

## ON SOME PSEUDOSYMMETRY TYPE CURVATURE CONDITION

Dedicated to Professor Dr. Witold Roter on his seventieth birthday

By

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**Abstract.** We prove that semi-Riemannian manifolds satisfying some curvature condition of pseudosymmetry type are semisymmetric. We give also some inverse theorems.

### 1. Introduction

Recently, in [8] a curvature property of pseudosymmetry type of Einstein manifold has been found. Namely we have

**THEOREM 1.1** ([8], Theorem 3.1). *On any semi-Riemannian Einstein manifold  $(M, g)$ ,  $n \geq 4$ , the following identity is fulfilled*

$$R \cdot C - C \cdot R = \frac{\kappa}{(n-1)n} Q(g, R) = \frac{\kappa}{(n-1)n} Q(g, C).$$

For precise definition of the symbols used we refer to Section 2 of this paper. Recently, a review of results on curvature conditions of pseudosymmetry type was presented in [3].

Motivated by the above theorem we introduced in [8] a family of curvature conditions of pseudosymmetry type. In particular, in [8] we investigated curvature properties of non-Einstein and non-conformally flat semi-Riemannian manifolds of dimension  $\geq 4$  satisfying the condition

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(\*)<sub>1</sub> the tensors  $R \cdot C - C \cdot R$  and  $Q(g, C)$  are linearly dependent at every point of  $M$ .

It is clear that (\*)<sub>1</sub> is equivalent on the set  $\mathcal{U}_C = \{x \in M \mid C \neq 0 \text{ at } x\}$  to

$$R \cdot C - C \cdot R = L_1 Q(g, C),$$

where  $L_1$  is some function on  $\mathcal{U}_C$ . In this paper we study curvature properties of manifolds fulfilling the next condition

(\*)<sub>2</sub> the tensors  $R \cdot C - C \cdot R$  and  $Q(g, R)$  are linearly dependent at every point of  $M$ .

It is clear that (\*)<sub>2</sub> is equivalent on  $\mathcal{U}_R = \{x \in M \mid R - (\kappa/(n(n-1)))G \neq 0 \text{ at } x\}$  to

$$R \cdot C - C \cdot R = L_2 Q(g, R), \tag{1}$$

where  $L_2$  is some function on  $\mathcal{U}_R$ .

The paper is organized as follows. In Section 2 we fix the notations and present auxiliary results. Section 3 is devoted to quasi-Einstein manifolds satisfying (1). We prove that such a manifold always is Ricci-simple and fulfills a few more strong curvature identities. Section 4 deals with the non-quasi-Einstein case. It is shown that in this case (1) forces the very special form of the curvature tensor. Theorem 4.2 states that any manifold satisfying (1) is semisymmetric ( $R \cdot R = 0$ ; [1], [14]). Finally, in Section 5, basing on the inverse theorems (Proposition 3.2 and Proposition 4.2) we give examples of Ricci-simple as well as non-quasi-Einstein warped products satisfying (1).

## 2. Preliminaries

Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class  $C^\infty$ .

Let  $(M, g)$  be a connected  $n$ -dimensional,  $n \geq 3$ , semi-Riemannian manifold of class  $C^\infty$ . We denote by  $\nabla$ ,  $R$ ,  $C$ ,  $S$  and  $\kappa$  the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of  $(M, g)$ , respectively. The Ricci operator  $\mathcal{S}$  is defined by  $g(\mathcal{S}X, Y) = S(X, Y)$ , where  $X, Y \in \Xi(M)$ ,  $\Xi(M)$  being the Lie algebra of vector fields on  $M$ . More detailed information on the basical notions used in this paper we can find, for instance, in [1] and [13]. Further, we define the endomorphisms  $X \wedge_A Y$ ,  $\mathcal{R}(X, Y)$  and  $\mathcal{C}(X, Y)$  of  $\Xi(M)$  by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y,$$

$$\mathcal{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z,$$

$$\mathcal{C}(X, Y)Z = \mathcal{R}(X, Y)Z - \frac{1}{n-2} \left( X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y \right) Z,$$

respectively, where  $X, Y, Z \in \Xi(M)$  and  $A$  is a symmetric  $(0, 2)$ -tensor. Now the Riemann-Christoffel curvature tensor  $R$ , the Weyl conformal curvature tensor  $C$  and the  $(0, 4)$ -tensor  $G$  of  $(M, g)$  are defined by

$$R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4),$$

$$C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4),$$

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge_g X_2)X_3, X_4),$$

respectively, where  $X, Y, Z, X_1, X_2, \dots \in \Xi(M)$ .

Let  $\mathcal{B}(X, Y)$  be a skew-symmetric endomorphism of  $\Xi(M)$  and let  $B$  be a  $(0, 4)$ -tensor associated with  $\mathcal{B}(X, Y)$  by

$$B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2)X_3, X_4). \quad (2)$$

According to [12], the tensor  $B$  is said to be a generalized curvature tensor if the following conditions are fulfilled

$$B(X_1, X_2, X_3, X_4) + B(X_2, X_3, X_1, X_4) + B(X_3, X_1, X_2, X_4) = 0,$$

$$B(X_1, X_2, X_3, X_4) = B(X_3, X_4, X_1, X_2) = 0.$$

Clearly, the tensors  $R$ ,  $C$  and  $G$  are generalized curvature tensors. Further, for symmetric  $(0, 2)$ -tensors  $E$  and  $F$  we define their Kulkarni-Nomizu tensor  $E \wedge F$  by

$$\begin{aligned} (E \wedge F)(X_1, X_2, X_3, X_4) &= E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4) \\ &\quad - E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3). \end{aligned}$$

It is easy to see that  $E \wedge F$  is also a generalized curvature tensor.

Let  $\mathcal{B}(X, Y)$  be a skew-symmetric endomorphism of  $\Xi(M)$  and let  $B$  be the tensor defined by (2). We extend the endomorphism  $\mathcal{B}(X, Y)$  to derivation  $\mathcal{B}(X, Y) \cdot$  of the algebra of tensor fields on  $M$ , assuming that it commutes with contractions and  $\mathcal{B}(X, Y) \cdot f = 0$  for any smooth function on  $M$ . Now for a  $(0, k)$ -tensor field  $T$ ,  $k \geq 1$ , we can define the  $(0, k+2)$ -tensor  $B \cdot T$  by

$$\begin{aligned}
(B \cdot T)(X_1, \dots, X_k; X, Y) \\
&= (\mathcal{B}(X, Y) \cdot T)(X_1, \dots, X_k; X, Y) \\
&= -T(\mathcal{B}(X, Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, \mathcal{B}(X, Y)X_k).
\end{aligned}$$

In addition, if  $A$  is a symmetric  $(0, 2)$ -tensor field then we define the  $(0, k + 2)$ -tensor  $Q(A, T)$  by

$$\begin{aligned}
Q(A, T)(X_1, \dots, X_k; X, Y) \\
&= (X \wedge_A Y \cdot T)(X_1, \dots, X_k; X, Y) \\
&= -T((X \wedge_A Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, (X \wedge_A Y)X_k).
\end{aligned}$$

In particular, in this way, we obtain the  $(0, 6)$ -tensors  $B \cdot B$  and  $Q(A, B)$ .

Setting in the above formulas  $\mathcal{B} = \mathcal{R}$  or  $\mathcal{B} = \mathcal{C}$ ,  $T = R$  or  $T = C$  or  $T = S$ ,  $A = g$  or  $A = S$ , we get the tensors  $R \cdot R$ ,  $R \cdot C$ ,  $C \cdot R$ ,  $R \cdot S$ ,  $C \cdot S$ ,  $Q(g, R)$ ,  $Q(S, R)$ ,  $Q(g, C)$  and  $Q(g, S)$ .

We note that the tensor  $C$  can be presented also in the following form

$$C = R - \frac{1}{n-2}g \wedge S + \frac{\kappa}{(n-1)(n-2)}G.$$

Let  $(M, g)$  be covered by a system of charts  $\{W; x^k\}$ . We denote by  $g_{ij}$ ,  $R_{hijk}$ ,  $S_{ij}$ ,  $G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}$  and

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) + \frac{\kappa}{(n-1)(n-2)}G_{hijk} \quad (3)$$

the local components of the metric tensor  $g$ , the Riemann-Christoffel curvature tensor  $R$ , the Ricci tensor  $S$ , the tensor  $G$  and the Weyl tensor  $C$ , respectively. Further, we denote by  $S_{ij}^2 = S_{ir}S_j^r$  and  $S_i^j = g^{jr}S_{ir}$  the local components of the tensor  $S^2$  defined by  $S^2(X, Y) = S(\mathcal{S}X, Y)$ , and of the Ricci operator  $\mathcal{S}$ , respectively.

Let  $(R \cdot C)_{hijklm}$  and  $(C \cdot R)_{hijklm}$  denote the local components of the tensors  $R \cdot C$  and  $C \cdot R$ , respectively. Thus, by definition, we have

$$(R \cdot C)_{hijklm} = g^{rs}(C_{rijk}R_{shlm} + C_{hrjk}R_{sil m} + C_{hirk}R_{sjlm} + C_{hijr}R_{sklm}), \quad (4)$$

$$(C \cdot R)_{hijklm} = g^{rs}(R_{rijk}C_{shlm} + R_{hrjk}C_{sil m} + R_{hirk}C_{sjlm} + R_{hijr}C_{sklm}), \quad (5)$$

respectively. We have also the following identities

$$g^{rs}Q(g, R)_{hrsklm} = g_{hl}S_{km} + g_{kl}S_{hm} - g_{hm}S_{kl} - g_{km}S_{hl} = Q(g, S)_{hkml}, \quad (6)$$

$$g^{rs}Q(g, R)_{rijksl} = g_{kl}S_{ij} - g_{jl}S_{ik} - (n-1)R_{lijk}, \quad (7)$$

$$g^{rs}Q(S, R)_{hrsklm} = A_{lkhm} - A_{lhmk} - A_{mkhl} + A_{mhlk}, \quad (8)$$

$$g^{rs}Q(S, R)_{rijksl} = A_{lijk} - A_{iljk} - A_{jilk} - A_{kijl} - \kappa R_{lijk} + S_{kl}S_{ij} - S_{jl}S_{ik}, \quad (9)$$

$$A_{mijk} = S_m^s R_{sijk}. \quad (10)$$

Applying (3) in (4) and (5) we get

$$\begin{aligned} (R \cdot C)_{hijklm} &= (R \cdot R)_{hijklm} - \frac{1}{n-2} (g_{ij}(A_{hkml} + A_{khlm}) + g_{hk}(A_{ijlm} + A_{jilm}) \\ &\quad - g_{ik}(A_{hjlm} + A_{jhlm}) - g_{hj}(A_{iklm} + A_{kilm}), \\ (C \cdot R)_{hijklm} &= (R \cdot R)_{hijklm} - \frac{1}{n-2} Q(S, R)_{hijklm} + \frac{\kappa}{(n-1)(n-2)} Q(g, R)_{hijklm} \\ &\quad - \frac{1}{n-2} (g_{hl}A_{mijk} - g_{hm}A_{lijk} - g_{il}A_{mhjk} + g_{im}A_{lhjk} \\ &\quad + g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi}), \end{aligned} \quad (11)$$

where  $(R \cdot R)_{hijklm}$ ,  $Q(S, R)_{hijklm}$ ,  $Q(g, R)_{hijklm}$  and  $Q(g, C)_{hijklm}$  are the local components of the tensors  $R \cdot R$ ,  $Q(S, R)$ ,  $Q(g, R)$  and  $Q(g, C)$ , respectively. Using the two last identities we obtain

$$\begin{aligned} (n-2)(R \cdot C - C \cdot R)_{hijklm} &= Q(S, R)_{hijklm} - \frac{\kappa}{n-1} Q(g, R)_{hijklm} + g_{hl}A_{mijk} - g_{hm}A_{lijk} \\ &\quad - g_{il}A_{mhjk} + g_{im}A_{lhjk} + g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} \\ &\quad + g_{km}A_{ljhi} - g_{ij}(A_{hkml} + A_{khlm}) - g_{hk}(A_{ijlm} + A_{jilm}) \\ &\quad + g_{ik}(A_{hjlm} + A_{jhlm}) + g_{hj}(A_{iklm} + A_{kilm}). \end{aligned} \quad (12)$$

Contracting this with  $g^{ij}$  and using (6) and (8) we obtain

$$g^{rs}(R \cdot C - C \cdot R)_{hrsklm} = g_{hl}D_{km} + g_{kl}D_{hm} - g_{km}D_{hl} - g_{hm}D_{kl} - A_{hkml} - A_{khlm}, \quad (13)$$

where  $D = \frac{1}{n-2} \left( S^2 - \frac{\kappa}{n-1} S \right)$ .

Now we present some results which will be used in the next sections.

LEMMA 2.1 ([7], Lemma 2.2). *Let  $(M, g)$ ,  $n \geq 3$ , be a semi-Riemannian*

manifold. Let  $a$  be a nonzero covector and  $\mathcal{B}$  a generalized curvature tensor at a point  $x$  of  $M$  satisfying the equality  $Q(a \otimes a, \mathcal{B}) = 0$ . Then at  $x$  we have

$$\sum_{X, Y, Z} a(X) \mathcal{B}(Y, Z) = 0, \quad X, Y, Z \in T_x(M).$$

LEMMA 2.2 (cf. [6], Lemma 3.4). Let  $(M, g)$ ,  $n \geq 3$ , be a semi-Riemannian manifold. Let at a point  $x \in M$  be given a nonzero symmetric  $(0, 2)$ -tensor  $A$  and a generalized curvature tensor  $\mathcal{B}$  such that at  $x$  the following condition is satisfied:  $Q(A, \mathcal{B}) = 0$ . Moreover, let  $V$  be a vector at  $x$  such that the scalar  $\rho = a(V)$  is nonzero, where  $a$  is a covector defined by  $a(X) = A(X, V)$ ,  $X \in T_x(M)$ .

(i) If  $A = \frac{1}{\rho} a \otimes a$  then at  $x$  we have  $\sum_{X, Y, Z} a(X) \mathcal{B}(Y, Z) = 0$ , where  $X, Y, Z \in T_x(M)$ .

(ii) If  $A - \frac{1}{\rho} a \otimes a$  is nonzero then at  $x$  we have  $B = \frac{\gamma}{2} A \wedge A$ ,  $\gamma \in \mathbf{R}$ . Moreover, in both cases, at  $x$  we have  $B \cdot B = Q(\text{Ric}(\mathcal{B}), B)$ .

LEMMA 2.3 ([6], Lemma 3.5). Let  $(M, g)$ ,  $n \geq 3$ , be a semi-Riemannian manifold. Let at a point  $x \in M$  be given a symmetric  $(0, 2)$ -tensor  $A$  and two generalized curvature tensors  $\mathcal{B}_1$  and  $\mathcal{B}_2$  such that  $B_1 = \frac{1}{2} A \wedge A$  and  $B_2 = g \wedge A$ , respectively. Then at  $x$  we have  $Q(A, G) = -Q(g, B_2)$  and  $Q(A, B_2) = -Q(g, B_1)$ .

THEOREM 2.1 ([6], Theorem 4.2). Let  $(M, g)$ ,  $n \geq 3$ , be a semi-Riemannian manifold. If at a point  $x \in \mathcal{U}_S \cap \mathcal{U}_C$  its curvature tensor  $R$  is of the form  $R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \eta G$ ,  $\phi, \mu, \eta \in \mathbf{R}$ , then at  $x$  we have

$$R \cdot R = L_R Q(g, R) = Q(S, R) + \left( L_R + \frac{\mu}{\phi} \right) Q(g, C),$$

$$L_R = (n - 2) \left( \frac{\mu}{\phi} \left( \mu - \frac{1}{n - 2} \right) - \eta \right).$$

We finish this section with two algebraic lemmas.

LEMMA 2.4. Let  $F, E$  and  $D$  be symmetric  $(0, 2)$ -tensors on a semi-Riemannian manifold  $(M, g)$ ,  $n \geq 3$ , such that  $R \cdot F = Q(E, D)$  on  $M$ . Then on  $M$  we have

$$R(\mathcal{F} X_1, X, X_2, X_3) + R(\mathcal{F} X_2, X, X_3, X_1) + R(\mathcal{F} X_3, X, X_1, X_2) = 0,$$

where  $\mathcal{F}$  is the  $(1, 1)$ -tensor on  $M$  defined by  $g(\mathcal{F} X, Y) = F(X, Y)$ .

PROOF. In local coordinates our assumption takes the form

$$F_h^r R_{rijk} + F_i^r R_{rhjk} = E_{hj} D_{ik} + E_{ij} D_{hk} - E_{hk} D_{ij} - E_{ik} D_{hj},$$

where  $F_{hj}$ ,  $E_{hj}$  and  $D_{hj}$  are local components of the tensors  $F$ ,  $E$  and  $D$ , respectively, and  $F_h^r = g^{rs} F_{hs}$ . Summing the above equality cyclically in  $h, j, k$  we obtain

$$F_h^r R_{rijk} + F_j^r R_{rikh} + F_k^r R_{rihj} = 0,$$

completing the proof.

**LEMMA 2.5.** *Let  $(M, g)$ ,  $n \geq 4$ , be a semi-Riemannian manifold and  $A$  be the  $(0, 2)$ -tensor at  $x \in M$  defined by  $A = \alpha g + \beta w \otimes w$ ,  $w \in T_x^*(M)$ ,  $\alpha, \beta \in \mathbf{R}$ . If at  $x$  the curvature tensor  $R$  is expressed by  $R = \frac{\gamma}{2} A \wedge A$ ,  $\gamma \in \mathbf{R}$ , then the Weyl tensor  $C$  vanishes at  $x$ .*

**PROOF.** Our assumptions imply the equality

$$R_{hijk} = \alpha \gamma (\beta (g_{hk} w_i w_j + g_{ij} w_h w_k - g_{hj} w_i w_k - g_{ik} w_h w_j) + \alpha G_{hijk}). \quad (14)$$

Contracting (14) with  $g^{hk}$  we obtain

$$\alpha \beta \gamma w_i w_j = \frac{1}{n-2} S_{ij} - \left( \frac{n-1}{n-2} \alpha + \frac{\beta w^r w_r}{n-2} \right) \alpha \gamma g_{ij}.$$

Substituting this equality and (14) into (3) we get  $C_{hijk} = \lambda G_{hijk}$ , for some  $\lambda \in \mathbf{R}$ . From the last relation, by contraction with  $g^{hk}$  we get  $\lambda = 0$ , which reduces this relation to  $C = 0$ , completing the proof.

**REMARK 2.1.** *In the same manner we can prove that if at a point  $x \in M$  we have  $R = \gamma(g \wedge w \otimes w) + \eta G$  then  $C$  vanishes at  $x$ .*

### 3. Quasi-Einstein Manifolds

A semi-Riemannian manifold  $(M, g)$ ,  $n \geq 3$ , is said to be an *Einstein manifold* if on  $M$  we have  $S = \frac{\kappa}{n} g$ . Einstein manifolds form a natural subclass of the class of *quasi-Einstein manifolds*. A semi-Riemannian manifold  $(M, g)$ ,  $n \geq 3$ , is called a quasi-Einstein manifold if at every point  $x \in M$  we have  $S = \alpha g + \beta w \otimes w$ ,  $w \in T_x^*(M)$ ,  $\alpha, \beta \in \mathbf{R}$ . Another subclass of quasi-Einstein manifolds form *Ricci-simple manifolds*, i.e. semi-Riemannian manifolds having the Ricci tensor of rank at most one ([2]).

Let  $(M, g)$ ,  $n \geq 3$ , be a non-Einstein quasi-Einstein manifold. It is easy to see that on the set  $\mathcal{U}_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$  we have the following decom-

position of its Ricci tensor  $S = \alpha g + \beta w \otimes w$ , where  $w$  is a covector field on  $\mathcal{U}_S$  and  $\alpha$  and  $\beta$  are some function on  $\mathcal{U}_S$ . Evidently,  $\beta$  and  $w$  are nonzero at every point of  $\mathcal{U}_S$ . It is also easy to check that if  $S$  has on  $\mathcal{U}_S$  another decomposition of the form  $S = \tilde{\alpha}g + \tilde{\beta}v \otimes v$  then on  $\mathcal{U}_S$  we have  $\alpha = \tilde{\alpha}$  and  $v = \rho w$ , where  $\rho$  is some function on  $\mathcal{U}_S$ .

Let  $(M, g)$ ,  $n \geq 4$ , be a semi-Riemannian manifold. With respect to the Theorem 1.1, in the following we restrict our considerations to the subset  $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$ . It is clear that  $\mathcal{U} \subset \mathcal{U}_R$ .

**PROPOSITION 3.1.** *Let  $(M, g)$ ,  $n \geq 4$ , be a semi-Riemannian manifold satisfying on the set  $\mathcal{U} \subset M$  (1) and*

$$S = \alpha g + \beta w \otimes w, \quad (15)$$

where  $w$  is a 1-form on  $\mathcal{U}$  and  $\alpha, \beta$  are some functions on  $\mathcal{U}$ . Then  $w$  is an isotropic 1-form on  $\mathcal{U}$ .

**PROOF.** Applying to (1) the identity (12) we obtain

$$\begin{aligned} Q(S, R)_{hijklm} - \left( \frac{\kappa}{n-1} + (n-2)L_2 \right) Q(g, R)_{hijklm} + g_{hl}A_{mijk} - g_{hm}A_{lijk} \\ - g_{il}A_{mhjk} + g_{im}A_{lhjk} + g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi} \\ - g_{ij}(A_{hklm} + A_{khlm}) - g_{hk}(A_{ijlm} + A_{jilm}) + g_{ik}(A_{hjlm} + A_{jhlm}) \\ + g_{hj}(A_{iklm} + A_{kilm}) = 0. \end{aligned}$$

This, by making use of (15), yields

$$\begin{aligned} \left( 2\alpha - \frac{\kappa}{n-1} - (n-2)L_2 \right) Q(g, R)_{hijklm} + \beta Q(w \otimes w, R)_{hijklm} \\ + \beta(g_{hl}w_m w^s R_{sijk} - g_{hm}w_l w^s R_{sijk} - g_{il}w_m w^s R_{shjk} + g_{im}w_l w^s R_{shjk} + g_{jl}w_m w^s R_{skhi} \\ - g_{jm}w_l w^s R_{skhi} - g_{kl}w_m w^s R_{sjhi} + g_{km}w_l w^s R_{sjhi} - g_{ij}(w_h w^s R_{sklm} + w_k w^s R_{shlm}) \\ - g_{hk}(w_i w^s R_{sjlm} + w_j w^s R_{silm}) + g_{ik}(w_h w^s R_{sjlm} + w_j w^s R_{shlm}) \\ + g_{hj}(w_i w^s R_{sklm} + w_k w^s R_{silm})) = 0, \quad (16) \end{aligned}$$

where  $w^s = g^{rs}w_r$ . Contracting (16) with  $g^{ij}$  we get

$$(n-2)(w_h w^s R_{sklm} + w_k w^s R_{shlm}) = \beta \phi(g_{hl}w_k w_m + g_{kl}w_h w_m - g_{hm}w_k w_l - g_{km}w_h w_l), \quad (17)$$



which, by (15) yields

$$R \cdot S = \frac{\phi}{n-2} Q(g, S), \quad (18)$$

where

$$\phi = 2\alpha - \frac{\kappa}{n-1} - (n-2)L_2 + \beta^2 w^r w_r. \quad (19)$$

Further, transvecting (17) with  $w^h$  we obtain

$$w^r w_r w^s R_{sklm} = \frac{\phi\beta}{n-2} w^r w_r (w_m g_{kl} - w_l g_{km}). \quad (20)$$

Suppose that there exists a point  $x \in \mathcal{U}$  such that  $w^r w_r \neq 0$  at  $x$ .

Now we see that (20) turns into

$$w^s R_{sklm} = \phi_1 (w_m g_{kl} - w_l g_{km}), \quad (21)$$

where  $\phi_1 = \frac{\phi\beta}{n-2}$ . Thus (16) reduces to

$$Q\left(\left(2\alpha - (n-2)L_2 - \frac{\kappa}{n-1}\right)g + \beta w \otimes w, R\right) = 0. \quad (22)$$

Since  $\beta$  is nonzero at every point of  $\mathcal{U}$  we have two possibilities:

$$\text{rank}\left(\left(2\alpha - (n-2)L_2 - \frac{\kappa}{n-1}\right)g + \beta w \otimes w\right) = 1 \quad (23)$$

or

$$\text{rank}\left(\left(2\alpha - (n-2)L_2 - \frac{\kappa}{n-1}\right)g + \beta w \otimes w\right) > 1. \quad (24)$$

We suppose that (23) holds at  $x$ . Thus we have

$$\left(2\alpha - (n-2)L_2 - \frac{\kappa}{n-1}\right)g + \beta w \otimes w = \rho z \otimes z,$$

where  $z \in T_x^*(M)$  and  $\rho \in \mathbf{R}$ . The last relation implies

$$2\alpha - (n-2)L_2 - \frac{\kappa}{n-1} = 0.$$

Now (22) reduces to

$$Q(w \otimes w, R) = 0, \quad (25)$$

which, in view of Lemma 2.1, gives

$$w_l R_{hijk} + w_j R_{hikl} + w_k R_{hilj} = 0. \quad (26)$$

Transvecting this with  $w^l$  and using (21) we obtain

$$R_{hijk} = \phi_1(w^r w_r)^{-1}(w_h w_k g_{ij} + w_i w_j g_{hk} - w_j w_h g_{ik} - w_i w_k g_{hj}).$$

But this, in view of Remark 2.1, implies  $C = 0$ , a contradiction. Thus the case (23) cannot occur and (24) must be fulfilled at  $x$ . Now (22), in virtue of Lemma 2.2 gives

$$R = \frac{\rho}{2} A \wedge A, \quad A = \left( 2\alpha - (n-2)L_2 - \frac{\kappa}{n-1} \right) g + \beta w \otimes w, \quad \rho \neq 0.$$

Applying Lemma 2.5 we obtain again  $C = 0$ , a contradiction.

Therefore  $w$  is an isotropic 1-form on  $\mathcal{U}$ . This completes the proof.

**THEOREM 3.1.** *Let  $(M, g)$ ,  $n \geq 4$ , be a semi-Riemannian manifold satisfying (1) and (15). Then we have the following curvature identities on  $\mathcal{U}$ :*

$$S = \beta w \otimes w, \quad \kappa = 0,$$

$$R \cdot R = 0, \quad C \cdot R = 0, \quad \sum_{X, Y, Z} w(X) \mathcal{C}(Y, Z) = 0.$$

**PROOF.** To prove the first equality, in view of (15), we must show that  $\alpha = 0$  on  $\mathcal{U}$ . Taking into account Proposition 3.1 we observe that (15) and (19) imply

$$\alpha = \frac{\kappa}{n} \quad \text{and} \quad \phi = (n-2) \left( \frac{\alpha}{n-1} - L_2 \right).$$

First we assert that

$$L_2 = \frac{\alpha}{n-1} = \frac{\kappa}{n(n-1)} \quad (27)$$

on  $\mathcal{U}$ . Suppose that  $L_2 \neq \frac{\alpha}{n-1}$  (i.e.  $\phi \neq 0$ ) at some point  $x \in \mathcal{U}$ . From (16), by contraction with  $g^{hm}$ , we obtain

$$\begin{aligned} & \phi(-(n-1)R_{lij} + g_{kl}S_{ij} - g_{jl}S_{ik}) + \beta(w_l w^r R_{rij} - w_i w^r R_{rlj}) \\ & + w_j w^r R_{rik} + w_k w^r R_{ril} + w_l(w_k S_{ij} - w_j S_{ik}) + \beta(w_l w^r R_{rij}) \\ & - n w_l w^r R_{rij} - g_{jl}B_{ik} + g_{kl}B_{ij} - g_{ij}(B_{kl} - \alpha w_k w_l) \\ & - w_i w^r R_{rjl} - w_j w^r R_{ril} + g_{ik}(B_{jl} - \alpha w_j w_l) + w_i w^r R_{rkj} + w_k w^r R_{rik} = 0, \quad (28) \end{aligned}$$

where  $B_{ij} = w^r w^s R_{rijs}$ . Further, contracting (17) with  $g^{hm}$  and using (15) we get

$$B_{kl} = \rho w_k w_l, \quad \rho = \alpha - \frac{n\beta\phi}{n-2}. \quad (29)$$

Now, applying in (28) Lemma 2.4, (15) and (17) we find

$$\begin{aligned} (n-1)\phi R_{lijk} &= \alpha\phi G_{lijk} + \beta\phi(g_{kl}w_i w_j - g_{jl}w_i w_k) \\ &+ 2\alpha\beta(g_{ij}w_l w_k - g_{ik}w_l w_j) + \beta(-2w_i w^r R_{rljk} - n w_l w^r R_{rijk}) \\ &+ \beta\rho(g_{kl}w_i w_j - g_{jl}w_i w_k - g_{ij}w_k w_l + g_{ik}w_l w_j). \end{aligned} \quad (30)$$

This, by transvection with  $w^l$  and making use of  $w_r w^r = 0$ , turns into

$$\phi \left( w^r R_{rijk} - \frac{\alpha}{n-1} (w_k g_{ij} - w_j g_{ik}) \right) = 0,$$

whence

$$w^r R_{rijk} = \frac{\alpha}{n-1} (w_k g_{ij} - w_j g_{ik}).$$

Substituting this into (17) we get

$$\frac{n-2}{n-1} \alpha = \beta\phi, \quad (31)$$

whence it follows that  $\alpha$  is nonzero at  $x$ . Now (29) yields  $B_{ij} = -\frac{\alpha}{n-1} w_i w_j$  and  $\rho = -\frac{\alpha}{n-1}$ . Applying this and (31) to (30) we obtain

$$R_{lijk} = \alpha G_{lijk} + \frac{n-1}{n-2} \beta^2 (g_{ij}w_l w_k + g_{lk}w_i w_j - g_{ik}w_l w_j - g_{lj}w_i w_k).$$

But this, in view of Lemma 2.5, implies  $C = 0$  at  $x$ , a contradiction.

Thus we have (27). Now (17) reduces to

$$w_h w^r R_{rklm} + w_k w^r R_{rhlm} = 0$$

which implies  $w^r R_{rklm} = 0$ . Further, applying (27) and the above equality to (16) and (18) we obtain

$$R \cdot S = 0, \quad (32)$$

(25) and, consequently, (26). But (26), in view of Theorem 1 of [5], implies  $R \cdot R = Q(S, R)$  which, by (25), turns into  $R \cdot R = \alpha Q(g, R)$ . This, by a suitable contraction, gives  $R \cdot S = \alpha Q(g, S)$ , whence, by (32), we get  $\alpha Q(g, S) = 0$ , and in

a consequence,  $\alpha = 0$ . This, in virtue of (27) implies  $\kappa = 0$  and  $L_2 = 0$ . As we have seen in the last part of the previous proof we obtain also (26) and  $R \cdot R = 0$ . This equality, by  $L_2 = 0$  and (1) implies  $C \cdot R = 0$ . Finally, (26), by making use of  $\kappa = 0$  and  $S = \beta w \otimes w$ , leads to the last identity of our assertion. This completes the proof.

Taking into account Proposition 4.2 of [8] we immediately obtain the following inverse statement

**PROPOSITION 3.2.** *Let  $(M, g)$ ,  $n \geq 4$ , be a semi-Riemannian manifold satisfying*

$$S = \beta w \otimes w, \quad \kappa = 0,$$

$$\sum_{X, Y, Z} w(X) \mathcal{C}(Y, Z) = 0.$$

*Then on the set  $\mathcal{U}$  we have  $R \cdot R = 0$ ,  $C \cdot R = 0$ , and consequently  $R \cdot C - C \cdot R = 0$ .*

#### 4. Non-Quasi-Einstein Manifolds

First we prove some general formula for semi-Riemannian manifolds satisfying  $(*)_2$ .

**PROPOSITION 4.1.** *Let  $(M, g)$ ,  $n \geq 4$ , be a semi-Riemannian manifold fulfilling  $(*)_2$ . Then on the set  $\mathcal{U}$  the following identity is satisfied*

$$Q(S, R) - Q\left(\left(\frac{\kappa}{n-1} - L_2\right)g, R\right) + \frac{1}{2(n-2)} Q(g, S \wedge S) = 0. \quad (33)$$

**PROOF.** We can write (1) in the form

$$(n-2)(R \cdot C - C \cdot R)_{hijklm} = (n-2)L_2 Q(g, R)_{hijklm}. \quad (34)$$

Contracting this with  $g^{ij}$ , in virtue of (13) and (6), we get

$$A_{hklm} + A_{khlm} = g_{hl} F_{km} + g_{kl} F_{hm} - g_{hm} F_{kl} - g_{km} F_{hl}, \quad (35)$$

where

$$F = \frac{1}{n-2} S^2 - \left(\frac{\kappa}{(n-1)(n-2)} + L_2\right) S. \quad (36)$$

Further, summing cyclically (35) in  $h, l, m$  we get

$$A_{hklm} + A_{lkmh} + A_{mkhl} = 0.$$

Contracting now (34) with  $g^{hm}$  and using the above equality, (7) and (9) we obtain

$$\begin{aligned} (n-2)A_{lijk} &= S_{kl}S_{ij} - S_{lj}S_{ik} + g_{kl}\left(E_{ij} - \frac{\kappa}{n-1}S_{ij} - (n-2)L_2S_{ij} + 2F_{ij}\right) \\ &\quad - g_{lj}\left(E_{ik} - \frac{\kappa}{n-1}S_{ik} - (n-2)L_2S_{ik} + 2F_{ik}\right) + g_{ik}(E_{jl} - S_{jl}^2 + 2F_{jl}) \\ &\quad - g_{ij}(E_{lk} - S_{lk}^2 + 2F_{lk}) + (n-1)(n-2)L_2R_{lijk}. \end{aligned}$$

Contracting (35) with  $g^{hm}$  we have  $E - S^2 = \text{tr}(F)g - nF$  and taking into account (36) we get

$$E - \frac{\kappa}{n-1}S - (n-2)L_2S + 2F = \text{tr}(F)g, \quad E - S^2 + 2F = \text{tr}(F)g - (n-2)F.$$

Thus

$$A_{lijk} = \frac{1}{n-2}(S_{kl}S_{ij} - S_{lj}S_{ik}) + g_{ij}F_{lk} - g_{ik}F_{lj} + (n-1)L_2R_{lijk}. \quad (37)$$

Substituting (34) into (12) we get

$$\begin{aligned} Q(S, R)_{hijklm} - \frac{\kappa}{n-1}Q(g, R)_{hijklm} - (n-2)L_2Q(g, R)_{hijklm} \\ = g_{ij}(A_{hklm} + A_{hkml}) + g_{hk}(A_{ijlm} + A_{jilm}) - g_{ik}(A_{hjlm} + A_{jhlm}) - g_{hj}(A_{iklm} + A_{kilm}) \\ - (g_{hl}A_{mijk} - g_{hm}A_{lijk} - g_{il}A_{mhjk} + g_{im}A_{lhjk} + g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjni} \\ + g_{km}A_{ljni}). \end{aligned}$$

In view of (35) and (37) the right-hand side of this equality is equal to

$$-\frac{1}{n-2}Q\left(g, \frac{1}{2}S \wedge S\right) - (n-1)L_2Q(g, R).$$

But this completes the proof.

**THEOREM 4.1.** *Let  $(M, g)$ ,  $n \geq 4$ , be a non-quasi-Einstein semi-Riemannian manifold fulfilling (1). Then at every point  $x \in \mathcal{U}$  the curvature tensor  $R$  is of the form*

$$R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \eta G, \quad \phi, \mu, \eta \in \mathbf{R}, \quad (38)$$

where

$$\mu \left( \mu - \frac{1}{n-2} \right) = \phi \eta. \quad (39)$$

Consequently,  $R \cdot R = 0$  on  $\mathcal{U}$ .

PROOF. Let  $x \in \mathcal{U}$ . We consider two cases.

(A)  $L_2 = \frac{\kappa}{n-1}$ .

Applying now Lemma 2.3 we obtain  $Q(S, R) = -\frac{1}{2(n-2)} Q(g, S \wedge S) = \frac{1}{n-2} Q(S, g \wedge S)$  and

$$Q \left( S, R - \frac{1}{n-2} g \wedge S \right) = 0.$$

Taking into account Lemma 2.2 and our assumption we get  $\text{rank } S > 1$  and, consequently, also

$$R = \frac{\gamma}{2} S \wedge S + \frac{1}{n-2} g \wedge S. \quad (40)$$

(B)  $\frac{\kappa}{n-1} - L_2 \neq 0$ .

Denoting  $\tau = \frac{\kappa}{n-1} - L_2$ , in view of the obvious identity  $Q(S, S \wedge S) = 0$ , (33) can be written in the form

$$Q \left( S - \tau g, R - \frac{1}{2\tau(n-2)} S \wedge S \right) = 0.$$

As in the previous case, in view of Lemma 2.2, we obtain  $\text{rank}(S - \tau g) > 1$  and

$$R - \frac{1}{2\tau(n-2)} S \wedge S = \frac{\gamma}{2} (S - \tau g) \wedge (S - \tau g).$$

The last identity can be written in the form

$$R = \frac{1}{2} \left( \gamma + \frac{1}{\tau(n-2)} \right) S \wedge S - \gamma \tau g \wedge S + \gamma \tau^2 G.$$

It is obvious that (40) and the last equality are of the form (38). Moreover, it is easy to see that in both cases (39) is also satisfied. Applying now Theorem 2.1 we see that  $R \cdot R = 0$ . This completes the proof.

We have also the following inverse statement.

**PROPOSITION 4.2.** *Let  $(M, g)$ ,  $n \geq 4$ , be a semi-Riemannian manifold. If at a point  $x \in \mathcal{U}$  the curvature tensor  $R$  is of the form (38) and (39) is satisfied then at  $x$  we have  $R \cdot R = 0$  and  $C \cdot R = -L_2 Q(g, R)$ ,  $L_2 = \frac{\kappa}{n-1} + \frac{(n-2)\mu-1}{(n-2)\phi}$ . Consequently,  $R \cdot C - C \cdot R = L_2 Q(g, R)$  at  $x$ .*

**PROOF.** First we observe that (38) and (39), in view of Theorem 2.1, imply  $R \cdot R = 0$ . (38) in local coordinates takes the form

$$R_{hijk} = \phi(S_{hk}S_{ij} - S_{hj}S_{ik}) + \mu(g_{hk}S_{ij} + g_{ij}S_{hk} - g_{hj}S_{ik} - g_{ik}S_{hj}) + \eta(g_{hk}g_{ij} - g_{hj}g_{ik}). \quad (41)$$

Contracting (41) with  $g^{ij}$  we get

$$S^2 = \alpha S + \beta g, \quad \alpha = \kappa + \frac{(n-2)\mu-1}{\phi}, \quad \beta = \frac{\mu\kappa + (n-1)\eta}{\phi}. \quad (42)$$

Transvecting now (41) with  $S_i^h$  and using (10), (42) and the equality  $\alpha\mu + \eta = \beta\phi$ , which is equivalent to (39), we get

$$A = \frac{\alpha\phi + \mu}{2} S \wedge S + \beta\phi g \wedge S + \beta\mu G.$$

Substituting this into (11) and using (38),  $R \cdot R = 0$  and Lemma 2.3, after standard but somewhat lengthy calculations, we get

$$C \cdot R = -\left(\frac{\kappa}{n-1} + \frac{(n-2)\mu-1}{(n-2)\phi}\right) Q(g, R).$$

This completes the proof.

**REMARK 4.1.** *We note that if at  $x \in M$  the curvature tensor  $R$  and the Ricci tensor  $S$  are of the form (38) and (15), respectively, then (see Remark 2.1) the Weyl conformal curvature tensor  $C$  vanishes at  $x$ . Therefore, if  $x \in \mathcal{U}$  then  $S$  cannot be of the form (15).*

Taking into account Theorem 3.1 and Theorem 4.1 we obtain

**THEOREM 4.2.** *Let  $(M, g)$ ,  $n \geq 4$ , be a semi-Riemannian manifold fulfilling (1). Then  $R \cdot R = 0$  on  $\mathcal{U}$ .*

### 5. Examples

We present examples of semisymmetric warped product manifolds satisfying  $(*)_2$ .

Let now  $(\bar{M}, \bar{g})$  and  $(\tilde{N}, \tilde{g})$ ,  $\dim \bar{M} = p$ ,  $\dim \tilde{N} = n - p$ ,  $1 \leq p < n$ , be semi-Riemannian manifolds covered by systems of charts  $\{\bar{U}; x^a\}$  and  $\{\tilde{V}; y^\alpha\}$ , respectively. Let  $F$  be a positive smooth function on  $\bar{M}$ . The warped product  $\bar{M} \times_F \tilde{N}$  of  $(\bar{M}, \bar{g})$  and  $(\tilde{N}, \tilde{g})$  is the product manifold  $\bar{M} \times \tilde{N}$  with the metric  $g = \bar{g} \times_F \tilde{g}$ , defined by  $\bar{g} \times_F \tilde{g} = \pi_1^* \bar{g} + (F \circ \pi_1) \pi_2^* \tilde{g}$ , where  $\pi_1 : \bar{M} \times \tilde{N} \rightarrow \bar{M}$  and  $\pi_2 : \bar{M} \times \tilde{N} \rightarrow \tilde{N}$  are the natural projections on  $\bar{M}$  and  $\tilde{N}$ , respectively. Let  $\{\bar{U} \times \tilde{V}; x^1, \dots, x^p, x^{p+1} = y^1, \dots, x^n = y^{n-p}\}$  be a product chart for  $\bar{M} \times \tilde{N}$ . The local components of the metric  $g = \bar{g} \times_F \tilde{g}$  with respect to this chart are the following  $g_{hk} = \bar{g}_{ab}$  if  $h = a$  and  $k = b$ ,  $g_{hk} = F \tilde{g}_{\alpha\beta}$  if  $h = \alpha$  and  $k = \beta$ , and  $g_{hk} = 0$  otherwise, where  $a, b, c, d \in \{1, \dots, p\}$  and  $\alpha, \beta \in \{p+1, \dots, n\}$ . We will denote by bars (resp., by tildes) tensors formed from  $\bar{g}$  (resp.,  $\tilde{g}$ ). For more detailed information about warped products see to [13].

**EXAMPLE 5.1** (see [4], Example 4.1, Example 5.1 and Corollary 5.2). *Let  $(\tilde{N}, \tilde{g})$ , be a 1-dimensional Riemannian manifold. Let  $\bar{M}$  be a non-empty open connected subset of  $\mathbf{R}^p$ ,  $p = n - 1 \geq 3$ , equipped with the standard metric  $\bar{g}$ ,  $\bar{g}_{ab} = \varepsilon_a \delta_{ab}$ ,  $\varepsilon_a = \pm 1$ . We set  $F = F(x^1, \dots, x^p) = k \exp(\xi_a x^a)$ , where  $\xi_1, \dots, \xi_p$  and  $k$  are constants such that  $\xi_1^2 + \dots + \xi_p^2 > 0$ ,  $\bar{g}^{ab} \xi_a \xi_b = 0$  and  $k > 0$ . We consider the warped product  $\bar{M} \times_F \tilde{N}$ . Now the formulas (22)–(25) of [4] turn into*

$$\begin{aligned} R_{abcd} &= 0, & R_{nabn} &= -\frac{1}{4} \xi_a \xi_b g_{nn}, & S_{ab} &= -\frac{1}{4} \xi_a \xi_b, & S_{nn} &= 0, \\ \kappa &= 0, & T_{ab} &= \frac{F}{2} \xi_a \xi_b, & \operatorname{tr}_{\tilde{g}}(T) &= 0, & \Delta_1 F &= 0, \end{aligned} \quad (43)$$

respectively. The local components of the Weyl tensor  $C_{rstu}$  of  $\bar{M} \times_F \tilde{N}$  are the following

$$\begin{aligned} C_{abcd} &= \frac{1}{4(n-2)} (g_{ad} \xi_b \xi_c + g_{bc} \xi_a \xi_d - g_{ac} \xi_b \xi_d - g_{bd} \xi_a \xi_c), \\ C_{annn} &= -\frac{n-3}{4(n-2)} \xi_a \xi_d g_{nn}. \end{aligned} \quad (44)$$

Using these formulas we can check that on  $\bar{M} \times_F \tilde{N}$  we have  $R \cdot R = C \cdot R = 0$ .

**EXAMPLE 5.2** ([11], Example 3.1). *Let  $\bar{M}$  be a nonempty open connected*



subset of  $\mathbf{R}^p$ ,  $p \geq 2$ , equipped with the standard metric  $\bar{g}$ ,  $\bar{g}_{ab} = \varepsilon_a \delta_{ab}$ ,  $\varepsilon_a = \pm 1$ , where  $a, b \in \{1, \dots, p\}$ . We set  $F = F(x^1, \dots, x^p) = k \exp(\xi_a x^a)$ , where  $k, \xi_1, \dots, \xi_p \in \mathbf{R}$ ,  $\xi_1^2 + \dots + \xi_p^2 > 0$  and  $k > 0$ . Further, let  $\tilde{N}$  be a nonempty open connected subset of  $\mathbf{R}^{n-p}$ ,  $n \geq 4$ , equipped with the standard metric  $\tilde{g}$ ,  $\tilde{g}_{\alpha\beta} = \varepsilon_\alpha \delta_{\alpha\beta}$ ,  $\varepsilon_\alpha = \pm 1$ , where  $\alpha, \beta \in \{p+1, \dots, n\}$ . We consider the warped product  $\bar{M} \times_F \tilde{N}$  of the manifolds  $(\bar{M}, \bar{g})$  and  $(\tilde{N}, \tilde{g})$  with the warping function  $F$  defined above. Using again (22)–(25) of [4] we can check that on  $\bar{M} \times_F \tilde{N}$  we have

$$R_{abcd} = 0, \quad R_{\alpha bc\beta} = -\frac{1}{4} \xi_b \xi_c g_{\alpha\beta}, \quad S_{ab} = -\frac{n-p}{4} \xi_a \xi_b$$

$$S_{\alpha\beta} = \left( -\frac{\text{tr } T}{2} - \frac{n-p-1}{4F} \Delta_1 F \right) \tilde{g}_{\alpha\beta}, \quad T_{ab} = \frac{F}{2} \xi_a \xi_b,$$

$$\Delta_1 F = F^2 \xi^f \xi_f, \quad \text{tr}_{\tilde{g}}(T) = \frac{F}{2} \xi^f \xi_f, \quad \kappa = -\frac{(n-p)(n-p+1)}{4} \xi^f \xi_f, \quad (45)$$

where  $\xi^f = \bar{g}^{ef} \xi_e$ . If  $p = 1$  then  $\bar{M} \times_F \tilde{N}$  is a conformally flat manifold. If  $p \geq 2$  then  $\bar{M} \times_F \tilde{N}$  is a non-conformally flat semisymmetric manifold. Further, using the above relations we can easily check that on  $\bar{M} \times_F \tilde{N}$  we have

$$\kappa R = \frac{n-p+1}{2(n-p)} S \wedge S. \quad (46)$$

It is clear that there exist the constants  $\varepsilon$ ,  $\varepsilon_a$ , and  $\xi_a$  such that  $\xi^f \xi_f$  is non-zero. Thus the scalar curvature  $\kappa$  of  $\bar{M} \times_F \tilde{N}$  is nonzero. From (46), in view of Theorem 3.2 of [10], it follows that on  $\bar{M} \times_F \tilde{N}$  we have  $C \cdot R = -(p-2)\kappa / ((n-2)(n-1)(n-p+1))Q(g, R)$ .

We note that semisymmetric manifolds satisfying  $C \cdot R = 0$  as well as warped products fulfilling  $R = \frac{\beta}{2} S \wedge S$  were investigated in [9] and [10], respectively.

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