}^{t-r} \left(1 + \sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right) \left(\frac{\beta+2k}{k}\right)^{-q/2} F_{(q-2)/2}(t-r,\beta)\, d\beta,$$

$$J_{22}(r,t) = \int_{-k}^{t-r} \left(1 + \sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right) \left(\frac{\beta+2k}{k}\right)^{-q(p-3)/2} F_{(q-2)/2}(t-r,\beta)\, d\beta, \tag{6.11}$$

where we denote a characteristic function of a set S by χ_S. Applying Lemma 3.1 with $P = p/2 > 1$ to J_{11}, we have

$$J_{11}(r,t) \le Ck. \tag{6.12}$$

On the other hand, since

$$F_{(q-2)/2}(t-r,\beta) = \left(\frac{\beta+2k}{k}\right)^{(4-q)/2}, \tag{6.13}$$

we get

$$J_{21}(r,t) = \int_{-k}^{t-r} \left(1 + \sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right) \left(\frac{\beta+2k}{k}\right)^{2-q} d\beta \tag{6.14}$$

as in the previous case. Therefore Lemma 3.1 yields again that

$$J_{21}(r,t) \le Ck. \tag{6.15}$$

In order to estimate J_{22}, we need the following fact.

Remark 6.1. It follows from Remark 4.4 that

$$\frac{3-q}{2} + 1 - q\frac{p-3}{2} = -\mu + q(p-1)F(p,q). \tag{6.16}$$

This means that, in (p,q)-plane, a curve $\{1 - q(p-3)/2 = 0\}$ is over a curve $\{F(p,q) = 0\}$. Both two curves meet only on a line $\{p = q\}$. Because,

$$\mu + \frac{3-q}{2} = \frac{(p-q)(pq+1)}{p(pq-1)} \le 0 \tag{6.17}$$

holds by (4.18).

Due to Remark 6.1, one can find

$$J_{22}(r,t) = \int_{-k}^{t-r} \left(1 + \sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right) \left(\frac{\beta+2k}{k}\right)^{-1+q(p-1)F(p,q)} d\beta \tag{6.18}$$

because of $1/2 - \mu \le 0$. Hence we have

$$J_{22}(r,t) \le Ck. \tag{6.19}$$

Summing up three estimates, we obtain

$$\begin{aligned} \|u_n\|_{S_{11}} &\le 2C_{f_1,g_1}\varepsilon, \\ \|u_n\|_{S_{12}} &\le 2M_1\varepsilon^p, \\ \|v_n\| &\le 2C_{f_2,g_2}\varepsilon \end{aligned} \tag{6.20}$$

if the following inequalities hold.

$$\begin{aligned} &Ck^2(2C_{f_2,g_2}\varepsilon)^p\left(\frac{t-r+2k}{k}\right)^{(4-p)/2} \le C_{f_1,g_1}\varepsilon && \text{in } S_{11}, \\ &C_{f_1,g_1}(2K_1)^{(p-4)/2}\varepsilon^p + Ck^2(2C_{f_2,g_2}\varepsilon)^p \le 2M_1\varepsilon^p && \text{in } S_{12}, \\ &Ck^2\{(2C_{f_1,g_1}\varepsilon)^q + (2M_1\varepsilon^p)^q\} \le C_{f_2,g_2}\varepsilon && \text{in } \mathbf{R}^2 \times [0,T]. \end{aligned} \tag{6.21}$$

The first and second lines of (6.21) are always valid by definition of each constants in (6.6). Therefore the boundedness of a sequence $\{(u_n, v_n)\}_{\mathbf{N}}$ in this weighted L^∞ space is obtained for any $T > 0$ if ε satisfies the third line of (6.21) and (6.7). Because we can take

$$\begin{aligned} \|u_0\|_{S_{11}} &\le C_{f_1,g_1}\varepsilon, \\ \|u_0\|_{S_{12}} &\le M_1\varepsilon^p, \\ \|v_0\| &\le C_{f_2,g_2}\varepsilon. \end{aligned} \tag{6.22}$$

Next we shall estimate the differences under (6.21). Similarly to above case, by (6.5), (6.6) and (6.7), we have

$$\begin{aligned} \|u_{n+1} - u_n\|_{S_{11}} &\le 2^{p-1}pCk^2(2C_{f_2,g_2}\varepsilon)^{p-1}(2K_1)^{(4-p)/2}\varepsilon^{1-p}\|v_n - v_{n-1}\| \\ \|u_{n+1} - u_n\|_{S_{12}} &\le 2^{p-1}pCk^2(2C_{f_2,g_2}\varepsilon)^{p-1}\|v_n - v_{n-1}\| \\ \|v_{n+1} - v_n\| &\le 2^{q-1}qCk^2(2C_{f_1,g_1}\varepsilon)^{q-1}\|u_n - u_{n-1}\|_{S_{11}} \\ &\quad + 2^{q-1}qCk^2(2M_1\varepsilon^p)^{q-1}\|u_n - u_{n-1}\|_{S_{12}}. \end{aligned} \tag{6.23}$$

Therefore a convergence of $\{(u_n, v_n)\}$ follows from

$$\begin{cases} \|u_{n+1} - u_n\|_{S_{11}} \le 4^{-1}\|u_{n-1} - u_{n-2}\|_{S_{11}} + 4^{-1}\|u_{n-1} - u_{n-2}\|_{S_{12}} \\ \|u_{n+1} - u_n\|_{S_{12}} \le 4^{-1}\|u_{n-1} - u_{n-2}\|_{S_{11}} + 4^{-1}\|u_{n-1} - u_{n-2}\|_{S_{12}} \\ \|v_{n+1} - v_n\| \le 2^{-1}\|v_{n-1} - v_{n-2}\|, \end{cases} \tag{6.24}$$

provided all the following inequalities hold.

$$C\varepsilon^{q-1} \le 4^{-1}, \quad C\varepsilon^{p(q-1)} \le 4^{-1}$$
$$C\varepsilon^{p+q-2} \le 4^{-1}, \quad C\varepsilon^{pq-1} \le 4^{-1}. \tag{6.25}$$

In (6.25), $C = C(f_1, f_2, g_1, g_2, p, q, k) > 0$ may be a different constant from each other. In fact, we have by (6.24)

$$\|u_{n+1} - u_n\|_{S_{11}} + \|u_{n+1} - u_n\|_{S_{12}}$$
$$\le 2^{-1}\{\|u_{n-1} - u_{n-2}\|_{S_{11}} + \|u_{n-1} - u_{n-2}\|_{S_{12}}\}. \tag{6.26}$$

(6.25) is always valid for sufficiently small ε. In this case, the proof is completed by taking K_1 as

$$K_1 = (2^{p-(p-4)/2} C k^2 C_{f_2,g_2}^p C_{f_1,g_1}^{-1})^{2/(p-4)}. \tag{6.27}$$

7. Proof for $3 < p < 4$ and $0 < \mu < 1/2$ except for the Case $F(p,q) = 0$ with $p \neq q$

We shall put

$$\|v_n\|_{S_{21}} = \|w v_n\|_{L^\infty(S_{21})}, \quad \|v_n\|_{S_{22}} = \|w_2 v_n\|_{L^\infty(S_{22})}. \tag{7.1}$$

First we define a closed subset Y of X by

$$Y = \{(u,v) \in X : \|u\|_{S_{11}} \le 2C_{f_1,g_1}\varepsilon, \|u\|_{S_{12}} \le 2M_1\varepsilon^p,$$
$$\|v\|_{S_{21}} \le 2C_{f_2,g_2}\varepsilon, \|v\|_{S_{22}} \le 2M_2\varepsilon^q\}. \tag{7.2}$$

Then we presumably assume that

$$\|u_{n-1}\|_{S_{11}} \le 2C_{f_1,g_1}\varepsilon,$$
$$\|u_{n-1}\|_{S_{12}} \le 2M_1\varepsilon^p,$$
$$\|v_{n-1}\|_{S_{21}} \le 2C_{f_2,g_2}\varepsilon,$$
$$\|v_{n-1}\|_{S_{22}} \le 2M_2\varepsilon^q. \tag{7.3}$$

where M_1, S_{11}, S_{12} are defined in (6.6), (6.5) and

$$S_{21} = \{(x,t) \in \mathbf{R}^2 \times [0,T]; -k \le t - |x| \le K_2\varepsilon^{-L_2}k\},$$
$$S_{22} = \{(x,t) \in \mathbf{R}^2 \times [0,T]; t - |x| \ge K_2\varepsilon^{-L_2}k\}. \tag{7.4}$$

Here we put

$$
\begin{aligned}
K_2 &\le (2^{q-(\mu-1/2)}Ck^2C_{f_1,g_1}^{p}C_{f_2,g_2}^{-1})^{1/(\mu-1/2)},\\
L_2 &= \frac{q-1}{1/2-\mu},\\
M_2 &= 2C_{f_2,g_2}K_2^{\mu-1/2} + 2Ck^2(2C_{f_1,g_1})^{q}.
\end{aligned}
\tag{7.5}
$$

K_1, K_2 will be determined later. For the sake of simplicity, we put E_1, E_2 by

$$
\begin{aligned}
E_1(T) &\equiv F_{1-p(q-1)F(p,q)}(T,-k)\\
&= \begin{cases} 1 & \text{if } F(p,q)<0\\ \log 2\dfrac{T+2k}{k} & \text{in } F(p,q)=0,\ p=q\\ \left(\dfrac{T+2k}{k}\right)^{p(q-1)F(p,q)} & \text{if } F(p,q)>0, \end{cases}
\end{aligned}
\tag{7.6}
$$

$$
\begin{aligned}
E_2(T) &\equiv F_{1-q(p-1)F(p,q)}(T,-k)\\
&= \begin{cases} 1 & \text{if } F(p,q)<0\\ \log 2\dfrac{T+2k}{k} & \text{in } F(p,q)=0,\ p=q\\ \left(\dfrac{T+2k}{k}\right)^{q(p-1)F(p,q)} & \text{if } F(p,q)>0. \end{cases}
\end{aligned}
\tag{7.7}
$$

Now, we assume that

$$
\min_{i=1,2} K_i\varepsilon^{-L_i} \ge 2. \tag{7.8}
$$

Then it follows from Theorem 1 and (7.3) that

$$
\begin{aligned}
w_1(r,t)|u_n(x,t)| &\le C_{f_1,g_1}\varepsilon\left(\frac{t-r+2k}{k}\right)^{(p-4)/2}\\
&\quad + Ck\{(2C_{f_2,g_2}\varepsilon)^{p}J_{11}(r,t) + (2M_2\varepsilon^{q})^{p}J_{12}(r,t)\},\\
w_2(r,t)|v_n(x,t)| &\le \left(\frac{t-r+2k}{k}\right)^{\mu-1/2}\\
&\quad \times \{C_{f_2,g_2}\varepsilon + Ck(2C_{f_1,g_1}\varepsilon)^{q}J_{21}(r,t) + Ck(2M_1\varepsilon^{p})^{q}J_{22}(r,t)\},
\end{aligned}
\tag{7.9}
$$

where J_{11}, J_{21} and J_{22} are the one in (6.10) and (6.11). J_{12} is defined by

$$J_{12}(r,t) = \int_{-k}^{t-r} \left(1 + \sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right)\left(\frac{\beta+2k}{k}\right)^{-p\mu} d\beta. \tag{7.10}$$

Due to Remark 4.4 together with Lemma 3.1, we get

$$J_{12}(r,t) \le CkE_1(T). \tag{7.11}$$

J_{11} is estimated in the completely same way as in the previous case. Hence we obtain

$$J_{11}(r,t) \le Ck. \tag{7.12}$$

For J_{21} and J_{22}, we need to divide into the following three cases.

Case 1, $q > 4$.

In this case it follows from

$$F_{(q-2)/2}(t-r,\beta) = \left(\frac{\beta+2k}{k}\right)^{(4-q)/2} \tag{7.13}$$

that

$$\begin{aligned} J_{21}(r,t) &= \int_{-k}^{t-r} \left(1 + \sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right)\left(\frac{\beta+2k}{k}\right)^{(4-q)/2-q/2} d\beta, \\ J_{22}(r,t) &= \int_{-k}^{t-r} \left(1 + \sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right)\left(\frac{\beta+2k}{k}\right)^{(4-q)/2-q(p-3)/2} d\beta. \end{aligned} \tag{7.14}$$

Hence we simply obtain by Remark 6.1 and Lemma 3.1 that

$$J_{21}(r,t) \le Ck,$$

$$J_{22}(r,t) \le Ck \begin{cases} \left(\dfrac{t-r+2k}{k}\right)^{q(p-1)F(p,q)+1/2-\mu} & \text{when } F(p,q) > 0, \\ \left(\dfrac{t-r+2k}{k}\right)^{1/2-\mu} & \text{when } F(p,q) \le 0. \end{cases} \tag{7.15}$$

Case 2, $q = 4$.

In this case we have

$$F_{(q-2)/2}(t-r,\beta) = \log 2\frac{t-r+2k}{\beta+2k}. \tag{7.16}$$

Hence it follows from Lemma 3.1 that

$$\left(\frac{t-r+2k}{k}\right)^{\mu-1/2} J_{21}(r,t) \le Ck\left(\frac{t-r+2k}{k}\right)^{\mu-1/2} \log 2\frac{t-r+2k}{k} \le Ck \quad (7.17)$$

because of $0<\mu<1/2$.

When $1-q(p-3)/2>0$, one can find a small δ satisfying $1-q(p-3)/2-\delta>0$. Hence we have

$$J_{22}(r,t) \le C\left(\frac{t-r+2k}{k}\right)^{\delta} \int_{-k}^{t-r}\left(1+\sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right)\left(\frac{\beta+2k}{k}\right)^{-q(p-3)/2-\delta} d\beta \quad (7.18)$$

which implies that, by Remark 6.1,

$$\left(\frac{t-r+2k}{k}\right)^{\mu-1/2} J_{22}(r,t) \le Ck\left(\frac{t-r+2k}{k}\right)^{q(p-1)F(p,q)}. \quad (7.19)$$

When $1-q(p-3)/2\le 0$, we have

$$J_{22}(r,t) \le \log 2\frac{t-r+2k}{k}\int_{-k}^{t-r}\left(1+\sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right)\left(\frac{\beta+2k}{k}\right)^{-q(p-3)/2} d\beta. \quad (7.20)$$

Hence we obtain

$$\left(\frac{t-r+2k}{k}\right)^{\mu-1/2} J_{22}(r,t) \le Ck. \quad (7.21)$$

Case 3, $3<q<4$.

In this case we have

$$F_{(q-2)/2}(t-r,\beta) = \left(\frac{t-r+2k}{k}\right)^{(4-q)/2}. \quad (7.22)$$

Hence it follows from Lemma 3.1 that

$$\left(\frac{t-r+2k}{k}\right)^{\mu-1/2} J_{21}(r,t) \le Ck\left(\frac{t-r+2k}{k}\right)^{\mu+(3-q)/2} \le Ck \quad (7.23)$$

because Remark 6.1 implies that

$$\mu+\frac{3-q}{2}=\frac{(p-q)(pq+1)}{p(pq-1)}\le 0. \quad (7.24)$$

For J_{22} we get

$$J_{22}(r,t) \le \left(\frac{t-r+2k}{k}\right)^{(4-q)/2}\int_{-k}^{t-r}\left(1+\sqrt{\frac{t-r+2k}{t-r-\beta}}\right)\left(\frac{\beta+2k}{k}\right)^{-q(p-3)/2} d\beta. \quad (7.25)$$

When $1 - q(p-3) > 0$, Remark 6.1 yields that

$$\left(\frac{t-r+2k}{k}\right)^{\mu-1/2} J_{22}(r,t) \le Ck\left(\frac{t-r+2k}{k}\right)^{q(p-1)F(p,q)}. \tag{7.26}$$

When $1 - q(p-3) = 0$, we have

$$\left(\frac{t-r+2k}{k}\right)^{\mu-1/2} J_{22}(r,t) \le Ck\left(\frac{t-r+2k}{k}\right)^{\mu+(3-q)/2} \log\frac{t-r+2k}{k}. \tag{7.27}$$

Hence (7.24) implies that

$$\left(\frac{t-r+2k}{k}\right)^{\mu-1/2} J_{22}(r,t) \le Ck\begin{cases} 1 & \text{when } p \neq q, \\ \log\dfrac{T+2k}{k} & \text{when } p = q. \end{cases} \tag{7.28}$$

We note that $p = q$ in this case means $F(p,q) = 0$ by Remark 4.4 and Remark 6.1. When $1 - q(p-3)/2 < 0$, the situation is the same as J_{21}. We get

$$\left(\frac{t-r+2k}{k}\right)^{\mu-1/2} J_{22}(r,t) \le Ck. \tag{7.29}$$

Now, three cases above are combined and the following estimates hold.

$$\begin{aligned} &J_{11}(r,t) \le Ck, \\ &J_{12}(r,t) \le CkE_1(T), \\ &\left(\frac{t-r+2k}{k}\right)^{\mu-1/2} J_{21}(r,t) \le Ck, \\ &\left(\frac{t-r+2k}{k}\right)^{\mu-1/2} J_{22}(r,t) \le CkE_2(T). \end{aligned} \tag{7.30}$$

Therefore we obtain

$$\begin{aligned} \|u_n\|_{S_{11}} &\le 2C_{f_1,g_1}\varepsilon, \\ \|u_n\|_{S_{12}} &\le 2M_1\varepsilon^p, \\ \|v_n\|_{S_{21}} &\le 2C_{f_2,g_2}\varepsilon, \\ \|v_n\|_{S_{22}} &\le 2M_2\varepsilon^q \end{aligned} \tag{7.31}$$

if the following inequalities hold.

$$\begin{aligned}
&Ck^2\{(2C_{f_2,g_2}\varepsilon)^p + (2M_2\varepsilon^q)^p E_1(T)\}(2K_1\varepsilon^{-L_1})^{(4-p)/2} \le C_{f_1,g_1}\varepsilon,\\
&2Ck^2(2M_2\varepsilon^q)^p E_1(T) \le M_1\varepsilon^p\\
&Ck^2\{(2C_{f_1,g_1}\varepsilon)^q + (2M_1\varepsilon^p)^q E_2(T)\}(2K_2\varepsilon^{-L_2})^{1/2-\mu} \le C_{f_2,g_2}\varepsilon,\\
&2Ck^2(2M_1\varepsilon^p)^q E_2(T) \le M_2\varepsilon^q.
\end{aligned} \tag{7.32}$$

Hence the boundedness of a sequence $\{(u_n, v_n)\}_{\mathbf{N}}$ in this weighted L^∞ space is obtained if (7.8) and (7.32) hold. By definition of all constants in (6.6) and (7.5), one can see that (7.32) follows from

$$\begin{aligned}
&(2M_2)^p\varepsilon^{p(q-1)}E_1(T) \le \max\{(2C_{f_2,g_2})^p, (Ck^2)^{-1}M_1\},\\
&(2M_1)^q\varepsilon^{q(p-1)}E_2(T) \le \max\{(2C_{f_1,g_2})^q, (Ck^2)^{-1}M_2\}.
\end{aligned} \tag{7.33}$$

This proves Proposition 4.3 except for the case $F(p,q) = 0$ with $p \ne q$ because we can take

$$\begin{aligned}
&\|u_0\|_{S_{11}} \le C_{f_1,g_1}\varepsilon,\\
&\|u_0\|_{S_{12}} \le M_1\varepsilon^p,\\
&\|v_0\|_{S_{21}} \le C_{f_2,g_2}\varepsilon,\\
&\|v_0\|_{S_{22}} \le M_2\varepsilon^q.
\end{aligned} \tag{7.34}$$

Next we shall estimate the differences under (7.32). Similarly to the above case, by the definition of the devided domain and (7.8), we have

$$\begin{aligned}
\|u_{n+1} - u_n\|_{S_{11}} &\le 2^{p-1}(2K_1)^{(4-p)/2}\varepsilon^{1-p}\{pCk^2(2C_{f_2,g_2}\varepsilon)^{p-1}\|v_n - v_{n-1}\|_{S_{21}}\\
&\quad + pCk^2E_1(K_1\varepsilon^{-L_1}k)(2M_2\varepsilon^q)^{p-1}\|v_n - v_{n-1}\|_{S_{22}}\}\\
\|u_{n+1} - u_n\|_{S_{12}} &\le 2^{p-1}pCk^2(2C_{f_2,g_2}\varepsilon)^{p-1}\|v_n - v_{n-1}\|_{S_{21}}\\
&\quad + 2^{p-1}pCk^2E_1(T)(2M_2\varepsilon^q)^{p-1}\|v_n - v_{n-1}\|_{S_{22}}\\
\|v_{n+1} - v_n\|_{S_{21}} &\le 2^{q-1}(2K_2)^{1/2-\mu}\varepsilon^{1-q}\{qCk^2(2C_{f_1,g_1}\varepsilon)^{q-1}\|u_n - u_{n-1}\|_{S_{11}}\\
&\quad + qCk^2E_2(K_2\varepsilon^{-L_2}k)(2M_1\varepsilon^p)^{q-1}\|u_n - u_{n-1}\|_{S_{12}}\}\\
\|v_{n+1} - v_n\|_{S_{22}} &\le 2^{q-1}qCk^2(2C_{f_1,g_1}\varepsilon)^{q-1}\|u_n - u_{n-1}\|_{S_{11}}\\
&\quad + 2^{q-1}qCk^2E_2(T)(2M_1\varepsilon^p)^{q-1}\|u_n - u_{n-1}\|_{S_{12}}
\end{aligned} \tag{7.35}$$

Therefore a convergence of $\{(u_n, v_n)\}$ follows from (6.26) provided

$$\begin{aligned}
&2^{p+q-2}pqC^2k^4(2K_1)^{(4-p)/2}(2K_2)^{1/2-\mu}(2C_{f_2,g_2})^{p-1}(2C_{f_1,g_1})^{q-1} \\
&\qquad + C\varepsilon^{q-p+q(p-1)}E_1(K_1\varepsilon^{-L_1}k) \le 4^{-1} \\
&C\varepsilon^{1-q+p(q-1)}E_2(K_2\varepsilon^{-L_2}k) + C\varepsilon^{1-p+p(q-1)+q(p-1)}E_1(K_1\varepsilon^{-L_1}k)E_2(T) \le 4^{-1} \\
&C\varepsilon^{p-1} + C\varepsilon^{pq-1}E_1(T) \le 4^{-1} \\
&C\varepsilon^{p-q+p(q-1)}E_2(K_2\varepsilon^{-L_2}k) + C\varepsilon^{p(q-1)+q(p-1)}E_1(T)E_2(T) \le 4^{-1} \\
&2^{p+q-2}pqC^2k^4(2K_1)^{(4-p)/2}(2K_2)^{1/2-\mu}(2C_{f_2,g_2})^{p-1}(2C_{f_1,g_1})^{q-1} \\
&\qquad + C\varepsilon^{p-q+p(q-1)}E_2(K_2\varepsilon^{-L_2}k) \le 4^{-1} \\
&C\varepsilon^{1-p+q(p-1)}E_1(K_1\varepsilon^{-L_1}k) + C\varepsilon^{1-q+p(q-1)+q(p-1)}E_1(T)E_2(K_2\varepsilon^{-L_2}k) \le 4^{-1} \\
&C\varepsilon^{q-1} + C\varepsilon^{pq-1}E_1(T) \le 4^{-1} \\
&C\varepsilon^{q-p+q(p-1)}E_1(K_1\varepsilon^{-L_1}k) + C\varepsilon^{q(p-1)+p(q-1)}E_1(T)E_2(T) \le 4^{-1}.
\end{aligned} \tag{7.36}$$

Here $C = C(f_1, f_2, g_1, g_2, p, q, k) > 0$ may be a different constant from each other.

Now we can fix K_1 and K_2 to satisfy (6.6) and (7.5), so that the first term of the first and fifth inequalities in (7.36) is less than 8^{-1}. When $F(p,q) < 0$ or $F(p,q) = 0$ with $p = q$, the other inequalities in (7.36) are valid for a sufficiently small $\varepsilon > 0$. When $F(p,q) > 0$, they follow from the same requirement to (7.32) and the following Lemma with a sufficiently small ε. We note that all powers of ε of indistinct terms in the left-hand side of (7.36) are summarized in the following quantities.

$$\begin{aligned}
P_1 &\equiv 1 - p - L_1p(q-1)F(p,q) + q(p-1), \\
P_2 &\equiv 1 - q - L_2q(p-1)F(p,q) + q(p-1).
\end{aligned} \tag{7.37}$$

Lemma 7.1. *Let $3 < p < 4$, $F(p,q) > 0$ and $p \le q$. Then the following inequalities hold.*

$$P_1 > 0, \quad P_2 > 0. \tag{7.38}$$

Proof. First we prove the first inequality. By the definition of L_1 in (6.6), we see that

$$P_1 = (p-1)(q-1)\left\{1 - \frac{2pF(p,q)}{4-p}\right\}. \tag{7.39}$$

Then, it follows from another expression of $F(p,q)$ such as

$$F(p,q) = \frac{1}{p} + \frac{2(p+1)}{p(pq-1)} - \frac{1}{2} \tag{7.40}$$

that $\partial_q F(p,q) < 0$ is always valid for a fixed p. Making use of this fact together with

$$\begin{aligned}(p-1)(4-p-2pF(p,p)) &= (p-1)(4-p) + p^2 - 3p - 2 \\ &= 2(p-3) > 0,\end{aligned} \tag{7.41}$$

we have

$$F(p,q) \le F(p,p) < \frac{4-p}{2p}. \tag{7.42}$$

Therefore $P_1 > 0$ holds.

Next we prove the second inequality. Remark 4.4 implies that

$$1 - L_2 F(p,q) = \frac{p-2}{2p(1/2-\mu)}. \tag{7.43}$$

Hence we get

$$P_2 = \frac{q(p-1)(p-2) - p(q-1) + 2p(q-1)\mu}{2p(1/2-\mu)}. \tag{7.44}$$

We note that, when $p, q > 3$,

$$\begin{aligned}\mu &= \frac{q(p-3)}{2p} + \frac{q-p}{2p}\left(1 - \frac{4}{pq-1}\right) \\ &> \frac{q(p-3)}{2p} + \frac{q-p}{4p}.\end{aligned} \tag{7.45}$$

Then the numerator of P_2 in (7.44) is estimated from below in the following way.

$$\begin{aligned}&q(p-1)(p-2) - p(q-1) + q(p-3)(q-1) + \frac{(q-p)(q-1)}{2} \\ &> (p-3)q^2 + (p^2-5p+5)q + p + \frac{(q-p)(q-1)}{2} \\ &> (p-3)q^2 - q + p + \frac{(q-p)(q-1)}{2} \\ &= (p-3)q^2 + (q-p)\frac{q-3}{2}.\end{aligned} \tag{7.46}$$

The third line in (7.46) follows from the fact that $p^2 - 5p + 5$ is monotonously increasing in $3 < p < 4$. Therefore $P_2 > 0$ holds.

8. Proof for $F(p,q) = 0$ with $p \neq q$

In this case, it follows from Remark 4.4 that

$$3 < p < 4 \quad \text{and} \quad 0 < \mu = \frac{1}{p} < \frac{1}{2}. \tag{8.1}$$

A key fact is that the strict inequality in (7.24) holds here. We shall put

$$\|v_n\|_{S_1} = \|wv_n\|_{L^\infty(S_1)}, \quad \|v_n\|_{S_2} = \|w_2 v_n\|_{L^\infty(S_2)}. \tag{8.2}$$

First we define a closed subspace Y of X by

$$\begin{aligned} Y = \{(u,v) : \|u\|_{S_{11}} \le 2C_{f_1,g_1}\varepsilon, \|u\|_{S_{12}} \le 2M_1\varepsilon^p, \\ \|v\|_{S_1} \le 2C_{f_2,g_2}\varepsilon, \|v\|_{S_2} \le 2M\varepsilon^q\}. \end{aligned} \tag{8.3}$$

Then we presumably assume that

$$\begin{aligned} &\|u_{n-1}\|_{S_{11}} \le 2C_{f_1,g_1}\varepsilon, \\ &\|u_{n-1}\|_{S_{12}} \le 2M_1\varepsilon^p, \\ &\|v_{n-1}\|_{S_1} \le 2C_{f_2,g_2}\varepsilon, \\ &\|v_{n-1}\|_{S_2} \le 2M\varepsilon^q, \end{aligned} \tag{8.4}$$

where M_1, S_{11}, S_{12} are defined in (6.6), (6.5) and

$$\begin{aligned} S_1 &= \left\{(x,t) \in \mathbf{R}^2 \times [0,T]; \left(\frac{t-r+2k}{k}\right)^{\mu-1/2}\left(\log\frac{t-r+3k}{k}\right)^{\nu} \ge K\varepsilon^{q-1}\right\}, \\ S_2 &= \left\{(x,t) \in \mathbf{R}^2 \times [0,T]; \left(\frac{t-r+2k}{k}\right)^{\mu-1/2}\left(\log\frac{t-r+3k}{k}\right)^{\nu} \le K\varepsilon^{q-1}\right\}. \end{aligned} \tag{8.5}$$

Here we put

$$\begin{aligned} K &\le 2Ck^2(2C_{f_1,g_1})^q C_{f_2,g_2}^{-1}, \\ M &= C_{f_2,g_2}K + Ck^2(2C_{f_1,g_1})^q. \end{aligned} \tag{8.6}$$

K_1, K will be determined later.

Now, we assume that

$$K_1\varepsilon^{-L_1} \ge 3, \quad (\log 2)^{\nu} \ge K\varepsilon^{q-1}. \tag{8.7}$$

Then it follows from Theorem 1 and (8.4) that

$$\begin{aligned} w_1(r,t)|u_n(x,t)| &\le C_{f_1,g_1}\varepsilon\left(\frac{t-r+2k}{k}\right)^{(p-4)/2} \\ &\quad + Ck\{(2C_{f_2,g_2}\varepsilon)^p J_{11}(r,t) + (2M\varepsilon^q)^p J_1(r,t)\}, \\ w_2(r,t)|v_n(x,t)| &\le \left(\frac{t-r+2k}{k}\right)^{\mu-1/2}\left(\log\frac{t-r+3k}{k}\right)^{\nu} \\ &\quad \times \{C_{f_2,g_2}\varepsilon + Ck(2C_{f_1,g_1}\varepsilon)^q J_{21}(r,t) \\ &\quad + Ck(2M_1\varepsilon^p)^q J_{22}(r,t)\}, \end{aligned} \tag{8.8}$$

where J_{11} is the one in (6.10) and J_{21}, J_{22} are already defined by (6.11). Here we have

$$J_1(r,t) \le \int_{-k}^{t-r}\left(1+\sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right)\left(\frac{\beta+2k}{k}\right)^{-1}\left(\log\frac{\beta+3k}{k}\right)^{-p\nu} d\beta. \tag{8.9}$$

In order to estimate J_1, we need the following lemma.

LEMMA 8.1. *For $l<1$, $b>0$ and $b\ge a\ge -k$, it holds that*

$$J' \equiv \int_a^b \sqrt{\frac{b+2k}{b-\beta}}\left(\frac{\beta+2k}{k}\right)^{-1}\left(\log\frac{\beta+3k}{k}\right)^{-l} d\beta \le Ck\left(\log\frac{b+3k}{k}\right)^{1-l}. \tag{8.10}$$

PROOF. This is almost the same as Lemma 3.1. When $b\ge a\ge b/2-k$, we have

$$\frac{J'}{\sqrt{b+2k}} \le \left(\frac{b/2+k}{k}\right)^{-1}\left(\log\frac{b/2+2k}{k}\right)^{-l}\int_{b/2-k}^{b}\frac{d\beta}{\sqrt{b-\beta}} \tag{8.11}$$

which implies that

$$J' \le Ck\left(\log\frac{b/2+3k}{k}\right)^{-l}. \tag{8.12}$$

When $b/2-k\ge a$, we have

$$\begin{aligned} \frac{J'}{\sqrt{b+2k}} &\le \frac{1}{\sqrt{b/2+k}}\int_a^{b/2-k}\left(\frac{\beta+2k}{k}\right)^{-1}\left(\log\frac{\beta+3k}{k}\right)^{-l} d\beta \\ &\quad + \left(\frac{b/2+k}{k}\right)^{-1}\left(\log\frac{b/2+2k}{k}\right)^{-l}\int_{b/2-k}^{b}\frac{d\beta}{\sqrt{b-\beta}}. \end{aligned} \tag{8.13}$$

Therefore the lemma follows.

The application to J_1 is trivial by $l = p\nu < 1$. We note again that Remark 6.1 implies

$$1 - q\frac{p-3}{2} > 0 \quad \text{when } F(p,q) = 0 \text{ with } p \neq q. \tag{8.14}$$

Hence it follows from estimates in the previous case of $0 < \mu < 1/2$ that

$$\begin{aligned}
&J_{11}(r,t) \leq Ck, \\
&J_1(r,t) \leq Ck\left(\log\frac{T+3k}{k}\right)^{1-p\nu}, \\
&\left(\frac{t-r+2k}{k}\right)^{\mu-1/2}\left(\log\frac{t-r+3k}{k}\right)^{\nu} J_{21}(r,t) \leq Ck, \\
&\left(\frac{t-r+2k}{k}\right)^{\mu-1/2} J_{22}(r,t) \leq Ck.
\end{aligned} \tag{8.15}$$

The difference can be found in only the estimate for J_{21}. The logarithm term disappears by (7.24) with $p \neq q$.

Therefore we obtain

$$\begin{aligned}
&\|u_n\|_{S_{11}} \leq 2C_{f_1,g_1}\varepsilon, \\
&\|u_n\|_{S_{12}} \leq 2M_1\varepsilon^p, \\
&\|v_n\|_{S_1} \leq 2C_{f_2,g_2}\varepsilon, \\
&\|v_n\|_{S_2} \leq 2M\varepsilon^q
\end{aligned} \tag{8.16}$$

if the following inequalities hold.

$$\begin{aligned}
&Ck^2\left\{(2C_{f_2,g_2}\varepsilon)^p + (2M\varepsilon^q)^p\left(\log\frac{T+3k}{k}\right)^{1-p\nu}\right\}(2K_1\varepsilon^{-L_1})^{(4-p)/2} \leq C_{f_1,g_1}\varepsilon, \\
&2Ck^2(2M\varepsilon^q)^p\left(\log\frac{T+3k}{k}\right)^{1-p\nu} \leq M_1\varepsilon^p \\
&Ck^2\left\{(2C_{f_1,g_1}\varepsilon)^q + (2M_1\varepsilon^p)^q\left(\log\frac{T+3k}{k}\right)^{\nu}\right\}(K\varepsilon^{q-1})^{-1} \leq C_{f_2,g_2}\varepsilon, \\
&C_{f_2,g_2}K\varepsilon^q + Ck^2\left\{(2C_{f_1,g_1}\varepsilon)^q + (2M_1\varepsilon^p)^q\left(\log\frac{T+3k}{k}\right)^{\nu}\right\} \leq 2M\varepsilon^q.
\end{aligned} \tag{8.17}$$

Hence the boundedness of a sequence $\{(u_n, v_n)\}_{\mathbf{N}}$ in this weighted L^∞ space is obtained if (8.7) and (8.17) hold. By definition of all constants, one can see that (8.17) follows from

$$
\begin{aligned}
&(2M)^p\varepsilon^{p(q-1)}\left(\log\frac{T+3k}{k}\right)^{1-pv} \le \max\{(2C_{f_2,g_2})^p, (Ck^2)^{-1}M_1\},\\
&(2M_1)^q\varepsilon^{q(p-1)}\left(\log\frac{T+3k}{k}\right)^{v} \le \max\{(2C_{f_1,g_1})^q, (Ck^2)^{-1}M\}.
\end{aligned}
\tag{8.18}
$$

This completes the proof of Proposition 4.3 by Remark 4.4 because we can take

$$
\begin{aligned}
&\|u_0\|_{S_{11}} \le C_{f_1,g_1}\varepsilon,\\
&\|u_0\|_{S_{12}} \le M_1\varepsilon^p,\\
&\|v_0\|_{S_1} \le C_{f_2,g_2}\varepsilon,\\
&\|v_0\|_{S_2} \le M\varepsilon^q.
\end{aligned}
\tag{8.19}
$$

Next we shall estimate the differences under (8.17). Similarly to above case, by the definition of the each domain and (8.7), we have

$$
\begin{aligned}
\|u_{n+1}-u_n\|_{S_{11}} \le{}& 2^{p-1}(2K_1)^{(4-p)/2}\varepsilon^{1-p}\{pCk^2(2C_{f_2,g_2}\varepsilon)^{p-1}\|v_n-v_{n-1}\|_{S_1}\\
&+pCk^2(\log 2K_1\varepsilon^{-L_1})^{1-pv}(2M\varepsilon^q)^{p-1}\|v_n-v_{n-1}\|_{S_2}\}\\
\|u_{n+1}-u_n\|_{S_{12}} \le{}& 2^{p-1}pCk^2(2C_{f_2,g_2}\varepsilon)^{p-1}\|v_n-v_{n-1}\|_{S_1}\\
&+2^{p-1}pCk^2\left(\log\frac{T+3k}{k}\right)^{1-pv}(2M\varepsilon^q)^{p-1}\|v_n-v_{n-1}\|_{S_2}\\
\|v_{n+1}-v_n\|_{S_1} \le{}& 2^{q-1}K^{-1}\varepsilon^{1-q}\{qCk^2(2C_{f_1,g_1}\varepsilon)^{q-1}\|u_n-u_{n-1}\|_{S_{11}}\\
&+qCk^2(2M_1\varepsilon^p)^{q-1}\|u_n-u_{n-1}\|_{S_{12}}\}\\
\|v_{n+1}-v_n\|_{S_2} \le{}& 2^{q-1}qCk^2(2C_{f_1,g_1}\varepsilon)^{q-1}\|u_n-u_{n-1}\|_{S_{11}}\\
&+2^{q-1}qCk^2\left(\log\frac{T+3k}{k}\right)^{v}(2M_1\varepsilon^p)^{q-1}\|u_n-u_{n-1}\|_{S_{12}}.
\end{aligned}
\tag{8.20}
$$

Therefore a convergence of $\{(u_n, v_n)\}$ follows by (6.26) provided

$$
\begin{aligned}
&2^{p+q-2}pqC^2k^4(2K_1)^{(4-p)/2}K^{-1}(2C_{f_2,g_2})^{p-1}(2C_{f_1,g_1})^{q-1}\\
&\qquad + C\varepsilon^{p(q-1)}(\log 2K_1\varepsilon^{-L_1})^{1-pv} \le 4^{-1}\\
&C\varepsilon^{(p-1)(q-1)} + C\varepsilon^{1-p+q(p-1)+p(q-1)}(\log 2K_1\varepsilon^{-L_1})^{1-pv}\left(\log\frac{T+3k}{k}\right)^{v} \le 4^{-1}\\
&C\varepsilon^{p-1} + C\varepsilon^{pq-1}\left(\log\frac{T+3k}{k}\right)^{1-pv} \le 4^{-1}\\
&C\varepsilon^{q(p-1)} + C\varepsilon^{q(p-1)+p(q-1)}\left(\log\frac{T+3k}{k}\right)^{1-pv+v} \le 4^{-1}\\
&2^{p+q-2}pqC^2k^4(2K_1)^{(4-p)/2}K^{-1}(2C_{f_2,g_2})^{p-1}(2C_{f_1,g_1})^{q-1} + C\varepsilon^{q(p-1)} \le 4^{-1}\\
&C\varepsilon^{(p-1)(q-1)}(\log 2K_1\varepsilon^{-L_1})^{1-pv} + C\varepsilon^{1-q+p(q-1)+q(p-1)}\left(\log\frac{T+3k}{k}\right)^{1-pv} \le 4^{-1}\\
&C\varepsilon^{q-1} + C\varepsilon^{pq-1}\left(\log\frac{T+3k}{k}\right)^{v} \le 4^{-1}\\
&C\varepsilon^{p(q-1)}(\log 2K_1\varepsilon^{-L_1})^{1-pv} + C\varepsilon^{q(p-1)+p(q-1)}\left(\log\frac{T+3k}{k}\right)^{1-pv+v} \le 4^{-1}
\end{aligned}
\tag{8.21}
$$

where $C = C(f_1, f_2, g_1, g_2, p, q, k) > 0$ may be a different constant from each other.

Now we can fix K_1 and K to satisfy (6.6) and (8.6), so that the first term of the first and fifth inequalities in (8.21) is less than 8^{-1}. The other inequalities follow from (8.17) with a sufficiently small ε.

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